

UC IRVINE SUMMER SCHOOL 2026

Homogenization and Selection Problems for Hamilton–Jacobi Equations

A four-lecture introduction

LECTURES BY

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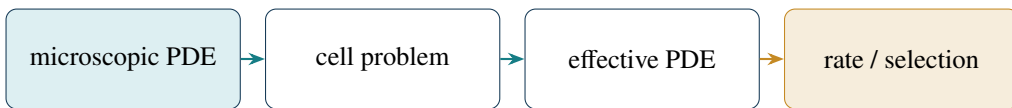
Lecture notes prepared from the recorded board lectures,
with formulas cross-checked against the cited papers.

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About these notes

These notes accompany the four-video series *Homogenization and Selection Problems for Hamilton–Jacobi Equations*, recorded at the 2026 UC Irvine Summer School on Theory and Applications of Elliptic PDE. They are a mathematical reconstruction of the lectures rather than a word-for-word transcript. The order, motivation, and proof strategy follow the board presentations; notation has been standardized, intermediate steps have been supplied, and a small number of slips corrected in context.

The intended reader knows basic nonlinear PDE and viscosity solutions but need not know homogenization. The main narrative is deliberately simple:



The first two lectures build the cell problem and then show that the vanishing viscosity selection of correctors can fail. The third treats the critical viscous homogenization problem and its sharp rates. The fourth explains the dimension-dependent first-order theory and the geometric reason convexity improves the rate to order ε .

Video links and a detailed timestamp concordance appear in [chapter B](#). The final appendix records the normalization and sign choices used throughout.

Editorial convention. The words “guiding idea” and “from the board” identify intuition emphasized in the lectures. Statements marked “open problem” are not theorems; dates and scope are given whenever that distinction matters.

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Roadmap and standing notation

The four questions

Fix a continuous Hamiltonian

$$H : \mathbb{T}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},$$

periodic in its first variable and coercive in momentum, uniformly in the fast variable:

$$\lim_{R \rightarrow \infty} \inf_{\substack{y \in \mathbb{T}^n \\ |p| \geq R}} H(y, p) = +\infty. \quad (0.1)$$

The lectures revolve around four related questions.

Q1. First-order homogenization. For

$$u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0, \quad u^\varepsilon(\cdot, 0) = g, \quad (0.2)$$

identify the limit as $\varepsilon \downarrow 0$.

Q2. Critical viscous homogenization. Do the same for

$$u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = \varepsilon \Delta u^\varepsilon, \quad u^\varepsilon(\cdot, 0) = g. \quad (0.3)$$

Q3. Vanishing-viscosity selection. If an inviscid cell problem has many correctors, does the small-viscosity approximation select one of them as $\delta \downarrow 0$?

Q4. Quantitative convergence. How large can $\|u^\varepsilon - u\|_\infty$ be, and which features of H determine the optimal exponent?

The first two problems have the same effective equation,

$$u_t + \overline{H}(Du) = 0, \quad u(\cdot, 0) = g, \quad (0.4)$$

but generally different effective Hamiltonians because their cell problems are different.

Two cell problems

For each frozen slope $p \in \mathbb{R}^n$, the first-order corrector equation is

$$H(y, p + Dv_p(y)) = \bar{H}(p) \quad \text{in } \mathbb{T}^n. \quad (0.5)$$

At the critical viscous scale, it becomes

$$H(y, p + Dv_p(y)) - \Delta v_p(y) = \bar{H}(p) \quad \text{in } \mathbb{T}^n. \quad (0.6)$$

In either case, the constant $\bar{H}(p)$ is the macroscopic quantity. The corrector v_p carries the microscopic oscillation.

Guiding idea

The cell equation is not an auxiliary afterthought. It is the mechanism that cancels the order-one oscillation when a macroscopic test function is corrected by $\varepsilon v_p(x/\varepsilon)$. This cancellation is the core of both the qualitative proof and every sharp quantitative argument in the lectures.

Rate map

The results discussed in Lectures III and IV can be organized as follows. Here “optimal” refers to the worst-case raw L^∞ error over the stated class, on a fixed macroscopic time interval.

Equation / structure	Rate	Status
First order, convex in p	$O(\varepsilon)$	Sharp, all dimensions; uniform in time
First order, general, $n = 1$	$O(\varepsilon^{1/2})$	Sharp
First order, general, $n = 2$	$O(\varepsilon^{1/3})$	Power sharp; logarithmic gap remains
First order, general, $n \geq 3$	$O(\varepsilon^{1/3})$	Sharp
Viscous critical scale, general	$O(\varepsilon^{1/2})$	Sharp over the general class
Viscous quadratic mechanical	$O(\varepsilon \log \varepsilon)$ globally	Sharp; for locally semiconcave data, $O(\varepsilon)$ at twice-differentiability points

The rates are not contradictory. They concern different structures. Convexity gives an action

representation and a bounded-error stable-norm theorem. Uniform ellipticity regularizes the family of correctors but, without additional structure, the simultaneous homogenization and vanishing-viscosity error can still be $\sqrt{\varepsilon}$. Quadratic structure provides an exact Hopf–Cole linearization and exposes a sharper Gaussian prefactor.

Notation

$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$	the unit flat torus; functions of y are $\mathbb{Z}^n$ -periodic
$y = x/\varepsilon$	fast variable
D, D^2, Δ	derivatives in the displayed variable unless a subscript is shown
$\overline{H}$	effective Hamiltonian; its definition depends on the cell problem in use
v_p	a periodic corrector at slope p
$\text{osc } f$	$\sup f - \inf f$
C	a positive constant whose value may change from line to line

Unless a theorem says otherwise, g is bounded and Lipschitz. Viscosity solutions are understood throughout. A compact review of the particular viscosity-solution tools used in the notes appears in [chapter A](#).

From microscopic motion to the cell problem

The first lecture answers a basic structural question: why does periodic homogenization reduce to an equation on one cell? We begin with moving fronts, then derive the cell equations by separating the slow and fast variables, and finish with the discounted construction of the effective Hamiltonian.

1.1 Level sets turn motion into a PDE

Let a moving hypersurface be represented as the zero level set

$$\Gamma_t = \{x : u(x, t) = 0\},$$

with $u < 0$ on one side and $u > 0$ on the other. At a regular point, the chosen unit normal is

$$v = \frac{Du}{|Du|}.$$

Suppose a trajectory $x(t) \in \Gamma_t$ moves normally with speed $a(x/\varepsilon) > 0$:

$$\dot{x}(t) = a\left(\frac{x(t)}{\varepsilon}\right) v(x(t), t).$$

Because $u(x(t), t) = 0$, the chain rule gives

$$0 = \frac{d}{dt}u(x(t), t) = u_t + Du \cdot \dot{x} = u_t + a\left(\frac{x}{\varepsilon}\right) |Du|.$$

Thus the geometric motion is encoded by the eikonal-type equation

$$u_t^\varepsilon + a\left(\frac{x}{\varepsilon}\right) |Du^\varepsilon| = 0. \tag{1.1}$$

This derivation is robust: the zero set is unchanged if u is replaced by any strictly increasing function of u . That invariance is one reason Hamilton–Jacobi equations are the natural language for front propagation.

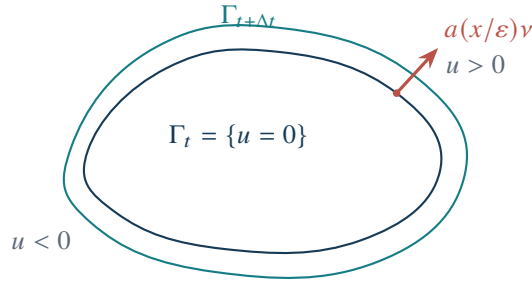


Figure 1.1: The level-set function converts a normal front velocity into a Hamilton–Jacobi equation. The function remembers the front even when the microscopic interface becomes highly wrinkled.

1.1.1 Curvature and degenerate ellipticity

Add a curvature correction at the same scale as the microscopic period. With mean curvature

$$\kappa = \operatorname{div} \left(\frac{Du}{|Du|} \right)$$

and normal velocity $(a(x/\varepsilon) - \varepsilon\kappa)\nu$, the level-set equation becomes

$$u_t^\varepsilon + a\left(\frac{x}{\varepsilon}\right) |Du^\varepsilon| - \varepsilon |Du^\varepsilon| \operatorname{div} \left(\frac{Du^\varepsilon}{|Du^\varepsilon|} \right) = 0. \quad (1.2)$$

For $p \neq 0$,

$$|Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) = \operatorname{tr} \left[\left(I - \frac{Du \otimes Du}{|Du|^2} \right) D^2 u \right]. \quad (1.3)$$

The matrix

$$A(p) = I - \frac{p \otimes p}{|p|^2}$$

is nonnegative but has a zero eigenvalue in the p -direction. The diffusion acts tangentially to the level set and not normally to it. The equation is therefore degenerate elliptic, not uniformly parabolic.

