

THE ART OF FORCING AND ITS APPLICATIONS

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Date: July 6, 2022.

For the enemy, and his tiny, small epsilons.

ABSTRACT

In 1931, Kurt Gödel proved that there are statements of mathematics that can neither be proved nor disproved [Gödel, 1931]. Such a statement is said to be *independent*. A statement can be shown to be independent by constructing two models of mathematics, one in which it is true (demonstrating that the statement cannot be disproved within mathematics), and one in which the statement is false (demonstrating that no proof of the statement exists within mathematics).

The standard way of constructing these models is through *forcing*, which was developed by Paul Cohen in the 1960's [Cohen, 1963]. This paper is intended to serve as an introduction to the art of forcing to an advanced undergraduate with little to no background in formal set theory. We shall also investigate how forcing can be used to prove theorems within mathematics.

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Declaration: I certify that this project report has been written by me, is a record of work carried out by me, and is essentially different from work undertaken for any other purpose or assessment.

Isabella ☺

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1. INTRODUCTION

As mentioned in the abstract, the purpose of forcing in mathematics is twofold - proving independence and proving theorems. The paper will focus on the latter, although the former is too fundamental to the development of forcing to be skipped entirely. I like to think of these two uses for forcing as corresponding to two ways of doing mathematics.

On the one hand, there are “mathematicians-as-therapists” who use tools about mathematics to prove theorems *about* mathematics itself – what can and cannot be proved, how the structure of a set of axioms affects the structure of the resulting theory, and the like. This is a fairly new approach to mathematics and is deeply tied to the more philosophical questions concerning the appropriate foundation for mathematics. The introspective ability of mathematics is really quite incredible and much work has been done in independence results and the so-called reverse mathematics.

On the other hand, there are “mathematicians-as-surgeons” who use mathematical tools to prove theorems *within* mathematics and thus uncover the structure of mathematics from the inside out. This is the more traditional approach to mathematics and the one that we shall primarily concern ourselves with here.

Here is a brief roadmap of what lays ahead:

Section 2 lays out the mathematical background not typically covered at the undergraduate level that will be needed throughout the paper. It is intended primarily as a reference section, and readers may wish to skip it and refer back to it as necessary.

Section 3 focusses on forcings and related definitions, while Section 4 develops the theory of a specific class of forcings, those that preserve cardinalities. These sections will be fairly technical, but Section 5 should bring the theory to light, by using it to prove the independence of the Continuum Hypothesis.

Section 6 will then go into detail on how one uses the machinery of forcing to prove theorems within mathematics. We will finish with a well-known result of Todorćević about compact subsets of the space of Baire class one functions whose only known proof is through forcing.

2. PRELIMINARIES

We will now set out the framework in which we will be working throughout this paper. Knowledge of set theory is not assumed, so we hope this section will provide adequate background for the rest of the paper. It just brushes the surface of a lot of interesting mathematics and in particular, omits precise definitions of some terms for which an intuitive understanding ought to suffice. For a more thorough introduction, see [Jech, 2002] or [Enderton, 1977].

2.1. Axioms of ZFC. We start with the axioms of set theory. They owe their origins to Ernst Zermelo [Zermelo, 1908] in the early 20th century, and are theoretically the foundation to which any proof in mathematics can be traced. There are many equivalent formulations of the axioms; I have chosen to follow [Jech, 2002]. Note that *set* and the binary membership relation $\in$ are primitive terms.

Definition 2.1.1 (Zermelo-Fraenkel set theory). The following axioms define *Zermelo-Fraenkel set theory*. Throughout the paper, I will refer to them as “the axioms of ZFC”. By “ZFC”, I will denote the smallest consistent theory containing the axioms of ZFC and their consequences.

- (i) **Axiom of Extensionality:** Let x and y be sets with the same elements. Then $x = y$.

- (ii) **Axiom of Pairing:** For any two sets x and y , there is a set $\{x, y\}$ whose elements are just x and y .
- (iii) **Axiom Schema of Separation:** Let P be a property and x a set. Then there is some set y whose elements are exactly those sets in x that satisfy P ; i.e., $y = \{z \in x \mid P(z)\}$.
- (iv) **Axiom of Union:** For any set x , there is some set $y = \cup x$, consisting of exactly those sets which are elements of elements of x .
- (v) **Axiom of Power Set:** Given a set x , there is some set $y = \mathcal{P}(x)$ which contains all subsets of x .
- (vi) **Axiom of Infinity:** There is a set x with $\emptyset \in x$ and $y \cup \{y\} \in x$ for every $y \in x$. (Note that this postulates the existence of an infinite set.)
- (vii) **Axiom Schema of Replacement:** The image of a set under a function class is a set.
- (viii) **Axiom of Regularity:** Let x be a nonempty set. Then x has a $\in$ -minimal element.
- (ix) **Axiom of Choice:** Given a set x whose elements are nonempty pairwise disjoint sets. Then there is some set y containing exactly one element from each element of x .

While it is instructive to have seen all the axioms once, they will never be explicitly used throughout the text. For a deeper explanation of each axiom, see [Jech, 2002].

It is worth pointing out that the final axiom, the Axiom of Choice, seems less intuitive and “basic” than the others. Historically, there was considerable controversy as to whether to accept the Axiom of Choice, particularly as it can lead to some seemingly paradoxical results. It has been shown that it cannot be deduced from the other axioms and is now commonly accepted by almost all working mathematicians. We shall be assuming it throughout this paper.

2.2. Ordinals and Cardinals. Ordinals and cardinals generalise counting to the infinite setting and can both be seen as extensions of the natural numbers. Intuitively, ordinal numbers represent the ordering of a set, and cardinal numbers represent the size of the set.

However, before we can present a more formal definition, we will need a few basic definitions from order theory.

Definition 2.2.1 (Partial order). A *partially ordered set* or *poset* is a set X equipped with a binary relation $\leq$ satisfying

- (i) **Reflexivity:** For every $x \in X$, $x \leq x$.
- (ii) **Antisymmetry:** For every $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.
- (iii) **Transitivity:** For every $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Example 2.2.2. Consider $\mathbb{N}$ and write $x \leq y$ if and only if $x \mid y$. Then $\leq$ partially orders $\mathbb{N}$.

Figure 1 shows the *Hasse diagram* for this ordering on the numbers 1 to 10. If there is an upwards path from x to y in the diagram, then $x \leq y$. For example, $1 \leq x$ for every x from 1 to 10.

Two elements $x, y \in X$ such that neither $x \leq y$, nor $y \leq x$ are said to be *incomparable*. In the previous example, 5 and 3 are incomparable.

An *antichain* is a set of pairwise incomparable elements. $\{4, 6, 9, 10\}$ is an antichain in Example 2.2.2.

Definition 2.2.3 (Total order). A *totally ordered set* or *linearly ordered set* is a set X equipped with a binary relation $\leq$ satisfying

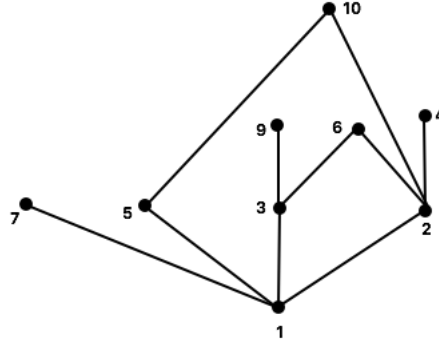


FIGURE 1. This shows the ordering of the numbers 1 to 10 from Example 2.2.2.

- (i) **Totality:** For every $x, y \in X$, $x \leq y$ or $y \leq x$.
- (ii) **Antisymmetry:** For every $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.
- (iii) **Transitivity:** For every $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Note that the totality condition implies reflexivity, so this is a direct generalisation of a partial order.

Example 2.2.4. The integers equipped with the usual ordering is a totally ordered set. Figure 2 shows the Hasse diagram for the integers.

Definition 2.2.5 (Well order). A *well order* $\leq$ on a set X is a total order such that every subset of X has a minimal element with respect to $\leq$.

Note that the totality and antisymmetry conditions implies that for any subset, this minimal element is unique.

Example 2.2.6. The natural numbers equipped with the usual ordering is a well-ordered set. Figure 3 shows a Hasse diagram for the natural numbers.

There are several different ways to define ordinals, but we shall follow John von Neumann, see [von Neumann, 1923].

Definition 2.2.7 (Ordinal numbers). A set α is an *ordinal* or *ordinal number* if and only if all of the following hold:

- (i) for all $\beta \in \alpha$, $\beta \subsetneq \alpha$
- (ii) for all $\beta, \gamma \in \alpha$, one of the following holds:
 - (a) $\beta = \gamma$
 - (b) $\beta \in \gamma$
 - (c) $\gamma \in \beta$
- (iii) for all $\beta \subsetneq \alpha$, there is some $\gamma \in \alpha$ with $\gamma \cap \beta = \emptyset$

We will write ON for the class of ordinal numbers.



FIGURE 2. This shows the ordering of the integers from Example 2.2.4.

Note that I carefully used the word “class”, instead of “set” for the collection of all ordinal numbers. If the ordinal numbers did form a set, we would run into the *Burali-Forti paradox*, which results from attempting to assign an ordinal to the collection of all ordinals. This is an important metamathematical distinction, but will not be central to the discussion in this paper.



FIGURE 3. This shows the ordering of the natural numbers from Example 2.2.6.

The following example provides a standard construction of the ordinal numbers, and will hopefully shed some light on the previous definition.

We see that the ordinal numbers properly extend the natural numbers.

Example 2.2.8 (Constructing the ordinal numbers). We construct the ordinal numbers as follows:

$$0 = \emptyset$$

$$\alpha^+ = \alpha \cup \{\alpha\}$$

α^+ is called the *successor ordinal* of α .

$$\alpha = \bigcup_{\beta < \alpha} \beta$$

Such an α is called a *limit ordinal*.

So, in particular, the first few ordinals are

$$0 = \emptyset$$

$$1 = \{\emptyset\} = \{0\}$$

$$2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$$

...

$$\omega = \{0, 1, 2, \dots\}$$

$$\omega + 1 = \{0, 1, 2, \dots, \{0, 1, 2, \dots\}\}$$

...

Recalling the axioms of ZFC, Definition 2.1.1, we now see that the Axiom of Infinity merely asserted the existence of ω , the set of natural numbers.

Note that each ordinal forms a linearly ordered set with respect to $\in$ and contains every ordinal smaller than it. Say a bijection $\phi : X \rightarrow Y$ of ordered sets is an *order-isomorphism* if it preserves the ordering; i.e., if $x \leq x'$ in X , then $\phi(x) \leq \phi(x')$ in Y . Two ordered sets are *order isomorphic* if there is some order isomorphism between them. Then we have the following proposition, which implies that any well-ordered set is order-isomorphic to some ordinal.

Proposition 2.2.9. *Let X and Y be two well-ordered sets. Then one of the following is true:*

- (i) X is order-isomorphic to Y .
- (ii) X is order-isomorphic to an initial segment of Y .
- (iii) Y is order-isomorphic to an initial segment of X .

We said that cardinal numbers represent the *size* of a set, but what does size mean in the infinite setting? The following definition captures the intuition that two sets are the same size if their elements can be matched up in a one-to-one correspondence.

Definition 2.2.10 (Equinumerous). Let X and Y be sets. Then X and Y are *equinumerous* or *of the same cardinality* if and only if there is some bijection $\phi : X \rightarrow Y$.

The cardinal numbers are a canonical choice of representative for each equivalence class of equinumerous sets.

Definition 2.2.11 (Cardinal number). The *cardinality* of a set X is the least ordinal α such that α and X are equinumerous.

An ordinal α for which there exists a set of cardinality α is called a *cardinal*.

We write $0, 1, 2, \dots, \aleph_0, \aleph_1, \dots, \aleph_\omega, \dots$ for the cardinal numbers.

Note that every cardinal is an ordinal. In particular,

$$\aleph_0 = \omega = \mathbb{N}$$

$$\aleph_1 = \omega_1$$

$$\aleph_2 = \omega_2$$

etc.

Note how the ordinals and the cardinals both generalise distinct properties of the natural numbers.

We will need a few more specific definitions associated with cardinals.

Definition 2.2.12 (Cofinality, Regular cardinal). The *cofinality* of a cardinal κ is the least cardinality of an unbounded subset of κ .

A cardinal is called *regular* if and only if it is equal to its own cofinality. In other words, a cardinal κ is regular if and only if every unbounded subset of κ has cardinality κ .

Example 2.2.13. The cofinality of $\aleph_0$ is $\aleph_0$ since any unbounded set in $\aleph_0$ must be infinite and cannot be larger than $\aleph_0$.

Hence $\aleph_0$ is a regular cardinal.

Example 2.2.14. Since $\mathbb{N}$ is an unbounded subset of $\mathbb{R}$, the cofinality of $|\mathbb{R}|$ is $\aleph_0$. Since it can be shown that $|\mathbb{R}| > \aleph_0$, $|\mathbb{R}|$ is not a regular cardinal.

2.3. Model Theory. Recall that our goal is to build *models* of ZFC satisfying certain properties. In order to understand to what this means, we will need some background in model theory.

Intuitively a *model* is a structure satisfying some axioms. These axioms will be written using certain allowable symbols called a *language*. We will mostly be concerned with models of ZFC, but it is good to be aware of the more general definitions.

However, model theory is a very rich subject that has come to fruition in recent years, and such a short exposition fails to do it justice. The curious reader should consult [Chang and Keisler, 2012] or [Marker, 2002].

