Some new methods for Hamilton–Jacobi type nonlinear partial differential equations

by

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Abstract

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I present two recent research directions in this dissertation. The first direction is on the study of the Nonlinear Adjoint Method for Hamilton–Jacobi equations, which was introduced recently by Evans [27]. The main feature of this new method consists of the introduction of an additional equation to derive new information about the solutions of the regularized Hamilton–Jacobi equations. More specifically, we linearize the regularized Hamilton–Jacobi equations first and then introduce the corresponding adjoint equations. Looking at the behavior of the solutions of the adjoint equations and using integration by parts techniques, we can prove new estimates, which could not be obtained by previous techniques.

We use the Nonlinear Adjoint Method to study the eikonal-like Hamilton–Jacobi equations [79], and Aubry–Mather theory in the non convex settings [10]. The latter one is based on joint work with Filippo Cagnetti and Diogo Gomes. We are able to relax the convexity conditions of the Hamiltonians in both situations and achieve some new results.

The second direction, based on joint work with Hiroyoshi Mitake [69, 68], concerns the study of the properties of viscosity solutions of weakly coupled systems of Hamilton–Jacobi equations. In particular, we are interested in cell problems, large time behavior, and homogenization results of the solutions, which have not been studied much in the literature.

We obtain homogenization results for weakly coupled systems of Hamilton–Jacobi equations with fast switching rates and analyze rigorously the initial layers appearing naturally in the systems. Moreover, some properties of the effective Hamiltonian are also derived.

Finally, we study the large time behavior of the solutions of weakly coupled systems of Hamilton–Jacobi equations and prove the results under some specific conditions. The general case is still open up to now.

To my wife and my daughter,
Van Hai Van and An My-Ngoc Tran.

Contents

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Chapter 1

Viscosity solutions of Hamilton–Jacobi equations

In this Chapter, we give a short introduction to the theory of viscosity solutions of first order Hamilton–Jacobi equations, which was introduced by Crandall and Lions [22] (see also Crandall, Evans, and Lions [21]). Most of this short introduction is taken from the book of Evans [28]. Let us for simplicity only consider initial-value problem of Hamilton–Jacobi equations:

$$\text{(C)} \qquad \begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n, \end{cases}$$

where the Hamiltonians $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given, as is the initial function $g : \mathbb{R}^n \to \mathbb{R}$.

The original approach [22, 21] is to consider the following approximated equation

$$(\mathrm{C}_\varepsilon) \qquad \begin{cases} u^\varepsilon_t + H(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon & \text{in } \mathbb{R}^n \times (0,T), \\ u^\varepsilon(x,0) = g(x) & \text{on } \mathbb{R}^n, \end{cases}$$

for $\varepsilon > 0$. The term $\varepsilon\Delta$ in (C_ε) regularizes the Hamilton–Jacobi equations, and this is the method of *vanishing viscosity*. We then let $\varepsilon \to 0$ and study the limit of the family $\{u^\varepsilon\}_{\varepsilon>0}$. It is often the case that $\{u^\varepsilon\}_{\varepsilon>0}$ is bounded and locally equicontinuous on $\mathbb{R}^n \times (0,T)$. We hence can use the Arzela-Ascoli theorem to deduce that

$$u^{\varepsilon_j} \to u, \quad \text{locally uniformly in } \mathbb{R}^n \times (0,T),$$

for some subsequence $\{u^{\varepsilon_j}\}$ and some limit function $u \in C(\mathbb{R}^n \times (0,T))$. We expect that u is some kind of solution of (C) but we only have that u is continuous and absolutely no information about Du and u_t. Also as (C) is fully nonlinear and not of the divergence structure, we cannot use integration by parts and weak convergence techniques to justify that u is the weak solution in such sense. We instead use the maximum principle to obtain the notion of weak solution, which is viscosity solutions.

The term *viscosity solutions* is used in honor of the vanishing viscosity technique. In the modern approach, the existence of viscosity solutions can be obtained by using Perron's method. We can see later that the definition of viscosity solutions does not involve viscosity of any kind but the name remains because of the history of the subject.

1.1 Definitions

Definition 1.1.1 (Viscosity subsolutions, supersolutions, solutions)**.** A bounded, uniformly continuous function u is called a viscosity subsolution of the intial-value problem (C) provided that

- $u(\cdot, 0) = g$ on $\mathbb{R}^n$.
- For each $v \in C^1(\mathbb{R}^n \times (0,T))$, if $u - v$ has a local maximum at $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$ then
$$v_t(x_0, t_0) + H(x_0, Dv(x_0, t_0)) \leq 0.$$

A bounded, uniformly continuous function u is called a viscosity supersolution of the intial-value problem (C) provided that

- $u(\cdot, 0) = g$ on $\mathbb{R}^n$.
- For each $v \in C^1(\mathbb{R}^n \times (0,T))$, if $u - v$ has a local minimum at $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$ then
$$v_t(x_0, t_0) + H(x_0, Dv(x_0, t_0)) \geq 0.$$

A bounded, uniformly continuous function u is called a viscosity solution of the intial-value problem (C) if u is both a subsolution, and a supersolution of (C).

Remark 1.1.2. In Definition 1.1.1, a local maximum (resp., minimum) can be replaced by a maximum (resp., minimum) or even by a strict maximum (resp., minimum). Besides, a C^1 test function v can be replaced by a C^∞ test function v as well.

1.2 Existence

Theorem 1.2.1. *Let u^ε be the solution of (C_ε) for $\varepsilon > 0$. Assume that there exists a subsequence $\{u^{\varepsilon_j}\}$ such that*

$$u^{\varepsilon_j} \to u, \quad \textit{locally uniformly in } \mathbb{R}^n \times [0,T]$$

for some $u \in C(\mathbb{R}^n \times [0,T])$ bounded and uniformly continuous. Then u is a viscosity solution of (C).

Proof. It is enough to prove that u is a viscosity subsolution of (C). Take any $v \in C^\infty(\mathbb{R}^n \times (0,T))$ and assume that $u - v$ has a strict maximum at $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$.

Recall that $u^{\varepsilon_j} \to u$ locally uniformly as $j \to \infty$. For j large enough, $u^{\varepsilon_j} - v$ has a local maximum at (x_j, t_j) and
$$(x_j, t_j) \to (x_0, t_0), \quad \text{as } j \to \infty.$$

We have $Du^{\varepsilon_j}(x_j,t_j) = Dv(x_j,t_j)$, $u_t^{\varepsilon_j}(x_j,t_j) = v_t(x_j,t_j)$, and $-\Delta u^{\varepsilon_j}(x_j,t_j) \geq -\Delta v(x_j,t_j)$. Hence,

$$\begin{aligned} v_t(x_j,t_j) + H(x_j, Dv(x_j,t_j)) &= u_t^{\varepsilon_j}(x_j,t_j) + H(x_j, Du^{\varepsilon_j}(x_j,t_j)) \\ &= \varepsilon \Delta u^{\varepsilon_j}(x_j,t_j) \leq \varepsilon_j \Delta v(x_j,t_j). \end{aligned}$$

Let $j \to \infty$ to imply that

$$v_t(x_0,t_0) + H(x_0, Dv(x_0,t_0)) \leq 0.$$

□

Remark 1.2.2. Let us emphasize that obtaining viscosity solutions through the vanishing viscosity approach is the classical approach. This method does not work for second order equations. In general, we can use Perron's method to prove the existence of viscosity solutions. However, we do not present Perron's method here in this short introduction to the theory of viscosity solutions.

1.3 Consistency

We here prove that the notion of viscosity solutions is consistent with that of classical solutions.

Firstly, it is quite straightforward to see that if $u \in C^1(\mathbb{R}^n \times [0,T])$ solves (C) and u is also bounded and continuous, then u is a viscosity solution of (C).

Next, we show that if a viscosity solution is differentiable at some point, then it solves (C) there. We need the following Lemma

Lemma 1.3.1 (Touching by a C^1 function)**.** *Assume $u : \mathbb{R}^m \to \mathbb{R}$ is continuous and differentiable at some point x_0. There exists $v \in C^1(\mathbb{R}^m)$ such that $u(x_0) = v(x_0)$ and $u - v$ has a strict local maximum at x_0.*

Proof. Without loss of generality, we assume first that

$$x_0 = 0, \; u(0) = 0, \text{ and } Du(0) = 0. \tag{1.3.1}$$

We use (1.3.1) and the differentiability of u at 0 to deduce that

$$u(x) = |x|\omega(x) \tag{1.3.2}$$

where $\omega : \mathbb{R}^m \to \mathbb{R}$ is continuous with $\omega(0) = 0$. For each $r > 0$, we define

$$\rho(r) = \max_{x \in B_r(0)} |\omega(x)|.$$

We see that $\rho : [0,\infty) \to [0,\infty)$ is continuous, increasing, and $\rho(0) = 0$.

We define

$$v(x) = \int_{|x|}^{2|x|} \rho(r)dr + |x|^2, \quad \text{for } x \in \mathbb{R}^m. \tag{1.3.3}$$

It is clear that $|v(x)| \le |x|\rho(2|x|) + |x|^2$, which implies

$$v(0) = 0, \ Dv(0) = 0.$$

Besides, for $x \neq 0$, explicit computations give us that

$$Dv(x) = \frac{2x}{|x|}\rho(2|x|) - \frac{x}{|x|}\rho(|x|) + 2x,$$

and hence $v \in C^1(\mathbb{R}^m)$.

Finally for every $x \neq 0$,

$$\begin{aligned} u(x) - v(x) &= |x|\omega(x) - \int_{|x|}^{2|x|} \rho(r)dr - |x|^2 \\ &\le |x|\rho(|x|) - |x|\rho(|x|) - |x|^2 \le 0 = u(0) - v(0). \end{aligned}$$

The proof is complete. □

Lemma 1.3.1 immediately implies the following.

Theorem 1.3.2 (Consistency of viscosity solutions)**.** *Let u be a viscosity solution of (C) and suppose that u is differentiable at $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$, then*

$$u_t(x_0, t_0) + H(x_0, Du(x_0, t_0)) = 0.$$

1.4 Stability

It is really important to mention that viscosity solutions remain stable under the L^∞-norm. The following proposition shows this basic fact.

Proposition 1.4.1. *Let $\{H_k\}_{k\in\mathbb{N}} \subset C(\mathbb{R}^n \times \mathbb{R}^n)$ and $\{g_k\}_{k\in\mathbb{N}} \subset C(\mathbb{R}^n)$. Assume that $H_k \to H$, $g_k \to g$ locally uniformly in $\mathbb{R}^n \times \mathbb{R}^n$ as $k \to \infty$ for some $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$ and $g \in C(\mathbb{R}^n)$. Let $\{u_k\}_{k\in\mathbb{N}}$ be viscosity solutions of the intial-value Hamilton–Jacobi equations corresponding to $\{H_k\}_{k\in\mathbb{N}}$ with $u_k(\cdot, 0) = g_k$. Assume furthermore that $u_k \to u$ locally uniformly in $\mathbb{R}^n \times [0, T]$ as $k \to \infty$ for some u bounded and uniformly continuous. Then u is a viscosity solution of (C).*

Proof. It is enough to prove that u is a viscosity subsolution of (C). Take $\phi \in C^1(\mathbb{R}^n \times (0, T))$ and assume that $u - \phi$ has a strict maximum at $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$. By the hypothesis, for k large enough, $u_k - \phi$ has a maximum at some point $(x_k, t_k) \in \mathbb{R}^n \times (0, T)$ and $(x_k, t_k) \to (x_0, t_0)$ as $k \to \infty$. By definition of viscosity subsolutions, we have

$$\phi_t(x_k, t_k) + H_k(x_k, D\phi(x_k, t_k)) \le 0.$$

We let $k \to \infty$ to obtain the result. □

1.5 Uniqueness

We now establish the uniqueness of a viscosity solution of (C).

Lemma 1.5.1 (Extrema at a terminal time)**.** *Assume that u is a viscosity subsolution (resp., supersolution) of (C) and $u-v$ has a local maximum (resp., minimum) at a point $(x_0,t_0)\in\mathbb{R}^n\times(0,T]$ for some $v\in C^1(\mathbb{R}^n\times[0,T])$. Then*

$$v_t(x_0,t_0)+H(x_0,Dv(x_0,t_0))\le 0(\ge 0).$$

The point here is that terminal time $t_0=T$ is allowed.

Proof. We just need to verify the case of subsolution. Assume $u-v$ has a strict maximum at (x_0,T). We define

$$\overline{v}(x,t)=v(x,t)+\frac{\varepsilon}{T-t},\quad \text{for } (x,t)\in\mathbb{R}^n\times(0,T).$$

For $\varepsilon>0$ small enough, $u-\overline{v}$ has a local maximum at $(x_\varepsilon,t_\varepsilon)\in\mathbb{R}^n\times(0,T)$ and $(x_\varepsilon,t_\varepsilon)\to(x_0,T)$ as $\varepsilon\to 0$. By definition of viscosity subsolutions, we have

$$\overline{v}_t(x_\varepsilon,t_\varepsilon)+H(x_\varepsilon,D\overline{v}(x_\varepsilon,t_\varepsilon))\le 0$$

which is equivalent to

$$v_t(x_\varepsilon,t_\varepsilon)+\frac{\varepsilon}{(T-t_\varepsilon)^2}+H(x_\varepsilon,Dv(x_\varepsilon,t_\varepsilon))\le 0.$$

Hence

$$v_t(x_\varepsilon,t_\varepsilon)+H(x_\varepsilon,Dv(x_\varepsilon,t_\varepsilon))\le 0.$$

We let $\varepsilon\to 0$ to achieve the result. □

We now assume further that the Hamiltonian H satisfies

(H1) There exist a positive constant C such that

$$|H(x,p)-H(x,q)|\le C|p-q|,\ |H(x,p)-H(y,p)|\le C|x-y|(1+|p|),\ \text{for } (x,y,p,q)\in(\mathbb{R}^n)^4.$$

Theorem 1.5.2 (Comparison Principle for (C))**.** *Assume that (H1) holds. If u, $\tilde{u}$ are viscosity subsolution, and supersolution of (C) respectively, then $u\le\tilde{u}$.*

Proof. We assume by contradiction that

$$\sup_{\mathbb{R}^n\times[0,T]}(u-\tilde{u})=\sigma>0.$$

For $\varepsilon,\lambda\in(0,1)$, we define

$$\Phi(x,y,t,s)=u(x,t)-\tilde{u}(y,s)-\lambda(t+s)-\frac{1}{\varepsilon^2}(|x-y|^2+(t-s)^2)-\varepsilon(|x|^2+|y|^2),\ \text{for } x,y\in\mathbb{R}^n,\ t,s\ge 0.$$

There exists a point $(x_0, y_0, t_0, s_0) \in \mathbb{R}^{2n} \times [0,T]^2$ such that

$$\Phi(x_0, y_0, t_0, s_0) = \max_{\mathbb{R}^{2n}\times[0,T]^2} \Phi(x, y, t, s).$$

For ε, λ small enough, we have $\Phi(x_0, y_0, t_0, s_0) \geq \sigma/2$.

We use $\Phi(x_0, y_0, t_0, s_0) \geq \Phi(0,0,0,0)$ to get

$$\lambda(t_0+s_0)+\frac{1}{\varepsilon^2}(|x_0-y_0|^2+(t_0-s_0)^2)+\varepsilon(|x_0|^2+|y_0|^2) \leq u(x_0,t_0)-\tilde{u}(y_0,s_0)-u(0,0)+\tilde{u}(0,0) \leq C. \tag{1.5.1}$$

Hence

$$|x_0 - y_0| + |t_0 - s_0| \leq C\varepsilon,\ |x_0| + |y_0| \leq \frac{C}{\varepsilon^{1/2}}. \tag{1.5.2}$$

We next use $\Phi(x_0, y_0, t_0, s_0) \geq \Phi(x_0, x_0, t_0, t_0)$ to deduce that

$$\frac{1}{\varepsilon^2}(|x_0 - y_0|^2 + (t_0 - s_0)^2) \leq \tilde{u}(x_0, t_0) - \tilde{u}(y_0, s_0) + \lambda(t_0 - s_0) + \varepsilon(x_0 - y_0)\cdot(x_0 + y_0).$$

In view of (1.5.2) and the uniformly continuity of $\tilde{u}$, we get

$$|x_0 - y_0| + |t_0 - s_0| = o(\varepsilon). \tag{1.5.3}$$

By (1.5.2) and (1.5.3), we can take $\varepsilon > 0$ small enough so that $s_0, t_0 \geq \mu > 0$ for some $\mu > 0$.

Notice that $(x,t) \mapsto \Phi(x, y_0, t, s_0)$ has a maximum at (x_0, t_0). In view of the definition of Φ, $u - v$ has a maximum at (x_0, t_0) for

$$v(x,t) = \tilde{u}(y_0, s_0) + \lambda(t + s_0) + \frac{1}{\varepsilon^2}(|x - y_0|^2 + (t - s_0)^2) + \varepsilon(|x|^2 + |y_0|^2).$$

By definition of viscosity subsolutions,

$$\lambda + \frac{2(t_0 - s_0)}{\varepsilon^2} + H(x_0, \frac{2(x_0 - y_0)}{\varepsilon^2} + 2\varepsilon x_0) \leq 0. \tag{1.5.4}$$

Similarly, by using the fact that $(y,s) \mapsto \Phi(x_0, y, t_0, s)$ has a maximum at (y_0, s_0), we obtain that

$$-\lambda + \frac{2(t_0 - s_0)}{\varepsilon^2} + H(y_0, \frac{2(x_0 - y_0)}{\varepsilon^2} - 2\varepsilon y_0) \geq 0. \tag{1.5.5}$$

Subtract (1.5.5) from (1.5.4)

$$2\lambda \leq H(y_0, \frac{2(x_0 - y_0)}{\varepsilon^2} - 2\varepsilon y_0) - H(x_0, \frac{2(x_0 - y_0)}{\varepsilon^2} + 2\varepsilon x_0) \leq C\varepsilon(|x_0| + |y_0|) + C|x_0 - y_0|. \tag{1.5.6}$$

We let $\varepsilon \to 0$ to discover that $\lambda \leq 0$, which is the contradiction. □

By using the comparison principle above, we obtain the following uniqueness result immediately.

Theorem 1.5.3 (Uniqueness of viscosity solution)**.** *Under assumption (H1) there exists at most one viscosity solution of (C).*

Chapter 2

The Nonlinear Adjoint Method for Hamilton–Jacobi equations

2.1 Introduction to The Nonlinear Adjoint Method

In this Chapter and the next Chapter, we use the Nonlinear Adjoint Method, introduced by Evans [27], to study the properties of solutions of Hamilton–Jacobi equations, and to study Aubry–Mather theory, in which the Hamiltonians are non convex.

In order to describe clearly the Nonlinear Adjoint Method, let us focus on the following time-dependent Hamilton–Jacobi equation studied by Evans [27]

$$\begin{cases} u_t^\varepsilon + H(Du^\varepsilon) = \varepsilon\Delta u^\varepsilon & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon = g & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{2.1.1}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is a given smooth Hamiltonian and g is a given smooth initial data, and for simplicity, we assume further that

$$g : \mathbb{R}^n \to \mathbb{R} \text{ is smooth and has compact support.}$$

It is well-known from the theory of viscosity solution [22, 21, 59] that for any given $T > 0$, there exists a constant $C = C(T)$ independent of ε so that

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}^n\times[0,T])} + \|Du^\varepsilon\|_{L^\infty(\mathbb{R}^n\times[0,T])} + \|u_t^\varepsilon\|_{L^\infty(\mathbb{R}^n\times[0,T])} \le C,$$

and u^ε converges locally uniformly to u, which is the unique viscosity solution of the Hamilton–Jacobi equation

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{2.1.2}$$

However, the theory of viscosity solution does not give us the information about the structures of the singularities of u as well as the understanding about the convergence $u^\varepsilon \to 0$, e.g. whether $Du^\varepsilon \to Du$ almost everywhere or not.

Evans [27] introduced a new idea of using the Nonlinear Adjoint Method to provide some new understanding about the above issues as follows. First of all, we can see that the formal linearized operator of (2.1.1) is

$$L^\varepsilon v = v_t + DH(Du^\varepsilon) \cdot Dv - \varepsilon \Delta v.$$

We then can introduce the adjoint equation of the linearized operator L^ε: For each $T > 0$ and a probability measure α on $\mathbb{R}^n$, we study

$$\begin{cases} -\sigma^\varepsilon_t - \operatorname{div}(DH(Du^\varepsilon)\sigma^\varepsilon) = \varepsilon\Delta\sigma^\varepsilon & \text{in } \mathbb{R}^n \times (0,\infty), \\ \sigma^\varepsilon = \alpha & \text{on } \mathbb{R}^n \times \{t = T\}. \end{cases} \tag{2.1.3}$$

Then $\sigma^\varepsilon = \sigma^{\varepsilon,\alpha,T}$ were used, for various of choices of the terminal data α, to extract more information about the vanishing viscosity process. It is straightforward by using Maximum Principle and integration by parts to see that $\sigma^\varepsilon \geq 0$ and for each $t \in [0,T]$

$$\int_{\mathbb{R}^n} \sigma^\varepsilon(x,t)\, dx = 1.$$

The new and important inequality that Evans derived is that

$$\varepsilon \int_0^T \int_{\mathbb{R}^n} (|D^2 u^\varepsilon|^2 + |Du^\varepsilon_t|^2)\sigma^\varepsilon \, dx\, dt \leq C, \tag{2.1.4}$$

for some constant $C > 0$ independent of ε. Notice that (2.1.4) tells us that we have better control on the second derivative of u^ε on the support of σ^ε, as usually we only have $\varepsilon|\Delta u^\varepsilon|$ is bounded on $\mathbb{R}^n \times [0,T]$ by (2.1.1).

