

UNIVERSITY OF WISCONSIN-MADISON

A Topic Course on Viscosity Solutions

First-order equations, fully nonlinear PDE,
and level-set mean curvature flow

LECTURES BY

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A Topic Course on Viscosity Solutions

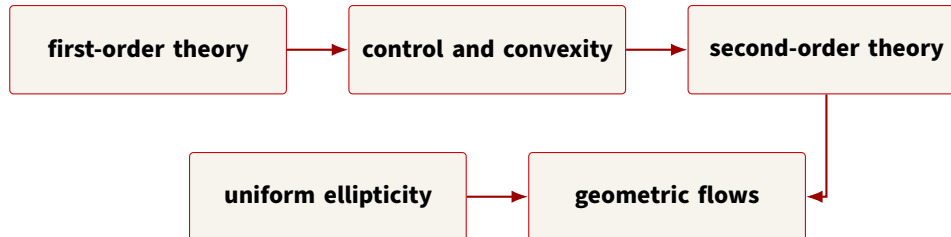
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These notes are intended for teaching and study. Corrections and comments are welcome.

About these notes

These notes reconstruct the Math 821 topic course on viscosity solutions from five sets of handwritten class notes, supplemented by the instructor's handwritten outline of the second-order theory. They are not a line-by-line transcription. The order and the principal proof ideas follow the lectures, while notation has been standardized, hypotheses have been stated explicitly, intermediate arguments have been supplied, and sign and normalization issues have been corrected.

The course has a natural five-part architecture:



The first two parts develop the basic viscosity method and its variational meaning. The third part introduces the theorem on sums, the central nonlinear replacement for differentiating two nonsmooth functions at a common maximum. The fourth part explains how uniform ellipticity leads to the Pucci extremal operators and the Krylov–Safonov regularity mechanism. The last part derives the level-set mean curvature equation and uses the viscosity framework to continue the motion through singularities.

Guiding idea The unifying principle is simple. A nonlinear PDE should be tested only at points where a smooth function touches the candidate solution. At such a point the touching function carries precisely the first- and second-order information that remains meaningful under uniform limits, dynamic programming, singular perturbations, and geometric degeneration.

Editorial conventions

- The words *from the lectures* identify a motivation, example, or proof strategy emphasized in the handwritten notes.
- A theorem quoted from the literature is stated with complete hypotheses and an exact reference. Core comparison, stability, dynamic-programming, and barrier arguments are proved in the text.

- For a proper operator, the convention is

$$F(x, r, p, X) \leq F(x, s, p, Y) \quad \text{when } r \leq s \text{ and } Y \leq X.$$

Thus F is nondecreasing in the scalar variable and nonincreasing in the matrix variable.

- For mean curvature flow we use

$$u_t - |Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) = 0,$$

so that a sphere with outward-pointing gradient shrinks according to $R(t)^2 = R_0^2 - 2(n-1)t$.

- Matrix inequalities are in the quadratic-form sense. The space of real symmetric $n \times n$ matrices is denoted by $\mathbb{S}^n$.

Acknowledgment of the source notes

The original class notes were written by Hieu Nguyen during Math 821. The additional handwritten pages supplied by the instructor focus on proper degenerate elliptic operators, semicontinuous envelopes, the theorem on sums, sup-convolution, Jensen's lemma, and the second-order comparison argument. They serve as an important cross-check for Part III of this book.

A handwritten lecture record is necessarily compressed. In particular, quantifiers in the theorem on sums, signs in the Pucci operators, and the definition of the level-set operator at $Du = 0$ cannot be safely inferred from a mnemonic formula alone. Those points are written out in full below.

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Roadmap and standing notation

The five questions

Q1. What is a solution?

How should one interpret a first- or second-order nonlinear PDE when the natural solution is not differentiable?

Q2. Why is the definition useful?

Which features of classical solutions survive: comparison, uniqueness, stability, and existence?

Q3. Where do solutions come from?

How do value functions, variational formulas, vanishing viscosity, and Perron's method produce viscosity solutions?

Q4. What does ellipticity buy?

How do the extremal Pucci operators encode every uniformly elliptic equation, and how does this lead to interior regularity?

Q5. Can geometry pass through singularities?

How does the level-set PDE continue mean curvature flow after a classical hypersurface pinches, merges, or changes topology?

Notation

Symbol	Meaning
$\mathbb{S}^n$	real symmetric $n \times n$ matrices, ordered by quadratic forms
$\text{USC}(\Omega), \text{LSC}(\Omega)$	upper and lower semicontinuous functions
u^*, u_*	upper and lower semicontinuous envelopes
$J^{2,+}u(x), J^{2,-}u(x)$	second-order superjet and subjet
$\bar{J}^{2,+}u(x), \bar{J}^{2,-}u(x)$	closures of the semijets
$\mathcal{P}^{2,+}u(x, t)$	parabolic superjet
$\mathcal{P}_{\lambda, \Lambda}^+, \mathcal{P}_{\lambda, \Lambda}^-$	Pucci maximal and minimal operators in the convention of chapter 13
C	a positive constant whose value may change from line to line
$B_r(x)$	Euclidean ball of radius r centered at x
$\text{osc}_E u$	$\sup_E u - \inf_E u$

Unless stated otherwise, domains are open, Hamiltonians are continuous, and all viscosity inequalities are understood at interior touching points. On unbounded domains, comparison statements include the growth assumptions needed to keep the doubled-variable maximum from escaping to infinity.

From the lectures The handwritten course moves repeatedly between a formal smooth calculation and its viscosity replacement. We retain that rhythm. Each major nonlinear argument begins with the smooth heuristic, identifies the step that fails for nonsmooth functions, and then introduces the precise viscosity tool that repairs it.

Part I

First-order viscosity solutions

1

Why generalized solutions are unavoidable

Source correspondence: Math 821, Part 1; introductory examples and the first homework set.

A first-order Hamilton–Jacobi equation has the form

$$u_t + H(x, Du) = 0 \quad \text{or} \quad H(x, Du) = f(x). \quad (1.1)$$

Even when the data are smooth, a classical solution may cease to exist. The obstruction is not a defect of the equation. It is often the geometry one wishes to describe: characteristics intersect, optimal trajectories switch, and distance functions develop corners.

1.1 The elementary eikonal example

Consider

$$|u'(x)| = 1 \quad \text{in } (-1, 1), \quad u(-1) = u(1) = 0. \quad (1.2)$$

There is no C^1 solution. Indeed, a classical solution would satisfy $u'(x) \in \{-1, 1\}$ for every x . Since u' is continuous and $(-1, 1)$ is connected, u' would be constant. Then $u(1) - u(-1) = \pm 2$, contradicting the boundary values.

The natural candidate is $u(x) = 1 - |x|$. It satisfies the equation wherever it is differentiable and has the correct boundary values. The only issue is the corner at $x = 0$.

Guiding idea At a corner one should not choose an artificial derivative. Instead, one asks what slopes can occur for smooth functions touching the graph from above or below. The PDE is imposed on those touching functions.

Suppose $\phi \in C^1((-1, 1))$ and $u - \phi$ has a local maximum at 0 . Then

$$\phi(h) - \phi(0) \geq u(h) - u(0) = -|h|.$$

For $h > 0$ this gives $[\phi(h) - \phi(0)]/h \geq -1$, and for $h < 0$ it gives $[\phi(h) - \phi(0)]/h \leq 1$. Passing to the limit, $-1 \leq \phi'(0) \leq 1$, so $|\phi'(0)| - 1 \leq 0$. This is the correct subsolution inequality.

There is no C^1 function touching u from below at the strict cusp maximum 0 . If such a ϕ existed, then

$$\phi(h) - \phi(0) \leq -|h|,$$

and the right and left difference quotients would force simultaneously $\phi'(0) \leq -1$ and $\phi'(0) \geq 1$. Thus the supersolution condition at 0 is vacuous.

Example 1.1 (The wrong corner). The function $v(x) = |x| - 1$ also satisfies $|v'| = 1$ away from o and has the same boundary values. It is not a viscosity solution. The constant function $\phi \equiv -1$ touches v from below at o , but $|\phi'(o)| - 1 = -1 < 0$, violating the supersolution inequality.

1.2 Characteristics and the formation of corners

For the Cauchy problem

$$u_t + H(Du) = 0, \quad u(\cdot, 0) = g, \quad (1.3)$$

assume temporarily that H and u are smooth. Writing $p = Du$, the Hamiltonian system is

$$\dot{x} = D_p H(p), \quad \dot{p} = 0$$

when H is independent of x . Thus the slope is transported with velocity $D_p H(p)$. Different slopes generally travel at different speeds. If two characteristic families meet, the classical gradient becomes multivalued and the solution develops a corner.

For $H(p) = |p|^2/2$, the characteristics have velocity p . A region with a larger initial slope moves faster and can overtake a region with a smaller slope. The viscosity solution continues uniquely after this crossing; the characteristic description does not.

1.3 Three requirements for a satisfactory notion

A useful generalized solution should satisfy the following.

1. **Consistency.** Every classical solution is a generalized solution, and a smooth generalized solution satisfies the PDE pointwise.
2. **Stability.** If u^ε solves an approximating equation and $u^\varepsilon \rightarrow u$ locally uniformly, then u solves the limiting equation.
3. **Comparison.** Subsolutions lie below supersolutions when they are ordered on the parabolic or elliptic boundary.

Distributional weak solutions are excellent for divergence-form equations. Hamilton–Jacobi equations are usually not in divergence form, and their nonlinear dependence on Du is not compatible with weak convergence of gradients. The touching-test definition is built instead around uniform convergence, exactly the compactness supplied by control formulas and vanishing viscosity.

1.4 Formal maximum-principle calculation

Let $F(x, r, p, X)$ be a second-order operator and suppose $u \in C^2(\Omega)$ solves $F(x, u, Du, D^2u) = 0$. If $u - \phi$ has a local maximum at x_o , then

$$u(x_o) = \phi(x_o), \quad Du(x_o) = D\phi(x_o), \quad D^2u(x_o) \leq D^2\phi(x_o).$$

If F is nonincreasing in its matrix argument, then

$$0 = F(x_o, u(x_o), Du(x_o), D^2u(x_o)) \geq F(x_o, u(x_o), D\phi(x_o), D^2\phi(x_o)).$$

This is the model for the viscosity subsolution definition.

The sign of degenerate ellipticity depends on how the PDE is written. In these notes $F = 0$ is degenerate elliptic when $X \leq Y$ implies $F(x, r, p, X) \geq F(x, r, p, Y)$. Thus F is nonincreasing in X . For example, $F(X) = -\operatorname{tr} X$ is degenerate elliptic.

Chapter summary

The viscosity method does not smooth a corner and does not assign it a derivative. It encodes the PDE through every smooth function that can touch the solution. Upward and downward corners are treated differently, exactly as required by comparison.

2

Touching tests, semicontinuity, and first-order jets

Source correspondence: Math 821, Part 1; viscosity sub- and supersolutions and semicontinuous envelopes.

2.1 Semicontinuous functions

Let $\Omega \subset \mathbb{R}^n$ be open. A function $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous if

$$\limsup_{y \rightarrow x} u(y) \leq u(x).$$

Lower semicontinuity is defined by reversing the inequality. For a locally bounded u , define

$$u^*(x) = \lim_{r \downarrow 0} \sup_{B_r(x) \cap \Omega} u,$$

$$u_*(x) = \lim_{r \downarrow 0} \inf_{B_r(x) \cap \Omega} u.$$

Lemma 2.1 (Envelope properties). *The function u^* is the least upper semicontinuous majorant of u , and u_* is the greatest lower semicontinuous minorant. Moreover,*

$$u_* \leq u \leq u^*, \quad (u^*)^* = u^*, \quad (u_*)_* = u_*.$$

The function u is continuous exactly when $u^* = u_* = u$.

Proof. The pointwise inequalities are immediate. To prove upper semicontinuity, take $x_k \rightarrow x$ and $\eta > 0$. Choose $r > 0$ with $\sup_{B_{2r}(x)} u \leq u^*(x) + \eta$. For large k , $B_r(x_k) \subset B_{2r}(x)$, so $u^*(x_k) \leq u^*(x) + \eta$. Hence $\limsup_{y \rightarrow x} u(y) \leq u^*(x)$. If v is upper semicontinuous and $v \geq u$, then $u^*(x) \leq \limsup_{y \rightarrow x} v(y) \leq v(x)$, proving minimality. Apply this argument to $-u$ for u_* . The remaining statements follow. ■

2.2 Stationary first-order equations

Consider

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega, \tag{2.1}$$

where F is continuous.

Definition 2.2 (Viscosity solution). An upper semicontinuous function u is a viscosity subsolution of (2.1) if every $\phi \in C^1(\Omega)$ for which $u - \phi$ has a local maximum at x_o satisfies

$$F(x_o, u(x_o), D\phi(x_o)) \leq 0.$$

A lower semicontinuous function v is a viscosity supersolution if every $\phi \in C^1(\Omega)$ for which $v - \phi$ has a local minimum at x_o satisfies

$$F(x_o, v(x_o), D\phi(x_o)) \geq 0.$$

A continuous function is a viscosity solution if it is both.

Proposition 2.3 (Strict touching is equivalent). *It is equivalent to test only strict local maxima for subsolutions and strict local minima for supersolutions.*

Proof. If $u - \phi$ has a possibly non-strict maximum at x_o , use $\phi_\delta(x) = \phi(x) + \delta|x - x_o|^2$. Then $u - \phi_\delta$ has a strict maximum at x_o and $D\phi_\delta(x_o) = D\phi(x_o)$. For supersolutions subtract the quadratic term. ■

Proposition 2.4 (C^1 and C^2 tests). *For a first-order equation, the definition is unchanged if one uses only C^2 test functions.*

Proof. Suppose $u - \phi$ has a strict local maximum at x_o with $\phi \in C^1$. There is a smooth nonnegative function $\omega(r) = o(r)$ such that

$$|\phi(x) - \phi(x_o) - D\phi(x_o) \cdot (x - x_o)| \leq \omega(|x - x_o|)$$

near x_o . A smooth radial majorant of 2ω produces

$$\psi(x) = \phi(x_o) + D\phi(x_o) \cdot (x - x_o) + 2\omega(|x - x_o|)$$

with $u - \psi$ having a local maximum at x_o and $D\psi(x_o) = D\phi(x_o)$. A local smoothing away from x_o makes ψ C^2 without changing its first derivative there. The minimum case is analogous. ■

2.3 First-order semijets

Define

$$D^+u(x) = \{p : u(y) \leq u(x) + p \cdot (y - x) + o(|y - x|)\},$$

$$D^-u(x) = \{p : u(y) \geq u(x) + p \cdot (y - x) + o(|y - x|)\}.$$

Proposition 2.5 (Tests and semijets). *A vector p belongs to $D^+u(x)$ exactly when there is $\phi \in C^1$ touching u from above at x with $D\phi(x) = p$. The analogous statement holds for $D^-u(x)$ and lower touching.*

Proof. A touching function gives the expansion by Taylor's theorem. Conversely, from $p \in D^+u(x)$ choose a modulus $\rho(r) \downarrow 0$ with $u(y) \leq u(x) + p \cdot (y - x) + |y - x|\rho(|y - x|)$. Choose a smooth increasing majorant $\tilde{\rho}$ and set $\eta(r) = \int_0^{2r} \tilde{\rho}(s) ds$. Then $\eta(r) \geq r\rho(r)$ and $\eta'(0) = 0$, so $\phi(y) = u(x) + p \cdot (y - x) + \eta(|y - x|)$ touches from above and has gradient p at x . ■

Thus the subsolution condition is equivalent to

$$F(x, u(x), p) \leq 0 \quad (p \in D^+u(x)).$$

If u is differentiable at x and either $D^+u(x)$ or $D^-u(x)$ is nonempty, every element equals $Du(x)$.

2.4 Evolution equations

For

$$u_t + F(x, t, u, Du) = 0 \quad \text{in } \Omega \times (0, T), \quad (2.2)$$

a subsolution satisfies

$$\phi_t(x_0, t_0) + F(x_0, t_0, u(x_0, t_0), D\phi(x_0, t_0)) \leq 0$$

whenever $u - \phi$ has a local maximum at (x_0, t_0) . The supersolution condition is the reverse inequality at local minima. Initial data are imposed as

$$u^*(x, 0) \leq g(x), \quad v_*(x, 0) \geq g(x).$$

Proposition 2.6 (Classical consistency). *Every classical solution is a viscosity solution. Conversely, a C^1 viscosity solution solves the equation pointwise.*

Proof. At a contact point between two C^1 functions their gradients agree. Conversely, take the test function $\phi = u$; it touches both from above and below at every point. ■

Chapter summary

Viscosity inequalities are local and one-sided. The definition can be stated with touching functions or semijets, with strict or non-strict contacts, and with C^1 or C^2 tests for first-order equations.

3

Stability and the doubling-variable comparison method

Source correspondence: Math 821, Part 1; stability of maximum points, comparison, and contraction for Cauchy problems.

3.1 Stability of touching points

Lemma 3.1 (Stability of strict maxima). *Let $u_k \in \text{USC}(\Omega)$ converge locally uniformly to u , and suppose $u - \phi$ has a strict local maximum at x_o . Then there are $x_k \rightarrow x_o$ such that $u_k - \phi$ has a local maximum at x_k .*

Proof. Choose $r > 0$ with $\overline{B}_r(x_o) \subset \Omega$ and $(u - \phi)(x_o) > \max_{\partial B_r(x_o)} (u - \phi)$. Let x_k maximize $u_k - \phi$ on $\overline{B}_r(x_o)$. Uniform convergence implies x_k is interior for large k . Any limit point $\bar{x}$ satisfies $(u - \phi)(\bar{x}) \geq (u - \phi)(x_o)$, so strictness gives $\bar{x} = x_o$. ■

Theorem 3.2 (Stability). *If $F_k \rightarrow F$ locally uniformly, u_k is a subsolution of $F_k(x, u_k, Du_k) = 0$, and $u_k \rightarrow u$ locally uniformly, then u is a subsolution of $F = 0$. The supersolution statement is analogous.*

Proof. Make a touching maximum strict by adding $|x - x_o|^4$. Use [theorem 3.1](#) to obtain $x_k \rightarrow x_o$ and apply the subsolution inequality at x_k . Passing to the limit gives the desired inequality. ■

For a locally bounded family u^ε , define the half-relaxed limits

$$\overline{u}(x) = \limsup_{\substack{\varepsilon \rightarrow 0 \\ y \rightarrow x}} u^\varepsilon(y), \quad \underline{u}(x) = \liminf_{\substack{\varepsilon \rightarrow 0 \\ y \rightarrow x}} u^\varepsilon(y).$$

The same proof, choosing points that realize the limsup or liminf, gives:

Theorem 3.3 (Half-relaxed-limit stability). *If $F^\varepsilon \rightarrow F$ locally uniformly and u^ε is locally uniformly bounded, then the upper half-relaxed limit of subsolutions is a subsolution of $F = 0$, and the lower half-relaxed limit of supersolutions is a supersolution.*

3.2 The quadratic penalization lemma

Lemma 3.4. *Let $u \in \text{USC}(\overline{\Omega})$ and $v \in \text{LSC}(\overline{\Omega})$ be bounded. If $(x_\varepsilon, y_\varepsilon)$ maximizes*

$$u(x) - v(y) - \frac{|x - y|^2}{2\varepsilon}$$

on $\overline{\Omega}^2$, then

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \rightarrow 0.$$

Every limit point is $(\hat{x}, \hat{x})$, where $\hat{x}$ maximizes $u - v$.