Guiding idea

The factor ε in front of curvature is critical. A spatial oscillation of wavelength ε has second derivative of size ε^{-1} after it is placed in a corrector of amplitude ε . Consequently, εD^2 survives at order one, exactly alongside the Hamiltonian.

One may also add a periodic incompressible drift W :

$$u_t^\varepsilon + \left(a\left(\frac{x}{\varepsilon}\right) - \varepsilon \operatorname{div} \frac{Du^\varepsilon}{|Du^\varepsilon|} \right) |Du^\varepsilon| + W\left(\frac{x}{\varepsilon}\right) \cdot Du^\varepsilon = 0.$$

This is a geometric relative of the G -equation used in turbulent combustion. The lectures use it as motivation rather than pursue its homogenization theory.

1.2 Two scales and one effective slope

We now turn to the first-order Cauchy problem

$$u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0, \quad u^\varepsilon(\cdot, 0) = g. \quad (1.4)$$

Write $y = x/\varepsilon$ and formally seek

$$u^\varepsilon(x, t) = u_0(x, y, t) + \varepsilon u_1(x, y, t) + \varepsilon^2 u_2(x, y, t) + \cdots, \quad (1.5)$$

with each u_j periodic in y . The total spatial derivative is

$$D_x^{\text{tot}} = D_x + \frac{1}{\varepsilon} D_y.$$

The leading momentum contains $\varepsilon^{-1} D_y u_0$. Uniform coercivity (0.1) prevents this from remaining bounded unless

$$D_y u_0 = 0.$$

Thus the leading profile is macroscopic: $u_0 = u_0(x, t)$.

At order one, the expansion yields

$$(u_0)_t + H(y, D_x u_0 + D_y u_1) = 0. \quad (1.6)$$

Freeze (x, t) and set

$$p = D_x u_0(x, t), \quad c = -(u_0)_t(x, t), \quad v(y) = u_1(x, y, t).$$

Then (1.6) becomes the cell problem

$$H(y, p + Dv(y)) = c \quad \text{in } \mathbb{T}^n. \quad (1.7)$$

Only one value of c admits a periodic solution. Calling it $\bar{H}(p)$ gives the effective equation

$$(u_0)_t + \bar{H}(Du_0) = 0.$$

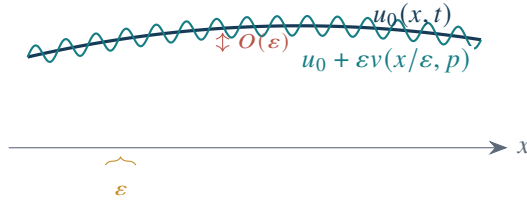


Figure 1.2: The corrector has microscopic wavelength and macroscopic amplitude $O(\varepsilon)$. Its gradient is order one, so it can cancel the oscillation in $H(x/\varepsilon, Du^\varepsilon)$.

Point to keep straight

The expansion is a derivation, not yet a proof of homogenization. The effective solution is only Lipschitz in general, so $v(x/\varepsilon, Du)$ cannot be differentiated classically. The rigorous replacement is the perturbed-test-function method in [section 3.2](#).

1.3 The periodic cell theorem

The basic existence theorem goes back to Lions, Papanicolaou, and Varadhan [9].

Theorem 1.1 (Periodic cell problem). *Assume $H \in C(\mathbb{T}^n \times \mathbb{R}^n)$ and the uniform coercivity (0.1). For every $p \in \mathbb{R}^n$, there is a unique constant $c \in \mathbb{R}$ for which*

$$H(y, p + Dv) = c \quad \text{in } \mathbb{T}^n$$

admits a periodic viscosity solution. We define $\bar{H}(p) = c$.

The constant is unique; the corrector need not be unique, even after quotienting by additive constants.

The existence argument uses the discounted equation

$$\lambda v^\lambda + H(y, p + Dv^\lambda) = 0 \quad \text{in } \mathbb{T}^n, \quad \lambda > 0. \quad (1.8)$$

The zero-order term makes comparison strict and gives a unique periodic solution.

1.3.1 Uniform estimates and compactness

Let $C_0 = \|H(\cdot, p)\|_\infty$. The constant functions $\pm C_0/\lambda$ are barriers, so comparison yields

$$\|\lambda v^\lambda\|_{L^\infty(\mathbb{T}^n)} \leq C_0. \quad (1.9)$$

Coercivity then gives a Lipschitz estimate

$$\|Dv^\lambda\|_{L^\infty(\mathbb{T}^n)} \leq C(p), \quad (1.10)$$

independent of small λ . Normalize

$$w^\lambda(y) = v^\lambda(y) - v^\lambda(0).$$

The family $\{w^\lambda\}$ is uniformly bounded and equi-Lipschitz. Along a sequence $\lambda_j \downarrow 0$,

$$w^{\lambda_j} \longrightarrow v \quad \text{uniformly,} \quad -\lambda_j v^{\lambda_j}(0) \longrightarrow c.$$

Rewrite (1.8) as

$$\lambda_j w^{\lambda_j} + H(y, p + Dw^{\lambda_j}) = -\lambda_j v^{\lambda_j}(0).$$

Stability of viscosity solutions gives $H(y, p + Dv) = c$.

1.3.2 Why the ergodic constant is unique

Suppose v_i solve the cell equation with constants c_i and $c_1 < c_2$. Choose a constant K , and then choose $\delta > 0$ sufficiently small that

$$\delta(v_1 + K) + c_1 < \delta v_2 + c_2 \quad \text{on } \mathbb{T}^n.$$

Then $v_1 + K$ is a strict subsolution and v_2 a supersolution of the same discounted equation. Comparison implies $v_1 + K \leq v_2$. Taking K larger than $\max(v_2 - v_1)$ gives a contradiction. Hence $c_1 = c_2$.

From the board

The proof is worth remembering as a template: use a small discount to compare objects that are only defined up to constants, and exploit the freedom to shift one corrector. The discount breaks the additive invariance just long enough to force uniqueness of the constant.

1.4 Why the viscous cell problem is different

For the critical equation

$$u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = \varepsilon \Delta u^\varepsilon, \tag{1.11}$$

insert the same first corrector $\varepsilon u_1(x, x/\varepsilon, t)$. Two fast derivatives give

$$\varepsilon \Delta_x(\varepsilon u_1(x, x/\varepsilon, t)) = \Delta_y u_1 + O(\varepsilon).$$

The order-one equation is therefore

$$(u_0)_t + H(y, D_x u_0 + D_y u_1) - \Delta_y u_1 = 0,$$

and the cell equation becomes

$$H(y, p + Dv_p) - \Delta v_p = \bar{H}(p) \quad \text{in } \mathbb{T}^n. \quad (1.12)$$

Uniform ellipticity gives much more rigidity than in (1.7): after a normalization, the corrector is unique, and its dependence on p is regular under standard smoothness assumptions on H . This difference drives the quantitative theory of Lecture III.

Lecture summary

- A moving periodic front becomes a Hamilton–Jacobi equation through the level-set chain rule.
- Coercivity forces the leading two-scale profile to be independent of the fast variable.
- Freezing the macroscopic slope produces the cell problem and defines $\bar{H}$.
- Discounted equations give existence; a second use of discount gives uniqueness of the ergodic constant.
- At the scale $\varepsilon\Delta$, diffusion enters the cell problem at order one and regularizes the corrector family.

Questions to test understanding

1. In the formal expansion (1.5), locate precisely where coercivity is used. What could go wrong for a noncoercive Hamiltonian?
2. Verify (1.3) and identify all eigenvalues of $A(p)$.
3. For $H(y, p) = a(y)|p|$ in one dimension, write the cell equation and derive the usual harmonic-mean formula for the effective speed on a monotone branch.
4. Explain why uniqueness of $\bar{H}(p)$ does not imply uniqueness of v_p .

Corrector selection and dancing wells

The cell theorem determines $\overline{H}(p)$ uniquely, but it need not determine a unique corrector. Lecture II asks whether a small viscosity resolves this ambiguity. In a one-dimensional mechanical model the answer is, surprisingly, no: two wells can exchange dominance infinitely often as viscosity vanishes.

