

Computability in group theory – DRAFT

I. Scott

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1 Background

We assume familiarity with basic model theory, computability theory, and group theory. For more details on these, we recommend [CK90], [Soa16], and [LS15] respectively.

We will adhere to the following conventions throughout the class:

- Models are implicitly assumed to be countable.
- Unless otherwise mentioned, “computable” and its derivatives will always refer to Turing computability.
- L is the language $\{1, \cdot, \bullet^{-1}\}$ for group theory. (Here $\bullet^{-1}$ indicates that we use the notation g^{-1} for “ g -inverse”.)

Discussion. We do need to include a constant symbol for 1. We do not need to include a function symbol for inversion, but it will simplify our arguments without losing any generality.

Most notation and definitions are standard; however, we point out that we use “computably presentable” to refer to computable structures, rather than group presentations (see Definitions 1.13 and 1.24).

1.1 Notation for formulas

Definition 1.1. We establish the following conventions for L -formulas. The only language we will refer to is the language of group theory, $L = \{1, \cdot, \bullet^{-1}\}$.

- An *atomic formula* is a quantifier-free formula $\varphi(\bar{x})$ which does not include conjunctions, disjunctions, or negations. Relative to the axioms of group theory, any atomic formula in the language of group theory is equivalent to “ $w(\bar{x}) = e$ ”, where $w(\bar{x})$ is a *word* in $\bar{x}$ and its inverses.
- If $\varphi(\bar{x}) = “\bigwedge_{i \in I} \psi_i(\bar{x}) \wedge \bigwedge_{j \in J} \neg \psi_j(\bar{x})”$, where I and J are finite, and $(\psi_i)_{i \in I}$ and $(\psi_j)_{j \in J}$ are atomic formulas, then we write

$$\varphi^+(\bar{x}) = \{\psi_i(\bar{x}) : i \in I\}$$

and call this the *positive part* of $\varphi(\bar{x})$ or the *equations* of $\varphi(\bar{x})$.

Analogously, we write

$$\varphi^-(\bar{x}) = \{\psi_j(\bar{x}) : j \in J\}$$

and call this the *negative part* of $\varphi(\bar{x})$ or the *inequations* of $\varphi(\bar{x})$.

- If $\Phi = \{\varphi_i : i < n\}$ is a finite set of formulas, we write $\bigwedge \Phi$ for $\bigwedge_{i < n} \varphi_i$.
- Let M be an L -structure. For a quantifier-free $\varphi(\bar{x}, \bar{m})$ with parameters $\bar{m} \in M$, say the L -structure N *satisfies* φ over M if $N \geq M$ and $N \models \exists \bar{x} \varphi(\bar{x}, \bar{m})$. For $\bar{n} \in N$ with $N \models \varphi(\bar{n}, \bar{m})$, we will say that $\bar{n}$ *satisfies* $\varphi(\bar{x}, \bar{m})$ in N .

The following easy proposition will simplify our arguments showing that a group M is existentially closed. It says that to check existential closure, it suffices to check finite systems of equations and inequations.

Proposition 1.2. *For a group M , the following are equivalent:*

1. *M is existentially closed; i.e., for every quantifier-free formula $\varphi(\bar{x}, \bar{m})$ with parameters from M , if there is a group $N \geq M$ with $N \models \exists \bar{x} \varphi(\bar{x}, \bar{m})$, then $M \models \exists \bar{x} \varphi(\bar{x}, \bar{m})$.*
2. *For every conjunction $\varphi(\bar{x}, \bar{m})$ of atomic and negated atomic formulas with parameters from M , if there is a group $N \geq M$ with $N \models \exists \bar{x} \varphi(\bar{x}, \bar{m})$, then $M \models \exists \bar{x} \varphi(\bar{x}, \bar{m})$.*

Proof sketch. Clearly (1) implies (2). For the other direction, let $\varphi(\bar{x}, \bar{m}) = \bigvee_{i < n} \varphi_i(\bar{x}, \bar{m})$ be a quantifier-free formula in disjunctive normal form which is realised in some group $N \geq M$. Thus, N models at least one of the disjuncts, say $\varphi_i(\bar{x}, \bar{m})$. Applying (2) to this φ_i , we obtain that $M \models \exists \bar{x} \varphi_i(\bar{x}, \bar{m})$ and hence $M \models \exists \bar{x} \varphi(\bar{x}, \bar{m})$. $\square$

1.2 Computability Theory

While the details of the model of computation will not be directly relevant to this course, it is important to be conversant in the basics of computability theory.

1.2.1 Computation

Definition 1.3. A *Turing machine* is a tuple $M = (Q, S, \delta)$ where $Q \neq \emptyset$ is a finite set of *states*, $S \neq \emptyset$ is a finite set of *symbols*, and $\delta : Q \times S \rightarrow Q \times S \times \{R, L\}$ is a function.

The intuition is that M codes a machine with a single head reading a single cell in a single tape of order type $\mathbb{Z}$. Each cell contains a symbol from S . The head may be in different “states”, corresponding in human terms to “states of mind”, which determine how the head interprets the symbol it is reading. This is formalised by δ , which can be thought of as M ’s “programme”: $(q, s, q', s', \varepsilon)$ is interpreted as specifying that if the reading head is in state q , observing symbol s , then it transitions to state q' , overwrites s with s' and moves one cell to the ε .

We need to fix conventions. Our eventual notions of “computability” and “relative computability” will be independent of the choice of these conventions, but we cannot proceed without making some decisions.

Convention 1.4. Henceforth, $S = \{0, 1\}$. We say that a configuration of the tape *represents* n , if it consists of $n + 1$ adjacent 1s and 0s everywhere else.

We call such a configuration a *standard configuration*.

Definition 1.5. Fix a $q_i \in Q$ called the *initial state* and a $q_f \in Q$ called the *halting state*. We allow $q_i = q_f$. M *halts on input* n if, M , when started in state q_i , with its reading head on the leftmost 1 in the standard configuration for n , eventually enters the state q_f . If, upon entering q_f for the first time, M is in the standard configuration for m , we say that M *halts on input* n *with output* m . This is written

$$M(n) \downarrow = m.$$

Notice that there is at most one such m . If $\exists m (M(n) \downarrow = m)$, we write $M(n) \downarrow$.

If M does not halt, or, when it enters q_f for the first time, the tape is not in a standard configuration, write

$$M(n) \uparrow$$

and say that $M(n)$ *is undefined*.

Say that M *represents the function*

$$M(n) = \begin{cases} m & \text{if } M(n) \downarrow = m \\ \text{undefined} & \text{otherwise} \end{cases}$$

Notice that M is a (partial) function.

We now arrive at what it means to be computable.

Definition 1.6. A partial function $f : \omega \rightarrow \omega$ is *computable* if there is a Turing machine M which represents f . Confusingly, we call such functions *partial computable functions* (rather than the more correct, computable partial functions).

A set X is *computable* if χ_X is computable.

Note that we can easily upgrade to functions from ω^n to ω^m . We leave the details to the reader.

Now, each partial computable function corresponds to a Turing machine, which is just a triple (Q, S, δ) satisfying a few simple properties. After fixing some Gödel coding for tuples, we obtain an enumeration $(\Phi_e)_{e < \omega}$ of all Turing machines. (More precisely, given some $n \in \omega$, we can determine if it is the Gödel code of a Turing machine or not; and, if so, we can “follow” the computation of the machine on a given input). Thus, we have the following.

Theorem 1.7 (Enumeration Theorem). *There is a partial computable $f : \omega^2 \rightarrow \omega$ such that $f(e, x) = \Phi_e(x)$. Thus, in particular, there is i such that $f(e, x) = \Phi_i(e, x)$.*

Remark. There is no such effective enumeration of all *total* computable functions.

The Enumeration Theorem proves the existence of a *universal Turing machine* – i.e., a Turing machine which simulates all Turing machines.

Henceforth, we shall be fairly lax about checking that functions are computable. While one could in principle check that any function that we assert is computable is in fact computable, we will often as best resort to intuitive arguments about computation, rather than referring to the specific model.

To build intuition, we show that unbounded searches can be carried out by Turing machines. More formally, we have the following.

Proposition 1.8. *Let $\theta(x, y) : \omega^2 \rightarrow \omega$ be computable. Then*

$$\psi(x) = \mu y (\theta(x, y) \downarrow = 1 \wedge \forall z < y (\theta(x, z) \downarrow \neq 1)).$$

is also partial computable.

Proof. Fix x . Run $\theta(x, y)$ on each y in turn. If it ever halts and gives 1, output this y . Otherwise, the machine will run forever on input x , so $\psi(x) \uparrow$. $\square$

1.2.2 Enumeration

We wish to distinguish between objects which can be computed, and objects which can be “effectively enumerated”. As a starting point, consider the proposition at the end of the last section. Given x , we could tell if a given y satisfies $\psi(x) \downarrow = y$. Thus, if $\psi(x) \downarrow$, we are able to effectively determine that this is the case in a finite amount of time. However, if $\psi(x) \uparrow$, we may never know; we may think that we just have not yet seen the corresponding y .

Definition 1.9. Let $\psi : \omega \rightarrow \omega$ be a partial computable function. Write

$$\text{dom}(\psi) = \{x \in \omega \mid \psi(x) \downarrow\}.$$

Say that $W \subseteq \omega$ is *computably enumerable* if there is $e < \omega$ such that

$$W = \Phi_e.$$

Notice that this gives an effective enumeration $(W_e)_{e < \omega}$ of the c.e. sets. In particular, the following set is c.e.:

$$K_0 = \{\langle e, x \rangle \mid x \in W_e\}$$

This is a very important set. It’s our first (explicit) example of a set which is not computable. We do this by showing that a seemingly simpler set K is already not computable, and then arguing that if we could compute K_0 , then we could compute K .

$$K = \{x \in \omega \mid x \in W_x\}$$

K is called the *halting set*.

Theorem 1.10. *K and K_0 are not computable.*

Proof. Suppose that K were computable. Then by diagonalisation, we may produce a function f that is partial computable, but is not Φ_e for any e .

$$f(x) = \begin{cases} \Phi_x(x) + 1 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

This is a contradiction.

Further, if K_0 were computable, then we could compute K : $x \in K$ iff $\langle x, x \rangle \in K_0$. $\square$

This argument: that from K_0 , we can compute K , suggests that computation can be *relativised*.

1.2.3 Relativisation

We now develop a theory of relative computation. Intuitively, X is *reducible to* Y if there is a Turing machine, which “given Y as an oracle”, can compute X .

This can be formalised by, for example, adding a second, read-only tape to our models of Turing machines, together with a second head. The characteristic function of Y will be on the second tape. We will again have a nonempty set Q of states, and now two sets of symbols: $S_1 = \{1, B\}$ for the “working tape” and $S_2 = \{B, 0, 1\}$ for the “oracle tape”. The programme will be a function $\delta : Q \times S_1 \times S_2 \rightarrow S_1 \times S_2 \times \{R, L\} \times \{R, L\}$. *Halting*, etc are defined as before.

If machine Φ , on input n , and with oracle Y , halts in the standard configuration for m , we write

$$\Phi^Y(n) \downarrow = m.$$

We may similarly define $\Phi(n) \uparrow$, etc. Thus, we may define what it means to be computable relative to Y .

Definition 1.11. A partial function f is *computable relative to* Y (written $X \leq_T Y$) if there is a Turing machine Φ such that $\Phi^Y = f$.

X is *computable relative to* Y or *reducible to* Y if χ_X is computable relative to Y . In this case, we write $X \leq_T Y$.

Write $X \equiv_T Y$ if $X \leq_T Y$ and $Y \leq_T X$.

1.3 Computable structure theory

Definition 1.12. A *copy* or ω -*presentation* of and L -structure M is a model $N \cong M$ with $\text{dom}(N) = \omega$. Then, the *atomic diagram* of N , written $D(N)$, is the collection of (Gödel numbers of) atomic sentences of $L(\omega)$ which hold in N .

Similarly $D^c(N)$ refers to the *complete diagram* of N , the collection of first order sentences of $L(N)$ which hold in N .

Here we will always be working with countably infinite structures, so, as we see below, there is no loss in generality in working with ω -presentations. (See [Mon21] for further details.)

In this course, we will focus on the computability-theoretic power of $D(M)$, as this is more standard both in group theory (where, for groups, it corresponds to the word problem) and in computability theory. The definitions below go through equally well for the complete diagram, but will not be as relevant for the paper.

In general, the computability-theoretic power of $D(M)$ may depend on the choice of copy of M .

Definition 1.13. Let M be an L -structure.

- We write $\text{Spec}(M)$ for $\{\mathbf{a} : \mathbf{a} \geq_T D(N) \text{ for some countable } N \cong M\}$. If $\mathbf{a} \in \text{Spec}(M)$, then we will say M is *$\mathbf{a}$ -computable* or *has an $\mathbf{a}$ -computable copy*.
- In particular, M is *computable* or *has a computable copy* when $0 \in \text{Spec}(M)$.
- The (*Turing*) *degree* of M is the minimum element $\mathbf{a}$ of $\text{Spec}(M)$, if it exists.

Knight showed in [Kni86] that for nontrivial M , $\text{Spec}(M)$ is upward-closed in the Turing degrees. Since every M we consider here is nontrivial, we may replace the $\geq_T$ with $\equiv_T$ in the definition of $\text{Spec} M$.

A stronger notion, which we also investigate here, is the *degree spectrum of a class of models*.

Definition 1.14. The *degree spectrum* of a class $\mathcal{K}$ of models is given by

$$\text{DegSpec}(\mathcal{K}) = \{\mathbf{a} \mid \mathbf{a} \text{ is the degree of } M \text{ and } M \in \mathcal{K}\}$$

In words, $\text{DegSpec}(\mathcal{K})$ is the collection of degrees, in the sense above, of models in $\mathcal{K}$.

1.4 Constructions in group theory

For much of the basic material on groups, we follow [LS15].

1.4.1 Free groups

Perhaps the most important class of groups are the *free groups*.

Definition 1.15. Let X be a possibly infinite set of constants and let X^{-1} be the set obtained by formally inverting each element of X . (We require in particular that $X \cap X^{-1} = \emptyset$.)

- A *word* on X is an element of $(X \sqcup X^{-1})^{<\omega}$.
- For a word w in n letters, we will write $w(\bar{x})$ for the corresponding word in the free variables $\bar{x}$, $|\bar{x}| = n$. Similarly, for constants $\bar{g}$, $|\bar{g}| = n$, write $w(\bar{g})$ for the word where each letter is replaced by the corresponding element of $\bar{g}$. Write $\text{Words}(X)$ for the set of words on alphabet X .
- A word w is *reduced* if it does not contain a consecutive subword of the form $c \cdot c^{-1}$ or $c^{-1} \cdot c$. For a word $w \in \text{Words}(X)$, write $\text{Red}(w)$ for the result of applying all possible reductions to w . The set of all equivalence classes words on a set X , under the reducibility relation, forms a group with multiplication given by concatenation, called the *free group* with *basis* X and written F_X .
- A group is *free* if it is isomorphic to F_X for some X . In this case, we sometimes say the image of X under this isomorphism *freely generates* F .
- If $w \in \text{Words}(X)$, then $|w|$ denotes the length of $\text{Red}(w)$.

It is sometimes useful to use the *universal property for free groups*.

Theorem 1.16. A group F is free with basis X iff for any group G and any function $f : X \rightarrow Y$, where $Y \subseteq G$, there is a unique homomorphism $\hat{f} : F \rightarrow G$.

The cardinality of X determines F_X up to isomorphism.

Proposition 1.17. $F_X \cong F_Y$ iff $|X| = |Y|$.

The cardinality of a basis of a free group is called the *rank* of F .

Perhaps surprisingly, any two countable free groups each embed into the other. To prove this, we need a sufficient condition for a subset $X \subseteq G$ to freely generate a free subgroup of G .

Notation 1.18. For a subset X of a group G , write $\langle X \rangle$ for the smallest subgroup of G containing X . $\langle X \rangle$ is called the *subgroup generated by* X .

Lemma 1.19 (Nielsen). Let $F(X)$ be a free group and $U \subseteq \text{Words}(X^{\pm 1})$ satisfy the following:

- (a) $\forall x \in U^{\pm 1} (x \neq 1)$
- (b) $\forall x, y \in U^{\pm 1} (xy \neq 1 \implies |xy| \geq \max\{|x|, |y|\})$
- (c) $\forall x, y, z \in U^{\pm 1} ((xy \neq 1 \wedge yz \neq 1) \implies |xyz| > |x| - |y| + |z|)$

Then $\langle U \rangle$ is a free group with basis U .

Proposition 1.20. *Let F_n and F_m be free groups on n and m respectively, where $1 < m, n \leq \omega$. Then*

$$F_n \leq F_m \quad \text{and} \quad F_n \leq F_m$$

Proof. Evidently, $F_2 \leq F_n$ for all such n , so it suffices to show that $F_n \leq F_2$.

Let $1 < n \leq \omega$. Define the subset U_n of $F_2 = \langle a, b \rangle$ as follows

$$U_n = \{b^{-i}ab^i \mid 1 \leq i \leq n\}.$$

We check that U_n satisfies the conditions of Lemma 1.19. (a) is immediate. Let $x = b^{-i}a^\varepsilon b^i$ and $y = b^{-j}a^{\varepsilon'} b^j$. Then $|b^{-i}a^\varepsilon b^{i-j}a^{\varepsilon'} b^j|$

$$|b^{-i}a^\varepsilon b^{i-j}a^{\varepsilon'} b^j| = \begin{cases} 2i+2 & \text{if } i > j \text{ or } (i = j \text{ and } \varepsilon = \varepsilon') \\ 2j+2 & \text{if } i < j \\ 0 & \text{if } i = j \text{ and } \varepsilon = \varepsilon' \end{cases}$$

In the last case, $x = y^{-1}$, so (b) is satisfied.

(c), follows from a similar, albeit slightly more involved, analysis. □

1.4.2 Presentations

One of the reasons that free groups are so important is that they are ubiquitous: every group is isomorphic to the quotient of a free group.

Theorem 1.21. *Let G be a group. Then there is a free group F and a normal subgroup N of F such that $G \cong F/N$.*

Proof. Let X be such that $|X| = |G|$ and let $f : X \rightarrow G$ be a bijection between them. Then, by the universal property of free groups, f has a unique extension $\hat{f} : F_X \rightarrow G$ that is a homomorphism. Evidently, $\hat{f}$ is surjective. Let $N \trianglelefteq F_X$ be $\ker \hat{f}$. Then $G \cong F_X/N$. □

If you prefer the more “constructive” version of free groups, then G can be thought of as $\text{Words}(G)$ up to free reduction (this is the “free part”) modulo the relations $g_0g_1g_2^{-1}$ for every $g_0g_1 = g_2$ in the atomic diagram of G .

Thus, G is determined up to isomorphism by the rank of the free group F and a subset R of F such that N is the smallest normal subgroup containing R . This gives rise to the notion of group presentation. Notice that neither the size of X nor the choice of generators of N are uniquely determined.

Definition 1.22. A *presentation* $\langle X \mid R \rangle$ for a group G consists of a set of *generators* X and a set R of words on X called *relations* or *relators* such that G is isomorphic to a quotient of the free group on X by the normal subgroup generated by R . In particular, the group given by a presentation $\langle X \mid R \rangle$ is the group satisfying exactly the equations $w(X) = v(X)$ provable from the axioms of group theory and the sentences $r(X) = e$ for every $r \in R$. (See [Rot99, Chapter 11] for more details.)

Example 1.23. Show that the following group is trivial:

$$\langle a, b \mid a^{-1}bab^{-2}; b^{-1}aba^{-2} \rangle$$

Definition 1.24. We establish the following conventions for group presentations.