Proof. Comparing with (z, z) gives

$$u(x_\varepsilon) - v(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \geq u(z) - v(z).$$

Boundedness first gives $|x_\varepsilon - y_\varepsilon|^2 = O(\varepsilon)$. At a convergent subsequence, semicontinuity and the preceding inequality identify the diagonal limit as a maximizer of $u - v$. The maximal values converge to $\max(u - v)$; since $u(x_\varepsilon) - v(y_\varepsilon)$ has upper limit at most the same number, the nonnegative penalty converges to zero. ■

3.3 Stationary comparison

Consider

$$u + H(x, Du) = f(x) \quad \text{in } \Omega, \tag{3.1}$$

where Ω is bounded and

$$|H(x, p) - H(y, p)| \leq C(1 + |p|)|x - y|. \tag{3.2}$$

Theorem 3.5 (Comparison). *If $u \in \text{USC}(\overline{\Omega})$ is a subsolution, $v \in \text{LSC}(\overline{\Omega})$ is a supersolution, and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.*

Proof. Suppose $M = \max_{\overline{\Omega}}(u - v) > 0$. Let $(x_\varepsilon, y_\varepsilon)$ maximize the doubled expression. By the boundary condition and **theorem 3.4**, both points are interior for small ε . Put $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$. The functions

$$x \mapsto v(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon}, \quad y \mapsto u(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon}$$

touch u from above and v from below, respectively, and both have gradient p_ε . Therefore

$$\begin{aligned} u(x_\varepsilon) + H(x_\varepsilon, p_\varepsilon) &\leq f(x_\varepsilon), \\ v(y_\varepsilon) + H(y_\varepsilon, p_\varepsilon) &\geq f(y_\varepsilon). \end{aligned}$$

Subtracting and using (3.2),

$$u(x_\varepsilon) - v(y_\varepsilon) \leq f(x_\varepsilon) - f(y_\varepsilon) + C|x_\varepsilon - y_\varepsilon| + C\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}.$$

The right side tends to zero, while the left side tends to $M > 0$, a contradiction. ■

From the lectures The identity

$$D_x \frac{|x - y|^2}{2\varepsilon} = -D_y \frac{|x - y|^2}{2\varepsilon}$$

supplies the same momentum to the subsolution and the supersolution. For second-order equations the separate Hessians contain terms of size $1/\varepsilon$ that do not cancel. The theorem on sums repairs precisely this defect.

3.4 Evolutionary comparison

Assume H satisfies the corresponding uniform continuity hypotheses. For bounded uniformly continuous sub- and supersolutions of

$$u_t + H(x, Du) = 0$$

with ordered initial data, maximize

$$u(x, t) - v(y, t) - \frac{|x - y|^2}{2\varepsilon} - \delta(|x|^2 + |y|^2) - \eta t.$$

The maximum has positive time if the ordering fails. The two time derivatives differ by η , while the spatial gradients converge to the same vector. The viscosity inequalities yield $\eta \leq o(1)$, a contradiction.

Theorem 3.6 (Cauchy comparison and contraction). *If $u(\cdot, 0) \leq v(\cdot, 0)$, then $u \leq v$ for later times. Consequently, solutions satisfy*

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_\infty \leq \|g_1 - g_2\|_\infty.$$

Proof. The comparison argument is the one just described. For contraction, set $K = \|g_1 - g_2\|_\infty$. Translation invariance in u makes $u_2 + K$ a supersolution and $u_2 - K$ a subsolution. Apply comparison twice. ■

Chapter summary

Stability follows by moving touching points to nearby approximating solutions. Comparison follows by doubling the spatial variable and penalizing the distance between the two copies. The quadratic penalty makes the two touching gradients equal and its energy vanishes in the limit.

4

Perron's method and approximation

Source correspondence: Math 821, Part 1; Perron's construction, vanishing viscosity, and elementary a priori estimates.

Consider the Dirichlet problem

$$\begin{cases} F(x, u, Du) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

where F is continuous, proper in u , and comparison holds. Assume there are a subbarrier $\underline{u}$ and a superbarrier $\bar{u}$ with $\underline{u} \leq \bar{u}$ in $\bar{\Omega}$ and $\underline{u} = \bar{u} = g$ on $\partial\Omega$.

Let $\mathcal{S}$ be the set of subsolutions w satisfying $\underline{u} \leq w \leq \bar{u}$, and set $U(x) = \sup_{w \in \mathcal{S}} w(x)$.

Lemma 4.1 (Supremum of subsolutions). *The upper envelope U^* is a subsolution.*

Proof. If $U^* - \phi$ has a strict maximum at x_o , choose $w_j \in \mathcal{S}$ and $x_j \rightarrow x_o$ with $w_j(x_j) \rightarrow U^*(x_o)$. Let y_j maximize $w_j - \phi$ in a small ball. Strictness gives $y_j \rightarrow x_o$ and $w_j(y_j) \rightarrow U^*(x_o)$. Apply the subsolution inequality to w_j at y_j and pass to the limit. ■

Lemma 4.2 (Perron bump). *The lower envelope U_* is a supersolution.*

Proof. Assume, to the contrary, that a smooth function ϕ touches U_* from below at x_o and

$$F(x_o, U_*(x_o), D\phi(x_o)) < 0.$$

Make the contact strict by replacing ϕ with $\phi - |x - x_o|^4$; this does not change its value or gradient at x_o . Continuity of F then gives $r > 0$ and $\eta > 0$ such that $\bar{B}_r(x_o) \subset \Omega$,

$$F(x, \phi(x) + \eta, D\phi(x)) < 0 \quad \text{in } B_r(x_o),$$

and

$$\phi + \eta < U_* \quad \text{in a neighborhood of } \partial B_r(x_o).$$

The strict contact also lets us choose points $x_j \rightarrow x_o$ with $U(x_j) \rightarrow U_*(x_o)$ and, for all large j ,

$$\phi(x_j) + \eta > U(x_j).$$

Define

$$\tilde{U}(x) = \begin{cases} \max\{U(x), \phi(x) + \eta\}, & x \in B_r(x_o), \\ U(x), & x \notin B_r(x_o). \end{cases}$$

Near $\partial B_r(x_o)$ the second branch is inactive, so no gluing problem occurs. Moreover,

$$\tilde{U}^* = \max\{U^*, \phi + \eta\} \quad \text{in } B_r(x_o).$$

The maximum of two upper semicontinuous subsolutions of a proper equation is again a subsolution; hence $\tilde{U}^*$ is a subsolution in Ω . It agrees with U^* near $\partial\Omega$, so $\tilde{U}^* \leq \bar{u}$ on $\partial\Omega$. Comparison with the superbarrier $\bar{u}$ gives $\tilde{U}^* \leq \bar{u}$ in Ω , while $\tilde{U}^* \geq U \geq \underline{u}$. Thus $\tilde{U}^* \in \mathcal{S}$. But at x_j ,

$$\tilde{U}^*(x_j) \geq \phi(x_j) + \eta > U(x_j),$$

contradicting the definition of U as the pointwise supremum of $\mathcal{S}$. ■

Theorem 4.3 (Perron's method). *The function U is continuous and is the unique solution of (4.1).*

Proof. The preceding lemmas show that U^* is a subsolution and U_* is a supersolution. The barriers give the boundary data. Comparison yields $U^* \leq U_*$, while the reverse inequality is automatic. Thus $U^* = U_* = U$. ■

4.1 Vanishing viscosity

Approximate

$$u + H(x, Du) = f(x)$$

by

$$u^\varepsilon + H(x, Du^\varepsilon) = f(x) + \varepsilon \Delta u^\varepsilon.$$

If the regularized solutions satisfy local bounds

$$\|u^\varepsilon\|_\infty + \|Du^\varepsilon\|_\infty \leq C,$$

Arzelà–Ascoli gives a locally uniformly convergent subsequence. Stability identifies the limit as a viscosity solution. Comparison for the limiting equation removes the subsequence.

Proof map

The argument has four independent pieces: solve the elliptic approximation, obtain bounds independent of ε , use compactness, and identify the limit by stability and comparison. Only the a priori estimate is equation-specific.

On a compact manifold without boundary, a maximum-point calculation gives the elementary bound

$$\|u^\varepsilon\|_\infty \leq \|f\|_\infty + \|H(\cdot, 0)\|_\infty.$$

Gradient bounds require structure and are developed in [chapter 12](#).

4.2 Vanishing discount

The equation

$$\delta v^\delta + H(x, Dv^\delta) = o$$

has a positive zero-order term and hence a strong comparison principle. Constants show $|\delta v^\delta| \leq \|H(\cdot, o)\|_\infty$. If coercivity gives an oscillation bound, then $v^\delta - v^\delta(x_o)$ is compact as $\delta \downarrow o$. This is the starting point of the cell-problem theory in [chapter 7](#).

Chapter summary

Perron's method converts comparison into existence by taking the largest admissible subsolution. Vanishing viscosity and vanishing discount instead produce solutions by approximation. Semicontinuous envelopes and stability close both arguments.



Control, convexity, and variational formulas

5

Dynamic programming and Hamilton–Jacobi–Bellman equations

Source correspondence: Math 821, Part 2; infinite- and finite-horizon control, dynamic programming, and the viscosity verification of the Bellman equation.

Let A be a compact metric space. For a measurable control $\alpha : [0, \infty) \rightarrow A$, let $X^{x, \alpha}$ solve

$$\dot{X}(s) = b(X(s), \alpha(s)), \quad X(0) = x. \quad (5.1)$$

Assume, uniformly in $a \in A$,

$$\begin{aligned} |b(x, a) - b(y, a)| &\leq L_b |x - y|, & |b(x, a)| &\leq C_b(1 + |x|), \\ |f(x, a) - f(y, a)| &\leq L_f |x - y|, & |f(x, a)| &\leq C_f. \end{aligned}$$

For $\lambda > 0$, define

$$J(x; \alpha) = \int_0^\infty e^{-\lambda s} f(X^{x, \alpha}(s), \alpha(s)) ds, \quad u(x) = \inf_\alpha J(x; \alpha). \quad (5.2)$$

Then $|u| \leq C_f/\lambda$.

Lemma 5.1 (Dependence on initial data). *For the same control,*

$$|X^{x, \alpha}(s) - X^{y, \alpha}(s)| \leq e^{L_b s} |x - y|.$$

If $\lambda > L_b$, then

$$|u(x) - u(y)| \leq \frac{L_f}{\lambda - L_b} |x - y|.$$

Proof. The trajectory estimate is Gronwall's inequality. Comparing the two costs generated by the same control gives

$$|J(x; \alpha) - J(y; \alpha)| \leq L_f |x - y| \int_0^\infty e^{-(\lambda - L_b)s} ds.$$

Take the infimum in both directions. ■

5.1 Dynamic programming

Theorem 5.2 (Dynamic programming principle). *For every $t > 0$,*

$$u(x) = \inf_{\alpha} \left\{ \int_0^t e^{-\lambda s} f(X(s), \alpha(s)) ds + e^{-\lambda t} u(X(t)) \right\}. \quad (5.3)$$

Proof. Split any cost at time t . The continuation cost is at least $u(X(t))$, giving one inequality. Conversely, concatenate an arbitrary control on $[0, t]$ with an η -optimal control for the value at $X(t)$, then let $\eta \downarrow 0$. ■

Define

$$H(x, p) = \sup_{a \in A} \{-b(x, a) \cdot p - f(x, a)\}. \quad (5.4)$$

The Hamiltonian is convex in p .

Theorem 5.3 (The Bellman equation). *The value function is the unique bounded uniformly continuous viscosity solution of*

$$\lambda u + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n. \quad (5.5)$$

Proof. Subsolution. Let $u - \phi$ have a local maximum at x_0 and fix a constant control a on $[0, h]$. The DPP and $u \leq \phi$ near x_0 give

$$0 \leq \int_0^h e^{-\lambda s} f(X(s), a) ds + e^{-\lambda h} \phi(X(h)) - \phi(x_0).$$

Divide by h and let $h \downarrow 0$:

$$\lambda u(x_0) - b(x_0, a) \cdot D\phi(x_0) - f(x_0, a) \leq 0.$$

Since a is arbitrary, $\lambda u + H(x, D\phi) \leq 0$.

Supersolution. Let $u - \phi$ have a local minimum at x_0 . Choose a control on $[0, h]$ that is h^2 -optimal in the DPP. Then

$$u(x_0) + h^2 \geq \int_0^h e^{-\lambda s} f(X_h(s), \alpha_h(s)) ds + e^{-\lambda h} \phi(X_h(h)).$$

The occupation measures of α_h on $[0, h]$ have a weakly convergent subsequence to a probability measure μ on A . Dividing by h and passing to the limit gives

$$\lambda u(x_0) - \int_A (b(x_0, a) \cdot D\phi(x_0) + f(x_0, a)) d\mu(a) \geq 0.$$

The integral is at least the infimum over a , hence $\lambda u + H(x, D\phi) \geq 0$.

Uniqueness follows from comparison. ■

Remark 5.4. The supersolution proof naturally produces a relaxed control. If the velocity–cost set

$$C(x) = \{(b(x, a), f(x, a)) : a \in A\}$$

is convex, the averaged pair is represented by an ordinary control value. Otherwise relaxed controls provide the cleanest formulation and yield the same Hamiltonian.

5.2 Finite horizon

For $0 \leq t \leq T$, define

$$U(x, t) = \inf_{\alpha} \left\{ \int_t^T f(X(s), \alpha(s)) ds + g(X(T)) \right\},$$

where $X(t) = x$. For $t < s \leq T$,

$$U(x, t) = \inf_{\alpha} \left\{ \int_t^s f(X(r), \alpha(r)) dr + U(X(s), s) \right\}.$$

The same touching argument gives

$$-U_t + H(x, DU) = 0, \quad U(\cdot, T) = g.$$

Setting $u(x, \tau) = U(x, T - \tau)$ yields the forward equation $u_\tau + H(x, Du) = 0$ with initial data g .

Guiding idea The DPP is an exact finite-increment identity. The viscosity proof extracts its first-order consequence at a touching point. No optimal control and no value function needs to be differentiable.

Chapter summary

A value function has three equivalent descriptions: a global minimization problem, a local recursive identity, and a Hamilton–Jacobi–Bellman equation. Dynamic programming and viscosity testing are the bridges between them.

6

Convex duality and the Hopf–Lax formula

Source correspondence: Math 821, Part 2; Legendre transforms, Lagrangian action, and the Hopf–Lax representation.

Assume $H : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and superlinear. Its Legendre transform is

$$L(q) = \sup_{p \in \mathbb{R}^n} \{p \cdot q - H(p)\}. \quad (6.1)$$

Then L is convex, finite, and superlinear.

Theorem 6.1 (Fenchel–Young inequality). *For all p, q ,*

$$p \cdot q \leq H(p) + L(q).$$

If H is differentiable, equality holds exactly when $q = DH(p)$.

Proof. The inequality is the definition of L . Equality means that p maximizes $r \mapsto r \cdot q - H(r)$, equivalently $q \in \partial H(p)$. ■

Example 6.2 (Power Hamiltonians). For $H(p) = |p|^m/m, m > 1$, one obtains $L(q) = |q|^{m'}/m'$, where $m' = m/(m-1)$. The maximizer is parallel to q and has magnitude $|q|^{1/(m-1)}$.

6.1 Least action

For an absolutely continuous curve $\gamma : [0, t] \rightarrow \mathbb{R}^n$, define $A_t(\gamma) = \int_0^t L(\dot{\gamma}) \, ds$. If $\gamma(0) = y$ and $\gamma(t) = x$, Jensen's inequality gives

$$A_t(\gamma) \geq tL\left(\frac{x-y}{t}\right).$$

Equality is attained by the straight line. Thus the least action from y to x in time t is

$$A_t(y, x) = tL\left(\frac{x-y}{t}\right). \quad (6.2)$$

6.2 The Hopf–Lax semigroup

For $g \in \text{BUC}(\mathbb{R}^n)$, set

$$(S_t g)(x) = \inf_y \left\{ g(y) + tL\left(\frac{x-y}{t}\right) \right\}. \quad (6.3)$$

Superlinearity ensures that minimizers exist.

Proposition 6.3 (Semigroup property). $S_t(S_s g) = S_{t+s}g$.

Proof. For fixed x, y , convexity gives

$$\begin{aligned} sL\left(\frac{z-y}{s}\right) + tL\left(\frac{x-z}{t}\right) \\ \geq (s+t)L\left(\frac{x-y}{s+t}\right). \end{aligned}$$

Equality holds for $z = (ty + sx)/(s+t)$. Take the infimum first in z and then in y . ■

Proposition 6.4 (Initial trace). $S_t g \rightarrow g$ uniformly as $t \downarrow 0$.

Proof. Choosing $y = x$ gives the upper bound. For the lower bound, uniform continuity controls $g(y) - g(x)$ when $|x - y|$ is small, while superlinearity makes the action larger than $2\|g\|_\infty$ when $|x - y|$ is not small and t is small. ■

Theorem 6.5 (Hopf–Lax formula). The function $u(x, t) = S_t g(x)$ is the unique bounded uniformly continuous viscosity solution of

$$u_t + H(Du) = 0, \quad u(\cdot, 0) = g.$$

Proof. The semigroup property gives, for $0 < h < t$,

$$u(x, t) = \inf_z \left\{ u(z, t-h) + hL\left(\frac{x-z}{h}\right) \right\}.$$

If $u - \phi$ has a local maximum at (x_0, t_0) , choose $z = x_0 - hq$ for arbitrary q . Then

$$0 \leq \phi(x_0 - hq, t_0 - h) - \phi(x_0, t_0) + hL(q).$$

After division by h , $\phi_t + D\phi \cdot q - L(q) \leq 0$. Taking the supremum in q gives $\phi_t + H(D\phi) \leq 0$.