2.1 Mechanical Hamiltonians and flat effective energies

Consider

$$H(y, p) = \frac{1}{2}|p|^2 - V(y), \quad V \geq 0, \quad (2.1)$$

with V periodic. The inviscid cell problem is

$$\frac{1}{2}|p + Dv_p|^2 - V(y) = \overline{H}(p). \quad (2.2)$$

The effective Hamiltonian is continuous and convex because the microscopic Hamiltonian is convex in p . Strict convexity, however, can be lost: $\overline{H}$ may have a flat minimum. The flat region records the fact that several microscopic calibrations can have the same long-time cost.

There is also a useful min–max characterization,

$$\overline{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{y \in \mathbb{T}^n} H(y, p + D\phi(y)). \quad (2.3)$$

It makes convexity of $\overline{H}$ transparent, while also warning us that a minimizing corrector need not vary continuously with p .

2.1.1 Two wells in one dimension

Take $n = 1$, $p = 0$, and let V vanish exactly at 0 and $a = \frac{1}{2}$ on $\mathbb{T}$. Then $\overline{H}(0) = 0$, and

$$\frac{1}{2}(v')^2 - V = 0, \quad \text{hence} \quad v' = \pm\sqrt{2V}. \quad (2.4)$$

Periodicity imposes $\int_0^1 v' = 0$, but it does not prescribe the sign branch pointwise. In the symmetric case, one may choose the positive branch between the wells and the negative branch

after the second well, or reverse these choices. The resulting tent and inverted-tent profiles are not translates by a constant.

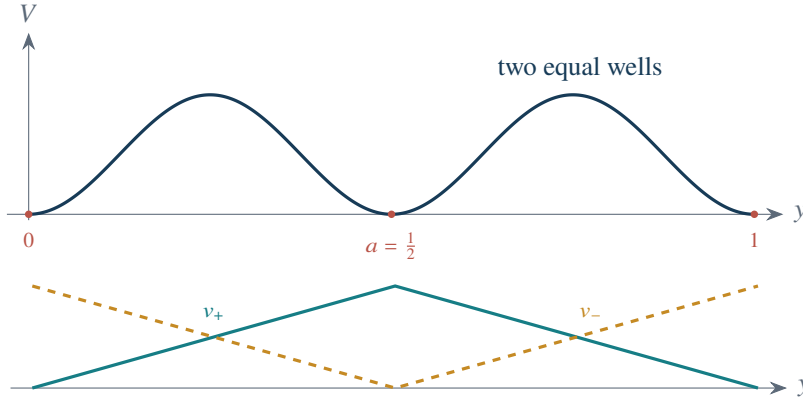


Figure 2.1: Equal wells permit two distinct branch choices in (2.4). The drawing is schematic; the slopes are $\pm\sqrt{2V(y)}$, not constant.

The wells are the places where a branch switch from $-\sqrt{2V}$ to $+\sqrt{2V}$ can occur without violating the viscosity equation. A switch in the opposite direction may form a downward corner even where $V > 0$. The two smooth branch choices drawn above switch at the wells, which form the relevant projected Aubry set. This example already contains the selection problem: which of these correctors should a vanishing viscosity choose?

2.2 The vanishing-viscosity cell problem

To separate the viscosity parameter from the homogenization scale, write it as δ :

$$H(y, p + Dv_p^\delta) - \delta \Delta v_p^\delta = \bar{H}^\delta(p), \quad \delta \downarrow 0. \quad (2.5)$$

For each $\delta > 0$, ellipticity gives a unique normalized corrector. Compactness produces inviscid subsequential limits. The selection question asks whether the entire normalized family converges and, if so, how the limit is characterized.

For (2.1) in one dimension at $p = 0$,

$$\frac{1}{2}(\phi'_\delta)^2 - V(y) - \delta \phi''_\delta = c(\delta), \quad \phi_\delta(0) = 0. \quad (2.6)$$

This is related to the steady forced viscous Burgers equation. Indeed, if $w = \phi'_\delta$, then

$$\left(\frac{1}{2}w^2 - \delta w_x - V \right)_x = 0.$$

The issue is selection among periodic steady states, not entropy selection for a Cauchy problem; this is the Burgers/cell-problem connection behind the question of Jauslin, Kreiss, and Moser [8].

Theorem 2.1 (Nonexistence of a vanishing-viscosity limit [10]). *There exists $F \in C^3(\mathbb{T})$ such that the normalized solutions of*

$$\frac{1}{2}|D\phi^\delta|^2 + F(x) - \delta\Delta\phi^\delta = c(\delta), \quad \phi^\delta(0) = 0, \quad (2.7)$$

do not converge in $C(\mathbb{T})$ as $\delta \downarrow 0$.

This is the result of Liu, Tran, and Yu discussed in Lecture II. Their paper also explains how to obtain any prescribed finite regularity C^k by increasing the order of the oscillatory perturbation [10]. The stated theorem does not assert an analytic counterexample.

2.3 Hopf–Cole turns selection into localization

Set $F = -V$ in (2.7) and define

$$U_\delta(x) = \exp\left(-\frac{\phi^\delta(x)}{2\delta}\right). \quad (2.8)$$

A direct calculation gives

$$\left(-2\delta^2 \frac{d^2}{dx^2} + V(x)\right)U_\delta = -c(\delta)U_\delta. \quad (2.9)$$

Thus $-c(\delta)$ is the principal eigenvalue of a periodic semiclassical Schrödinger operator, and U_δ is its positive ground state. Corrector selection has become a ground-state localization problem.

Guiding idea

The logarithm magnifies localization. If the ground state is exponentially larger near one well than another, then $\phi^\delta = -2\delta \log U_\delta$ records an order-one preference. To destroy selection, it suffices to make the preferred well alternate along two sequences $\delta_j \downarrow 0$.

2.4 The dancing-well construction

Let $a = \frac{1}{2}$ and, for small $r > 0$, define the slowly oscillating function

$$\omega(r) = \sin(\log \log(1/r)). \quad (2.10)$$

Without loss of generality take $A > 0$, and construct a nonnegative periodic potential V , positive away from exactly two zeros, such that near the two wells

$$V(x) = \frac{1}{2}x^2 + Ax^4\omega(|x|), \quad x \approx 0, \quad (2.11)$$

$$V(x) = \frac{1}{2}(x-a)^2 - A(x-a)^4\omega(|x-a|), \quad x \approx a. \quad (2.12)$$

The wells have the same quadratic part and opposite quartic perturbations. Because ω oscillates arbitrarily slowly, the perturbation looks almost constant on the natural ground-state scale $|x| \sim \sqrt{\delta}$.

Let $E_0^D(\delta)$ and $E_a^D(\delta)$ denote local Dirichlet ground energies in small neighborhoods of the two wells. Harmonic approximation gives

$$E_0^D(\delta) = \delta + 3A\omega(\sqrt{\delta})\delta^2 + o(\delta^2), \quad (2.13)$$

$$E_a^D(\delta) = \delta - 3A\omega(\sqrt{\delta})\delta^2 + o(\delta^2). \quad (2.14)$$

Here is the scaling behind the coefficient. Near 0, put $x = \sqrt{\delta}z$. The local operator is

$$\delta \left[-2 \frac{d^2}{dz^2} + \frac{1}{2}z^2 + \delta A z^4 \omega(\sqrt{\delta}|z|) \right].$$

The unperturbed normalized ground state

$$\psi_0(z) = (2\pi)^{-1/4} e^{-z^2/4}$$

has energy 1 and fourth moment $\int_{\mathbb{R}} z^4 \psi_0(z)^2 dz = 3$. Slow variation of ω then yields (2.13); the sign reversal gives (2.14).

Choose the explicit sequences

$$\delta_n^+ = \exp \left[-2 \exp \left(\frac{\pi}{2} + 2\pi n \right) \right], \quad \omega(\sqrt{\delta_n^+}) = 1, \quad (2.15)$$

$$\delta_n^- = \exp \left[-2 \exp \left(\frac{3\pi}{2} + 2\pi n \right) \right], \quad \omega(\sqrt{\delta_n^-}) = -1. \quad (2.16)$$

Along one sequence the well at a has lower local energy; along the other, the well at 0 does.