- G is *finitely generated* if it has a presentation $\langle X \mid R \rangle$ with X finite, and *finitely presented* if it has a presentation $\langle X \mid R \rangle$ with both X and R finite. To emphasise that a given tuple $\bar{g}$ generates G , we will sometimes write $\langle \bar{g} \mid R \rangle$, and to emphasise that R is a set of words whose letters are contained in $\bar{g}$, we may write $R(\bar{g})$.
- If G is a subgroup of H and $\bar{g} \in H$ generates G , we will write $G = \langle \bar{g} \rangle$.
- G has an *$\mathbf{a}$ -computable set of relators* if there is a presentation $\langle X \mid R \rangle$ with X finite and $R \leq_{\mathbf{T}} \mathbf{a}$. When $\mathbf{a} = 0$, we say that G has a *computable set of relators*.

Remark. Every group with a c.e. set of relators also has a computable one: suppose G is such a group with c.e. list of relations $\langle R_i(\bar{g}) : i < \omega \rangle$. Then G is computably presented via $\langle R_i(\bar{g})e^i : i < \omega \rangle$.

Indeed, to check if some word is a relator in the new relating set, first check that it is of the form $w(\bar{g})1^i$, where w has no terminal 1s. Then, to check if this is a relator, go through the computable enumeration of the old relating set up to position i . This will be enough to determine if $w(\bar{g})1^i$ corresponds to a relator in the old relating set.

This is called Craig's Trick.

Definition 1.25. We will write $H \leq G$ to denote that H is a *subgroup* of G ¹. When $H \leq G$, call G an *overgroup* of H .

We will make use of several constructions from group theory. Throughout, we use $\bar{g}, \bar{h}$ to denote the concatenation of $\bar{g}$ and $\bar{h}$.

Definition 1.26. Let $G = \langle \bar{g} \mid R \rangle$ and $H = \langle \bar{h} \mid S \rangle$ be finitely generated groups.

- The *direct product* of G and H is given by

$$G \times H = \langle \bar{g}, \bar{h} \mid R \cup S \cup \{[g, h] \mid g \in \bar{g}, h \in \bar{h}\} \rangle$$

- The *free product* of G and H is given by

$$G * H = \langle \bar{g}, \bar{h} \mid R \cup S \rangle.$$

¹Note that this notation is similar to, but distinct from, the notation for computability introduced in an earlier subsection. Throughout, undecorated $\leq$ will refer to the subgroup relation or the ordering on the natural numbers, while decorated $\leq$ will denote a computability-theoretic reduction.

- Suppose G and H both embed some group F , generated by $\bar{f}$, with $\alpha : F \hookrightarrow G$ and $\beta : F \hookrightarrow H$. Then the *free product with amalgamation of G and H over F* is given by

$$G *_F H = \langle \bar{g}, \bar{h} \mid R \cup S \cup \{\alpha(f)\beta^{-1}(f) : f \in \bar{f}\} \rangle.$$

- If G contains two copies of F with an isomorphism, say $\alpha : F \cong F'$ between them, then the *HNN extension of G over α* is given by

$$G *_\alpha = \langle \bar{g}, t \mid R \cup \{tft^{-1}\alpha(f)^{-1} : f \in \bar{f}\} \rangle$$

where t is a letter not appearing in $\bar{g}$.

The last definition may seem unmotivated. The key property of HNN extensions is that $G *_\alpha$ is an overgroup of G (as we see below) in which the isomorphism between H and $\tilde{H}$ is now realised as a (particularly simple) definable automorphism.

1.5 Word problems

Definition 1.27. For a finitely generated group $G = \langle X \mid R \rangle$, let the *word problem* of G be

$$W(G, X) = \{w : w \text{ a word on } X \text{ and } w = 1 \text{ in } G\}.$$

Remark. Note that $\langle X \mid W(G, X) \rangle$ is always a presentation of G .

We can somewhat explicitly describe the word problem of a group from its presentation.

Proposition 1.28. *Let $G = \langle X \mid R \rangle$ be a group, where X is computable. Then $W(G, X)$ can be effectively enumerated from R .*

For now (until we meet enumeration reducibility), what this proposition tells us is that if R is computable (or c.e.) then $W(G, X)$ is computably enumerable.

Proof. Recall that $G \cong F_X / \langle\langle R \rangle\rangle$. Let $U = \{x^{-1}rx \mid x \in X \sqcup X^{-1}, r \in R\}$. Then

$$\langle\langle R \rangle\rangle = \text{Words}(U \sqcup U^{-1}) / \sim$$

where $\sim$ is the reduction relation used in the definition of the free group.

Thus, with a computable enumeration of X , relative to any enumeration of R , we may effectively enumerate the elements of $\langle\langle R \rangle\rangle$. $\square$

The following easy proposition justifies the use of the phrase *the* degree of the word problem for G .

Proposition 1.29. *Let G be finitely generated. Then the degree of $W(G)$ is independent of the choice of finite generating set for G .*

Proof. Let G be a finitely generated group and let $\bar{g}$ and $\bar{h}$ be finite generating sets for G , say via the presentations

$$G \cong \langle \bar{g} \mid R \rangle \cong \langle \bar{h} \mid S \rangle.$$

Let $\hat{\alpha}$ be an isomorphism between $F_{\bar{g}} / \langle\langle R \rangle\rangle$ and $F_{\bar{h}} / \langle\langle S \rangle\rangle$. Then define $\alpha : \bar{g} \rightarrow \text{Words } \bar{h}$ to be the map sending g_i to a representative of $\hat{\alpha}(g_i)$. Notice that this function is finite and hence computable.

Further, $w(\bar{g}) = 1$ in $F_{\bar{g}} / \langle\langle R \rangle\rangle$ iff $w(\alpha(\bar{g})) = 1$ in $F_{\bar{h}} / \langle\langle S \rangle\rangle$. Thus, $W(G, \bar{g}) \leq_T W(G, \bar{h})$.

The converse is proved analogously.

Thus, $\deg_T(W(G, \bar{g})) \in \text{Spec } G$. $\square$

Hence, from now on, for finitely generated groups, we write $W(G)$ for $W(G, X)$, where X is a finite generating set.

1.6 The complexity of a group

We've now seen two ways to measure the computability-theoretic content of a group – the spectrum of the group (i.e., the degrees computing atomic diagrams of isomorphic copies) and the word problem. We show now that these are equivalent.

We start with the simple observation that the degree of a copy of G is equivalent to the degree of the multiplication function.

Proposition 1.30. *Let G be an ω -presentation of a countable group G . Then $\deg_T(G) = \deg_T(\cdot_G)$.*

Proof. It suffices to show that the inverse function $\bullet^{-1}$ is computable from $\cdot$. But $h = g^{-1}$ iff $g \cdot h = 1$. Since every element has an inverse, we can search G (i.e., ω) for such an element. $\square$

We now verify that we can obtain a copy of G from a presentation. Recall that in identifying G with $F_X/\langle\langle R \rangle\rangle$, we assume that the inverse map $x \mapsto x^{-1}$ on X is computable.

Proposition 1.31. *Suppose $G = \langle X \mid R \rangle$. Then $\deg_T(W(G, X)) \in \text{Spec}(G)$.*

Proof. Without loss of generality, we may take $X \sqcup X^{-1} \subseteq \omega$. Thus we may represent words in $X \sqcup X^{-1}$ as elements of ω using some fixed computable Gödel coding, so that G has domain equivalence classes of elements of ω . Notice that the union of these equivalence classes (which is a subset of ω) is computable from $X \sqcup X^{-1}$, since the Gödel coding is effective.

Note that $X \leq_T W(G, X)$: $x \in X$ iff $x \cdot x^{-1} \in W(G, X)$.

From G , we build a copy $\tilde{G}$ of G by taking as elements of $\tilde{G}$ the least element of each equivalence class. This set is computable in $W(G, X)$ since $[g] \equiv_R [h]$ iff $gh^{-1} \in W(G, X)$. (The argument in Proposition 1.30 implies that h^{-1} may be found computably in $X \oplus W(G, X)$.) Now, this is a group isomorphic to G with $\text{dom } \tilde{G} \subseteq \omega$. Now, let $(g_i)_{i < \omega}$ be the $\tilde{G}$ -computable increasing enumeration of $\tilde{G}$ and let α be the map sending i to g_i . Then $\tilde{\tilde{G}} = \alpha^{-1}(\tilde{G})$ is a group under the multiplication $\cdot_{\tilde{\tilde{G}}} = \cdot_{\tilde{G}} \circ \alpha^{-1}$. Further, $n \cdot_{\tilde{\tilde{G}}} m = \ell$ iff $\alpha(n) \cdot_{\tilde{G}} \alpha(m) = \alpha(\ell)$, which is computable in $W(G, X)$, as described above. $\square$

The following proposition finishes the argument that, for a finitely generated group G , $\deg_T(W(G, \bar{g}))$, which by Proposition 1.29 does not depend on the choice of finite generating set, witnesses the degree of G .

Proposition 1.32. *Let $G = \langle \bar{g} \mid R \rangle$ be a finitely generated group, and let $\tilde{G}$ be a copy of G . Then $\deg_T(W(\bar{g}, R)) \leq_T \cdot_{\tilde{G}}$.*

Proof. Let $\hat{\alpha} : G \rightarrow \tilde{G}$ be an isomorphism and let $\alpha = \hat{\alpha} \upharpoonright \bar{g}$ be a finite (hence computable) map. Then

$$w(\bar{g}) \in W(G, \bar{g}) \iff w(\alpha(\bar{g})) = 1 \text{ in } \tilde{G}$$

$\square$

Thus the degree of a group generalises the degree of the word problem to possibly infinitely-generated groups. Furthermore, by Proposition 1.46, every Turing degree is the degree of the word problem of a finitely generated group. Before we prove this, we need some ways of constructing groups.

1.7 Normal Forms

One pleasant property of all of these constructions is that elements of each of these groups admit *normal forms*. We will meet these after we discuss word problems.

Proposition 1.33 (Normal Form Theorem for Direct Products). *Every word in $\text{Words}(\bar{g}, \bar{h})$ is equivalent in $G \times H$ to a unique word of the form*

$$uv$$

where $u \in G$ and $v \in H$.

(Notice the uniqueness is up to a choice of word in each equivalence class of $\text{Words}(\bar{g})$ and $\text{Words}(\bar{h})$.)

Thus we obtain the following corollaries.

Corollary 1.34. $G, H \leq G \times H$

Corollary 1.35. $W(G \times H) \cong_T W(G) \oplus W(H)$

Proposition 1.36 (Normal Form Theorem for Free Products). *Every word $w \in \text{Words}(G \sqcup H)$ is equivalent in $G * H$ to a unique word of the form*

$$u_0 v_0 u_1 v_1 \cdots u_{n-1} v_{n-1}$$

where:

- (a) $u_i \in G$ and $v_i \in H$
- (b) $i > 0 \implies u_i \neq 1$ and $i < n - 1 \implies v_i \neq 1$

Again, we obtain similar corollaries.

Corollary 1.37. $G, H \leq G * H$.

Corollary 1.38. $W(G * H) \cong_T W(G) \oplus W(H)$

Proof. Take a word in $\text{Words}(G \sqcup H)$. Using $W(G)$ and $W(H)$, we may reduce it to its normal form. Then $w = 1$ in $G * H$ iff its normal form is 1. $\square$

Proposition 1.39 (Normal Form Theorem for Free Products with Amalgamation). *Let $w \in \text{Words}(G \sqcup H)$. Then w is equivalent in $G *_F H$ to a word of the form*

$$s u_0 v_0 \cdots u_{n-1} v_{n-1}$$

where

- (a) $u_i \in G \setminus \alpha(F)$ and $v_i \in H \setminus \beta(F)$
- (b) $i > 0 \implies u_i \neq 1$ and $i < n - 1 \implies v_i \neq 1$
- (c) $s \in \alpha(F)$ is arbitrary

In this case, we only obtain uniqueness of the normal form up to a choice of coset representatives for $\alpha(F)$ and $\beta(F)$ in G and H respectively.

Corollary 1.40. $G, H, F \leq G *_F H$

Corollary 1.41. $W(G *_F H) \leq_T W(G) \oplus W(H) \oplus \alpha \oplus \alpha^{-1} \oplus \beta \oplus \beta^{-1} \oplus W(\alpha(F) \subseteq G) \oplus W(\beta(F) \subseteq H)$

In practice, many of these terms will end up being computable.

Proposition 1.42 (Normal Form Theorem for HNN extensions). *Let $G *_\alpha$ be an HNN extension with associated subgroups H and $\tilde{H}$. Then every word $w \in (G \cup \{t\})^{<\omega}$ is equivalent in $G *_\alpha$ to a word of the form*

$$v = g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n$$

where $\varepsilon_i = 1 \implies g_i \notin H$ and $\varepsilon_i = -1 \implies g_i \notin \tilde{H}$.

Further, if we pick representatives of the right cosets $\{Ag\}_{g \in G}$ and $\{Bg\}_{g \in G}$, then v is unique subject to the constraints:

- (a) $\varepsilon_i = 1 \implies g_i$ is the choice of representative of a coset of A
- (b) $\varepsilon_i = -1 \implies g_i$ is the choice of representative of a coset of B
- (c) $g_0 \in G$ is arbitrary

There are a number of very important applications of HNN extensions, all of which rely on the Normal Form Theorem.

Theorem 1.43 (Britton's Lemma). *Let $G *_\alpha$ be an HNN extension, and $w \in (\bar{g} \cup \{t\})^{<\omega}$ be a word with no subwords of the form $t^{-1}ft$ or $t\alpha(f)t^{-1}$, for $f \in F$. Then $w \neq 1$ in $G *_\alpha$*

Proof. Take such a word w . Then, putting w into a normal form from left to right does not change the number of t s. Hence $w \neq 1$ in $G *_\alpha$. $\square$

Corollary 1.44 (Higman–Neumann–Neumann). $G \leq G *_\alpha$ via the identity map.

This is not the original proof of HNN, who did not have access to Britton's Lemma.

Proof. If w does not contain t , then w is its own normal form. Thus, for $w \in G$, $w = 1$ in $G *_\alpha$ iff $w = 1$ in G . $\square$

Corollary 1.45.

$$W(G *_\alpha) \equiv_T W(G) \oplus W(H \subseteq G) \oplus W(\tilde{H} \subseteq G) \oplus \alpha \oplus \alpha^{-1}$$

We can now show that even finitely generated groups can be arbitrarily complicated. (Recall that we cannot hope for the same from finitely presented groups, which must be c.e.)

Proposition 1.46. *For every X , there is a finitely generated group G such that $W(G) \equiv_T X$.*

Proof. Let $X \subseteq \omega$. Recall that $U = \{b^{-i}ab^i \mid i \in \omega\}$ is a basis for a free subgroup of $F_{\langle a, b \rangle}$. Consider the HNN extension

$$H = \langle a, b, t \mid \{t^{-1}b^{-i}ab^i t = b^{-i}ab^i : i \in X\} \rangle$$

which forces t to commute with $b^{-i}ab^i$ if $i \in X$.

We claim that $[t, b^{-i}ab^i] = 1$ in H iff $i \in X$. For the nontrivial direction, consider $[t, b^{-i}ab^i]$ for $i \notin X$. By Britton's Lemma, this is not equal to 1 in H . Thus $X \leq_T W(H)$.

On the other hand, by the Normal Form Theorem for HNN extensions,

$$H \leq_T \text{id} \oplus \text{id}^{-1} \oplus W(F_{\langle a, b \rangle}) \oplus W(\langle \{b^{-i}ab^i : i \in X\} \rangle) \subseteq H).$$

Each term on the RHS is computable except $W(\langle \{b^{-i}ab^i : i \in X\} \rangle) \subseteq H) \leq_T X$. $\square$

Theorem 1.47 (Higman–Neumann–Neumann). *Every countable group can be embedded in a two-generator group.*

Proof. Let G be a countable group and $\bar{g} = (g_0, g_1, \dots)$ a possibly infinite set of generators for G . Let $F_2 = \langle a, b \mid \rangle$. Let $H = G * F_2$. Then we claim the following both freely generate F_ω in H :

$$X_0 = \{a, b^{-1}ab, b^{-2}ab^2, \dots\}$$

and

$$X_1 = \{b, g_0a^{-1}ba, g_1a^{-2}ba^2, \dots\}.$$

That X_0 freely generates a free group follows from Lemma 1.19. For X_1 , consider the homomorphism from $X_1 \rightarrow F_{\langle a, b \rangle}$ generated by $f(a) = a$, $f(b) = b$, and $f(g_i) = 1$. Then $f(X_1)$ is a free basis for a free subgroup of $F_{\langle a, b \rangle}$ by Lemma 1.19 again. Hence, if there were a nontrivial relation amongst the elements of X_1 , this would push forward to a nontrivial relation amongst the elements of $f(X_1)$, a contradiction.

Thus, the map α which sends $a \mapsto b$ and $b^{-(i+1)}ab^{i+1} \mapsto g_ib^{-(i+1)}ab^{i+1}$ extends to an isomorphism between subgroups of H . (We also denote this isomorphism by α .) Consider the HNN extension

$$H*_\alpha = \langle \bar{g}, a, b, t \mid W(\bar{g}) \cup \{t^{-1}at = b\} \cup \{t^{-1}b^{-(i+1)}ab^{i+1}t = g_ia^{-(i+1)}ba^{i+1}\} \rangle.$$

By the Normal Form Theorem, H , and hence G , embeds into $H*_\alpha$, and $H*_\alpha$ is evidently generated by a and t . $\square$

Notice that the argument above depends on the enumeration of the generators of G . We note an interesting result of Thomas that the argument above cannot be made uniform, in an effective way.

Theorem 1.48 (S. Thomas ([Tho09])). *There is no Borel, isomorphism-invariant map $\psi : \mathcal{G} \rightarrow \mathcal{G}_2$ such that $G \leq \psi(G)$.*

2 Early theorems in group theory and computability

In this section, we focus on two classical theorems connecting group theory and computability theory. The first, due to Sergei Adian ([Adi55]) and, independently, Michael Rabin ([Rab58]), states that recognising whether the group given by a finite presentation has one of a very large class of properties is undecidable (and in fact decides 0'). The second is Higman's celebrated Embedding Theorem which characterises finitely generated groups with computable set of relators as exactly those which embed into a finitely presented group. The proofs we give of both theorems are modern; for the Adian–Rabin Theorem we follow the proof in [LS15], while for Higman's Embedding Theorem, we follow [LS15].

2.1 Adian–Rabin Theorem

We start by specifying the “large class of properties” that the theorem covers. The name is in honour of Andrey Markov, Jr, who proved a result analagous to the Adian–Rabin Theorem for semigroups ([Mar51]).

Definition 2.1. A *Markov property for finitely presented groups* is a property P such that:

- (a) There is a finitely presented group G_+ satisfying P
- (b) There is a finitely presented group G_- such that any overgroup $H \geq G_-$ does not satisfy P

The Adian–Rabin Theorem states that for any Markov property P there is no algorithm which decides whether a given presentation satisfies P .

Theorem 2.2 (Adian–Rabin Theorem). *Let P be a Markov property. Then*

$$\{[\langle X \mid R \rangle] \mid \langle X \mid R \rangle \text{ is a finite presentation and } \langle X \mid R \rangle \text{ satisfies } P\} \geq_T 0'$$

Remark. This theorem can be thought of as a “sibling theorem” to Rice’s Theorem in computability, which states that any nontrivial index set; i.e., any set of c.e. sets which is neither empty nor all of them, computes $0'$.

More precisely, an index set I is a subset of ω such that $e \in I$ and $W_e = W_{e'} \implies e' \in I$. Then Rice’s Theorem says that $I \neq \emptyset, \omega$ implies that $I \geq_T 0'$.

In order to prove this theorem, we need one fact which we will obtain later, as a consequence of the Higman Embedding Theorem. It was originally due to Pyotor Novikov ([Nov55]) and, independently, William Boone ([Boo58]).

Theorem 2.3 (Novikov–Boone Theorem). *There is a finitely presented group F with unsolvable word problem; indeed, there is such F with $W(F) \geq_T 0'$.*

Novikov and Boone did not refer to the halting set in their papers, but the stronger result follows from their arguments. Further, observe that since any finitely presented group has c.e. word problem, we automatically get that $W(F) \equiv_T 0'$.

In fact, Clapham has shown that Higman’s Embedding Theorem is sufficiently effective to obtain the following strengthening of Theorem 2.3 ([Cla67]).