Definition 2.3.1 (Language). A *language* $\mathcal{L}$ is a collection of constant symbols, m -ary relation symbols, and n -ary function symbols, where m and n depend on the relation or function respectively. We write

$$\mathcal{L} = \langle (c_i)_i, (R_j)_j, (F_k)_k \rangle$$

where $(c_i)_i$ are the constant symbols, $(R_j)_j$ are the relation symbols, and $(F_k)_k$ are the function symbols.

It will be implicitly assumed that $\mathcal{L}$ will contain a binary relation symbol $=$, denoting equality.

Example 2.3.2 (Language of groups).

$$\mathcal{L} = \langle \mathbf{1}, \bullet, \text{inv} \rangle,$$

where $\mathbf{1}$ is a constant symbol, $\cdot$ is a binary function symbol, and inv is a unary function symbol.

Example 2.3.3 (Language of Set Theory).

$$\mathcal{L} = \langle \in \rangle,$$

where $\in$ is a binary relation symbol.

Definition 2.3.4 (Model). A *model* M for a language $\mathcal{L}$ is a tuple $(U, \mathcal{I})$, where

- U is a set, called the *universe* of M
- $\mathcal{I}$ is function, called the *interpretation function*, such that
 - $\mathcal{I}$ maps constant symbols to elements of U
 - $\mathcal{I}$ maps relation symbols to relations on U
 - $\mathcal{I}$ maps function symbols to functions on U .

Example 2.3.5 (Groups). A *group* $\mathcal{G} = (G, \mathcal{I})$ is a model for the language of groups satisfying the following axioms:

- $\forall x, y, z \in G : (x \cdot y) \cdot z = x \cdot (y \cdot z)$
- $\forall x \in G : 1 \cdot x = x \cdot 1 = x$
- $\forall x \in G : x \cdot x^{-1} = x^{-1} \cdot x = 1$

where 1 is the interpretation of $\mathbf{1}$, $\cdot$ is the interpretation of $\bullet$, $\cdot^{-1}$ is the interpretation of inv .

Definition 2.3.6 ($\models$). Let M be a model for a language $\mathcal{L}$ and φ a sentence of $\mathcal{L}$. We write

$$M \models \varphi$$

if and only if φ holds in M .

Example 2.3.7. Define $\varphi(a, b)$ to be $a \cdot b = b \cdot a$. Then a group $\mathcal{G}$ with universe set G is abelian if and only if for every a, b in G ,

$$\mathcal{G} \models \varphi(a, b).$$

Example 2.3.8 (Models of ZFC). A model of ZFC, $M = (U, \mathcal{I})$, is a model for the language of set theory that satisfies all the axioms of ZFC (which can be restated in the language of set theory).

A model M of ZFC is *transitive* if for every $x \in U$ and $y \in x$, we have that $y \in U$.

We have already mentioned that we will be using forcing to construct *model extensions* of ZFC. Much as the name suggests, a model extension M of a ground model N merely extends the interpretation function in N . In particular, if N is a model of some set of sentences Σ , then M will be a larger model of Σ .

Definition 2.3.9 (Submodel, model extension). A model $N = (V, \mathcal{J})$ for $\mathcal{L}$ is a *submodel* of $M = (U, \mathcal{I})$ for $\mathcal{L}$ if and only if

- $V \subseteq U$ and $V \neq \emptyset$
- $\mathcal{I} = \mathcal{J}$ on the constant symbols of $\mathcal{L}$.
- for any m -ary relation symbol with P its interpretation in M and P' the corresponding interpretation in N , $P(x_0, \dots, x_m) = P'(x_0, \dots, x_m)$ for all $x_0, \dots, x_m \in V$.

- for any n -ary function symbol with G its interpretation in M and G' the corresponding interpretation in N , $G(x_0, \dots, x_n) = G'(x_0, \dots, x_n)$, for all $x_0, \dots, x_n \in V$.

In this case, M is a *model extension* of N .

Example 2.3.10. Consider $(\mathbb{R}, +)$ as a model of the axioms of a group. Then $(\mathbb{Z}, +)$ as a model of the axioms of a group is a submodel of $\mathbb{R}$.

However, if we were to interpret the binary function $\bullet$ as multiplication in $\mathbb{R}$ and addition in $\mathbb{Z}$, then $\mathbb{Z}$ is no longer a submodel of $\mathbb{R}$, as their interpretations of $\mathbf{1}$ are different.

In any area of mathematics, it is important to know when we have two different representations of essentially the same object. In the world of models, an *isomorphism* between two models is a bijection of their universes that preserves the structure of the interpretation function.

Definition 2.3.11 (Isomorphic as models). Two models $M = (U, \mathcal{I})$ and $N = (V, \mathcal{J})$ for a language $\mathcal{L}$ are *isomorphic* if and only if there is some function $\phi : U \rightarrow V$ such that

- ϕ is a bijection
- $\phi(a) = a'$, where a is the interpretation of a constant symbol in M and a' is the corresponding interpretation of the constant symbol in N .
- $P(x_0, \dots, x_m) = P'(\phi(x_0), \dots, \phi(x_m))$, where P is the interpretation of a m -ary relation symbol in M and P' is the corresponding interpretation of the relation symbol in N .
- $G(x_0, \dots, x_n) = G'(\phi(x_0), \dots, \phi(x_n))$, where G is the interpretation of a n -ary function symbol in M and G' is the corresponding interpretation of the function symbol in N .

In this case ϕ is said to be an *isomorphism of models*, or merely an *isomorphism* if the context is clear.

Example 2.3.12. Isomorphism of models generalises the notion of an isomorphism of groups.

Let $\mathcal{G} = (G, \mathcal{I})$, and $\mathcal{H} = (H, \mathcal{J})$ be groups. Suppose $\mathcal{G}$, and $\mathcal{H}$ are isomorphic as models, and let ϕ be the function witnessing the isomorphism. Then $\phi : G \rightarrow H$ is a group isomorphism between $\mathcal{G}$ and $\mathcal{H}$.

It is natural to ask, given any axiom system one can dream up, whether a model of that axiom system exists. In particular, is there even such a thing as a model of ZFC? Clearly, if we were to postulate $x \neq x$ for some x , there is no model satisfying this axiom. In fact, Leon Henkin showed that such contradictory statements are actually the only obstruction to a model existing [Henkin, 1949].

Definition 2.3.13 (Consistent). Call a set of axioms *inconsistent* if one can derive a contradiction from them, and *consistent* otherwise.

Theorem 2.3.14 (Model Existence Theorem). *For any set of axioms Σ over a language $\mathcal{L}$, there is a model satisfying all the axioms in Σ if and only if Σ is consistent.*

This is, in many senses, the strongest result we could have hoped for. While it is not constructive, it allows us to postulate the existence of a model satisfying an axiom system with only the information that the axiom system is consistent.

The following result shows that any set of axioms with an infinite model actually has a countable model. In particular, if we can show ZFC is consistent, it will have a countable model. This will be useful later on, as it turns out to be much easier to work with countable models.

Theorem 2.3.15 (Löwenheim-Skolem Theorem). *Let Σ be a collection of axioms over some first order¹ language $\mathcal{L}$. If Σ has a model of some infinite size, then Σ has a model of every infinite size.*

However, things are not as rosy as they might seem. It turns out the consistency of a sufficiently expressive, consistent theory, like ZFC, is an example of an *independent* statement; it cannot be proved or disproved within that theory [Gödel, 1931].

Definition 2.3.16 (Independent). Let Σ be a collection of sentences over some language $\mathcal{L}$. Then a sentence φ of $\mathcal{L}$ is *independent* (of Σ) if and only if neither it nor its negation is provable from Σ .

Example 2.3.17 (The Parallel Postulate). Euclid’s Parallel Postulate is known to be independent of the other axioms of Euclidean geometry.

This was shown by constructing a model of Euclidean geometry in which the Parallel Postulate holds (classical Euclidean geometry) and one in which it does not hold (hyperbolic geometry).

Throughout this paper, however, Σ will almost always be the axioms of ZFC.

Theorem 2.3.18 (Gödel’s Second Incompleteness Theorem, see [Gödel, 1931]). *Let Σ be a consistent collection of axioms over some language $\mathcal{L}$ strong enough to express arithmetic. Then the statement “ Σ is consistent” is not provable within Σ .*

The hypothesis the theory be “strong enough to express arithmetic” may seem a little odd, but it comes from the fact that the proof relies upon finding a canonical encoding of statements of the theory as natural numbers. This encoding relies on the tools from arithmetic, like the notion of prime and exponentiation.

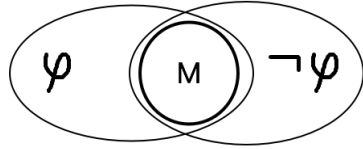


FIGURE 4. The ground model M and the two extensions showing φ is independent

However, ZFC is certainly strong enough to express arithmetic, so Theorem 2.3.18 says that ZFC cannot prove its own consistency, and so, by Theorem 2.3.14, it cannot prove that there is a model of ZFC. We shall assume the consistency of ZFC and hence the existence of a countable model of ZFC, although it is important to note that this is not something we can actually prove within ZFC. As we shall see later, this will pose a slight inconvenience as we develop the theory of forcing.

On the other hand, the notion of independence strikes at the heart of why forcing exists. “ZFC is consistent” is not the only statement which is independent of ZFC. As discussed in Example 2.3.17, showing a statement is independent of a set of axioms involved building two models of those axioms – one in which the statement holds and one in which its negation holds (see Figure 4). Forcing currently the only way we have of building these models.

¹The first order condition is a technical condition which essentially stipulates that one can quantify over variables, but not propositions. We will be working with first order classical logic throughout this paper, which is the standard logic for doing mathematics. While the first order assumption is necessary, it will not materially affect your understanding of the paper if you read over it.

3. DEFINING A FORCING

Recall that the motivation behind forcing is extending our ground model, M , of ZFC to larger models that satisfy certain statements. What forcing will ultimately do is specify a new object, G , that is outside of our ground model. The model extension, $M[G]$, will be the smallest model extension of M containing G (see Figure 5.)

In this section, we will define forcings and see how they can be used to build a model extension. We will largely be following [Jech, 2002]. Chapters 3-5 of [Moore, NA], Chapter 1 of [Shelah, 1998], [Jackson, NA], and [Easwaran, 2007] have also been useful.

3.1. Basic Definitions.

Definition 3.1.1 (Forcing). A *forcing* Q is a poset with a greatest element, usually denoted 1_Q (or merely 1 when the poset is clear from context.)

The elements of Q are called *forcing conditions* and p is said to be *stronger than* q or p *explains* q whenever $p \leq q$. We think of the smallest conditions as those with the most information about the generic object we are adding to the ground model.

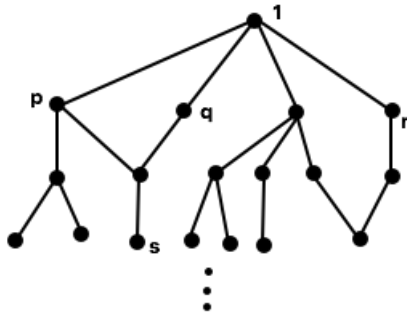


FIGURE 6. An example of a forcing

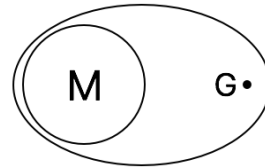


FIGURE 5. The ground model M and the generic extension $M[G]$

The following example will be our canonical example of a forcing throughout this section:

Example 3.1.2. Let $\mathcal{B}$ denote the Borel sets of positive Lebesgue measure on $[0, 1] \subseteq \mathbb{R}$ ordered by inclusion. Then $(\mathcal{B}, \subseteq)$ is a forcing with maximum element $1_{\mathcal{B}} = [0, 1]$.

Similarly, let $\mathcal{B}_I$ denote the collection of Borel sets of positive measure on $[0, 1]^I$ (where each $[0, 1]$ is given the Lebesgue measure and $[0, 1]^I$ the product measure).

In this case, the strongest elements are the smallest sets, as knowing a point is in a small set gives more information on what the point is. As we shall see

in Section 5, the generic object of interest that we add to the ground model is a specific point that we approximate by stronger and stronger elements of $\mathcal{B}_I$.

Example 3.1.3. Let κ be a cardinal, and let $F(\kappa)$ be the set of injective, partial functions, f , from κ into $\mathbb{R}$ such that $|\text{dom} f| < \kappa$. Then $(F(\kappa), \leq)$ is a forcing, where $f \leq g$ if and only if $\text{dom} g \subseteq \text{dom} f$ and $f = g$ on $\text{dom} f \cap \text{dom} g$.

Here the strongest elements are the partial functions for which we know the most about their values. As we shall see in Section 5, the generic object that we end up studying is a function that we approximate by stronger and stronger elements of $F(\kappa)$.

Definition 3.1.4 (Incompatible). Two conditions p and q of a forcing Q are said to be *compatible* if and only if there is some r in Q with $r \leq p$ and $r \leq q$. Otherwise they are *incompatible*. We write $p \perp q$ if p and q are incompatible.

Remark 3.1.5. Comparable implies compatible (just take as r the least of p and q), and so by contraposition, incompatible implies incomparable. However, comparability and compatibility are not equivalent.

In Figure 6, p and s are compatible and comparable. p and q are compatible, but not comparable. p and r are neither compatible nor comparable.

The motivation behind this definition is that the forcing will eventually be used to encode a possible world. $\mathbf{1}$ will *force* that everything in ZFC holds, so $\mathbf{1}$ will have to be in any possible world.