In particular, we can use (2.1.4) to derive the rate of convergence of u^ε to u,

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0,T]} \leq C\varepsilon^{1/2},$$

in a very beautiful way by proving that $|\frac{\partial u^\varepsilon}{\partial \varepsilon}(x,t)| \leq \frac{C}{\varepsilon^{1/2}}$ for $(x,t) \in \mathbb{R}^n \times [0,T]$.

Deeper analysis using the Nonlinear Adjoint Method gives us some new phenomenon, which is the *matrix of dissipation measures*. The matrix of dissipation measures contains some hidden information about the jumps of the gradients of Du along the characteristics as described in [27].

We will use the Nonlinear Adjoint Method to study the eikonal-like Hamilton–Jacobi equations in the next Section, which is taken from [79]. We derive the rate of convergence of u^ε to u and also relax the convexity of the Hamiltonians. Next Chapter devotes to the study of Aubry–Mather theory in the non convex setting. The results are taken from [10]. We construct Aubry–Mather measures for the non convex Hamiltonians and prove that these measures may fail to be invariant under the Hamiltonian flow because of the appearance of the dissipation measures. However, in the important case of uniformly quasi-convex Hamiltonians the dissipation measures vanish, and as a consequence the invariance is guaranteed.

We refer the readers to [27, 79, 10, 33, 9, 11] for the progress in recent years of the Nonlinear Adjoint Method.

2.2 Eikonal-like equation in bounded domain

We study the following of Eikonal-like Hamilton–Jacobi equation in a given bounded domain U with smooth boundary

$$\begin{cases} H(Du(x)) = 0 & \text{in } U, \\ u(x) = 0 & \text{on } \partial U. \end{cases} \tag{2.2.1}$$

Crandall and Lions studied this equation in sense of viscosity solution first in [22]. See also [59]. After that, Fleming and Souganidis studied it in more details and also gave some asymptotic series of the solutions of the regularized problem in [42]. Then Ishii gave a simple and direct proof of the uniqueness of the solution in [49]. We here base on the conditions given in [42, 59] and we refer the readers to [49, 42, 59] for more details.

Our approach, as usual, is to consider the following regularized problem

$$\begin{cases} H(Du^\varepsilon(x)) = \varepsilon\Delta u^\varepsilon(x) & \text{in } U, \\ u^\varepsilon(x) = 0 & \text{on } \partial U. \end{cases} \tag{2.2.2}$$

Our goal here is twofold. First, we use the nonlinear adjoint method to study the speed of convergence of u^ε to u as ε tends to 0. Secondly, we relax the convexity of the Hamiltonian H, which was often required in the study of (2.2.1) in the literature. We replace the convexity condition by some weaker and more natural condition as follows.

We assume the Hamiltonian H satisfies the following conditions

(H1) H smooth and $H(0) < 0$.

(H2) H is superlinear, i.e. $\lim_{|p|\to\infty} \dfrac{H(p)}{|p|} = \infty$.

(H3) There exist $\gamma, \delta > 0$ such that $DH(p) \cdot p - \gamma H(p) \geq \delta > 0$ for all $p \in \mathbb{R}^n$.

Condition (H3) is used to replace the convexity condition and will be discussed later. We just make an obvious observation that if H is convex then (H3) holds with $\gamma = 1$ and $\delta = -H(0)$.

Theorem 2.2.1. *There exists a constant $C > 0$ independent of ε such that*

$$\|u^\varepsilon\|_{L^\infty},\ \|Du^\varepsilon\|_{L^\infty} \leq C. \tag{2.2.3}$$

Proof. In the case where H is convex then this theorem was proved in [42] by Lemmas 1.1 and 1.2 or in [59]. We here follow the proof in [42] and just need to slightly modify some estimates using the convexity of H.

By Lemma 1.1 and the first part of Lemma 1.2 in [42], there exists a constant $C > 0$ such that $0 \leq u^\varepsilon \leq C$ in $\overline{U}$ and $|Du^\varepsilon| \leq C$ on ∂U. To complete the proof, we will only need to bound $|Du^\varepsilon|$ in U.

Using the same ideas like in [42, 59], let $w = |Du^\varepsilon| - \mu u^\varepsilon$, where μ is to be a suitably chosen constant. Suppose that w has a positive maximum at an interior point $x_0 \in U$. At x_0 we have

$$0 = w_{x_i} = \frac{\sum_k u^\varepsilon_{x_k} u^\varepsilon_{x_k x_i}}{|Du^\varepsilon|} - \mu u^\varepsilon_{x_i},$$

which implies that

$$\sum_i \Big(\sum_k u^\varepsilon_{x_k} u^\varepsilon_{x_k x_i}\Big)^2 = \mu^2 |Du^\varepsilon|^4.$$

Furthermore,

$$0 \le -\varepsilon\Delta w = \frac{\varepsilon\sum_i(\sum_k u^\varepsilon_{x_k} u^\varepsilon_{x_k x_i})^2}{|Du^\varepsilon|^3} - \frac{\varepsilon\sum_{i,k}(u^\varepsilon_{x_k x_i})^2}{|Du^\varepsilon|} + \frac{\sum_k u^\varepsilon_{x_k}(-\varepsilon\Delta u^\varepsilon)_{x_k}}{|Du^\varepsilon|} + \mu(\varepsilon\Delta u^\varepsilon),$$

By using the inequality $\dfrac{(\Delta u^\varepsilon)^2}{n} \le \sum_{i,k}(u^\varepsilon_{x_i x_k})^2$ and (2.2.2), we derive that

$$0 \le \varepsilon\mu^2|Du^\varepsilon| - \frac{H^2}{n\varepsilon|Du^\varepsilon|} - \mu DH\cdot Du^\varepsilon + \mu H.$$

Besides, (H3) implies

$$\mu DH\cdot Du^\varepsilon - \mu\gamma H > \delta\mu > 0,$$

Thus,

$$\frac{H^2}{|Du^\varepsilon|^2} \le n\mu^2\varepsilon^2 + n\varepsilon(\mu - \mu\gamma)\frac{H}{|Du^\varepsilon|} \le n\mu^2\varepsilon^2 + n\varepsilon\mu(1+\gamma)\frac{|H|}{|Du^\varepsilon|}.$$

Choose $\mu = \dfrac{1}{2n(1+\gamma)}$ then for $\varepsilon < 1$, we get the estimate:

$$\frac{H^2}{|Du^\varepsilon|^2} \le 1 + \frac{|H|}{|Du^\varepsilon|}.$$

By the superlinearity condition (H2) we finally get $|Du^\varepsilon|$ is bounded independently of ε. □

Remark 2.2.2. The existence of the solution of (2.2.2) then follows directly from [42] with some changes and adaptations similar to the proof of Theorem 2.2.1 above.

Now we discuss the uniqueness of the viscosity solution u of (2.2.1). For $p \in \mathbb{R}^n$, let us consider $\phi : (0,\infty) \to \mathbb{R}$ as following

$$\phi(t) = t^{-\gamma}H(tp) \quad \forall\ t > 0,$$

then

$$\phi'(t) = t^{-\gamma-1}(DH(tp)\cdot(tp) - \gamma H(tp)) > t^{-\gamma-1}\delta > 0.$$

Hence ϕ is strictly increasing and for $t < 1$ we have furthermore:

$$\phi(1) - \phi(t) = \int_t^1 \phi'(s)\,ds > \int_t^1 s^{-\gamma-1}\delta\,ds = \frac{\delta}{\gamma+1}(t^{-\gamma} - 1) > 0,$$

Thus,

$$H(tp) \le t^\gamma H(p) - \frac{\delta}{\gamma+1}(1 - t^\gamma) = t^\gamma H(p) + \frac{-\delta}{(\gamma+1)H(0)}(1 - t^\gamma)H(0).$$

Notice that $H(0) < 0$ by (H1). So H satisfies all the conditions (H1)-(H3) and (H4)' in [49] with $\varphi = 0$. Therefore, (2.2.1) has a unique viscosity solution.

The proof of the uniqueness of u^ε is quite complicated and follows the key idea of this Section. Therefore, we put it at the end of this Section.

Our main theorem of this Section is

Theorem 2.2.3. *There exists a constant $C > 0$ independent of ε such that*

$$\|u^\varepsilon - u\|_{L^\infty} \le C\varepsilon^{1/2}. \tag{2.2.4}$$

Adjoint method. The formal linearized operator of (2.2.2) is

$$L^\varepsilon v = DH(Du^\varepsilon) \cdot Dv - \varepsilon\Delta v.$$

We now introduce the adjoint equation of the above operator. For each $x_0 \in U$, we consider the following PDE

$$\begin{cases} -\text{div}(DH(Du^\varepsilon)\sigma^\varepsilon) = \varepsilon\Delta\sigma^\varepsilon + \delta_{x_0} & \text{in } U, \\ \sigma^\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{2.2.5}$$

The adjoint equation here is very nice, natural and similar to the one that Evans introduced in [27] to study the time-dependent Hamilton–Jacobi equations. We can use σ^ε and integration by parts techniques to extract more properties of u^ε as well as u, which are our very important goals especially in the case that H is not convex in p.

In order to derive the properties of σ^ε, we need to use the adjoint equation of (2.2.5). For each $f \in C^\infty(\overline{U})$ and $f \ge 0$, we consider the following equation

$$\begin{cases} DH(Du^\varepsilon) \cdot Dv^\varepsilon = \varepsilon\Delta v^\varepsilon + f & \text{in } U, \\ v^\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{2.2.6}$$

By Maximum principle, we derive that $v^\varepsilon \ge 0$. It is moreover straightforward to see that $v^\varepsilon = 0$ when $f = 0$ by using Maximum Principle again. Hence by Fredholm alternative, both equations (2.2.6) and (2.2.5) have unique solutions. By the theory of distributions (see Chapter 5 in [72]), $\sigma^\varepsilon \in C^\infty(U \setminus \{x_0\})$.

Lemma 2.2.4. *The following fact holds*

$$\int_U f\sigma^\varepsilon\, dx = v^\varepsilon(x_0) \geq 0. \tag{2.2.7}$$

In particular, $\sigma^\varepsilon \geq 0$ *in* $U \setminus \{x_0\}$.

Proof. By (2.2.5) and (2.2.6),

$$\int_U f\sigma^\varepsilon\, dx = \int_U \left(DH(Du^\varepsilon)\cdot Dv^\varepsilon\sigma^\varepsilon - \varepsilon\Delta v^\varepsilon\sigma^\varepsilon\right) dx \tag{2.2.8}$$
$$= \int_U (-\mathrm{div}(DH(Du^\varepsilon)\sigma^\varepsilon) - \varepsilon\Delta\sigma^\varepsilon)v^\varepsilon\, dx = v^\varepsilon(x_0) \geq 0.$$

The proof is complete. □

From the above Lemma, we can easily derive some following properties of σ^ε.

Lemma 2.2.5. ***Properties of*** σ^ε

(i) $\sigma^\varepsilon \geq 0$ *in* $U \setminus \{x_0\}$. *In particular,* $\dfrac{\partial\sigma^\varepsilon}{\partial n} \leq 0$ *on* ∂U.

(ii) $\displaystyle\int_{\partial U} \varepsilon\frac{\partial\sigma^\varepsilon}{\partial n} dS = -1.$

Lemma 2.2.6. *Let* $w^\varepsilon = \dfrac{|Du^\varepsilon|^2}{2}$ *then* w^ε *satisfies:*

$$DH(Du^\varepsilon)\cdot Dw^\varepsilon = \varepsilon\Delta w^\varepsilon - \varepsilon|D^2u^\varepsilon|^2. \tag{2.2.9}$$

The proof of Lemma 2.2.6 is quite standard, hence omitted.

Lemma 2.2.7. *There exists a constant* $C > 0$ *such that*

$$\int_U \varepsilon|D^2u^\varepsilon|^2\sigma^\varepsilon\, dx \leq C. \tag{2.2.10}$$

This is one of the key Lemma of this Section and the inequality (2.2.10) is a new inequality in the theory of viscosity solutions, which was discovered first by Evans [27].

Proof. By (2.2.9), we have

$$\int_U (DH(Du^\varepsilon)\cdot Dw^\varepsilon - \varepsilon\Delta w^\varepsilon)\sigma^\varepsilon\, dx = -\int_U \varepsilon|D^2u^\varepsilon|^2\sigma^\varepsilon\, dx. \tag{2.2.11}$$

Integrate by parts the left hand side of the above equality to derive

$$\begin{aligned}&\int_U (DH(Du^\varepsilon)\cdot Dw^\varepsilon - \varepsilon\Delta w^\varepsilon)\sigma^\varepsilon\,dx\\ =&\int_U (-\mathrm{div}(DH(Du^\varepsilon)\sigma^\varepsilon)w^\varepsilon - \varepsilon\Delta\sigma^\varepsilon w^\varepsilon)\,dx + \int_{\partial U}\varepsilon\frac{\partial\sigma^\varepsilon}{\partial n}w^\varepsilon dS\\ =&\int_U (-\mathrm{div}(DH(Du^\varepsilon)\sigma^\varepsilon) - \varepsilon\Delta\sigma^\varepsilon)w^\varepsilon\,dx + \int_{\partial U}\varepsilon\frac{\partial\sigma^\varepsilon}{\partial n}w^\varepsilon\,dS = w(x_0) + \int_{\partial U}\varepsilon\frac{\partial\sigma^\varepsilon}{\partial n}w^\varepsilon\,dS.\end{aligned}$$

The results of Theorem 2.2.1 and Lemma 2.2.5 then yield the conclusion. □

As usual, if we can bound $\int_U \sigma^\varepsilon\,dx$ independently of ε then Theorem 2.2.3 follows immediately by using Lemma 2.2.7 as one can see later by using the same arguments as in [27]. However, it is not easy to bound $\int_U \sigma^\varepsilon dx$ here. We will show the reasons why in the following discussions.

Choose $f = 1$ then (2.2.6) reads

$$\begin{cases} DH(Du^\varepsilon)\cdot Dv^\varepsilon = \varepsilon\Delta v^\varepsilon + 1 & \text{in } U,\\ v^\varepsilon = 0 & \text{on } \partial U.\end{cases} \tag{2.2.12}$$

And also Lemma 2.2.4 reads

$$\int_U \sigma^\varepsilon\,dx = v^\varepsilon(x_0) \geq 0. \tag{2.2.13}$$

Hence, in order to bound $\int_U \sigma^\varepsilon\,dx$, we need to bound $v^\varepsilon(x_0)$. And since x_0 may vary, $\max_U v^\varepsilon$ should be bounded uniformly independently of ε. It turns out that this fact is not true for general H. For example, when $DH(p) = 0$ for all p, the above fact is no longer true, i.e. we will no longer have the uniformly bound for $\max_U v^\varepsilon$ by the following explicit example.

Let us consider the following ODE:

$$\begin{cases} \varepsilon\Delta v^\varepsilon + 1 = 0 & \text{in } (0,1),\\ v^\varepsilon(0) = v^\varepsilon(1) = 0.\end{cases} \tag{2.2.14}$$

Then $v^\varepsilon(x) = \dfrac{1}{2\varepsilon}(x - x^2)$, which implies $\max_{[0,1]} v^\varepsilon = \dfrac{1}{8\varepsilon}$. So $\max_{[0,1]} v^\varepsilon$ blows up as ε tends to 0. Heuristically, this counter-example shows that we need to have some conditions on the gradient of the Hamiltonian H that allow us to control v^ε.

We introduce next the second example, where we have some growth control on $DH(p)$, as following

$$\begin{cases} (v^\varepsilon)' = \varepsilon\Delta v^\varepsilon + 1 & \text{in } (0,1),\\ v^\varepsilon(0) = v^\varepsilon(1) = 0.\end{cases} \tag{2.2.15}$$

Explicit computations give us that

$$v^\varepsilon(x) = x - \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}.$$

Hence, $\max_{[0,1]} v^\varepsilon \le 1$, which provides us the uniformly boundedness of $\max_U v^\varepsilon$ independent of ε. While the first example fails, the second one intuitively shows that if we can control the growth of $DH(p)$ in an appropriate way, we will have such uniform bound.

Based upon the above examples and discussions, we introduce condition (H3), which is weaker than the convexity condition, but still allows us to have the uniform bound of $\max_U v^\varepsilon$ independent of ε. Let us recall (H3).

(H3) There exist $\gamma, \delta > 0$ so that $DH(p) \cdot p - \gamma H(p) \ge \delta > 0$ for all $p \in \mathbb{R}^n$.

In particular, if H is convex then (H3) follows with $\gamma = 1, \delta = -H(0)$. In fact, the required condition (H3) is similar to the homogenous condition. It is natural and it works well for a lot of cases where H is not convex. For example, for $n = 1$, if we take

$$H(p) = (p^2 - 1)^2 - 2 = p^4 - 2p^2 - 1,$$

then H is not convex, but

$$DH(p) \cdot p - 2H(p) = (4p^4 - 4p^2) - 2(p^4 - 2p^2 - 1) = 2p^4 + 2 \ge 2 > 0.$$

It is easy to check that H satisfies (H1)–(H3) and H is not convex.

The following lemma shows the way to bound $\max_U v^\varepsilon$.

Lemma 2.2.8. *Let $\alpha, \beta \in \mathbb{R}$ and $z(x) = \alpha x \cdot Du^\varepsilon(x) + \beta u^\varepsilon(x)$ then*

$$DH(Du^\varepsilon) \cdot Dz - \varepsilon \Delta z = (\alpha + \beta) DH(Du^\varepsilon) \cdot Du^\varepsilon - (2\alpha + \beta)\varepsilon \Delta u^\varepsilon. \tag{2.2.16}$$

Proof. It's enough to work with $z(x) = x \cdot Du^\varepsilon(x) = x_i u^\varepsilon_{x_i}$. We compute

$$z_{x_k} = u^\varepsilon_{x_k} + x_i u^\varepsilon_{x_i x_k},\ z_{x_k x_k} = u^\varepsilon_{x_k x_k} + u^\varepsilon_{x_k x_k} + x_i u^\varepsilon_{x_k x_k x_i},$$

to get

$$Dz = Du^\varepsilon + x_i D u^\varepsilon_{x_i},\ \Delta z = 2\Delta u^\varepsilon + x_i \Delta u^\varepsilon_{x_i}.$$

Next, we differentiate (2.2.2) with respect to x_i

$$DH(Du^\varepsilon) \cdot Du^\varepsilon_{x_i} = \varepsilon \Delta u^\varepsilon_{x_i}, \tag{2.2.17}$$

and combine all the above computations to derive that

$$\begin{aligned} DH(Du^\varepsilon) \cdot Dz - \varepsilon \Delta z &= DH(Du^\varepsilon) \cdot Du^\varepsilon - 2\varepsilon \Delta u^\varepsilon + x_i (DH(Du^\varepsilon) \cdot Du^\varepsilon_{x_i} - \varepsilon \Delta u^\varepsilon_{x_i}) \\ &= DH(Du^\varepsilon) \cdot Du^\varepsilon - 2\varepsilon \Delta u^\varepsilon. \end{aligned}$$

□

This lemma gives us an idea to obtain a bound of $\max_U v^\varepsilon$ by finding a supersolution φ of (2.2.12) of the type z, and then performing Comparison Principle to get $v^\varepsilon \le \varphi$.

We can choose appropriate α, β such that $\alpha + \beta > 0$ and $\dfrac{2\alpha+\beta}{\alpha+\beta} = \gamma$. By using this relation and (H3),

$$\begin{aligned} &DH(Du^\varepsilon)\cdot Dz - \varepsilon\Delta z = (\alpha+\beta)(DH(Du^\varepsilon)\cdot Du^\varepsilon - \gamma\varepsilon\Delta u^\varepsilon) \\ \ge&(\alpha+\beta)(\gamma H(Du^\varepsilon) + \delta - \gamma\varepsilon\Delta u^\varepsilon) = (\alpha+\beta)\delta > 0. \end{aligned} \tag{2.2.18}$$

Let $k = \dfrac{1}{(\alpha+\beta)\delta}$ and let $\varphi(x) = kz(x) + M$ with $M > 0$ large enough so that $\varphi|_{\partial U} \ge 0$. Then by (2.2.18), φ is a supersolution of (2.2.12), i.e.

$$DH(Du^\varepsilon)\cdot D\varphi - \varepsilon\Delta\varphi \ge 1. \tag{2.2.19}$$

By Comparison Principle, we easily get:

$$0 \le v^\varepsilon \le \varphi. \tag{2.2.20}$$

Therefore, there exists $C > 0$ such that $0 \le v^\varepsilon \le C$. Notice that the boundedness of U plays the crucial role here since it implies the boundedness of $z(x) = \alpha x\cdot Du^\varepsilon(x) + \beta u^\varepsilon(x)$. If U is not bounded then z may not be bounded.