Semiclassical localization shows that a gap of order δ^2 dominates the exponentially small tunnelling between the two wells. Since the normalization $\phi^\delta(0) = 0$ fixes $U_\delta(0) = 1$ rather than its L^2 norm, set

$$\widehat{U}_\delta = \frac{U_\delta}{\|U_\delta\|_{L^2(\mathbb{T})}}.$$

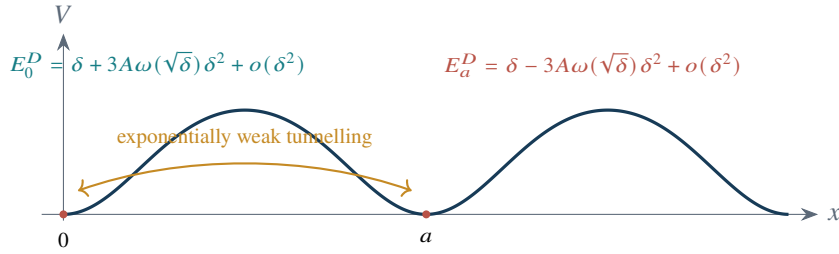


Figure 2.2: The wells agree to quadratic order, while their small energy corrections oscillate with opposite signs. A polynomial energy gap dominates the exponentially small tunnelling between separated wells.

Then the normalized ground-state measures satisfy

$$(\widehat{U}_{\delta_n^+})^2 dx \rightarrow \delta_a, \quad (\widehat{U}_{\delta_n^-})^2 dx \rightarrow \delta_0. \quad (2.17)$$

Since

$$\phi^\delta(x) = -2\delta \log \frac{U_\delta(x)}{U_\delta(0)},$$

the values at a have opposite limiting behavior along the two subsequences. The normalized correctors therefore have no full limit.

2.4.1 Arbitrary finite regularity

Replace the perturbation $x^4\omega(|x|)$ by $x^{2m}\omega(|x|)$. Then the potential can be made C^{2m-1} , and

$$\begin{aligned} E_0^D(\delta) &= \delta + c_m A\omega(\sqrt{\delta})\delta^m + o(\delta^m), \\ E_a^D(\delta) &= \delta - c_m A\omega(\sqrt{\delta})\delta^m + o(\delta^m), \end{aligned}$$

where $c_m = (2m - 1)!!$. As explained in the cited paper, the localization argument can be run for each fixed m , so any prescribed finite smoothness is compatible with nonselection. Extending the mechanism to C^∞ potentials was suggested informally in the lecture but is not part of the stated theorem; the analytic case is more rigid and was left as a separate question.

2.5 What the counterexample teaches

The mechanism separates three notions that are easy to conflate.

1. The ergodic constant $c(\delta)$ is unique for every $\delta > 0$.
2. The inviscid ergodic constant $\overline{H}(0)$ is also unique.
3. The normalized eigenfunction, and hence the normalized corrector, need not have a unique

zero-viscosity limit.

Uniqueness of an eigenvalue is not uniqueness of its singular-limit profile. The counterexample is deterministic and finely tuned. Lecture II closes by asking whether random space–time forcing with finite-range dependence might restore almost-sure selection. That is presented as a conjectural direction, not as a theorem.

Random forcing and generic selection

For a one-space, one-time stationary random potential with short-range dependence, determine whether the normalized small-viscosity solutions converge almost surely. If they do, identify the mechanism that suppresses dancing-well instability; if not, find a robust random analogue of the alternating localization construction.

Lecture summary

- Mechanical cell problems can have many correctors even though the effective constant is unique.
- Hopf–Cole identifies the quadratic viscous cell problem with a semiclassical principal-eigenfunction problem.
- Two equal wells with opposite, slowly oscillating higher-order terms exchange energetic dominance infinitely often.
- Localization converts that exchange into nonconvergence of normalized correctors.

Questions to test understanding

1. Check the Hopf–Cole computation from (2.6) to (2.9), including the factor 2δ .
2. Why must the two wells agree at quadratic order for the oscillating quartic term to decide the preferred well?
3. Verify the identities in (2.15)–(2.16).
4. Explain why a polynomial energy gap can dominate exponentially small tunnelling even when the polynomial coefficient tends to zero slowly.

Critical viscous homogenization and sharp rates

Lecture III returns to the Cauchy problem

$$u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = \varepsilon \Delta u^\varepsilon, \quad u^\varepsilon(\cdot, 0) = g. \quad (3.1)$$

The first goal is qualitative homogenization. The second is quantitative: why is the general rate $\sqrt{\varepsilon}$, and why does a quadratic Hamiltonian improve it to $\varepsilon |\log \varepsilon|$?

3.1 The viscous corrector family

For each slope $p \in \mathbb{R}^n$, solve

$$H(y, p + D_y v(y, p)) - \Delta_y v(y, p) = \bar{H}(p) \quad \text{on } \mathbb{T}^n. \quad (3.2)$$

The corrector is unique modulo constants. Fixing, for example, $v(0, p) = 0$ makes it unique. Under the regularity assumptions used in the quantitative theorem, v depends locally Lipschitz-continuously on p :

$$\|v(\cdot, p) - v(\cdot, q)\|_{L^\infty(\mathbb{T}^n)} \leq C_R |p - q|, \quad |p|, |q| \leq R. \quad (3.3)$$

This is the decisive second-order advantage. For first-order cell equations in dimension at least two, a continuous global selection may not exist.

The formal local approximation is

$$u^\varepsilon(x, t) \approx u(x, t) + \varepsilon v\left(\frac{x}{\varepsilon}, Du(x, t)\right). \quad (3.4)$$

It is local because the parameter $p = Du(x, t)$ changes from point to point. The real work lies in gluing these local microscopic profiles without differentiating the generally nonsmooth effective solution.

3.2 Perturbed test functions

We give the qualitative argument because it is the cleanest expression of the cell problem's role. This is Evans's perturbed-test-function method [4, 5]. Define the upper and lower half-relaxed

limits

$$\begin{aligned}\bar{u}(x, t) &= \limsup_{\substack{\varepsilon \rightarrow 0 \\ (y, s) \rightarrow (x, t)}}^* u^\varepsilon(y, s), \\ \underline{u}(x, t) &= \liminf_{\substack{\varepsilon \rightarrow 0 \\ (y, s) \rightarrow (x, t)}}^* u^\varepsilon(y, s).\end{aligned}$$

We show that $\bar{u}$ is a subsolution of $u_t + \bar{H}(Du) = 0$. The supersolution argument for $\underline{u}$ is analogous.

Let $\phi \in C^2$ touch $\bar{u}$ from above at a strict local maximum (x_0, t_0) , and set $p = D\phi(x_0, t_0)$. Insert the cell oscillation into the test function:

$$\phi^\varepsilon(x, t) = \phi(x, t) + \varepsilon v\left(\frac{x}{\varepsilon}, p\right). \quad (3.5)$$

By the definition of the upper relaxed limit, $u^\varepsilon - \phi^\varepsilon$ has nearby maxima $(x_\varepsilon, t_\varepsilon) \rightarrow (x_0, t_0)$. The viscosity subsolution inequality gives

$$\begin{aligned}\phi_t(x_\varepsilon, t_\varepsilon) + H\left(\frac{x_\varepsilon}{\varepsilon}, D\phi(x_\varepsilon, t_\varepsilon) + D_y v\left(\frac{x_\varepsilon}{\varepsilon}, p\right)\right) \\ \leq \varepsilon \Delta \phi(x_\varepsilon, t_\varepsilon) + \Delta_y v\left(\frac{x_\varepsilon}{\varepsilon}, p\right).\end{aligned} \quad (3.6)$$

Since $D\phi(x_\varepsilon, t_\varepsilon) \rightarrow p$, continuity of H and the cell equation cancel the fast terms. Sending $\varepsilon \rightarrow 0$ yields

$$\phi_t(x_0, t_0) + \bar{H}(p) \leq 0.$$

Thus $\bar{u}$ is a subsolution. Comparison for the effective equation gives $\bar{u} \leq \underline{u}$; the reverse inequality is automatic, so both equal the unique effective solution u .

Guiding idea

The perturbed test function supplies exactly the microscopic derivative that a macroscopic test function lacks. The identity

$$\varepsilon \Delta_x [\varepsilon v(x/\varepsilon, p)] = \Delta_y v(x/\varepsilon, p)$$

is the critical-scale cancellation. No classical differentiability of u^ε is used.

3.3 The general $\sqrt{\varepsilon}$ theorem

The qualitative proof freezes the momentum at a single contact point. A rate requires a quantitative way to compare two different points and two nearby momenta.

Theorem 3.1 (Qian–Sprekeler–Tran–Yu [12]). *Under the periodicity, quantitative coercivity,*

and local Lipschitz assumptions of the general viscous theory, with Lipschitz initial data g , the solutions of (3.1) and the effective equation satisfy, for every $T > 0$,

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0, T])} \leq C(1 + T)\sqrt{\varepsilon}. \quad (3.7)$$

No convexity in p is required. The exponent $1/2$ is optimal for this general class.