Theorem 2.4. *Let W be a c.e. set. Then there is a finitely presented group F such that $W(F) \equiv_T W$.*

We postpone the proof of Theorem 2.3 until after the proof of Higman’s Embedding Theorem. Before we prove the Adian–Rabin Theorem, we give some examples.

Corollary 2.5. *The following properties of finite presentations $\langle X \mid R \rangle$ are undecidable (and in general require $0'$):*

- (a) $\langle X \mid R \rangle$ is trivial;
- (b) $\langle X \mid R \rangle$ is finite;
- (c) $\langle X \mid R \rangle$ is abelian;
- (d) $\langle X \mid R \rangle$ is free;
- (e) $\langle X \mid R \rangle$ is computable;
- (f) $\langle X \mid R \rangle$ has Turing degree $\mathbf{a}$, for a fixed c.e. degree $\mathbf{a}$;
- (g) $\langle X \mid R \rangle$ is simple.

Proof. For most of these, it is easy to check that the given property is Markov. We do the last three, and leave the remainder as exercises.

For (e), let G_+ be $\langle a \mid \rangle$ and G_- be the group F given by Theorem 2.3. Similarly, for (f), we may take G_+ as the group given in Theorem 2.4 and G_- as the finitely presented group in Novikov–Boone Theorem. Evidently, this argument generalises, and there are many other Markov properties related to the degree of the word problem of a group.

Finally, for (g), we can take G_+ to be any finite simple group; e.g. A_5 . G_- is a little trickier – certainly non-simple groups may have simple subgroups, like, for example $A_5 * F_2$. We use an observation due to Kuznetsov that finitely presented simple groups are always computable ([Kuz58]).

Let $K = \langle X \mid R \rangle$ be a finitely presented simple group. If $K \cong \{1\}$, then we are done. Else, let $w \in \text{Words}(X^{\pm 1})$ be such that $w \neq 1$ in K . Then we can enumerate $\overline{W(K, X)}$ as follows: $v \in \text{Words}(X^{\pm 1})$ is not equivalent to 1 in K iff $w \in \langle\langle v \rangle\rangle$. Since $\langle\langle v \rangle\rangle$ can be computably enumerated, this is a c.e. property of v .

On the other hand, because K is finitely presented, $W(K)$ is c.e. Hence $W(K)$ is computable. $\square$

This last observation deserves its own signposting. Notice that the argument only uses that $W(K)$ is c.e.

Proposition 2.6. *Let K be a simple group with c.e. word problem. Then $W(K)$ is computable.*

Before we prove Adian–Rabin, we note one further corollary in topology.

Corollary 2.7. *The problem of deciding, for a given finite simplicial complex which represents a manifold, whether that manifold is simply connected, is undecidable.*

Proof. Given a finitely presented group, one may effectively construct a simplicial complex S which represents a compact manifold M such that $\pi_1(M) \cong G$ (e.g. Chapter 1 of [Hat02]). Thus, if we could decide if the manifold for S were simply connected, we would be able to decide if the original presentation represented the trivial group, contradicting Adian–Rabin. $\square$

We now present the key lemma in Miller’s proof of Theorem 2.2.

Lemma 2.8. *Let $H = \langle X \mid R \rangle$. Let $X = \{x_i\}_{i \leq n}$, where $n \leq \omega$ – notice this is $\leq$ and not $<!$ For a word $w \in \text{Words}(X)$, let H_w be the quotient of $H * F_{\langle a, b, c \rangle}$ given by adding the following relations:*

$$a^{-1}ba = c^{-1}b^{-1}cbc \tag{1}$$

$$a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2 \tag{2}$$

$$a^{-3}[w, b]a^3 = c^{-3}bc^3 \tag{3}$$

$$a^{-(4+i)}x_i ba^{4+i} = c^{-(4+i)}bc^{4+i} \tag{4}$$

Then:

- (a) *If $w \neq 1$ in H , then $H \leq H_w$ via the identity.*
- (b) *If $w = 1$ in H , then H_w is trivial.*

Note further that if H is finitely presented, so is H_w ; and if H is finitely generated, then so is H_w .

Exercise 2.9. Show that H_w is generated by b and ca^{-1} . Thus, H_w is finitely generated, even if H is not.

Proof of Lemma. (a) Assume $w \neq 1$. The trick is that in this case, H_w is secretly a free product with amalgamation. Let U be the set of words on the RHS and V the set of words on the LHS. Then U is Nielsen-reduced in $F_{\langle b, c \rangle}$, is a basis for the free subgroup $\langle U \rangle \leq F_{\langle b, c \rangle}$.

We check that V also is a free bases for $\langle V \rangle \leq H * F_{\langle a, b \rangle}$. Indeed, consider a word $v(V^\pm) = v_0 v_1 \cdots v_{n-1} = 1$ in $H * F_{\langle a, b \rangle}$. Then, by Proposition 1.36, it must be that adjacent elements are inverses — otherwise we do not get cancellation of the a 's and hence we end up with a normal form with positive length.

Thus, H_w is the free product of $F_{\langle b, c \rangle}$ and $H * F_{\langle a, b \rangle}$ with these free subgroups identified. Hence, by the Normal Form Theorem for free products with amalgamation (Proposition 1.39), $H \leq H_w$.

(b) Assume that $w = 1$. Then, using Eq. (3), we have

$$w = 1 \implies [w, b] = 1 \implies c^{-3}bc^3 = 1 \implies b = 1.$$

By Eq. (1) and Eq. (2), we then get

$$b = 1 \implies c = 1 \implies a = 1.$$

Finally, by Eq. (4), we have that $x_i = 1$ for all i .

□

We can now prove the Adian–Rabin Theorem.

Proof of Theorem 2.2. Let $F = \langle X \mid R \rangle$ be a finitely presented group with $W(F) \geq_T 0'$ as given by the Novikov–Boone Theorem. Let P be a Markov property and G_- and G_+ the corresponding finitely presented groups. For each word $w \in \text{Words}(X^{\pm 1})$, let $P(w)$ be the standard presentation for $(F * G_-)_w * G_+$ and $H(w)$ be the corresponding group. Then $w = 1$ in F implies that $(F * G_-)_w \cong \{1\}$, so $H(w) \cong G_+$ and hence satisfies P . On the other hand, if $w \neq 1$ in F , then $(F * G_-)_w \geq F * G_-$, so $H(w) \geq G_-$, and hence does not satisfy P . Thus

$$\{[\langle X \mid R \rangle] \mid \langle X \mid R \rangle \text{ is a finite presentation and } \langle X \mid R \rangle \text{ satisfies } P\} \geq_T W(F) \geq_T 0'.$$

□

2.2 Higman Embedding Theorem

In this section, we sketch a proof of Higman's Embedding Theorem. We follow the streamlined treatment in [LS15], although [Hig61] remains remarkably readable.

We start with some applications.

Corollary 2.10. *Every group with a computable set of relators embeds in a finitely presented group.*

Proof. Check that the proof of Theorem 1.47 actually produces a group with computable set of relators from a group with a computable set of relators. □

Corollary 2.11. *There is a universal finitely presented group.*

Proof. Effectively list all finite presentations. Take the free product. This has computable set of relators. By the previous corollary, it embeds into a finitely presented group. $\square$

Corollary 2.12. *There is a finitely presented group with unsolvable word problem.*

Proof. Let $X \subseteq \omega$ be c.e. Then the group G with $G \equiv_T X$ from Proposition 1.46 has c.e. presentation. Thus, by Craig's trick, it has a computable set of relators. Hence it can be embedded in a finitely presented group F . Since it's finitely generated, $W(G) \leq_T W(F)$, as required. $\square$

Theorem 2.13 (Boone–Higman Theorem). *A f.g. group G has computable word problem iff G embeds in a simple subgroup of a finitely presented group*

Proof. Firstly, suppose that G is finitely generated and $G \leq K \leq F$ where K is simple and $F = \langle X \mid R \rangle$ is finitely presented. Then $W(F)$ and hence $W(G)$ are c.e. On the other hand, we can effectively list $\overline{W(G)}$ as follows: fix some $v \in K$ with $v \neq 1$. Then $w \notin W(K)$ (and in particular $w \notin W(G)$) iff $v \in \langle\langle w \rangle\rangle_F$ iff $v \in \langle\langle \{w\} \cup R \rangle\rangle$, which is itself c.e. (See Proposition 2.6). Thus, $W(G)$ is c.e. and co-c.e., as required.

Now, let G be a f.g. group, say with generators $\bar{g}$, with computable word problem. Consider $G * \langle x \rangle$, the group obtain by freely adding a new generator x . Then, by Proposition 1.36, for every $v, w \in \text{Words}(\bar{g}^\pm)$ with $w, v \neq 1$ in G , both $vx^{-1}vx$ and $x^{-1}vxw$ have infinite order, and hence, $\langle vx^{-1}vx \rangle \cong \langle x^{-1}vxw \rangle$. Thus, we may take the HNN extension $(G * \langle x \rangle)_{vx^{-1}vx \mapsto x^{-1}vxw}$. In this HNN extension, we have that

$$w = x^{-1}v^{-1}xt^{-1}vx^{-1}vxt \in \langle\langle v \rangle\rangle.$$

Thus, taking countably many HNN extensions, we can build a group

$$G^* = \langle \bar{g} \cup \{t_{v,w} : v, w \in \text{Words}(\bar{g}^\pm)\} \mid \text{Rel}(G, \bar{g}) \cup \{t_{u,v}^{-1}vx^{-1}vxt_{u,v} = x^{-1}vxw : v, w \in \text{Words}(\bar{g}^\pm)\} \rangle.$$

In G^* , every pair of words $w, v \in G$ with $v, w \neq 1$ satisfy that $w \in \langle\langle v \rangle\rangle$. Hence, performing this countably many times, we obtain

$$G \leq G^* \leq G^{**} \leq \dots$$

Let K be the union of this chain. Since the increasing union of groups is a group, K is a group, and by construction, K is simple. Further, since G has solvable word problem, K has a computable presentation, and hence, by Higman's Embedding Theorem, embeds in a finitely presented group. $\square$

The following example (due to Higman) shows that it is not sufficient for a group to have computable set of relators, or even for every finitely generated subgroup to have computable set of relators, to embed into a finitely presented group.

Proposition 2.14. *There is a group G such that G has computable set of relators and every finitely generated subgroup $H \leq G$ embeds into a finitely presented group, but G itself does not.*

Proof sketch. Let $\tilde{G}$ be the group constructed in Proposition 1.46 for a c.e. set S . Let X generate G and note that G has no torsion (this can be checked by Britton's Lemma). Then let G be the group given by the presentation:

$$G = \langle X, (y_w)_{w \in \text{Words}(X^{\pm 1})}, t \mid W(G), \{u_w^{-1}wu_w = t : w \neq 1 \text{ in } G\} \rangle.$$

Then G is obtained from $\tilde{G}$ by a series of HNN extensions so $\tilde{G} \leq G$. Further, any f.g. subgroup H of G is a subgroup of $\langle X, (u_w)_{w \leq n}, t \rangle$, which is f.g. and has computable set of relators, and hence embeds in a finitely presented group. However, G does not; else $\overline{W(G)} = \{\text{words conjugate to } t\}$ would be c.e. which would imply that $W(G)$ is computable. $\square$

2.2.1 A proof of Higman's Embedding Theorem

Again, we follow the proof given in [LS15, IV.7], the key parts of which they attribute to Valiev ([Val68], with later unpublished simplifications).

For a subgroup $H \leq G$, write $G *_H$ for $G *_{\text{id}_H}$.

Definition 2.15 (Higman). A normal subgroup N of a finitely generated group G is *benign* if $G *_N$ embeds into a finitely presented group.

Theorem 2.16 (Higman's trick). *If N is a benign subgroup of G , then G/N embeds in a finitely presented group.*

Proof. Let $N \trianglelefteq G$ be benign, and let $F = \langle \bar{f} \mid R \rangle$ be a finitely presented group with $G *_N \hookrightarrow F$. We assume that G is generated by the elements $\bar{g}$, that $\bar{g} \subseteq \bar{f}$, and that $G *_N$ has stable letter t . Further, let G' and N' be isomorphic copies of G and N , with $\bar{g}'$ generating G' .

Let $H = \langle G, t^{-1}Gt \rangle \leq G *_N$. Then H is isomorphic to the free product of G and $t^{-1}Gt$, amalgamated over N .

We pause to verify this claim. Clearly both groups have the same generators, so it suffices to show they have the same word problems. Firstly, note that every relation of $G *_N t^{-1}Gt$ holds in H . Thus, take a word $w \in \text{Words}((\bar{g} \cup \{t\})^{\pm 1})$ and suppose that $w \neq 1$ in $G *_N t^{-1}Gt$. By the Normal Form Theorem for Free Products with Amalgamation (Proposition 1.39), it follows that, in $G *_N t^{-1}Gt$, w is equivalent to a word of the form $c_0 \cdot c_1 \cdot \dots \cdot c_{n-1}$, where $c_i \in N$ iff $i = 0$ and, for every $i > 0$, $c_i \in G$ iff $c_{i+1} \in t^{-1}Gt$. Further, since $w \neq 1$, we have that $n > 0$. Since H is a quotient of $G *_N t^{-1}Gt$ (because it satisfies all the relations of $G *_N t^{-1}Gt$), it follows that w is equivalent to this word in H as well. Now, since H is a subgroup of an HNN extension, we may apply Britton's Lemma (Theorem 1.43), which implies that $w \neq 1$ in H .

Thus, the map $\varphi : H \rightarrow G'/N'$ given by $\varphi(w) = w'$ and $\varphi(t^{-1}wt) = 1$ is a well-defined homomorphism.

Working inside $F \times G'/N'$, let $\psi : H \times \{1\} \rightarrow F \times G'/N'$ be defined by $\psi((h, 1)) = (h, \varphi(h))$. By the first coordinate, this is injective, and thus we may form the HNN extension $K = (F \times G'/N') *_{\psi}$. We claim that K is finitely presented.

Indeed, F is finitely presented, G' is finitely generated, and H is finitely generated, so, if the relations of G'/N' follow from these, then K is finitely presented by the standard HNN presentation (without the relations for G'/N'). Suppose $w' = 1$ in G'/N' . Then $w' \in N'$, so $w \in N$ and hence, in $G *_N$, $t^{-1}wt = w$. Thus $\psi((w, 1)) = (w, w')$, while $\psi((t^{-1}wt, 1)) = (t^{-1}wt, 1)$. But $t^{-1}wt = w$, so it follows that $w' = 1$, as required. $\square$

Thus, we need to show that every c.e. normal subgroup of a free group is benign.

We skip the following lemma, which is Lemma 7.7 of [LS15].

Lemma 2.17. *Let G be a f.g. group which embeds into a finitely presented group. Then:*

- (a) *Every f.g. subgroup of G is benign in G .*
- (b) *If H and K are benign in G , then $H \cap K$ and $\langle H, K \rangle$ are both benign in G .*
- (c) *If K is benign in G and H is a finitely generated group with $K \leq H \leq G$, then K is benign in H .*

The following lemma shows that c.e. subgroups of a particular form are benign.

Lemma 2.18 (Principal Lemma (Valiev)). *Let $S \subseteq \omega$ be c.e. Then*

$$G_S = \langle a_0^x b_0 c_0^x \mid x \in S \rangle \leq F_{\{a_0, b_0, c_0\}}$$

is benign in $F_{\{a_0, b_0, c_0\}}$.

See [LS15] for the proof, and for how to deduce Higman’s Embedding Theorem from the Principal Lemma.

3 Relativisation of Higman’s Embedding Theorem

In this section, we show that Higman’s Embedding Theorem *relativises* – namely that groups H which embed in a group which is *finitely presented over* some fixed group G , can be characterised computability-theoretically. The relevant notion of computability theory is the *enumeration degrees*.

3.1 Enumeration reducibility

The idea of enumeration reducibility is that it focuses on *positive information*. Whereas in Turing reducibility, we can answer positive and negative questions of our set (i.e., questions of the form $x \in X$ and $x \notin X$, respectively) by a series of positive and negative queries to our oracle, enumeration reducibility requires merely that we are able to answer positive questions, but we are only allowed positive questions to our oracle. Another way to conceptualise this is that X is *enumeration reducible* to Y if from any enumeration of Y , we can effectively (and uniformly) produce an enumeration of X ².

Note that *on sets*, Turing reducibility and enumeration reducibility are incomparable, although we will see below that the Turing degrees naturally embed in the enumeration degrees.

We fix a computable pairing function $\langle \cdot, \cdot \rangle \rightarrow \omega$ and a computable enumeration $(D_u)_{u \in \omega}$ of the finite subsets of ω .

Definition 3.1. Let $X, Y \subseteq \omega$. Then X is *enumeration reducible to Y* , written $X \leq_e Y$ iff there is a c.e. set W such that

$$x \in X \iff \exists u (\langle x, u \rangle \in W \wedge D_u \subseteq Y).$$

The elements $\langle x, u \rangle \in W$ are called (*enumeration*) *axioms*.

If $W = W_e$, we write $X = \Gamma_e(Y)$, and call Γ_e the *eth enumeration functional* or the *eth enumeration operator*.

Thus, we can determine if $x \in X$ by computably listing the elements $\langle y, u \rangle$ of W , and waiting for one of the form $\langle x, u \rangle$. We then go through the enumeration of Y and, if we see each element of D_u , we can conclude that $x \in X$. This gives an enumeration of X . At no point can we conclude that $x \notin X$.

It is not hard to check that $\leq_e$ is reflexive and transitive, and thus we can define the *enumeration degrees* $\mathcal{D}_e$ as the set of equivalence classes on $\mathcal{P}(\omega)$ under the induced equivalence relation. Like $\mathcal{D}_T$, the enumeration degrees form an *upper semilattice* with join $\oplus$.

It is worth noting, although we will not need it in this class, that enumeration reducibility can be defined in terms of the notion of “c.e. in”. The following was proved by Alan Selman in his PhD thesis ([Sel71]).

²Many of the arguments in this section come from Soskova et al’s unpublished book “Computing with Incomplete Information” on the enumeration degrees.

Theorem 3.2 (Selman's Theorem). *For $X \subseteq \omega$, write $\mathcal{CE}(X)$ for $\{Z \mid X \text{ is c.e. in } Z\}$. Then*

$$X \leq_e Y \iff \mathcal{CE}(X) \supseteq \mathcal{CE}(Y).$$

In a sense, enumeration reducibility generalises Turing reducibility.

Proposition 3.3. *Let $X, Y \subseteq \omega$. Then*

$$X \leq_T Y \iff X \oplus \overline{X} \leq_e Y \oplus \overline{Y}.$$

Further, X is c.e. in Y iff $X \leq_e Y \oplus \overline{Y}$.

Sets, like those of the form $X \oplus \overline{X}$, for which $\overline{Y} \leq_e Y$, are called *total* and enumeration degrees containing total sets are also called total. Every total enumeration degree contains a set of the form $X \oplus \overline{X}$. (Naturally, every Turing degree contains a total set, since $X \equiv_T X \oplus \overline{X}$.)

The previous proposition implies that the poset of the Turing degrees embeds into the poset of the enumeration degrees via the map $\iota : X \rightarrow X \oplus \overline{X}$. By Selman's Theorem, the enumeration degree of X is determined by the set of total degrees above it.

We make explicit another connection between enumeration reducibility and a concept we have already seen.

Proposition 3.4. *There is a minimum enumeration degree 0_e consisting of exactly the c.e. sets.*

Further, as in the Turing degrees, there is a notion of the *jump* of an enumeration degree.

Definition 3.5. Let $X \subseteq \omega$. Then

$$K_X := \bigoplus_{e < \omega} \Gamma_e(X)$$

The *enumeration jump* of X is the set $X'_e = K_X \oplus \overline{K_X}$.

Note that $\overline{K_X} = \bigoplus_{e < \omega} \overline{\Gamma_e(X)}$ is the set whose columns consist of the complements of sets enumeration reducible to X .

Proposition 3.6. *Let $X, Y \subseteq \omega$. Then:*

- (a) $X \equiv_e K_X$.
- (b) $X \leq_e Y \implies X'_e \leq_e Y'_e$. *So, in particular, the enumeration jump is degree-invariant.*
- (c) $X <_e X'_e$.