In order to prove Theorem 2.2.3, we prove the following theorem

Theorem 2.2.9. *There exists $C > 0$ such that*

$$|u^\varepsilon_\varepsilon(x)| \le C\varepsilon^{-1/2},$$

where $u^\varepsilon_\varepsilon(x) := \dfrac{\partial u^\varepsilon}{\partial\varepsilon}(x)$.

Proof. Differentiate (2.2.2) with respect to ε to get

$$\begin{cases} DH(Du^\varepsilon)\cdot Du^\varepsilon_\varepsilon = \varepsilon\Delta u^\varepsilon_\varepsilon + \Delta u^\varepsilon & \text{in } U, \\ u^\varepsilon_\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{2.2.21}$$

Pick a point $x_0 \in U$ so that

$$|u^\varepsilon_\varepsilon(x_0)| = \max_U |u^\varepsilon_\varepsilon(x)| \ge 0.$$

Multiply (2.2.21) by σ^ε and then integrate by parts over U to achieve

$$u^\varepsilon_\varepsilon(x_0) = \int_U \Delta u^\varepsilon \sigma^\varepsilon\, dx.$$

By Holder's inequality, Lemma 2.2.6 and the boundedness of $\int_U \sigma^\varepsilon\, dx$, we finally get that

$$|u^\varepsilon_\varepsilon(x_0)| \le \left(\int_U |D^2u^\varepsilon|^2\sigma^\varepsilon\, dx\right)^{1/2}\left(\int_U \sigma^\varepsilon\, dx\right)^{1/2} \le C\varepsilon^{-1/2}.$$

□

Finally, we end this Chapter by giving the proof of the uniqueness of the solution u^ε of equation (2.2.2).

Theorem 2.2.10. *If u and v are the solutions of (2.2.2) then we get $u = v$.*

Proof. It is enough to prove that $u \le v$. The strategy is to find a sequence of functions $\{z^\theta\}$ such that z^θ converges uniformly to v as $\theta \to 0$ and

$$H(Du) - \varepsilon\Delta u < H(Dz^\theta) - \varepsilon\Delta z^\theta \text{ in } U; \text{ and } u \le z^\theta \text{ on } \partial U.$$

By Remark 2.2.2, for $t > 1$ we have

$$H(tp) \ge t^\gamma H(p) + \frac{\delta}{\gamma+1}(t^\gamma - 1).$$

Let $z = sv + t(x \cdot Dv + M)$ where $M > 0$ is to be a suitable chosen constant. We can see that the function z here is similar to the one in Lemma 2.2.8, and

$$Dz = (s+t)Dv + tx_i Dv_{x_i},\ \Delta z = (s+2t)\Delta v + tx_i \Delta v_{x_i}.$$

For s close to 1, for $t > 0$ close to 0 and $s + t > 1$,

$$\begin{aligned}
&H(Dz) - \varepsilon\Delta z = H((s+t)Dv + tx_i Dv_{x_i}) - \varepsilon(s+2t)\Delta v + t\varepsilon x_i \Delta v_{x_i}\\
&= H((s+t)Dv) + tDH((s+t)Dv)\cdot(x_i Dv_{x_i}) + t^2 O(1) - \varepsilon(s+2t)\Delta v + t\varepsilon x_i \Delta v_{x_i}\\
&= H((s+t)Dv) + tDH(Dv)\cdot(x_i Dv_{x_i}) + t((s+t)-1)O(1) + t^2 O(1) -\\
&\quad - \varepsilon(s+2t)\Delta v + t\varepsilon x_i \Delta v_{x_i}\\
&\ge (s+t)^\gamma H(Dv) + \frac{\delta}{\gamma+1}((s+t)^\gamma - 1) + t((s+t)-1)O(1) + t^2 O(1) - \varepsilon(s+2t)\Delta v.
\end{aligned}$$

For $\theta > 0$, let $t = (1+\theta)^\gamma - (1+\theta)$ and $s = 2(1+\theta) - (1+\theta)^\gamma$. Let $z^\theta = sv + t(x \cdot Dv + M)$ corresponding to s, t chosen. Notice that z^θ converges uniformly to v as $\theta \to 0$. Furthermore, $(s+t)^\gamma = s + 2t = (1+\theta)^\gamma$ and for θ small enough

$$\frac{\delta}{\gamma+1}((s+t)^\gamma - 1) + t((s+t)-1)O(1) + t^2 O(1) > 0.$$

Hence we get $H(Dz^\theta) - \varepsilon\Delta z^\theta > 0$. Finally, choose M large enough to guarantee $z^\theta \ge u$ on ∂U. The proof is complete. $\square$

Chapter 3

Aubry–Mather Measures in the Non Convex setting

3.1 Introduction

Let us consider a periodic Hamiltonian system whose energy is described by a smooth Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$. Here $\mathbb{T}^n$ denotes the n-dimensional torus, $n \in \mathbb{N}$. It is well known that the time evolution $t \mapsto (\mathbf{x}(t), \mathbf{p}(t))$ of the system is obtained by solving the Hamilton's ODE

$$\begin{cases} \dot{\mathbf{x}} = -D_p H(\mathbf{x}, \mathbf{p}), \\ \dot{\mathbf{p}} = D_x H(\mathbf{x}, \mathbf{p}). \end{cases} \tag{3.1.1}$$

Assume now that, for each $P \in \mathbb{R}^n$, there exists a constant $\overline{H}(P)$ and a periodic function $u(\cdot, P)$ solving the following time independent Hamilton-Jacobi equation

$$H(x, P + D_x u(x, P)) = \overline{H}(P). \tag{3.1.2}$$

Suppose, in addition, that both $u(x, P)$ and $\overline{H}(P)$ are smooth functions. Then, if the following relations

$$X = x + D_P u(x, P), \qquad p = P + D_x u(x, P), \tag{3.1.3}$$

define a smooth change of coordinates $X(x, p)$ and $P(x, p)$, the ODE (3.1.1) can be rewritten as

$$\begin{cases} \dot{\mathbf{X}} = -D_P \overline{H}(\mathbf{P}), \\ \dot{\mathbf{P}} = 0. \end{cases} \tag{3.1.4}$$

Since the solution of (3.1.4) is easily obtained, solving (3.1.1) is reduced to inverting the change of coordinates (3.1.3). Unfortunately, several difficulties arise.

Firstly, it is well known that the solutions of the nonlinear PDE (3.1.2) are not smooth in the general case. For the convenience of the reader, we recall the definition of viscosity solution.

Definition 3.1.1. We say that u is a *viscosity solution* of (3.1.2) if for each $v \in C^\infty(\mathbb{R}^n)$

- If $u - v$ has a local maximum at a point $x_0 \in \mathbb{R}^n$ then
$$H(x_0, P + Dv(x_0)) \leq \overline{H}(P);$$
- If $u - v$ has a local minimum at a point $x_0 \in \mathbb{R}^n$ then
$$H(x_0, P + Dv(x_0)) \geq \overline{H}(P).$$

One can anyway solve (3.1.2) in this weaker sense, as made precise by the following theorem, due to Lions, Papanicolaou and Varadhan.

Theorem 3.1.2 (See [60])**.** *Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be smooth such that*

$$\lim_{|p|\to+\infty} H(x, p) = +\infty. \tag{3.1.5}$$

Then, for every $P \in \mathbb{R}^n$ there exists a unique $\overline{H}(P) \in \mathbb{R}$ such that (3.1.2) *admits a continuous $\mathbb{T}^n$-periodic viscosity solution $u(\cdot, P)$.*

We call (3.1.2) the *cell problem*. It can be easily proved that all the viscosity solutions of the cell problem are Lipschitz continuous by using the coercivity of H.

A second important issue is that the solution $u(\cdot, P)$ of (3.1.2) may not be unique, even modulo addition of constants. Indeed, a simple example is given by the Hamiltonian $H(x, p) = p \cdot (p - D\psi(x))$, where $\psi : \mathbb{T}^n \to \mathbb{R}$ is a smooth fixed function. In this case, for $P = 0$ and $\overline{H}(0) = 0$, the cell problem is

$$Du \cdot D(u - \psi) = 0,$$

which admits both $u \equiv 0$ and $u = \psi$ as solutions. Therefore, smoothness of $u(x, P)$ in P cannot be guaranteed.

Finally, even in the particular case in which both $u(x, P)$ and $\overline{H}(P)$ are smooth, relations (3.1.3) may not be invertible, or the functions $X(x, p)$ and $P(x, p)$ may not be smooth or globally defined.

Therefore, in order to understand the solutions of Hamilton's ODE (3.1.1) in the general case, it is very important to exploit the functions $\overline{H}(P)$ and $u(x, P)$, and to extract any possible information "encoded" in $\overline{H}(P)$ about the dynamics.

Classical Results: the convex case

Classically, the additional hypotheses required in literature on the Hamiltonian H are:

(i) $H(x, \cdot)$ is strictly convex;

(ii) $H(x,\cdot)$ is superlinear, i.e.

$$\lim_{|p|\to+\infty}\frac{H(x,p)}{|p|}=+\infty.$$

A typical example is the mechanical Hamiltonian

$$H(x,p)=\frac{|p|^2}{2}+V(x),$$

where V is a given smooth $\mathbb{T}^n$-periodic function. Also, one restricts the attention to a particular class of trajectories of (3.1.1), the so-called one sided *absolute minimizers* of the action integral. More precisely, one first defines the Lagrangian $L:\mathbb{T}^n\times\mathbb{R}^n\to\mathbb{R}$ associated to H as the Legendre transform of H:

$$L(x,v):=H^*(x,v)=\sup_{p\in\mathbb{R}^n}\{-p\cdot v-H(x,p)\}\quad \text{for every } (x,v)\in\mathbb{T}^n\times\mathbb{R}^n. \tag{3.1.6}$$

Here the signs are set following the Optimal Control convention (see [41]). Then, one looks for a Lipschitz curve $\mathbf{x}(\cdot)$ which minimizes the action integral, i.e. such that

$$\int_0^T L(\mathbf{x}(t),\dot{\mathbf{x}}(t))\,dt\le\int_0^T L(\mathbf{y}(t),\dot{\mathbf{y}}(t))\,dt \tag{3.1.7}$$

for each time $T>0$ and each Lipschitz curve $\mathbf{y}(\cdot)$ with $\mathbf{y}(0)=\mathbf{x}(0)$ and $\mathbf{y}(T)=\mathbf{x}(T)$. Under fairly general conditions such minimizers exist, are smooth, and satisfy the Euler-Lagrange equations

$$\frac{d}{dt}\left[D_vL(\mathbf{x}(t),\dot{\mathbf{x}}(t))\right]=D_xL(\mathbf{x}(t),\dot{\mathbf{x}}(t)),\quad t\in(0,+\infty). \tag{3.1.8}$$

It may be shown that if $\mathbf{x}(\cdot)$ solves (3.1.7) (and in turn (3.1.8)), then $(\mathbf{x}(\cdot),\mathbf{p}(\cdot))$ is a solution of (3.1.1), where $\mathbf{p}(\cdot):=-D_vL(\dot{\mathbf{x}}(\cdot),\mathbf{x}(\cdot))$. This is a consequence of assumptions (i) and (ii), that in particular guarantee a one to one correspondence between Hamiltonian space and Lagrangian space coordinates, through the one to one map $\Phi:\mathbb{T}^n\times\mathbb{R}^n\to\mathbb{T}^n\times\mathbb{R}^n$ defined as

$$\Phi(x,v):=(x,-D_vL(x,v)). \tag{3.1.9}$$

There are several natural questions related to the trajectories $\mathbf{x}(\cdot)$ satisfying (3.1.7), in particular in what concerns ergodic averages, asymptotic behavior and so on. To address such questions it is common to consider the following related problem.

In 1991 John N. Mather (see [64]) proposed a relaxed version of (3.1.7), by considering

$$\min_{\nu\in\mathcal{D}}\int_{\mathbb{T}^n\times\mathbb{R}^n}L(x,v)\,d\nu(x,v), \tag{3.1.10}$$

where $\mathcal{D}$ is the class of probability measures in $\mathbb{T}^n\times\mathbb{R}^n$ that are invariant under the Euler-Lagrange flow. In Hamiltonian coordinates the property of invariance for a measure ν can be written more conveniently as:

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\{\phi,H\}\,d\mu(x,p)=0,\qquad \text{for every } \phi\in C_c^1(\mathbb{T}^n\times\mathbb{R}^n),$$

where $\mu = \Phi_{\#}\nu$ is the push-forward of the measure ν with respect to the map Φ, i.e., the measure μ such that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,p)\,d\mu(x,p) = \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,-D_vL(x,v))\,d\nu(x,v),$$

for every $\phi \in C_c(\mathbb{T}^n\times\mathbb{R}^n)$. Here the symbol $\{\cdot,\cdot\}$ stands for the Poisson bracket, that is

$$\{F,G\} := D_pF\cdot D_xG - D_xF\cdot D_pG, \qquad \text{for every } F,G\in C^1(\mathbb{T}^n\times\mathbb{R}^n).$$

Denoting by $\mathcal{P}(\mathbb{T}^n\times\mathbb{R}^n)$ the class of probability measures on $\mathbb{T}^n\times\mathbb{R}^n$, we have

$$\mathcal{D} = \left\{\nu\in\mathcal{P}(\mathbb{T}^n\times\mathbb{R}^n) : \int_{\mathbb{T}^n\times\mathbb{R}^n} \{\phi,H\}\,d\Phi_{\#}\nu(x,p) = 0, \quad \text{for every } \phi\in C^1_c(\mathbb{T}^n\times\mathbb{R}^n)\right\}. \tag{3.1.11}$$

The main disadvantage of problem (3.1.10) is that the set (3.1.11) where the minimization takes place depends on the Hamiltonian H and thus, in turn, on the integrand L. For this reason, Ricardo Mañe (see [63]) considered the problem

$$\min_{\nu\in\mathcal{F}} \int_{\mathbb{T}^n\times\mathbb{R}^n} L(x,v)\,d\nu(x,v), \tag{3.1.12}$$

where

$$\mathcal{F} := \left\{\nu\in\mathcal{P}(\mathbb{T}^n\times\mathbb{R}^n) : \int_{\mathbb{T}^n\times\mathbb{R}^n} v\cdot D\psi(x)\,d\nu(x,v) = 0, \quad \text{for every } \psi\in C^1(\mathbb{T}^n)\right\}.$$

Measures belonging to $\mathcal{F}$ are called *holonomic* measures. Notice that, in particular, to every trajectory $\mathbf{y}(\cdot)$ of the original problem (3.1.7) we can associate a measure $\nu_{\mathbf{y}(\cdot)}\in\mathcal{F}$. Indeed, for every $T>0$ we can first define a measure $\nu_{T,\mathbf{y}(\cdot)}\in\mathcal{P}(\mathbb{T}^n\times\mathbb{R}^n)$ by the relation

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,v)\,d\nu_{T,\mathbf{y}(\cdot)}(x,v) := \frac{1}{T}\int_0^T \phi(\mathbf{y}(t),\dot{\mathbf{y}}(t))\,dt \quad \text{for every } \phi\in C_c(\mathbb{T}^n\times\mathbb{R}^n).$$

Then, from the fact that

$$\operatorname{supp}\nu_{T,\mathbf{y}(\cdot)} \subset \mathbb{T}^n\times[-M,M], \quad \text{for every } T>0, \qquad (M = \text{Lipschitz constant of } \mathbf{y}(\cdot))$$

we infer that there exists a sequence $T_j\to\infty$ and a measure $\nu_{\mathbf{y}(\cdot)}\in\mathcal{P}(\mathbb{T}^n\times\mathbb{R}^n)$ such that $\nu_{T_j,\mathbf{y}(\cdot)} \overset{*}{\rightharpoonup} \nu_{\mathbf{y}(\cdot)}$ in the sense of measures, that is,

$$\lim_{j\to\infty}\frac{1}{T_j}\int_0^{T_j} \phi(\mathbf{y}(t),\dot{\mathbf{y}}(t))\,dt = \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,v)\,d\nu_{\mathbf{y}(\cdot)}(x,v) \quad \text{for every } \phi\in C_c(\mathbb{T}^n\times\mathbb{R}^n). \tag{3.1.13}$$

Choosing $\phi(x,v) = v\cdot D\psi(x)$ in (3.1.13) it follows that $\nu_{\mathbf{y}(\cdot)}\in\mathcal{F}$, since

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} v\cdot D\psi(x)\,d\nu_{\mathbf{y}(\cdot)}(x,v) = \lim_{j\to\infty}\frac{1}{T_j}\int_0^{T_j} \dot{\mathbf{y}}(t)\cdot D\psi(\mathbf{y}(t))\,dt = \lim_{j\to\infty}\frac{\psi(\mathbf{y}(T_j)) - \psi(\mathbf{y}(0))}{T_j} = 0.$$

In principle, since $\mathcal{F}$ is much larger than the class of measures $\mathcal{D}$, we could expect the last problem not to have the same solution of (3.1.10). However, Mañe proved that every solution of (3.1.12) is also a minimizer of (3.1.10).

A more general version of (3.1.12) consists in studying, for each $P \in \mathbb{R}^n$ fixed,

$$\min_{\nu \in \mathcal{F}} \int_{\mathbb{T}^n \times \mathbb{R}^n} (L(x,v) + P \cdot v)\, d\nu(x,v), \tag{3.1.14}$$

referred to as *Mather problem*. Any minimizer of (3.1.14) is said to be a *Mather measure*. An interesting connection between the Mather problem and the time independent Hamilton-Jacobi equation (3.1.2) is established by the identity:

$$-\overline{H}(P) = \min_{\nu \in \mathcal{F}} \int_{\mathbb{T}^n \times \mathbb{R}^n} (L(x,v) + P \cdot v)\, d\nu(x,v). \tag{3.1.15}$$

Notice that problems (3.1.12) and (3.1.14) have the same Euler-Lagrange equation, but possibly different minimizers, since the term $P \cdot v$ is a null Lagrangian. The following theorem gives a characterization of Mather measures in the convex case.

Theorem 3.1.3. *Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying (i) and (ii), and let $P \in \mathbb{R}^n$. Then, $\nu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ is a solution of* (3.1.14) *if and only if:*

$$(a) \quad \int_{\mathbb{T}^n \times \mathbb{R}^n} H(x,p)\, d\mu(x,p) = \overline{H}(P) = H(x,p) \qquad \mu\text{-a.e.};$$

$$(b) \quad \int_{\mathbb{T}^n \times \mathbb{R}^n} (p - P) \cdot D_pH(x,p)\, d\mu(x,p) = 0;$$

$$(c) \quad \int_{\mathbb{T}^n \times \mathbb{R}^n} D_pH(x,p) \cdot D\phi(x)\, d\mu(x,p) = 0, \qquad \text{for every } \phi \in C^1(\mathbb{T}^n),$$

where $\mu = \Phi_\# \nu$ and $\overline{H}(P)$ is defined by Theorem 3.1.2.

Before proving Theorem 3.1.3 we state the following proposition, which is a consequence of the results in [63], [35], [37], [34], [36] and [32].

Proposition 3.1.4. *Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying (i) and (ii). Let $P \in \mathbb{R}^n$, let $\nu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ be a minimizer of* (3.1.14) *and set $\mu = \Phi_\# \nu$. Then,*

(1) μ is invariant under the Hamiltonian dynamics, i.e.

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \{\phi, H\}\, d\mu(x,p) = 0 \qquad \text{for every } \phi \in C^1_c(\mathbb{T}^n \times \mathbb{R}^n);$$

(2) μ is supported on the graph

$$\Sigma := \{(x,p) \in \mathbb{T}^n \times \mathbb{R}^n : p = P + D_xu(x,P)\},$$

where u is any viscosity solution of (3.1.2).

We observe that property (2), also known as the *graph theorem*, is a highly nontrivial result. Indeed, by using hypothesis (ii) one can show that any solution $u(\cdot, P)$ of (3.1.2) is Lipschitz continuous, but higher regularity cannot be expected in the general case.

Proof of Theorem 3.1.3. To simplify, we will assume $P = 0$.

Let ν be a minimizer of (3.1.14). By the previous proposition, we know that properties (1) and (2) hold; let us prove that $\mu = \Phi_{\#}\nu$ satisfies (a)–(c). By (3.1.15), we have

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} L(x,v)\,d\nu(x,v) = -\overline{H}(0).$$

Furthermore, because of (2)

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} H(x,p)\,d\mu(x,p) = \overline{H}(0),$$

that is, (a). Since $H(x,p) = -L(x, -D_pH(x,p)) + p\cdot D_pH(x,p)$, this implies that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} p\cdot D_pH(x,p)\,d\mu(x,p) = 0,$$

and so (b) holds. Finally, (c) follows directly from the fact that $\nu \in \mathcal{F}$.