At a local minimum, choose an h^2 -minimizer z_h and set $q_h = (x_0 - z_h)/h$. Superlinearity keeps q_h bounded. Along a subsequence $q_h \rightarrow q$, and the same calculation with reversed inequalities gives $\phi_t + D\phi \cdot q - L(q) \geq 0$. Since $D\phi \cdot q - L(q) \leq H(D\phi)$, the supersolution inequality follows. Comparison gives uniqueness. ■

If H is strictly convex and smooth, an interior minimizer satisfies $Dg(y) = DL((x-y)/t)$, or $x = y + tDH(Dg(y))$. This is the characteristic map. Several competing minimizers correspond exactly to a corner in the viscosity solution.

Chapter summary

Convex duality turns a Hamiltonian into a running cost. Jensen's inequality identifies straight lines as least-action curves. Dynamic programming for that action yields the

Hopf–Lax formula and the viscosity solution of the Cauchy problem.

7

Maximal subsolutions, intrinsic metrics, and discounted cell problems

Source correspondence: Math 821, Part 2; maximal subsolutions, metric problems, and periodic discount equations.

Assume $H(x, p)$ is continuous, convex in p , and coercive uniformly in x . Fix μ such that $K_\mu(x) = \{p : H(x, p) \leq \mu\}$ is nonempty for every x . Define

$$\mathcal{A}_\mu = \{w \in \text{Lip}(\mathbb{R}^n) : H(x, Dw) \leq \mu \text{ a.e.}\}$$

and

$$m_\mu(y, x) = \sup_{w \in \mathcal{A}_\mu} \{w(x) - w(y)\}. \quad (7.1)$$

Proposition 7.1 (Metric properties). *There is C_μ such that $|m_\mu(y, x)| \leq C_\mu|x - y|$. Moreover,*

$$m_\mu(x, x) = 0, \quad m_\mu(y, x) \leq m_\mu(y, z) + m_\mu(z, x).$$

If $H(x, p) = H(x, -p)$, then m_μ is symmetric.

Proof. Coercivity bounds $|Dw|$ uniformly for $w \in \mathcal{A}_\mu$, giving the Lipschitz estimate. The triangle inequality follows by splitting $w(x) - w(y)$ at z and taking the supremum. Evenness sends w to $-w$ and gives symmetry. ■

Theorem 7.2 (The metric equation). *For fixed y , $m(x) = m_\mu(y, x)$ is a viscosity subsolution of $H(x, Dm) = \mu$ in $\mathbb{R}^n$ and a viscosity solution in $\mathbb{R}^n \setminus \{y\}$.*

Proof. For convex Hamiltonians, a Lipschitz a.e. subsolution is a viscosity subsolution: mollify, use convexity in p and continuity in x , and pass to the limit. The supremum-stability argument then shows that m is a subsolution.

If m failed to be a supersolution at $x_o \neq y$, a smooth function ϕ would touch from below there with $H(x_o, D\phi(x_o)) < \mu$. After shrinking a ball disjoint from y and adding a small constant, ϕ becomes a strict subsolution that lies below m on the boundary but above m at an interior point. Replacing m by $\max\{m, \phi\}$ in the ball gives a larger admissible subsolution normalized by $m(y) = 0$, contradicting maximality. ■

Let

$$\sigma_\mu(x, q) = \sup\{p \cdot q : H(x, p) \leq \mu\}.$$

Then

$$m_\mu(y, x) = \inf_{\gamma(o)=y, \gamma(1)=x} \int_0^1 \sigma_\mu(\gamma, \dot{\gamma}) ds. \quad (7.2)$$

One inequality follows by integrating $Dw \cdot \dot{\gamma} \leq \sigma_\mu(\gamma, \dot{\gamma})$; the reverse follows because the action is itself the maximal subsolution.

7.1 Periodic discounted cell problems

Let $H(y, p)$ be periodic in y . For fixed p and $\delta > 0$, consider

$$\delta v^\delta + H(y, p + Dv^\delta) = 0 \quad \text{in } \mathbb{T}^n. \quad (7.3)$$

Theorem 7.3 (Uniform estimates). *Equation (7.3) has a unique periodic solution, and*

$$\|\delta v^\delta\|_\infty + \text{osc}_{\mathbb{T}^n} v^\delta + \|Dv^\delta\|_\infty \leq C_p$$

independently of $\delta \in (0, 1]$.

Proof. Comparison follows from the positive zero-order term; Perron's method gives existence. Constants give $|\delta v^\delta| \leq M_p := \max_y |H(y, p)|$. Coercivity supplies R_p such that

$$H(y, q) > M_p + 1 \quad \text{for every } y \in \mathbb{T}^n \text{ and } |q| \geq R_p.$$

Set $K = R_p + |p| + 1$, and let $d_{\mathbb{T}^n}$ denote the geodesic distance on the flat torus. We claim that

$$v^\delta(x) - v^\delta(y) \leq K d_{\mathbb{T}^n}(x, y).$$

If not, the upper semicontinuous function

$$v^\delta(x) - v^\delta(y) - K d_{\mathbb{T}^n}(x, y)$$

has a positive maximum at a pair $x \neq y$. Choose $k \in \mathbb{Z}^n$ such that, for suitable lifts of x and y ,

$$d_{\mathbb{T}^n}(x, y) = |x - y + k|.$$

Since $d_{\mathbb{T}^n}(z, y) \leq |z - y + k|$ near x , the smooth function

$$z \mapsto v^\delta(y) + K|z - y + k| + C$$

touches v^δ from above at x for the appropriate constant C . Its gradient there is $q = K(x - y + k)/|x - y + k|$, and hence $|p + q| \geq K - |p| > R_p$. The subsolution inequality gives

$$H(x, p + q) \leq -\delta v^\delta(x) \leq M_p,$$

contradicting the definition of R_p . Interchanging x and y proves the opposite inequality and therefore the Lipschitz bound. The oscillation estimate follows from the finite diameter of the torus. ■

Normalize $w^\delta = v^\delta - v^\delta(o)$. Compactness gives, along a subsequence, $w^\delta \rightarrow w$ and $-\delta v^\delta(o) \rightarrow c$. Stability yields

$$H(y, p + Dw) = c.$$

The constant is unique. Indeed, suppose periodic v, w solve

$$H(y, p + Dv) = c, \quad H(y, p + Dw) = d.$$

For $\eta > 0$, maximize

$$v(x) - w(y) - \frac{|x - y|^2}{2\eta}$$

on the torus, using compatible lifts near a maximizing pair. Put $q_\eta = (x_\eta - y_\eta)/\eta$. The penalization lemma gives $|x_\eta - y_\eta|^2/\eta \rightarrow 0$. The subsolution inequality for v and the supersolution inequality for w imply

$$H(x_\eta, p + q_\eta) \leq c, \quad H(y_\eta, p + q_\eta) \geq d.$$

Coercivity and the first inequality keep q_η bounded. Uniform continuity of H on bounded momentum sets therefore yields $d - c \leq o(1)$, so $d \leq c$. Interchanging v and w gives $c \leq d$. Thus $c = d$. Denote the common constant by $\bar{H}(p)$. Hence

$$-\delta v^\delta(y) \rightarrow \bar{H}(p) \quad \text{uniformly in } y.$$

Guiding idea The discount fixes the additive constant missing from the cell problem. The functions v^δ may diverge like $1/\delta$, but their oscillations remain bounded, and the scalar quantity $-\delta v^\delta$ converges to the unique ergodic constant.

Chapter summary

Maximal subsolutions produce intrinsic metrics and solve the Hamilton–Jacobi equation away from a vertex. Discounted periodic problems produce correctors and a unique additive eigenvalue. Both are nonlinear forms of Perron’s construction.

Part 

Fully nonlinear elliptic and parabolic equations

8

Proper degenerate elliptic equations and second-order jets

Source correspondence: Math 821, Part 3, and the instructor's handwritten supplement; properness, semicontinuous envelopes, viscosity definitions, and second-order semijets.

We now consider

$$F(x, u, Du, D^2u) = o \quad \text{in } \Omega \subset \mathbb{R}^n, \quad (8.1)$$

where

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \longrightarrow \mathbb{R}.$$

The correct monotonicity is dictated by the maximum principle.

8.1 Degenerate ellipticity and properness

Definition 8.1 (Degenerate ellipticity). The operator F is *degenerate elliptic* if

$$X \leq Y \implies F(x, r, p, X) \geq F(x, r, p, Y). \quad (8.2)$$

It is *proper* if, in addition,

$$r \leq s, \quad Y \leq X \implies F(x, r, p, X) \leq F(x, s, p, Y). \quad (8.3)$$

Thus a proper operator is nondecreasing in r and nonincreasing in X .

The matrix order is the quadratic-form order: $X \leq Y$ means $\xi \cdot X\xi \leq \xi \cdot Y\xi$ for every $\xi \in \mathbb{R}^n$.

Example 8.2 (Linear equations). Let $A(x) \in \mathbb{S}^n$, $A(x) \geq o$, and $c(x) \geq o$. Then

$$F(x, r, p, X) = -\operatorname{tr}(A(x)X) + b(x) \cdot p + c(x)r - f(x)$$

is proper. Indeed, if $Y - X \geq o$, then

$$\operatorname{tr}(A(Y - X)) = \operatorname{tr}(A^{1/2}(Y - X)A^{1/2}) \geq o,$$

so $-\operatorname{tr}(AX) \geq -\operatorname{tr}(AY)$.

Example 8.3 (Bellman and Isaacs operators). Suppose $F_{\alpha\beta}$ is proper for every α, β and the resulting quantities are finite. Then

$$F = \sup_{\alpha} \inf_{\beta} F_{\alpha\beta} \quad \text{and} \quad G = \inf_{\beta} \sup_{\alpha} F_{\alpha\beta}$$

are proper. Monotonicity is preserved by infima and suprema. These operators arise from stochastic control and differential games.

Proposition 8.4 (Elementary closure properties). *If F and G are proper and $a, b \geq 0$, then $aF + bG$ is proper. If $\{F_{\alpha}\}$ is a family of proper operators and the pointwise supremum or infimum is finite, that extremum is proper.*

Proof. Every assertion follows by applying the defining inequality before taking the nonnegative linear combination, supremum, or infimum. ■

8.2 The viscosity definition

Definition 8.5 (Second-order viscosity solutions). Let F be continuous and proper.

- An upper semicontinuous function u is a viscosity subsolution of (8.1) if, whenever $\phi \in C^2(\Omega)$ and $u - \phi$ has a local maximum at x_0 ,

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

- A lower semicontinuous function v is a viscosity supersolution if, whenever $\phi \in C^2(\Omega)$ and $v - \phi$ has a local minimum at x_0 ,

$$F(x_0, v(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

- A continuous function is a viscosity solution if it is both a subsolution and a supersolution.

Proposition 8.6 (Consistency). *Every C^2 classical solution of a proper equation is a viscosity solution. Conversely, every C^2 viscosity solution satisfies the equation pointwise.*

Proof. Suppose u is classical and $u - \phi$ has a local maximum at x_0 . Then

$$Du(x_0) = D\phi(x_0), \quad D^2u(x_0) \leq D^2\phi(x_0).$$

Degenerate ellipticity yields

$$F(x_0, u, Du, D^2\phi) \leq F(x_0, u, Du, D^2u) = 0.$$

This is the subsolution inequality. The minimum case is identical. Conversely, use the test function $\phi = u$ at each point. ■

Strict and non-strict touching are equivalent, as before: add or subtract $\delta|x - x_0|^4$, which has zero gradient and zero Hessian at x_0 .

8.3 Discontinuous operators

Some natural equations, including level-set mean curvature flow at $p = 0$, have an operator that is discontinuous in (p, X) . For a locally bounded function F on its state space, let

$$F^*(z) = \limsup_{z' \rightarrow z} F(z'), \quad F_*(z) = \liminf_{z' \rightarrow z} F(z').$$

Definition 8.7 (Discontinuous viscosity convention). For a possibly discontinuous F , an upper semicontinuous u is a subsolution if every upper test satisfies

$$F_*(x, u, D\phi, D^2\phi) \leq 0.$$

A lower semicontinuous v is a supersolution if every lower test satisfies

$$F^*(x, v, D\phi, D^2\phi) \geq 0.$$

Remark 8.8 (Why the envelopes are crossed). Suppose continuous $F_k \rightarrow F$ only pointwise and $u_k \rightarrow u$. At nearby touching points $z_k \rightarrow z$, the limit of the inequalities $F_k(z_k) \leq 0$ can control only the lower relaxed limit of the operators. Thus the stable subsolution inequality uses F_* . The supersolution inequality uses F^* for the same reason with inequalities reversed.

8.4 Second-order semijets

Definition 8.9 (Semijets). For $u : \Omega \rightarrow \mathbb{R}$, define

$$J^{2,+}u(x) = \left\{ (p, X) \in \mathbb{R}^n \times \mathbb{S}^n : u(x+h) \leq u(x) + p \cdot h + \frac{1}{2}h \cdot Xh + o(|h|^2) \right\},$$

$$J^{2,-}u(x) = \left\{ (p, X) \in \mathbb{R}^n \times \mathbb{S}^n : u(x+h) \geq u(x) + p \cdot h + \frac{1}{2}h \cdot Xh + o(|h|^2) \right\}.$$

The matrix component is not unique. If $(p, X) \in J^{2,+}u(x)$ and $Y \geq X$, then $(p, Y) \in J^{2,+}u(x)$. For subjects the direction is reversed.

Proposition 8.10 (Tests and semijets). *The following are equivalent:*

1. $(p, X) \in J^{2,+}u(x)$;
2. there is $\phi \in C^2$ such that $u - \phi$ has a local maximum at x and

$$D\phi(x) = p, \quad D^2\phi(x) = X.$$

The analogous statement holds for $J^{2,-}$ and local minima.

Proof. A touching test gives the jet inequality by Taylor's formula. Conversely, write

$$u(x+h) \leq u(x) + p \cdot h + \frac{1}{2}h \cdot Xh + |h|^2\rho(|h|),$$

where $\rho(r) \rightarrow 0$. Choose a smooth nonnegative majorant $\tilde{\rho}$ with $\tilde{\rho}(r) \rightarrow 0$ and define

$$\phi(x+h) = u(x) + p \cdot h + \frac{1}{2}h \cdot Xh + |h|^2\tilde{\rho}(|h|).$$

The final term can be chosen C^2 with value, gradient, and Hessian zero at $h = 0$. Then $u \leq \phi$ near x , with equality at x , and the prescribed derivatives hold. ■

Thus the subsolution condition is equivalent to

$$F(x, u(x), p, X) \leq 0 \quad \text{for every } (p, X) \in J^{2,+}u(x). \quad (8.4)$$

8.5 Closed semijets

A limiting argument usually produces jets at nearby points rather than at the point itself.

Definition 8.11 (Closure of the semijets). We say $(p, X) \in \bar{J}^{2,+}u(x)$ if there are $x_j \rightarrow x, u(x_j) \rightarrow u(x)$, and $(p_j, X_j) \in J^{2,+}u(x_j)$ such that $(p_j, X_j) \rightarrow (p, X)$. Define $\bar{J}^{2,-}$ analogously.

If F is continuous, the viscosity inequality automatically extends from $J^{2,+}$ to $\bar{J}^{2,+}$. This is why the theorem on sums may return closed jets.

Example 8.12 (A corner). For $u(x) = |x|$ on $\mathbb{R}$, one has

$$J^{2,-}u(o) = \{(p, X) : |p| < 1, X \in \mathbb{R}\} \cup \{(-1, X) : X \leq o\} \cup \{(1, X) : X \leq o\},$$

while $J^{2,+}u(o) = \emptyset$. The exact boundary cases follow by comparing the quadratic remainder on the two sides. This asymmetry is the second-order version of the eikonal corner from [chapter 1](#).

8.6 Stability in the second-order setting

The proofs of [theorems 3.2](#) and [3.3](#) apply verbatim with C^2 test functions. At a nearby maximizing point x_k , the same fixed test supplies $D\phi(x_k)$ and $D^2\phi(x_k)$, both of which converge. We record the result.

Theorem 8.13 (Second-order stability). Let $F_k \rightarrow F$ locally uniformly, and suppose u_k is a viscosity subsolution of

$$F_k(x, u_k, Du_k, D^2u_k) = o.$$

If $u_k \rightarrow u$ locally uniformly, then u is a subsolution of $F = o$. More generally, for a locally bounded family, the upper half-relaxed limit is a subsolution. The corresponding lower-limit statement holds for supersolutions.

Chapter summary

Properness is the nonlinear maximum-principle structure: increase the value of the function or decrease its Hessian and the operator cannot decrease. Second-order semijets encode every smooth quadratic touching profile, and their closures are stable under approximation.

9

Sup-convolution, semiconvexity, and Jensen's lemma

Source correspondence: Math 821, Part 3, and pages 5–8 of the instructor's handwritten supplement; sup-convolution, the magic property, Alexandrov differentiability, and Jensen's perturbation lemma.

The theorem on sums is proved by regularizing upper semicontinuous functions from above. The correct regularization is the sup-convolution, because it preserves maxima and produces semiconvex functions without averaging the equation.

9.1 Sup- and inf-convolution

Let u be bounded and upper semicontinuous on an open set Ω . For $\varepsilon > 0$, define

$$u^\varepsilon(x) = \sup_{y \in \Omega} \left\{ u(y) - \frac{|x - y|^2}{2\varepsilon} \right\}. \quad (9.1)$$

For a bounded lower semicontinuous v , define

$$v_\varepsilon(x) = \inf_{y \in \Omega} \left\{ v(y) + \frac{|x - y|^2}{2\varepsilon} \right\}. \quad (9.2)$$

Proposition 9.1 (Elementary properties). *Let $M = \text{osc}_\Omega u$.*

1. $u \leq u^\varepsilon$, and u^ε decreases pointwise to u as $\varepsilon \downarrow 0$.
2. If $\hat{y}$ attains the supremum in (9.1), then

$$|x - \hat{y}| \leq \sqrt{2M\varepsilon}.$$

3. The function $x \mapsto u^\varepsilon(x) + |x|^2/(2\varepsilon)$ is convex. Thus u^ε is semiconvex with constant $1/\varepsilon$.
4. On every compact $K \Subset \Omega$, $u^\varepsilon \rightarrow u$ uniformly if u is uniformly continuous on a neighborhood of K .