The sharp lower example may have no genuine spatial oscillation in H ; its error comes from smoothing a Lipschitz kink by small viscosity. Thus theorem 3.1 is a worst-case theorem for the full class, not a claim that every oscillatory medium produces a $\sqrt{\varepsilon}$ error.

The lecture also compares this theorem with forced graphical mean-curvature homogenization. An analogous $O(\sqrt{\varepsilon})$ estimate is available in an appropriate regime, while more specialized results give an $\varepsilon^{1/8}$ -type rate under additional hypotheses. Neither is presented as the optimal rate; indeed, outside suitable regimes even homogenization can fail. The sharp quantitative theory for this degenerate geometric equation remains open.

3.3.1 Why the naive two-scale approximation is illegal

If u were smooth, one might insert (3.4) directly. But differentiating it twice introduces D^2u and D^3u , while the effective solution generally develops shocks and is only Lipschitz. The quantitative proof instead creates the needed slopes by penalization.

One convenient tripled-variable auxiliary function is

$$\begin{aligned} \Phi(x, y, z, t, s) = & u^\varepsilon(x, t) - u(y, s) - \varepsilon v\left(\frac{x}{\varepsilon}, \frac{z - y}{\sqrt{\varepsilon}}\right) \\ & - \frac{|x - y|^2 + |x - z|^2 + |t - s|^2}{2\sqrt{\varepsilon}} - K(t + s) - \gamma\sqrt{1 + |x|^2}. \end{aligned} \quad (3.8)$$

The final term localizes the maximum; K is chosen at the small scale required by the comparison. At a maximizer $(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{s})$, the penalization yields

$$|\hat{x} - \hat{z}| \leq C\varepsilon, \quad (3.9)$$

$$|\hat{x} - \hat{y}| + |\hat{y} - \hat{z}| + |\hat{t} - \hat{s}| \leq C\sqrt{\varepsilon}. \quad (3.10)$$

The momentum supplied to the corrector is

$$p_\varepsilon = \frac{\hat{z} - \hat{y}}{\sqrt{\varepsilon}},$$

while the viscosity inequalities produce a nearby slope. Their difference is $O(\sqrt{\varepsilon})$, and the Lipschitz dependence (3.3) absorbs it. The competing localization and parameter errors balance at scale $\sqrt{\varepsilon}$.

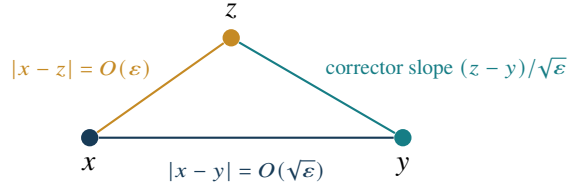


Figure 3.1: The third spatial variable decouples the slope fed to the corrector from the contact point of u^ε . This is the geometric content of the tripling-variable method.

3.4 Quadratic mechanics and the logarithmic rate

Now impose the special structure

$$u_t^\varepsilon + \frac{1}{2}|Du^\varepsilon|^2 + V\left(\frac{x}{\varepsilon}\right) = \frac{\varepsilon}{2}\Delta u^\varepsilon. \quad (3.11)$$

The factors $1/2$ on both the Hamiltonian and diffusion are important: they make the Hopf–Cole transform exact.

Theorem 3.2 (Liu–Tran–Yu [11]). *For (3.11) with periodic $V \in \text{Lip}(\mathbb{T}^n)$ and $g \in W^{1,\infty}(\mathbb{R}^n)$,*

$$|u^\varepsilon(x, t) - u(x, t)| \leq \varepsilon \left(C + \frac{n}{2} \log \frac{\max\{t, \varepsilon\}}{\varepsilon} \right). \quad (3.12)$$

The global order $\varepsilon |\log \varepsilon|$ on fixed positive time intervals is sharp. If, in addition, g is locally semiconcave, then at every point where $u(\cdot, t)$ is twice differentiable the error is $O(\varepsilon)$; consequently this pointwise improvement holds almost everywhere.

3.4.1 Hopf–Cole and the ballistic kernel

Introduce microscopic variables and set

$$u^\varepsilon(x, t) = -\varepsilon \log w\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right). \quad (3.13)$$

Then w solves the linear periodic equation

$$w_s - \frac{1}{2}\Delta_y w - V(y)w = 0, \quad w(y, 0) = \exp\left(-\frac{g(\varepsilon y)}{\varepsilon}\right). \quad (3.14)$$

Let $K(s, x, y)$ denote its positive kernel. For large time s and ballistic displacement $-qs$, a periodic Bloch–Floquet/Doob-transform analysis yields

$$C_1 s^{-n/2} e^{-s\bar{L}(q)} \leq K(s, 0, -qs) \leq C_2 s^{-n/2} e^{-s\bar{L}(q)}, \quad (3.15)$$

uniformly for q in compact sets. The exponential rate $\bar{L}$ is the Legendre transform of $\bar{H}$.

At $(x, t) = (0, 1)$, for example,

$$u^\varepsilon(0, 1) = -\varepsilon \log \left[\varepsilon^{-n} \int_{\mathbb{R}^n} K\left(\frac{1}{\varepsilon}, 0, \frac{y}{\varepsilon}\right) e^{-g(y)/\varepsilon} dy \right]. \quad (3.16)$$

The exponential in (3.15) gives the effective Hopf–Lax action. The prefactor $s^{-n/2}$ contributes

$$-\varepsilon \log s^{-n/2} = \frac{n}{2} \varepsilon \log s, \quad s \sim \frac{t}{\varepsilon},$$

which is exactly the logarithm in (3.12).

Guiding idea

The logarithm is not a defect of the proof. It is the logarithm of the Gaussian normalization in a ballistic heat kernel. Under locally semiconcave data, local second-order information at a twice-differentiability point improves the error to $O(\varepsilon)$.

3.5 Beyond the quadratic transform

The lecture ends with the one-dimensional quartic model

$$u_t^\varepsilon + \frac{1}{4} |u_x^\varepsilon|^4 - V\left(\frac{x}{\varepsilon}\right) = \varepsilon u_{xx}^\varepsilon, \quad (3.17)$$

where the sign of V is a convention. A scalar point transform cannot cancel a quartic gradient term: the chain rule for a Laplacian creates a quadratic gradient correction, which is precisely why Hopf–Cole is special to quadratic Hamiltonians.

Numerical and scaling evidence discussed in the lecture suggests an $\varepsilon^{2/3}$ law for quartic kinetics in relevant regimes. A nontrivial periodic potential may change the effective local convexity and the observed rate, but whether it does so is part of the open problem. The general strictly convex, nonquadratic periodic theory remains open.

A separate model emphasized at the board keeps the critical periodic viscous setting but replaces the quadratic kinetic energy $|p|^2/2$ by a nonquadratic, uniformly convex function $K(p)$. The lecture again proposes $\varepsilon^{2/3}$ as the rate to test. Here the challenge is not degeneracy of D^2K but the loss of the exact linear Hopf–Cole representation.

A nonlinear replacement for Hopf–Cole

For smooth nonconstant periodic V in (3.17), determine the sharp global rate and the regular-point rate. A successful method should explain both the conjectured homogeneous quartic $\varepsilon^{2/3}$ scaling and whether periodic cell dynamics change that law. For the uniformly convex nonquadratic model, seek a parallel sharp theorem. Promising routes emphasized in the lecture are an SDE/Legendre-drift representation and, ultimately, a direct PDE argument that does not depend on an exact transform.

For the homogeneous Hamiltonian $H(p) = |p|^\gamma/\gamma$, a rarefaction through $p = 0$ suggests the scale

$$\varepsilon^{\gamma/[2(\gamma-1)]}. \quad (3.18)$$

Indeed, $p \sim (x/t)^{1/(\gamma-1)}$ near the center, while diffusion smooths on $x \sim \sqrt{\varepsilon}$; integrating the resulting gradient scale gives (3.18). For $\gamma = 4$ this is $\varepsilon^{2/3}$. This is a heuristic, not a theorem for the periodic problem.

Lecture summary

- The perturbed-test-function proof turns the viscous cell equation into an exact cancellation at a contact point.
- Quantitative gluing uses three spatial variables; the optimal balance in the general class is $\sqrt{\varepsilon}$.
- Quadratic mechanics linearizes under Hopf–Cole. A ballistic Gaussian prefactor produces the sharp $\varepsilon |\log \varepsilon|$ global error.
- Nonquadratic strictly convex kinetics lack an exact linearization and form a central open direction.