Let now $\mu \in \mathcal{P}(\mathbb{T}^n\times\mathbb{R}^n)$ satisfy (a)–(c), and let us show that $\nu = (\Phi^{-1})_{\#}\mu$ is a minimizer of (3.1.14). First of all, observe that $\nu\in\mathcal{F}$. Indeed, by using (c) for every $\psi \in C^1(\mathbb{T}^n)$

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} v\cdot D\psi(x)\,d\nu(x,v) = -\int_{\mathbb{T}^n\times\mathbb{R}^n} D_pH(x,p)\cdot D\psi(x)\,d\mu(x,p) = 0.$$

Let now prove that ν is a minimizer.

Integrating equality $H(x,p) = -L(x,-D_pH(x,p)) + p\cdot D_pH(x,p)$ with respect to μ, and using (a) and (b) we have

$$\begin{aligned}\overline{H}(0) &= \int_{\mathbb{T}^n\times\mathbb{R}^n} H(x,p)\,d\mu(x,p)\\ &= -\int_{\mathbb{T}^n\times\mathbb{R}^n} L(x,-D_pH(x,p))\,d\mu(x,p) + \int_{\mathbb{T}^n\times\mathbb{R}^n} p\cdot D_pH(x,p)\,d\mu(x,p)\\ &= -\int_{\mathbb{T}^n\times\mathbb{R}^n} L(x,-D_pH(x,p))\,d\mu(x,p) = -\int_{\mathbb{T}^n\times\mathbb{R}^n} L(x,v)\,d\nu(x,v).\end{aligned}$$

By (3.1.15), ν is a minimizer of (3.1.14).

□

The Non Convex Case

The main goal of this Chapter is to use the techniques of [27] and [79] to construct Mather measures under fairly general hypotheses, when the variational approach just described cannot be used. Indeed, when (i) and (ii) are satisfied H coincides with the Legendre transform of L, that is, identity $H = H^{**}$ holds. Moreover, L turns out to be convex and superlinear as well, and relation (3.1.9) defines a smooth diffeomorphism, that allows to pass from Hamiltonian to Lagrangian coordinates.

First of all, we extend the definition of Mather measure to the non convex setting, without making use of the Lagrangian formulation.

Definition 3.1.5. We say that a measure $\mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ is a *Mather measure* if there exists $P \in \mathbb{R}^n$ such that properties (a)–(c) are satisfied.

The results exposed in the previous subsection show that, modulo the push-forward operation, this definition is equivalent to the usual one in literature (see e.g. [38], [63], [64]). We would like now to answer the following natural questions:

- **Question 1:** Does a Mather measure exist?
- **Question 2:** Let μ be a Mather measure. Are properties (1) and (2) satisfied?

We just showed that in the convex setting both questions have affirmative answers. Before addressing these issues, let us make some hypotheses on the Hamiltonian H. We remark that without any coercivity assumption (i.e. without any condition similar to (ii)), there are no a priori bounds for the modulus of continuity of periodic solutions of (3.1.2). Indeed, for $n = 2$ consider the Hamiltonian

$$H(x,p) = p_1^2 - p_2^2 \qquad \text{for every } p = (p_1, p_2) \in \mathbb{R}^2.$$

In this case, equation (3.1.2) for $P = 0$ and $\overline{H}(P) = 0$ becomes

$$u_x^2 - u_y^2 = 0. \tag{3.1.16}$$

Then, for every choice of $f : \mathbb{R} \to \mathbb{R}$ of class C^1, the function $u(x,y) = f(x-y)$ is a solution of (3.1.16). Clearly, there are no uniform Lipschitz bounds for the family of all such functions u. We assume that

(H1) H is smooth;

(H2) $H(\cdot, p)$ is $\mathbb{T}^n$-periodic for every $p \in \mathbb{R}^n$;

(H3) $\lim_{|p|\to+\infty} \left(\frac{1}{2}|H(x,p)|^2 + D_x H(x,p) \cdot p \right) = +\infty$ uniformly in x.

Note that if hypothesis (ii) of the previous subsection holds uniformly in x and we have a bound on $D_xH(x,p)$, e.g. $|D_xH(x,p)| \le C(1+|p|)$, then (H3) holds.

First we consider, for every $\varepsilon > 0$, a regularized version of (3.1.2), showing existence and uniqueness of a constant $\overline{H}^{\varepsilon}(P)$ such that

$$-\frac{\varepsilon^2}{2}\Delta u^{\varepsilon}(x) + H(x, P + Du^{\varepsilon}(x)) = \overline{H}^{\varepsilon}(P) \tag{3.1.17}$$

admits a $\mathbb{T}^n$-periodic viscosity (in fact smooth) solution (see Theorem 3.2.1).

Thanks to (H3), we can establish a uniform bound on $\|Du^{\varepsilon}\|_{L^\infty}$ and prove that, up to subsequences, $\overline{H}^{\varepsilon}(P) \to \overline{H}(P)$ and $u^{\varepsilon}(\cdot, P)$ converges uniformly to $u(\cdot, P)$ as $\varepsilon \to 0$, where $\overline{H}(P)$ and $u(\cdot, P)$ solve equation (3.1.2).

Observe that, in particular, this shows that Theorem 3.1.2 still holds true under assumption (H3) when (3.1.5) does not hold, as for instance when $n = 1$ and

$$H(x,p) = p^3 + V(x), \quad V \text{ smooth and } \mathbb{T}^n\text{-periodic.}$$

On the other hand (3.1.5) does not imply (H3), see the Hamiltonian

$$H(x,p) = p^2\left(3 + \sin(e^{p^2}(\cos 2\pi x))\right)$$

(here again $n = 1$). Thus, although (H3) seems to be a technical assumption strictly related to the particular choice of the approximating equations (3.1.17), it is not less general than (3.1.5), as just clarified by the previous examples. Anyway, it is not clear at the moment if the results we prove in this Chapter are still true for Hamiltonians satisfying (3.1.5) but not (H3).

Once suitable properties for the sequence $\{u^{\varepsilon}\}$ are proved, for every $\varepsilon > 0$ we define the perturbed Hamilton SDE (see Section 3.3) as

$$\begin{cases} d\mathbf{x}^{\varepsilon} = -D_pH(\mathbf{x}^{\varepsilon}, \mathbf{p}^{\varepsilon})\,dt + \varepsilon\,dw_t, \\ d\mathbf{p}^{\varepsilon} = D_xH(\mathbf{x}^{\varepsilon}, \mathbf{p}^{\varepsilon})\,dt + \varepsilon D^2u^{\varepsilon}\,dw_t, \end{cases} \tag{3.1.18}$$

where w_t is a n-dimensional Brownian motion. The main reason why we use a stochastic approach, is that in this way we emphasize the connection with the convex setting by averaging functions along trajectories. Nevertheless, our techniques can also be introduced in a purely PDE way (see Section 3.3 for a sketch of this approach).

In the second step, as just explained, in analogy to what is done in the convex setting we encode the long-time behavior of the solutions $t \mapsto (\mathbf{x}^{\varepsilon}(t), \mathbf{p}^{\varepsilon}(t))$ of (3.1.18) into a family of probability measures $\{\mu^{\varepsilon}\}_{\varepsilon>0}$, defined by

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,p)\,d\mu^{\varepsilon}(x,p) := \lim_{T_j\to\infty}\frac{1}{T_j}E\left[\int_0^{T_j}\phi(\mathbf{x}^{\varepsilon}(t), \mathbf{p}^{\varepsilon}(t))\,dt\right] \quad \text{for every } \phi \in C_c(\mathbb{T}^n\times\mathbb{R}^n),$$

where with $E[\cdot]$ we denote the expected value and the limit is taken along appropriate subsequences $\{T_j\}_{j\in\mathbb{N}}$ (see Section 3.3).

Using the techniques developed in [27], we are able to provide some bounds on the derivatives of the functions u^ε. More precisely, defining σ_{μ^ε} as the projection on the torus $\mathbb{T}^n$ of the measure μ^ε (see Section 3.3), we give estimates on the $(L^2, d\sigma_{\mu^\varepsilon})$-norm of the second and third derivatives of u^ε, uniformly w.r.t. ε (see Proposition 3.4.1).

In this way, we show that there exist a Mather measure μ and a nonnegative, symmetric $n \times n$ matrix of Borel measures $(m_{kj})_{k,j=1,\ldots,n}$ such that μ^ε converges weakly to μ up to subsequences and

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \{\phi, H\}\, d\mu + \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi_{p_k p_j}\, dm_{kj} = 0, \qquad \forall\, \phi \in C^2_c(\mathbb{T}^n \times \mathbb{R}^n), \tag{3.1.19}$$

with sum understood over repeated indices (see Theorem 3.5.1). As in [27], we call m_{kj} the *dissipation measures*. Relation (3.1.19) is the key point of our work, since it immediately shows the differences with the convex case. Indeed, the Mather measure μ is invariant under the Hamiltonian flow if and only the dissipation measures m_{kj} vanish. When $H(x, \cdot)$ is convex, this is guaranteed by an improved version of the estimates on the second derivatives of u^ε (see Proposition 3.4.1, estimate (3.4.4)). We give in Section 3.10 a one dimensional example showing that, in general, the dissipation measures $(m_{kj})_{k,j=1,\ldots,n}$ do not disappear.

We study property (2) in Section 3.8. In particular, we show that if (3.1.2) admits a solution $u(\cdot, P)$ of class C^1, which is a rather restrictive condition, then the corresponding Mather measure μ given by Theorem 3.1.2 satisfies

$$D_pH(x, P + D_xu(x, P)) \cdot (p - P - D_xu(x, P)) = 0$$

in the support of μ (see Corollary 3.8.2). Observe that this single relation is not enough to give us (2) in general, e.g. $n \geq 2$.

Finally, we are able to provide some examples of non-convex Hamiltonians (see Section 3.9), for which both properties (1) and (2) are satisfied. We observe that the case of strictly quasiconvex Hamiltonians, which appears among our examples, could also be studied using duality (see Section 3.9).

3.2 Elliptic regularization of the cell problem

We start by quoting a classical result concerning an elliptic regularization of equation (3.1.2). This, also called vanishing viscosity method, is a well known tool to study viscosity solutions. In the context of Mather measures this procedure was introduced by Gomes in [45], see also [1], [2], [55].

Theorem 3.2.1. *For every $\varepsilon > 0$ and every $P \in \mathbb{R}^n$, there exists a unique number $\overline{H}^\varepsilon(P) \in \mathbb{R}$ such that the equation*

$$-\frac{\varepsilon^2}{2}\Delta u^\varepsilon(x) + H(x, P + Du^\varepsilon(x)) = \overline{H}^\varepsilon(P) \tag{3.2.1}$$

admits a unique (up to constants) $\mathbb{T}^n$-periodic viscosity solution. Moreover, for every $P \in \mathbb{R}^n$

$$\lim_{\varepsilon \to 0^+} \overline{H}^{\varepsilon}(P) = \overline{H}(P), \qquad \textit{and} \qquad u^{\varepsilon} \to u \textit{ uniformly} \qquad \textit{(up to subsequences),}$$

where $\overline{H}(P) \in \mathbb{R}$ and $u : \mathbb{T}^n \to \mathbb{R}$ are such that (3.1.2) *is satisfied in the viscosity sense.*

We call (3.2.1) the *stochastic cell problem.*

Definition 3.2.2. Let $\varepsilon > 0$ and $P \in \mathbb{R}^n$. The *linearized operator* $L^{\varepsilon,P} : C^2(\mathbb{T}^n) \to C(\mathbb{T}^n)$ associated to equation (3.2.1) is defined as

$$L^{\varepsilon,P} v(x) := -\frac{\varepsilon^2}{2}\Delta v(x) + D_pH(x, P + Du^{\varepsilon}(x)) \cdot Dv(x),$$

for every $v \in C^2(\mathbb{T}^n)$.

Sketch of the Proof. We mimic the proof in [60]. For every $\lambda > 0$, let's consider the following problem

$$\lambda v^{\lambda} + H(x, P + Dv^{\lambda}) = \frac{\varepsilon^2}{2}\Delta v^{\lambda}.$$

The above equation has a unique smooth solution v^{λ} in $\mathbb{R}^n$ which is $\mathbb{Z}^n$-periodic.
We will prove that $\|\lambda v^{\lambda}\|_{L^\infty}, \|Dv^{\lambda}\|_{L^\infty} \leq C$, for some positive constant C independent on λ and ε. By using the viscosity property with $\varphi = 0$ as a test function, we get $\|\lambda v^{\lambda}\|_{L^\infty} \leq C$. Let now $w^{\lambda} = \frac{|Dv^{\lambda}|^2}{2}$. Then we have

$$2\lambda w^{\lambda} + D_pH \cdot Dw^{\lambda} + D_xH \cdot Dv^{\lambda} = \frac{\varepsilon^2}{2}\Delta w^{\lambda} - \frac{\varepsilon^2}{2}|D^2 v^{\lambda}|^2.$$

Notice that for $\varepsilon < 1/\sqrt{n}$

$$\frac{\varepsilon^2}{2}|D^2 v^{\lambda}|^2 \geq \frac{\varepsilon^4}{4}|\Delta v^{\lambda}|^2 = (\lambda v^{\lambda} + H)^2 \geq \frac{1}{2}H^2 - C.$$

Therefore,

$$2\lambda w^{\lambda} + D_pH \cdot Dw^{\lambda} + D_xH \cdot Dv^{\lambda} + \frac{1}{2}H^2 - C \leq \frac{\varepsilon^2}{2}\Delta w^{\lambda}.$$

At $x_1 \in \mathbb{T}^n$ where $w^{\lambda}(x_1) = \max_{\mathbb{T}^n} w^{\lambda}$

$$2\lambda w^{\lambda}(x_1) + D_xH \cdot Dv^{\lambda}(x_1) + \frac{1}{2}H^2 \leq C.$$

Since $w^{\lambda}(x_1) \geq 0$, using condition (H3) we deduce that w^{λ} is bounded independently of λ, ε. Finally, considering the limit $\lambda \to 0$ we conclude the proof.

$\square$

Remark 3.2.3. Bernstein method and (H3) were used in the proof to deduce the uniform bound on $\|Dv^\lambda\|_{L^\infty}$, which is one of the key properties we need along our derivation. See [59, Appendix 1] for conditions similar to (H3).

The classical theory (see [59]) ensures that the functions $u^\varepsilon(\cdot, P)$ are C^∞. In addition, the previous proof shows that they are Lipschitz, with Lipschitz constant independent of ε.

3.3 Stochastic dynamics

We now introduce a stochastic dynamics associated with the stochastic cell problem (3.2.1). This will be a perturbation to the Hamiltonian dynamics (3.1.1), which describes the trajectory in the phase space of a classical mechanical system.

Let $(\mathbb{T}^n, \sigma, P)$be a probability space, and let w_t be a n-dimensional Brownian motion on $\mathbb{T}^n$. Let $\varepsilon > 0$, and let u^ε be a $\mathbb{T}^n$-periodic solution of (3.2.1). To simplify, we set $P = 0$. Consider now the solution $\mathbf{x}^\varepsilon(t)$ of

$$\begin{cases} d\mathbf{x}^\varepsilon = -D_pH(\mathbf{x}^\varepsilon, Du^\varepsilon(\mathbf{x}^\varepsilon))\, dt + \varepsilon\, dw_t, \\ \mathbf{x}^\varepsilon(0) = \overline{x}, \end{cases} \tag{3.3.1}$$

with $\overline{x} \in \mathbb{T}^n$ arbitrary. Accordingly, the momentum variable is defined as

$$\mathbf{p}^\varepsilon(t) = Du^\varepsilon(\mathbf{x}^\varepsilon(t)).$$

Remark 3.3.1. From Remark 3.2.3 it follows that

$$\sup_{t>0} |\mathbf{p}^\varepsilon(t)| < \infty.$$

Let us now recall some basic fact of stochastic calculus. Suppose $\mathbf{z} : [0, +\infty) \to \mathbb{R}^n$ is a solution to the SDE:

$$d\mathbf{z}_i = a_i\, dt + b_{ij}\, w_t^j \qquad\qquad i = 1, \ldots, n,$$

with a_i and b_{ij} bounded and progressively measurable processes. Let $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a smooth function. Then, $\varphi(\mathbf{z}, t)$ satisfies the *Itô formula*:

$$d\varphi = \varphi_{z_i}\, dz_i + \left(\varphi_t + \frac{1}{2} b_{ij} b_{jk} \varphi_{z_i z_k}\right) dt. \tag{3.3.2}$$

An integrated version of the Itô formula is the *Dynkin's formula*:

$$E\left[\phi(\mathbf{z}(T)) - \phi(\mathbf{z}(0))\right] = E\left[\int_0^T \left(a_i D_{z_i}\phi(\mathbf{z}(t)) + \frac{1}{2} b_{ij} b_{jk} D^2_{z_i z_k}\phi(\mathbf{z}(t))\right) dt\right].$$

Here and always in the sequel, we use Einstein's convention for repeated indices in a sum. In the present situation, we have

$$a_i = -D_{p_i}H(\mathbf{x}^\varepsilon, Du^\varepsilon), \qquad b_{ij} = \varepsilon\delta_{ij}.$$

Hence, recalling (3.3.1) and (3.3.2)

$$\begin{aligned} dp_i &= u^\varepsilon_{x_i x_j}\, dx^\varepsilon_j + \frac{\varepsilon^2}{2}\Delta(u^\varepsilon_{x_i})\, dt = -L^{\varepsilon,P} u^\varepsilon_{x_i} dt + \varepsilon u^\varepsilon_{x_i x_j}\, dw^j_t \\ &= D_{x_i} H\, dt + \varepsilon u^\varepsilon_{x_i x_j}\, dw^j_t, \end{aligned}$$

where in the last equality we used identity (3.4.9). Thus, $(\mathbf{x}^\varepsilon, \mathbf{p}^\varepsilon)$ satisfies the following stochastic version of the Hamiltonian dynamics (3.1.1):

$$\begin{cases} d\mathbf{x}^\varepsilon = -D_p H(\mathbf{x}^\varepsilon, \mathbf{p}^\varepsilon)\, dt + \varepsilon\, dw_t, \\ d\mathbf{p}^\varepsilon = D_x H(\mathbf{x}^\varepsilon, \mathbf{p}^\varepsilon)\, dt + \varepsilon D^2 u^\varepsilon\, dw_t. \end{cases} \tag{3.3.3}$$

We are now going to study the behavior of the solutions u^ε of equation (3.2.1) along the trajectory $\mathbf{x}^\varepsilon(t)$. Thanks to the Itô formula and relations (3.3.3) and (3.2.1):

$$\begin{aligned} du^\varepsilon(\mathbf{x}^\varepsilon(t)) &= Du^\varepsilon d\mathbf{x}^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon\, dt = -L^{\varepsilon,P} u^\varepsilon dt + \varepsilon Du^\varepsilon\, dw_t \\ &= \big(H - \overline{H}^\varepsilon - Du^\varepsilon \cdot D_p H\big)\, dt + \varepsilon Du^\varepsilon\, dw_t. \end{aligned} \tag{3.3.4}$$

Using Dynkin's formula in (3.3.4) we obtain

$$E\big(u^\varepsilon(\mathbf{x}^\varepsilon(T)) - u^\varepsilon(\mathbf{x}^\varepsilon(0))\big) = E\left[\int_0^T \big(H - \overline{H}^\varepsilon - Du^\varepsilon \cdot D_p H\big)\, dt\right].$$

We observe that in the convex case, since the Lagrangian L is related with the Hamiltonian by the relation

$$L = p \cdot D_p H - H,$$

we have

$$u^\varepsilon(\mathbf{x}^\varepsilon(0)) = E\left[\int_0^T (L + \overline{H}^\varepsilon)\, dt + u^\varepsilon(\mathbf{x}^\varepsilon(T))\right].$$

Phase space measures

We will encode the asymptotic behaviour of the trajectories by considering ergodic averages. More precisely, we associate to every trajectory $(\mathbf{x}^\varepsilon(\cdot), \mathbf{p}^\varepsilon(\cdot))$ of (3.3.3) a probability measure $\mu^\varepsilon \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ defined by

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x,p)\, d\mu^\varepsilon(x,p) := \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \phi(\mathbf{x}^\varepsilon(t), \mathbf{p}^\varepsilon(t))\, dt\right], \tag{3.3.5}$$

for every $\phi \in C_c(\mathbb{T}^n \times \mathbb{R}^n)$. In the expression above, the definition makes sense provided the limit is taken over an appropriate subsequence. Moreover, no uniqueness is asserted, since by choosing a different subsequence one can in principle obtain a different limit measure μ^ε. Then, using Dynkin's formula we have, for every $\phi \in C_c^2(\mathbb{T}^n \times \mathbb{R}^n)$,

$$E\left[\phi(\mathbf{x}^\varepsilon(T), \mathbf{p}^\varepsilon(T)) - \phi(\mathbf{x}^\varepsilon(0), \mathbf{p}^\varepsilon(0))\right] = E\left[\int_0^T \Big(D_p\phi \cdot D_x H - D_x\phi \cdot D_p H\Big)\, dt\right]$$
$$+ E\left[\int_0^T \left(\frac{\varepsilon^2}{2}\phi_{x_i x_i} + \varepsilon^2 u^\varepsilon_{x_i x_j}\phi_{x_i p_j} + \frac{\varepsilon^2}{2} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_i x_j} \phi_{p_k p_j}\right) dt\right]. \tag{3.3.6}$$

Dividing last relation by T and passing to the limit as $T \to +\infty$ (along a suitable subsequence) we obtain

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \{\phi, H\}\, d\mu^\varepsilon + \int_{\mathbb{T}^n\times\mathbb{R}^n} \left[\frac{\varepsilon^2}{2}\phi_{x_i x_i} + \varepsilon^2 u^\varepsilon_{x_i x_j}\phi_{x_i p_j} + \frac{\varepsilon^2}{2} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_i x_j}\phi_{p_k p_j}\right] d\mu^\varepsilon = 0. \tag{3.3.7}$$

Projected measure

We define the *projected measure* $\sigma_{\mu^\varepsilon} \in \mathcal{P}(\mathbb{T}^n)$ in the following way:

$$\int_{\mathbb{T}^n} \varphi(x)\, d\sigma_{\mu^\varepsilon}(x) := \int_{\mathbb{T}^n\times\mathbb{R}^n} \varphi(x)\, d\mu^\varepsilon(x,p), \qquad \forall\, \varphi \in C(\mathbb{T}^n).$$

Using test functions that do not depend on the variable p in the previous definition we conclude from identity (3.3.7) that

$$\int_{\mathbb{T}^n} D_p H \cdot D\varphi\; d\sigma_{\mu^\varepsilon} = \frac{\varepsilon^2}{2}\int_{\mathbb{T}^n\times\mathbb{R}^n} \Delta\varphi\, d\sigma_{\mu^\varepsilon}, \qquad \forall\, \varphi \in C^2(\mathbb{T}^n). \tag{3.3.8}$$

PDE Approach

The measures μ^ε and σ_{μ^ε} can be defined also by using standard PDE methods from (3.3.8). Indeed, given u^ε we can consider the PDE

$$-\frac{\varepsilon^2}{2}\Delta\sigma^\varepsilon - \operatorname{div}\left(D_p H(x, Du^\varepsilon)\,\sigma^\varepsilon\right) = 0,$$

which admits a unique non-negative solution σ^ε with

$$\int_{\mathbb{T}^n} \sigma^\varepsilon(x)\, dx = 1,$$

since it is not hard to see that 0 is the principal eigenvalue of the following elliptic operator in $C^2(\mathbb{T}^n)$:

$$v \longmapsto -\frac{\varepsilon^2}{2}\Delta v - \operatorname{div}(D_p H(x, Du^\varepsilon)\, v).$$

Then μ^ε can be defined as a unique measure such that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \psi(x,p)\, d\mu^\varepsilon(x,p) = \int_{\mathbb{T}^n} \psi(x, Du^\varepsilon(x))\, d\sigma^\varepsilon(x),$$

for every $\psi \in C_c(\mathbb{T}^n \times \mathbb{R}^n)$. Finally, identity (3.3.7) requires some work but can also be proved in a purely analytic way.