The dual statements hold for v_ε .

Proof. Choosing $y = x$ gives $u^\varepsilon(x) \geq u(x)$. If $0 < \varepsilon_1 < \varepsilon_2$, the penalty is larger for ε_1 , hence $u^{\varepsilon_1} \leq u^{\varepsilon_2}$. If a maximizer $\hat{y}$ satisfies

$$u(\hat{y}) - \frac{|x - \hat{y}|^2}{2\varepsilon} \geq u(x),$$

then the oscillation bound gives the second assertion.

Expanding the square,

$$u^\varepsilon(x) + \frac{|x|^2}{2\varepsilon} = \sup_y \left\{ u(y) - \frac{|y|^2}{2\varepsilon} + \frac{x \cdot y}{\varepsilon} \right\},$$

a supremum of affine functions of x , hence a convex function. Finally, the distance estimate confines maximizing points to $o(1)$ of x ; uniform continuity then gives

$$u(x) \leq u^\varepsilon(x) \leq u(x) + \omega(\sqrt{2M\varepsilon}),$$

where ω is a modulus of continuity. ■

Definition 9.2 (Semiconvexity). A function w is semiconvex with constant C if $w(x) + C|x|^2/2$ is convex. Equivalently, in the viscosity sense,

$$D^2w \geq -CI.$$

A function is semiconcave if its negative is semiconvex.

Semiconvex functions are locally Lipschitz and twice differentiable almost everywhere. The second assertion is Alexandrov's theorem.

Theorem 9.3 (Alexandrov). A locally semiconvex function on $\mathbb{R}^m$ has a second-order Taylor expansion at almost every point: for almost every x there are p and $X \in \mathbb{S}^m$ such that

$$w(x+h) = w(x) + p \cdot h + \frac{1}{2}h \cdot Xh + o(|h|^2).$$

We use Alexandrov's theorem as a classical finite-dimensional result. A proof may be found in [4, Appendix A] or [7, Appendix].

9.2 The magic property

The crucial feature of sup-convolution is that a jet of u^ε transfers to a jet of the original function at a related maximizing point.

Theorem 9.4 (Magic property). Let $(p, X) \in J^{2,+}u^\varepsilon(x)$. Then the supremum in (9.1) is attained at the unique point

$$\hat{x} = x + \varepsilon p, \tag{9.3}$$

and

$$u^\varepsilon(x) = u(\hat{x}) - \frac{\varepsilon}{2}|p|^2, \quad (p, X) \in J^{2,+}u(\hat{x}). \tag{9.4}$$

Proof. Let ϕ touch u^ε from above at x , with $D\phi(x) = p$ and $D^2\phi(x) = X$. Choose a maximizing point $\hat{x}$ in (9.1). For z near x ,

$$u^\varepsilon(z) \geq u(\hat{x}) - \frac{|z - \hat{x}|^2}{2\varepsilon}.$$

Since $u^\varepsilon - \phi$ has a local maximum at x , the function

$$z \mapsto u(\hat{x}) - \frac{|z - \hat{x}|^2}{2\varepsilon} - \phi(z)$$

has a local maximum at x . Differentiating gives

$$-\frac{x - \hat{x}}{\varepsilon} - D\phi(x) = o,$$

which is (9.3). Thus every maximizing point is the same, proving uniqueness and the value identity.

To transfer the jet, fix y near $\hat{x}$ and set $z = y - \varepsilon p$. Then z is near x and $z - y = -\varepsilon p$. By the definition of the sup-convolution,

$$u(y) - \frac{\varepsilon}{2}|p|^2 \leq u^\varepsilon(y - \varepsilon p).$$

The upper touching expansion at x gives

$$\begin{aligned} u^\varepsilon(y - \varepsilon p) &\leq u^\varepsilon(x) + p \cdot [(y - \varepsilon p) - x] \\ &\quad + \frac{1}{2}[(y - \varepsilon p) - x] \cdot X[(y - \varepsilon p) - x] + o(|y - \hat{x}|^2). \end{aligned}$$

Because $x = \hat{x} - \varepsilon p$, the displacement in brackets is $y - \hat{x}$. Using the value identity and cancelling $\varepsilon|p|^2/2$ yields

$$u(y) \leq u(\hat{x}) + p \cdot (y - \hat{x}) + \frac{1}{2}(y - \hat{x}) \cdot X(y - \hat{x}) + o(|y - \hat{x}|^2).$$

Hence $(p, X) \in J^{2,+}u(\hat{x})$. ■

Remark 9.5. The location formula has a useful geometric meaning. At a differentiability point of u^ε ,

$$\hat{x} = x + \varepsilon Du^\varepsilon(x).$$

The regularized graph at x is supported by a paraboloid whose vertex is tied to the original graph at $\hat{x}$.

9.3 Preservation of viscosity inequalities

Let

$$\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \sqrt{2M\varepsilon}\}.$$

For $x \in \Omega_\varepsilon$, every maximizing point in the sup-convolution remains inside Ω .

Proposition 9.6 (Approximate preservation). *Assume u is a bounded subsolution of*

$$F(x, u, Du, D^2u) = o$$

and F is continuous. Then u^ε is a subsolution in Ω_ε of

$$F_\varepsilon(x, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon) \leq o, \tag{9.5}$$

where

$$F_\varepsilon(x, r, p, X) = \inf_{|y-x| \leq \sqrt{2M\varepsilon}} F\left(y, r + \frac{|x-y|^2}{2\varepsilon}, p, X\right).$$

In particular, $F_\varepsilon \rightarrow F$ locally uniformly when F is locally uniformly continuous.

Proof. Let $(p, X) \in J^{2,+}u^\varepsilon(x)$. By [theorem 9.4](#), with $\hat{x} = x + \varepsilon p$,

$$(p, X) \in J^{2,+}u(\hat{x}), \quad u(\hat{x}) = u^\varepsilon(x) + \frac{|x - \hat{x}|^2}{2\varepsilon}.$$

The subsolution inequality for u gives

$$F\left(\hat{x}, u^\varepsilon(x) + \frac{|x - \hat{x}|^2}{2\varepsilon}, p, X\right) \leq 0.$$

The maximizing-point estimate puts $\hat{x}$ in the infimum defining F_ε . ■

9.4 Jensen's perturbation lemma

At a strict maximum of a semiconvex function, small linear perturbations have maxima at a set of positive measure. This allows one to choose a maximizing point where Alexandrov differentiability holds.

Lemma 9.7 (Jensen). *Let w be semiconvex in $B_r(o)$ and suppose w has a strict maximum at o relative to $\overline{B_r}$. For every sufficiently small $\delta > 0$, the set*

$$E_\delta = \{x \in B_r : \text{for some } p \in B_\delta, x \text{ maximizes } w(z) + p \cdot z \text{ on } \overline{B_r}\}$$

has positive Lebesgue measure. More quantitatively, if $D^2w \geq -CI$ in the semiconvex sense, then

$$|E_\delta| \geq c_m \left(\frac{\delta}{C+1} \right)^m$$

for small δ , with $c_m > 0$ depending only on the dimension.

Proof. Strict maximality gives $\eta > 0$ such that

$$w(o) \geq \max_{\partial B_r} w + 2\eta.$$

If $|p| \leq \delta$ and $\delta r < \eta$, every maximizer of $w(z) + p \cdot z$ lies in B_r . At a differentiability point x that is such a maximizer,

$$Dw(x) + p = 0.$$

Let E_δ be the collection of all these interior maximizers. Then

$$B_\delta \subset -Dw(E_\delta)$$

up to a null set: each p has a maximizer, and semiconvex functions are differentiable almost everywhere; a standard arbitrarily small quadratic perturbation removes exceptional nonunique maxima.

Set $g(x) = w(x) + C|x|^2/2$, which is convex. At points of twice differentiability,

$$0 \leq D^2g = D^2w + CI.$$

At a local maximum of $w + p \cdot x$, one also has $D^2w \leq 0$, so

$$0 \leq D^2g \leq CI.$$

Thus the Jacobian of the map $x \mapsto -Dw(x) = Cx - Dg(x)$ is bounded in absolute value by a dimensional multiple of $(C+1)^m$ on E_δ . The area formula therefore gives

$$|B_\delta| \leq C_m(C+1)^m |E_\delta|,$$

which is the stated estimate. ■

The measure statement, not merely the existence of one perturbed maximum, is essential. Alexandrov differentiability holds almost everywhere, so a positive-measure family of candidate points must contain points where the regularized functions have genuine Hessians.

9.5 How the ingredients fit together

At a maximum of $u(x) + v(y) - \phi(x, y)$, regularize u and v by sup-convolution. Their sum remains semiconvex. Jensen's lemma supplies nearby maxima at which both regularizations are twice differentiable. The ordinary second-derivative test then gives a block matrix inequality. Finally, the magic property transfers the resulting jets back to the original functions. This is the proof mechanism behind the theorem on sums.

Chapter summary

Sup-convolution is a nonlinear regularization by supporting paraboloids. It produces semiconvexity, preserves the viscosity inequality up to a controlled perturbation, and transfers jets back to the original function. Jensen's lemma lets us place a maximum at a point of twice differentiability.

10

The theorem on sums

Source correspondence: Math 821, Part 3, and pages 3–9 of the instructor’s handwritten supplement; the failed separate-variable calculation, the block Hessian heuristic, and the sup-convolution–Jensen proof mechanism.

For first-order comparison, a quadratic penalty gives the same gradient to a subsolution and a supersolution. For second-order comparison, testing the two variables separately produces Hessians of size $1/\varepsilon$ with the wrong sign. The theorem on sums retains the full block structure of the doubled-variable maximum and recovers the matrix ordering seen in the smooth calculation.

10.1 Why separate testing fails

Suppose one compares sub- and supersolutions of

$$u + H(Du) - \Delta u = f, \quad v + H(Dv) - \Delta v = f,$$

using

$$\Phi_\varepsilon(x, y) = u(x) - v(y) - \frac{|x - y|^2}{2\varepsilon}.$$

At a maximum $(x_\varepsilon, y_\varepsilon)$, testing u with $|x - y_\varepsilon|^2/(2\varepsilon)$ gives Hessian I/ε . Testing v with $-|x_\varepsilon - y|^2/(2\varepsilon)$ gives Hessian $-I/\varepsilon$. The resulting Laplacian terms contribute quantities of order n/ε rather than canceling.

If u, v were C^2 , one would differentiate the full function Φ_ε and obtain

$$\begin{pmatrix} D^2u(x_\varepsilon) & \circ \\ \circ & -D^2v(y_\varepsilon) \end{pmatrix} \leq \frac{1}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \quad (10.1)$$

Testing this inequality on (ξ, ξ) yields

$$\xi \cdot (D^2u - D^2v)\xi \leq 0,$$

so $D^2u \leq D^2v$. This is the information needed by degenerate ellipticity.

Guiding idea The theorem on sums is a nonsmooth block second-derivative test. It does not assign Hessians to u and v . It produces matrices belonging to their closed semijets and couples those matrices by the same block inequality that holds in the smooth case.

10.2 Statement of the theorem

Let $\Omega_i \subset \mathbb{R}^{n_i}$ be open, $u_i \in \text{USC}(\Omega_i)$, and

$$z = (x_1, \dots, x_k) \in \Omega_1 \times \dots \times \Omega_k.$$

Write $\text{diag}(X_1, \dots, X_k)$ for the block diagonal matrix.

Theorem 10.1 (Theorem on sums; Crandall–Ishii). *Let $\phi \in C^2(\Omega_1 \times \dots \times \Omega_k)$ and suppose*

$$\sum_{i=1}^k u_i(x_i) - \phi(x_1, \dots, x_k)$$

has a local maximum at $\hat{z} = (\hat{x}_1, \dots, \hat{x}_k)$. Put

$$A = D^2\phi(\hat{z}).$$

Then, for every $\mu > 0$, there exist matrices $X_i \in \mathbb{S}^{n_i}$ such that

$$(D_{x_i}\phi(\hat{z}), X_i) \in \bar{J}^{2,+} u_i(\hat{x}_i) \quad (1 \leq i \leq k) \quad (10.2)$$

and

$$-\left(\frac{1}{\mu} + \|A\|\right)I \leq \text{diag}(X_1, \dots, X_k) \leq A + \mu A^2. \quad (10.3)$$

This is the finite-dimensional maximum principle in [7, Theorem 3.2]. The exact constants are less important than three facts: the matrices are genuine limiting jets, they are bounded below, and their block diagonal is controlled above by the full Hessian of the coupling function.

10.3 Proof mechanism

We record the proof mechanism because it explains every component of (10.3). The technical finite-dimensional argument is standard; the matrix identities used in the passage from regularized Hessians to (10.3) are collected in chapter A.

Proof map

After localization and an arbitrarily small strict quadratic perturbation, the proof consists of the following steps.

1. Replace each u_i by its sup-convolution u_i^ε . The sum of the regularizations is semiconvex on the product space.
2. Use Jensen's lemma to perturb the maximum by a small linear function and choose nearby maximizing points at which all u_i^ε are twice differentiable.
3. At such a point, ordinary calculus gives

$$\text{diag}(D^2 u_1^\varepsilon, \dots, D^2 u_k^\varepsilon) \leq D^2 \phi + o(1).$$

Semiconvexity gives the lower bound $D^2 u_i^\varepsilon \geq -\varepsilon^{-1}I$.

4. The magic property transfers the gradients and Hessians to superjets of the original u_i at nearby points.
5. Choose the convolution scale as a function of μ and pass to a subsequence. The resolvent identity for the quadratic sup-convolution converts the upper bound into $A + \mu A^2$; the semiconvexity bound becomes the left side of (10.3).

For completeness, we spell out the limiting step. The regularization produces points $x_i^j \rightarrow \hat{x}_i$, values $u_i(x_i^j) \rightarrow u_i(\hat{x}_i)$, and jets $(p_i^j, X_i^j) \in J^{2,+}u_i(x_i^j)$ with

$$p_i^j \rightarrow D_{x_i} \phi(\hat{x}_i)$$

and a uniform matrix bound of the form

$$-C_\mu I \leq \text{diag}(X_1^j, \dots, X_k^j) \leq A_j + \mu A_j^2 + o(1), \quad A_j \rightarrow A.$$

Finite-dimensional compactness gives a convergent subsequence $X_i^j \rightarrow X_i$. By definition, the limits belong to the closed semijets, and passage to the limit gives (10.3). The preceding four steps are precisely the sup-convolution, magic-property, Alexandrov, and Jensen calculations of chapter 9. No differentiability of the original u_i is used.

Remark 10.2 (The resolvent form). An equivalent sharp form, convenient in the sup-convolution proof, is

$$-\frac{1}{\kappa} I \leq \text{diag}(X_1, \dots, X_k) \leq A(I - \kappa A)^{-1},$$

provided $0 < \kappa < \|A\|^{-1}$. Since A commutes with every polynomial in A ,

$$A(I - \kappa A)^{-1} = A + \kappa A^2(I - \kappa A)^{-1}.$$

Choosing κ sufficiently small in terms of μ yields (10.3). The standard form avoids carrying the invertibility restriction.

10.4 The quadratic doubling corollary

Take $k = 2$, $u_1 = u$, $u_2 = -v$, and

$$\phi(x, y) = \frac{\alpha}{2} |x - y|^2.$$

Then

$$D_x \phi = \alpha(x - y), \quad D_y \phi = -\alpha(x - y),$$

and

$$A = \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad A^2 = 2\alpha A, \quad \|A\| = 2\alpha. \quad (10.4)$$

Choose $\mu = 1/\alpha$. The theorem gives $X, Y \in \mathbb{S}^n$ with

$$(\alpha(x - y), X) \in \bar{J}^{2,+} u(x), \quad (10.5)$$

$$(\alpha(x - y), Y) \in \bar{J}^{2,-} v(y), \quad (10.6)$$

and

$$-3\alpha I_{2n} \leq \begin{pmatrix} X & \circ \\ \circ & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \quad (10.7)$$

Here the subjet statement for v follows from

$$(p, -Y) \in \bar{J}^{2,+}(-v) \iff (-p, Y) \in \bar{J}^{2,-}v,$$

with the sign of the y -gradient taken into account.

Corollary 10.3 (Ordering of the two Hessian jets). *Under (10.7),*

$$X \leq Y.$$

Proof. Test the upper matrix inequality on $(\xi, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. The right-hand side vanishes because

$$\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = \circ.$$

Therefore

$$\xi \cdot (X - Y)\xi \leq 0$$

for every ξ . ■

10.5 General coupling functions

The quadratic penalty is sufficient for many equations, but the full theorem allows

$$\phi(x, y) = \frac{\alpha}{2}|x - y|^2 + \rho(\psi(x) + \psi(y)),$$

where ψ confines the maximum or keeps it away from the boundary. It also permits anisotropic penalties, several interacting functions, and the parabolic theorem on sums. The price is that the matrix estimate must retain the complete Hessian $D^2\phi$ rather than only its diagonal blocks.

One cannot replace the block inequality by the two separate estimates $X \leq D_{xx}^2\phi$ and $-Y \leq D_{yy}^2\phi$. Those estimates lose the mixed derivatives $D_{xy}^2\phi$, and the mixed derivatives are exactly what force $X \leq Y$.

Chapter summary

The theorem on sums is the central second-order comparison tool. It assigns compatible closed jets to all functions participating in a maximum and preserves the block Hessian information of the coupling penalty. For the quadratic doubled-variable penalty, it yields the decisive ordering $X \leq Y$.

11

Second-order comparison, Perron's method, and parabolic equations

Source correspondence: Math 821, Part 3; comparison after the theorem on sums, Perron's method for proper equations, and the parabolic viscosity framework.

We now close the basic second-order theory: comparison gives uniqueness, Perron's method gives existence, and the same arguments extend to parabolic equations.

11.1 The structural continuity condition

Let $F : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$ be continuous and proper. We assume strict monotonicity in the scalar variable: there is $\gamma > 0$ such that

$$F(x, r, p, X) - F(x, s, p, X) \geq \gamma(r - s) \quad \text{whenever } r \geq s. \quad (11.1)$$

We also assume that there is a modulus ω , with $\omega(0) = 0$, such that whenever $\alpha > 0$ and

$$-3\alpha I_{2n} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad (11.2)$$

then

$$F(y, r, \alpha(x - y), Y) - F(x, r, \alpha(x - y), X) \leq \omega(\alpha|x - y|^2 + |x - y|). \quad (11.3)$$

This is the standard Crandall–Ishii–Lions structure condition.