Proof sketch. Evidently, $X \leq_e K_X$, since $X = \Gamma_e(X)$ for some e . Then $x \in X$ iff $\langle e, x \rangle \in K_X$. On the other hand, for every $e, x < \omega$, we have $\langle e, x \rangle \in K_X$ iff $x \in \Gamma_e(X)$. Because there is a single c.e. set whose e th column is the e th c.e. set, this can be determined from an enumeration of X . This completes the proof of (a).

Assume that $X \leq_e Y$. Then every column of K_X appears as a column of K_Y . Further, that the proof of transitivity of $\leq_e$ produces a total computable function f such that if $Z \leq_e X$ via Γ_e , then $X \leq Y$ via $\Gamma_{f(e)}$. Thus, the $f(e)$ th column of K_Y is equal to the e th column of K_X , and we have that $K_X \leq_e K_Y$. Similarly, $\overline{K_X} \leq_e \overline{K_Y}$. This proves (b).

We show that $\overline{K_X} \not\leq_e X$. The result then follows. Assume that $\overline{K_X} \leq_e X$. Then $\overline{K_X^*} \leq_e X$ where $\overline{K_X^*} = \{e \mid \langle e, e \rangle \in \overline{K_X}\}$. Suppose that $\overline{K_X^*} \leq_e X$ via Γ_e . Then

$$\begin{aligned} e &\in \overline{K_X^*} \\ \iff \langle e, e, \rangle &\in K_X \\ \iff \langle e, e, \rangle &\notin \overline{K_X} \\ \iff \langle e, e \rangle &\notin \overline{K_X^*}, \end{aligned}$$

a contradiction. □

3.1.1 Enumeration reduction and group theory

Here we discuss how the enumeration degrees interact with group theory.

Proposition 3.7. *Let $\langle X \mid R \rangle \cong \langle Y \mid S \rangle$ be presentations for a group G with X and Y both finite. Then $W(G, X) \equiv_e W(G, Y)$.*

Proof. As in the Turing-reducibility case: since X is finite, the map $\hat{\iota} : X \rightarrow W(G, Y)$ which determines the group isomorphism $\iota : G \rightarrow H$ is computable. Hence to determine if $w \in \text{Words}(X^\pm)$ is 1 in G , wait for its image to appear in $W(H, Y)$. □

Proposition 3.8. *For every group $G = \langle X \mid R \rangle$, we have that $W(G, X) \leq_e R$.*

Proof. $W(G, X) = \langle \langle R \rangle \rangle$ in F_X . This is enumerable from R . □

Proposition 3.9. *For every $S \subseteq \omega$, there is a finitely generated group G with $W(G) \equiv_e S$.*

Proof. Recall the argument from Proposition 1.46: In that proof, we formed the group

$$H = \langle a, b, t \mid \{[t, b^{-i}ab^i] : i \in S\} \rangle$$

and argued that $W(H) \equiv_T S$. By Proposition 3.8, we have $W(H) \leq_e R$. On the other hand, $i \in S$ iff $[t, b^{-i}ab^i] = 1$ in H , so $R \leq_e W(H)$. □

Proposition 3.10. *Every simple group has total word problem*

Proof. This is essentially Kuznetsov's argument in Proposition 2.6. Fix K simple and without loss of generality, assume that $K \neq \{1\}$. Take $v \in K$ such that $v \neq 1$. Then, from an enumeration of $W(K)$, we can determine if $v \in \langle \langle w \rangle \rangle$. Then $w \in \overline{W(K)}$ iff $v \in \langle \langle w \rangle \rangle$. (Note that in general is it possible to computably enumerate $\langle \langle w \rangle \rangle$. The use of the enumeration oracle $W(K)$ is to determine if one of these words in $\langle \langle w \rangle \rangle$ is equal to v in K .) □

3.2 Statement and applications of the relativisation

With this background on enumeration reducibility, we are almost ready to state Ziegler's relativisation of Higman's Embedding Theorem. We need one further definition.

Definition 3.11. Let G be a group. A group F is *finitely presented over G* or *finitely presented relative to G* if there is a finite set of generators X and relators R such that

$$F = \langle \text{gen}(G), X \mid \text{rel}(G), R \rangle$$

and the subgroup $\langle \text{gen}(G) \rangle$ of F is isomorphic to G via the natural map.

The following proposition is an immediate consequence of Proposition 3.8.

Proposition 3.12. *Let H be finitely generated. If H is a subgroup of a group which is finitely presented over G , then $W(H) \leq_e W(G, X)$ for any set of generators X of G .*

Martin Zielger proved the converse to this proposition in [Zie80].

Theorem 3.13 (Relativisation of Higman's Embedding Theorem). *Let G and H be finitely generated groups. Then the following are equivalent:*

- (a) $W(H) \leq_e W(G)$
- (b) H is a subgroup of a group which is finitely presented over G .

Notice that we recover Higman's Embedding Theorem by taking $G = \{1\}$.

Before we prove Theorem 3.13, we note that we obtain relativised versions of the corollaries to Higman's Embedding Theorem.

Corollary 3.14. *Every group H such that $W(H, X) \leq_e W(G)$ embeds in a group which is finitely presented over G .*

Proof. It suffices to check that the two-generated overgroup from Theorem 1.47 is e -equivalent to H . $\square$

Corollary 3.15. *Let $\mathbf{a}$ be an enumeration degree. Then there is a universal group for being finitely generated and having word problem enumeration reducible to $\mathbf{a}$.*

Proof. Fix $\mathbf{a}$. Let H be a finitely generated group with $W(H) \leq_e \mathbf{a}$. Enumerate the sets R of words on finite alphabets with $R \leq \mathbf{a}$. Then the free product of all these groups (on formally disjoint alphabets) is e -reducible to $W(H)$. Thus it embeds in a group F finitely presented over H and in particular finitely generated. Since F embeds in a group finitely presented over H , we have $W(F) \leq_e W(H)$, but since H has finitely generated subgroups with word problem of e -degree $\mathbf{a}$, we also have $W(F) \geq_e \mathbf{a} \equiv_e W(H)$. $\square$

The following is a relativisation of the Boone–Higman Theorem.

Corollary 3.16. *A f.g. group G has total word problem iff G embeds in a simple subgroup of a group which is finitely presented over G .*

Proof. Suppose that $G \leq K \leq F$ where F is finitely presented over G with the new relations R . Fix $v \notin W(K)$. Then $w \neq 1$ in G iff $\langle\langle w \rangle\rangle_F \ni v$ iff $\langle\langle \{w\} \cup R \cup W(G) \rangle\rangle_F$ (the normal closure of this set in the group F), which is e -reducible to $W(G)$.

On the other hand, if G has total word problem, then by an argument analogous to that of Theorem 2.13, we can build a simple group K such that the relators of K are $\leq_e G$. By Corollary 3.14, we get that K embeds in a group which is finitely presented over G . $\square$

3.3 Proof of Theorem 3.13

In this section, we show that for finitely generated groups G and H , $W(H) \leq_e W(G)$ iff H is a subgroup of a group finitely presented over G .

The right-to-left direction follows from Proposition 3.8.

For the remainder of this section, we prove the left-to-right direction.

Let $\bar{g}$ and $\bar{h}$ generate G and H respectively. Let Γ be the c.e. set witnessing $W(H) \leq_e W(G)$, and let

$$\mathcal{I}(\bar{x}, \bar{y}) = \left\{ \bigwedge_{w \in D_u} w(\bar{x}) = 1 \rightarrow v(\bar{y}) = 1 \mid \langle u, v \rangle \in \Gamma \right\}.$$

Then, for any group K , we have that $K \models \mathcal{I}(\bar{g}, \bar{h})$ and $\langle \bar{g} \rangle \cong G$, then $\bar{h} \cong H$.

The following claim establishes that each of the implications in $\mathcal{I}$ being satisfied in K is equivalent to the satisfiability of a Σ_1^0 sentence in overgroups of K .

Claim 3.16.1. Let K be a group. For each n , let $I_n(x_0, \dots, x_n)$ be the implication $\bigwedge_{1 \leq i \leq n} x_i = 1 \rightarrow x_0 = 1$. Then for each n , there is a formula $\varphi_n(\bar{z}; \bar{x})$, where $n = |\bar{x}|$, such that for every $a_0, \dots, a_n \in K$, $K \models I(\bar{a})$ exactly when $\exists \bar{z} \varphi_n(\bar{z}; \bar{a})$ is satisfied in some overgroup of K .

Furthermore, φ_n can be chosen of the form $\varphi_n = "w_n(\bar{z}; a_1, \dots, a_n, a_0) = 1"$

The reader could be forgiven for wondering whether this is really progress towards our goal; after all, we have gone from a set of first order implications to a set of Σ_1^1 conditions. However, we will show that these conditions can be combined into a single quantifier-free $L_{\omega_1, \omega}^c$ formula which is satisfied in some overgroup of K iff $\mathcal{I}(\bar{g}, \bar{h})$ holds. By Higman's Embedding Theorem, we will be able to simplify this to a formula of first order logic, and we will be able to read the new relations R off of this formula.

Proof of Claim. The proof proceeds by induction on n . Suppose $G \models a_0 = 1 \rightarrow a_1 = 1$. We show this is equivalent to $\varphi_1(x, y) = "a_1 = [[a_0, x], y]"$ being solvable over K .

The backwards direction is straightforward.

In the other direction, suppose $G \models a_1 = 1 \rightarrow a_0 = 1$. This is equivalent to $G \models a_1 \neq 1 \vee a_0 = 1$. If $a_0 = 1$, then $x = y = 1$ solves φ_1 in G . So suppose $a_1 \neq 1$. In the first case, $a_0 = a_1$. Let $\langle a' \rangle$ be such that $\langle a' \rangle \cong \langle a_0 \rangle$. Consider $G \times \langle a' \rangle$. Then let H_0 be the HNN extension $(G \times \langle a' \rangle)_{a_0 \leftrightarrow a'}$. Then $t^{-1}a_0t = a'$ and the order of $a_0^{-1}a'$ equals the order of a' , so we may take a further HNN extension $H_1 = H_0_{a_0^{-1}a' \leftrightarrow a'}$. Then

$$\begin{aligned} s^{-1}a_0^{-1}a's &= a' \\ s^{-1}a_0^{-1}t^{-1}a_0ts &= t^{-1}a_0t \\ a_0 &= [[a_0, t], s]. \end{aligned}$$

Now, suppose $a_1 \neq a_0$, and $a_0, a_1 \neq 1$. Then let H be a nontrivial group and $h \in H_0 \setminus \{1\}$. Then, in $G * H_0$, both $[a_0, h]$ and $a_1[a_0, h]$ have infinite order. (Here we use that $a_0 \neq a_1$.) Thus, they are conjugate in the HNN extension $H_1 = (G * H_0)_{[a_0, h] \leftrightarrow a_1[a_0, h]}$, as required. This completes the $n = 1$ case.

Now suppose that φ_{n-1} has been defined and satisfies the conclusions of the claim. Notice that $I_n(a_0, \dots, a_n)$ is equivalent to $a_n = e \rightarrow I_{n-1}(a_0, \dots, a_{n-1})$. By hypothesis, $I_{n-1}(a_0, \dots, a_{n-1})$ is equivalent to the solvability of $\exists \bar{z} \varphi(\bar{z}; \bar{a})$ in some overgroup of K . Let H be any such overgroup. In H , we have that $a_n = 1 \rightarrow \varphi_{n-1}(a_0, \dots, a_{n-1})$. Further, $\varphi_{n-1}(a_0, \dots, a_{n-1})$ is of the form

$w_{n-1}(a_0, \dots, a_{n-1}) = 1$ by the induction hypothesis. Applying the $n = 1$ case, we get that $a_n = 1 \rightarrow w_{n-1}(a_0, \dots, a_{n-1}) = 1$ in H iff $w_{n-1}(a_0, \dots, a_{n-1}) = [[a_n, z_0], z_1]$ is solvable in an overgroup of H . Thus, if $K \models I_n(\bar{g})$ then $\exists z_0, z_1 (w_{n-1}(a_0, \dots, a_{n-1}) = [[a_n, z_0], z_1])$ is solvable in an overgroup of K . The converse is clear.

This finishes the proof of the claim. $\dashv$

Now, returning to the main proof, write $\theta_{u,v}$ for the implication $\bigwedge_{w \in D_u} w(\bar{x}) = 1 \rightarrow v(\bar{y}) = 1$. Then, what we have so far is that $K \models \mathcal{I}(\bar{g}, \bar{h})$ iff $\bigwedge_{\langle u,v \rangle \in \Phi} \exists \bar{z}_{\langle u,v \rangle} \theta_{u,v}(\bar{z}_{u,v}, \bar{w}(\bar{g}), v(\bar{h}))$ is satisfied in an overgroup of K . (Here we use that if H and H' are overgroups of K , then there is a further overgroup $H *_K H'$ of K containing both H and H' (and hence satisfying all the Σ_1^0 sentences of both H and H').)

The next claim shows that we may in fact choose a finite collection of $\bar{z}$ witnessing the existential. Suppose that there is some $L \geq K$ which satisfies $\bigwedge_{\langle u,v \rangle \in \Phi} \exists \bar{z}_{\langle u,v \rangle} \varphi_{u,v}(\bar{z}_{u,v}, \bar{w}(\bar{g}), v(\bar{h}))$. Let $(a_i)_{i < \omega}$ be an enumeration of the witnesses for the existentials. Then the claim shows that, at the expense of moving to a further overgroup, there are b_0 and b_1 such that a_i can be expressed as a word in b_0 and b_1 .

Claim 3.16.2. There is a computable sequence of words $\{v_i(x, y) \mid i \in \omega\}$ such that for every sequence $\{a_i \mid i < \omega\}$ from a countable group L , the system of equations $\{v_i(x, y) = a_i \mid i < \omega\}$ has a solution over L .

Proof. This follows by tracing through the proof of the Higman–Neumann–Neumann Embedding Theorem (Theorem 1.47), choosing an enumeration $(k_i)_{i < \omega}$ of K where $k_{2i+1} = a_i$.

(Neumann obtains the simpler expression $t_i(x, y) = [[x, y^{2i+1}], x]$ in [?].) $\dashv$

Hence, $L \models \bigwedge_{\langle u,v \rangle \in \Phi} \exists \bar{z}_{\langle u,v \rangle} \theta_{u,v}(\bar{z}_{u,v}, \bar{w}(\bar{g}), v(\bar{h}))$ iff there is some $L' \geq L$ satisfying

$$L' \models \exists z_0, z_1 \left(\bigwedge_{\langle u,v \rangle \in \Phi} w_{u,v}(\bar{t}_{u,v}(z_0, z_1), \bar{w}(\bar{g}), v(\bar{h})) = 1 \right)$$

Let $\psi(\bar{x}, \bar{y})$ be the formula $\bigwedge_{\langle u,v \rangle \in \Phi} w_{u,v}(\bar{t}_{u,v}(z_0, z_1), \bar{w}(\bar{g}), v(\bar{h})) = 1$. Notice that $\{w_{u,v} \mid \langle u, v \rangle \in \Phi\}$ is c.e. Let

$$\tilde{F} = \langle a_0, a_1, \bar{g}', \bar{h}', \bar{f} \mid R(a_0, a_1, \bar{g}', \bar{h}', \bar{f}) \rangle$$

be a finitely presented group embedding the group

$$\langle z_0, z_1, \bar{g}, \bar{h} \mid \{w_{u,v}(\bar{t}_{u,v}(z_0, z_1), \bar{w}(\bar{g}), v(\bar{h})) : \langle u, v \rangle \in \Gamma\} \rangle,$$

as provided by Higman's Embedding Theorem.

We need one further trick to finish the proof.

Claim 3.16.3. For every n , there is a formula η_n (uniform in n) such that for every group M and every $\bar{a}, \bar{b} \subseteq M$ with $|\bar{a}| = |\bar{b}| = n$, the map $\bar{a} \mapsto \bar{b}$ is a homomorphism iff $\exists \bar{x} \eta_n(\bar{a}, \bar{b}, \bar{x})$ is solvable over M .

Proof. Let $\langle \bar{a} \rangle, \langle \bar{b} \rangle \subseteq G$. Suppose $\langle \bar{a} \rangle \twoheadrightarrow \langle \bar{b} \rangle$ and let $\langle \bar{a}' \rangle \cong \langle \bar{a} \rangle$. Then in $G \times \langle \bar{a}' \rangle$, the map $\bar{a}' \mapsto (\bar{b}, \bar{a})$ generates an isomorphism. Hence, taking two HNN extensions, we can find an overgroup H which

has elements s and t such that

$$\begin{aligned} t^{-1}\bar{a}t &= \bar{a}' \\ s^{-1}\bar{a}s &= \bar{a}'\bar{b} = t^{-1}\bar{a}t\bar{b} \end{aligned}$$

Let

$$\eta_n(\bar{a}, \bar{b}, x, y) = \bigwedge y^{-1}\bar{a}y = x^{-1}\bar{a}x\bar{b} \wedge \bigwedge [\bar{b}, x^{-1}ax] = 1.$$

By construction, if $\langle \bar{a} \rangle \twoheadrightarrow \langle \bar{b} \rangle$, then η_n is solved in an extension.

On the other hand, suppose $H \geq G$ solves η_n and suppose $w(\bar{a}) = 1$. Then

$$\begin{aligned} s^{-1}w(\bar{a})s &= 1 \\ w(s^{-1}\bar{a}s) &= 1 \\ w(t^{-1}\bar{a}t\bar{b}) &= 1 \\ t^{-1}w(\bar{a})tw(\bar{b}) &= 1 \\ w(\bar{b}) &= 1 \end{aligned}$$

where the last line follows since $w(\bar{a}) = 1$. -1

We claim that

$$E(\bar{x}, \bar{y}, z_0, z_1, \bar{x}', \bar{y}', z'_0, z'_1, \bar{c}, \bar{t}) = R(\bar{x}', \bar{y}', z'_0, z'_1, \bar{c}') \cup \eta_n(\bar{x}'\bar{y}; \bar{x}', \bar{y}'; \bar{t})$$

is the desired relating set. More precisely, set

$$F = \langle a_0, a_1, \bar{g}', \bar{h}', \bar{f}, \bar{g}, \bar{b} \mid R(a_0, a_1, \bar{g}', \bar{h}', \bar{f}), \eta(\bar{g}, \bar{g}', \bar{b}), W(G, \bar{g}) \rangle.$$

Let's pause briefly to recollect where each piece of F comes from, and how it serves our purposes.

Firstly, the finite set of relations R is provided by Higman's Embedding Theorem and ensures that each of the words $w_{\langle u, v \rangle}$ coding the enumeration reduction $W(H) \leq_e W(G)$ is equal to 1 in F . Hence, in F , we have that $W(\langle h' \rangle) \subseteq \Gamma(W(\langle g' \rangle))$. Thus, if $\langle g' \rangle$ is isomorphic to a quotient of G (i.e., $W(\langle g' \rangle) \subseteq W(G, \bar{g}')$), then we will have that $\langle h' \rangle$ is isomorphic to a quotient of H .

Recall that $\eta(\bar{g}, \bar{g}', \bar{b})$ was defined so that the map $g_i \mapsto g'_i$ is a homomorphism (and thus in fact an epimorphism). Hence, in the presence of $W(G, \bar{g})$, we have that $\eta(\bar{g}, \bar{g}', \bar{b})$ guarantees that $\langle g' \rangle$ is in fact isomorphic to a quotient of G , and hence that $\langle \bar{h}' \rangle$ is isomorphic to a quotient of H , by above.

To finish the proof, we need to verify firstly, that $G \leq F$ via the identity map (so that F is finitely presented relative to G), and secondly, that $H \leq F$. We already know that $\langle g \rangle$ and $\langle h' \rangle$ are isomorphic to quotients of G and H respectively. Thus, since F is the free-est group (i.e., has the fewest relations) of any group satisfying $R(a_0, a_1, \bar{g}', \bar{h}', \bar{f}) \cup \eta(\bar{g}, \bar{g}', \bar{b}) \cup W(G, \bar{g})$, it suffices to show that there is a group satisfying all of these relations, and such that $\langle h' \rangle \cong H$ and $\langle G \rangle \cong G$.