3.4 Uniform estimates

In this section we derive several estimates that will be useful when passing to the limit as $\varepsilon \to 0$. We will use here the same techniques as in [27] and [79].

Proposition 3.4.1. *We have the following estimates:*

$$\varepsilon^2 \int_{\mathbb{T}^n} |D^2_{xx} u^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon} \leq C, \tag{3.4.1}$$

$$\varepsilon^2 \int_{\mathbb{T}^n} |D^2_{Px} u^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon} \leq \int_{\mathbb{T}^n} |D_P u^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon} + \int_{\mathbb{T}^n} |D_p H - D_P \overline{H}^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon}, \tag{3.4.2}$$

$$\varepsilon^2 \int_{\mathbb{T}^n} |D u^\varepsilon_{x_i x_i}|^2 \, d\sigma_{\mu^\varepsilon} \leq C \left(1 + \int_{\mathbb{T}^n} |D^2_{xx} u^\varepsilon|^3 \, d\sigma_{\mu^\varepsilon} \right), \qquad i = 1, \ldots, n. \tag{3.4.3}$$

In addition, if H is uniformly convex in p, inequalities (3.4.1) *and* (3.4.2) *can be improved to:*

$$\int_{\mathbb{T}^n} |D^2_{xx} u^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon} \leq C, \tag{3.4.4}$$

$$\int_{\mathbb{T}^n} |D^2_{Px} u^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon} \leq C \operatorname{trace}(D^2_{PP} \overline{H}^\varepsilon), \tag{3.4.5}$$

respectively. Here C denotes a positive constant independent of ε.

Remark 3.4.2. Estimate (3.4.4) was already proven in [27] and [79].

To prove the proposition we first need an auxiliary lemma. In the following, we denote by β either a direction in $\mathbb{R}^n$ (i.e. $\beta \in \mathbb{R}^n$ with $|\beta| = 1$), or a parameter (e.g. $\beta = P_i$ for some $i \in \{1, \ldots, n\}$). When $\beta = P_i$ for some $i \in \{1, \ldots, n\}$ the symbols H_β and $H_{\beta\beta}$ have to be understood as H_{p_i} and $H_{p_i p_i}$, respectively.

Lemma 3.4.3. *We have*

$$\varepsilon^2 \int_{\mathbb{T}^n} |D_x u^\varepsilon_\beta|^2 \, d\sigma_{\mu^\varepsilon} = 2 \int_{\mathbb{T}^n} u^\varepsilon_\beta (\overline{H}^\varepsilon_\beta - H_\beta) \, d\sigma_{\mu^\varepsilon}, \tag{3.4.6}$$

$$\int_{\mathbb{T}^n} (\overline{H}^\varepsilon_{\beta\beta} - H_{\beta\beta} - 2 D_p H_\beta \cdot D_x u^\varepsilon_\beta - D^2_{pp} H \, D_x u^\varepsilon_\beta \cdot D_x u^\varepsilon_\beta) \, d\sigma_{\mu^\varepsilon} = 0, \tag{3.4.7}$$

$$\varepsilon^2 \int_{\mathbb{T}^n} |D_x u^\varepsilon_{\beta\beta}|^2 \, d\sigma_{\mu^\varepsilon} = 2 \int_{\mathbb{T}^n} u^\varepsilon_{\beta\beta} (\overline{H}^\varepsilon_{\beta\beta} - H_{\beta\beta} - 2 D_p H_\beta \cdot D_x u^\varepsilon_\beta - D^2_{pp} H : D_x u^\varepsilon_\beta \otimes D_x u^\varepsilon_\beta \, d\sigma_{\mu^\varepsilon}. \tag{3.4.8}$$

Proof. By differentiating equation (3.2.1) with respect to β and recalling Definition 3.2.2 we get

$$L^{\varepsilon,P} u^\varepsilon_\beta = \overline{H}^\varepsilon_\beta - H_\beta, \tag{3.4.9}$$

so that

$$\frac{1}{2} L^{\varepsilon,P}(|u^\varepsilon_\beta|^2) = u^\varepsilon_\beta L^{\varepsilon,P} u^\varepsilon_\beta - \frac{\varepsilon^2}{2} |D_x u^\varepsilon_\beta|^2 = u^\varepsilon_\beta (\overline{H}^\varepsilon_\beta - H_\beta) - \frac{\varepsilon^2}{2} |D_x u^\varepsilon_\beta|^2.$$

Integrating w.r.t. σ_{μ^ε} and recalling (3.3.8) we get (3.4.6).

To prove (3.4.7), we differentiate (3.4.9) w.r.t. β obtaining

$$L^{\varepsilon,P} u^\varepsilon_{\beta\beta} = \overline{H}^\varepsilon_{\beta\beta} - H_{\beta\beta} - 2D_pH_\beta \cdot D_x u^\varepsilon_\beta - D^2_{pp}H : D_x u^\varepsilon_\beta \otimes D_x u^\varepsilon_\beta. \tag{3.4.10}$$

Integrating w.r.t. σ_{μ^ε} and recalling (3.3.8) equality (3.4.7) follows. Finally, using (3.4.10)

$$\begin{aligned}\frac{1}{2}L^{\varepsilon,P}(|u^\varepsilon_{\beta\beta}|^2) &= u^\varepsilon_{\beta\beta} L^{\varepsilon,P} u^\varepsilon_{\beta\beta} - \frac{\varepsilon^2}{2}|D_x u^\varepsilon_{\beta\beta}|^2 \\ &= u^\varepsilon_{\beta\beta}(\overline{H}^\varepsilon_{\beta\beta} - H_{\beta\beta} - 2D_pH_\beta \cdot D_x u^\varepsilon_\beta - D^2_{pp}H : D_x u^\varepsilon_\beta \otimes D_x u^\varepsilon_\beta) - \frac{\varepsilon^2}{2}|D_x u^\varepsilon_{\beta\beta}|^2.\end{aligned}$$

Once again, we integrate w.r.t. σ_{μ^ε} and use (3.3.8) to get (3.4.8). □

We can now proceed to the proof of Proposition 3.4.1.

Proof of Proposition 3.4.1. Summing up the n identities obtained from (3.4.6) with $\beta = x_1, \ldots, x_n$ respectively, we have

$$\varepsilon^2 \int_{\mathbb{T}^n} |D^2_{xx} u^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon} = -2 \int_{\mathbb{T}^n} D_x u^\varepsilon \cdot D_x H \, d\sigma_{\mu^\varepsilon}.$$

Thanks to Remark 3.2.3, (3.4.1) follows. Analogously, relation (3.4.2) is obtained by summing up (3.4.6) with $\beta = P_1, P_2, \ldots, P_n$, which yields

$$\varepsilon^2 \int_{\mathbb{T}^n} |D^2_{Px} u^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon} = 2 \int_{\mathbb{T}^n} D_P u^\varepsilon \cdot \left[D_P \overline{H}^\varepsilon - D_p H\right] \, d\sigma_{\mu^\varepsilon}.$$

Let us show (3.4.3). Thanks to (3.4.8)

$$\begin{aligned}&\varepsilon^2 \int_{\mathbb{T}^n} |D_x u^\varepsilon_{x_i x_i}|^2 \, d\sigma_{\mu^\varepsilon} \\ &\quad = -2 \int_{\mathbb{T}^n} u^\varepsilon_{x_i x_i}(H_{x_i x_i} + 2D_pH_{x_i} \cdot D_x u^\varepsilon_{x_i} + D^2_{pp}H : D_x u^\varepsilon_{x_i} \otimes D_x u^\varepsilon_{x_i}) \, d\sigma_{\mu^\varepsilon}.\end{aligned}$$

Since the functions u^ε are uniformly Lipschitz, we have

$$|H_{x_i x_i}|,\ |D_p H_{x_i}|,\ |D^2_{pp}H| \le C, \qquad \text{on the support of } \sigma_{\mu^\varepsilon}.$$

Hence,

$$\begin{aligned}\varepsilon^2 \int_{\mathbb{T}^n} |D_x u^\varepsilon_{x_i x_i}|^2 \, d\sigma_{\mu^\varepsilon} &\le C\left[\int_{\mathbb{T}^n} |D^2_{xx} u^\varepsilon| \, d\sigma_{\mu^\varepsilon} + \int_{\mathbb{T}^n} |D^2_{xx} u^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon} + \int_{\mathbb{T}^n} |D^2_{xx} u^\varepsilon|^3 \, d\sigma_{\mu^\varepsilon}\right] \\ &\le C\left(1 + \int_{\mathbb{T}^n} |D^2_{xx} u^\varepsilon|^3 \, d\sigma_{\mu^\varepsilon}\right).\end{aligned}$$

Finally, assume that H is uniformly convex. Thanks to (3.4.7) for every $i = 1, \dots, n$

$$0 = \int_{\mathbb{T}^n} (H_{x_i x_i} + 2D_p H_{x_i} \cdot D_x u^\varepsilon_{x_i} + D^2_{pp} H D_x u^\varepsilon_{x_i} \cdot D_x u^\varepsilon_{x_i})\, d\sigma_{\mu^\varepsilon}$$
$$\geq \int_{\mathbb{T}^n} \left(H_{x_i x_i} + 2D_p H_{x_i} \cdot D_x u^\varepsilon_{x_i}\right)\, d\sigma_{\mu^\varepsilon} + \alpha \|D_x u^\varepsilon_{x_i}\|^2_{L^2(\mathbb{T}^n; d\sigma_{\mu^\varepsilon})},$$

for some $\alpha > 0$. Thus, using Cauchy's and Young's inequalities, for every $\eta \in \mathbb{R}$

$$\alpha \|D_x u^\varepsilon_{x_i}\|^2_{L^2(\mathbb{T}^n; d\sigma_{\mu^\varepsilon})} \leq -\int_{\mathbb{T}^n} H_{x_i x_i}\, d\sigma_{\mu^\varepsilon} + 2\|D_p H_{x_i}\|_{L^2(\mathbb{T}^n; d\sigma_{\mu^\varepsilon})} \|D_x u^\varepsilon_{x_i}\|_{L^2(\mathbb{T}^n; d\sigma_{\mu^\varepsilon})}$$
$$\leq -\int_{\mathbb{T}^n} H_{x_i x_i}\, d\sigma_{\mu^\varepsilon} + \frac{1}{\eta^2}\|D_p H_{x_i}\|^2_{L^2(\mathbb{T}^n; d\sigma_{\mu^\varepsilon})} + \eta^2 \|D_x u^\varepsilon_{x_i}\|^2_{L^2(\mathbb{T}^n; d\sigma_{\mu^\varepsilon})}.$$

Finally,

$$(\alpha - \eta^2)\|D_x u^\varepsilon_{x_i}\|^2_{L^2(\mathbb{T}^n; d\sigma_{\mu^\varepsilon})} \leq -\int_{\mathbb{T}^n} H_{x_i x_i}\, d\sigma_{\mu^\varepsilon} + \frac{1}{\eta^2}\|D_p H_{x_i}\|^2_{L^2(\mathbb{T}^n; d\sigma_{\mu^\varepsilon})}.$$

Choosing $\eta^2 < \alpha$ we get (3.4.4).

Let $i \in \{1, \dots, n\}$ and let us integrate w.r.t. σ_{μ^ε} relation (3.4.10) with $\beta = P_i$:

$$0 = \int_{\mathbb{T}^n} (\overline{H}^\varepsilon_{P_i P_i} - H_{p_i p_i} - 2D_p H_{p_i} \cdot D_x u^\varepsilon_{P_i} - D^2_{pp} H D_x u^\varepsilon_{P_i} \cdot D_x u^\varepsilon_{P_i})\, d\sigma_{\mu^\varepsilon}.$$

Since $D^2_{pp} H$ is positive definite,

$$\alpha \int_{\mathbb{T}^n} |D_x u^\varepsilon_{P_i}|^2\, d\sigma_{\mu^\varepsilon} \leq \int_{\mathbb{T}^n} (\overline{H}^\varepsilon_{P_i P_i} - H_{p_i p_i} - 2D_p H_{p_i} \cdot D_x u^\varepsilon_{P_i})\, d\sigma_{\mu^\varepsilon}$$
$$\leq \int_{\mathbb{T}^n} (\overline{H}^\varepsilon_{P_i P_i} - 2D_p H_{p_i} \cdot D_x u^\varepsilon_{P_i})\, d\sigma_{\mu^\varepsilon}.$$

Using once again Cauchy's and Young's inequalities and summing up with respect to $i = 1, \dots, n$ (3.4.5) follows. □

3.5 Existence of Mather measures and dissipation measures

We now look at the asymptotic behavior of the measures μ^ε as $\varepsilon \to 0$, proving existence of Mather measures. The main result of the section is the following.

Theorem 3.5.1. *Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying conditions (H1)–(H3), and let $\{\mu^\varepsilon\}_{\varepsilon>0}$ be the family of measures defined in Section 3.3. Then there exist a Mather measure μ and a nonnegative, symmetric $n \times n$ matrix $(m_{kj})_{k,j=1,\dots n}$ of Borel measures such that*

$$\mu^\varepsilon \overset{*}{\rightharpoonup} \mu \quad \textit{in the sense of measures up to subsequences,} \tag{3.5.1}$$

and

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \{\phi, H\}\, d\mu + \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi_{p_k p_j}\, dm_{kj} = 0, \qquad \forall\, \phi \in C_c^2(\mathbb{T}^n \times \mathbb{R}^n). \tag{3.5.2}$$

Moreover,

$$\operatorname{supp}\mu \textit{ and } \operatorname{supp} m \textit{ are compact.} \tag{3.5.3}$$

We call the matrix m_{kj} the dissipation measure.

Proof. First of all, we notice that since we have a uniform (in ε) Lipschitz estimate for the functions u^ε, there exists a compact set $K \subset \mathbb{T}^n \times \mathbb{R}^n$ such that

$$\operatorname{supp}\mu^\varepsilon \subset K, \qquad \forall\, \varepsilon > 0.$$

Moreover, up to subsequences, we have (3.5.1), that is

$$\lim_{\varepsilon\to 0} \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi\, d\mu^\varepsilon \to \int_{\mathbb{T}^n\times\mathbb{R}^n} \phi\, d\mu,$$

for every function $\phi \in C_c(\mathbb{T}^n \times \mathbb{R}^n)$, for some probability measure $\mu \in \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$, and this proves (3.5.1). From what we said, it follows that

$$\operatorname{supp}\mu \subset K.$$

To show (3.5.2), we need to pass to the limit in relation (3.3.7). First, let us focus on the second term of the aforementioned formula:

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \left[\frac{\varepsilon^2}{2}\phi_{x_i x_i} + \varepsilon^2 u^\varepsilon_{x_i x_j}\phi_{x_i p_j} + \frac{\varepsilon^2}{2} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_i x_j}\phi_{p_k p_j}\right] d\mu^\varepsilon. \tag{3.5.4}$$

By the bounds of the previous section,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}^n\times\mathbb{R}^n} \left[\frac{\varepsilon^2}{2}\phi_{x_i x_i} + \varepsilon^2 u^\varepsilon_{x_i x_j}\phi_{x_i p_j}\right] d\mu^\varepsilon = 0.$$

However, as in [27], the last term in (3.5.4) does not vanish in the limit. In fact, through a subsequence, for every $k, j = 1, \dots, n$ we have

$$\frac{\varepsilon^2}{2}\int_{\mathbb{T}^n\times\mathbb{R}^n} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_i x_j}\psi(x,p)\, d\mu^\varepsilon(x,p) \longrightarrow \int_{\mathbb{T}^n\times\mathbb{R}^n} \psi(x,p)\, dm_{kj}(x,p) \quad \forall\, \psi \in C_c(\mathbb{T}^n \times \mathbb{R}^n),$$

for some nonnegative, symmetric $n \times n$ matrix $(m_{kj})_{k,j=1,\dots n}$ of Borel measures. Passing to the limit as $\varepsilon \to 0$ in (3.3.7) condition (3.5.2) follows. From Remark 3.3.1 we infer that $\operatorname{supp} m \subset K$, so that (3.5.3) follows.

Let us show that μ satisfies conditions (a)–(c) with $P = 0$. As in [27] and [79], consider

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \left(H(x,p) - \overline{H}^\varepsilon\right)^2 d\mu^\varepsilon(x,p) = \frac{\varepsilon^4}{4}\int_{\mathbb{T}^n\times\mathbb{R}^n} |\Delta u^\varepsilon(x)|^2\, d\mu^\varepsilon(x,p) \longrightarrow 0$$

as $\varepsilon \to 0$, where we used (3.2.1) and (3.4.1). Therefore, (a) follows. Let us consider relation (3.3.7), and let us choose as test function $\phi = \varphi(u^\varepsilon)$. We get

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \varphi'(u^\varepsilon) D_x u^\varepsilon \cdot D_p H \, d\mu^\varepsilon + \varepsilon^2 \int_{\mathbb{T}^n\times\mathbb{R}^n} \left(\varphi'(u^\varepsilon) u^\varepsilon_{x_i x_i} + \varphi''(u^\varepsilon)(u^\varepsilon_{x_i})^2\right) \, d\mu^\varepsilon = 0.$$

Passing to the limit as $\varepsilon \to 0$, we have

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \varphi'(u) \, p \cdot D_p H \, d\mu = 0.$$

Choosing $\varphi(u) = u$ we get (b). Finally, relation (c) follows by simply choosing in (3.5.2) test functions ϕ that do not depend on the variable p.

□

We conclude the section with a useful identity that will be used in Section 3.9.

Proposition 3.5.2. *For every* $\lambda \in \mathbb{R}$

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} e^{\lambda H} \left(\lambda H_{p_k} H_{p_j} + H_{p_k p_j}\right) \, dm_{kj} = 0. \tag{3.5.5}$$

Proof. First recall that for any function $f : \mathbb{R} \to \mathbb{R}$ of class C^1

$$\{H, f(H)\} = 0,$$

and, furthermore, for any $\psi \in C^1(\mathbb{T}^n \times \mathbb{R}^n)$

$$\{H, \psi f(H)\} = \{H, \psi\} \, f(H).$$

Let now $\lambda \in \mathbb{R}$. By choosing in (3.5.2) $\phi = \psi f(H)$ with $f(z) = e^{\lambda z}$ and $\psi \equiv 1$ we conclude the proof. □

3.6 Support of the dissipation measures

We discuss now in a more detailed way the structure of $\operatorname{supp} m$.