An element p will force everything that weaker elements force, in addition to some new statements. The collection of statements forced by a single element is necessarily consistent and closed under consequence. In particular, two compatible elements can be in the same possible world, since the element stronger than both of them ensures the statements they force are mutually consistent. Two incompatible elements, on the other hand, cannot, as it is possible one will force some statement φ while the other will force $\neg\varphi$.

Example 3.1.6. In the case when $Q = \mathcal{B}$, then p and q are incompatible just when they intersect on a null set.

Example 3.1.7. In the case when $Q = F(\kappa)$, then f and g are incompatible just when they have empty intersection.

Definition 3.1.8 (Open). A subset D of a forcing Q is *open* if and only if p in D and $q \leq p$ implies q is in D .

In fact, this defines a topology on Q . The following “definition” merely fleshes out what it means for a set to be dense in topology and it an easy exercise to see that the two are equivalent.

Definition 3.1.9 (Dense). A subset D of a forcing Q is *dense* if and only if for every q in Q , there is some d in D with $d \leq q$ (see Figure 7).

Example 3.1.10. The set

$$D_\epsilon = \{p \in \mathcal{B}_i \mid \lambda^I(p) < \epsilon\}$$

is dense for every $0 < \epsilon < 1$.

Example 3.1.11. The set

$$D_i = \{f \in F(\kappa) \mid i \in \text{dom} f\}$$

is dense for every $i \in \kappa$.

Intuitively, dense sets contain the conditions that give the most information.

Definition 3.1.12 (Filter). A *filter* is a nonempty subset $G \subseteq Q$ such that:

- (i) if p is in G , then q is in G for every $q \geq p$ in Q . (G is *upward closed*)
- (ii) if p and q are in G , then there is some r in G with $r \leq p, q$. (G is *downward pointing*)

(See Figure 8.)

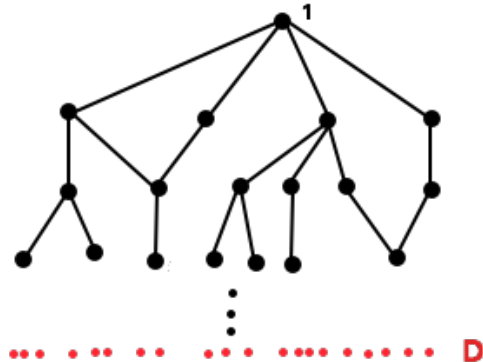


FIGURE 7. Artist's impression of a dense set

Choosing a filter corresponds to picking a possible world that extends the ground model. Condition 1 ensures that the filter is closed under taking weaker conditions, and captures the rough intuition that if we know, for example, $x \in [\frac{1}{6}, \frac{1}{5}]$, we certainly know that $x \in [0, \frac{1}{2}]$. Condition 2 guarantees that all elements of a filter are compatible. Together, these imply that the class of statements forced by a filter is a consistent (possibly trivial) extension of those true in the ground model, as we shall soon see.

Definition 3.1.13 (Generic). A filter $G \subseteq Q$ is *generic* iff it has nonempty intersection with every dense subset.

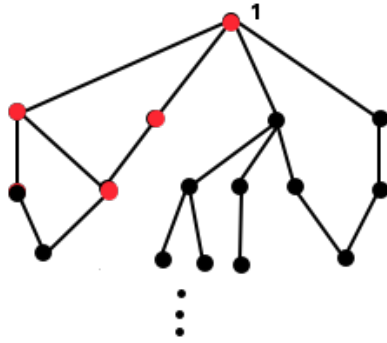


FIGURE 8. The red dots form a filter

Intuitively, dense subsets are “large” in that they account for all elements of Q . In a similar vein, as generic sets intersect every large set, elements of generic sets do not have any special, distinguishing properties. We may think of them as representing the average or expected properties of elements of the forcing.

You may wonder whether the definition of generic is so strong that there is in fact no set satisfying it. Indeed, if G has nonempty intersection with every dense set and is upward closed, then $Q \subseteq G$, and so $G = Q$. But this holds

for any forcing Q , including those with incompatible elements, so $G \neq Q$ in general.

However, this argument merely shows that G need not exist *within* the model M . In fact, it is not difficult to show that G exists in general, assuming that ZFC is consistent, and hence that M exists.

Theorem 3.1.14 (Existence of a generic filter). *Let M be a countable model of ZFC, and let Q be a forcing in M . Then there is a generic filter G for Q (not necessarily in M).*

Proof. Since M is countable, there are only countably many dense sets of Q . Enumerate them as $D_0, D_1, D_2, \dots$ and define a sequence $(p_i)_{i \in \omega}$ as follows:

$$\begin{aligned} p_0 &\in D_0 \\ p_1 &\in D_1 \text{ and } p_1 \leq p_0 \\ &\dots \\ p_i &\in D_i \text{ and } p_i \leq p_{i-1} \\ &\dots \end{aligned}$$

Let G be the upward closure of $\bigcup_{i \in \omega} \{p_i\}$. In other symbols,

$$G = \{q \mid \exists p_i : p_i \leq q\}.$$

Then G is a filter:

- **Upward closed:** Let $q \in G$ and $q \leq r$. Then, there is some $p_i \in q$ with $p_i \leq q \leq r$, as required.

- **Downward pointing:** Let $q, r \in G$. Let p_i and p_j be such that $p_i \leq q$ and $p_j \leq r$. Define $p = p_{\min(i,j)+1}$. Then $p \leq p_i \leq q$ and $p \leq p_j \leq r$, as required. $\square$

Note that the preceding proof relies heavily on the countability of M , which is not generally provable within M . In particular, the enumeration of the dense sets in Q takes place outside of M and hence G will not usually be in M . This is what we want since G will be the new object that we add to our ground model. However, in practice, we will often be interested in other objects we can form from G , like $\cap G$ and $\cup G$, rather than the filter itself.

The following lemma says that if the forcing is *atomless*, then G does not belong to the ground model.

Definition 3.1.15 (Atomless). A forcing Q is *atomless* if and only if for every $p \in Q$, there are some $q, r \leq p$ with $q \perp r$.

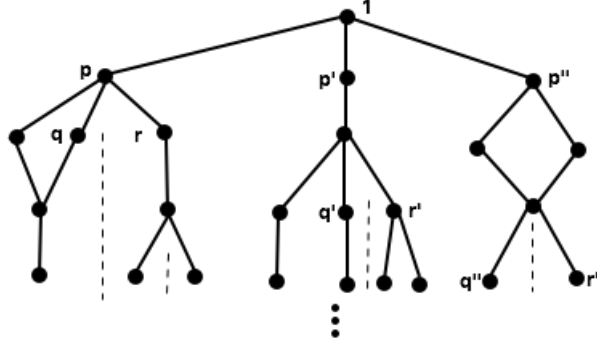


FIGURE 9. This represents the weakest elements of an atomless forcing. The dotted lines signify that the two branches are incompatible – there will be no single stronger element on both branches.

Intuitively, a forcing is atomless if and only if every element can be extended to at least two distinct possible worlds. In other words, we cannot pin down a unique possible world with knowledge of a single element.

Lemma 3.1.16. *Let M be a countable model of ZFC and Q an atomless forcing in M . Then there exists some generic filter G over Q and $G \notin M$.*

Proof. The existence of G follows from Theorem 3.1.14.

We now show $G \notin M$.

Let F be a filter and consider the set

$$D_F = \{p \in Q \mid p \notin F\}.$$

If we show D_F is dense for every filter in the ground model (*), then $G \cap D_F$ is nonempty for every filter. In particular, G contains some element $p_F \notin F$ for every F and thus G is not a filter in the ground model.

To show a set D_F is dense, we have to show that for any $p \in Q$, there is some $q \leq p$ with $q \in D_F$.

Let $p \in Q$. Since Q is atomless, there are some $q, r \leq p$ with $q \perp r$. But since q and r are incompatible, they cannot both be in F . Hence, q or $r \in D_F$, as required. $\square$

The key step in which we used the properties of the ground model was the phrase “for every filter in the ground model” (see *). In particular, the forcing itself is not completely contained within the ground model and may have more filters that

the ground model does not see. So G must be some object outside of the ground model, as we desired.

Figure 10 is an artist's impression of the situation. Note that the model itself sits inside the larger mathematical universe (it is just a mathematical object itself), although I have not attempted to draw that.

3.2. Building Generic Extensions.

We will now work on understanding how we can use these generic filters to build new models of ZFC, and how the forcing conditions “force” statements to hold in that extension.

Definition 3.2.1 (Forcing language).

The *forcing language* is the collection of all first order formulas in $\mathcal{L} \cup \{p_i\}_{i \in I}$, for some index set I , where each p_i is a constant symbol not already in $\mathcal{L}$ associated to an element of the forcing.

Let M be a countable transitive ground model of ZFC². We now seek to establish some sort of a relation between a forcing Q in M and the forcing language. In particular, we want a way of using the forcing to pick out the statements in the forcing language which will be true in the generic extension. Note that we also wish that all statements true in the ground model to hold in the generic extension and that we are able to verify which statements are true in the extension internally to M . In particular, we need to be able to build the new constants of $M[G]$ and evaluate the truth of formulas in $M[G]$ using only the resources available in M . To do this, we define a relation $\Vdash$ between the conditions of the forcing and the forcing language.

We are able to name each element of the extension $M[G]$ using a coding system entirely within M . Each element is “labelled” with a sequence of elements from Q which will then determine the “behaviour” of the element in the extension depending on which generic filter is chosen.

Definition 3.2.2 (Q -name). We define Q -names inductively as follows:

- (i) $N_0 = \emptyset$
- (ii) $N_{\alpha+1} = \{(\dot{x}, p) : \dot{x} \text{ is in } N_\alpha \text{ and } p \text{ is in } Q.\}$
- (iii) For limit ordinals β , $N_\beta = \bigcup \{N_\alpha : \alpha < \beta\}$.

The class of all Q -names in M , denoted M^Q is $\bigcup_{\alpha \in ON} N_\alpha$. Note that this is defined entirely within M and does not depend on our choice of generic filter.

Aside 3.2.3. For those that have seen von Neumann's construction of the set theoretic universe, note the parallels between that and this construction of the class of Q -names. In a sense, we are constructing a “sub-universe” using the forcing conditions.

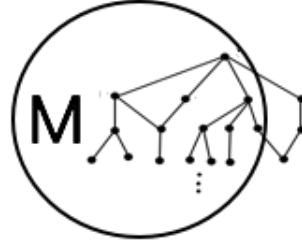


FIGURE 10. A forcing that is not contained entirely within the ground model

²We used Löwenheim-Skolem and our assumption that ZFC is consistent to postulate the existence of a countable model of ZFC. However, this model need not be transitive, so the existence of a countable, *transitive* model of ZFC is a further assumption that we will need that is not provable in ZFC. One can do forcing without transitivity, but defining elements of the extended model becomes significantly more complicated.

There are certain names that deserve special mention for their utility in forcing in general and in getting an intuition for how names work.

Definition 3.2.4 (Check name). Let x be a set. Then the *check name* of x is given by

$$\check{x} = \{(\check{y}, 1_Q) : y \in x\}.$$

The motivation behind the definition of a check name is that for any x in the ground model, $\check{x}$ is a copy of x in the generic extension. In particular, the generic extension contains a copy of the ground model, as given by these check names.

Example 3.2.5. $\check{\emptyset} = \check{\emptyset} = \emptyset = 0$

$$\check{1} = \{\check{\emptyset}\} = \{(\check{\emptyset}, 1_Q)\} = \{(\emptyset, 1_Q)\} = \{(0, 1_Q)\}$$

$$\check{2} = \{0, 1\} = \{(\check{\emptyset}, 1_Q), (\check{1}, 1_Q)\} = \{(0, 1_Q), (\{0, 1_Q\}, 1_Q)\}$$

...

Definition 3.2.6 (Name for Generic Filter).

$$\dot{G} = \{(\check{q}, q) : q \in Q\}.$$

Definition 3.2.7 (Interpretation). The *interpretation* of a name $\dot{x}$ is given by

$$\emptyset^G = \emptyset$$

$$\dot{x}^G = \{\dot{y}^G : \text{for some } q \in G, (y, q) \in \dot{x}\}$$

This is an inductive definition based on the *rank* of the set x . The rank of x can be thought of as a measure of the “complexity” of x - How many steps it takes to construct it from $\emptyset$? How long of a $\in$ -chain does it contain?

I will prove a small lemma about interpretations in the hope that this will help clarify the definition and provide some intuition. However, having a precise understanding of rank will not be necessary in the remainder of the paper.

Example 3.2.8. $\check{x}^G = x$

Proof. We proceed by transfinite induction.

Base case: $\check{\emptyset}^G = \emptyset^G = \emptyset$

Inductive hypothesis: Suppose that $\check{y}^G = y$ for all sets y where $\text{rank } y \leq \alpha$ for all ordinals $\alpha < \beta$.

Inductive step: Let x be a set with rank β . We want to show that $\check{x}^G = x$. We have

$$\begin{aligned} \check{x}^G &= \{\dot{y}^G \mid \text{there is some } q \in G \text{ with } (y, q) \in \check{x}\} \\ &= \{\check{y}^G \mid y \in x\} \\ &= \{y \mid y \in x\} \quad \text{by the inductive hypothesis, since the rank of } y \text{ is less than } \beta \\ &= x \end{aligned}$$

□

Example 3.2.9. $\dot{G}^G = G$

Proof. Recall

$$\dot{G} = \{(\check{q}, q) : q \in Q\}.$$

Hence

$$\begin{aligned} \dot{G}^G &= \{p^G \mid \text{there is some } q \in G \text{ with } (p, q) \in \dot{G}\} \\ &= \{\check{q}^G \mid q \in G\} && \text{by definition of } G \\ &= \{q \mid q \in G\} && \text{by Example 3.2.8} \\ &= G \end{aligned}$$

□

These interpretations bring us to some of the most important definitions in this section.