Consider $L = G * (\tilde{F}/W(G, \bar{g}'))$. Then, evidently $G \leq L$. Further, by definition of $\tilde{F}$, we have that $R(a_0, a_1, \bar{g}', \bar{h}', \bar{f})$ holds in L . Moreover, the map $g_i \mapsto g'_i$ is a homomorphism since we have quotiented out by $W(G, \bar{g}')$, and so there is an overgroup $L' \geq L$ and $\bar{b} \in L'$ such that $\eta(\bar{g}, \bar{g}', \bar{b})$ holds in L' .

The argument for H is similar.

4 Ziegler's Definability Theorem

Higman's Embedding Theorem and its relativisation only code positive data (i.e., words which are equal to 1) in the group which is being embedded. That we get an embedding and not a quotient comes down to the fact that we are mapping into the free-est group satisfying that some collection of words is equal to the identity.

However, we have already seen that this operation can hide computational complexity; for example, there are finitely presented groups whose word problem computes the halting set.

In this section, we prove a theorem, due to Martin Ziegler, that upgrades the “algebraic” side of the Higman Embedding Theorems to capture the full *atomic diagram*, namely the word and co-word problem of the group. The computability-theoretic side uses *Ziegler-reducibility*, which was introduced by Martin Ziegler for this purpose. It is worth noting that Ziegler-reducibility extends the known Q -reducibility (or quasi-reducibility) which had been studied on the c.e. sets. CITE!!

We will state Ziegler's Theorem here and provide definitions for undefined terms in the upcoming sections.

Theorem 4.1 (Ziegler's Definability Theorem). *Let G and H be finitely generated groups. Then the following are equivalent:*

- (a) $W(H) \leq^* W(G)$
- (b) *There is a quantifier-free formula $\varphi(\bar{x}, \bar{h}, \bar{g})$ of group theory, for which there is a group $F \geq G$ satisfying $\exists \bar{x} \varphi(\bar{x}, \bar{h}, \bar{g})$, and such that for any such group, with $\langle \bar{g} \rangle \cong G$, we must have $\langle \bar{h} \rangle \cong H$.*

The “unrelativised case”, i.e., when $G = \{1\}$ states that a f.g. group H has solvable word problem iff there is a single existential formula which guarantees that a given set of parameters $\bar{h}$ in a group F is isomorphic to H . Thus, groups with solvable word problem are in some sense definable (at the expense of going to an overgroup). This was first stated by Rips in [Rip82] but was probably known earlier.

4.1 Ziegler reducibility

Fix as before a computable enumeration D_u of the finite subsets of ω and a computable enumeration D_v^1 of the finite subsets of ω of cardinality at most 1.

Definition 4.2. Let X, Y be subsets of ω . Then we define $X \leq_e^1 Y^3$ if there is a c.e. set W such that

$$x \in X \iff \exists \langle x, u, v \rangle \in W \ (D_u \subseteq Y \wedge D_v^1 \subseteq \bar{Y}).$$

Then $X \leq^* Y$ (pronounced “ X is *Ziegler-reducible* to Y ” or “ X is **-reducible* to Y ” iff

$$X \leq_e Y \quad \text{and} \quad \bar{X} \leq_e^1 Y$$

First things first, we verify that this relation is reflexive and transitive.

Proposition 4.3. *Let $X, Y, Z \subseteq \omega$. Then*

- (a) $X \leq^* X$

³This is misleading notation as this relation is not transitive

$$(b) \ X \leq^* Y \leq^* Z \implies X \leq^* Z$$

Proof sketch. The first claim is clear.

Towards the second, we have that $X \leq_e Z$ since enumeration reducibility is transitive. To check that $x \notin X$, we ask Y finitely many positive questions and one negative question per axiom. The one negative question can be converted, via the 1- e -reduction from $\bar{Y}$ to Z to finitely many positive questions and one negative question being put to Z . Thus, $\bar{X} \leq_e^1 Z$. $\square$

Thus, $\leq^*$ induces an ordering on the equivalence classes given by $X \equiv^* Y$ iff $X \leq^* Y$ and $Y \leq^* X$. We call this structure the *Ziegler degrees* and denote it $\mathcal{D}^*$.

Proposition 4.4. *$\mathcal{D}^*$ is an upper semilattice with least element $\mathbf{0}_*$, which consists of exactly the computable sets, and join given by $X \oplus Y$.*

We further note that Ziegler reducibility in general properly refines Turing reducibility and enumeration reducibility.

Proposition 4.5. *Let $X, Y \subseteq \omega$. Then $X \leq^* Y$ implies $X \leq_T Y$ and $X \leq_e Y$.*

Further, via the usual argument, we have the following.

Proposition 4.6. *Let G and H be a finitely generated groups. Then*

- (a) *if $\bar{g}$ is a finite generating set for G and X is any generating set, then $W(G, \bar{g}) \leq^* W(G, X)$*
- (b) *if $G \leq H$, then $W(G) \leq^* W(H)$*

Finally, the following is due to Ziegler in CITE. It is more complicated than the traditional argument (as, indeed, there are m -degrees that do not contain the word problem of a finitely generated group.)

Theorem 4.7. *Every $*$ -degree contains the word problem of a finitely generated group.*

Exercise 4.8. Show that $W(G \times H) \equiv^* W(G) \oplus W(H)$.

4.2 Warm-up to the proof of Ziegler's definability theorem

In this section, we give an example of Ziegler's definability theorem for a group with a solvable word problem. We then sketch the proof of the theorem for groups with solvable word problem, following [Rip82].

Example 4.9. We will find the formula φ which isolates $\mathbb{Z}$.

Consider the group

$$F = \langle a, b, c, d \mid a^{-1}ba - b^2, b^{-1}cb = c^2, c^{-1}dc = d^2, d^{-1}ad = a^2 \rangle.$$

Higman showed in CITE that this group is infinite and has no finite quotients. For us, it will be enough to show that $\langle a \rangle_F \cong \mathbb{Z}$ and any quotient of F for which $a \neq 1$ still has $\langle a \rangle \cong \mathbb{Z}$. Then the formula for $\mathbb{Z}$ is

$$\varphi(x, y, z, a) = "a^{-1}xa - x^2 \wedge x^{-1}yx = y^2 \wedge y^{-1}zy = z^2 \wedge z^{-1}az = a^2 \wedge a \neq 1".$$

F can be built from HNN extensions and free products with amalgamation. Let $G_0 = \langle a, b \mid a^{-1}ba = b^2 \rangle \cong \langle b \mid \rangle *_{b \mapsto b^2}$. Similarly, we may define G_1, G_2 , and G_3 to be the corresponding groups of generators $\langle b, c \rangle, \langle c, d \rangle$, and $\langle d, a \rangle$ respectively. Then by the normal form theorem for HNN extensions, we have that $\langle b \rangle_{G_0} \cong \langle b \rangle_{G_1}$, so we may form the amalgamated free product $H_0 = G_0 *_{\langle b \rangle} G_1$. A similar argument gives the amalgamated free product $H_1 = G_2 *_{\langle d \rangle} G_3$. Then the normal form theorem for free products guarantees that the subgroups $\langle a, c \rangle_{H_0}$ and $\langle a, c \rangle_{H_1}$ are freely generated by the given generators, and hence we may form a final amalgamated free product $F = H_0 *_{\langle a, c \rangle} H_1$. By construction, $\langle a \rangle \cong \mathbb{Z}$ in F .

Now, we finish by showing that if $\tilde{F}$ is a quotient of F and $a \neq 1$ in $\tilde{F}$, then $\langle a \rangle_{\tilde{F}} \cong \mathbb{Z}$. Suppose that $a \neq 1$, but that a has order n for some finite n (i.e., $a^n = 1$ and $a^m \neq 1$ for every $0 < m < n$). If n is even, then by the last relation, $a^{\frac{n}{2}} = 1$, contradicting minimality, so n is odd.

Furthermore, the order of a being finite implies that the order of every generator is finite. Since a has order n , we get that $b = a^{-n}ba^n = b^{2^n}$, so b has order dividing $2^n - 1$, and in particular has odd order. That the orders of the other generators are finite follows similarly.

We finish by obtaining a contradiction. Let n_a , respectively n_b , etc, be the orders of a, b , etc, and p_a, p_b , etc, be the least prime multiples of n_a, n_b , etc. By the argument above, we have that

$$a^{2^{n_d}-1} = 1,$$

so n_a , and hence p_a , divides $2^{n_d} - 1$. Thus $2^{n_d} \equiv_{p_a} 1$. Let k be the order of 2 in $(\mathbb{Z}/p_a\mathbb{Z})^\times$. Note that $k > 1$ since p_a is odd. Then $k \mid n_d$ and by Lagrange's Theorem $k \mid p_a - 1$. Hence $p_d \leq k < p_a$.

A similar series of arguments show that $p_a < p_b < p_c < p_d < p_a$, a contradiction.

Hence, any group satisfying $\exists x, y, z (a^{-1}xa - x^2 \wedge x^{-1}yx = y^2 \wedge y^{-1}zy = z^2 \wedge z^{-1}az = a^2 \wedge a \neq 1)$ must have $\langle a \rangle \cong \mathbb{Z}$.

4.3 Proof of Ziegler's definability theorem

We start with what has been the “easy” direction in our earlier embedding theorems.

Theorem 4.10 (First half of Ziegler's Theorem). *Let G and H be f.g. groups with such that there is a sentence $\exists \bar{x} \varphi(\bar{x}, \bar{g}, \bar{h})$, where φ is quantifier-free and consists of conjunctions of atomics and negative atomics, which is satisfied in some group $F \geq G$, and for every such group F , we have that $\langle \bar{h} \rangle \cong H$.*

Proof. Note that $\varphi(\bar{f}, \bar{g}, \bar{h})$ states that some collection of words in $\bar{f}, \bar{g}, \bar{h}$ is equal to the identity, and some collection of words in the same elements are not. Let $\varphi^+(\bar{f}, \bar{g}, \bar{h})$ consist of only the equations.

Then, since φ is consistent with G , the group

$$F = \langle \bar{f}, \bar{g}, \bar{h} \mid \varphi^+(\bar{f}, \bar{g}, \bar{h}) \cup W(G, \bar{g}) \rangle$$

satisfies φ . Hence $\langle \bar{h} \rangle_F \cong H$.

Thus, by Proposition 3.8, $W(H) \leq_e \varphi^+(\bar{f}, \bar{g}, \bar{h}) \cup W(G)$.

On the other hand, $w \notin W(H)$ iff $\langle \langle w(\bar{h}) \cup \varphi^+(\bar{f}, \bar{g}, \bar{h}) \cup W(G, \bar{g}) \rangle \rangle$. Using $W(G)$ as an enumeration oracle, one can enumerate both this subgroup, say as $(n_i)_{i < \omega}$ and $W(F)$. If there is some $n_i \cdot v(\bar{f}, \bar{g}, \bar{h})^{-1}$ and $v \neq 1$ is a conjunct of φ^{-1} , then $w(\bar{h}) \notin W(H)$. Further, if there is $n_i \cdot v(\bar{g})^{-1}$, and $v(\bar{g}) \in W(G)$, then $w(\bar{h}) \in W(H)$. $\square$

The other direction follows similarly to the proof of Theorem 3.13 (the relativisation of Higman's Embedding Theorem).

Theorem 4.11 (Second half of Ziegler's Theorem). *Let G and H be finitely generated groups with $W(G) \leq^* W(H)$. Then there is a quantifier-free formula $\varphi(\bar{x}, \bar{g}, \bar{h})$ which is solved in some overgroup of G and for any such overgroup F , we have that $\langle \bar{h} \rangle_H \cong H$.*

Proof. Suppose that $W(H) \leq_e W(G)$ via the c.e. set Φ and $\overline{W(H)} \leq_e^1 W(G)$ via the c.e. set Ψ .

As in the proof of Theorem 3.13, we may form the sets of implications

$$\mathcal{I} = \left\{ \bigwedge_{w \in D_u} w(\bar{g}) = 1 \rightarrow v(\bar{h}) = 1 \mid \langle v, u \rangle \in \Phi \right\}$$

and

$$\mathcal{J} = \left\{ \bigwedge_{w \in D_u} w(\bar{g}) = 1 \wedge \bigwedge_{\tilde{w} \in D_t^1} \tilde{w}(\bar{g}) \neq 1 \rightarrow v(\bar{h}) \neq 1 \mid \langle v, u, t \rangle \in \Psi \right\}$$

Then in any group F with elements $\bar{g}, \bar{h} \subseteq F$ and $\langle \bar{g} \rangle_F \cong G$, and such that F satisfies $\mathcal{I}(\bar{g}, \bar{h}) \cup \mathcal{J}(\bar{g}, \bar{h})$, we have that $\langle \bar{h} \rangle \cong H$.

Notice that since $|D_t^1| \leq 1$, each implication in $\mathcal{J}$ is logically equivalent to one of the form

$$\bigwedge_{w \in D_u} w(\bar{g}) = 1 \wedge v(\bar{h}) = 1 \rightarrow \tilde{w}(\bar{g}) = 1$$

or

$$\bigwedge_{w \in D_u} w(\bar{g}) = 1 \wedge v(\bar{h}) = 1 \rightarrow \perp$$

Let c be a new symbol, and suppose we are in a group with the relation $c \neq 1$. Then we can rewrite the second form as

$$\bigwedge_{w \in D_u} w(\bar{g}) = 1 \wedge v(\bar{h}) = 1 \rightarrow c = 1$$

Now we have a c.e. set of implications of the form $\bigwedge_{w \in D_u} w(\bar{g}, \bar{h}) = 1 \rightarrow v(\bar{g}, \bar{h}, c) = 1$.

Exactly as in the proof of the relativisation of Higman's Embedding Theorem (Theorem 3.13), we obtain a finite set of equations $\hat{R}(\bar{a}, \bar{b}, c, c'\bar{g}, \bar{h}, \bar{f}) = R(\bar{a}, \bar{g}', c'\bar{h}, \bar{f}) \cup \eta(\bar{g}, c, \bar{g}', c', \bar{b})$ (in the notation of that proof). The desired formula is

$$\varphi = \bigwedge_{r \in \hat{R}} r = 1 \wedge c \neq 1.$$

□

5 Existentially closed groups

Ziegler proved his definability theorem in the context of *existentially closed groups*, where he solved several open questions concerning the possible classes of finitely generated subgroups of e.c. groups and properties which are first-order definable in the class of e.c. groups.

We start with the definitions.

5.1 Existential closure

We work in this section in the language of group theory, but everything below can be done for an arbitrary theory, for an arbitrary language. For some of the theorems to go through as stated, one needs to assume that the language is countable or that T is a $\forall\exists$ theory.

Definition 5.1. A group M is existentially closed if for any $\bar{m} \subseteq M$ and any quantifier-free formula $\varphi(\bar{x}, \bar{m})$ such that there is $N \supseteq M$ and $\bar{n} \subseteq N$ with $(\bar{n}, \bar{m})$, then there is already $\bar{a} \subseteq M$ with $\varphi(\bar{a}, \bar{m})$.

For the purposes of this definition, a quantifier-free formula can be thought of as a conjunction of atoms of the form $w(\bar{x}, \bar{m}) = 1$ and negated atoms of the form $w(\bar{x}, \bar{n}) \neq 1$.

We start with some simple observations about existentially closed groups, to get used to the definition.

Proposition 5.2. *Let M be a countable e.c. group. Then:*

- (a) M is infinite
- (b) M is not finitely generated
- (c) M is not abelian
- (d) M contains as a subgroup every f.g. computable group
- (e) M is simple
- (f) M is ω -homogeneous; i.e., any isomorphism between finitely generated subgroups of M extends to an automorphism of M . (In fact, the automorphism can be taken to be conjugation.)

Proof. (a) Let $m_0, \dots, m_{n-1} \in M$. Then $\exists x (x \neq m_0 \wedge \dots \wedge x \neq m_{n-1})$ is solved in $M * \langle a \rangle$ by a , and so there is some $m_n \in M$ with $m_n \neq m_0, \dots, m_n \neq m_{n-1}$. Thus $M \neq \{m_0, \dots, m_{n-1}\}$.

(b) Let $m_0, \dots, m_{n-1} \in M$. Then $\exists x, y ([x, m_0] = 1 \wedge \dots \wedge [x, m_{n-1}] = 1 \wedge [x, y] \neq 1)$ is solved in the HNN extension of M associated to the map $\text{id} : \langle m_0, \dots, m_{n-1} \rangle \rightarrow \langle m_0, \dots, m_{n-1} \rangle$.

(c) Exercise

(d) By the “unrelativised” Ziegler definability theorem, for any f.g. computable group G , there is a quantifier-free $\varphi(\bar{x}, \bar{y})$ such that there is a group satisfying φ and such that any group satisfying $\varphi(\bar{h}, \bar{g})$ has $\langle \bar{g} \rangle \cong G$. Let H be a group satisfying $\exists \bar{x} \bar{y} \varphi(\bar{x}, \bar{y})$. Then $H * M$ also satisfies $\exists \bar{x}, \bar{y} \varphi$, so M satisfies $\exists \bar{x}, \bar{y} \varphi$. Hence M contains an isomorphic copy of G .

(e) Let $m_0, m_1 \in M$. Then, in $M * \langle s \rangle$, the elements $[m_0, s]m_1$ and $[m_0, s]$ both have infinite order, so we may take the HNN extension along the isomorphism which sends $[m_0, s]m_1 \mapsto [m_0, s]$. In this overgroup, there is t such that $t^{-1}[m_0, s]m_1t = [m_0, s]$, whence there are $a, b \in M$ with $b^{-1}[m_0, a]m_1b = [m_0, a]$, witnessing that $m_1 \in \langle \langle m_0 \rangle \rangle$.

(f) Exercise (Hint: use HNN extensions)

□

Proposition 5.3. *The class of e.c. groups is not first order.*

Proof. This proof is beyond the scope of this course, but I include it here for readers who know about ultraproducts, or are inspired by the argument to learn about them.

Let M be existentially closed. Then, by the last part of the previous proposition, any two elements of infinite order in M are conjugate. Let $M^{\mathcal{U}}$ be a non-principal ultrapower of M . By Łoś's Theorem, $M^{\mathcal{U}} \equiv M$. However, we show that $M^{\mathcal{U}}$ cannot be e.c. by finding two elements of infinite order that are not conjugate. Indeed, take an element $\mathbf{g} = (g_1, g_2, g_3, \dots)$ where g_i has order i , and an element $\mathbf{h} = (h_1, h_2, h_3, \dots)$, where h_i has order ω . (Why do such elements exist?) Then, again by Łoś's Theorem, the order of $\mathbf{g}$ and $\mathbf{h}$ are both infinite. But $M \models \neg \exists t (t^{-1} g_i t = h_i)$ for all i , and hence $\mathbf{g}$ and $\mathbf{h}$ are not conjugate in $M^{\mathcal{U}}$. $\square$

Further, as an immediate consequence of Ziegler's definability theorem, we get the following result.

Theorem 5.4. *Let G and H be finitely generated groups. If $W(H) \leq^* W(G)$, then H embeds into every e.c. group containing G .*

Proof. Let $W(H) \leq^* W(G)$ and let $\varphi(\bar{x}, \bar{g})$ be the quantifier-free formula from Theorem 4.1. Let M be an e.c. group with $G \leq M$. Then $\exists \bar{x} \varphi(\bar{x}, \bar{g})$ is solved in some $F \geq G$ by the consistency condition in Theorem 4.1. Thus, φ is solved in $F * M$, and hence, by existential closure, also in M , say by $\bar{a}$. Thus $H \cong \langle \bar{a} \rangle \cong \bar{a}$. $\square$

In due course, we will prove a converse to this theorem. But first, we should make sure that e.c. groups actually exist!