Proposition 3.6.1. *We have*

$$\operatorname{supp} m \subset \overline{\bigcup_{x\in\mathbb{T}^n} \operatorname{co} G(x)} =: K, \tag{3.6.1}$$

where with $\operatorname{co} G(x)$ *we denote the convex hull in* $\mathbb{R}^n$ *of the set* $G(x)$*, and*

$$G(\overline{x}) := \operatorname{supp} \mu \cap \{(x,p) \in \mathbb{T}^n \times \mathbb{R}^n : x = \overline{x}\}, \quad \overline{x} \in \mathbb{T}^n.$$

Remark 3.6.2. We stress that the convex hull of the set $G(x)$ is taken only with respect to the variable p, while the closure in the right-hand side of (3.6.1) is taken in *all* $\mathbb{T}^n \times \mathbb{R}^n$.

Sketch of the proof. For $\tau > 0$ sufficiently small, we can choose an open set K_τ in $\mathbb{T}^n \times \mathbb{R}^n$ such that $K \subset K_\tau$, $\operatorname{dist}(\partial K_\tau, K) < \tau$, and $K_\tau(x) := \{p \in \mathbb{R}^n : (x,p) \in K_\tau\}$ is convex for every $x \in \mathbb{T}^n$.

Also, we can find a smooth open set $K_{2\tau} \subset \mathbb{T}^n \times \mathbb{R}^n$ such that, for every $x \in \mathbb{T}^n$, $K_{2\tau}(x) := \{p \in \mathbb{R}^n : (x,p) \in K_{2\tau}\}$ is *strictly convex*, $K_{2\tau}(x) \supset K_\tau(x)$, and $\operatorname{dist}(\partial K_{2\tau}(x), K_\tau(x)) < \tau$.

Finally, we can construct a smooth function $\eta_\tau : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for every $x \in \mathbb{T}^n$:

- $\eta_\tau(x,p) = 0$ for $p \in K_\tau(x)$.
- $p \mapsto \eta_\tau(x,p)$ is convex.
- $p \mapsto \eta_\tau(x,p)$ is *uniformly convex* on $\mathbb{R}^n \setminus K_{2\tau}(x)$.

In this way, $\eta_\tau(x,p) = 0$ on $K_\tau \supset K \supset \operatorname{supp}\mu$. Therefore

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \{\eta_\tau, H\} d\mu = 0.$$

Combining with (3.5.2),

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} (\eta_\tau)_{p_k p_j} dm_{kj} = 0,$$

which implies $\operatorname{supp} m \subset \bigcup_{x \in \mathbb{T}^n} K_{2\tau}(x)$. Letting $\tau \to 0$, we finally get the desired result. □

As a consequence, we have the following corollary.

Corollary 3.6.3.

$$\operatorname{supp} m \subset \overline{\operatorname{co}\{H(x,p) \le \overline{H}\}}.$$

Proof. The proof follows simply from the fact that for every $x \in \mathbb{T}^n$ we have

$$G(x) \subset \{H(x,p) \le \overline{H}\}.$$

□

3.7 Averaging

In this section we prove some additional estimates concerning averaging with respect to the process (3.1.17). When necessary, to avoid confusion we will explicitly write the dependence on P. Let us start with a definition.

Definition 3.7.1. We define the *rotation number* ρ_0 associated to the measures μ and m as

$$\rho_0 := \lim_{\varepsilon \to 0} \lim_{T \to +\infty} E\left[\frac{\mathbf{x}^\varepsilon(T) - \mathbf{x}^\varepsilon(0)}{T}\right],$$

where the limit is taken along the same subsequences as in (3.3.5) and (3.5.1).

The following theorem gives a formula for the rotation number.

Theorem 3.7.2. *There holds*

$$\rho_0 = \int_{\mathbb{T}^n \times \mathbb{R}^n} D_p H \, d\mu. \tag{3.7.1}$$

Moreover, defining for every $\varepsilon > 0$ the variable $\mathbf{X}^\varepsilon := \mathbf{x}^\varepsilon + D_P u^\varepsilon(\mathbf{x}^\varepsilon)$, we have

$$E\left[\frac{\mathbf{X}^\varepsilon(T) - \mathbf{X}^\varepsilon(0)}{T}\right] = -D_P \overline{H}^\varepsilon(P), \tag{3.7.2}$$

and

$$\lim_{T \to +\infty} E\left[\frac{\left(\mathbf{X}^\varepsilon(T) - \mathbf{X}^\varepsilon(0) + D_P \overline{H}^\varepsilon(P) T\right)^2}{T}\right] \leq 2\, n\, \varepsilon^2 + 2 \int_{\mathbb{T}^n} |D_P u^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon}$$
$$+ 2 \int_{\mathbb{T}^n} |D_p H - D_P \overline{H}^\varepsilon|^2 \, d\sigma_{\mu^\varepsilon}.$$

Proof. Choosing $\phi(x) = x_i$ with $i = 1, 2, 3$ in (3.3.6) we obtain

$$E\left[\frac{\mathbf{x}^\varepsilon(T) - \mathbf{x}^\varepsilon(0)}{T}\right] = -E\left[\frac{1}{T}\int_0^T D_p H(\mathbf{x}^\varepsilon(t), \mathbf{p}^\varepsilon(t))\, dt\right].$$

Passing to the limit as $T \to +\infty$

$$\rho_\varepsilon := \lim_{T \to +\infty} E\left[\frac{\mathbf{x}^\varepsilon(T) - \mathbf{x}^\varepsilon(0)}{T}\right] = \int_{\mathbb{T}^n \times \mathbb{R}^n} D_p H \, d\mu^\varepsilon.$$

We get (3.7.1) by letting ε go to zero.

To prove (3.7.2), recalling Itô's formula (3.3.2) we compute

$$\begin{aligned} d\mathbf{X}^\varepsilon &= d\mathbf{x}^\varepsilon + D^2_{Px} u^\varepsilon(\mathbf{x}^\varepsilon)\, d\mathbf{x}^\varepsilon + \frac{\varepsilon^2}{2} D_P \Delta u^\varepsilon(\mathbf{x}^\varepsilon)\, dt \\ &= \left(-D_p H(\mathbf{x}^\varepsilon, \mathbf{p}^\varepsilon)(I + D^2_{Px} u^\varepsilon(\mathbf{x}^\varepsilon)) + \frac{\varepsilon^2}{2} D_P \Delta u^\varepsilon(\mathbf{x}^\varepsilon)\right) dt + \varepsilon(I + D^2_{Px} u^\varepsilon(\mathbf{x}^\varepsilon))\, dw_t, \end{aligned}$$

where in the last equality we used (3.3.1). By differentiating equation (3.2.1) w.r.t. P we obtain

$$-D_p H(\mathbf{x}^\varepsilon, \mathbf{p}^\varepsilon)(I + D^2_{Px} u^\varepsilon(\mathbf{x}^\varepsilon)) + \frac{\varepsilon^2}{2} D_P \Delta u^\varepsilon(\mathbf{x}^\varepsilon) = -D_P \overline{H}^\varepsilon(P), \tag{3.7.3}$$

so that

$$d\mathbf{X}^\varepsilon = -D_P \overline{H}^\varepsilon(P)\, dt + \varepsilon(I + D^2_{Px} u^\varepsilon(\mathbf{x}^\varepsilon))\, dw_t. \tag{3.7.4}$$

Using the fact that

$$E\left[\int_0^T \varepsilon(I + D^2_{Px} u^\varepsilon(\mathbf{x}^\varepsilon))\, dw_t\right] = 0,$$

(3.7.2) follows.

Finally, using once again Itô's formula (3.3.2) and relation (3.7.4) we can write

$$\begin{aligned}
&d\left[\left(\mathbf{X}^\varepsilon(t)-\mathbf{X}^\varepsilon(0)+D_P\overline{H}^\varepsilon(P)t\right)^2\right]\\
&=2\left(\mathbf{X}^\varepsilon(t)-\mathbf{X}^\varepsilon(0)+D_P\overline{H}^\varepsilon(P)t\right)\left(d\,\mathbf{X}^\varepsilon+D_P\overline{H}^\varepsilon(P)\,dt\right)+\varepsilon^2|I+D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)|^2\,dt\\
&=2\,\varepsilon\left(\mathbf{X}^\varepsilon(t)-\mathbf{X}^\varepsilon(0)+D_P\overline{H}^\varepsilon(P)t\right)\left(I+D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)\right)dw_t+\varepsilon^2|I+D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)|^2\,dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&E\left[\left(\mathbf{X}^\varepsilon(T)-\mathbf{X}^\varepsilon(0)+D_P\overline{H}^\varepsilon(P)T\right)^2\right]\\
&=E\left[\int_0^T 2\,\varepsilon\left(\mathbf{X}^\varepsilon(t)-\mathbf{X}^\varepsilon(0)+D_P\overline{H}^\varepsilon(P)t\right)\left(I+D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)\right)dw_t\right]\\
&\qquad+E\left[\int_0^T \varepsilon^2|I+D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)|^2\,dt\right]\\
&=E\left[\int_0^T \varepsilon^2|I+D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)|^2\,dt\right].
\end{aligned}$$

Dividing by T and letting T go to infinity

$$\begin{aligned}
&\lim_{T\to+\infty}E\left[\frac{\left(\mathbf{X}^\varepsilon(T)-\mathbf{X}^\varepsilon(0)+D_P\overline{H}^\varepsilon(P)T\right)^2}{T}\right]=\lim_{T\to+\infty}E\left[\int_0^T\frac{\varepsilon^2|I+D^2_{Px}u^\varepsilon(\mathbf{x}^\varepsilon)|^2}{T}\,dt\right]\\
&=\varepsilon^2\int_{\mathbb{T}^n}|I+D^2_{Px}u^\varepsilon|^2\,d\sigma_{\mu^\varepsilon}\le 2\,n\,\varepsilon^2+2\,\varepsilon^2\int_{\mathbb{T}^n}|D^2_{Px}u^\varepsilon|^2\,d\sigma_{\mu^\varepsilon}\\
&\le 2\,n\,\varepsilon^2+2\int_{\mathbb{T}^n}|D_Pu^\varepsilon|^2\,d\sigma_{\mu^\varepsilon}+2\int_{\mathbb{T}^n}|D_pH-D_P\overline{H}^\varepsilon|^2\,d\sigma_{\mu^\varepsilon},
\end{aligned}$$

where we used (3.4.2).

□

We conclude the section with a proposition which shows in a formal way how much relation (3.1.3) is "far" from being an actual change of variables. Let us set $w^\varepsilon(x,P):=P\cdot x+u^\varepsilon(x,P)$, where $u^\varepsilon(x,P)$ is a $\mathbb{T}^n$-periodic viscosity solution of (3.1.17), and let $k\in\mathbb{T}^n$. We recall that in the convex setting the following weak version of the change of variables (3.1.3) holds [32, Theorem 9.1]:

$$\lim_{h\to 0}\int_{\mathbb{T}^n}\Phi\left(D^h_Pu(x,P)\right)\,d\sigma_\mu=\int_{\mathbb{T}^n}\Phi\left(X\right)\,dX,$$

for each continuous $\mathbb{T}^n$-periodic function $\Phi:\mathbb{R}^n\to\mathbb{R}$, where

$$D^h_Pu(x,P):=\left(\frac{u(x,P+he_1)-u(x,P)}{h},\ldots,\frac{u(x,P+he_n)-u(x,P)}{h}\right),$$

$e_1, \ldots, e_n$ being the vectors of the canonical basis in $\mathbb{R}^n$. The quoted result was proven by the authors by considering the Fourier series of Φ, and then analyzing the integral on the left-hand side mode by mode. The next proposition shows what happens for a fixed mode in the non convex case.

Proposition 3.7.3. *The following inequality holds:*

$$\begin{aligned}(k \cdot D_P \overline{H}^{\varepsilon}) &\int_{\mathbb{T}^n} e^{2\pi i k \cdot D_P w^{\varepsilon}} \, d\sigma_{\mu^{\varepsilon}} \\ &\leq 2\pi |k|^2 \left(\varepsilon^2 + \int_{\mathbb{T}^n} |D_P u^{\varepsilon}|^2 \, d\sigma_{\mu^{\varepsilon}} + \int_{\mathbb{T}^n} |D_p H - D_P \overline{H}^{\varepsilon}|^2 \, d\sigma_{\mu^{\varepsilon}} \right).\end{aligned}$$

Proof. Recalling identity (3.3.8) with

$$\varphi(x) = e^{2\pi i k \cdot D_P w^{\varepsilon}(x,P)}$$

we obtain

$$\begin{aligned}0 &= \int_{\mathbb{T}^n} L^{\varepsilon,P} e^{2\pi i k \cdot D_P w^{\varepsilon}} \, d\sigma_{\mu^{\varepsilon}} \\ &= 2\pi i \int_{\mathbb{T}^n} e^{2\pi i k \cdot D_P w^{\varepsilon}} \left[L^{\varepsilon,P} (k \cdot D_P w^{\varepsilon}) - \pi i \varepsilon^2 |D_x (k \cdot D_P w^{\varepsilon})|^2 \right] d\sigma_{\mu^{\varepsilon}} \\ &= 2\pi i \int_{\mathbb{T}^n} e^{2\pi i k \cdot D_P w^{\varepsilon}} \left[k \cdot D_P \overline{H}^{\varepsilon} - \pi i \varepsilon^2 |D_x (k \cdot D_P w^{\varepsilon})|^2 \right] d\sigma_{\mu^{\varepsilon}},\end{aligned}$$

where we used (3.4.9) and the fact that $w^{\varepsilon} = P \cdot x + u^{\varepsilon}$. Thus, thanks to estimate (3.4.2)

$$\begin{aligned}&\left| (k \cdot D_P \overline{H}^{\varepsilon}) \int_{\mathbb{T}^n} e^{2\pi i k \cdot D_P w^{\varepsilon}} \, d\sigma_{\mu^{\varepsilon}} \right| \leq \pi \varepsilon^2 \int_{\mathbb{T}^n} |D_x (k \cdot D_P w^{\varepsilon})|^2 \, d\sigma_{\mu^{\varepsilon}} \\ &\leq 2\pi |k|^2 \left(\varepsilon^2 + \varepsilon^2 \int_{\mathbb{T}^n} |D^2_{Px} u^{\varepsilon}|^2 \, d\sigma_{\mu^{\varepsilon}} \right) \\ &\leq 2\pi |k|^2 \left(\varepsilon^2 + \int_{\mathbb{T}^n} |D_P u^{\varepsilon}|^2 \, d\sigma_{\mu^{\varepsilon}} + \int_{\mathbb{T}^n} |D_p H - D_P \overline{H}^{\varepsilon}|^2 \, d\sigma_{\mu^{\varepsilon}} \right).\end{aligned}$$

□

Remark 3.7.4. When H is uniformly convex, thanks to (3.4.5) the last chain of inequalities becomes

$$\left| (k \cdot D_P \overline{H}^{\varepsilon}) \int_{\mathbb{T}^n} e^{2\pi i k \cdot D_P w^{\varepsilon}} \, d\sigma_{\mu^{\varepsilon}} \right| \leq C |k|^2 \varepsilon^2 \big(1 + \operatorname{trace} (D^2_{PP} \overline{H}^{\varepsilon}) \big).$$

Thus, if $\operatorname{trace}(D^2_{PP} \overline{H}^{\varepsilon}) \leq C$, the right-hand side vanishes in the limit as $\varepsilon \to 0$, and we recover [32, Theorem 9.1].

3.8 Compensated compactness

In this section, some analogs of compensated compactness and Div-Curl lemma introduced by Murat and Tartar in the context of conservation laws (see [31], [78]) will be studied, in order to better understand the support of the Mather measure μ. Similar analogs are also considered in [27], to investigate the shock nature of non-convex Hamilton-Jacobi equations.
What we are doing here is quite different from the original Murat and Tartar work (see [78]), since we work on the support of the measure σ_{μ^ε}. Besides, our methods work on arbitrary dimensional space $\mathbb{R}^n$ while usual compensated compactness and Div-Curl lemma in the context of conservation laws can only deal with the case $n = 1, 2$. However, we can only derive one single relation and this is not enough to characterize the support of μ as in the convex case. To avoid confusion, when necessary we will explicitly write the dependence on the P variable.

Let ϕ be a smooth function from $\mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$, and let

$$\rho^\varepsilon = \{\phi, H\}\sigma_{\mu^\varepsilon} + \frac{\varepsilon^2}{2}\phi_{p_j p_k} u^\varepsilon_{x_i x_j} u^\varepsilon_{x_i x_k} \sigma_{\mu^\varepsilon}.$$

By (3.3.7) and (3.4.1), there exists $C > 0$ such that

$$\int_{\mathbb{T}^n} |\rho^\varepsilon| dx \le C.$$

So, up to passing to some subsequence, if necessary, we may assume that $\rho^\varepsilon \overset{*}{\rightharpoonup} \rho$ as a (signed) measure.
By (3.5.2), $\rho(\mathbb{T}^n) = 0$. We have the following theorem.

Theorem 3.8.1. *The following properties are satisfied:*

(i) for every $\phi \in C(\mathbb{T}^n \times \mathbb{R}^n)$

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} D_p H \cdot (p - P)\, \phi(x, p)\, d\mu = \int_{\mathbb{T}^n} u\, d\rho; \tag{3.8.1}$$

(ii) for every $\phi \in C(\mathbb{T}^n \times \mathbb{R}^n)$ *and for every* $\eta \in C^1(\mathbb{T}^n)$,

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} D_p H \cdot D\eta\, \phi(x, p)\, d\mu = \int_{\mathbb{T}^n} \eta d\rho. \tag{3.8.2}$$

Proof. Let $w^\varepsilon = \phi(x, P + D_x u^\varepsilon)$. Notice first that

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} D_p H \cdot (p - P)\, \phi(x, p)\, d\mu = \lim_{\varepsilon \to 0} \int_{\mathbb{T}^n} D_p H(x, P + D_x u^\varepsilon) \cdot D_x u^\varepsilon w^\varepsilon d\sigma_{\mu^\varepsilon}.$$

Integrating by parts the right hand side of the above equality we obtain

$$\int_{\mathbb{T}^n} D_pH(x, P + D_xu^\varepsilon) \cdot D_xu^\varepsilon w^\varepsilon d\sigma_{\mu^\varepsilon} = -\int_{\mathbb{T}^n} u^\varepsilon \mathrm{div}(D_pHw^\varepsilon\sigma_{\mu^\varepsilon})dx$$
$$= -\int_{\mathbb{T}^n} u^\varepsilon(\mathrm{div}(D_pH\sigma_{\mu^\varepsilon})w^\varepsilon + D_pH \cdot D_xw^\varepsilon\sigma_{\mu^\varepsilon})dx \tag{3.8.3}$$
$$= \int_{\mathbb{T}^n} u^\varepsilon(\frac{\varepsilon^2}{2}\Delta\sigma_{\mu^\varepsilon}w^\varepsilon - D_pH \cdot D_xw^\varepsilon\sigma_{\mu^\varepsilon})dx.$$

After several computations, by using (3.2.1) we get

$$D_pH \cdot D_xw^\varepsilon = -\{\phi, H\} + \frac{\varepsilon^2}{2}\phi_{p_i}\Delta u^\varepsilon_{x_i}.$$

Hence

$$\frac{\varepsilon^2}{2}\Delta\sigma_{\mu^\varepsilon}w^\varepsilon - D_pH \cdot D_xw^\varepsilon\sigma_{\mu^\varepsilon} = \frac{\varepsilon^2}{2}\Delta\sigma_{\mu^\varepsilon}w^\varepsilon + \{\phi, H\}\sigma_{\mu^\varepsilon} - \frac{\varepsilon^2}{2}\phi_{p_i}\Delta u^\varepsilon_{x_i}\sigma_{\mu^\varepsilon}$$
$$=\frac{\varepsilon^2}{2}\Delta w^\varepsilon\sigma_{\mu^\varepsilon} + \frac{\varepsilon^2}{2}(\mathrm{div}(D_x\sigma_{\mu^\varepsilon}w^\varepsilon) - \mathrm{div}(D_xw^\varepsilon\sigma_{\mu^\varepsilon})) + \{\phi, H\}\sigma_{\mu^\varepsilon} - \frac{\varepsilon^2}{2}\phi_{p_i}\Delta u^\varepsilon_{x_i}\sigma_{\mu^\varepsilon}$$
$$=\frac{\varepsilon^2}{2}(\phi_{p_jp_k}u^\varepsilon_{x_ix_j}u^\varepsilon_{x_ix_k}+\phi_{p_jx_i}u^\varepsilon_{x_jx_i} + \phi_{x_ix_i} + \phi_{p_i}\Delta u^\varepsilon_{x_i})\sigma_{\mu^\varepsilon}$$
$$+ \frac{\varepsilon^2}{2}(\mathrm{div}(D_x\sigma_{\mu^\varepsilon}w^\varepsilon) - \mathrm{div}(D_xw^\varepsilon\sigma_{\mu^\varepsilon})) + \{\phi, H\}\sigma_{\mu^\varepsilon} - \frac{\varepsilon^2}{2}\phi_{p_i}\Delta u^\varepsilon_{x_i}\sigma_{\mu^\varepsilon}$$
$$=\rho^\varepsilon + \frac{\varepsilon^2}{2}\phi_{x_ix_i}\sigma_{\mu^\varepsilon}+\frac{\varepsilon^2}{2}\phi_{p_jx_i}u^\varepsilon_{x_jx_i} + \frac{\varepsilon^2}{2}(\mathrm{div}(D_x\sigma_{\mu^\varepsilon}w^\varepsilon) - \mathrm{div}(D_xw^\varepsilon\sigma_{\mu^\varepsilon})).$$

Therefore

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} D_pH \cdot (p - P)\,\phi(x, p)\,d\mu$$
$$= \lim_{\varepsilon\to 0}\int_{\mathbb{T}^n} u^\varepsilon\left[\rho^\varepsilon + \frac{\varepsilon^2}{2}\phi_{x_ix_i}\sigma_{\mu^\varepsilon}+\frac{\varepsilon^2}{2}\phi_{p_jx_i}u^\varepsilon_{x_jx_i} + \frac{\varepsilon^2}{2}(\mathrm{div}(D_x\sigma_{\mu^\varepsilon}w^\varepsilon) - \mathrm{div}(D_xw^\varepsilon\sigma_{\mu^\varepsilon}))\right]dx. \tag{3.8.4}$$

Since u^ε converges uniformly to u,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}^n} u^\varepsilon\rho^\varepsilon dx = \int_{\mathbb{T}^n} u\,d\rho.$$

The second term in the right hand side of (3.8.4) obviously converges to 0 as $\varepsilon \to 0$. The third term also tends to 0 by (3.4.1).