Example 11.1 (Semilinear uniformly continuous coefficients). For

$$F(x, r, p, X) = -\operatorname{tr}(A(x)X) + H(x, p) + c(x)r - f(x),$$

assume $c \geq \gamma > 0$, $A = \sigma\sigma^T$, and σ, H, f, c have suitable uniform continuity. The matrix inequality (11.2), tested on vectors $(\sigma(x)e_j, \sigma(y)e_j)$ and summed in j , controls

$$-\operatorname{tr}(A(x)X) + \operatorname{tr}(A(y)Y)$$

by $C\alpha|\sigma(x) - \sigma(y)|^2$. If σ is Lipschitz, this is bounded by $C\alpha|x - y|^2$. The remaining terms are controlled by their moduli of continuity. Thus (11.3) holds.

11.2 Elliptic comparison

Theorem 11.2 (Comparison for proper second-order equations). Assume (11.1)–(11.3). Let $u \in \text{USC}(\overline{\Omega})$ be a bounded subsolution and $v \in \text{LSC}(\overline{\Omega})$ a bounded supersolution of

$$F(x, w, Dw, D^2w) = o \quad \text{in } \Omega.$$

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ on $\overline{\Omega}$.

Proof. Suppose

$$M := \max_{\overline{\Omega}}(u - v) > o.$$

For $\alpha > o$, let (x_α, y_α) maximize

$$u(x) - v(y) - \frac{\alpha}{2}|x - y|^2$$

on $\overline{\Omega} \times \overline{\Omega}$. The penalization lemma gives, along a subsequence,

$$x_\alpha, y_\alpha \rightarrow \hat{x} \in \Omega, \quad \alpha|x_\alpha - y_\alpha|^2 \rightarrow o, \quad u(x_\alpha) - v(y_\alpha) \rightarrow M. \quad (11.4)$$

The boundary condition ensures that the limiting maximum point is interior, so both points are interior for large α .

Apply [theorem 10.1](#) to $u(x) + [-v(y)]$ and the quadratic penalty. There exist X_α, Y_α satisfying (11.2) and

$$\begin{aligned} (p_\alpha, X_\alpha) &\in \overline{J}^{2,+} u(x_\alpha), \\ (p_\alpha, Y_\alpha) &\in \overline{J}^{2,-} v(y_\alpha), \end{aligned}$$

where $p_\alpha = \alpha(x_\alpha - y_\alpha)$. Therefore

$$F(x_\alpha, u(x_\alpha), p_\alpha, X_\alpha) \leq o, \quad (11.5)$$

$$F(y_\alpha, v(y_\alpha), p_\alpha, Y_\alpha) \geq o. \quad (11.6)$$

For large α , $u(x_\alpha) > v(y_\alpha)$. By strict properness,

$$\begin{aligned} \gamma(u(x_\alpha) - v(y_\alpha)) &\leq F(x_\alpha, u(x_\alpha), p_\alpha, X_\alpha) - F(x_\alpha, v(y_\alpha), p_\alpha, X_\alpha) \\ &\leq F(y_\alpha, v(y_\alpha), p_\alpha, Y_\alpha) - F(x_\alpha, v(y_\alpha), p_\alpha, X_\alpha) \\ &\leq \omega(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|). \end{aligned}$$

The second line uses (11.5)–(11.6); the third uses the structural condition. Letting $\alpha \rightarrow \infty$ and using (11.4) gives $\gamma M \leq o$, a contradiction. ■

Remark 11.3 (Removing strict properness). If F is merely nondecreasing in r , comparison may still hold when the equation has another source of strictness. One can replace u by a strict subsolution, add a small barrier, or use a transformation such as $\tilde{u} = e^{-\lambda t} u$ in a parabolic problem. The exact device depends on the equation.

11.3 Perron's method in the second-order setting

Consider

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Assume comparison and the existence of a continuous subbarrier $\underline{u}$ and superbarrier $\bar{u}$ equal to g on the boundary.

Theorem 11.4 (Second-order Perron theorem). *The supremum of all subsolutions between $\underline{u}$ and $\bar{u}$, after semicontinuous regularization, is the unique continuous viscosity solution.*

Proof. The proof is identical in structure to [theorem 4.3](#). The upper envelope is a subsolution because a strict upper contact can be realized on one member of an approximating sequence. If the lower envelope were not a supersolution, a C^2 lower test would satisfy a strict subsolution inequality in a small ball. Raising the envelope by the maximum with this strict local subsolution produces a larger admissible subsolution. Properness ensures that maxima of subsolutions remain subsolutions. This contradicts maximality. Comparison then identifies the upper and lower envelopes and gives continuity and uniqueness. ■

11.4 Parabolic semijets

For a function $u(x, t)$, the parabolic superjet is

$$\mathcal{P}^{2,+}u(x, t) = \left\{ (a, p, X) : \begin{aligned} &u(x + h, t + s) \leq u(x, t) + as + p \cdot h \\ &\quad + \frac{1}{2}h \cdot Xh + o(|s| + |h|^2) \end{aligned} \right\}.$$

The parabolic subjet is defined by reversing the inequality. Notice that time is first order and space is second order.

For

$$u_t + F(x, t, u, Du, D^2u) = 0, \tag{11.7}$$

a subsolution satisfies

$$a + F(x, t, u, p, X) \leq 0 \quad \text{for every } (a, p, X) \in \mathcal{P}^{2,+}u(x, t).$$

11.5 The parabolic theorem on sums

The parabolic version of [theorem 10.1](#) gives spatial matrices with the same block bounds and time components whose sum equals the time derivative of the coupling function. We record the two-function corollary used in comparison.

Theorem 11.5 (Parabolic doubled-variable lemma). *Suppose*

$$u(x, t) - v(y, t) - \frac{\alpha}{2}|x - y|^2 - \eta t$$

has a local maximum at $(\hat{x}, \hat{y}, \hat{t})$ with $\hat{t} > 0$. Then there are numbers a, b and matrices X, Y such that

$$\begin{aligned} (a, \alpha(\hat{x} - \hat{y}), X) &\in \overline{\mathcal{P}}^{2,+}u(\hat{x}, \hat{t}), \\ (b, \alpha(\hat{x} - \hat{y}), Y) &\in \overline{\mathcal{P}}^{2,-}v(\hat{y}, \hat{t}), \end{aligned}$$

with

$$a - b = \eta$$

and the block matrix estimate (11.2).

The proof is obtained by adding a separate time variable for each function, penalizing their difference, applying the elliptic theorem on sums in space-time, and then sending the time-penalty parameter to infinity. See [7, Theorem 8.3].

11.6 Parabolic comparison

Theorem 11.6 (Parabolic comparison). *Assume F is proper and satisfies the spatial structure condition uniformly in t . Let u be a bounded upper semicontinuous subsolution and v a bounded lower semicontinuous supersolution of (11.7) in $\Omega \times (0, T)$. If*

$$u \leq v \quad \text{on } (\partial\Omega \times [0, T]) \cup (\overline{\Omega} \times \{0\}),$$

then $u \leq v$ in $\overline{\Omega} \times [0, T]$.

Proof. Assume the contrary. For small $\eta > 0$, the function $u - v - \eta t$ has a positive maximum at some positive time. Double the spatial variables and maximize

$$u(x, t) - v(y, t) - \frac{\alpha}{2}|x - y|^2 - \eta t.$$

For large α , the maximizer is interior and the spatial penalization vanishes as in (11.4). Apply theorem 11.5. The viscosity inequalities give

$$\begin{aligned} a + F(x, t, u, p, X) &\leq 0, \\ b + F(y, t, v, p, Y) &\geq 0. \end{aligned}$$

Since the doubled maximum is positive, $u(x, t) > v(y, t)$ for large α . Properness in the scalar variable gives $F(x, t, u, p, X) \geq F(x, t, v, p, X)$. Subtracting the viscosity inequalities and using $a - b = \eta$ therefore yields

$$\eta \leq F(y, t, v, p, Y) - F(x, t, u, p, X) \leq \omega(\alpha|x - y|^2 + |x - y|).$$

The right-hand side tends to zero while $\eta > 0$, a contradiction. ■

Corollary 11.7 (Parabolic contraction). *For equations invariant under addition of constants, two bounded solutions satisfy*

$$\|u(\cdot, t) - v(\cdot, t)\|_\infty \leq \|u(\cdot, 0) - v(\cdot, 0)\|_\infty.$$

Chapter summary

The theorem on sums restores the block Hessian ordering needed by ellipticity. Strict properness converts a positive gap $u - v$ into a fixed positive operator gap, while the structural continuity condition makes the doubled-variable error vanish. Perron's method and parabolic comparison then follow the same blueprint as in first-order theory.

12

Bernstein's method and gradient bounds

Source correspondence: Math 821, Part 3; Bernstein calculations for viscous Hamilton–Jacobi equations and the role of a priori Lipschitz estimates in compactness.

A viscosity compactness argument often reduces convergence to one quantitative estimate: a bound for the gradient independent of the parameter that is being sent to zero. Bernstein's method obtains such bounds by applying the maximum principle to $|Du|^2$.

12.1 The basic differential identity

Let $u \in C^3(\mathbb{T}^n)$ solve

$$\delta u - \nu \Delta u + H(Du) = f(x), \quad \delta > 0, \nu > 0, \quad (12.1)$$

where $H \in C^1(\mathbb{R}^n)$ and $f \in C^1(\mathbb{T}^n)$. Set

$$w = \frac{1}{2}|Du|^2.$$

Differentiate (12.1) in x_k , multiply by u_k , and sum over k . We obtain

$$\delta |Du|^2 - \nu \sum_k u_k \Delta u_k + D_p H(Du) \cdot Dw = Df \cdot Du. \quad (12.2)$$

Since

$$\Delta w = |D^2 u|^2 + \sum_k u_k \Delta u_k, \quad (12.3)$$

we have

$$2\delta w - \nu \Delta w + \nu |D^2 u|^2 + D_p H(Du) \cdot Dw = Df \cdot Du. \quad (12.4)$$

Let x_o maximize w . Then $Dw(x_o) = 0$ and $\Delta w(x_o) \leq 0$, and therefore

$$\delta |Du(x_o)|^2 + \nu |D^2 u(x_o)|^2 \leq \|Df\|_\infty |Du(x_o)|. \quad (12.5)$$

Remark 12.1. The drift $D_p H(Du)$ disappears at the maximum of w . This is the characteristic feature of the Bernstein calculation. The estimate is a maximum-principle argument, not an integration-by-parts argument.

12.2 A bound uniform in the discount

The direct consequence of (12.5), $\|Du\|_\infty \leq \|Df\|_\infty/\delta$, degenerates as $\delta \downarrow 0$. The equation itself provides a stronger estimate when H is coercive.

Theorem 12.2 (Discount-uniform gradient bound). *Assume the quantitative coercivity condition*

$$\frac{H(p)}{\sqrt{1+|p|}} \longrightarrow +\infty \quad \text{as } |p| \rightarrow \infty.$$

Let u solve (12.1), and suppose

$$\|\delta u\|_\infty \leq C_0 \tag{12.6}$$

with C_0 independent of $\delta \in (0, 1]$. Then

$$\|Du\|_\infty \leq C,$$

where C depends on H , ν , C_0 , $\|f\|_{C^1}$, and n , but not on δ .

Proof. Let x_0 maximize $|Du|$ and put $P = |Du(x_0)|$. By Cauchy–Schwarz for the eigenvalues of D^2u ,

$$|D^2u|^2 \geq \frac{1}{n}(\Delta u)^2.$$

Equation (12.1) gives

$$\nu \Delta u = \delta u + H(Du) - f.$$

Combining this with (12.5) and discarding the nonnegative term δP^2 yields

$$\frac{1}{n\nu}(\delta u(x_0) + H(Du(x_0)) - f(x_0))^2 \leq \|Df\|_\infty P. \tag{12.7}$$

Hence

$$H(Du(x_0)) \leq C_0 + \|f\|_\infty + \sqrt{n\nu\|Df\|_\infty P}. \tag{12.8}$$

The right-hand side grows at most like $P^{1/2}$, whereas the quantitative coercivity assumption makes the left-hand side dominate that square-root growth. More explicitly, choose R so large that

$$\inf_{|p| \geq R} H(p) > C_0 + \|f\|_\infty + \sqrt{n\nu\|Df\|_\infty |p|}$$

for $|p| \geq R$. Then (12.8) rules out $P \geq R$. ■

For the discounted equation on the torus, (12.6) follows directly from constant sub- and supersolutions.

Example 12.3 (Quadratic Hamiltonian). If $H(p) = |p|^2$, then (12.8) becomes

$$P^2 \leq C + CP^{1/2}.$$

Consequently $P \leq C$. The constant deteriorates as $\nu \downarrow 0$, which is expected: the estimate above is uniform in the discount, not simultaneously uniform in viscosity.

12.3 Oscillation and compactness

Once $\|Du\|_\infty \leq C$ on the torus,

$$\text{osc}_{\mathbb{T}^n} u \leq C \text{diam}(\mathbb{T}^n).$$

Normalize by $u(x_0) = 0$. The family is then uniformly bounded and equi-Lipschitz, so Arzelà–Ascoli gives compactness. This is the standard route from the discounted viscous equation to an ergodic problem.

Proposition 12.4 (Stability of the estimate under smooth approximation). *Suppose H and f are merely Lipschitz and the solution is viscosity rather than classical. Approximate the data smoothly and add a small uniformly elliptic term if needed. If the constants in the preceding calculation depend only on the displayed norms and coercivity modulus, the estimates survive the approximation. Viscosity stability then passes the bound to the limiting solution.*

Proof. For smooth approximations H_j, f_j with the same coercivity and uniformly bounded Lipschitz norms, solve the classical uniformly elliptic problems. The preceding estimate is uniform in j . A subsequence converges uniformly, and the gradients are uniformly bounded in the weak-* sense. The uniform limit is Lipschitz with the same constant and, by stability, solves the limiting viscosity equation. Uniqueness identifies it with the original solution. ■

12.4 A general quasilinear template

Let

$$u_t - a_{ij}(Du)u_{ij} = 0, \quad A(p) = (a_{ij}(p)) \geq 0.$$

Differentiating and setting $w = |Du|^2/2$ gives

$$w_t - a_{ij}(Du)w_{ij} = -a_{ij}(Du)u_{ki}u_{kj} + a_{ij,p_\ell}(Du)w_{\ell}u_{ij}. \quad (12.9)$$

At a spatial maximum of w , the last term vanishes and the first is nonpositive. Therefore

$$\max_x |Du(x, t)| \leq \max_x |Du(x, 0)|.$$

This template will be used for the regularized level-set mean curvature equation in [chapter 17](#).

A Bernstein estimate must be matched to the parameter regime. The calculation above is uniform as $\delta \downarrow 0$ with $\nu > 0$ fixed. It does not by itself give a bound uniform as $\nu \downarrow 0$. When discount and viscosity vanish together, one needs additional barriers, control representations, adjoint identities, or equation-specific structure.

Chapter summary

Bernstein's method differentiates the equation and applies the maximum principle to $|Du|^2$. Coercivity and the equation convert the positive Hessian term into a discount-uniform gradient bound. The method is powerful but parameter-sensitive.

Part IV

Uniform ellipticity and interior regularity

13

Uniform ellipticity and the Pucci extremal operators

Source correspondence: Math 821, Part 4; uniform ellipticity, Pucci's operators, and the inequalities satisfied by differences of solutions.

Degenerate ellipticity records only monotonicity. Uniform ellipticity quantifies how strongly the operator responds when a positive matrix is added. The Pucci operators are the extremal responses compatible with fixed ellipticity constants.

13.1 The extremal operators

Fix $0 < \lambda \leq \Lambda$. Let

$$\mathcal{A}_{\lambda,\Lambda} = \{A \in \mathbb{S}^n : \lambda I \leq A \leq \Lambda I\}.$$

For $M \in \mathbb{S}^n$, define

$$\mathcal{P}_{\lambda,\Lambda}^+(M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AM), \quad (13.1)$$

$$\mathcal{P}_{\lambda,\Lambda}^-(M) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AM). \quad (13.2)$$

These operators are increasing in the matrix order.

Proposition 13.1 (Eigenvalue formulas). *If $e_1, \dots, e_n$ are the eigenvalues of M , then*

$$\mathcal{P}_{\lambda,\Lambda}^+(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i, \quad (13.3)$$

$$\mathcal{P}_{\lambda,\Lambda}^-(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i. \quad (13.4)$$

In particular,

$$\mathcal{P}_{\lambda,\Lambda}^-(M) = -\mathcal{P}_{\lambda,\Lambda}^+(-M). \quad (13.5)$$

Proof. Let q_i be an orthonormal eigenbasis of M . For every $A \in \mathcal{A}_{\lambda,\Lambda}$,

$$\operatorname{tr}(AM) = \sum_i e_i (q_i \cdot A q_i).$$

Each Rayleigh quotient lies in $[\lambda, \Lambda]$. To maximize the sum, choose the coefficient Λ when $e_i > 0$ and λ when $e_i < 0$. This choice is attained by taking A diagonal in the basis $\{q_i\}$. The minimum is obtained by reversing the choices. The duality follows immediately. ■

Proposition 13.2 (Algebraic properties). *For $M, N \in \mathbb{S}^n$ and $t \geq 0$,*

$$\begin{aligned}\mathcal{P}_{\lambda,\Lambda}^+(tM) &= t\mathcal{P}_{\lambda,\Lambda}^+(M), \\ \mathcal{P}_{\lambda,\Lambda}^-(tM) &= t\mathcal{P}_{\lambda,\Lambda}^-(M), \\ \mathcal{P}_{\lambda,\Lambda}^+(M+N) &\leq \mathcal{P}_{\lambda,\Lambda}^+(M) + \mathcal{P}_{\lambda,\Lambda}^+(N), \\ \mathcal{P}_{\lambda,\Lambda}^-(M+N) &\geq \mathcal{P}_{\lambda,\Lambda}^-(M) + \mathcal{P}_{\lambda,\Lambda}^-(N), \\ \mathcal{P}_{\lambda,\Lambda}^-(N) &\leq \mathcal{P}_{\lambda,\Lambda}^+(M+N) - \mathcal{P}_{\lambda,\Lambda}^+(M) \leq \mathcal{P}_{\lambda,\Lambda}^+(N).\end{aligned}$$

Proof. Positive homogeneity and subadditivity of the supremum follow directly from (13.1); the statements for the infimum are dual. For the last line, write

$$\mathcal{P}_{\lambda,\Lambda}^+(M+N) = \sup_A \{\operatorname{tr}(AM) + \operatorname{tr}(AN)\}.$$

The upper bound follows by taking separate suprema. For the lower bound, choose matrices A_j approaching the supremum for M and use $\operatorname{tr}(A_j N) \geq \mathcal{P}_{\lambda,\Lambda}^-(N)$ before passing to the limit. ■

13.2 Uniform ellipticity in the proper sign convention

Recall that our operator F is nonincreasing in X . We call F uniformly elliptic with constants (λ, Λ) if, for every $N \geq 0$,

$$-\Lambda \operatorname{tr} N \leq F(x, r, p, X+N) - F(x, r, p, X) \leq -\lambda \operatorname{tr} N. \quad (13.6)$$

Theorem 13.3 (Pucci characterization). *Condition (13.6) is equivalent to*

$$-\mathcal{P}_{\lambda,\Lambda}^+(X-Y) \leq F(x, r, p, X) - F(x, r, p, Y) \leq -\mathcal{P}_{\lambda,\Lambda}^-(X-Y) \quad (13.7)$$

for all $X, Y \in \mathbb{S}^n$.