Definition 3.2.10 (Ground model, generic extension). Let M be a countable transitive model of ZFC, Q a forcing in M and G a generic filter over Q . Then the *generic extension* $M[G]$ of the *ground model* M is given by

$$M[G] = \{\dot{x}^G \mid \dot{x} \text{ is a } Q\text{-name for an element of } M\}.$$

Examples 3.2.8 and 3.2.9 show that $M[G]$ is indeed an extension of M containing G .

The following theorem says that $M[G]$ behaves as we had hoped. It is a model of ZFC, and the smallest extension of M that contains G .

Theorem 3.2.11 (Generic Model Theorem). *Let M be a countable transitive model of ZFC, and let $(Q, \leq)$ be a forcing in M . Let G be a generic filter over Q . Then there exists a transitive model $M[G]$ such that:*

- (i) $M \subseteq M[G]$ and G is in $M[G]$
- (ii) $M[G]$ is a transitive model of ZFC
- (iii) $M[G]$ has the same ordinals as M
- (iv) $M[G]$ is the smallest extension of M containing G . In other words, if N is a transitive model of ZFC with $M \subseteq N$ and G in N , then $M[G] \subseteq N$.

We now lay out a relation $\Vdash$ between elements of the forcing and statements in the forcing language. In this way, we are able to pick out the true statements in $M[G]$ using G . Note that while M^Q , the class of Q -names, does not depend on the generic filter, the forcing relation - and hence the true statements in the extension - do. Thus, while the elements of $M[G]$ can all be constructed within M , the truth values of statements in $M[G]$ depends on the choice of possible world given by G .

The following definition is quite involved, but I attempt to explain how each property does satisfy some intuitive concept that a forcing relation ought to satisfy. This definition follows [Jech, 2002].

Definition 3.2.12 (Forcing relation). The forcing relation $\Vdash$ is a relation between elements of the generic set and sentences of the forcing language such that the following conditions hold:

- (i) for any q in Q , the collection of formulas related to q by $\Vdash$ contains the axioms of ZFC and the axioms of first order logic, is closed under logical deduction, and is consistent.
(All the properties of the ground model to hold in the extension, and we can “safely” apply the rules of logic to make deductions.)
- (ii) if $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$.
(Stronger elements give at least as much information about the extension as weaker elements.)
- (iii) for every φ and every p , there is some $q \leq p$ with $p \Vdash \varphi$ or $p \Vdash \neg\varphi$.
(In this case we say that p decides φ . Every statement is decided by some element of the filter.)
- (iv) for any p in Q and any φ in the forcing language, $p \Vdash \neg\varphi$ if and only if there is no $q \leq p$ with $q \Vdash \varphi$.
- (v) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$.
- (vi) $p \Vdash \forall x \varphi$ iff $p \Vdash \varphi(\dot{x})$ for every $\dot{x} \in M^Q$.
- (vii) $p \Vdash \exists x \varphi$ iff for every $q \leq p$, there exists some $r \leq q$ with $r \Vdash \varphi$ or $r \Vdash \psi$
(If p forces that φ or ψ holds, then in every extension of p , either φ or ψ is forced to hold. Note that different extensions may have different opinions about which one does hold.)

- (viii) $p \Vdash \exists x \varphi$ iff for every $q \leq p$, there exists some $r \leq q$ and some $\dot{x} \in M^Q$ such that $r \Vdash \varphi(\dot{x})$.

(If p forces that there is some x for which φ holds, then in every extension of p , it is forced that φ holds for a specific $\dot{x}$. Note that the actual $\dot{x}$ that satisfies φ may depend on the extension.)

This definition allows us to evaluate the truth value of any statement in $M[G]$ by building it up inductively from smaller, atomic statements. The next proposition collects some useful properties of the forcing relation that we will use later on.

Proposition 3.2.13 (Properties of the forcing relation). *Let M be a countable transitive model of ZFC and Q a forcing in M . Then the following hold:*

- (i) *if $\dot{x}$ is a Q -name, then the collection of all Q -names $\dot{y}$ such that for some q in Q , $q \Vdash \dot{y} \in \dot{x}$ forms a set.*
($\Vdash$ preserves sets.)
- (ii) *for any p in Q and sets x and y , $p \Vdash \dot{x} \in \dot{y}$ if and only if $x \in y$.*
($\Vdash$ preserves inclusion)
- (iii) *for any x and any Q -name $\dot{y}$, if $p \Vdash \dot{y} \in \dot{x}$, then there is some $q \leq p$ and some z such that $q \Vdash \dot{y} = \check{z}$.*
(In this case, we say q decides x (to be z). We are able to pinpoint all the elements of sets.)
- (iv) *if, for some p in Q , $p \Vdash \text{"}\dot{\alpha} \text{ is an ordinal"}$ then there is some $q \leq p$ in Q and an ordinal β such that $q \Vdash \dot{\alpha} = \check{\beta}$.*
($\Vdash$ preserves ordinals.)

The following theorem shows that $\Vdash$ truly respects and extends the traditional satisfaction relation. In other words, the intuition we have been using about using the forcing to determine which statements are true in the generic extension is a valid one; a statement is true in the generic extension just in the case that it is forced by an element of G .

Theorem 3.2.14 (Forcing Theorem). *Let M be some ground model and let $(Q, \leq)$ be a forcing in M . Then for any sentence φ of the forcing language and any generic filter G over Q ,*

$$M[G] \models \varphi \quad \text{iff} \quad \exists p \in G : p \Vdash \varphi$$

We will put this theory into practice after a few lemmata on an especially useful class of forcings which preserve cardinalities.

4. PRESERVING CARDINALITIES

It is not always the case for a cardinality κ , that it is *preserved* in the generic extension. In other words, it is not necessarily true that $\dot{\kappa}$, the Q -name for the cardinality κ is forced to be equal to $\check{\kappa}$, the Q -name for the set κ .

Example 4.0.1 (A generic extension that does not preserve cardinals). Let κ be an uncountable cardinal and consider the set $\kappa^{<\omega}$ of all finite sequences of κ . For $p, q \in \kappa^{<\omega}$, we say $p \leq q$ if p extends q .

Claim 4.0.2.

$$1 \Vdash \dot{\kappa} = \check{\aleph}_0$$

Proof of Claim. Let G be a generic filter over $\kappa^{<\omega}$. We will show that $\cup G$ is a ω -length sequence that contains every element of κ .

To see that $\cup G$ is a sequence of length ω , let $i \in \omega$ and consider

$$D_i = \{p \in \kappa^{<\omega} \mid \text{the length of } p \text{ is at least } i\}.$$

In other words, D_i consists of those sequences that are defined at i .

If we can show D_i is dense, then, since G is generic, there will be some $p \in G \cap D_i$. Thus p is defined at i , and so $\cup G$ defined at i .

To show D_i is dense, let $p \in \kappa^{<\omega}$. We want to find some $q \leq p$ with $q \in D_i$.

If $p \in D_i$, we are done, so suppose the length of p is n , where $n < i$. Construct q as follows:

$$q(j) = \begin{cases} p(j) & \text{for } j \in \text{dom } p \\ 0 & \text{for } n < j \leq i \end{cases}$$

Now we show that $\cup G$ contains every element of κ . Let $i \in \kappa$ and consider

$$R_i = \{p \in \kappa^{<\omega} \mid i \in \text{ran } p\}$$

If we can show this is dense, then there is some $p \in G \cap R_i$, and hence $i \in \text{ran } p$. Thus $i \in \text{ran } \cup G$.

To show R_i is dense, let $p \in \kappa^{<\omega}$ and we seek a $q \leq p$ with $q \in R_i$. Suppose p has length n . Define q by

$$q(j) = \begin{cases} p(j) & \text{for } j \in \text{dom } p \\ i & \text{for } j = n + 1 \end{cases}$$

This shows R_i is dense, so $\cup G$ is an ω -length sequence containing all the elements of κ and hence κ has cardinality no more than ω . $\square$

4.1. Countable Chain Condition. The *countable chain condition* is a condition that bounds the size of antichains and is one property that guarantees the preservation of cardinals.

Definition 4.1.1 (Countable Chain Condition). A forcing $(Q, \leq)$ is said to have the *countable chain condition* (c.c.c.) if and only if every antichain contains at most countably many elements.

Recall an *antichain* $A \subseteq Q$ is a collection of pairwise incompatible conditions.

Example 4.1.2. $\mathcal{B}$ is a c.c.c. forcing.

Proof. Let $\mathcal{A}$ be an antichain in $\mathcal{B}_I$. Recall that $p \perp q$ in $\mathcal{B}_I$ iff they intersect on a set of zero measure.

On one hand, we have that

$$\lambda^I(\cup_{a \in \mathcal{A}} a) \leq 1.$$

However, we also have that

$$\lambda^I(\cup_{a \in \mathcal{A}} a) = \sum_{a \in \mathcal{A}} \lambda^I(a),$$

where each $\lambda^I(a)$ is positive for all a . Hence, there are at most finitely many $a \in \mathcal{A}$. $\square$

The following theorem is the main result of this section, and will be instrumental in the following chapters.

Theorem 4.1.3. Let Q be a c.c.c. forcing and G a generic filter. Then $M[G]$ preserves cardinalities. In particular, for any ordinal α

$$1 \Vdash \check{\aleph}_\alpha = \check{\aleph}_\alpha.$$

The full proof of Theorem 4.1.3 is slightly beyond the scope of these notes, but we shall prove the special case when the cardinal is regular. It can be shown that Theorem 4.1.3 follows from this and a standard result in set theory that any cardinal is the supremum of a set of regular cardinals.

Recall that a cardinal κ is *regular* if and only if every unbounded subset of κ has cardinality κ .

Lemma 4.1.4. *Let κ be a regular cardinal and let Q be a c.c.c. forcing. Then*

$$\mathbf{1} \Vdash \dot{\kappa} = \check{\kappa}$$

Proof. Let $\dot{f}$ and $\dot{\lambda}$ be Q names and let $p \in Q$ be such that

$$p \Vdash \dot{\lambda} < \dot{\kappa} \quad \text{and} \quad p \Vdash \dot{f} : \dot{\lambda} \rightarrow \dot{\kappa}.$$

By Proposition 3.2.13 part (iii), we can assume, extending p if necessary, that

$$p \Vdash \dot{\lambda} = \check{\lambda} \quad \text{and} \quad p \Vdash \dot{f} : \check{\lambda} \rightarrow \dot{\kappa}$$

for some $\lambda \in \kappa$ (i.e., $\lambda < \kappa$) in the ground model.

Thus, we want to show that it is forced that $\dot{f}$ is not a surjection.

First note that if κ is countable, then λ is finite and so, by Proposition 3.2.13 part (iii), $\dot{f}$ is decided by elements of Q and hence $\dot{f}$ is not surjective. This is true in general and does not rely on the forcing being c.c.c.

For each $\alpha \in \lambda$, let $P(\alpha)$ be the set of possible values for $\dot{f}(\check{\alpha})$ given p . In other words,

$$P(\alpha) = \{\beta \in \kappa \mid \exists q \leq p : q \Vdash \dot{f}(\check{\alpha}) = \check{\beta}\}.$$

Note that $P(\alpha)$ is a function in the ground model.

We will now show that P is bounded away from κ in the ground model, which will prove the lemma.

First we show that $P(\alpha)$ is countable for any α . Note that if β and β' are distinct elements of $P(\alpha)$, then there must be two incompatible elements $q, q' \in Q$ with

$$q \Vdash \dot{f}(\check{\alpha}) = \check{\beta} \quad \text{and} \quad q' \Vdash \dot{f}(\check{\alpha}) = \check{\beta}'.$$

If q and q' were compatible, then there would be some $r \leq q, q'$ with

$$r \Vdash \check{\beta} = \dot{f}(\check{\alpha}) = \check{\beta}'$$

contradicting that $\beta \neq \beta'$.

Thus, $P(\alpha)$ corresponds to an antichain in Q and hence is countable.

Because κ is an uncountable regular cardinal, any countable subset of κ must be bounded. Let $g(\alpha)$ the bound for $P(\alpha)$, i.e.,

$$\beta < g(\alpha) \text{ for every } \beta \in P(\alpha).$$

This defines a function $g : \lambda \rightarrow \kappa$ in the ground model. Using the regularity of κ again and the fact that $|\text{rang}| < \kappa$, we have that g is bounded, say by some $\mu \in \kappa$.

Thus it suffices to show that

$$p \Vdash \forall \alpha \in \check{\lambda} : \dot{f}(\check{\alpha}) \in \check{P}(\check{\alpha})$$

as this implies that

$$p \Vdash \forall \alpha \in \check{\lambda} : \dot{f}(\check{\alpha}) < \check{\mu} < \check{\kappa}.$$

This should be fairly obvious by the definition of $P(\alpha)$, but we will present the proof nonetheless.