Let us look at the last term. We have

$$\begin{aligned}
&\left|\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{2}\int_{\mathbb{T}^n} u^\varepsilon(\operatorname{div}(D_x\sigma_{\mu^\varepsilon}w^\varepsilon)-\operatorname{div}(D_x w^\varepsilon\sigma_{\mu^\varepsilon}))dx\right|\\
=&\left|\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{2}\int_{\mathbb{T}^n} -D_x u^\varepsilon\cdot D_x\sigma_{\mu^\varepsilon}w^\varepsilon + D_x u^\varepsilon\cdot D_x w^\varepsilon\sigma_{\mu^\varepsilon}dx\right|\\
=&\left|\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{2}\int_{\mathbb{T}^n} \operatorname{div}(D_x u^\varepsilon w^\varepsilon)\sigma_{\mu^\varepsilon} + D_x u^\varepsilon\cdot D_x w^\varepsilon\sigma_{\mu^\varepsilon}dx\right|\\
=&\left|\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{2}\int_{\mathbb{T}^n} (\Delta u^\varepsilon w^\varepsilon + 2D_x u^\varepsilon\cdot D_x w^\varepsilon)\sigma_{\mu^\varepsilon}dx\right|\\
\leq&\lim_{\varepsilon\to 0} C\varepsilon^2\int_{\mathbb{T}^n}|D^2_{xx}u^\varepsilon|\sigma_{\mu^\varepsilon}dx \leq \lim_{\varepsilon\to 0}C\varepsilon = 0,
\end{aligned}$$

which implies (3.8.1). Relation (3.8.2) can be derived similarly. □

As a consequence, we have the following corollary.

Corollary 3.8.2. *Let $u(\cdot, P)$ be a classical solution of* (3.1.2)*, and let μ be the corresponding Mather measure given by Theorem 3.1.2. Then,*

$$D_pH\cdot(p-P-D_xu)=0 \quad \text{in supp}\,\mu.$$

Proof. By (3.8.1) and (3.8.2)

$$\int_{\mathbb{T}^n} D_pH\cdot(p-P-D_xu)\,\phi\,d\mu = 0,$$

for all ϕ. Therefore, the conclusion follows. □

3.9 Examples

In this section, we study non-trivial examples where the Mather measure μ is invariant under the Hamiltonian dynamics. Notice that, by (3.5.2), the Mather measure μ is invariant under the Hamiltonian dynamics if and only if the dissipation measures (m_{kj}) vanish. An example in Section 3.10 shows that this is not always guaranteed. As explained in [27], the dissipation measures m_{kj} record the jump of the gradient D_xu along the shock lines.

We investigate now under which conditions we still have the invariance property (1). We provide some partial answers by studying several examples, which include the important class of strongly quasiconvex Hamiltonians (see [39]).

H is uniformly convex

There exists $\alpha > 0$ so that $D^2_{pp}H \geq \alpha > 0$.
Let $\lambda = 0$ in (3.5.5) then

$$0 = \int_{\mathbb{T}^n\times\mathbb{R}^n} H_{p_kp_j}dm_{kj},$$

which implies $m_{kj} = 0$ for all $1 \leq k, j \leq n$. We then can follow the same steps as in [32] to get that μ also satisfies (2).

Uniformly convex conservation law

Suppose that there exists $F(p, x)$, strictly convex in p, such that $\{F, H\}$=0. Then $m = 0$.

Some special non-convex cases

The cases we consider here are somehow variants of the uniformly convex case.

Suppose there exists ϕ uniformly convex and a smooth real function f such that either $\phi = f(H)$ or $H = f(\phi)$. Then, by (3.5.2) we have $m_{kj} = 0$ for all k, j. In particular, if $H = f(\phi)$ with f increasing, then H is quasiconvex.
One explicit example of the above variants is $H(x, p) = (|p|^2 + V(x))^2$, where $V : \mathbb{T}^n \to \mathbb{R}$ is smooth and may take negative values. Then $H(x, p)$ is not convex in p anymore. Anyway, we can choose $\phi(x, p) = |p|^2 + V(x)$, so that $H(x, p) = (\phi(x, p))^2$ and ϕ is uniformly convex in p. Therefore, μ is invariant under the Hamiltonian dynamics.

The case when $n = 1$

Let's consider the case $H(x, p) = H(p) + V(x)$.
In this particular case, property (H3) implies that $|H(x, p)| \to \infty$ as $|p| \to +\infty$. Let us suppose that

$$\lim_{|p| \to +\infty} H(p) = +\infty.$$

Assume also that there exists $p_0 \in \mathbb{R}$ such that $H'(p) = 0$ if and only if $p = p_0$ and $H''(p_0) \neq 0$. Notice that $H(p)$ does not need to be convex. Obviously, uniform convexity of H implies this condition.
We will show that $m_{11} = 0$, which implies that μ is invariant under the Hamiltonian dynamics. From our assumptions, we have that $H'(p) > 0$ for $p > p_0$, $H'(p) < 0$ for $p < p_0$ and hence $H''(p_0) > 0$. Then there exists a neighborhood $(p_0 - r, p_0 + r)$ of p_0 such that

$$H''(p) > \frac{H''(p_0)}{2}, \qquad \forall\, p \in (p_0 - r, p_0 + r).$$

And since the support of m_{11} is bounded, we may assume

$$\text{supp}(m_{11}) \subset \mathbb{T} \times [-M, M],$$

for some $M > 0$ large enough. We can choose M large so that $(p_0 - r, p_0 + r) \subset (-M, M)$. Since $|H'(p)|^2 > 0$ for $p \in [-M, M] \setminus (p_0 - r, p_0 + r)$ and $[-M, M] \setminus (p_0 - r, p_0 + r)$ is compact, there exists $\gamma > 0$ such that

$$|H'(p)|^2 \geq \gamma > 0, \qquad \forall\, p \in [-M, M] \setminus (p_0 - r, p_0 + r).$$

Hence, by choosing $\lambda \gg 0$

$$\lambda|H'(p)|^2 + H''(p) \geq \frac{H''(p_0)}{2}, \qquad \forall\, p \in [-M, M],$$

which shows $m_{11} = 0$ by (3.5.5).

Case in which there are more conserved quantities

Let's consider

$$H(x,p) = H(p) + V(x_1 + ... + x_n),$$

where $V : \mathbb{T} \to \mathbb{R}$ is smooth.
For $k \neq j$, define $\Phi^{kj} = p_k - p_j$. It is easy to see that $\{H, \Phi^{kj}\} = 0$ for any $k \neq j$.
Therefore $\{H, (\Phi^{kj})^2\} = 0$ for any $k \neq j$.
For fixed $k \neq j$, let $\phi = (\Phi^{kj})^2$ in (3.5.2) then

$$2\int_{\mathbb{T}^n\times\mathbb{R}^n} (m_{kk} - 2m_{kj} + m_{jj})\, dxdp = 0.$$

The matrix of *dissipation measures* (m_{kj}) is non-negative definite, therefore $m_{kk} - 2m_{kj} + m_{jj} \geq 0$. Thus, $m_{kk} - 2m_{kj} + m_{jj} = 0$ for any $k \neq j$.
Let $\varepsilon \in (0,1)$ and take $\xi = (\xi_1, ..., \xi_n)$, where $\xi_k = 1+\varepsilon, \xi_j = -1$ and $\xi_i = 0$ otherwise. We have

$$0 \leq m_{kj}\xi_k\xi_j = (1+\varepsilon)^2 m_{kk} - 2(1+\varepsilon)m_{kj} + m_{jj} = 2\varepsilon(m_{kk} - m_{kj}) + \varepsilon^2 m_{kk}.$$

Dividing both sides of the inequality above by ε and letting $\varepsilon \to 0$,

$$m_{kk} - m_{kj} \geq 0.$$

Similarly, $m_{jj} - m_{kj} \geq 0$. Thus, $m_{kk} - m_{kj} = m_{jj} - m_{kj} = 0$ for all $k \neq j$.
Hence, there exists a non-negative measure m such that

$$m_{kj} = m \geq 0, \qquad \forall\, k, j.$$

Therefore, (3.5.5) becomes

$$0 = \int_{\mathbb{T}^n\times\mathbb{R}^n} e^{\lambda H}\Big(\lambda\big(\sum_j H_{p_j}\big)^2 + \sum_{j,k} H_{p_j p_k}\Big)\, dm.$$

We here point out two cases which guarantee that $m = 0$. In the first case, assuming additionally that $H(p) = H_1(p_1) + ...H_n(p_n)$ and $H_2, ..., H_n$ are convex, but not necessarily uniformly convex (their graphs may have flat regions) and H_1 is uniformly convex, then we still have $m = 0$.

In the second case, suppose that $H(p) = H(|p|)$, where $H : [0, \infty) \to \mathbb{R}$ is smooth, $H'(0) = 0, H''(0) > 0$ and $H'(s) > 0$ for $s > 0$. Notice that H is not necessarily convex. This example is similar to the example above when $n = 1$. Then for $p \neq 0$

$$\lambda\big(\sum_j H_{p_j}\big)^2 + \sum_{j,k} H_{p_jp_k} = n\frac{H'}{|p|} + \frac{(p_1 + ... + p_n)^2}{|p|^2}\left(\lambda(H')^2 + H'' - \frac{H'}{|p|}\right),$$

and at $p = 0$

$$\lambda\big(\sum_j H_{p_j}(0)\big)^2 + \sum_{j,k} H_{p_jp_k}(0) = nH''(0) > 0.$$

So, we can choose $r > 0$, small enough, so that for $|p| < r$

$$\lambda\big(\sum_j H_{p_j}\big)^2 + \sum_{j,k} H_{p_jp_k} > \frac{n}{2}H''(0) > 0.$$

Since the support of m is bounded, there exists $M > 0$ large enough

$$\operatorname{supp} m \subset \mathbb{T}^n \times \{p : |p| \leq M\}.$$

Since $\min_{s\in[r,M]} H'(s) > 0$, by choosing $\lambda \gg 0$, we finally have for $|p| \leq M$

$$\lambda\big(\sum_j H_{p_j}\big)^2 + \sum_{j,k} H_{p_jp_k} \geq \beta > 0,$$

for $\beta = \dfrac{n}{2}\min\left\{H''(0), \dfrac{\min_{s\in[r,M]} H'(s)}{M}\right\}$.
Thus $m = 0$, and therefore μ is invariant under the Hamiltonian dynamics.

Quasiconvex Hamiltonians: a special case

Let's consider

$$H(x, p) = H(|p|) + V(x),$$

where $H : [0, \infty) \to \mathbb{R}$ is smooth, $H'(0) = 0, H''(0) > 0$ and $H'(s) > 0$ for $s > 0$.
Once again, notice that H is not necessarily convex. We here will show that $(m_{jk}) = 0$.
For $p \neq 0$ then

$$(\lambda H_{p_j}H_{p_k} + H_{p_jp_k})m_{jk} = \frac{H'}{|p|}(m_{11} + ... + m_{nn}) + \left(\lambda(H')^2 + H'' - \frac{H'}{|p|}\right)\frac{p_jp_km_{jk}}{|p|^2}.$$

For any symmetric, non-negative definite matrix $m = (m_{jk})$ we have the following inequality

$$0 \leq p_jp_km_{jk} \leq |p|^2 \operatorname{trace} m = |p|^2(m_{11} + ... + m_{nn}).$$

There exists $r > 0$ small enough so that for $|p| < r$

$$\frac{H'}{|p|} > \frac{3}{4}H''(0); \qquad \left|\frac{H'}{|p|} - H''\right| < \frac{1}{4}H''(0).$$

Hence for $|p| < r$

$$(\lambda H_{p_j} H_{p_k} + H_{p_j p_k}) m_{jk} \geq \frac{1}{2} H''(0)(m_{11} + ... + m_{nn}).$$

Since the support of (m_{jk}) is bounded, there exists $M > 0$ large enough

$$\operatorname{supp} m_{jk} \subset \mathbb{T}^n \times \{p : |p| \leq M\}, \qquad \forall\, j, k.$$

Since $\min_{s\in[r,M]} H'(s) > 0$, by choosing $\lambda \gg 0$ we finally have for $|p| \leq M$

$$(\lambda H_{p_j} H_{p_k} + H_{p_j p_k}) m_{jk} \geq \beta(m_{11} + ... + m_{nn}),$$

for $\beta = \min\left\{\frac{H''(0)}{2}, \frac{\min_{s\in[r,M]} H'(s)}{M}\right\} > 0$.
We then must have $m_{11} + ... m_{nn} = 0$, which implies $(m_{jk}) = 0$. Thus, μ is invariant under the Hamiltonian dynamics in this case.
We now derive the property (2) of μ rigorously. Since the support of μ is also bounded, we can use a similar procedure as above to show that $\phi(x, p) = e^{\lambda H(x,p)}$ is uniformly convex in $\mathbb{T}^n \times \bar{B}(0, M) \supset \operatorname{supp}(\mu)$ for some λ large enough.
More precisely,

$$\phi_{p_j p_k} \xi_j \xi_k \geq e^{\lambda H} \beta |\xi|^2, \quad \xi \in \mathbb{R}^n,\ (x, p) \in \mathbb{T}^n \times \bar{B}(0, M),$$

for β chosen as above. Then doing the same steps as in [32], we get μ satisfies (2).
There is another simple approach to prove (2) by using the properties we get in this non-convex setting. Let's just assume that u is C^1 on the support of μ.
By Remark 3.8.2, it follows that $D_p H.(p - P - Du) = 0$ on support of μ. And since $D_p H(x, p) = H'(|p|)\frac{p}{|p|}$ for $p \neq 0$ and $H'(|p|) > 0$, we then have $p.(p - P - Du) = 0$ on support of μ. Hence $|p|^2 = p.(P + Du)$ on $\operatorname{supp}(\mu)$.
Besides, $H(x, p) = H(x, P + Du(x)) = \overline{H}(P)$ on $\operatorname{supp}(\mu)$ by property (a) of Mather measure and the assumption that u is C^1 on $\operatorname{supp}(\mu)$. It follows that $H(|p|) = H(|P + Du|)$. Therefore, $|p| = |P + Du|$ by the fact that $H(s)$ is strictly increasing.
So we have $|p|^2 = p.(P + Du)$ and $|p| = |P + Du|$ on $\operatorname{supp}(\mu)$, which implies $p = P + Du$ on $\operatorname{supp}(\mu)$, which is the property (2) of μ.

Quasiconvex Hamiltonians

We treat now the general case of uniformly quasiconvex Hamiltonians. We start with a definition.

Definition 3.9.1. A smooth set $A \subset \mathbb{R}^n$ is said to be *strongly convex with convexity constant* c if there exists a positive constant c with the following property. For every $p \in \partial A$ there exists an orthogonal coordinate system $(q_1, \ldots, q_n)$ centered at p, and a coordinate rectangle $R = (a_1, b_1) \times \ldots \times (a_n, b_n)$ containing p such that $T_p \partial A = \{q_n = 0\}$ and $A \cap R \subset \{q \in R : c \sum_{i=1}^{n-1} |q_i|^2 \leq q_n \leq b_n\}$.

The previous definition can be stated in the following equivalent way, by requiring that for every $p \in \partial A$

$$(\mathbf{B}_p \mathbf{v}) \cdot \mathbf{v} \geq c|\mathbf{v}|^2 \quad \text{for every } \mathbf{v} \in T_p \partial A,$$

where $\mathbf{B}_p : T_p\partial A \times T_p\partial A \to \mathbb{R}$ is the second fundamental form of ∂A at p.

We consider in this subsection strongly quasiconvex Hamiltonians. That is, we assume that there exists $c > 0$ such that

(j) $\{p \in \mathbb{T}^n : H(x,p) \leq a\}$ is strongly convex with convexity constant c for every $a \in \mathbb{R}$ and for every $x \in \mathbb{T}^n$.

In addition, we suppose that there exists $\alpha \in \mathbb{R}$ such that for every $x \in \mathbb{T}^n$

(jj) There exists unique $\overline{p} \in \mathbb{R}^n$ s.t. $D_pH(x, \overline{p}) = 0$, and

$$D^2_{pp}H(x, \overline{p}) \geq \alpha.$$

Notice that the special case presented in Section 3.9, where the level sets are spheres, fits into this definition. We will show that under hypotheses (j)–(jj) there exists $\lambda > 0$ such that

$$\lambda \, D_pH \otimes D_pH + D^2_{pp}H \quad \text{is positive definite.}$$

From this, thanks to relation (3.5.5), we conclude that $m_{kj} = 0$. First, we state a well-known result. We give the proof below, for the convenience of the reader.

Proposition 3.9.2. *Let (j)–(jj) be satisfied, and let* $(x^*, p^*) \in \mathbb{T}^n \times \mathbb{R}^n$ *be such that* $D_pH(x^*, p^*) \neq 0$. *Then*

$$D_pH(x^*, p^*) \perp T_{p^*}\mathcal{C} \quad \text{and} \quad D^2_{pp}H(x^*, p^*) = |D_pH(x^*, p^*)|\mathbf{B}_{p^*}, \tag{3.9.1}$$

where $\mathbf{B}_{p^*}$ *denotes the second fundamental form of the level set*

$$\mathcal{C} := \{p \in \mathbb{R}^n : H(x^*, p) = H(x^*, p^*)\}$$

at the point p^*.