Proof. The implication from (13.7) to (13.6) follows by taking $X = Y + N$ and observing that, for $N \geq 0$,

$$\mathcal{P}_{\lambda,\Lambda}^+(N) = \Lambda \operatorname{tr} N, \quad \mathcal{P}_{\lambda,\Lambda}^-(N) = \lambda \operatorname{tr} N.$$

Conversely, put $M = X - Y$ and decompose spectrally $M = M^+ - M^-$, where $M^+, M^- \geq 0$ and $M^+ M^- = 0$. Move from Y to $Y - M^-$ and then to $Y - M^- + M^+ = X$. Applying (13.6) at each step gives

$$\begin{aligned}F(X) - F(Y) &\leq \Lambda \operatorname{tr} M^- - \lambda \operatorname{tr} M^+ = -\mathcal{P}_{\lambda,\Lambda}^-(M), \\ F(X) - F(Y) &\geq \lambda \operatorname{tr} M^- - \Lambda \operatorname{tr} M^+ = -\mathcal{P}_{\lambda,\Lambda}^+(M).\end{aligned}$$

■

Many references define elliptic operators to be increasing in X . In that convention the Pucci inequalities appear without the leading minus signs. Here $F = -\Delta$ is elliptic, so the signs in (13.7) are the correct ones.

13.3 Differences of solutions

The nonlinear equation itself may not be convex or concave. Nevertheless, the difference of a subsolution and a supersolution satisfies an inequality governed by the Pucci maximal operator.

Theorem 13.4 (Difference inequality). *Let $F(X, x)$ be uniformly elliptic in X . Suppose*

$$F(D^2u, x) \leq f(x), \quad F(D^2v, x) \geq g(x)$$

in the viscosity sense. Then $w = u - v$ is a viscosity subsolution of

$$-\mathcal{P}_{\lambda, \Lambda}^+(D^2w) \leq f - g + \omega(|Dw|, x), \quad (13.8)$$

where the final modulus records any dependence of F on x, r, p . If $F = F(X)$ is independent of those variables, this reduces to

$$\mathcal{P}_{\lambda, \Lambda}^+(D^2w) \geq g - f. \quad (13.9)$$

Proof. We prove the clean case. Let ϕ touch w from above at x_o . For $\varepsilon > 0$, maximize

$$u(x) - v(y) - \phi(x) - \frac{|x - y|^2}{2\varepsilon}$$

near (x_o, x_o) . The theorem on sums gives jets (p_x, X) for u and (p_y, Y) for v , with

$$X - Y \leq D^2\phi(x_o) + o(1).$$

The viscosity inequalities and uniform ellipticity yield

$$\begin{aligned} f(x_o) - g(x_o) + o(1) &\geq F(X) - F(Y) \\ &\geq -\mathcal{P}_{\lambda, \Lambda}^+(X - Y) \\ &\geq -\mathcal{P}_{\lambda, \Lambda}^+(D^2\phi(x_o) + o(1)). \end{aligned}$$

The last inequality uses monotonicity of $\mathcal{P}_{\lambda, \Lambda}^+$. Letting $\varepsilon \downarrow 0$ gives $-\mathcal{P}_{\lambda, \Lambda}^+(D^2\phi) \leq f - g$. ■

Corollary 13.5 (The class $S(\lambda, \Lambda)$). *If u and v solve the same translation-invariant uniformly elliptic equation, then $w = u - v$ satisfies*

$$\mathcal{P}_{\lambda, \Lambda}^-(D^2w) \leq 0 \leq \mathcal{P}_{\lambda, \Lambda}^+(D^2w) \quad (13.10)$$

in the viscosity sense.

Proof. Apply **theorem 13.4** to $u - v$ to obtain the right inequality. Apply it to $v - u = -w$ and use (13.5) to obtain the left inequality. ■

The class of functions satisfying (13.10) is stable under rescaling and compactness. It is the universal regularity class for uniformly elliptic equations.

13.4 A linear interpretation

At a point where a smooth uniformly elliptic operator is differentiable in the matrix variable,

$$F(X) - F(Y) = \int_0^1 D_X F(Y + t(X - Y)) : (X - Y) dt.$$

The matrix $-D_X F$ has eigenvalues between λ and Λ . Thus the difference behaves like a linear equation with measurable coefficients. Pucci's operators encode the best and worst such coefficients without assuming differentiability of F .

Chapter summary

The Pucci operators are the extremal traces over all coefficient matrices between λI and ΛI . They give an exact algebraic characterization of uniform ellipticity and control differences of solutions, reducing nonlinear regularity to estimates for a universal extremal class.

14

The ABP estimate and the Krylov–Safonov mechanism

Source correspondence: Math 821, Part 4; the ABP maximum principle, Harnack inequalities, oscillation decay, and interior Hölder regularity.

Uniform ellipticity turns pointwise curvature information into measure information. The Alexandrov–Bakelman–Pucci estimate is the basic bridge. The Krylov–Safonov theory combines that estimate with a covering argument to obtain Harnack inequalities and Hölder continuity for equations with merely measurable coefficients.

14.1 Upper contact sets

Let Ω be bounded and $u \in C(\overline{\Omega})$. A point $x \in \Omega$ belongs to the upper contact set if there is an affine function touching u from above at x throughout Ω . Equivalently, x lies in the contact set of u with its concave envelope.

If $u \in C^2$ and x is an upper contact point, then

$$D^2u(x) \leq 0.$$

At such a point, a uniformly elliptic inequality controls every eigenvalue of $-D^2u$ through its trace.

14.2 The ABP maximum principle

Theorem 14.1 (ABP estimate). *Let $\Omega \subset \mathbb{R}^n$ be bounded, $d = \text{diam } \Omega$, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

Assume

$$-\mathcal{P}_{\lambda, \Lambda}^+(D^2u) \leq f \quad \text{in } \Omega. \quad (14.1)$$

Let Γ^+ be the upper contact set of u with its concave envelope. Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \frac{d}{n\lambda \omega_n^{1/n}} \|f^+\|_{L^n(\Gamma^+)}, \quad (14.2)$$

where $\omega_n = |B_1|$.

Proof. Subtract the boundary constant and replace u by $u - \sup_{\partial\Omega} u^+$. It is enough to treat $u \leq 0$ on $\partial\Omega$. Let $M = \sup_{\Omega} u$. If $M \leq 0$, there is nothing to prove. Choose $x_0 \in \Omega$ with $u(x_0) = M$.

For every $p \in B_{M/d}(\mathfrak{o})$, consider

$$x \mapsto u(x) - p \cdot x.$$

If $y \in \partial\Omega$, then

$$[u(x_{\mathfrak{o}}) - p \cdot x_{\mathfrak{o}}] - [u(y) - p \cdot y] \geq M - |p| |x_{\mathfrak{o}} - y| > \mathfrak{o}.$$

Thus the maximum is attained at an interior point x_p . The affine function with slope p touching there lies above u on $\overline{\Omega}$, so $x_p \in \Gamma^+$ and

$$p = Du(x_p).$$

Consequently,

$$B_{M/d}(\mathfrak{o}) \subset Du(\Gamma^+). \quad (14.3)$$

At an upper contact point, $D^2u \leq \mathfrak{o}$. By the eigenvalue formula,

$$-\mathcal{P}_{\lambda, \Lambda}^+(D^2u) = \lambda \operatorname{tr}(-D^2u).$$

The differential inequality therefore gives

$$\lambda \operatorname{tr}(-D^2u) \leq f^+ \quad \text{on } \Gamma^+. \quad (14.4)$$

The arithmetic–geometric mean inequality for the nonnegative eigenvalues of $-D^2u$ gives

$$\det(-D^2u) \leq \left(\frac{\operatorname{tr}(-D^2u)}{n} \right)^n \leq \left(\frac{f^+}{n\lambda} \right)^n. \quad (14.5)$$

Applying the area formula to the gradient map on the contact set and using (14.3),

$$\begin{aligned} \omega_n \left(\frac{M}{d} \right)^n &\leq |Du(\Gamma^+)| \\ &\leq \int_{\Gamma^+} |\det D^2u| \, dx \\ &\leq \frac{1}{(n\lambda)^n} \int_{\Gamma^+} (f^+)^n \, dx. \end{aligned}$$

Taking the n th root yields (14.2). ■

Remark 14.2 (Viscosity version). The C^2 assumption is removable. One may apply the argument to sup-convolutions, use Alexandrov’s theorem on the contact set, and pass to the limit. Equivalently, one works directly with the concave envelope and second-order jets. The estimate remains valid for viscosity subsolutions.

14.3 A measure-to-point principle

The ABP estimate is combined with explicit barriers to prove the following growth lemma. This is the quantitative heart of the Krylov–Safonov theory.

Theorem 14.3 (Growth lemma). *There exist constants $\theta \in (0, 1)$ and $\eta \in (0, 1)$, depending only on n , λ , and Λ , with the following property. If $u \geq 0$ in B_1 ,*

$$\mathcal{P}_{\lambda, \Lambda}^-(D^2 u) \leq 1 \quad \text{in } B_1,$$

and

$$|\{u \leq 1\} \cap B_1| \geq \eta |B_1|,$$

then

$$\sup_{B_{1/2}} u \leq \theta^{-1}.$$

After scaling the right-hand side and the level, the same implication holds on every ball.

The proof slides paraboloids of fixed opening from below, records their contact points, and applies the ABP estimate to an auxiliary function. The contact-point map has a controlled Jacobian by uniform ellipticity. A Calderón–Zygmund cube decomposition then iterates the one-step measure estimate. Full proofs are given in [4, Chapters 4–5] and [17, Chapter 9].

From the lectures The lecture notes emphasize the logic rather than the covering book-keeping:

many low points $\implies$ a controlled contact set $\implies$ a pointwise upper bound.

The ABP estimate is what converts the size of the contact set into a bound for the height of the solution.

14.4 Harnack's inequality

Theorem 14.4 (Krylov–Safonov Harnack inequality). *Let $u \geq 0$ in B_1 and suppose*

$$\mathcal{P}_{\lambda, \Lambda}^-(D^2 u) \leq f, \quad \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) \geq -f \quad \text{in } B_1 \tag{14.6}$$

in the viscosity sense, with $f \in L^n(B_1)$. Then

$$\sup_{B_{1/2}} u \leq C \left(\inf_{B_{1/2}} u + \|f\|_{L^n(B_1)} \right), \tag{14.7}$$

where C depends only on n , λ , and Λ .

The proof iterates [theorem 14.3](#) on dyadic cubes and uses a weak Harnack estimate for supersolutions. The remarkable point is that no continuity of the coefficient matrix is required.

14.5 Oscillation decay and Hölder continuity

We now derive Hölder continuity from Harnack's inequality. This deduction is elementary and important.

Theorem 14.5 (Interior Hölder estimate). *Suppose u satisfies*

$$\mathcal{P}_{\lambda, \Lambda}^-(D^2 u) \leq \mathfrak{o} \leq \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) \quad \text{in } B_1.$$

Then there are $\alpha \in (0, 1)$ and $C > 0$, depending only on n, λ , and Λ , such that

$$[u]_{C^\alpha(B_{1/2})} \leq C \|u\|_{L^\infty(B_1)}. \quad (14.8)$$

Proof. Let

$$M = \sup_{B_1} u, \quad m = \inf_{B_1} u,$$

and on $B_{1/2}$ set

$$b = \sup_{B_{1/2}} u, \quad a = \inf_{B_{1/2}} u.$$

The nonnegative functions $u - m$ and $M - u$ both belong to the Pucci class. Harnack's inequality gives

$$\begin{aligned} b - m &\leq C_H(a - m), \\ M - a &\leq C_H(M - b). \end{aligned}$$

Put $A = a - m$ and $B = M - b$. Adding the two inequalities in the equivalent forms $M - m - B \leq C_H A$ and $M - m - A \leq C_H B$ yields

$$A + B \geq \frac{2}{C_H + 1}(M - m).$$

Therefore

$$\text{osc}_{B_{1/2}} u = b - a = (M - m) - (A + B) \leq \vartheta \text{osc}_{B_1} u, \quad (14.9)$$

where

$$\vartheta = 1 - \frac{2}{C_H + 1} \in (0, 1).$$

The equation is invariant under translations and dilations. Iterating (14.9) gives

$$\text{osc}_{B_{2^{-k}}(x_0)} u \leq \vartheta^k \text{osc}_{B_1} u$$

whenever $B_{2^{-k+1}}(x_0) \subset B_{3/4}$. Choose $\alpha > 0$ by $2^{-\alpha} = \vartheta$. If $2^{-(k+1)} < r \leq 2^{-k}$, then

$$\text{osc}_{B_r(x_0)} u \leq C r^\alpha \text{osc}_{B_1} u.$$

This Campanato oscillation estimate is equivalent to (14.8). ■

14.6 Higher regularity: the next steps

For a translation-invariant uniformly elliptic equation

$$F(D^2u) = 0,$$

difference quotients satisfy Pucci inequalities. A compactness and improvement-of-flatness argument then gives

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}. \quad (14.10)$$

This is the nonlinear analogue of obtaining gradient Hölder continuity for harmonic functions. If, in addition, F is convex or concave, the Evans–Krylov theorem yields interior $C^{2,\alpha}$ estimates. These deeper results require an additional iteration beyond the scope of the handwritten Part 4; precise proofs are in [4, Chapters 5 and 8].

Chapter summary

The ABP estimate bounds the maximum through the determinant of the Hessian on a contact set. Sliding paraboloids and a covering argument turn this into the Krylov–Safonov Harnack inequality. Harnack applied to both $u - m$ and $M - u$ yields geometric oscillation decay and therefore interior Hölder regularity.

Part V

The level-set method for mean curvature flow

15

Geometry and the level-set mean curvature equation

Source correspondence: Math 821, Part 5; derivation of the level-set equation, graphical mean curvature flow, geometric invariance, and basic explicit motions.

A smooth mean curvature flow is a family of hypersurfaces moving in the normal direction with speed equal to minus their mean curvature. Classical motion may form singularities in finite time. The level-set method represents the hypersurface as a zero set of a function and evolves the function by a degenerate parabolic PDE.

15.1 Sign conventions

Let $\Gamma \subset \mathbb{R}^n$ be a smooth oriented hypersurface with unit normal ν . We define its scalar mean curvature by

$$H = \operatorname{div} \nu.$$

For a sphere of radius R with outward normal, $H = (n - 1)/R > 0$. Mean curvature flow is

$$V_\nu = -H, \tag{15.1}$$

where V_ν is the normal velocity relative to ν . Thus convex spheres move inward.

Only the normal component matters. Adding an arbitrary tangential velocity changes the parametrization but not the evolving hypersurface.

15.2 Derivation of the level-set PDE

Let $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ be smooth, and represent

$$\Gamma_t = \{x : u(x, t) = 0\}.$$

At a regular point $Du \neq 0$, choose

$$\nu = \frac{Du}{|Du|}.$$

If $X(t) \in \Gamma_t$, then

$$0 = \frac{d}{dt} u(X(t), t) = u_t + Du \cdot \dot{X} = u_t + |Du| V_\nu.$$

Using (15.1),

$$u_t - |Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) = 0. \tag{15.2}$$

A direct differentiation gives

$$|Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) = \operatorname{tr} \left[\left(I - \frac{Du \otimes Du}{|Du|^2} \right) D^2 u \right]. \quad (15.3)$$

Hence the level-set equation is

$$u_t - \operatorname{tr} \left[\left(I - \frac{Du \otimes Du}{|Du|^2} \right) D^2 u \right] = 0. \quad (15.4)$$

The matrix in parentheses is the orthogonal projection onto the tangent space of the level set.

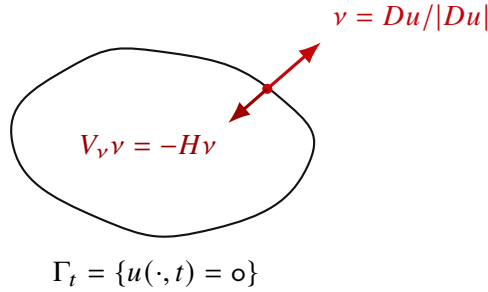


Figure 15.1: The level-set function supplies the normal direction; the PDE prescribes only the normal motion.

15.3 Why the equation is degenerate

For $p \neq 0$, set

$$F(p, X) = -\operatorname{tr} \left[\left(I - \frac{p \otimes p}{|p|^2} \right) X \right]. \quad (15.5)$$

Then (15.4) is $u_t + F(Du, D^2 u) = 0$. The projection has eigenvalue 0 in the normal direction and eigenvalue 1 in every tangent direction. Thus the equation diffuses along level sets but not across them. It is degenerate parabolic.

The formula is undefined at $p = 0$. This is not a removable pointwise singularity: the limit depends on the direction in which p approaches zero. The viscosity envelopes at $p = 0$ are computed in [chapter 16](#).

15.4 Geometric invariance

A geometric evolution should not depend on which function represents the same level sets.

Proposition 15.1 (Invariance under increasing changes of level). *Suppose u is a smooth solution of (15.4) with $Du \neq 0$, and let $\theta \in C^2(\mathbb{R})$ satisfy $\theta' > 0$. Then $v = \theta(u)$ also solves (15.4).*

Proof. We have

$$v_t = \theta'(u)u_t, \quad Dv = \theta'(u)Du,$$

and

$$D^2 v = \theta'(u)D^2 u + \theta''(u)Du \otimes Du.$$

The projection determined by Dv is the same as the one determined by Du . It annihilates $Du \otimes Du$, so

$$\begin{aligned} & \operatorname{tr} \left[\left(I - \frac{Dv \otimes Dv}{|Dv|^2} \right) D^2 v \right] \\ &= \theta'(u) \operatorname{tr} \left[\left(I - \frac{Du \otimes Du}{|Du|^2} \right) D^2 u \right]. \end{aligned}$$

Both terms in the PDE are multiplied by $\theta'(u)$. ■

The viscosity theory extends this invariance to continuous nondecreasing θ . Consequently, all level sets evolve consistently, not only the zero set.