Suppose this is not the case. Then there is some $\alpha \in \lambda$ with

$$\neg(p \Vdash \dot{f}(\check{\alpha}) \in \check{P}(\check{\alpha})).$$

Hence, here is some $\alpha \in \lambda$ and some $q \leq p$ with

$$q \Vdash \dot{f}(\check{\alpha}) \notin \check{P}(\check{\alpha}).$$

But then, by Theorem 3.2.13, there is some $r \leq q$ and some $\beta \in \kappa$ with

$$r \Vdash \dot{f}(\check{\alpha}) = \check{\beta}$$

and so $\beta \in P(\alpha)$ contradicting that q forced $\dot{f}(\check{\alpha}) \notin \check{P}(\check{\alpha})$, as required. $\square$

4.2. κ -Closure. κ -closure stipulates the existence of a lower bound on all decreasing sequences of length κ . This ensures that cardinalities are preserved up to κ .

Definition 4.2.1 (κ -closure). A forcing Q is κ -closed if and only if for every sequence $(q_i)_{i < \kappa}$ in Q with $q_i \leq q_j$ for $i \geq j$, there exists some $q \in Q$ with $q \leq q_i$ for all i .

Example 4.2.2. $F(\lambda)$ is κ -closed for $\lambda > \kappa$.

Proof. Let $(f_i)_{i < \kappa}$ be a decreasing sequence of elements of $F(\lambda)$. Then we claim $f = \bigcup_{i < \kappa} f_i \in F(\lambda)$ with $f \in Q$ and $f \leq f_i$ for all i .

f clearly extends each f_i , since it contains all of them by construction. f is a function since all the f_i 's merely extend one another, while f is injective since all the f_i 's are injective. Finally, $|\text{dom } f| \leq \kappa < \lambda$ since f is the κ union of sets of size less than κ . $\square$

Proposition 4.2.3. Let Q be a κ -closed forcing and G a generic filter. Then $M[G]$ preserves cardinalities up to κ , i.e.,

$$\mathbf{1} \Vdash \dot{\aleph}_\alpha = \check{\aleph}_\alpha$$

for every cardinal $\aleph_\alpha < \kappa$.

Proof. Let $\dot{f}$ be the Q -name for a function from some cardinal $\lambda < \kappa$ to some set X , so

$$\mathbf{1} \Vdash \dot{f} : \check{\lambda} \rightarrow \check{X}.$$

Equivalently,

$$\mathbf{1} \Vdash \forall i < \check{\lambda} : \dot{f}(i) \in \check{X}.$$

For each $p \in Q$, we will use transfinite induction to construct a decreasing sequence of $(p_i)_{i \leq \lambda}$ such that each p_i decides $\dot{f}(i)$.

Let $p_0 = p$ and suppose we have constructed $(p_i)_{i < \gamma}$.

For $\gamma < \lambda$ (in which case we are not at the end of the sequence), use κ -closedness to find some $r_\gamma \leq p_i$ for all $i < \gamma$. Using part (iii) of Proposition 3.2.13 and the fact that every condition forces that $\forall i < \kappa : \dot{f}(i) \in \check{X}$, we have that there is some p_γ with

$$p_\gamma \Vdash \dot{f}(i) = \check{x}_i$$

for some $x_i \in X$.

For $\gamma = \lambda$ (in which case we are at the last step of our sequence), we find r_γ as before, but this time set $p_\gamma = p_\lambda = r_\gamma$. Since $p_\lambda \leq p_i$ for all i , p_λ decides $\dot{f}(i)$ for all i . Hence,

$$p_\lambda \Vdash \dot{f} = \check{g}$$

for some g in the ground model.

Observe that $\{p_\lambda : p \leq q\}$ is dense in Q , since for any $r \in Q$, $r_\lambda \leq r$ and so any generic filter must contain some p_λ . In particular, the function $\dot{f}$ in any generic extension must have already existed in the ground model. $\square$

Remark 4.2.4. This last point is worth emphasising. Given a κ -closed forcing and function in the generic extension defined on some subset of κ , this function must already have existed in the ground model. In other words, we do not get any new functions defined on a subset of κ in the generic extension.

5. INTERLUDE: THE INDEPENDENCE OF THE CONTINUUM HYPOTHESIS

In fact, the machinery preserving cardinalities brings us so close to the proof of the independence of the Continuum Hypothesis that I will present the argument in this section. While this is not important for the rest of the paper, it does provide an instructive glimpse at the machinery of forcing in the context of the problems it was created to solve.

5.1. The Continuum Hypothesis. The *Continuum Hypothesis* (CH) is a statement about the size of the real numbers. Recall that $\aleph_0 = |\mathbb{N}|$ is the smallest cardinal number, $\aleph_1$ is the smallest cardinal number that is bigger than $\aleph_0$, and so on.

But where does $|\mathbb{R}|$ fall into this cardinal hierarchy? Georg Cantor, who developed the theory of ordinals and cardinals in the late 19th century, conjectured in [Cantor, 1879] that

$$|\mathbb{R}| = \aleph_1.$$

He would spend the rest of his career attempting to prove or disprove this Continuum Hypothesis, but with no avail. This was for good reason, as it would later be shown to be independent of ZFC. In 1938, Gödel showed in that CH is consistent [Gödel, 1938] and Paul Cohen developed forcing to show its negation is consistent in 1963 [Cohen, 1963], for which he won a Fields Medal.

The proof presented below of the consistency of the negation of CH is very similar to Cohen's original proof. However, the proof of the consistency of CH has been rewritten from the perspective of forcing which was not available to Gödel.

5.2. The Consistency of $\neg$ CH. We start by showing that $\neg$ CH is consistent with ZFC. To do this, we will show that $\mathcal{B}_I$ adds a generic object to the ground model that postulates the existence of I distinct real numbers. In particular, for $I = \aleph_2$, this shows that it is impossible to disprove the Continuum Hypothesis within ZFC.

Since $\mathcal{B}_I$ satisfies the c.c.c., we know that cardinalities in a generic extension will be preserved. Hence it suffices to show that

$$1 \Vdash |\dot{\mathbb{R}}| \geq \check{\aleph}_2.$$

Lemma 5.2.1. *Let $\bar{q}$ denote the closure of q for any set q and let $G \subseteq \mathcal{B}_I$ be a generic filter. Then*

$$1 \Vdash \text{"} \bigcap \{\bar{q} : q \in \dot{G}\} \text{ contains a unique element"}$$

Let $\dot{r}$ be the $\mathcal{B}_I$ -name for this element and let $\dot{r}_i$ be the $\mathcal{B}_I$ -name for the i th coordinate of r .

Note that the strongest elements of $\mathcal{B}_I$ give us the most information about r .

Proof of Lemma 5.2.1. Write X for $\bigcap \{\bar{q} : q \in \dot{G}\}$.

To show that X is nonempty, we recall a well-known result from topology stating that the intersection of any decreasing, nested sequence of closed sets is nonempty. Since $\mathcal{B}_I$ is a filter it must satisfy that for every $p, q \in \mathcal{B}_I$, there is some $r \in \mathcal{B}_I$ with $r \subseteq p \cap q$. This can be used to generate a decreasing sequence $\bar{r}_i$ of closed, nested elements of G with $X = \bigcap \bar{r}_i$.

Now suppose $x, y \in X$ with $x \neq y$. Then there exist open balls $B\left(x, \frac{|x-y|}{2}\right)$ and $B\left(y, \frac{|x-y|}{2}\right)$ around x and y of positive measure with

$$B\left(x, \frac{|x-y|}{2}\right) \cap B\left(y, \frac{|x-y|}{2}\right) = \emptyset.$$

But this contradicts that G is a filter. □

Now set $I = \omega_2$.

Define

$$D_{i,j} = \{q \in \mathcal{B}_I : q(i) \neq q(j)\}$$

where $q(i)$ denotes the projection of q into the i^{th} coordinate.

Claim 5.2.2. *For each $i, j \in \omega_2$, $D_{i,j}$ is dense in $\mathcal{B}_I$.*

The proof of this claim, which I have laid out below, requires some knowledge of measure theory and the construction of infinite product measures. If you are unfamiliar with this material, try to understand intuitively why the claim ought to be true, but do not worry if you do not understand the details of the proof.

Proof. Let $i, j \in \omega_2$ and let $q \in \mathcal{B}_I$ be arbitrary. We wish to find some $p \leq q$ with $p \in D_{i,j}$.

If $q(i) \neq q(j)$, then we are done, so suppose $q(i) = q(j)$. Since the measure of q is positive, $q(n)$ must be nonempty (and even infinite) for all n . Let $x \in q(i)$ and note that the set

$$X = [0, 1] \times \cdots \times [0, 1] \times [0, 1] \setminus \{x\} \times [0, 1] \times \cdots \times [0, 1] \subseteq [0, 1]^I,$$

where the $\{x\}$ is in the i^{th} coordinate, is measurable as the product of measurable sets, all but finitely many of which are $[0, 1]$.

So setting $p = q \cap X$, we have $p \in D_{i,j}$ and $p \leq q$, as required. $\square$

Hence, for a generic filter G over $\mathcal{B}_I$, $G \cap D_{i,j} \neq \emptyset$ for all i, j , so

$$\mathbf{1} \Vdash \forall i \in \omega_2 : \forall j \in \omega_2 : \dot{r}_i \neq \dot{r}_j.$$

In other words, it is forced that all ω_2 coordinates of r are distinct real numbers, from which we can easily deduce

$$\mathbf{1} \Vdash |\dot{\mathbb{R}}| \geq |\dot{I}| = \aleph_2 = \dot{\aleph}_2.$$

$\square$

5.3. The Consistency of CH. We will now show that CH is consistent with ZFC by constructing a cardinal-preserving model in which there is a bijection between $\mathbb{R}$ and $\aleph_1$. This demonstrates that no proof of $\neg\text{CH}$ is possible within ZFC.

We will be working with $F(\aleph_1)$, which preserves cardinalities up to $\aleph_1$.

Lemma 5.3.1. *Let $G \subseteq F(\aleph_1)$ be a generic filter. Then $g = \cup G$ is forced to be a bijective total function.*

Note that the strongest elements of $F(\aleph_1)$ are those that give the most information about g .

Proof. To show g is a function, we must show that, given any $i \in \aleph_1$, $g(i)$ has at most one possible value.

So let $i \in \aleph_1$ be arbitrary. Suppose that there was some $f_1, f_2 \in G$ with $f_1(i) \neq f_2(i)$. Then, since G is a filter, there is some $h \in G$ with $h \leq f_1$ and $h \leq f_2$. Thus $f_1(i) = h(i) = f_2(i)$, contradicting that $f_1(i) \neq f_2(i)$.

To show g is total, we must show that g is defined on every $i \in \aleph_1$.

So let $i \in \aleph_1$. Define

$$D_i = \{f \in F(\aleph_1) \mid i \in \text{dom } f\}.$$

If we can show D_i is dense then $G \cap D_i \neq \emptyset$, so $\cup G$ is defined on i .

Hence, let $f \in F(\aleph_1)$ be arbitrary. We want to find an extension of f that is in D_i .

Note that if $f \in D_i$, we are done, so assume not. Let y be such that $y \notin \text{ran} f$. (Such a y exists since f is only defined at countably many values and so only has a countable range.) Define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \text{dom} f \\ y & \text{if } x = i \end{cases}$$

Since $\tilde{f} \leq f$ and $i \in \text{dom} \tilde{f}$, g is total.

To show g is injective, just note that f is injective for every $f \in G$. Thus, for $i \neq j \in \aleph_1$, $f(i) \neq f(j)$ for every $f \in G$ with $i, j \in \text{dom} f$. Hence $g(i) \neq g(j)$.

To show g is surjective, we have to show that $i \in \text{ran} g$ for every $i \in \mathbb{R}$. So let $i \in \mathbb{R}$ and consider

$$R_i = \{f \in F(\aleph_1) \mid i \in \text{ran} f\}.$$

If we can show R_i is dense, then $G \cap R_i \neq \emptyset$, so i is in the range of $\cup G$.

Thus let $f \in F(\aleph_1)$ be arbitrary. We want to find an extension of f that is in R_i .

Note that if $f \in R_i$, we are done, so assume not. Let $y \in \aleph_1$ be such that $y \notin \text{dom} f$ and define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \text{dom} f \\ i & \text{if } x = y \end{cases}$$

Since $\tilde{f} \leq f$ and $i \in \text{ran} \tilde{f}$, g is surjective, as required. $\square$

So $\cup G$ is the required bijection between $\dot{\mathbb{R}}$ and $\check{\aleph}_1$. Since cardinalities are preserved, it is forced that $\check{\aleph}_1 = \dot{\aleph}_1$, and we are done. $\square$

6. APPLICATIONS OF FORCING

In this section, we will demonstrate how one can use forcing to prove results not about ZFC, but within ZFC. This will be closely related to the concept of *absoluteness* which guarantees that certain sentences are true within ZFC if and only if they are forced by some condition in a generic extension. As we shall see, it can be easier to work in generic extensions than in the ground model, and absoluteness provides the necessary link between the two.

It is worth noting that while we will be focussing on a particular paper in topology, a similar technique has also been used to prove results in Galois Theory ([Viale, NA]), Ramsey theory ([Shelah, 1987] and [Baumgartner and Hajnal, 1973]), covering the plane with graphs of functions ([Kubiś and Vejnar, 2012] and [Kunen, 2012]), fractals ([Komjáth, 1994]), and more.

6.1. Absoluteness.

Definition 6.1.1 (Generically Absolute). Let $\varphi(x_0, \dots, x_{n-1})$ be a sentence in $\mathcal{L}_\in$. Then φ is *generically absolute* just in the case that φ is true in ZFC iff there is some forcing extension $M[G]$ over a forcing Q and some condition $q \in Q$ with $q \Vdash \varphi$.