Questions to test understanding

1. Carry out the derivatives in (3.5) and verify every term in (3.6).
2. Why is a constant momentum $p = D\phi(x_0, t_0)$ used in the corrector, rather than $D\phi(x, t)$?
3. Starting from (3.11), derive (3.14).
4. Explain in one sentence why a scalar Hopf–Cole transform cannot linearize a quartic Hamiltonian.

First-order rates: dimension, convexity, and geometry

The fourth lecture studies the first-order equation

$$u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0, \quad u^\varepsilon(\cdot, 0) = g. \quad (4.1)$$

Two phenomena compete. For a general nonconvex Hamiltonian, correctors cannot be selected regularly in momentum and the worst rate is dimension-dependent. For a convex Hamiltonian, the equation has an optimal-control representation and a geometric curve-cutting argument improves the rate all the way to $O(\varepsilon)$.

4.1 Why first-order correctors are harder

The cell equation is

$$H(y, p + Dv(y, p)) = \overline{H}(p). \quad (4.2)$$

Correctors are merely Lipschitz, may be nonunique, and in dimension $n \geq 2$ may admit no continuous selection $p \mapsto v(\cdot, p)$, even after normalization. The direct analogue of the viscous tripling argument therefore lacks the parameter estimate (3.3).

The classical workaround uses discounted correctors:

$$\lambda v^\lambda + H(y, p + Dv^\lambda) = 0. \quad (4.3)$$

In a schematic quantitative form,

$$|\lambda v^\lambda(y, p) + \overline{H}(p)| \leq C\lambda, \quad (4.4)$$

$$\|v^\lambda(\cdot, p) - v^\lambda(\cdot, q)\|_\infty \leq \frac{C}{\lambda} |p - q|. \quad (4.5)$$

The discount creates a canonical selection but makes its momentum modulus blow up like λ^{-1} . Balancing the discount error, the momentum error, and the doubling-variable localization leads to the scale $\lambda \sim \varepsilon^{1/3}$.

Theorem 4.1 (Capuzzo-Dolcetta–Ishii [2]). *For a general periodic coercive Hamiltonian*

under the standard quantitative regularity assumptions,

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0, T])} \leq C_T \varepsilon^{1/3}. \quad (4.6)$$

For many years, the exponent $1/3$ was viewed mainly as the outcome of a proof balance. The recent theory discussed next shows that, in high dimension, it is also a genuine geometric obstruction.

4.2 The dimension-dependent nonconvex theorem

Theorem 4.2 (Jang–Sun–Tran–Yu [6]). *For Lipschitz, periodic, coercive Hamiltonians and Lipschitz data, the sharp worst-case first-order rates have the following form.*

- (i) *If $n = 1$, then $\|u^\varepsilon - u\|_\infty \leq C\varepsilon^{1/2}$, and the exponent is sharp.*
- (ii) *If $n \geq 3$, there are fixed examples with*

$$c\varepsilon^{1/3} \leq \|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0, 1])} \leq C\varepsilon^{1/3}.$$

- (iii) *If $n = 2$, the universal upper bound is $C\varepsilon^{1/3}$, and there is an example with*

$$\|u^\varepsilon - u\|_\infty \geq c \frac{\varepsilon^{1/3}}{\log^{4/3}(e + \varepsilon^{-1})}. \quad (4.7)$$

Thus the algebraic exponent is sharp, while the exact logarithmic modulus is open.

The one-dimensional improvement comes from order. One can choose normalized correctors so that, for $p < q$,

$$p + v_y(y, p) \leq q + v_y(y, q) \quad \text{a.e.}, \quad \|v(\cdot, p) - v(\cdot, q)\|_\infty \leq |p - q|. \quad (4.8)$$

The second inequality is the missing parameter regularity, recovered without ellipticity because one-dimensional slopes are ordered.

4.2.1 Where $1/3$ comes from geometrically

The high-dimensional lower construction uses two interlacing polygonal momentum structures at a small scale p_j . The associated support functions differ by about

$$p_j^2 |X|,$$

while the region in which the relevant branch remains active has size $|X| \lesssim p_j t$. Compatibility persists until

$$p_j^3 t \asymp 1. \quad (4.9)$$

Thus $p_j \asymp t^{-1/3}$ and the fast-time discrepancy grows like

$$p_j t \asymp t^{2/3}.$$

Homogenization at physical scale ε corresponds to fast time $t \asymp \varepsilon^{-1}$. Multiplying the discrepancy by ε gives

$$\varepsilon(\varepsilon^{-1})^{2/3} = \varepsilon^{1/3}.$$

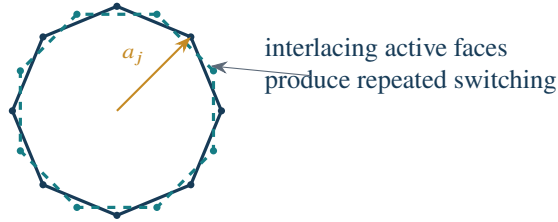


Figure 4.1: Schematic momentum geometry behind the nonconvex lower construction. The polygonal radius is $a_j = \frac{1}{4} + p_j$; the small parameter $p_j \asymp N_j^{-1}$ measures the radial/angular scale. The rigorous examples use a multiscale family; the two polygons shown here represent one active scale.

Guiding idea

The same exponent appears twice: the PDE proof balances penalization errors at $\varepsilon^{1/3}$, while the lower example balances polygonal compatibility at $p^3 t \sim 1$. A conceptual principle explaining why these two balances coincide is still missing.

4.3 Convexity changes the problem

Assume now that $p \mapsto H(y, p)$ is convex. Let

$$L(y, q) = \sup_{p \in \mathbb{R}^n} \{p \cdot q - H(y, p)\} \quad (4.10)$$

be the Lagrangian. The solution has the control representation

$$u^\varepsilon(x, t) = \inf_{\substack{\gamma \in AC([0, t]; \mathbb{R}^n) \\ \gamma(0) = x \\ \gamma(t) = x}} \left\{ g(\gamma(0)) + \int_0^t L\left(\frac{\gamma(s)}{\varepsilon}, \dot{\gamma}(s)\right) ds \right\}. \quad (4.11)$$

With $\gamma(s) = \varepsilon \eta(s/\varepsilon)$, the microscopic oscillation becomes a large-time action. Define

$$m(t, x, y) = \inf_{\substack{\eta(0) = x \\ \eta(t) = y}} \int_0^t L(\eta(s), \dot{\eta}(s)) ds. \quad (4.12)$$

The effective action is the stable limit

$$\overline{m}(t, x, y) = \lim_{k \rightarrow \infty} \frac{1}{k} m(kt, kx, ky). \quad (4.13)$$

Theorem 4.3 (Tran–Yu [13]). *Under the standard periodicity, coercivity, and convexity assumptions, with $g \in \text{BUC}(\mathbb{R}^n) \cap \text{Lip}(\mathbb{R}^n)$,*

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0, \infty))} \leq C\varepsilon. \quad (4.14)$$

The rate is optimal.

The quantitative heart is a bounded additive error in the action:

$$|m(k, 0, kz) - k\overline{m}(1, 0, z)| \leq C \quad (4.15)$$

for velocities z in the relevant compact range. Dividing by k gives $O(k^{-1})$; taking $k = \varepsilon^{-1}$ gives $O(\varepsilon)$.

4.4 Subadditivity, superadditivity, and curve cutting

The action satisfies the triangle inequality and integer translation invariance:

$$m(t + s, x, z) \leq m(t, x, y) + m(s, y, z), \quad (4.16)$$

$$m(t, x + \ell, y + \ell) = m(t, x, y), \quad \ell \in \mathbb{Z}^n. \quad (4.17)$$

Fix a compact velocity range $|y| \leq Rt$. For sufficiently large t there is a constant C_R , independent of t and y in this range, for which essential subadditivity gives

$$m(2t, 0, 2y) \leq 2m(t, 0, y) + C_R. \quad (4.18)$$

The difficult direction is essential superadditivity,

$$m(2t, 0, 2y) \geq 2m(t, 0, y) - C_R, \quad (4.19)$$

proved in the same compact velocity range once t is sufficiently large (the quantitative argument takes $t > n$). It says that a minimizing curve for a doubled displacement contains, after a bounded repair, enough pieces to build a competitor for half the displacement in half the time.