Theorem 5.5. *For every countable group G , there is an e.c. group M with $M \geq G$.*

For this theorem, it will be important that the theory of groups admits a $\forall \exists$ axiomatisation; namely the axioms of group theory are all of the form $\forall \bar{x} \exists \bar{y}$ (or simpler). For this, it is important that we have a constant symbol for 1.

Proof. Let G be a countable group and let Δ be the quantifier-free diagram of G with parameters from G ; that is, Δ is the set of quantifier-free sentences $\varphi(\bar{g})$. We will abuse notation and write G both for the group itself and for its underlying set. Let T_{gp} be the axioms of group theory. Let C be a countable set of new constant symbols. M will have domain $C \cup G$, and we will inductively build the quantifier-free diagram of M . Let $(\varphi_e(\bar{x}, \bar{c}_e, \bar{g}_e))_{e < \omega}$ be an enumeration of the quantifier-free formulas of group theory with parameters from $C \cup G$ and let $\psi_i(\bar{c}_i, \bar{g}_i)$ be an enumeration of the quantifier-free sentences of group theory with parameters from $C \cup G$.

Stage 0: Let $M_0 = \emptyset$.

Stage $s + 1$, $s \equiv_3 0$: Suppose that M_s has been defined, is a set of quantifier-free sentences in the language of group theory with parameters from $C \cup G$ such that $M_s \cup T_{\text{gp}} \cup \Delta$ is consistent.

Check if $M_s \cup T_{\text{gp}} \cup \Delta \cup \{\exists \bar{x} \varphi_{\frac{s}{3}}(\bar{x}, \bar{c}, \bar{g})\}$ is consistent. If not, $M_{s+1} = M_s$. If so, then let $\bar{d} \subseteq C$ be fresh (i.e., as yet unused) constants. Then set

$$M_{s+1} = M_s \cup \{\varphi_{\frac{s}{3}}(\bar{d}, \bar{c}, \bar{g})\}.$$

By hypothesis, this remains consistent.

Stage $s + 1$, $s \equiv_3 1$: Suppose that M_s has been defined, with the same conditions as above. Since $M_s \cup T_{\text{gp}} \cup \Delta$ is consistent, at least one of

$$M_s \cup \{\neg \psi_{\frac{s-1}{3}}(\bar{c}, \bar{g})\} \quad \text{or} \quad M_s \cup \{\psi_{\frac{s-1}{3}}(\bar{c}, \bar{g})\}$$

is consistent. Let M_{s+1} be one of these theories which is consistent.

Stage $s + 1$, $s \equiv_3 2$: Suppose that M_s has been defined, as above. For the first s elements of $C \cup G$ (with respect to a fixed enumeration of order type ω), say $a_0, \dots, a_{s-1}$, we set

$$M_{s+1} = M_s \cup \{a_i \cdot (a_j \cdot a_k) = (a_i \cdot a_j) \cdot a_k : i, j, k < s\} \cup \{a_i \cdot 1 = 1 \cdot a_i = a_i : i < s\} \\ \cup \{a_i \cdot a_i^* = a_i^* \cdot a_i = 1 \mid i < s\}$$

where a_i^* is a fresh constant.

Now, let $M = \bigcup_s M_s$. We claim that M is complete for quantifier-free formulas in the language of group theory with parameters from $C \cup G$ (i.e., for every such formula ψ , either $\psi \in M$ or $\neg\psi \in M$); that M is a group; that M is existentially closed; and that $G \leq M$.

We first show that M is quantifier-free complete. Indeed, let ψ be a quantifier-free formula in the language of groups with parameters from $G \cup C$. Then $\psi = \psi_i$ for some i , so at stage $s = 3(i + 1)$, we added either ψ_i or $\neg\psi_i$ to M . Thus M is quantifier-free complete.

Verification that M is a group is similar; at stages $s \equiv_3 2$, we guaranteed that all of the elements of M satisfy the group axioms. Here we use that the theory of groups is $\forall\exists$ — to guarantee inductively that $\forall\bar{x}\exists\bar{y}\theta(\bar{x}, \bar{y})$ sentence is satisfied, it suffices to ensure that for any finite subset $\bar{a}$ in the domain, there is some stage s and some finite tuple $\bar{b}$ in the domain such that $\theta(\bar{a}, \bar{b})$ has been added at stage s .

To check that M is existentially closed, let $\exists\bar{x}\varphi(\bar{x}, \bar{c}, \bar{g})$ be such that there is some $N \geq M$ such that N solves $\exists\bar{x}\varphi(\bar{x}, \bar{c}, \bar{g})$. There is some e such that $\varphi = \varphi_e$ and N witnesses that $M_{3e} \cup T_{\text{gp}} \cup \Delta \cup \{\exists\bar{x}\varphi_e(\bar{x}, \bar{c}, \bar{g})\}$ is consistent. Thus, at stage $3e$, we added some tuple $\bar{d}$ that solves φ_e in M .

Finally, to check that $G \leq M$, we note that the subset of M named by the constants of G is isomorphic to G . $\square$

Notice that this construction is in some sense very easy (or, at least, very robust) — at no point when adding sentences to M did we have to worry about possibly injuring later requirements on M . As long as our fairly mild induction hypotheses held, we were always able to add a sentence to further our goals of making M e.c., a group, and quantifier-free-complete.

To formalise this, we need the language of model-theoretic forcing.

5.2 Model-theoretic forcing

[*Note: This section was written while I was in graduate school, and has not been edited for consistency.*]

Throughout this subsection, let T be a first order theory in a countable language L and let C be a countable set of new constant symbols.

The following definitions are in direct analogue with their set-theoretic counterparts (see for example [Kun14]). They are due to Robinson in [BR70].

Definition 5.6. A (T) -condition is a finite collection of p of basic sentences of $L(C)$ such that $T \cup p$ is consistent.

We define a *forcing relation* $p \Vdash^* \varphi$ (read “ p *-forces φ ”), for a condition p and a sentence φ of $L(C)$, inductively as follows:

- If φ is basic, $p \Vdash^* \varphi$ if $\varphi \in p$
- If $\varphi = \psi \wedge \theta$, $p \Vdash^* \varphi$ if $p \Vdash^* \psi$ and $p \Vdash^* \theta$

- If $\varphi = \exists \bar{x} \psi(\bar{x})$, $p \Vdash^* \varphi$ if $p \Vdash^* \psi(t)$ for some closed term t of $L(C)$
- If $\varphi = \neg \psi$, $p \Vdash^* \varphi$ if there is no $q \supseteq p$ with $q \Vdash^* \psi$

We then define $p \Vdash \varphi$ (“ p forces φ ”) if $p \Vdash^* \neg \neg \varphi$.

Remark. It is worth taking the time here to unravel the relation $p \Vdash \varphi$. By definition, $p \Vdash \varphi$ iff $p \Vdash^* \neg \neg \varphi$. Tracing through the definition of $p \Vdash^* \neg \psi$, we get that $p \Vdash^* \neg \neg \varphi$ iff for every $q \supseteq p$, there is $r \supseteq q$ with $r \Vdash^* \varphi$.

The following lemmas are proved in exactly the same way they are proved in standard texts on set-theoretic forcing.

Lemma 5.7. *No condition forces φ and $\neg \varphi$.*

Lemma 5.8. *If $p \subseteq q$ are conditions and $p \Vdash \varphi$, then $q \Vdash \varphi$.*

Lemma 5.9. *If $p \nVdash \varphi$ then there is $q \supseteq p$ with $q \Vdash \neg \varphi$.*

As in set-theoretic forcing, the idea behind model-theoretic forcing is that the forcing relation should somehow build or pick out “special” structures. Again, the following definition is a transcription of the corresponding definition of a generic extension in set theory.

Definition 5.10. An $L(C)$ -structure M is *(T -)generic* if M is consistent with T (i.e., M embeds in a model of T), and for every sentence φ of $L(C)$,

$$M \models \varphi \iff \text{for some } p \subseteq \Delta^{qf}(M; M), p \Vdash \varphi.$$

We will sometimes also use *generic* to describe the reduct $M \upharpoonright L$.

Note that the requirement that M is consistent with T is equivalent to insisting that all $p \subseteq \Delta^{qf}(M; M)$ are in fact T -conditions.

This definition has a number of drawbacks. Firstly, it is not clear that generics always (or even, perhaps, ever) exist. Secondly, even if generic structures were to exist, the definition is very non-constructive — it gives no hint as to how one would go about building a generic structure. We will provide answers to both of these questions. However, we start with an example.

Example 5.11. Let $L = \{<\}$ and let T be the theory which says $<$ is a linear order with a least element.

$M = (\mathbb{Q}; <)$ is a T -generic structure.

M is consistent with T as one can adjoin an element which is less than every element of $\mathbb{Q}$ by compactness. The second requirement follows from quantifier elimination.

This example suggests another question: when, if ever, are T -generic structures models of T ? Are they determined by T (up to elementary equivalence or even isomorphism)?

We address the questions posed before the example first.

Theorem 5.12. *Let T be a theory. Then there is a T -generic structure M .*

Proof. We build a generic structure by specifying its quantifier-free diagram, proceeding in stages.

Order the quantifier-free $L(C)$ sentences as $\langle \varphi_i : i < \omega \rangle$.

At stage 0, let $p_0 = \emptyset$.

At stage $i + 1$, we will decide whether to include φ_i or its negation. Set $p_{i+1} = p_i \cup \{\varphi_i\}$ if $p \not\models \neg\varphi_i$ and $p_i \cup \{\neg\varphi_i\}$ otherwise.

One can check that each of the p_i s is a condition, so the union $\bigcup_i p_i$ is consistent and decides every quantifier-free formula in $L(C)$. Thus, this determines a countable model M built from the constants C .

We now check that M is generic.

Firstly, we need to verify that M embeds in a model of T . In other words, we check if the theory $T \cup \Delta^{qf}(M; M)$ is consistent. It suffices to check that $T \cup p$ is consistent for every finite subset $p \subseteq \Delta^{qf}(M; M)$. Let i be sufficiently large so that $p \subseteq p_i$. Since p_i is a T -condition, $T \cup p_i$ is consistent; hence $T \cup p$ is consistent.

Secondly, we show that $M \models \varphi$ iff $p \Vdash \varphi$ for some $p \subseteq \Delta^{qf}(M; MC)$. The result follows from a structural induction of formula complexity. The base case follows from the construction.

All of the inductive cases are fairly straightforward, so we only lay out the most interesting — negation — to give a flavour of the argument.

Let $\varphi = \neg\psi$ and suppose $M \models \varphi$. Then $M \not\models \psi$, so, by induction, there is no $p \subseteq \Delta^{qf}(M; C)$ with $p \Vdash \psi$. Thus $\emptyset \Vdash \neg\psi$, as required.

In the other direction, suppose there is some $p \subseteq \Delta^{qf}(M)$ with $p \Vdash \neg\psi$. Then for every $q \supseteq p$, $q \not\models \psi$. Hence $M \not\models \psi$, so $M \models \neg\psi$ as required. $\square$

Note that this answers both of our original concerns about the definition of generic structures — not only do they exist, but they can be built up inductively by taking “complete” sequences of conditions.

It motivates the following definition, first laid out in [Zie80] and further embellished with a Hodgesian whimsy in [Hod85].

Definition 5.13. Define a game between two players, $\forall$ (P1) and $\exists$ (P2) where each takes turns playing a T -condition satisfying that $p_{i+1} \supseteq p_i$ for all $i < \omega$, where p_i the condition played on the i th turn.

As above, we can always ensure that the plays give the quantifier-free diagram of a structure. Let $\bar{p} = \langle p_i : i < \omega \rangle$ and call this structure the *compiled structure*, denoted $M(\bar{p})$.

Let Cond be the set of conditions. Then a *strategy* is a function $\sigma : \text{Cond} \rightarrow \text{Cond}$ such that $\sigma(p) \supseteq p$. A player *plays* the strategy σ if they play $\sigma(p)$ upon seeing condition p .

Let P be a property of the sequence of plays or the compiled structure. $\exists$ has a *winning strategy* in $G(P)$ if there is a strategy σ such that by playing σ , $\exists$ ensures that P holds. Call a property P *enforceable* if $\exists$ has a winning strategy for P .

Define the forcing relation $p \text{ } G\text{-forces } \varphi$, for a condition p and an $L(C)$ -sentence φ , by $p \Vdash_G \varphi$ if $\exists$ has a winning strategy for $G(M \models \varphi)$ assuming $\forall$'s first move contains p .

Remark. We pause briefly to remark that this set-up bears similarity to Cohen's motivation for forcing as laid out in [Coh08]. There, he considers adding a new set to a countable transitive model of $V = L$ by working with a poset of conditions consisting of finite, consistent collections of sentences of the form $n \in a$ and $m \notin a$, where m and n are natural numbers and a is a new constant symbol corresponding to the set we are constructing. Picking a complete sequence of conditions corresponds to naming a subset of ω . Cohen shows that if the choice of conditions are “generic enough”, then the subset picked out is non-constructible and, in particular, not in the ground model.

The following proposition is elementary, but will ensure that structures we build using games are nontrivial.

Proposition 5.14. 1. *It is enforceable that each element of M is named by infinitely many constants in C .*

2. *If T has infinite models, then it is enforceable that M is infinite.*

3. *If $p \not\Vdash_G \varphi$, then there is $q \supseteq p$ with $q \Vdash \neg\varphi$.*

However, $\Vdash_G$ to be truly helpful in understanding model-theoretic forcing, we need to see how it relates to Robinson's original definition.

Proposition 5.15. *Let p be a T -condition and φ an $L(C)$ -sentence. Then*

$$p \Vdash_G \varphi \iff p \Vdash \varphi.$$

Proof. Let φ be atomic. Then $p \Vdash \varphi$ iff for every $q \supseteq p$, there is $r \supseteq q$ with $\varphi \in r$. We claim this is equivalent to $p \Vdash_G \varphi$. On the one hand, if for every $q \supseteq p$, there is $r \supseteq q$ with $\varphi \in r$, then $p \Vdash_G \varphi$, since $\exists$ can pick such an r when $\forall$ plays q . On the other hand, if there is $q \supseteq p$ such that for every $r \supseteq q$, $\varphi \notin r$, then we must have that φ is inconsistent with q . Hence, if $\forall$ plays q to start, $\exists$ does not have a winning strategy for $G(M \models \varphi)$.

Let $\varphi = \psi \wedge \theta$. Then $p \Vdash \varphi$ iff $p \Vdash \psi$ and $p \Vdash \theta$. By induction, this occurs iff $p \Vdash_G \psi$ and $p \Vdash_G \theta$. By interleaving $\exists$'s strategies for ψ and θ , one can show that the previous statement is equivalent to $p \Vdash_G \varphi$.

Let $\varphi = \exists x\psi(x)$. Then $p \Vdash \varphi$ iff for every $q \supseteq p$, there is $r \supseteq q$ with $r \Vdash^* \exists x\psi(x)$. Now, $r \Vdash^* \exists x\psi(x)$ iff $r \Vdash^* \psi(t)$ for some term t in $L(C)$. By induction, this is equivalent to $r \Vdash_G \psi(t)$. To finish, it suffices to show that the property

$$\text{for every } q \supseteq p, \text{ there is } r \supseteq q \text{ with } r \Vdash^* \exists x\psi(x)$$

is equivalent to $p \Vdash_G \exists x\psi(x)$.

The forward direction is clear. For the backwards direction, we know that $\exists$ can force that the compiled structure M models $\exists x\psi(x)$. Moreover, $\exists$ can also force that every element of M is named by a constant from C . By interleaving these strategies, we see that $\exists$ can force that $M \models \psi(c)$ for some constant c .

Finally, let $\varphi = \neg\psi$. Then $p \Vdash \varphi$ iff for every $q \supseteq p$, there is $r \supseteq q$ with $r \Vdash^* \neg\psi$. Thus, for every $s \supseteq r$, $s \not\Vdash^* \psi$. By induction, $s \not\Vdash^* \psi$ iff $s \not\Vdash_G \psi$. Hence, to conclude, we need to show that the property

$$\text{for every } q \supseteq p, \text{ there is } r \supseteq q \text{ such that for every } s \supseteq r, s \not\Vdash_G \psi$$

is equivalent to $p \Vdash_G \neg\psi$.

This follows from Lemma 5.14. □

Since both our definitions of a forcing relation are equivalent, we will now just use $\Vdash$.

Moreover, the following proposition is a generalisation of the procedure used in the previous argument to show that if $p \Vdash_G \psi$ and $p \Vdash_G \theta$, then $p \Vdash_G \psi \wedge \theta$.

Proposition 5.16 (Conjunction Lemma). *If $\{P_i : i < \omega\}$ is a countable collection of enforceable properties, then $\bigwedge_i P_i$ is also enforceable.*

Proof. Let σ_i be a winning strategy for $G(P_i)$. $\exists$ partitions ω into infinitely many infinite sets and plays σ_i on the i th partition. Since each set is cofinal in ω , $\exists$ can ensure that each P_i is satisfied. □

Tying together the models associated with each approach to forcing, it turns out that the property of being generic is enforceable.

Proposition 5.17. *For any theory T , the property “ M is T -generic” is enforceable.*

Proof. By construction M is consistent with T (see the proof of Theorem 5.12). Thus we only need to check that the property

$$“M \models \varphi \iff \text{for some } p \subseteq \Delta^{qf}(M; M), p \Vdash \varphi”$$

is enforceable.

The proof is by structural induction on the complexity of φ . □

Definition 5.18. The *forcing companion*, T^f of a theory T is

$$\{\varphi : \varphi \text{ is a sentence of } L \text{ and } \emptyset \Vdash \varphi\}.$$

Equivalently, T^f is the set of all L -sentences which hold in every generic model.

Note that T^f is a theory in the original language L .

So, to better understand model-theoretic forcing and the structures it builds, it will be useful to understand T^f . We have already seen an example — the theory of a linear order with a minimal element — where T is not contained in T^f . Let’s consider another example where it is.

Example 5.19. Let T be the theory of groups in the language $L = \{1, \cdot\}$. (Note that L typically also has a unary function $\bullet^{-1}$ for inversion, but for the purposes of this example, we omit it.)

Claim 5.19.1. Every T -generic structure is a group.

Proof. We show that the axioms of group theory are enforceable.

To show that $\forall x(x \cdot 1 = x)$ is enforceable, it suffices to prove that $c \cdot e = c$ is enforceable for every constant $c \in C$, as the compiled structure can be forced to be built from constants. So, order C as $\langle c_i : i < \omega \rangle$. At stage $2i$, $\exists$ plays $p_{2i} = p_{2i-1} \cup \{c_i \cdot 1 = c_i\}$. This will be a condition since at no point could $\forall$ have blocked it and stayed consistent with T .

The argument that associativity of $\cdot$ is enforceable is similar.

Finally, we show that $\forall x \exists y(x \cdot y = 1)$ is enforceable. Again, because the compiled structure can be forced to be built from constants, it suffices to show that for every $c \in C$, there is a $d \in C$ with $c \cdot d = 1$. At stage $2i$, let $p_{2i} = p_{2i-1} \cup \{c_i \cdot d = 1\}$ for some constant d which has not been used yet. Again, this is a condition because all earlier plays were required to be consistent with T .

Hence, by the Conjunction Lemma (Lemma 5.16), it is enforceable that M satisfies all the axioms of group theory. ⊥

So what went wrong with the theory of linear orders with a least element that didn’t go wrong for groups? The problem was that we could continue adding smaller elements without ever contradicting the theory — but, in the limit, we obtained a structure with no minimal element. In other words, the theory of linear orders with a least element is not closed under unions of chains. The next result shows that T being closed under unions of chains is enough to ensure that every generic structure is a model of T .

Proposition 5.20. $T^f \supseteq T_{\forall\exists}$

We omit the proof, which is very similar to the argument given in the previous example.

However, it is still possible that there is some other generic structure for the theory of linear orders with a least element that is a model of that theory. The next result shows that in fact, $(\mathbb{Q}; <)$ is the only countable generic model of that theory.

Recall that a theory T has the *joint embedding property* (JEP) if, whenever M and M' are models of T , there is a model N of T into which both M and M' embed.