This definition looks more confusing than it is. The intuition is that the truth of generically absolute statements cannot be changed by forcing. Clearly, one direction holds in general. If φ is true in ZFC then it holds in any forcing extension, as **1** forces all the axioms of ZFC and their consequences. The other direction merely says that if φ is generically absolute and is true in some generic extension, then we must have already had it in the ground model. This is the direction we will exploit to prove theorems within ZFC in this section.

Note that the class of generically absolute statements is closed under negation and logical consequence.

The game plan from here is to identify a class of statements that one can tell are absolute a priori, by just examining their structure. Then, to prove such a statement holds in ZFC, it suffices to carefully choose a generic extension and show it is true in that extension. We shall see, for example, a case in which the proof becomes much simpler if we are working in an extension in which CH holds. Finally, since the statement is absolute, proving it in the generic extension implies it holds within ZFC.

The earliest example I can find of this technique is [Solovay, 1967], only four years after Cohen's introduction of forcing to the mathematical world as a model-building tool.

Definition 6.1.2 (Nice Q -name). Let (X, d) be a metric space and Q be a forcing. A Q -name $\dot{x}$ is a *nice Q -name* if and only if there exists a countable collection of dense sets $\mathcal{D}$ such that for all $\mathcal{D}$ -generic filters over Q , $\dot{x}^G$ is in the completion of X .

The motivation behind this definition is that it often makes more sense for $\dot{x}$ to be a Q -name for the completion of $\check{X}$ and not the space $\check{X}$ itself. Recall that while a complete space X and the completion of X are not literally the same, as the completion of X consists of equivalence classes of Cauchy sequences of elements of X , they are canonically isomorphic.

Thus, the interpretation of a nice Q -name over a sufficiently generic filter will give an element of the completion of X rather than an element of X itself. The underlying intuition is that the Q -name $\dot{x}$ ought to give an indication of how to build the set x using forcing conditions and elements of X in much the same way that Cauchy sequences tell us how to approximate x using a sequence of points in X .

Definition 6.1.3 (Universally Baire Sets). Let (X, d) be a complete metric space. $A \subseteq X$ is *universally Baire* if and only if for all forcings Q , there exists a Q -name $\dot{A}$ such that for some nice Q -name $\dot{x}$ for an element of X and for some countable collection $\mathcal{D}$ of dense sets, the following hold:

- (i) $D_x = \{q \in Q \mid q \text{ decides } \dot{x}\}$ is in $\mathcal{D}$
- (ii) For every $\mathcal{D}$ -generic filter G on Q , $\dot{x}^G$ is in the completion of X
- (iii) $\dot{x}^G$ is in the completion of A if and only if $\dot{x} \in \dot{A}$

Note that universally Baire sets are only defined for *complete* metric spaces.

It is not clear from the definition that universally Baire sets are particularly “nice”, or even that they exist at all. The next result shows that universally Baire sets are in fact a quite natural to work with, and we shall see several examples of this in the coming sections.

Proposition 6.1.4. *The class of universally Baire subsets of a complete metric space (X, d) forms a σ -algebra containing all the Borel sets.*

Proposition 6.1.4 means that in practice, we no longer have to verify that a set, or collection of sets, is universally Baire from the definition. Instead, it suffices to show that it is Borel, which is far understood. This will make several proofs in the coming sections far more straightforward.

Theorem 6.1.5. *The statement that a countable Boolean combination of universally Baire sets is nonempty is generically absolute.*

In other words, if we can prove a countable Boolean combination of universally Baire sets is nonempty in some generic extension, then it is true in ZFC. Together with Proposition 6.1.4, this gives rise to a number of useful and natural statements that are generically absolute.

Proof. Let $(A_i)_{i \in \mathbb{N}}$ be a collection of universally Baire subsets of some complete metric space X . We will show that “ $\bigcap_{i \in \mathbb{N}} A_i \neq \emptyset$ ” is generically absolute. Showing “ $\bigcup_{i \in \mathbb{N}} A_i \neq \emptyset$ ” is similar.

First note that $\bigcap_{i \in \mathbb{N}} A_i$ is universally Baire by Proposition 6.1.4.

To show a statement φ is generically absolute, we must show that if φ is true in ZFC, then φ holds in every generic extension and that if φ holds in any generic extension, then it is true in ZFC.

As mentioned above, if φ is a theorem of ZFC, then it must be true in every forcing extension. So it suffices to show that if “ $\bigcap_{i \in \mathbb{N}} A_i \neq \emptyset$ ” holds in some forcing extension, it is true in ZFC.

Let Q be a forcing and suppose there is some $q \in Q$ such that

$$q \Vdash \bigcap_{i \in \mathbb{N}} \dot{A}_i \neq \emptyset.$$

Let $\dot{A}$ be the Q -name witnessing that $\bigcap_{i \in \mathbb{N}} A_i$ is universally Baire. Hence for every nice Q -name $\dot{x}$, there is some countable collection $\mathcal{D}$ of dense sets such that

- (i) $D_x = \{q \in Q \mid q \text{ decides } \dot{x}\}$ is in $\mathcal{D}$
- (ii) For every $\mathcal{D}$ -generic filter G on Q , $\dot{x}^G$ is in the completion of X
- (iii) $\dot{x}^G$ is in the completion of $\bigcap_{i \in \mathbb{N}} A_i$ if and only if $\dot{x} \in \dot{A}$

Since

$$q \Vdash \bigcap_{i \in \mathbb{N}} \dot{A}_i \neq \emptyset,$$

there is some $\dot{x}$ such that

$$q \Vdash \dot{x} \in \bigcap_{i \in \mathbb{N}} \dot{A}_i \neq \emptyset$$

and hence, by (iii), $\dot{x}^G$ is in the completion of $\bigcap_{i \in \mathbb{N}} A_i$ and hence $\bigcap_{i \in \mathbb{N}} A_i \neq \emptyset$. $\square$

6.2. Review of Topology. Before we can continue with our investigations in absoluteness and generic extensions, we will need to familiarise ourselves with some ideas from topology.

While a basic knowledge on par with a typical undergraduate course in point set topology is expected, we will provide some of the more advanced definitions and results needed for the next section here. For those who have not seen topology or who would like to understand the concepts more deeply, [Munkres, 2000] is an excellent resource.

Definition 6.2.1 (Base). A *base* for a topological space X is a collection $\mathcal{B}$ of subsets of X such that for every open set $U \subseteq X$ with $U \neq \emptyset$, there is some $B \in \mathcal{B}$ with $B \subseteq U$.

Definition 6.2.2 (Neighbourhood base). A *neighbourhood base* of a point x in a topological space X is a collection $\mathcal{B}$ of open neighbourhoods of x such that for any open neighbourhood U of x , there is some $B \in \mathcal{B}$ with $B \subseteq U$.

Note that this is exactly the definition of a base for a topology except that “open set” is replaced with “open neighbourhood of x ”.

Definition 6.2.3 (Compact). A topological space X is *compact* if and only if whenever there is a collection of open sets $\{U_i \mid i < \kappa\}$ of any cardinality with

$$\bigcup_{i < \kappa} U_i = X,$$

there is some finite $I \subseteq \kappa$ with

$$\bigcup_{i \in I} U_i = X.$$

In other words, every open cover has a finite subcover.

In a metric space, this is equivalent to every sequence having a convergent subsequence.

Example 6.2.4. $\mathbb{R} \cup \{\infty\}$ and $\mathbb{C} \cup \{\infty\}$ are compact.

Definition 6.2.5 (Separable). A topological space X is *separable* if and only if X has a countable dense subset.

Example 6.2.6. $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.

Definition 6.2.7. The polynomials form a countable dense subset of the continuous functions on the real numbers with the supremum norm.

Definition 6.2.8 (First countable). A topological space X is *first countable* if and only if for every $x \in X$, x has a countable neighbourhood base.

Example 6.2.9. Any metric space is first countable since $\{B(x, r) \mid r \in \mathbb{Q}^+\}$ is a countable neighbourhood base for any point x .

A lot of topology involves slowly stepping back from finite and countable objects and seeing how much structure still remains. Compactness, separability, and first countability each represent a way to capture that the space is “almost countable”.

Definition 6.2.10 (Hausdorff). A topological space is *Hausdorff* if and only if for any distinct points $x, y \in X$, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

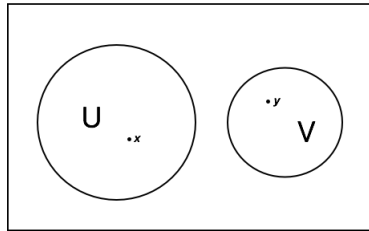


FIGURE 11. An illustration of a Hausdorff space

In other words, any two points of X can be separated into two disjoint open sets. (See Figure 11.)

Example 6.2.11. $\mathbb{R}$ is Hausdorff. In fact, any metrisable space is Hausdorff.

Definition 6.2.12 (Complete limit point). Let X be a topological space and $Y \subseteq X$. Recall that a point $x \in X$ is a *limit point* of Y if and only if every open neighbourhood of x contains an element of Y that is distinct from x .

A *complete limit point* of Y is a point $x \in X$ such that for any open neighbourhood U of x , $|U \cap Y| = |Y|$.

Example 6.2.13. Every element of $[0, 1]$ is a complete limit point of $[0, 1]$.

Proposition 6.2.14. Let X be an infinite compact topological space. Then every infinite subset of X has a complete limit point.

Proof. Let X be an infinite compact topological space and suppose $Y \subseteq X$ is infinite with no complete limit point. Then for every $x \in X$ and every open neighbourhood U_x of x , $|U_x \cap Y| < |Y|$.

Let $\mathcal{U}_x$ denote the family of open neighbourhoods of x . We have that $\bigcup_{x \in X} \bigcup_{U \in \mathcal{U}_x} U$ is an open cover for X and hence has a finite subcover, say $(U_i)_i$. But $|U_i| < |Y|$ for all i , and hence $|X| = |\bigcup_i U_i| < |Y|$, a contradiction. $\square$

6.3. Baire Class 1. In the section that follows, we will be proving results about compact subsets of the first Baire class, a generalisation of the space of continuous functions. However, we shall need a few more specialised definitions before we can proceed.

Definition 6.3.1 (Completely metrisable). A topological space X is *completely metrisable* if and only if it is homeomorphic to a complete metric space.

In other words, from a topological viewpoint, X essentially *is* a complete metric space.

Example 6.3.2. $\mathbb{N}$ with the discrete topology is completely metrisable.

Hint: Consider the metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Example 6.3.3. A finite product of completely metrisable spaces is completely metrisable.

Hint: Define a metric $d(x, y) = \left(\sum_{i \leq n} d_i(x, y)^i \right)^{\frac{1}{n}}$, where d_i is the metric on the i^{th} coordinate.

Proposition 6.3.4 (Alexandrov's Theorem). *Let X be a completely metrisable topological space and let $Y \subseteq X$ be a G_δ subset, i.e., one that can be expressed as a countable intersection of closed sets. Then Y is completely metrisable.*

See [Bruckner et al., 2008], Theorem 10.19, for a proof

Example 6.3.3 and Proposition 6.3.4 both provide ways to construct new completely metrisable spaces from old.

Definition 6.3.5 (Polish Space). A topological space X is a *Polish space* if and only if it is separable and completely metrisable.

These spaces are so-named as they were studied extensively by a school of Polish mathematicians including Waław Sierpiński, Alfred Tarski, and Kazimierz Kuratowski, during the inter-war period.

Example 6.3.6. $\mathbb{R}$ is the canonical example of a Polish space. In fact, some descriptive set theorists will refer to any Polish space as “the reals”. Indeed, a reader uncomfortable with Polish spaces will lose nothing but a little bit of generality by replacing all Polish spaces with $\mathbb{R}$.

Definition 6.3.7. A subset P of a topological space X is *perfect* if and only if P is closed and has no isolated points. Equivalently, P is perfect if and only if it is equal to its set of limit points.

Example 6.3.8. Any finite union of closed intervals is perfect.

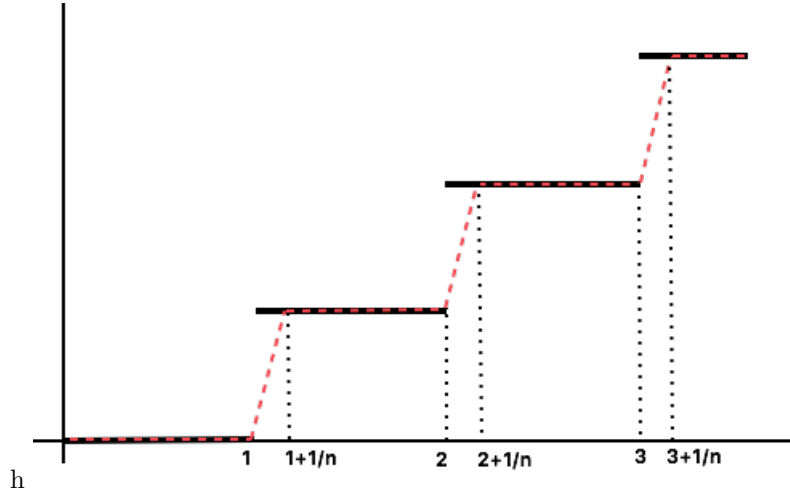


FIGURE 12.

Example 6.3.9. For readers familiar with basic fractal geometry, the Cantor set is perfect.

Definition 6.3.10 (Baire class one). Let X be a Polish space. Then the set of *Baire class one functions*, denoted $BC_1(X)$, is the collection of all pointwise limits of continuous functions.