Lemma 4.4 (Burago curve-cutting lemma [1]). *Let $\xi : [0, 1] \rightarrow \mathbb{R}^m$ be continuous. There exist pairwise disjoint intervals $[a_i, b_i]$, $1 \leq i \leq k$, with $k \leq (m + 1)/2$, such that*

$$\sum_{i=1}^k (\xi(b_i) - \xi(a_i)) = \frac{1}{2} (\xi(1) - \xi(0)). \quad (4.20)$$

The topological proof parametrizes partitions by a sphere, builds an odd map to $\mathbb{R}^m$, and applies Borsuk–Ulam. For the time-dependent action, one must apply the lemma to the space–time curve

$$\xi(s) = (\gamma(s), s) \in \mathbb{R}^{n+1}. \quad (4.21)$$

The selected pieces then have both half the total spatial displacement and half the total time.

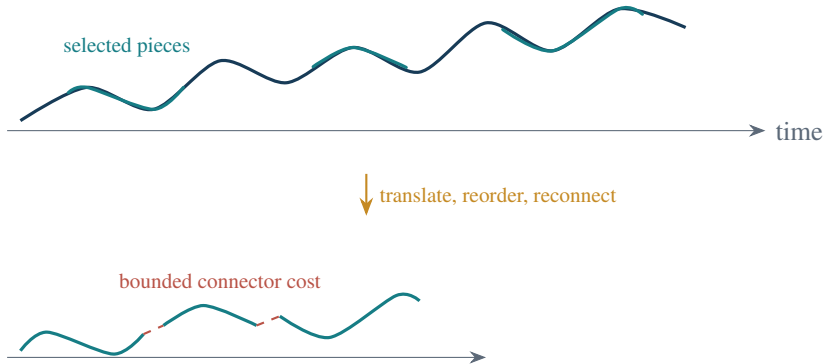


Figure 4.2: Burago’s lemma selects finitely many pieces with half the space–time displacement. Periodicity allows integer translates; bounded connectors turn the pieces into a legal competitor.

After selecting the pieces, translate them by integer vectors so that their microscopic environments are unchanged, reorder them, and compress a bounded-energy portion to make room. A standard controlled-growth reduction, together with bounded connector lengths and an upper bound for L on the relevant velocity range, bounds the cost of the finitely many connectors. This proves (4.19); together with (4.18), it yields (4.15).

4.5 Front propagation and stable shapes

For the eikonal front with positive periodic speed $a(x)$, let U_t be the reachable set from a bounded initial set. The stable action determines a convex limit shape $\tilde{U}$ such that

$$\frac{U_t}{t} \longrightarrow \tilde{U}, \quad d_H\left(\frac{U_t}{t}, \tilde{U}\right) \leq \frac{C}{t}. \quad (4.22)$$

This is the geometric counterpart of the $O(\varepsilon)$ convex homogenization rate. The boundary of $\tilde{U}$ is the unit sphere of a stable norm. Its corners, flat pieces, and differentiability encode the multiplicity and topology of long minimizing trajectories.

Curvature-forced front propagation

For the velocity vector

$$\mathbf{V} = (a(x) - \kappa)v,$$

characterize hypotheses that guarantee a deterministic large-time shape. Homogenization results in dimension two and counterexamples in dimensions at least three show that no unrestricted theorem is possible [3]. In regimes where homogenization holds, determine the optimal rate of convergence of U_t/t . Quantitative results for graphical laminar settings provide an important structured benchmark [7].

4.6 Open ends of the first-order theory

The lecture and the accompanying Jang–Sun–Tran–Yu paper leave two sharply posed questions [6].

Level-set quasiconvex Hamiltonians

Assume every sublevel set $\{p : H(y, p) \leq c\}$ is convex. What is the optimal homogenization rate in dimensions $n \geq 2$? One-dimensional lower examples show that a universal $O(\varepsilon)$ result is impossible. More explicitly, for a nonnegative one-dimensional example H_1 , the extension

$$H_n(y, p) = \max\{H_1(y_1, p_1), |p'|\}$$

is level-set quasiconvex, and transverse-independent data preserve the one-dimensional solution; hence the $\Omega(\sqrt{\varepsilon})$ obstruction embeds in every dimension. The general $O(\varepsilon^{1/3})$ estimate remains available. The natural challenge is to identify what part of convex action geometry survives at the level of nested sublevel sets.

The planar logarithmic gap

In dimension $n = 2$, is

$$\|u^\varepsilon - u\|_\infty = o(\varepsilon^{1/3})$$

for every fixed periodic Lipschitz Hamiltonian and datum? The present lower construction leaves only the logarithmic gap in (4.7); is that loss intrinsic, or merely a feature of the construction?

Lecture summary

- Discounted correctors produce the classical $\varepsilon^{1/3}$ balance but have momentum modulus of order λ^{-1} .

- One-dimensional order improves the rate to $\sqrt{\varepsilon}$; multiscale nonconvex geometry makes $\varepsilon^{1/3}$ sharp in dimensions at least three.
- Convexity converts the PDE into a large-time action problem.
- Burago curve cutting supplies essential superadditivity and a bounded stable-action error, hence the optimal $O(\varepsilon)$ rate.

Questions to test understanding

1. Explain why the estimate (4.5) becomes worse as the discounted equation approaches the cell equation.
2. Derive the physical exponent $1/3$ from the fast-time growth $t^{2/3}$.
3. Identify the two places where periodicity is used in the curve-cutting proof.
4. Why must time be included as one coordinate in (4.21)?

Viscosity solutions: a compact toolkit

This appendix isolates the viscosity facts used in the lectures. It is not a general introduction; its purpose is to make the perturbed-test-function proof self-contained at the level needed here. For the general theory, see the standard user's guide [14].

A.1 Test functions and semicontinuous envelopes

Consider

$$F(x, t, u, Du, D^2u) = 0. \quad (\text{A.1})$$

An upper semicontinuous function u is a viscosity subsolution if, whenever $\phi \in C^2$ and $u - \phi$ has a local maximum at (x_0, t_0) ,

$$F(x_0, t_0, u(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \leq 0.$$

A lower semicontinuous function is a supersolution when the reversed inequality holds at local minima. A continuous function that is both is a solution.

Given locally bounded u^ε , define

$$\bar{u}(x, t) = \limsup_{\substack{\varepsilon \rightarrow 0 \\ (y, s) \rightarrow (x, t)}}^* u^\varepsilon(y, s), \quad (\text{A.2})$$

$$\underline{u}(x, t) = \liminf_{\substack{\varepsilon \rightarrow 0 \\ (y, s) \rightarrow (x, t)}}^* u^\varepsilon(y, s). \quad (\text{A.3})$$

The upper limit is upper semicontinuous, the lower limit is lower semicontinuous, and $\underline{u} \leq \bar{u}$.

Lemma A.1 (Nearby maxima). *If ϕ touches $\bar{u}$ from above at a strict local maximum (x_0, t_0) , then there are $\varepsilon_j \downarrow 0$ and $(x_j, t_j) \rightarrow (x_0, t_0)$ such that $u^{\varepsilon_j} - \phi$ has a local maximum at (x_j, t_j) , after an arbitrarily small strictifying perturbation if needed.*

The same statement holds for minima and $\underline{u}$. This compactness lemma is what transfers a test of the limit to nearby tests of the approximating solutions.

A.2 Stability

Suppose $F_\varepsilon \rightarrow F$ locally uniformly and u^ε are subsolutions of $F_\varepsilon = 0$. Applying theorem A.1 and passing to the limit in the subsolution inequality shows that $\bar{u}$ is a subsolution of $F = 0$. This is the stability theorem used in the discounted-cell construction.

The perturbed-test-function method adds one twist: the test function itself depends on ε and oscillates. Its derivatives do not converge individually. Instead, the cell equation is used before the limit so that the oscillatory terms cancel.

A.3 Comparison closes the argument

Once homogenization has shown

$$\bar{u}_t + \bar{H}(D\bar{u}) \leq 0, \quad \underline{u}_t + \bar{H}(D\underline{u}) \geq 0,$$

and the initial trace is correct, comparison for the effective Cauchy problem gives $\bar{u} \leq \underline{u}$. Since the opposite inequality follows from their definitions,

$$\bar{u} = \underline{u} = u.$$

Equality of the half-relaxed limits plus local equiboundedness implies local uniform convergence of the original family.

A.4 Why nonsmooth correctors are still usable

In a formal calculation, one differentiates $v(x/\varepsilon, p)$. A first-order corrector is only Lipschitz, and even the viscous corrector may not be smooth under minimal assumptions. The rigorous viscosity implementation uses one of two equivalent devices:

1. approximate the corrector by a smooth sub/supersolution with a small cell error, or
2. use the theorem on sums / doubling variables to combine the jets of the macroscopic test and the corrector.