15.5 Graphical mean curvature flow

Let a hypersurface be the graph

$$\Gamma_t = \{(x, h(x, t)) : x \in \mathbb{R}^{n-1}\}.$$

The upward unit normal is

$$v = \frac{(-Dh, 1)}{\sqrt{1 + |Dh|^2}}.$$

Its normal velocity is $V_v = h_t / \sqrt{1 + |Dh|^2}$, and

$$H = -\operatorname{div} \left(\frac{Dh}{\sqrt{1 + |Dh|^2}} \right).$$

Thus $V_v = -H$ becomes

$$h_t = \sqrt{1 + |Dh|^2} \operatorname{div} \left(\frac{Dh}{\sqrt{1 + |Dh|^2}} \right). \quad (15.6)$$

Equivalently,

$$h_t = \left(\delta_{ij} - \frac{h_i h_j}{1 + |Dh|^2} \right) h_{ij}. \quad (15.7)$$

For bounded gradient this is uniformly parabolic.

15.6 Explicit solutions

Example 15.2 (Planes). For $u(x, t) = a \cdot x - c$, one has $D^2 u = 0$, so every hyperplane is stationary.

Example 15.3 (Shrinking spheres). Let

$$u(x, t) = |x|^2 - R_0^2 + 2(n-1)t. \quad (15.8)$$

Then $u_t = 2(n-1)$, $Du = 2x$, and $D^2 u = 2I$. For $x \neq 0$,

$$\operatorname{tr} \left[\left(I - \frac{x \otimes x}{|x|^2} \right) 2I \right] = 2(n-1).$$

Thus u solves the level-set equation. Its zero set is the sphere of radius

$$R(t) = \sqrt{R_0^2 - 2(n-1)t}, \quad (15.9)$$

which becomes extinct at $T = R_0^2 / [2(n-1)]$.

Example 15.4 (Cylinders). For a cylinder $S^{k-1}(R) \times \mathbb{R}^{n-k}$, only $k-1$ principal curvatures are nonzero. Hence

$$R'(t) = -\frac{k-1}{R(t)}, \quad R(t)^2 = R_0^2 - 2(k-1)t.$$

The signed distance to Γ_t generally does not remain a signed-distance function under the PDE. The level-set method evolves a representing function; geometric invariance ensures that its zero set is independent of the chosen initial representation.

Chapter summary

Mean curvature flow becomes a scalar degenerate parabolic equation by evolving all level sets at once. The projected Hessian measures tangential curvature, the normal direction is undiffused, and increasing reparametrizations of the level-set function leave the geometry unchanged.

16

Viscosity solutions for level-set mean curvature flow

Source correspondence: Math 821, Part 5; the singular operator at zero gradient, geometric viscosity solutions, comparison, invariance, and avoidance.

For $p \neq 0$, the level-set operator is

$$F(p, X) = -\operatorname{tr} X + \frac{p \cdot Xp}{|p|^2}.$$

Following Evans–Spruck, we extend it at $p = 0$ by $F(0, X) = -\operatorname{tr} X$ and then use its semi-continuous envelopes. This extension is exactly the one selected by the uniformly parabolic regularization in [chapter 17](#).

16.1 The operator at zero gradient

For a unit vector e ,

$$\lim_{\substack{p \rightarrow 0 \\ p/|p| \rightarrow e}} F(p, X) = -\operatorname{tr} X + e \cdot Xe.$$

The sequence $p = 0$ is also admissible in the relaxation. Therefore

$$F_*(0, X) = -\operatorname{tr} X + \min\{0, \lambda_{\min}(X)\}, \quad (16.1)$$

$$F^*(0, X) = -\operatorname{tr} X + \max\{0, \lambda_{\max}(X)\}. \quad (16.2)$$

For $p \neq 0$, $F_* = F^* = F$.

Definition 16.1 (Geometric viscosity solution). An upper semicontinuous function u is a subsolution of level-set mean curvature flow if every $\phi \in C^2$ touching from above at (x_0, t_0) satisfies

$$\phi_t + F_*(D\phi, D^2\phi) \leq 0.$$

A lower semicontinuous function v is a supersolution if every lower test satisfies

$$\phi_t + F^*(D\phi, D^2\phi) \geq 0.$$

At a critical touching point $D\phi = 0$, these become

$$\phi_t - \operatorname{tr} D^2\phi + \min\{0, \lambda_{\min}(D^2\phi)\} \leq 0 \quad \text{for a subsolution,} \quad (16.3)$$

$$\phi_t - \operatorname{tr} D^2\phi + \max\{0, \lambda_{\max}(D^2\phi)\} \geq 0 \quad \text{for a supersolution.} \quad (16.4)$$

Remark 16.2 (Equivalent directional form). Because

$$\min_{|\eta| \leq 1} \eta \cdot X\eta = \min\{0, \lambda_{\min}(X)\}, \quad \max_{|\eta| \leq 1} \eta \cdot X\eta = \max\{0, \lambda_{\max}(X)\},$$

(16.3) is equivalent to the existence of $\eta \in \mathbb{R}^n$, $|\eta| \leq 1$, such that

$$\phi_t \leq \operatorname{tr}[(I - \eta \otimes \eta)D^2\phi].$$

The supersolution condition is equivalent to the existence of such an η with the inequality reversed. This is the classical Evans–Spruck formulation.

16.2 Comparison: the quartic penalty

The ordinary quadratic penalty can produce zero gradient while retaining a nonzero Hessian. At $p = 0$, the lower and upper envelopes of F need not be ordered even if $X \leq Y$. The correct penalty is quartic:

$$\Psi_\alpha(x, y) = \frac{\alpha}{4}|x - y|^4. \quad (16.5)$$

Its gradient vanishes on the diagonal, and so does its Hessian.

Theorem 16.3 (Comparison for level-set mean curvature flow). *Let Ω be bounded. Suppose $u \in \operatorname{USC}(\overline{\Omega} \times [0, T])$ is a bounded subsolution and $v \in \operatorname{LSC}(\overline{\Omega} \times [0, T])$ a bounded supersolution of (15.4). If $u \leq v$ on the parabolic boundary, then $u \leq v$ in $\overline{\Omega} \times [0, T]$.*

Proof. Assume that $\sup(u - v) > 0$. Choose $\eta > 0$ so that $u - v - \eta t$ still has a positive maximum at a positive time. For $\alpha > 0$, maximize

$$u(x, t) - v(y, t) - \frac{\alpha}{4}|x - y|^4 - \eta t.$$

The usual penalization argument gives maximizing points $(x_\alpha, y_\alpha, t_\alpha)$ with

$$|x_\alpha - y_\alpha| \rightarrow 0, \quad \alpha|x_\alpha - y_\alpha|^4 \rightarrow 0, \quad t_\alpha > 0. \quad (16.6)$$

For large α the points are interior.

Set $z = x_\alpha - y_\alpha$ and

$$p_\alpha = \alpha|z|^2 z.$$

The spatial Hessian of $\alpha|z|^4/4$ is

$$B_\alpha = \alpha(|z|^2 I + 2z \otimes z),$$

and the full block Hessian is

$$A_\alpha = \begin{pmatrix} B_\alpha & -B_\alpha \\ -B_\alpha & B_\alpha \end{pmatrix}.$$

The parabolic theorem on sums gives time components a, b and matrices X, Y such that

$$a - b = \eta, \quad X \leq Y,$$

and

$$\begin{aligned} a + F_*(p_\alpha, X) &\leq 0, \\ b + F^*(p_\alpha, Y) &\geq 0. \end{aligned}$$

If $z \neq 0$, then $p_\alpha \neq 0$, the operator is continuous, and degenerate ellipticity gives

$$F(p_\alpha, X) \geq F(p_\alpha, Y).$$

Consequently,

$$\eta = a - b \leq F(p_\alpha, Y) - F(p_\alpha, X) \leq 0,$$

a contradiction.

It remains to consider $z = 0$. Then $p_\alpha = 0$ and $A_\alpha = 0$. The upper block estimate in the theorem on sums gives

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 0,$$

so $X \leq 0 \leq Y$. If $X \leq 0$, then

$$F_*(0, X) = -\operatorname{tr} X + \min\{0, \lambda_{\min}(X)\} = -\operatorname{tr} X + \lambda_{\min}(X) = -\sum_{i=2}^n \lambda_i(X) \geq 0,$$

where the eigenvalues are ordered increasingly. If $Y \geq 0$, then

$$F^*(0, Y) = -\operatorname{tr} Y + \max\{0, \lambda_{\max}(Y)\} = -\operatorname{tr} Y + \lambda_{\max}(Y) = -\sum_{i=1}^{n-1} \lambda_i(Y) \leq 0.$$

Thus again

$$\eta \leq F^*(0, Y) - F_*(0, X) \leq 0,$$

a contradiction. Comparison follows. ■

The quartic penalty is not an aesthetic choice. At a diagonal maximum it makes both the momentum and the coupling Hessian vanish. This restores the ordering needed between $F_*(0, X)$ and $F^*(0, Y)$.

16.3 Geometric invariance in the viscosity sense

Theorem 16.4 (Reparametrization invariance). *Let u be a viscosity solution of level-set mean curvature flow and let $\theta : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and nondecreasing. Then $\theta \circ u$ is also a viscosity solution.*

Proof. First assume $\theta \in C^2$ and $\theta' > 0$. Let $v = \theta(u)$ and suppose $v - \phi$ has a local maximum at (x_o, t_o) . Set $r_o = u(x_o, t_o)$ and define near the contact point

$$\psi = \theta^{-1}(\phi - \phi(x_o, t_o) + \theta(r_o)).$$

Then $u - \psi$ has a local maximum at (x_o, t_o) . The chain rule gives the same identities as in [theorem 15.1](#). The term containing $\theta'' Du \otimes Du$ is annihilated by the tangential projection; at zero gradient the envelope formulas are positively homogeneous and the same conclusion follows by continuity. Hence the subsolution inequality transfers from u to v . The supersolution argument is identical.

For a continuous nondecreasing θ , approximate it locally uniformly by smooth strictly increasing functions θ_j . Then $\theta_j(u) \rightarrow \theta(u)$ locally uniformly, and viscosity stability completes the proof. ■

16.4 Set evolution and independence of the defining function

Let $E_o \subset \mathbb{R}^n$ be closed. Choose a bounded uniformly continuous function u_o such that

$$E_o = \{u_o \geq 0\} \quad \text{and} \quad \partial E_o = \{u_o = 0\}.$$

Let u solve level-set mean curvature flow with initial data u_o , and define

$$E_t = \{u(\cdot, t) \geq 0\}, \quad \Gamma_t = \{u(\cdot, t) = 0\}.$$

If another initial function v_o has the same sign structure, monotone scalar functions can be used to trap v_o between reparametrizations of u_o . Reparametrization invariance and comparison then show that the resulting zero sets agree. Thus the geometric evolution is independent of the defining function.

16.5 Avoidance and inclusion

Corollary 16.5 (Inclusion principle). *If $E_o \subset F_o$, then the corresponding level-set flows satisfy $E_t \subset F_t$ for every $t \geq 0$.*

Proof. Choose defining functions ordered so that $u_o \leq v_o$ and the superlevel sets are E_o and F_o . Comparison gives $u(\cdot, t) \leq v(\cdot, t)$, which implies inclusion of the nonnegative superlevel sets. ■

Corollary 16.6 (Avoidance). *Two disjoint compact smooth mean curvature flows remain disjoint as long as both remain smooth.*

Proof. Represent one evolving region as a sublevel set and the other as a disjoint superlevel set. A small monotone reparametrization makes the initial defining functions strictly ordered. Comparison preserves the order, preventing contact. ■

Chapter summary

At zero gradient the level-set operator is interpreted through its lower and upper semicontinuous envelopes. Comparison requires a quartic spatial penalty, whose Hessian vanishes

when the doubled points coincide. Reparametrization invariance then makes the PDE a genuine geometric evolution of sets.

17

Existence by regularization, estimates, and singular behavior

Source correspondence: Math 821, Part 5; uniformly parabolic regularization, gradient estimates, passage to the limit, shrinking barriers, singularities, and fattening.

Comparison gives uniqueness. Existence is obtained by replacing the normal projection by a uniformly positive matrix, deriving estimates independent of the regularization parameter, and passing to a half-relaxed limit.

17.1 Uniformly parabolic approximation

For $\varepsilon > 0$, define

$$a_{ij}^\varepsilon(p) = \delta_{ij} - \frac{p_i p_j}{\varepsilon^2 + |p|^2}. \quad (17.1)$$

Consider

$$u_t^\varepsilon - a_{ij}^\varepsilon(Du^\varepsilon)u_{ij}^\varepsilon = 0, \quad u^\varepsilon(\cdot, 0) = u_0^\varepsilon. \quad (17.2)$$

The eigenvalues of $A^\varepsilon(p)$ are 1 on $p^\perp$ and

$$\frac{\varepsilon^2}{\varepsilon^2 + |p|^2}$$

in the direction of p . Thus the equation is uniformly parabolic for fixed ε on every region where $|Du^\varepsilon|$ is bounded.

Take smooth u_0^ε converging uniformly to $u_0 \in BUC(\mathbb{R}^n)$, with $\|Du_0^\varepsilon\|_\infty$ uniformly bounded when u_0 is Lipschitz. Classical quasilinear parabolic theory gives a smooth solution of (17.2); the estimates below prevent gradient blow-up and continue it for all time.

17.2 Maximum and gradient estimates

Constants solve (17.2). Comparison therefore gives

$$\inf u_0^\varepsilon \leq u^\varepsilon(x, t) \leq \sup u_0^\varepsilon. \quad (17.3)$$

Theorem 17.1 (Gradient maximum principle). *If u^ε is smooth and u_0^ε is Lipschitz, then*

$$\|Du^\varepsilon(\cdot, t)\|_\infty \leq \|Du_0^\varepsilon\|_\infty \quad (t \geq 0). \quad (17.4)$$

Proof. Set $w = |Du^\varepsilon|^2/2$ and abbreviate $a_{ij} = a_{ij}^\varepsilon(Du^\varepsilon)$. Differentiating the equation in x_k , multiplying by u_k^ε , and summing gives

$$w_t = a_{ij}w_{ij} - a_{ij}u_{ki}^\varepsilon u_{kj}^\varepsilon + a_{ij,p_\ell}w_{\ell}u_{ij}^\varepsilon.$$

At a spatial maximum of w , $Dw = 0$ and $(w_{ij}) \leq 0$. Since $A^\varepsilon \geq 0$,

$$a_{ij}w_{ij} \leq 0, \quad a_{ij}u_{ki}^\varepsilon u_{kj}^\varepsilon \geq 0.$$

Hence $w_t \leq 0$ at every spatial maximum. The parabolic maximum principle yields (17.4). $\blacksquare$

The initial trace can be controlled uniformly in ε . Assume $|u_0^\varepsilon| \leq M$ and that the functions u_0^ε share a modulus of continuity ω . Fix $x_0 \in \mathbb{R}^n$ and $r > 0$. The function

$$\Phi^+(x, t) = u_0^\varepsilon(x_0) + \omega(r) + \frac{2M}{r^2}|x - x_0|^2 + \frac{4nM}{r^2}t$$

dominates u_0^ε at $t = 0$: inside $B_r(x_0)$ this follows from the modulus, while outside $B_r(x_0)$ the quadratic term is at least $2M$. Moreover,

$$\Phi_t^+ - a_{ij}^\varepsilon(D\Phi^+)\Phi_{ij}^+ \geq \frac{4nM}{r^2} - \frac{4M}{r^2} \operatorname{tr} A^\varepsilon \geq 0,$$

because $0 \leq A^\varepsilon \leq I$. Thus Φ^+ is a supersolution. The analogous lower barrier gives

$$|u^\varepsilon(x_0, t) - u_0^\varepsilon(x_0)| \leq \omega(r) + \frac{4nM}{r^2}t. \quad (17.5)$$

First choose r small and then t small; the resulting initial-time modulus is independent of ε . Translation invariance and comparison also preserve the spatial modulus:

$$|u^\varepsilon(x + h, t) - u^\varepsilon(x, t)| \leq \omega(|h|).$$

17.3 The relaxed limit of the operators

Write (17.2) as

$$u_t^\varepsilon + F_\varepsilon(Du^\varepsilon, D^2u^\varepsilon) = 0,$$

where

$$F_\varepsilon(p, X) = -\operatorname{tr} X + \frac{p \cdot Xp}{\varepsilon^2 + |p|^2}. \quad (17.6)$$

If $p \neq 0$, then $F_\varepsilon(p, X) \rightarrow F(p, X)$. At $p = 0$, take sequences $\varepsilon_j \rightarrow 0$ and $p_j \rightarrow 0$. After a subsequence,

$$\frac{|p_j|^2}{\varepsilon_j^2 + |p_j|^2} \rightarrow a \in [0, 1], \quad \frac{p_j}{|p_j|} \rightarrow e$$

when $p_j \neq 0$. Hence the possible limits are

$$-\operatorname{tr} X + a(e \cdot Xe), \quad a \in [0, 1], \quad |e| = 1.$$

Their infimum and supremum are precisely

$$\begin{aligned} -\operatorname{tr} X + \min\{0, \lambda_{\min}(X)\} &= F_*(0, X), \\ -\operatorname{tr} X + \max\{0, \lambda_{\max}(X)\} &= F^*(0, X). \end{aligned}$$

Thus the regularization selects exactly the viscosity convention of [chapter 16](#).

17.4 Existence and convergence

Theorem 17.2 (Existence of the level-set solution). *For every $u_o \in BUC(\mathbb{R}^n)$, there is a unique viscosity solution*

$$u \in BUC(\mathbb{R}^n \times [0, T])$$

of level-set mean curvature flow for every $T > 0$. If $u^\varepsilon \rightarrow u_o$ uniformly and u^ε solves (17.2), then

$$u^\varepsilon \rightarrow u$$

locally uniformly on $\mathbb{R}^n \times [0, \infty)$.