Recall that the space of continuous functions is not closed under pointwise limits. $BC_1(X)$ contains all functions that can be approximated pointwise arbitrarily well by continuous functions.

Example 6.3.11 (Step functions). Step functions are $BC_1(X)$.

Figure 12 provides a pictorial indication that step functions are in $BC_1(X)$. In particular, any step function, represented by the black line, can be expressed as a limit of the continuous functions given by the red dotted line.

In particular, we shall be interested in compact subsets of $BC_1(X)$. However, in order to make sense of compactness, one needs to know which topology we will be considering for $BC_1(X)$.

Definition 6.3.12 (Topology of pointwise convergence). Let $\mathcal{M}(X, Y)$ be a collection of functions from X to Y . Then the *topology of pointwise convergence* on $\mathcal{M}$ is the topology induced by considering $\mathcal{M}$ as a subspace of $X \times Y$ with the product topology.

It can be shown that $f_n \rightarrow f$ in this topology if and only if $f_n(x) \rightarrow f(x)$ for every $x \in X$.

Definition 6.3.13 (Rosenthal Compactum). Let X be a Polish space. Then a *Rosenthal compactum* is a space that is homeomorphic to a pointwise compact subspace of $BC_1(X)$.

Rosenthal compacta owe their name to Haskell Rosenthal, who was the first to systematically study them in [Rosenthal, 1977].

Example 6.3.14 (One-point compactification). Let X be a Polish space and $x \in X$. Define

$$\delta_x(t) = \begin{cases} 1 & \text{if } t = x \\ 0 & \text{otherwise} \end{cases}$$

Then

$$R = \{\delta_x \mid x \in X \cup \{\infty\}\},$$

where δ_∞ is the constant 0 function, is a Rosenthal compactum.

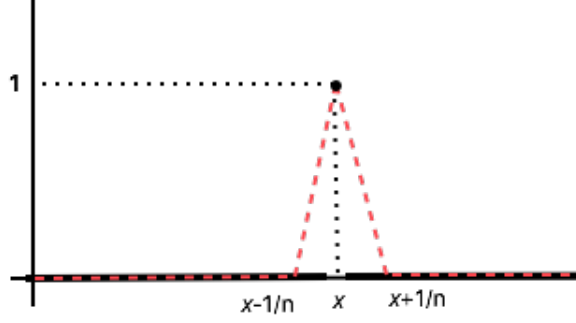


FIGURE 13.

Proof. Figure 13 indicates how one might show that $\delta_x \in BC_1(X)$. Again, the black function is δ_x and the red function suggests a sequence of continuous functions that converge to δ_x .

To show it is compact, let $(\delta_{x_n})_n$ be a sequence of functions in D . We want to find a convergent subsequence of $(\delta_{x_n})_n$.

If infinitely many of the x_n 's are the same, then this will be a convergent subsequence.

If not, then take a subsequence $(x_{n_k})_k$ of the x_n 's such that all elements of the subsequence are distinct. Then, for any t in X , $\delta_{x_{n_k}}(t)$ is eventually 0, and hence $\delta_{x_{n_k}} \rightarrow \delta_\infty$ as $k \rightarrow \infty$. $\square$

The following result highlights that Rosenthal compacta do satisfy a property held by all metrisable spaces; namely they are Hausdorff.

Proposition 6.3.15. *Let K be a Rosenthal compactum. Then K is Hausdorff in the topology of pointwise convergence.*

Since we have not explicitly given a base for K with the topology of pointwise convergence, we shall omit the proof.

The following theorem, due to René Baire, provides a useful characterisation of functions in $BC_1(X)$ that does not rely on finding a sequence of continuous functions [Baire, 1899].

Theorem 6.3.16 (Baire's Characterisation Theorem). *Let X be a Polish space and let $f : X \rightarrow \mathbb{R}$. Then $f \in BC_1(X)$ if and only if for all perfect sets P the restriction of f to P has a point of continuity.*

However, we shall be using the converse, which can be restated as follows.

Corollary 6.3.17 (Converse of Baire's Characterisation Theorem). *Let X be a Polish space and let $f : X \rightarrow \mathbb{R}$. Then $f \notin BC_1(X)$ if and only if there exists a perfect set P and countable dense subsets Y and Z of P such that for some rationals $\delta < \epsilon$ and all $y \in Y$ and $z \in Z$,*

$$f(y) < \delta \quad \text{and} \quad f(z) > \epsilon.$$

6.4. Forcing and Rosenthal Compacta. In [Todorćević, 1999], Todorćević deduces a number of properties of Rosenthal compacta by studying their behaviour in generic extensions and using absoluteness to transfer the results down to ZFC. While the results are important in and of themselves, what makes the paper truly novel is that it was one of the first to exploit absoluteness to prove something really new within ZFC.

We will present some of the results found in that paper here. The theorems are largely as stated as [Todorćević, 1999], while the proofs are based on those in [Todorćević, 1999] and [Moore, NA].

The main result (for our purposes) of [Todorćević, 1999] is that Rosenthal compacta are “almost metrisable” in that they contain a dense metrisable subspace. The rest of the paper will build up to that result.

The following lemma gives an indication as to why $BC_1(X)$ is so amenable to being studied by forcing.

Lemma 6.4.1. *Let X be a Polish space, $F \subseteq BC_1(X)$, and let f be a limit point of F . Then the assertion that “ $f \in BC_1(X)$ ” is generically absolute.*

Proof. Let $(f_n)_n$ be a sequence of functions on X converging pointwise to some $f : X \rightarrow \mathbb{R}$. Then the assertion that $f_n \rightarrow f$ pointwise can be expressed as

$$\bigcap_{\epsilon \in \mathbb{Q}^+} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{x \in X \mid |f_n(x) - f(x)| > \epsilon\} \neq \emptyset$$

In particular, since $\{x \in X \mid |f_n(x) - f(x)| > \epsilon\}$ is open and thus universally Baire for each n and ϵ , it follows that the statement that “ $(f_n)_n$ converges pointwise” is generically absolute.

Consider

$$S = \{(f_{n_i})_i \mid f_{n_i} \in C(X) \text{ and } (f_{n_i})_i \text{ converges pointwise to some } f_n \text{ for each } n \in \mathbb{N}\},$$

the set of sequences of $BC_1(X)$. It follows from Example 6.3.3 and Theorem 6.3.4 that $S \subseteq \mathbb{R}^X \times \mathbb{N} \times \mathbb{N}$ is completely metrisable, where each space is endowed with the discrete topology.

Consider some sequence $(f_n)_n$ in $BC_1(X)$ with pointwise limit f and approximate each function f_n by continuous functions $(f_{n_i})_i$. By Baire’s Characterisation Theorem, $f \notin BC_1(X)$ if and only if there is some closed set P with no isolated points, countable dense subsets $X, Y \subseteq P$, and rational ϵ and δ such that

$$f(x) < \delta \text{ and } f(y) > \epsilon.$$

Equivalently, f is not Baire class one if and only if

$$\bigcup_{\epsilon, \delta \in \mathbb{Q}^+} \bigcup_{N, I \in \mathbb{N}} \bigcap_{n \geq N, i \geq I} \{(x, y) \in X \times Y \mid f_{n_i}(x) < \delta \text{ and } f(y) > \epsilon\} \neq \emptyset$$

This is a countable Boolean combination of open subsets of S and so the result follows from Proposition 6.1.5. $\square$

The following definitions can be found in [Todorćević, 1990]. They are of interest as *free sequences* provide us with another way to measure how far our space is from being countable. Indeed, it can be shown that the maximum cardinality of a free sequence in a topological space corresponds to the *tightness* of the space, or how large of a set we need to approximate limit points [Rosenthal, 1977].

Definition 6.4.2 (Regular pair). Let X be a topological space and $F, G \subseteq X$. Then (F, G) is a *regular pair* if and only if F is closed, G is open, and $F \subseteq G$.

Definition 6.4.3 (Free sequence). A sequence of regular pairs $(F_i, G_i)_{i \leq \kappa}$ is *free* if and only if for any finite subsets $I, J \subseteq \kappa$ in which the maximum element of I is less than the minimum element of J ,

$$\bigcap_{i \in I} F_i \cap \bigcap_{j \in J} (X \setminus G_j) \neq \emptyset.$$

Proposition 6.4.4. *Every free sequence of regular pairs in a Rosenthal compactum is countable.*

In this way, Rosenthal compacta are not too big. This will allow us to work with these spaces more directly, as we can get a handle on their limit points.

Proof. Let K be a Rosenthal compactum and let $(F_i, G_i)_{i \leq \kappa}$ be a free sequence of regular pairs in K . Define a sequence $(x_i)_{i < \kappa}$ such that

$$x_i \in \bigcap_{j < i} F_j \cap \bigcap_{j \geq i} (X \setminus G_j)$$

Note that x_i exists by the definition of free sequences and compactness.

Suppose that κ is uncountable and hence, by compactness, $(x_i)_{i \in \kappa}$ has a complete limit point in K . Call this limit point x . By a lemma of Rosenthal ([Rosenthal, 1978], Lemma 3.9), there is some countable subset $I \subseteq \kappa$ with x in the closure of $(x_i)_{i \in I}$. (This is how we exploit that we can approximate limit points with countable sets, and countable sets are much easier to work with.) Note that since κ is uncountable, $\omega_1 \leq \kappa$, where ω_1 denotes the smallest uncountable ordinal, and $\omega_1 \geq i$ for all $i \in I$. (This is where we use our assumption that κ is uncountable.) Then for each $i \in I$, we have, by definition of x_i ,

$$x_i \in X \setminus G_{\omega_1}.$$

And so, since $X \setminus G_{\omega_1}$ is closed, and x is a limit point of the $(x_i)_{i \in I}$,

$$x \in X \setminus G_{\omega_1}.$$

But, for each $i \in \kappa$ with $i \geq \omega_1$, $x_i \in F_{\omega_1}$. Hence $X \setminus F_{\omega_1}$ is an open neighbourhood of x with at most countably many elements of $(x_i)_{i \in \kappa}$, which contradicts x being a complete limit point. $\square$

Theorem 6.4.5. *Every Rosenthal compactum contains a point with a countable neighbourhood base.*

In order to prove this statement, we need the following result of Čech and Pospíšil [Čech and Pospíšil, 1938].

Theorem 6.4.6 (Čech – Pospíšil). *Let K be a compact Hausdorff space with cardinality $\leq \aleph_1$. Then K contains a point with a countable neighbourhood base.*

Proof of 6.4.5. Let $Q = F(\aleph_1)$ and G a generic filter over Q . Recall that

$$F(\aleph_1) = \{f : \aleph_1 \rightarrow \mathbb{R} \mid \text{dom } f \text{ is countable and } f \text{ is injective}\}$$

where f is stronger than g if and only if f extends g . In particular, $|\dot{\mathbb{R}}| = \aleph_1$ in $M[G]$ since Q is $\aleph_1$ -closed.

Thus, the space of real valued continuous functions in $M[G]$ is also has cardinality $\aleph_1$ and so, for any uncountable Rosenthal compactum K ,

$$\mathbf{1} \Vdash |\dot{K}| = \aleph_1.$$

Let $K \subseteq \text{BC}_1(X)$ be compact and hence closed. By Lemma 6.4.1, the assertion that a limit point of K is in $\text{BC}_1(X)$ is generically absolute and hence,

$$\mathbf{1} \Vdash \text{“The closure of } \dot{K} \text{ is } \dot{K} \text{.”}$$

So,

$$\mathbf{1} \Vdash \check{K} \text{ is closed.} \quad \text{and} \quad \mathbf{1} \Vdash \check{K} \text{ is compact.}$$

Hence, by Theorem 6.4.6 above, there is some $\dot{g} \in \text{BC}_1(X)$ and some countable collection $(\dot{U}_i)_i$ of open sets of $\text{BC}_1(X)$ such that

$$\mathbf{1} \Vdash \dot{g} \in \check{K}$$

and

$$\mathbf{1} \Vdash "(\dot{U}_i)_i \text{ is countable neighbourhood base for } \dot{g}."$$

Recall we noted that the proof of Proposition 4.2.3 implied that Q adds no new functions on a subset of $\aleph_1$ to the ground model. Hence, there is some $q \in Q$, some $f \in \text{BC}_1(X)$, and some $(V_i)_i$, a countable collection of open subsets of $\text{BC}_1(X)$ with

$$q \Vdash \dot{g} = \check{f} \quad \text{and} \quad q \Vdash \dot{U}_i = \check{V}_i$$

for each $i \in \omega$.

In other words, the point in the Rosenthal compactum and its countable neighbourhood base we had chosen in the generic extension were in fact a point in the Rosenthal compactum and its countable neighbourhood base in the ground model, as required. $\square$

In fact, something stronger than Theorem 6.4.5 is true. We can show that the set of points with a countable neighbourhood base is dense; in other words, any Rosenthal compactum has a dense first countable subset. It will ultimately be shown that, under nice conditions, the existence of a dense countable neighbourhood base implies the existence of a dense metrisable subspace.

Before we get to the next result, we need the following definition.