The board calculation in section 3.2 displays the cancellation cleanly; the devices above justify it without assuming classical differentiability.

Point to keep straight

When ϕ touches an upper relaxed limit from above, $\bar{u} - \phi$ has a local maximum and is locally ≤ 0 after normalization. The resulting inequality is the subsolution inequality. Reversing either “above” or the sign reverses the viscosity inequality.

Video concordance

The notes were checked against the full board work and the spoken explanations in all four recordings. The timestamps below are intended to help a reader move between the written notes and the lectures. Times are approximate to within a few seconds.

B.1 Lecture I: models, scales, and cell problems

Homogenization and Selection Problems for Hamilton–Jacobi Equations I Runtime: 1:17:08

Time	Topic
00:00–08:05	First-order, critical viscous, and selection problems; prototype Hamiltonians.
11:14–20:50	Level-set derivation of the eikonal equation; normal velocity and the chain rule.
21:15–28:05	Microscopic scale versus far-field scaling; why a front remembers an averaged travel cost.
28:08–39:43	Forced mean curvature, tangential projection matrix, degenerate ellipticity, and advection.
40:13–43:49	Standing assumptions: periodicity, continuity, and uniform coercivity.
44:14–58:35	Two-scale expansion; coercivity forces $D_y u_0 = 0$; first-order cell problem.
58:37–62:43	Statement of the periodic cell theorem.
62:44–65:36	Critical viscous scaling and the second-order cell equation.
67:27–74:47	Discounted equation, uniform estimates, normalization, and compactness.
73:24–76:24	Q&A: terminology, coercivity, periodicity, and the critical diffusion coefficient.

B.2 Lecture II: nonuniqueness and nonselection

Homogenization and Selection Problems for Hamilton–Jacobi Equations II Runtime: 1:12:46

Time	Topic
00:00–12:58	Discount proof completed: periodicity, λv^λ bound, gradient bound, and compactness.
13:01–17:48	Uniqueness of the ergodic constant by a small-discount comparison and additive shift.
18:13–24:50	Mechanical Hamiltonian, convex effective Hamiltonian, flat minima, and lack of corrector selection.
24:05–32:53	One-dimensional two-well model; branch choices $v' = \pm\sqrt{2V}$; min–max formula.
33:09–46:34	Vanishing viscosity, the forced Burgers connection, and statement of the nonconvergence theorem.
47:24–52:16	Slowly oscillating two-well potential and the choice $\omega(r) = \sin(\log \log(1/r))$.
52:17–54:53	Hopf–Cole transform and the periodic semiclassical Schrödinger operator.
54:54–61:50	Local energy expansions, alternating subsequences, localization, and failure of convergence.
61:53–65:14	Random space–time selection conjecture.
65:38–67:03	Smoothness discussion: finite C^k is proved; C^∞ is suggested informally; the analytic case is posed as a question.
70:17–72:34	Q&A: principal eigenvalue simplicity and why Hopf–Cole is quadratic.

B.3 Lecture III: viscous homogenization and rates

Homogenization and Selection Problems for Hamilton–Jacobi Equations III Runtime: 1:09:02

Time	Topic
01:36–10:30	Critical viscous scaling, cell equation, normalization, and regular dependence on momentum.
10:30–19:16	Local corrector expansion, the gluing problem, and comparison with general nonlinear homogenization.
19:16–38:20	Half-relaxed limits and the perturbed-test-function proof.

Time	Topic
39:20–42:20	General sharp $C(1+T)\sqrt{\varepsilon}$ theorem; forced mean-curvature discussion.
42:30–52:15	Why the naive expansion fails; the tripling-variable construction and scale estimates.
52:20–60:00	Quadratic mechanical Hamiltonian, Hopf–Cole, ballistic heat kernel, and $\varepsilon \log \varepsilon $.
60:00–69:02	Quartic and nonquadratic open problems; numerical evidence and possible approaches.

B.4 Lecture IV: first-order quantitative theory

Homogenization and Selection Problems for Hamilton–Jacobi Equations IV Runtime: 1:12:06

Time	Topic
01:30–14:45	First-order corrector nonselection, discounted correctors, and the classical $\varepsilon^{1/3}$ proof balance.
14:45–17:30	Dimension-dependent optimal-rate theorem.
17:30–25:10	Interlacing polygon construction and the fast-time $t^{2/3}$ mechanism.
25:10–34:00	Convex $O(\varepsilon)$ theorem and the optimal-control representation.
34:00–41:30	Action metric, stable action, and the bounded additive-error target.
42:00–53:20	Subadditivity, periodic translations, and the difficult reverse inequality.
53:20–61:00	Burago curve-cutting lemma, Borsuk–Ulam, and space–time reassembly.
61:00–64:30	Stable norms, effective unit balls, corners, and flat pieces.
64:30–67:30	Curvature-forced propagation as a major open problem.
67:30–72:06	Q&A: quasiconvex and intermediate assumptions, and the conceptual $1/3$ mystery.

B.5 Series and event links

[Complete video playlist](#); [course abstract](#); and [summer-school site](#).

Normalization and board-work notes

The lectures were delivered at the board, where a sign or scale was sometimes corrected a few seconds after first being written. This appendix collects the choices used consistently in the manuscript.

C.1 Fast variables and corrector amplitude

The fast variable is $y = x/\varepsilon$. A first corrector appears with amplitude ε :

$$u^\varepsilon \approx u + \varepsilon v(x/\varepsilon, p).$$

Consequently,

$$D_x[\varepsilon v(x/\varepsilon, p)] = D_y v(x/\varepsilon, p), \quad \varepsilon \Delta_x[\varepsilon v(x/\varepsilon, p)] = \Delta_y v(x/\varepsilon, p).$$

There is no residual ε^{-1} in the second identity.

C.2 Mechanical-potential conventions

Lecture II uses

$$H(y, p) = \frac{1}{2}|p|^2 - V(y),$$

so the inviscid zero-slope cell equation is

$$\frac{1}{2}|Dv|^2 - V = 0.$$

The nonselection theorem is also written in the paper as

$$\frac{1}{2}|D\phi^\delta|^2 + F - \delta \Delta \phi^\delta = c(\delta), \quad F = -V.$$

With $U_\delta = e^{-\phi^\delta/(2\delta)}$, the corresponding Schrödinger equation is

$$\left(-2\delta^2 \Delta + V\right) U_\delta = -c(\delta) U_\delta.$$

The denominator is 2δ , and the operator contains $+V$.

Lecture III uses the different convention

$$u_t^\varepsilon + \frac{1}{2}|Du^\varepsilon|^2 + V(x/\varepsilon) = \frac{\varepsilon}{2}\Delta u^\varepsilon.$$

Both signs are legitimate; the effective Hamiltonian changes accordingly.

C.3 Curvature scaling

For the critical forced mean-curvature model, the curvature coefficient is ε :

$$u_t^\varepsilon + a(x/\varepsilon)|Du^\varepsilon| - \varepsilon|Du^\varepsilon| \operatorname{div} \left(\frac{Du^\varepsilon}{|Du^\varepsilon|} \right) = 0.$$

The factor was briefly omitted and then restored on the Lecture I board. The critical scaling is essential to the stated cell problem.

C.4 Corrector uniqueness

For the first-order cell problem, $\bar{H}(p)$ is unique, but v_p may be nonunique even modulo constants. For the uniformly elliptic viscous cell problem, the corrector is unique modulo constants under the standard assumptions. When a particular representative is needed, the notes use $v(0, p) = 0$.

C.5 The convex rescaling

In the control formula, the correct change of curve is

$$\gamma(s) = \varepsilon\eta(s/\varepsilon), \quad \dot{\gamma}(s) = \dot{\eta}(s/\varepsilon).$$

The curve-cutting lemma is applied to $\xi(s) = (\gamma(s), s) \in \mathbb{R}^{n+1}$, not merely to the spatial path. Including time ensures that the selected pieces have the correct duration as well as the correct displacement.

C.6 Scope of open-problem statements

- The general $\sqrt{\varepsilon}$ viscous rate is optimal over its full class; this does not imply that every strictly convex periodic Hamiltonian attains that rate.
- The $\varepsilon^{2/3}$ quartic statement in Lecture III is a conjectural direction supported by scaling and numerics, not a theorem for general nonconstant potentials.
- The proved dancing-well counterexample is C^3 , and the construction extends to any prescribed finite C^k . A C^∞ extension was suggested informally in the lecture but is not a theorem stated in the cited preprint; the analytic case was explicitly left open.

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