Proof. The L^∞ bound and the initial-time modulus make the family locally bounded and prevent loss of the initial condition. Define the half-relaxed limits

$$\bar{u} = \limsup_{\substack{\varepsilon \rightarrow 0 \\ (y,s) \rightarrow (x,t)}} u^\varepsilon(y,s), \quad \underline{u} = \liminf_{\substack{\varepsilon \rightarrow 0 \\ (y,s) \rightarrow (x,t)}} u^\varepsilon(y,s).$$

The relaxed-operator calculation above and the stability proof of [theorem 3.3](#) show that $\bar{u}$ is a geometric subsolution and $\underline{u}$ is a geometric supersolution. The initial barriers give

$$\bar{u}(\cdot, 0) \leq u_o \leq \underline{u}(\cdot, 0).$$

Comparison, [theorem 16.3](#), yields $\bar{u} \leq \underline{u}$. The reverse inequality is automatic, so the two functions coincide with a continuous solution u . Equality of all half-relaxed limits implies local uniform convergence of the entire family. Uniqueness follows from comparison. ■

17.5 Consistency with classical motion

Suppose a smooth compact hypersurface Γ_t exists classically on $[0, T_*)$. Choose a smooth defining function whose gradient is nonzero near the evolving surface. The classical calculation in [chapter 15](#) shows that it is a viscosity solution in a tubular neighborhood. Barriers extend the comparison to the two sides. Therefore the zero set of the global level-set solution agrees with Γ_t for $t < T_*$. The generalized flow is a continuation, not a different motion before the first singularity.

17.6 Extinction and shrinking barriers

Proposition 17.3 (Finite extinction of compact phases). *Let E_o be compact and contained in a ball $B_R(x_o)$. Then its level-set flow satisfies*

$$E_t \subset B_{\sqrt{R^2 - 2(n-1)t}}(x_o)$$

for $0 \leq t < R^2/[2(n-1)]$, and $E_t = \emptyset$ after the ball becomes extinct.

Proof. The ball evolves by the explicit shrinking-sphere solution (15.9). Since $E_o \subset B_R(x_o)$, the inclusion principle gives inclusion for all later times. Once the comparison ball is empty, so is E_t . ■

17.7 Topology change and fattening

The level-set function is unique for every continuous initial datum, but the zero set may develop interior. This phenomenon is called *fattening*:

$$\text{int}\{x : u(x, t) = 0\} \neq \emptyset$$

for some $t > 0$. Fattening can occur for nonsmooth initial sets. It reflects nonuniqueness of a sharp interface at that instant, not nonuniqueness of the viscosity solution. The positive and negative phases remain uniquely determined by the scalar PDE.

The source lectures emphasize the geometric lesson: a classical hypersurface may pinch, split, or merge, while the level-set function continues without choosing a parametrization. For discussions and examples of fattening and other pathologies, see [11, 6, 13].

17.8 A final comparison of the three viewpoints

Viewpoint	Strength	Limitation
Classical surface Graph equation	precise geometry and curvature uniformly parabolic under a gradient bound	may stop at singularities cannot represent arbitrary topology
Level-set viscosity solution	global, comparison-based, topology-free	zero sets may fatten and lose smoothness

Chapter summary

Uniformly parabolic regularization produces smooth approximate flows. The maximum principle preserves the Lipschitz constant, the relaxed operator at zero gradient gives the Evans–Spruck viscosity condition, and half-relaxed limits plus comparison yield a unique global flow through singularities.

Part VI

Appendices

A

Matrix lemmas used in the comparison theory

This appendix collects the finite-dimensional calculations that are repeatedly hidden inside the theorem on sums, the Pucci inequalities, and the ABP estimate.

A.1 Quadratic block matrices

Let

$$B = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \quad \text{on } \mathbb{R}^n \times \mathbb{R}^n.$$

Then

$$B \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi - \eta \\ \eta - \xi \end{pmatrix}.$$

The diagonal subspace $\{(\xi, \xi)\}$ is the kernel, and the anti-diagonal subspace $\{(\xi, -\xi)\}$ is the eigenspace with eigenvalue 2. Hence

$$B^2 = 2B, \quad \|B\| = 2. \quad (\text{A.1})$$

For $A = \alpha B$,

$$A^2 = 2\alpha A, \quad \|A\| = 2\alpha.$$

This gives the constants in (10.7).

Lemma A.1 (Diagonal test). *If*

$$\begin{pmatrix} X & \circ \\ \circ & -Y \end{pmatrix} \leq cB$$

for some $c \geq 0$, then $X \leq Y$.

Proof. Test on (ξ, ξ) , which lies in the kernel of B :

$$\xi \cdot (X - Y)\xi \leq 0.$$

■

A.2 Resolvent form of the matrix bound

Let $A \in \mathbb{S}^m$ and suppose $0 < \kappa < \|A\|^{-1}$. Then $I - \kappa A$ is positive definite and commutes with A . Spectral calculus gives

$$A(I - \kappa A)^{-1} = A + \kappa A^2(I - \kappa A)^{-1}. \quad (\text{A.2})$$

The correction term is positive semidefinite because its eigenvalues are

$$\frac{\kappa a^2}{1 - \kappa a} \geq 0.$$

If $\kappa\|A\| \leq 1/2$, then

$$0 \leq \kappa A^2(I - \kappa A)^{-1} \leq 2\kappa A^2.$$

Consequently, for any $\mu > 0$, choosing

$$\kappa \leq \min \left\{ \frac{1}{2\|A\|}, \frac{\mu}{2} \right\}$$

gives

$$A(I - \kappa A)^{-1} \leq A + \mu A^2. \quad (\text{A.3})$$

This explains the equivalence between the resolvent form appearing in sup-convolution proofs and the standard upper bound in [theorem 10.1](#).

A.3 Positive and negative parts

For $M \in \mathbb{S}^n$, let

$$M^+ = Q \operatorname{diag}(e_i^+) Q^T, \quad M^- = Q \operatorname{diag}(e_i^-) Q^T,$$

where $e_i^+ = \max\{e_i, 0\}$ and $e_i^- = \max\{-e_i, 0\}$. Then

$$M = M^+ - M^-, \quad M^+ M^- = 0, \quad |M| = M^+ + M^-.$$

The Pucci formulas can be written

$$\begin{aligned} \mathcal{P}_{\lambda, \Lambda}^+(M) &= \Lambda \operatorname{tr} M^+ - \lambda \operatorname{tr} M^-, \\ \mathcal{P}_{\lambda, \Lambda}^-(M) &= \lambda \operatorname{tr} M^+ - \Lambda \operatorname{tr} M^-. \end{aligned}$$

These forms are often more convenient than explicit eigenvalue sums.

A.4 Trace and determinant

If $N \geq 0$ has eigenvalues v_i , the arithmetic–geometric mean inequality gives

$$\det N = \prod_i v_i \leq \left(\frac{1}{n} \sum_i v_i \right)^n = \left(\frac{\operatorname{tr} N}{n} \right)^n. \quad (\text{A.4})$$

This is the only determinant estimate used in the ABP proof.

If $A, N \geq 0$ and $A \geq \lambda I$, then

$$\operatorname{tr}(AN) = \operatorname{tr}(N^{1/2} A N^{1/2}) \geq \lambda \operatorname{tr} N. \quad (\text{A.5})$$

The symmetric expression on the right of the first equality makes positivity transparent even when A and N do not commute.

A.5 A useful Schur-complement observation

Suppose

$$\begin{pmatrix} X & Z \\ Z^T & Y \end{pmatrix} \geq 0$$

and $X > 0$. Then

$$Y - Z^T X^{-1} Z \geq 0.$$

Indeed, test the block matrix on $(-X^{-1}Z\eta, \eta)$. This elementary fact is frequently used when converting block inequalities into bounds for individual matrices.

B

Exercises and selected hints

The exercises are arranged by part. The hints indicate the main tool without replacing the proof.

Part I: first-order viscosity solutions

Exercise B.1 (Corners). Determine which of the functions $|x|$, $-|x|$, and $1 - |x|$ are viscosity subsolutions, supersolutions, or solutions of $|u'| - 1 = 0$ near $x = 0$.

Hint. Compute the possible slopes of upper and lower touching functions.

Exercise B.2 (Strictification). Suppose $u - \phi$ has a non-strict local maximum at x_0 . Show that $u - \phi - \delta|x - x_0|^4$ has a strict local maximum at x_0 without changing the first two derivatives of the test function there.

Exercise B.3 (Maximum of subsolutions). Let F be proper. Prove that the upper semicontinuous envelope of $\max\{u_1, u_2\}$ is a subsolution whenever u_1, u_2 are subsolutions.

Hint. At a strict contact, select an active branch along a convergent sequence.

Exercise B.4 (Comparison with a zeroth-order coefficient). Adapt theorem 3.5 to

$$c(x)u + H(x, Du) = f(x), \quad c(x) \geq c_0 > 0.$$

Identify where c_0 enters the contradiction.

Exercise B.5 (Half-relaxed limits). Construct a bounded sequence u_k that has no pointwise limit but whose upper and lower half-relaxed limits are easy to compute. Verify their semicontinuity directly.

Part II: control and convexity

Exercise B.6 (A discounted control problem). Let $\dot{X} = \alpha$, $|\alpha| \leq 1$, and running cost $f(X, \alpha) = |X|^2 + |\alpha|^2$. Write the Hamiltonian and the discounted HJB equation.

Hint. Maximize $-\alpha \cdot p - |\alpha|^2$ over the unit ball.

Exercise B.7 (DPP and semigroup). Derive the semigroup property of the finite-horizon value operator directly by concatenating controls.

Exercise B.8 (Legendre transform). Compute the Legendre transform of $H(p) = a|p|^2/2 + b \cdot p + c$, where $a > 0$.

Exercise B.9 (Hopf–Lax minimizers). Assume $g \in C^1$ and H is strictly convex. Derive the characteristic relation satisfied by every differentiable minimizer in the Hopf–Lax formula.

Exercise B.10 (Metric domination). Let w satisfy $H(x, Dw) \leq \mu$ a.e. Prove from (7.2) that

$$w(x) - w(y) \leq m_\mu(y, x).$$

Show that equality along a curve is equivalent to equality in the support-function inequality almost everywhere.

Exercise B.11 (Uniqueness of the ergodic constant). Repeat the proof of the uniqueness of $\bar{H}(p)$ when H is only coercive and uniformly continuous, paying attention to the boundedness of the doubled-variable momentum.

Part III: second-order theory

Exercise B.12 (Semijets of a quadratic cusp). Compute $J^{2,+}u(o)$ and $J^{2,-}u(o)$ for $u(x) = |x|^\alpha$ in one dimension for the ranges $0 < \alpha < 1$, $\alpha = 1$, $1 < \alpha < 2$, and $\alpha = 2$.

Exercise B.13 (Magic property). Starting from the definition of u^ε , prove that if u^ε is differentiable at x , every maximizing point equals $x + \varepsilon Du^\varepsilon(x)$.

Exercise B.14 (Inf-convolution). State and prove the dual magic property for v_ε . Track every sign in the corresponding subjet relation.

Exercise B.15 (Block matrix). For the penalty $\alpha|x - y|^2/2$, verify $A^2 = 2\alpha A$ and derive (10.7) from theorem 10.1.

Exercise B.16 (A linear comparison proof). Use the theorem on sums to prove comparison for

$$c(x)u - \operatorname{tr}(A(x)D^2u) + b(x) \cdot Du = f(x),$$

assuming $c \geq c_o > 0$, $A = \sigma\sigma^T$, and σ, b, c, f are Lipschitz.

Exercise B.17 (Parabolic time penalization). Explain why subtracting ηt forces a contradiction in parabolic comparison but does not by itself keep a maximum away from $t = T$. Construct the standard additional terminal-time barrier.

Exercise B.18 (Discontinuous operators). Let F_k be continuous and converge to a discontinuous F in the relaxed sense. Reprove stability and identify exactly why subsolutions use F_* and supersolutions use F^* .

Part IV: uniform ellipticity

Exercise B.19 (Pucci extremizers). For a given matrix M , write explicitly a coefficient matrix A attaining the supremum and one attaining the infimum in the definitions of $\mathcal{P}_{\lambda,\Lambda}^+(M)$ and $\mathcal{P}_{\lambda,\Lambda}^-(M)$.

Exercise B.20 (Uniform ellipticity). Show directly that $F(X) = -\log \det X$ is uniformly elliptic on the set $\lambda_o I \leq X \leq \Lambda_o I$, and find admissible ellipticity constants.

Exercise B.21 (Difference class). Suppose u, v solve $F(D^2w) = 0$. Use [theorem 13.4](#) twice to prove (13.10) without assuming that F is differentiable.

Exercise B.22 (ABP geometry). Fill in the geometric proof of $B_{M/d} \subset Du(\Gamma^+)$ in (14.3), allowing u to have its maximum at more than one point.

Exercise B.23 (Oscillation decay). Starting from a one-step inequality $\text{osc}_{B_{1/2}} u \leq \vartheta \text{osc}_{B_1} u$, prove the full Hölder estimate with an explicit exponent $\alpha = -\log \vartheta / \log 2$.

Part V: mean curvature flow

Exercise B.24 (Projection identity). Derive (15.3) in index notation.

Exercise B.25 (Graph equation). Starting from the normal vector of a graph, derive both (15.6) and (15.7) and check their signs on a convex bowl.

Exercise B.26 (Critical-gradient envelopes). Starting from (17.6), prove the two relaxed-limit formulas at $p = 0$ by considering the parameter $a = |p|^2 / (\varepsilon^2 + |p|^2)$.

Exercise B.27 (Why quartic?). For the quadratic and quartic doubled-variable penalties, compute the gradient and block Hessian on the diagonal. Explain precisely why only the quartic penalty resolves the $p = 0$ comparison case.

Exercise B.28 (Shrinking cylinders). Construct a level-set function for $S^{k-1}(R(t)) \times \mathbb{R}^{n-k}$ and verify the radius law directly in the PDE.

Exercise B.29 (Gradient estimate). Verify every index in the identity used in [theorem 17.1](#). Explain why derivatives of $a_{ij}(Du)$ disappear at a maximum of $|Du|^2$.

Exercise B.30 (Extinction comparison). Let E_0 lie between two concentric balls. Use inclusion to obtain upper and lower bounds for its extinction time whenever the inner phase remains nonempty.

C

Source concordance and editorial corrections

This appendix records how the five handwritten parts were reorganized and identifies the mathematical normalizations that were checked during reconstruction.

C.1 Concordance with the handwritten course

Source	Main themes in the source	Location in this book
Part 1	motivation, first-order definitions, comparison, stability, Perron, approximation	Chapters 1–4
Part 2	deterministic control, DPP, HJB equations, Legendre duality, Hopf–Lax, metric problems, discount and cell problems	Chapters 5–7
Part 3	proper fully nonlinear equations, semijets, sup-convolution, Jensen, theorem on sums, elliptic and parabolic comparison, Bernstein estimates	Chapters 8–12
Part 4	uniform ellipticity, Pucci operators, ABP and Krylov–Safonov regularity	Chapters 13–14
Part 5	level-set mean curvature flow, singular operator, comparison, regularization, estimates, examples and singularities	Chapters 15–17
Instructor supplement	properness, semicontinuous envelopes, second-order doubling, theorem on sums, magic property and Jensen’s lemma	Chapters 8–11 and Appendix A

The course materials are available from the instructor’s Math 821 page and are cited as [22]. The five student-note files are titled *Math 821–Part 1* through *Math 821–Part 5*; the separate instructor supplement is cited as [23]. When handwriting or a compressed proof step was ambiguous, the statement was checked against the instructor’s book [21], the User’s Guide [7], and the subject-specific references listed in the bibliography.

C.2 Editorial principles

1. The conceptual sequence of the lectures was retained, but repeated calculations were consolidated.

2. Every theorem is stated with the assumptions actually used in its proof. Deep external results, such as Alexandrov's theorem and the Krylov–Safonov growth lemma, are explicitly identified and referenced rather than silently invoked.
3. Formal smooth calculations are separated from viscosity arguments. A formal calculation is never presented as a proof for a nonsmooth solution.
4. Sign conventions are fixed globally and checked against model equations: $F = -\Delta$ is proper elliptic, outward spheres have positive mean curvature, and mean curvature flow shrinks them.
5. The notation agrees as closely as possible with [21], while second-order notation follows [7, 4].

C.3 Corrections and normalizations

The following points are common sources of errors in handwritten viscosity-solution arguments. They have been written in a form that can be checked independently.

Properness.

The correct convention in this book is

$$r \leq s, \quad Y \leq X \quad \implies \quad F(x, r, p, X) \leq F(x, s, p, Y).$$

This makes F nondecreasing in r and nonincreasing in X .

The theorem on sums.

The matrix estimate is stated in the standard form

$$-(\mu^{-1} + \|A\|) I \leq \text{diag}(X_i) \leq A + \mu A^2.$$

The resolvent form involving $A(I - \kappa A)^{-1}$ is valid only when $I - \kappa A$ is positive definite. Appendix A explains the equivalence.

Mixed second derivatives.

In doubled-variable comparison, the mixed blocks of $D^2\phi(x, y)$ cannot be discarded. They are what yield $X \leq Y$.

Pucci signs.

The Pucci operators themselves are increasing. Because our proper operator F is decreasing in X , uniform ellipticity is expressed by

$$-\mathcal{P}_{\lambda, \Lambda}^+(X - Y) \leq F(X) - F(Y) \leq -\mathcal{P}_{\lambda, \Lambda}^-(X - Y).$$

Parabolic jets.

Time is first order and space is second order. In the parabolic theorem on sums the time components satisfy an additive relation, while only the spatial Hessians enter the block matrix estimate.

Mean curvature at $Du = o$.

The stable Evans–Spruck condition allows a vector η with $|\eta| \leq 1$. Equivalently, the relaxed operators are

$$F_*(o, X) = -\operatorname{tr} X + \min\{o, \lambda_{\min}(X)\},$$

and

$$F^*(o, X) = -\operatorname{tr} X + \max\{o, \lambda_{\max}(X)\}.$$

The unit-ball condition is selected by the regularized operator.

Regularization.

The uniformly parabolic approximation uses

$$I - \frac{Du \otimes Du}{\varepsilon^2 + |Du|^2},$$

not a denominator containing a square root.

Shrinking spheres.

With the sign convention of these notes,

$$R(t)^2 = R_o^2 - 2(n-1)t.$$

This formula is used as a sign check throughout the geometric part.

Fattening.

Uniqueness of the level-set function does not imply that its zero set has empty interior. Fattening concerns the sharp interface, not uniqueness of the viscosity solution.

C.4 Scope and further directions

The present text is designed to be read independently, while retaining the pace and proof strategy of the course. A later expanded edition could add full chapters on stochastic control, homogenization, L^p -viscosity solutions, Evans–Krylov theory, numerical approximation, and geometric flows beyond mean curvature.

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