Proposition 5.21. *Suppose T has the joint embedding property. Then T^f is complete.*

Proof. Suppose not, and let φ be a counterexample. Then, because $\emptyset \not\models \varphi$ and $\emptyset \not\models \neg\varphi$, there are p and q with $p \models \varphi$ and $q \models \neg\varphi$. Since φ is a sentence in L , we may assume that p and q have no witnesses in common. Moreover, as both p and q are consistent with T , there are models $N_p \models T \cup p$ and $N_q \models T \cup q$. Let N be a model of T containing both. Since p and q have no constants in common, N witnesses that $p \cup q$ is consistent with T and thus a condition. But $p \cup q \models \varphi \wedge \neg\varphi$, a contradiction. $\square$

The previous proposition shows that the forcing companions for both of the theories we have considered so far — that of linear orders with a minimal element and that of groups — are complete. We give one example of what happens when T does not have the JEP.

Example 5.22. Let T be the theory of fields. Then T^f is the theory of algebraically closed fields. Moreover, $\forall$ can determine a positive characteristic for the generic structure on the first turn, if he wishes.

We will now investigate the generic structures themselves, rather than just their theories. The next theorem justifies the development of the machinery of model-theoretic forcing in the context of studying existentially closed groups.

Theorem 5.23. *T -generic structures are existentially closed models of $T_{\forall\exists}$.*

Proof. We show that for every consistent sentence of the form $\exists \bar{x}\psi(\bar{x})$, where ψ is basic and may have parameters from C , it is enforceable that $M \models \exists \bar{x}\psi(\bar{x})$. When ψ has conjuncts or disjuncts, the argument is similar, but more complicated.

Enumerate all such $\exists_1$ sentences in $L(C)$ as $\langle \varphi_i : i < \omega \rangle$. At stage $2i$, if $\varphi_i = \exists \bar{x}\psi(\bar{x}, \bar{c})$ is consistent with p_{2i-1} , set $p_{2i} = p_{2i-1} \cup \{\psi(\bar{d}, \bar{c})\}$ for a new set of constants $\bar{d}$. Otherwise, let $p_{2i} = p_{2i-1}$.

Now let M be the compiled structure. By the Conjunction Lemma and the fact that “ $M \models T_{\forall\exists}$ ” is enforceable, we may assume that M is a model of $T_{\forall\exists}$. Now suppose that $\varphi = \exists \bar{x}\psi(\bar{x})$ is a sentence in $L(C)$ with ψ basic. Suppose $\varphi = \varphi_i$ in the list above. If φ is modelled by $N \supseteq M$ with $N \models T_{\forall\exists}$, then N witnesses that φ_i is consistent with p_{2i} at stage $2i$. Hence, by the construction, p_{2i+1} will contain a constant witnessing that $M \models \varphi$. $\square$

Thus, model-theoretic forcing gives us another way to build existentially closed groups.

However, not every existentially closed group is generic ([Mac72, Section 9]). Nonetheless, this is almost true; every existentially closed group is generic over some extension of group theory.

Theorem 5.24. *Let N be a countable model of T , and let $T_N = T \cup \Delta^{gf}(N; N)$. Then, in model-theoretic forcing over T_N , it is enforceable that the compiled structure embeds N .*

Moreover, if N is existentially closed for T , then in addition it is enforceable over T_N that the compiled structure is isomorphic to N .

Proof. Since we have added constants for every element of N , the first statement is clear.

To prove the second statement, we show that, for every $c \in C$, we can force that $c = n$ for some $n \in N$. Fix c and suppose that $\forall$ has just played p . Consider $\varphi(\bar{x}, \bar{n})$ defined by replacing the constants in $\bigwedge p$ from C with $\bar{x}$. Then $\exists \bar{x} \varphi(\bar{x}, \bar{n})$ is a quantifier-free $\exists_1$ sentence with parameters from N which is consistent in N . Thus, since N is existentially closed, there is a solution $\bar{m} \in N$. Let n be the element of $\bar{m}$ corresponding to c (if c did not appear in p , then n may be picked arbitrarily). Then, $p \cup \{c = n\}$ will guarantee that c names an element of N . $\square$

5.3 Ziegler's Embedding Theorem

We now prove a converse to Theorem 5.4. We state it as one theorem, and combine it with Theorem 4.1.

Theorem 5.25. *Let G and H be finitely generated groups. Then the following are equivalent.*

- (a) $W(H) \leq^* W(G)$
- (b) *For every e.c. M , if G embeds into M , then H embeds into M*
- (c) *There is a quantifier-free formula $\varphi(\bar{x}, \bar{y}, \bar{g})$ which is satisfied in some group F with $\langle \bar{g} \rangle_F \cong G$, and such that for any such group, the group generated by the interpretation of $\bar{y}$ is isomorphic to H .*

Proof. (a) $\iff$ (c) is just Theorem 4.1 and (a) $\implies$ (b) is Theorem 5.4. Thus, all that is left to check is (b) $\implies$ (a).

We will prove the contrapositive. Suppose that $W(H) \not\leq^* W(G)$. We will build an e.c. group M which contains G and not H . Let Δ be the quantifier-free diagram of G with constants from G and $T_G = T_{\text{gp}} \cup \Delta$. Since being an e.c. model of T_G is enforceable, and any such model is an e.c. group containing G , it suffices to show that $\langle \bar{c} \rangle \not\cong H$ is enforceable.

Let p be a condition and suppose that there is no condition $q \supseteq p$ and no quantifier-free $\psi(\bar{x})$ with $\neg\psi(\bar{c}) \in q$ and $\psi(\bar{h})$ true in H . Then, it must be that, for every $\psi(\bar{x})$, we have that $\bigwedge p \cup T_G \vdash \psi(\bar{c})$. Thus, let $\varphi(\bar{x}, \bar{y}, \bar{g})$ be the formula obtained by replacing the constants $\bar{c}$ in $\bigwedge p$ with $\bar{y}$ and the remainder of the constants with $\bar{x}$. Now φ defines H over G , contradicting that $W(H) \not\leq^* W(G)$. $\square$

5.4 Fraïssé's Theorem

In this subsection, we introduce another way of building structures that can be modified to yield existentially closed models. We start with a number of definitions. Again, we work with general languages and theories, but the familiar unfamiliar with this notion can always assume that $L = L_{\text{gp}}$ and $T = T_{\text{gp}}$.

Recall that a structure M is ω -homogeneous if for every isomorphism $\iota : G \rightarrow H$ of finitely generated submodels G and H of M , there is an automorphism $\alpha : M \rightarrow M$ with $\alpha \supseteq \iota$. As we showed in Proposition 5.2, every e.c. group is ω -homogeneous.

Fraïssé's Theorem characterises the finitely generated substructures of ω -homogeneous structures. Before we can state the theorem, we need a few definitions.

Definition 5.26. Fix a countable language L . Let $\mathcal{K}$ be a countable collection of (isomorphism types of) finitely generated L -structures. Then $\mathcal{K}$ has the:

1. *Hereditary Property (HP)*: for every $B \leq A \in \mathcal{K}$ with B finitely generated, we have $B \in \mathcal{K}$.
2. *Joint Embedding Property (JEP)*: for every $A, B \in \mathcal{K}$, there is some $C \in \mathcal{K}$ with $A, B \leq C$.
3. *Amalgamation Property (AP)*: for every $A, B, C \in \mathcal{K}$, and embeddings $\alpha : C \rightarrow A$ and $\beta : C \rightarrow B$, there is a $D \in \mathcal{K}$ and embeddings $\hat{\alpha} : A \rightarrow D$ and $\hat{\beta} : B \rightarrow D$ such that $\hat{\alpha} \supseteq \alpha$ and $\hat{\beta} \supseteq \beta$.

We call a $\mathcal{K}$ satisfying these properties a *Fraïssé class*.

It is not hard to see that if M is ω -homogeneous, then the collection of finitely generated substructures of M is a Fraïssé class. Fraïssé's Theorem is a converse to this.

Theorem 5.27 (Fraïssé's Theorem). *Let $\mathcal{K}$ be a countable collection Fraïssé class. Then there is a unique, countable, ω -homogeneous M such that $\mathcal{K}$ is exactly the collection of finitely generated substructures of M .*

Note that in the following proof, we use the amalgamation property to ensure that M is ω -homogeneous. If we drop the assumption that $\mathcal{K}$ has AP, we instead get a countable structure M with $\text{Sk}(M) = \mathcal{K}$, which may not be ω -homogeneous and may not be the unique countable structure with $\text{Sk}(M) = \mathcal{K}$.

Proof. We only prove the nontrivial direction. Let $(A_i)_{i < \omega}$ be an enumeration of $\mathcal{K}$.

Stage 0: Let $M_0 = A_0$.

Stage $s + 1$: Suppose that M_s has been constructed and is in $\mathcal{K}$. Let M_{s+1}^0 be a joint embedding of M_s . Then, for every $i, j \leq s$ with $A_i \leq A_j$ and every copy of A_i in M , let M_{s+1}^{k+1} be an amalgamation of A_j and M_{s+1}^k over A_i .

Let $M_{s+1} = \bigcup_k M_{s+1}^k$.

Finally, let $M = \bigcup_s M_s$.

Verification: It is clear from the construction that $\text{Sk}(M) = \mathcal{K}$ and that M is countable. Towards ω -homogeneity, let $\iota : A \cong B$ be an isomorphism between f.g. submodels of M . Then wait until a stage s such that $A, B \subseteq M_s$. We show how to do one step of the construction of an automorphism α of M by back-and-forth. A full proof can be found in Appendix A of [Sin17].

Suppose we have a partial automorphism α_s extending *iota*. Let $a \in M \setminus \text{dom } \alpha_s$. Then $\langle a, A \rangle_M$ is a finitely generated submodel of M and hence is A_j for some j . Wait for $t > s$ with $t > j$. Then $A_i \leq A_j$, and $A_i \cong B \leq M_{s+1}^k$. Thus, in M_{s+1}^{k+1} , there is an overmodel $C \geq B$ with $C \cong A_j$ via an isomorphism extending ι . Let b be the image of a under this isomorphism. We can extend the partial automorphism α_s to $\alpha_{s+1} = \alpha_s \cup \{(a, b)\}$.

Uniqueness of M follows from a similar back-and-forth argument. $\square$

Thus, every e.c. group is the Fraïssé limit of a Fraïssé class of finitely generated groups. However, not every Fraïssé class of groups gives rise to a Fraïssé limit.

Example 5.28. The class of finite groups forms a Fraïssé class. HP is immediate, and JEP follows from the direct product construction. AP is not obvious, but can be shown using Neumann's *permutational products* from [Neu60].

However, every e.c. group contains as a subgroup every finitely generated computable group; in particular $\mathbb{Z}$. Thus the Fraïssé limit of the class of finite groups is not existentially closed. What conditions do we need to add to the definition of a Fraïssé class to guarantee existential closure?

Definition 5.29. Let L be a countable language and $\mathcal{K}$ be a countable set of (isomorphism types of) finitely generated L -structures. Then $\mathcal{K}$ has *existential closure (EC)* if for every quantifier-free formula $\varphi(\bar{x}, \bar{g})$ with parameters from some $G \in \mathcal{K}$, and which is solved in some $\bar{G} \geq G$, there is some $H \in \mathcal{K}$ with $G \leq H$ and $H \models \exists \bar{x} \varphi(\bar{x}, \bar{g})$.

Theorem 5.30 (Fraïssé’s Theorem for existentially closed structures). *Let T be a $\forall\exists$ theory and $\mathcal{K}$ a countable collection of (isomorphism types of) finitely generated T structures. Then there is a countable, ω -homogeneous, existentially closed model M of T such that the finitely generated substructures of M are exactly the elements of $\mathcal{K}$ iff $\mathcal{K}$ is a Fraïssé class satisfying EC.*

Further, such M is unique.

The proof is very similar to the proof of Theorem 5.27.

Thus, every existentially closed group arises as a Fraïssé limit of an existentially closed Fraïssé class.

In fact, since e.c. groups are automatically ω -homogeneous, we get a stronger theorem in the case of groups.

Theorem 5.31. *A countable group M is e.c. iff $\text{Sk}(M)$ satisfies HP, JEP, and EC.*

Example 5.32.

$$\mathcal{K}_{\mathbf{0}_e} = \{G \mid G \text{ is f.g. and } W(G) \text{ is c.e.}\}$$

is an e.c. Fraïssé class of groups.

HP is clear, and JEP follows from the fact that $W(G * H) \equiv_e W(G) \oplus W(H)$. We are left with EC. Let $\bar{g} \subseteq G$ and $\varphi(\bar{x}, \bar{g})$ be solved in some $H \geq G$. Without loss of generality, we assume that $\bar{g}$ generates G . Then we claim that $\varphi(\bar{x}, \bar{g})$ is solved in

$$F = \langle \bar{g}, \bar{f} \mid W(G), \varphi^+(\bar{f}, \bar{g}) \rangle,$$

where $|\bar{f}| = |\bar{x}|$. Let $\bar{h} \subseteq H$ be a solution for φ in H . Then $\langle \bar{h}, \bar{g} \rangle_H$ is a quotient of F , and hence, since $\varphi^-(\bar{h}, \bar{g})$ holds in $\langle \bar{h}, \bar{g} \rangle_H$, it also holds in F . Thus, $\varphi(\bar{f}, \bar{g})$ holds in F ; as required. A similar argument guarantees that $G \leq F$.

To finish the proof, observe that $W(F) \leq_e W(G) \oplus \varphi^+(\bar{f}, \bar{g}) \equiv_e W(G)$, since φ^+ is finite.

In fact the previous example generalises to show that for every enumeration degree $\mathbf{a}$, the finitely generated groups G such that $W(G) \leq_e \mathbf{a}$ forms an e.c. ideal. In fact, we can leverage this argument to show that any countable ideal in the enumeration degree produces an e.c. Fraïssé class.

Proposition 5.33. *Let $\mathcal{I}$ be a countable ideal in $\mathcal{D}_e$, and let $\mathcal{K}_{\mathcal{I}}$ be the collection of f.g. groups G such that $W(G) \in \mathcal{I}^4$. Then $\mathcal{K}_{\mathcal{I}}$ is an e.c. Fraïssé class.*

Proof. Identical to above. □

5.5 Scott sets and e.c. groups

In this section, we show how to build e.c. groups from a *Scott set*. Scott sets are due to Dana Scott and were introduced in his study of nonstandard models of Peano Arithmetic [Sco62].

⁴Throughout this example, we abuse notation by writing “ $X \in \mathcal{I}$ ”, when more properly we should write “ $[X]_e \in \mathcal{I}$ ”, or “ $X \in \bigcup \mathcal{I}$ ”.

Let M be a countable nonstandard model of PA. Scott defines the *standard system* of a countable nonstandard model of PA as the collection of subsets of $\mathbb{N}$ which are coded by some element of M . More precisely, let $(p_n)_{n < \omega}$ be the increasing enumeration of the standard primes. For an element $a \in M$, define

$$X_a = \{n : M \models p_n \mid a\}$$

to be the subset of $\mathbb{N}$ coded by a . Then the *standard system* of M is

$$SS(M) = \{X_a \mid a \in M\}.$$

Scott gave an structural characterisation of the subsets of 2^ω which arise as standard systems of a countable nonstandard model of PA. We now call such subsets of 2^ω *Scott sets*. Before we start, we need the notion of a Π_1^0 -class.

Definition 5.34. A Π_1^0 -class is a subset $P \subseteq 2^\omega$ such that there is a c.e. set W_e such that

$$X \notin P \iff \exists \sigma \in W_e (\sigma \prec X)$$

Thus we can determine in a c.e. way that a set X is *not* in P . Since each σ corresponds to a basic open set in the product topology on 2^ω , the Π_1^0 -classes are exactly the *effectively closed sets*.

We can relativise the notion of Π_1^0 -class in the natural way.

Definition 5.35. Let $Y \subseteq \omega$. Then a Π_1^0 -class relative to Y , denoted $\Pi_1^{0,Y}$, is a subset $P \subseteq 2^\omega$ such that there is a set W , c.e. relative to Y such that

$$X \notin P \iff \exists \sigma \in W (\sigma \preceq X)$$

With these definitions, we can define Scott sets.

Definition 5.36. A subset $S \subseteq 2^\omega$ is a *Scott set* iff it satisfies the following:

- (a) If $X \in S$ and $Y \leq_T X$, then $Y \in S$;
- (b) If $X, Y \in S$, then $X \oplus Y \in S$; and
- (c) If $X \in S$ and P is a nonempty Π_1^0 -class relative to X , then there is $Y \in P \cap S$.

Thus, Scott's Theorem, in this language, is the following.

Theorem 5.37. $S \subseteq 2^\omega$ is a Scott set iff there is some countable nonstandard model of PA such that $S = SS(M)$.

Notice that the definition of Scott set is analagous in form to the definition of e.c. Fraïssé classes: condition (a) has a similar flavour to HP; (b) to JEP and AP, and (c) to EC. We formalise this in the following theorem.

Theorem 5.38. Let S be a Scott set and $\mathcal{K}_S$ the collection of finitely generated groups with $W(G) \in S$. Then $\mathcal{K}_S$ is an e.c. Fraïssé class.

Proof. HP follows quickly from (a) and JEP from (b). We now check EC. Let $\bar{g} \subseteq G \in \mathcal{K}_S$ and suppose that $\varphi(\bar{x}, \bar{g})$ is solved in $H \geq G$ by $\bar{h}$. Let $\bar{c}$ be a set of new constant symbols with $|\bar{c}| = |\bar{x}|$. Let $(\psi_i)_{i < \omega}$ be an enumeration of the quantifier-free sentences of $L_{gp}(\bar{c})$. Consider the $W(G)$ -computable binary tree T where $\sigma \in T$ iff

$$T_{gp} \cup W(G) \cup \varphi(\bar{c}, \bar{g}) \cup \{\psi_i^{\sigma(i)} \mid i < |\sigma|\}$$

is not contradictory by a proof of length $< |\sigma|$. Then $[T]$ is nonempty since $\langle \bar{h}, \bar{g} \rangle_H$ gives rise to a path on T . Thus, there is some path Y on T with $Y \in S$.

Y is the quantifier-free diagram of a group F generated by $\bar{g} \cup \bar{c}$ such that $F \geq G$ and F solves φ . Thus, $\mathcal{K}_S$ satisfies EC, as required. $\square$

6 Ideals in the Ziegler degrees

In this section, following [Zie80, III.3], we characterise the ideals in $\mathcal{D}^*$ which arise from skeletons of e.c. groups. The idea of the characterisation will be similar to the proof of Theorem 5.38; it will be necessary to give an analogue of *Scott sets* in $\mathcal{D}^*$.

6.1 Computably enumerable Hornclasses

Recall Definition 5.34 of a Π_1^0 -class: a subset $P \subseteq 2^\omega$ is a Π_1^0 class if there is a c.e. set of initial segments σ which witness that a set X is not in P . In other words, there is a c.e. set of finitary positive and negative questions that determine non-membership in P .

A c.e. Hornclass is a translation of this concept into the Ziegler degrees. Instead of a c.e. set of finitary conditions (on both X and $\bar{X}$) which determine whether $X \in P$, it will instead be a c.e. set of finitary conditions of the form: a finite set of restraints on X and at most one restraint on $\bar{X}$.

Recall that we have fixed a computable enumeration $(D_u)_{u < \omega}$ of the finite subsets of ω and a computable enumeration $(D_v^1)_{v \in V}$ of the subsets of ω of cardinality at most one.

Definition 6.1. A *c.e. Hornclass* is a subset $P \subseteq 2^\omega$ such that there is a c.e. set W_e such that

$$X \notin P \iff \exists \langle u, v \rangle \in W_e (D_u \subseteq X \wedge D_v^1 \cap X = \emptyset)$$

As with standard Π_1^0 -classes, we want to relativise this notion to an oracle Y . However, in keeping with Ziegler axioms, we require that we can determine non-membership of $X \in P$ by a c.e. set of finitely many positive questions and a single negative question about X and Y — in other words, we should only be able to ask a single question about $\bar{X}$ or $\bar{Y}$ for each axiom. Thus we obtain the following definition.