Definition 6.4.7 (Finite intersection property). Let X be a set and $\mathcal{A} = \{A_i \mid i \in I\}$ a collection of subsets of X . Then $\mathcal{A}$ has the *finite intersection property* (FIP) if and only if for every finite $J \subseteq I$,

$$\bigcap_{i \in J} A_i \neq \emptyset.$$

Proposition 6.4.8. *Let K be a Rosenthal compactum. Then there is some sequence $(F_i, G_i)_{i \leq \kappa}$ of regular pairs such that $\{\text{int} F_i \mid i \leq \kappa\}$ forms a base of K and such that whenever $\Omega \subseteq \kappa$ with $(F_i, G_i)_{i \in \Omega}$ has the finite intersection property, $(F_i, G_i)_{i \in \Omega}$ is free.*

The general idea of the proof, given a basis $(U_i)_{i < \lambda}$, is to inductively construct a sequence of sets with

$$\emptyset \neq \text{int} F_i \subseteq F_i \subseteq G_i \subseteq U_i,$$

carefully choosing the F_i and G_i so that the FIP will be satisfied.

Recall that a set U is *regular open* if and only if it is equal to the interior of its closure; i.e.,

$$U = \text{int}(\bar{U})$$

Proposition 6.4.9. *Let $K \subseteq \text{BC}_1(X)$ be compact and suppose $(F_i, G_i)_{i \leq \kappa}$ is a sequence of regular pairs of K satisfying the conclusion of Proposition 6.4.8. Let Q be the set of regular open sets of K , ordered by inclusion, and let G be a generic filter over Q . Write $\dot{x}$ for the unique element of $\cap G$. Then TFAE:*

- (i) *K has a base $\mathcal{U} = \bigcup_{n \in \mathbb{N}} U_n$, where each U_n is a collection of pairwise disjoint open sets. ($\mathcal{U}$ is σ -disjoint.)*
- (ii) *Some condition in Q forces that $\dot{x}$ has a countable neighbourhood base.*

(iii) *Some condition in Q forces that $\{i \in \check{\kappa} \mid \check{G}_i \in \dot{G}\}$ is countable.*

Note that $\dot{x}$ forced to exist by a similar argument to that used to prove Lemma 5.2.1.

Proof. We start by showing (i) $\Rightarrow$ (ii). Let $\mathcal{U}$ be the σ -disjoint base and write $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$ be a base for K , where each $\mathcal{U}_n$ is a collection of pairwise disjoint subsets of K . Let $\dot{\mathcal{V}}$ be a Q -name for $\{U \in \check{\mathcal{U}} \mid \dot{x} \in U\}$. This is our candidate for a countable neighbourhood base for $\dot{x}$.

To show $\dot{\mathcal{V}}$ is forced to be countable, note that

$$|\dot{\mathcal{V}}| = |\dot{\mathcal{V}} \cap \check{\mathcal{U}}| = |\dot{\mathcal{V}} \cap \bigcup_n \check{\mathcal{U}}_n| = \sum_n |\dot{\mathcal{V}} \cap \check{\mathcal{U}}_n| \leq \sum_n 1$$

since the elements of $\mathcal{U}_n$ being pairwise disjoint implies that $|\dot{\mathcal{V}} \cap \check{\mathcal{U}}_n| \leq 1$ for each n .

To show $\dot{\mathcal{V}}$ is forced to be a neighbourhood base for $\dot{x}$, let $p \in Q$ and let $\dot{W}$ be a Q -name for an open neighbourhood of $\dot{x}$. Thus, we want to find some $q \in Q$ and some $\check{U}$ with

$$q \Vdash \check{U} \in \dot{\mathcal{V}} \quad \text{and} \quad q \Vdash \check{U} \subseteq \dot{W}.$$

Without meaningful loss of generality, taking subsets if necessary, we may assume $\dot{W}$ is a basic open set. Since

$$p \Vdash \text{“}\dot{W} \text{ is a basic open set”},$$

there is some $p_0 \leq p$ which decides $\dot{W}$ by Property (vii) of the forcing relation. By Property (ii), $p_0 \subseteq W$. Since $\mathcal{U}$ is a base, there is some $U \in \mathcal{U}$ with $U \subseteq p_0 \cap W = p_0$. Let p_1 be a regular open set with $p_1 \subseteq U \subseteq p_0$. Then, using Property (vii) again, we have that

$$p_1 \Vdash \check{U} \in \dot{\mathcal{V}} \quad \text{and} \quad p_1 \Vdash \check{U} \subseteq \dot{W}$$

and so it is forced that $\dot{\mathcal{V}}$ is a countable neighbourhood base for $\dot{x}$.

We now show (ii) $\Leftrightarrow$ (iii).

To show (iii) $\Rightarrow$ (ii), note that $\{\text{int} F_i \mid i < \kappa \text{ and } \dot{x} \in \text{int} F_i\}$ is forced to be a neighbourhood base for $\dot{x}$, but that $\{(F_i, G_i) \mid i < \kappa \text{ and } \dot{x} \in \text{int} F_i\}$ is a free sequence of regular pairs. Hence, by 6.4.4, it is forced that $\{\text{int} F_i \mid i < \kappa \text{ and } \dot{x} \in \text{int} F_i\}$ is countable, as required.

To show (ii) $\Rightarrow$ (iii), we have that $\{(F_i, G_i) \mid i < \kappa \text{ and } \dot{x} \in \text{int} F_i\}$ is a free sequence of regular pairs and $\{\text{int} F_i \mid i < \kappa \text{ and } \dot{x} \in \text{int} F_i\}$ is forced to be a neighbourhood base for $\dot{x}$. But since $(F_i, G_i)_i$ is free, $\{\text{int} F_i \mid i < \kappa \text{ and } \dot{x} \in \text{int} F_i\}$ must be a neighbourhood base of minimal cardinality and hence $\{i \in \check{\kappa} \mid \check{G}_i \in \dot{G}\}$ is forced to be countable.

Finally, we show (iii) $\Rightarrow$ (i). We assume that it is forced that $\{i \in \check{\kappa} \mid \check{G}_i \in \dot{G}\}$ is countable and we want to construct a σ -disjoint base for K .

Let $(\dot{i}_n)_{n \in \omega}$ be a sequence such that it is forced that

$$\{\dot{i}_n\} = \{i \in \kappa \mid \dot{x} \in G_i\}.$$

For each n , let $\mathcal{U}_n$ be a maximal antichain such that the elements of $\mathcal{U}_n$ decide $\dot{i}_n$ and let $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$. $\mathcal{U}$ is σ -disjoint by construction and hence it merely remains for us to show it is a base for K .

Thus, take some open set $V \subseteq K$ and want to find an element $U \in \mathcal{U}$ with $U \subseteq V$. Let p be some nonempty regular open set with $p \subseteq V$. Then there exists some $\dot{n}$ such that $p \Vdash G_{\dot{i}_n} \subseteq V$. Let $U \in \mathcal{U}$ be such that U decides $\dot{n}$.

Then $U \subseteq G_{\dot{i}_n} \subseteq V$, as required. $\square$

In fact, one, and hence all the conclusions of Proposition 6.4.9 hold. In particular, any Rosenthal compactum K has a σ -base and thus every dense subset of K has a σ -base.

The following lemma of H. E. White provides the last step we need in proving the main theorem. It connects the results in Propositions 6.4.8 and 6.4.9 with the existence of a dense metrisable subspace.

Lemma 6.4.10. ([White, 1982], Corollary 2.2). *Let K be a first countable Hausdorff space and suppose K has a σ -disjoint base. Then K has a dense metrisable subspace.*

The main result now follows easily from our observations.

Theorem 6.4.11. *Every nonempty Rosenthal compactum contains a dense metrisable subspace.*

Proof. Let K be a Rosenthal compactum. By Theorem 6.4.5, K has a point with a countable neighbourhood base. We observed that this implies K has a dense set of points with a countable neighbourhood base and thus has a dense first countable subspace, say D . By Proposition 6.4.9 and following discussion, K has a σ -disjoint base, and so every dense subspace of K has a σ -disjoint base. Furthermore, K is Hausdorff by Proposition 6.3.15 and hence D is Hausdorff as well. Thus Lemma 6.4.10 applies, and D is a dense metrisable subspace, as required. $\square$

ACKNOWLEDGEMENTS

This has been a wonderful experience and a suitable capstone to my learning experience at St Andrews. I believe I owe some particular thanks you's to the following people:

Thank you to Professor Parnell for organising the honours projects. This was a great excuse to indulge my curiosity in set theory and get credit for it at the same time!

Thank you Dr Bleak for being an amazing supervisor. You gave me lots of freedom and independence (for lack of a better word) to explore whatever looked interesting to me and were always willing to offer advice on mathematics, chess, and the small subset of life that does not involve mathematics.

Thank you to Surabhi Desai for your patience, company, and supply of good food.

Thank you to Tim Bailey for proofreading these notes and asking lots of interesting questions. If I were as generous as Dr Fraser, you would have a lifetime supply of magic beans.

Thank you to the entire staff and studentry of the mathematics department for making my time at St Andrews so fun and full of learning.

Finally, thank you to my ground models for being so patient with me as I continued forcing generic objects down your throats.

REFERENCES

- [Baire, 1899] Baire, R. (1899). *Sur les fonction de variables réelles*. Bernardoni de C. Rebeschini & Company.
- [Baumgartner and Hajnal, 1973] Baumgartner, J. and Hajnal, A. (1973). A proof (involving Martin’s Axiom) of a partition relation. *Fundamenta Mathematicae*, 78:193 – 203.
- [Bruckner et al., 2008] Bruckner, A. M., Bruckner, J. B., and Thomson, B. S. (2008). *Real Analysis*. Prentice Hall, 2 edition.
- [Cantor, 1879] Cantor, G. (1879). Über unendliche, lineare Punktmannichfaltigkeiten. *Mathematische Annalen*.
- [Chang and Keisler, 2012] Chang, C. C. and Keisler, J. H. (2012). *Model Theory*. Dover Publications, 3 edition.
- [Cohen, 1963] Cohen, P. J. (1963). The independence of the Continuum Hypothesis. *National Academy of Sciences*, 50:1153 – 1158.
- [Easwaran, 2007] Easwaran, K. (2007). A cheerful introduction to forcing and the continuum hypothesis.
- [Enderton, 1977] Enderton, H. B. (1977). *Elements of Set Theory*. Academic Press.
- [Gödel, 1931] Gödel, K. (1931). On formally undecidable propositions of “Principia Mathematica”. *Monatshefte für Mathematik und Physik*.
- [Gödel, 1938] Gödel, K. (1938). The consistency of the axiom of choice and of the Generalized Continuum-Hypothesis. *Proceedings of the National Academy of Sciences of the United States of America*, pages 555–556.
- [Henkin, 1949] Henkin, L. (1949). The completeness of the first-order functional calculus. *The Journal of Symbolic Logic*, 14(3):159–166.
- [Jackson, NA] Jackson, S. (N/A). Introduction to forcing.
- [Jech, 2002] Jech, T. (2002). *Set Theory*. Springer, 3 edition.
- [Komjáth, 1994] Komjáth, P. (1994). A decomposition theorem for $\mathbb{R}^n$. *Proceedings of the American Mathematical Society*, 120:921 – 927.
- [Kubiś and Vejnar, 2012] Kubiś, W. and Vejnar, B. (2012). Covering an uncountable square by countably many continuous functions. *Proceedings of the American Mathematical Society*, 140:4359 – 4368.
- [Kunen, 2012] Kunen, K. (2012). Forcing and differentiable functions. Unpublished, but can be found at “http://www.math.wisc.edu/~kunen/papers_2010_when.html”.
- [Marker, 2002] Marker, D. (2002). *Model Theory: An Introduction*. Graduate Texts in Mathematics. Springer.
- [Moore, NA] Moore, J. (N/A). The method of forcing. <http://www.math.cornell.edu/~justin/Ftp/forcing.pdf>.
- [Munkres, 2000] Munkres, J. (2000). *Topology*. Prentice Hall, Inc, 2 edition.
- [Rosenthal, 1977] Rosenthal, H. P. (1977). Point-wise compact subsets of the first Baire class. *American Journal of Mathematics*, 99(2):362 – 378.
- [Rosenthal, 1978] Rosenthal, H. P. (1978). Some recent discoveries in the isomorphic theory of Banach spaces. *Bulletin of the American Mathematical Society*, 84(5).
- [Shelah, 1987] Shelah, S. (1987). Consistency of positive partition theorems for graphs and models. 1401:167 – 193.
- [Shelah, 1998] Shelah, S. (1998). *Proper and Improper Forcing*. Berlin: Springer-Verlag, 2 edition.
- [Solovay, 1967] Solovay, R. (1967). A nonconstructible δ_3^1 set of integers. *Transactions of the American Mathematical Society*, 127:50–75.
- [Todorćević, 1990] Todorćević, S. (1990). Free sequences. *Topology and its Applications*, 35:235 – 238.
- [Todorćević, 1999] Todorćević, S. (1999). Compact subsets of the first Baire Class. *Journal of the American Mathematical Society*, 12:1179–1212.
- [Čech and Pospíšil, 1938] Čech, E. and Pospíšil, B. (1938). Sur les espaces compacts. *Faculty of Science at Masaryk Brno*.
- [Viale, NA] Viale, M. (N/A). Forcing a weak form of schanuel’s conjecture. Unpublished preprint found at “http://www.logicatorino.altervista.org/matteo_viale/”.
- [von Neumann, 1923] von Neumann, J. (1923). Zur Einführung der transfiniten Zahlen. *Acta litterarum ac scientiarum Ragiae Universitatis Hungaricae Francisco-Josephinae, Sectio scientiarum mathematicarum*, pages 199–208.
- [White, 1982] White, H. E. (1982). Variants of Blumberg’s theorem. *Illinois Journal of Mathematics*, 26(3):359–373.
- [Zermelo, 1908] Zermelo, E. (1908). Untersuchungen ber die Grundlagen der Mengenlehre I. *Mathematische Annalen*, 65:261–281.