Definition 6.2. Let $Y \subseteq \omega$. Then a *c.e. Hornclass relative to Y* , or, more succinctly, a *Y -c.e. Hornclass* is a subset $P \subseteq 2^\omega$ such that there is a c.e. set W_e such that

$$X \notin P \iff \exists \langle u, v \rangle \in W_e (D_u \subseteq X \oplus Y \wedge D_v^1 \cap X \oplus Y = \emptyset)$$

Exercise 6.3. Rewrite the definition of $\Pi_1^{0,Y}$ -class in this form.

Lemma 6.4. Let P be an X -c.e. Hornclass and $Y \equiv^* X$. Then P is a Y -c.e. Hornclass.

Now, we have a notion of Scott set in the Ziegler degrees.

Definition 6.5. A subset $\mathcal{I} \subseteq 2^\omega$ is a *closed ideal* if:

- (a) $X \in \mathcal{I}$ and $Y \leq^* X$ implies $Y \in \mathcal{I}$
- (b) $X \in \mathcal{I}$ and $Y \in \mathcal{I}$ implies $X \oplus Y \in \mathcal{I}$
- (c) $X \in \mathcal{I}$ and P a nonempty c.e. Hornclass relative to X implies $P \cap \mathcal{I} \neq \emptyset$

6.2 A computability-theoretic characterisation of the skeleta of e.c. groups

As Ziegler showed in [Zie80, IV.3.12], closed ideals exactly characterise the e.c. Fraïssé classes of groups. To simplify the statement, we introduce some definitions. Note that we follow our standard tradition of abusing notation by confusing a Ziegler degree $\mathbf{a}$ and its union.

Definition 6.6. For a collection $\mathcal{K}$ of finitely generated groups let

$$\deg^*(\mathcal{K}) = \{\mathbf{a} \in \mathcal{D}^* \mid \exists G \in \mathcal{K} (W(G) \equiv^* \mathbf{a})\}.$$

On the other hand let $\mathcal{I} \subseteq \mathcal{D}^*$. Then set

$$\mathcal{K}_{\mathcal{I}} = \{G \text{ a f.g. group} \mid W(G) \in \mathcal{I}\}$$

In this language, [Zie80, IV.3.12] is the following.

Theorem 6.7. *Let $\mathcal{K}$ be an e.c. Fraïssé class of groups. Then $\deg^*(\mathcal{K})$ is a closed ideal. On the other hand, let $\mathcal{I}$ be a closed ideal. Then $\mathcal{K}_{\mathcal{I}}$ is an e.c. Fraïssé class of groups.*

Further, by Theorem 5.25, if $\mathcal{K} = \text{Sk}(M)$ for some e.c. M , then $\mathcal{K}_{\deg^*(\mathcal{K})} = \mathcal{K}$. In other words, if $G \in \mathcal{K}$ and H is finitely generated with $W(H) \equiv^* W(G)$, then $H \in \mathcal{K}$.

Proof. Firstly, let $\mathcal{I}$ be a closed ideal. We show that $\mathcal{K} := \mathcal{K}_{\mathcal{I}}$ is an e.c. Fraïssé class. It suffices to show that $\mathcal{K}$ satisfies HP, JEP, and EC. HP follows from the first clause of Definition 6.5 and JEP from the second. Towards EC, let $\bar{x} \subseteq G \in \mathcal{K}$ and $\varphi(\bar{x}, \bar{g})$ be a formula solved in an overgroup $H \geq G$. The idea is similar to that in Theorem 5.38; however, we will proceed more carefully as Hornclasses are less intuitive than Π_1^0 -classes.

Let $\bar{c}$ be a new set of constant symbols with $|\bar{c}| = |\bar{x}|$ and fix an ordering $(w_i)_{i < \omega}$ of $\text{Words}((\bar{c})^\pm \cup (\bar{g})^\pm)$. Without loss of generality, we can take w_{2i+1} to be the formal inverse of w_{2i} , and ensure that $(w_j)_{j \equiv 0,1}$ is an enumeration of $\text{Words}(\bar{g})$. We can then think of a set $X \in 2^\omega$ as coding a subset of $\text{Words}(\bar{c} \cup \bar{g})$, which we would like to identify as the word problem of a group. Notice that, given indices i and j , it is computable to find the index k so that $w_k = w_i \cdot w_j$. Write $k = i \wedge j$ to indicate this index. Thus, the following is a c.e. set of 1-enumeration conditions:

$$\begin{aligned} &\{X(i \wedge (i+1)) = 1 \mid i \in \omega\} \cup \\ &\{X(i) = 1 \wedge X(i \wedge j) = 1 \rightarrow X(j) = 1, X(i) = 1 \wedge X(j) = 1 \rightarrow X(i \wedge j) = 1 \mid i, j \in \omega\} \end{aligned}$$

X satisfies these conditions iff it represents the word problem of a group generated by $\bar{c}, \bar{g}$.

Further, we need to ensure that X codes a group containing G ; this is where we use the $W(G)$ -oracle. Letting $Y = X \oplus W(G)$, we need to add the following c.e. set of 1-enumeration conditions:

$$\{Y(2\ell+1) = 1 \rightarrow Y(4\ell) = 1, Y(2\ell+1) = 1 \rightarrow Y(4\ell) = 1 \mid \ell \text{ even}\} \cup \\ \{Y(2\ell+1) = 1 \rightarrow Y(2\ell-2) = 1, Y(2\ell+1) = 1 \rightarrow Y(2\ell-2) = 1 \mid \ell \text{ odd}\}$$

Putting this together, we get a $W(G)$ -c.e. Horn class P of word problems X of groups extending G which solve $\varphi(\bar{x}, \bar{c})$. Further, since there is a group satisfying $\exists \bar{x} \varphi(\bar{x}, \bar{g})$, P is nonempty, and hence has a solution in $\mathcal{I}$, as required.

On the other hand, let $\mathcal{K}$ be an e.c. Fraïssé class and let $\mathcal{I} = \deg^*(\mathcal{K})$. We first check that $\mathcal{I}$ is an ideal: let $\mathbf{a} \in \mathcal{I}$ and $\mathbf{b} \leq^* \mathbf{a}$. Then, by choice of $\mathcal{K}$, we may take G be a f.g. group with $W(G) \in \mathbf{a}$ and by Theorem 4.7, we may take H a f.g. group with $W(H) \in \mathbf{b}$. By Theorem 5.25, since $G \leq M$, we also have $H \leq M$, so $\mathbf{b} \in \mathcal{I}$, as required.

Similarly, if $\mathbf{a}, \mathbf{b} \in \mathcal{I}$ witnessed by G and H respectively. Let $\tilde{G}$ and $\tilde{H}$ be subgroups of M isomorphic to G and H respectively and generated by $\bar{g}$ and $\bar{h}$. Then $F = \langle \bar{g}, \bar{h} \rangle_M$ is a finitely generated subgroup of M with $W(G), W(H) \leq^* W(F)$. Thus, $W(G) \oplus W(H) \leq^* W(F)$, and so, since $[W(F)]^* \in \mathcal{I}$, by the first part we have that $\mathbf{a} \vee \mathbf{b} \in \mathcal{I}$.

Finally, we need to check closure. Let P be a nonempty $\mathbf{a}$ -c.e. Horn class for some $\mathbf{a} \in \mathcal{I}$. (This is well-defined by Lemma 6.4.) Take $W(G) \in \mathbf{a}$ by definition of $\mathcal{I}$. Recall Claim 3.16.2: there is a computable sequence $(v_i(x, y))_i$ of words on two letters such that for any group H and any countable sequence $(a_i)_i \subseteq H$, there is a group $G \geq H$ and $b, c \in G$ with $v_i(b, c) = a_i$. We think of $X \in 2^\omega$ as coding whether $v_i(b, c) = 1$. Now, let W_e be the c.e. set such that

$$X \in P \iff \{D_u \subseteq X \oplus W(G) \wedge D_v^1 \cap X \oplus W(G) = \emptyset \mid \langle u, v \rangle \in W_e\}.$$

This is equivalent to

$$X \in P \iff \bigwedge_{\langle u, v \rangle \in W_e} \left(\bigwedge_{i \in D_u} X \oplus W(G) = 1 \rightarrow \bigvee_{j \in D_v^1} X \oplus W(G)(j) \right)$$

This corresponds to a c.e. set of Horn implications. Thus, as in the proof of Theorem 3.13 it's equivalent to $\exists \bar{z} \varphi(\bar{z}, b, c\bar{g})$ being solvable in some overgroup H of G . Furthermore, there is such an H , as P is nonempty. Hence (by JEP for groups), $\exists \bar{z}, x, y \varphi(\bar{z}, x, y\bar{g})$ is solved in M ; say by $\bar{a}, b, c$. Then $W(\langle b, c \rangle_M)$ computes a path in X by checking whether $v_i(a, b) \in W(\langle b, c \rangle_M)$. $\square$

7 Definable properties of e.c. groups

In this section, we study the first-order definability of certain properties in e.c. groups. We characterise them as the *arithmetic group properties*. This follows Chapter IV in [Zie80]. Fix throughout a computable bijection $\lceil \cdot \rceil$ between $(\{x_0, x_1, x_2, \dots\}^\pm)^*$.

Definition 7.1. Let $P(\bar{v}, \bar{x}, \bar{\rho})$ be a relation between number variables $\bar{v}$, group element variables $\bar{x}$, and group sequence variables $\bar{\rho}$. Then P is *arithmetic for groups* if it is of the form

$$\exists \bar{\tau}_1 \exists \bar{y}_1 \exists \bar{u}_1 \forall \bar{\tau}_2 \forall \bar{y}_2 \forall \bar{u}_2 \cdots R(\bar{u}_1, \dots, \bar{v}, \overline{\ell(\tau_1)}, \dots, \ell(\rho); W(\langle \bar{y}_1, \dots, \bar{x}, \bar{\tau}_1, \dots, \bar{\rho} \rangle))$$

where R is a quantifier-free formula in the language of arithmetic with $\in$.

Towards the first-order definability of the arithmetic properties of groups, we show that every e.c. groups interprets a weak second order arithmetic.

Theorem 7.2. *Let M be an e.c. group and $a \in M$ with infinite order. Then there is an interpretation of $(\mathbb{N}, \deg^*(\text{Sk}(M)))$ in $(M; a)$.*

Thus, in particular, any purely arithmetical property is first-order definable in any e.c. group.

Proof. Firstly, note that since $\mathbb{Z}$ is computable, Theorem 4.1 implies that every e.c. group has an element of infinite order, so the hypothesis in the theorem does not restrict the class of e.c. groups.

Let

$$\varphi_{\mathbb{Z}}(x, a) = "\forall y ([y, a] = 1 \rightarrow [y, x] = 1)".$$

We claim that $M \models \varphi_{\mathbb{Z}}(b, a)$ iff $b \in \langle a \rangle$. Indeed, if $b \in \langle a \rangle$, we certainly have $M \models \varphi_{\mathbb{Z}}(b, a)$. On the other hand, suppose that $b \notin \langle a \rangle$. Then there is $t \in M_{* \langle a \rangle}$ with $[t, a] = 1$ and $[t, b] \neq 1$. Thus, since M is e.c., we have $M \models \exists y ([y, a] = 1 \wedge [y, b] \neq 1)$, as required.

Thus, we will consider a^n as representing $n \in \mathbb{Z}$. We now define addition and multiplication on this copy of $\mathbb{Z}$.

Let

$$\varphi_+(x, y, z, a) = "\bigwedge_{* \in \{x, y, z\}} \varphi_{\mathbb{Z}}(*, a) \wedge x \cdot y = z"$$

and

$$\varphi_{\times}(x, y, z, a) = "\bigwedge_{* \in \{x, y, z\}} \varphi_{\mathbb{Z}}(*, a) \wedge \exists t (t^{-1}at = x \wedge t^{-1}yt = z)".$$

Evidently, $M \models \varphi_+(b, c, d, a)$ iff $b = a^n$ and $c = a^m$ for some $n, m \in \mathbb{Z}$, and $d = a^{n+m}$.

Further, suppose that $M \models \varphi_{\times}(b, c, d, a)$. Then, again $b = a^n$ and $c = a^m$ for some $n, m \in \mathbb{Z}$. The condition on t guarantees that the automorphism $g \mapsto t^{-1}gt$ restricted to $\langle a \rangle$ is the map $a \mapsto a^n$. Thus, $d = (a^m)^n = a^{m \cdot n}$. On the other hand, if $b = a^n$, $c = a^m$, and $d = a^{m \cdot n}$ for some $n, m \in \mathbb{Z}$, then, again using HNN extensions the fact that M is e.c., there is some $t \in M$ with $t^{-1}gt = g^n$ for every $g \in \langle a \rangle$. Thus, $M \models \varphi_{\times}(b, c, d, a)$.

For $b, c \in \langle a \rangle$, write $b \oplus c = d$ and $b \otimes c = d$ for $M \models \varphi_+(b, c, d, a)$ and $M \models \varphi_{\times}(b, c, d, a)$ respectively.

Finally, using Lagrange's four-square theorem, we may define $\omega \subseteq \mathbb{Z}$ as follows:

$$\varphi_{\omega}(x, a) = \varphi_{\mathbb{Z}}(x, a) \wedge \exists y, z, w, t (x = y \otimes y \oplus z \otimes z \oplus w \otimes w \oplus t \otimes t).$$

This gives the first order part of $(\omega; +, \times)$.

Towards the second order part, for each $b, c \in M$, let $X_{b,c} = \{a^n \mid [a^{-n}ba^n, c] = 1\}$ and $\hat{X}_{b,c} = \{n \mid a^n \in X_{b,c}\}$. We will show that $A \in \deg^*(\text{Sk}(M))$ iff $A = X_{b,c}$ for some $b, c \in M$.

For the backwards direction, let $b, c \in M$. Then $X_{b,c} \leq^* W(\langle a, b, c \rangle)$ and so, by Theorem 6.7, since $\deg^*(\text{Sk}(M))$ is an ideal, we have that $X_{b,c} \in \deg^*(\text{Sk}(M))$.

On the other hand, let $A \in \deg^*(\text{Sk}(M))$, and let $G \in \text{Sk}(M)$ with $A \equiv^* W(G)$. By Theorem 4.1 and ω -homogeneity of e.c. groups, there is $b \in M$ such that $\{a, b\}$ freely generates a copy of F_2 in M . Then, by 1.19, $B = \{a^{-n}ba^n \mid n \in \omega\}$ is a free basis for F_{ω} . Let $B' = \{a^{-n}ba^n \mid n \in A\}$. Let $U = \langle \rangle \leq F_{a,b}$. Then every word $w \in F_{a,b}$ can be written uniquely in the form a^mv for some $v \in \text{Words}(B^{\pm})$ and $m \in \mathbb{Z}$. Further, $w \in U$ if it can be written as above with $m = 0$ and $v \in \text{Words}((B')^{\pm})$, and so $W(U \subseteq F) \leq^* A$.

Consider the sets of implications:

$$I(x) = \{[x, w(a, b)] = 1 \mid w \in U\} \leq_e W(G)$$

and

$$J(x) = \{[x, w(a, b)] = 1 \rightarrow a = 1 \mid w \notin U\} \leq_e^1 W(G)$$

By the proof of Theorem 4.11, $I(x) \cup J(x)$ is equivalent over $\langle \bar{g} \rangle \cong G$ to an existential formula $\exists \bar{y} \varphi(\bar{y}, x, \bar{g})$. By existential closure of M , there is an element $c \in M$ satisfying φ and hence $I \cup J$. In particular c satisfies that $U = \{w \in F_{a,b} \mid [c, w] = 1\}$. Hence, $n \in A$ iff $a^{-n}ba^n \in U$ iff $[c, a^{-n}ba^n] = 1$ iff $a^n \in X_{b,c}$, as required. $\square$

We note that the parameter a may be removed using the formula from Example 4.9. Thus, as a corollary, we obtain the following.

Corollary 7.3. *For every formula $\varphi(\bar{v}, \bar{X})$ of second order arithmetic, there is a first order formula $\check{\varphi}(\bar{y}, \bar{b}, \bar{c}, x)$ such that for every e.c. group M , every $\bar{n} \in \omega$, and every $a, \bar{b}, \bar{c} \in M$,*

$$(\mathbb{N}, \{W(G) \mid G \in \text{Sk}(M)\}) \models \varphi \iff M \models \check{\varphi}$$

It will further be useful to provide another characterisation of the class of properties that are arithmetic for groups. In [Zie80, Chap IV], Ziegler calls this logic L^*

Lemma 7.4. *The class of arithmetic properties of groups is the smallest class containing the following atomic formulas:*

- (a) $g_i \cdot g_j = g_k$
- (b) $g_i^{-1} = g_j$
- (c) $g_i = e$
- (d) $g_i^{n_j} = g_k$
- (e) $\sigma_i(n_j) = g_k$
- (f) $\ell(\sigma_i) = n_j$

and closed under

- i. number quantifiers
- ii. group element quantifiers
- iii. group sequence quantifiers

Proof. We first show that each of these atomic formulas is arithmetic for groups. (a) follows since $g_i \cdot g_j = g_k$ iff $[x_i \cdot x_j \cdot x_k^{-1}] \in X$. (b), (c), and (d) are similar.

Towards (e), we note that from $[\sigma]$ and n , we can obtain $\sigma(n)$, and so $\sigma(n) = g_i$ iff $[\sigma(n) \cdot g_i^{-1}] \in X$. Finally, (f) is built into the definition.

Thus every atomic formula is arithmetic for groups. Since arithmetic for groups is closed under number-, group element-, and group sequence-quantifiers, every property in this class is arithmetic for groups.

For the other direction, it suffices to show that if $R(\bar{v}, \overline{\ell(\tau)}; W(\langle \bar{x}, \bar{\tau} \rangle))$ is quantifier-free, then it belongs to the class in the lemma.

Indeed, R must be of the form

$$S(u, w, \bar{v}) \wedge D_u \subseteq X \wedge D_w \subseteq \omega \setminus X$$

where S is a quantifier-free formula of first order arithmetic. □

Theorem 7.5. *Every property which is arithmetic for group is first-order definable in every e.c. group.*

Proof. To every arithmetic formula $\varphi(\bar{v}, \bar{x}, \bar{\rho})$, we build a first order formula $\hat{\varphi}(\bar{y})$ so that for every $\bar{n} \in \omega$, $\bar{g} \in M$, and $\bar{\sigma} \in M^{<\omega}$, so that

$$\varphi(\bar{n}, \bar{g}, \bar{\sigma}) \text{ holds of } M \iff M \models \hat{\varphi}(\bar{a}^{\bar{n}}, \bar{g}, \bar{\sigma})$$

We use the characterisation of properties that are arithmetic for groups in Lemma 7.4. The only ones that require work are (d), (e), (f), and iii.

Towards (d), we note that $g_i^{n_j} = g_k$ iff the map $a \mapsto g_i$ sends a^{n_j} to g_k . In other words, $g_i^{n_j} = g_k$ iff $\langle g_i, g_k \rangle$ is the homomorphic image of $\langle a, a^{n_j} \rangle$ via the order-preserving map of the generators. By Claim 3.16.3, this is first order definable by a formula of the form $\exists \bar{z} \eta(a, a^{n_j}, g_i, g_k, \bar{z})$.

Towards (e) and (f), we recall that there is a sequence $(w_i(x, y))_{i < \omega}$ such that for any group sequence $(a_i)_{i < \omega} \subseteq G$, we have that there is an overgroup H and $b, c \in H$ with $w_i(b, c) = a_i$. Thus, in e.c. groups, we may replace sequence variables with these words. To associate a sequence σ so defined with its length, we also introduce a new number variable m_i , which we will ensure is the length of σ . Hence, $(\sigma_i(n_j) = g_k)^\wedge = "(n_j < m_i \wedge w(b_i, b'_i) = g_k) \vee (g_k = 1)"$. Similarly, $(\ell(\sigma_i) = n_j)^\wedge = "n_j = m_i"$.

Group quantifiers are straightforward; number quantifiers are replaced with $\exists x$ ($\varphi_\omega(x) \wedge \dots$) and group sequence quantifiers are replaced with $\exists m_i \exists z_i, z'_i \dots$. □

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