Proof. By the smoothness of H, there exists a neighborhood $U \subset \mathbb{R}^n$ of p^* and n smooth functions $\nu : U \to \mathcal{S}^{n-1}$, $\tau_i : U \to \mathcal{S}^{n-1}$, $i = 1, \ldots, n-1$, such that for every $p \in U$ the vectors $\{\tau_1(p), \ldots, \tau_{n-1}(p), \nu(p)\}$ are a smooth orthonormal basis of $\mathbb{R}^n$, and for every $p \in U \cap \mathcal{C}$ $\tau_1(p), \ldots, \tau_{n-1}(p) \in T_p\mathcal{C}$. Let now $i, j \in \{1, \ldots, n-1\}$ be fixed. Since

$$H(x^*, p) = a \quad \forall \, p \in U,$$

differentiating w.r.t $\tau_i(p)$ we have

$$D_pH(x^*, p) \cdot \tau_i(p) = 0 \quad \forall \, p \in U \cap \mathcal{C}. \tag{3.9.2}$$

Computing last relation at $p = p^*$ we get that $D_pH(x^*, p^*) \perp T_{p^*}\mathcal{C}$. Differentiating (3.9.2) along the direction $\tau_j(p)$ and computing at $p = p^*$

$$\left(D^2_{pp}H(x^*, p^*)\tau_j(p^*)\right) \cdot \tau_i(p^*) + D_pH(x^*, p^*) \cdot (D_p\tau_i(p^*)\tau_j(p^*)) = 0. \tag{3.9.3}$$

Notice that by differentiating along the direction $\tau_j(p)$ the identity $\tau_i(p) \cdot \nu(p) = 0$ and computing at p^* we get

$$(D_p\tau_i(p^*)\tau_j(p^*)) \cdot \nu(p^*) = -\left(D_p\nu(p^*)\tau_j(p^*)\right) \cdot \tau_i(p^*).$$

Plugging last relation into (3.9.3), and choosing $\nu(p^*)$ oriented in the direction of $D_pH(x^*, p^*)$ we have

$$\begin{aligned}&\left(D^2_{pp}H(x^*, p^*)\tau_j(p^*)\right) \cdot \tau_i(p^*) = -|D_pH(x^*, p^*)|\left(D_p\tau_i(p^*)\tau_j(p^*)\right) \cdot \nu(p^*)\\&= |D_pH(x^*, p^*)|\left(D_p\nu(p^*)\tau_j(p^*)\right) \cdot \tau_i(p^*) = |D_pH(x^*, p^*)|\left(\mathbf{B}_{p^*}\tau_j(p^*)\right) \cdot \tau_i(p^*).\end{aligned}$$

□

For every vector $v \in \mathbb{R}^n$, we consider the decomposition

$$v = v_{\|}\mathbf{v}^{\|} + v_{\perp}\mathbf{v}^{\perp},$$

with $v^{\|}, v^{\perp} \in \mathbb{R}$, $|\mathbf{v}^{\|}| = |\mathbf{v}^{\perp}| = 1$, $\mathbf{v}^{\|} \in T_{p^*}\mathcal{C}$, and $\mathbf{v}^{\perp} \in (T_{p^*}\mathcal{C})^{\perp}$. By hypothesis (jj) and by the smoothness of H, there exist $\tau > 0$ and $\alpha' \in (0, \alpha)$, independent of (x, p), such that

$$D^2_{pp}H(x, p) \geq \alpha' \quad \text{for every } (x, p) \in \{|D_pH| \leq \tau\}.$$

Let us now consider two subcases:
Case 1: $(x, p) \in \{|D_pH| \leq \tau\}$

First of all, notice that

$$\lambda\, D_pH \otimes D_pHv \cdot v = \lambda|D_pH \cdot v|^2 = \lambda\, v_{\perp}^2\, |D_pH|^2.$$

Then, we have

$$(\lambda\, D_pH \otimes D_pH + D^2_{pp}H)v \cdot v = \lambda\, v_{\perp}^2\, |D_pH|^2 + (D^2_{pp}Hv \cdot v) \geq \alpha'|v|^2.$$

Case 2: $(x, p) \in \{|D_pH| > \tau\}$

In this case we have

$$D^2_{pp}H\mathbf{v}^{\|} \cdot \mathbf{v}^{\|} \geq c|D_pH|,$$

which then yields

$$\begin{aligned}D^2_{pp}Hv \cdot v &= v_{\|}^2(D^2_{pp}H\mathbf{v}^{\|} \cdot \mathbf{v}^{\|}) + 2v_{\|}v_{\perp}(D^2_{pp}H\mathbf{v}^{\|} \cdot \mathbf{v}^{\perp}) + v_{\perp}^2(D^2_{pp}H\mathbf{v}^{\perp} \cdot \mathbf{v}^{\perp})\\&\geq c\, v_{\|}^2|D_pH| + 2v_{\|}v_{\perp}(D^2_{pp}H\mathbf{v}^{\|} \cdot \mathbf{v}^{\perp}) + v_{\perp}^2(D^2_{pp}H\mathbf{v}^{\perp} \cdot \mathbf{v}^{\perp}).\end{aligned}$$

By (3.5.3) we have

$$|D^2_{pp}H| \leq C \quad \text{along } \operatorname{supp}\mu.$$

Thus,

$$\begin{aligned}
&(\lambda\, D_pH \otimes D_pH + D^2_{pp}H)v \cdot v \\
&\geq \lambda\, v_\perp^2\, |D_pH|^2 + c\, v_\parallel^2 |D_pH| + 2v_\parallel v_\perp (D^2_{pp}H\mathbf{v}^\parallel \cdot \mathbf{v}^\perp) + v_\perp^2 (D^2_{pp}H\mathbf{v}^\perp \cdot \mathbf{v}^\perp) \\
&\geq v_\perp^2 \left(\lambda\, |D_pH|^2 - C\right) - 2C|v_\parallel||v_\perp| + c\, v_\parallel^2 |D_pH| \\
&> v_\perp^2 \Big(\lambda\, \tau^2 - C\Big(1 + \frac{1}{\eta^2}\Big)\Big) + v_\parallel^2 (c\,\tau - C\eta^2\,).
\end{aligned}$$

Choosing first $\eta^2 < \frac{c\,\tau}{C}$, and then

$$\lambda > \frac{C}{\tau^2}\Big(1 + \frac{1}{\eta^2}\Big),$$

we obtain

$$(\lambda\, D_pH \otimes D_pH + D^2_{pp}H)v \cdot v \geq \alpha''|v|^2,$$

for some $\alpha'' > 0$, independent of (x, p).

General Case

In the general case, we have

$$(\lambda\, D_pH \otimes D_pH + D^2_{pp}H)v \cdot v \geq \gamma|v|^2,$$

where $\gamma := \min\{\alpha', \alpha''\}$.
Similar to the case above, we basically have that $\phi(x,p) = e^{\lambda H(x,p)}$ is uniformly convex on the support of μ for λ large enough. Hence, by repeating again the same steps as in [32], we finally get that μ satisfies (2). As already mentioned in the introduction, we observe that one could also study the case of uniformly convex Hamiltonians by duality, that is, by considering a function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(H(x,\cdot))$ is convex for each $x \in \mathbb{T}^n$. In this way, the dynamics can be seen as a reparametrization of the dynamics associated to the convex Hamiltonian $\Phi(H)$.

3.10 A one dimensional example of nonvanishing dissipation measure m

In this section we sketch a one dimensional example in which the dissipation measure m does not vanish. We assume that the zero level set of the Hamiltonian $H : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is the smooth curve in Figure 3.1, and that everywhere else in the plane (x, p) the signs of H are as shown in the picture. In addition, H can be constructed in such a way that $(D_xH, D_pH) \neq (0, 0)$ for every $(x, p) \in \{(x, p) \in \mathbb{T} \times \mathbb{R} : H(x, p) = 0\}$. That is, the

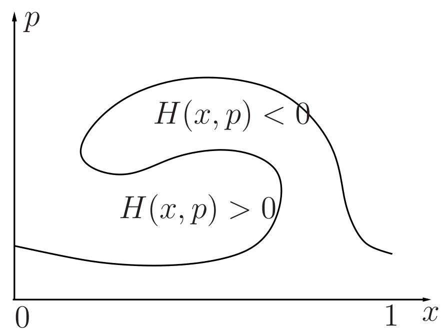

Figure 3.1: $\{H(x,p)=0\}$.

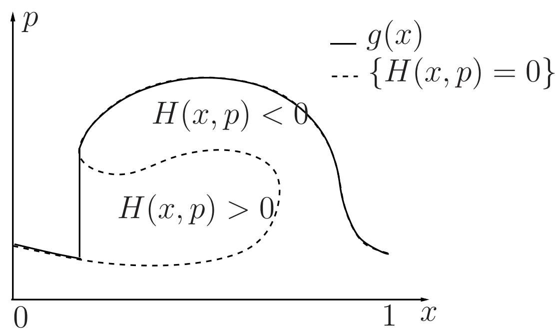

Figure 3.2: $g(x)$.

zero level set of H does not contain any equilibrium point. Consider now the piecewise continuous function $g:[0,1]\to\mathbb{R}$, with $g(0)=g(1)$, as shown in Figure 3.2. Then, set

$$P := \int_0^1 g(x)\,dx,$$

and define

$$u(x,P) := -Px + \int_0^x g(y)\,dy.$$

One can see that $u(\cdot,P)$ is the unique periodic viscosity solution of

$$H(x, P + D_x u(x,P)) = 0,$$

that is equation (3.1.2) with $\overline{H}(P)=0$. Assume now that a Mather measure μ exists, satisfying property (1). Then, the support of μ has necessarily to be concentrated on the graph of g, and not on the whole level set $\{H=0\}$. However, any invariant measure by the Hamiltonian flow will be supported on the whole set $\{H=0\}$, due to the non existence of equilibria and to the one-dimensional nature of the problem, thus giving a contradiction.

Chapter 4

Homogenization of weakly coupled systems of Hamilton–Jacobi equations with fast switching rates

4.1 Introduction

In this Chapter we study the behavior, as $\varepsilon(>0)$ tends to 0, of the viscosity solutions $(u_1^\varepsilon, u_2^\varepsilon)$ of the following weakly coupled systems of Hamilton–Jacobi equations

$$
(\mathrm{C}_\varepsilon) \qquad \begin{cases} (u_1^\varepsilon)_t + H_1(\frac{x}{\varepsilon}, Du_1^\varepsilon) + \frac{c_1}{\varepsilon}(u_1^\varepsilon - u_2^\varepsilon) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ (u_2^\varepsilon)_t + H_2(\frac{x}{\varepsilon}, Du_2^\varepsilon) + \frac{c_2}{\varepsilon}(u_2^\varepsilon - u_1^\varepsilon) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u_i^\varepsilon(x,0) = f_i(x) & \text{on } \mathbb{R}^n \text{ for } i = 1,2, \end{cases}
$$

where $T > 0$, c_1, c_2 are given positive constants and the Hamiltonians $H_i(\xi,p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are given continuous functions for $i = 1,2$, which are assumed throughout the Chapter to satisfy the followings.

(A1) The functions H_i are uniformly coercive in the ξ-variable, i.e.,

$$\lim_{r\to\infty} \inf\{H_i(\xi,p) \mid \xi \in \mathbb{R}^n, |p| \geq r\} = \infty.$$

(A2) The functions $\xi \mapsto H_i(\xi,p)$ are $\mathbb{T}^n$-periodic, i.e., $H_i(\xi+z,p) = H_i(\xi,p)$ for any $\xi, p \in \mathbb{R}^n$, $z \in \mathbb{Z}^n$ and $i = 1,2$.

The functions f_i are given continuously differentiable functions on $\mathbb{R}^n$ with $\|Df_i\|_{L^\infty(\mathbb{R}^n)}$ are bounded for $i = 1,2$, respectively. Here u_i^ε are the real-valued unknown functions on $\mathbb{R}^n \times [0,T]$ and $(u_i^\varepsilon)_t := \partial u_i^\varepsilon/\partial t$, $Du_i^\varepsilon := (\partial u_i^\varepsilon/\partial x_1, \ldots, \partial u_i^\varepsilon/\partial x_n)$ for $i = 1,2$, respectively.

Background: Randomly Switching Cost Problems

System (C_ε) arises as the dynamic programming for the optimal control of the system whose states are governed by certain ODEs, subject to random changes in the dynamics: the system randomly switches at a fast rate $1/\varepsilon$ among the two states. See [24, 25, 30]. Also see [71, 58, 15] for another switching cost problems. In order to explain the background more precisely, we assume in addition that the Hamiltonians H_i are convex in p here. We define the functions $u_i^\varepsilon \to \mathbb{R}^n \times [0,T] \to \mathbb{R}$ by

$$u_i^\varepsilon(x,t) := \inf\Big\{\mathbb{E}_i\Big(\int_0^t L_{\nu^\varepsilon(s)}(\frac{\eta(s)}{\varepsilon}, -\dot\eta(s))\,ds + f_{\nu^\varepsilon(t)}(\eta(t))\Big)\Big\}, \tag{4.1.1}$$

where $L_i : \mathbb{R}^{2n} \to \mathbb{R}$ are the Fenchel-Legendre transform of H_i, i.e., $L_i(\xi,q) := \sup_{p\in\mathbb{R}^n}(p\cdot q - H_i(\xi,p))$ for all $(\xi,q)\in\mathbb{R}^{2n}$ and the infimum is taken over $\eta \in \mathrm{AC}([0,t],\mathbb{R}^n)$ such that $\eta(0)=x$. Here $\mathbb{E}_i$ denotes the expectation of a process with $\nu^\varepsilon(0)=i$ where ν is a $\{1,2\}$-valued process which is a continuous-time Markov chain such that

$$\mathbb{P}\big(\nu^\varepsilon(s+\Delta s)=j \mid \nu^\varepsilon(s)=i\big) = \frac{c_i}{\varepsilon}\Delta s + o(\Delta s) \text{ as } \Delta s\to 0 \text{ for } i\neq j, \tag{4.1.2}$$

where $o:[0,\infty)\to[0,\infty)$ is a function which satisfies $o(r)/r\to 0$ as $r\to 0$. Formula (4.1.1) is basically the optimal control formula for the solution of (C_ε), where the random switchings among the two states are governed by (4.1.2).

We first give a formal proof that $(u_1^\varepsilon, u_2^\varepsilon)$ given by (4.1.1) is a solution of (C_ε). The rigorous derivation will be proved in Section 4.7 by using the dynamic programming principle. We suppose that $u_i \in C^1(\mathbb{R}^n\times[0,T])$. Set $u(x,i,t) := u_i(x,t)$ and $Y(s) := (\eta(s),\nu^\varepsilon(s))$ for $\eta\in\mathrm{AC}(\mathbb{R}^n)$ with $\eta(0)=x$ and let ν^ε be a Markov chain given by (4.1.2) with $\eta(0)=x$ and $\nu^\varepsilon(0)=i$. By Ito's formula for a jump process we have

$$\begin{aligned}
&\mathbb{E}_i\Big(u^\varepsilon(Y(t),0) - u^\varepsilon(Y(0),t)\Big)\\
=&\mathbb{E}_i\Big(\int_0^t -u_t^\varepsilon(Y(s),t-s) + Du^\varepsilon(Y(s),t-s)\cdot\dot\eta(s)\,ds\\
&\quad+\int_0^t\sum_{j=1}^2\big(u^\varepsilon(\eta(s),j,s) - u^\varepsilon(\eta(s),\nu^\varepsilon(s),s)\big)\cdot\frac{c_{\nu^\varepsilon(s)}}{\varepsilon}\,ds\Big)\\
\geq&\mathbb{E}_i\Big(\int_0^t -u_t^\varepsilon(Y(s),t-s) - H_{\nu^\varepsilon(s)}(\frac{\eta}{\varepsilon}, Du^\varepsilon) - L_{\nu^\varepsilon(s)}(\frac{\eta}{\varepsilon},-\dot\eta)\,ds\\
&\quad+\int_0^t\sum_{j=1}^2\big(u^\varepsilon(\eta(s),j,s) - u^\varepsilon(\eta(s),\nu^\varepsilon(s),s)\big)\cdot\frac{c_{\nu^\varepsilon(s)}}{\varepsilon}\,ds\Big)\\
=&-\mathbb{E}_i\Big(\int_0^t L_{\nu^\varepsilon(s)}(\frac{\eta}{\varepsilon},-\dot\eta)\,ds\Big).
\end{aligned}$$

Thus,

$$u^\varepsilon(x,i,t) \leq \mathbb{E}_i\Big(\int_0^t L_{\nu^\varepsilon(s)}(\frac{\eta}{\varepsilon},-\dot\eta)\,ds + u^\varepsilon(Y(t),0)\Big).$$

In the above inequality, the equality holds if $-\dot{\eta}(s) \in D_p^- H_{\nu^\varepsilon(s)}(\eta(s)/\varepsilon, Du(Y(s), s))$, where $D_p^- H_i$ denotes the subdifferential of H_i with respect to the p-variable.

Main Results

There have been extensively many important results on the study of homogenization of Hamilton–Jacobi equations. The first general result is due to Lions, Papanicolaou, and Varadhan [60] who studied the cell problems together with the effective Hamiltonian and established homogenization results under quite general assumptions on the Hamiltonians in the periodic setting. The next major contributions to the subject are due to Evans [30, 29] who introduced the perturbed test functions methods in the framework of viscosity solutions. The methods then have been adapted to study many different homogenization problems. Then Concordel [19, 18] achieved some first general results on the properties of the effective Hamiltonian concerning flat parts and non-flat parts. Afterwards Capuzzo-Dolceta and Ishii [16] combined the perturbed test functions with doubling variables methods to obtain the first results on the rate of convergence of u^ε to u. We refer to [12, 79] for some recent progress.

There have been some interesting results [76, 13, 14] on the study of homogenization for weakly coupled systems of Hamilton–Jacobi equations in the periodic settings or in the almost periodic settings. We refer the readers to [26, 52] for the complete theory of viscosity solutions for weakly coupled systems of Hamilton–Jacobi and Hamilton–Jacobi–Bellman equations. Since the maximum principle and comparison principle still hold, homogenization results can be obtained by using the perturbed test functions methods quite straightforwardly with some modifications. Let us call attention also to the new interesting direction on the large time behavior of weakly coupled sytems of Hamilton–Jacobi equations, which is related to homogenization through the cell problems. The authors [69], and Camilli, Ley, Loreti and Nguyen [15] obtained large time behavior results for some special cases but general cases still remain open.

Let us also refer to one of the main research directions in the study of homogenization, stochastic homogenization of Hamilton–Jacobi equations, which were first obtained by Souganidis [77], and Rezakhanlou and Tarver [73] independently. See [61, 56, 75, 62, 3] for more recent progress on the subject.

First we heuristically derive the behavior of solutions of (C_ε) as ε tends to 0. For simplicity, from now on, we *always* assume that $c_1 = c_2 = 1$. We consider the formal asymptotic expansions of the solutions $(u_1^\varepsilon, u_2^\varepsilon)$ of (C_ε) of the form

$$u_i^\varepsilon(x,t) := u_i(x,t) + \varepsilon v_i(\frac{x}{\varepsilon}) + O(\varepsilon^2).$$

Set $\xi := x/\varepsilon$. Plugging this into (C_ε) and performing formal calculations, we achieve

$$(u_i)_t + \ldots + H_i(\xi, D_x u_i + D_\xi v_i + \cdots) + \frac{1}{\varepsilon}(u_i - u_j) + (v_i - v_j) + \cdots = 0,$$

where we take $i, j \in \{1, 2\}$ such that $\{i, j\} = \{1, 2\}$. The above expansion implies that $u_1 = u_2 =: u$. Furthermore, if we let $P = Du(x,t)$ then (v_1, v_2) is a $\mathbb{T}^n$-periodic solution of the following cell problem

$$(\mathrm{E}_P) \qquad \begin{cases} H_1(\xi, P + Dv_1(\xi, P)) + v_1(\xi, P) - v_2(\xi, P) = \overline{H}(P) & \text{in } \mathbb{R}^n, \\ H_2(\xi, P + Dv_2(\xi, P)) + v_2(\xi, P) - v_1(\xi, P) = \overline{H}(P) & \text{in } \mathbb{R}^n, \end{cases}$$

where $\overline{H}(P)$ is a unknown constant. Because of the $\mathbb{T}^n$-periodicity of the Hamiltonians H_i, we can also consider the above cell problem on the torus $\mathbb{T}^n$, which is equivalent to consider it on $\mathbb{R}^n$ with $\mathbb{T}^n$-periodic solutions. By an argument similar to the classical one in [60], we have

Proposition 4.1.1 (Cell Problems)**.** *For any $P \in \mathbb{R}^n$, there exists a unique constant $\overline{H}(P)$ such that (E_P) admits a $\mathbb{T}^n$-periodic solution $(v_1(\cdot, P), v_2(\cdot, P)) \in C(\mathbb{R}^n)^2$. We call $\overline{H}$ the effective Hamiltonian assosiated with (H_1, H_2).*

See also [9, 69] for more details about the cell problems.

Our main goal in this Chapter is threefold. First of all, we want to demonstrate that u_i^ε converge locally uniformly to the same limit u in $\mathbb{R}^n \times (0, T)$ for $i = 1, 2$ and u solves

$$u_t + \overline{H}(Du) = 0 \quad \text{in } \mathbb{R}^n \times (0, T).$$

This part is a rather standard part in the study of homogenization of Hamilton–Jacobi equations by using the perturbed test functions method introduced by Evans [30] with some modifications. The only hard part comes from the fact that we do not have uniform bounds on the gradients of u_i^ε here because of the fast switching terms. We overcome this difficulty by introducing the barrier functions (see Lemma 4.2.1) and using the half-relaxed limits (see the proof of Theorem 4.1.2). The barrier functions furthermore give us the correct initial data for the limit u. Let $(u_1^\varepsilon, u_2^\varepsilon)$ be the solution of (C_ε) henceforth.

Theorem 4.1.2 (Homogenization Result)**.** *Then, u_i^ε converge locally uniformly in $\mathbb{R}^n \times (0, T)$ to the same limit $u \in C^{0,1}(\mathbb{R}^n \times [0, T])$ as $\varepsilon \to 0$ for $i = 1, 2$ and u solves*

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, 0) = \overline{f}(x) := \dfrac{f_1(x) + f_2(x)}{2} & \text{on } \mathbb{R}^n. \end{cases} \tag{4.1.3}$$

Formula (4.1.1) of solutions of (C_ε) actually gives us an intuitive explanation about the effective initial datum $\overline{f}$. As we send ε to 0, the switching rate becomes very fast and processes have to jump randomly very quickly between the two states with equal probability as given by (4.1.2) with $c_1 = c_2 = 1$. Therefore, it is relatively clear that $\overline{f}$ is the average of the given initial data f_i for $i = 1, 2$.

The second main part of this Chapter is the study of the initial layers appearing naturally in the problem as the initial data of u_i^ε and u are different in general. We first study the initial layers in a heuristic mode by finding inner and outer solutions, and using the matching asymptotic expansion method to identify matched solutions (see Section 3.1). We then combine the techniques of the matching asymptotic expansion method and of Capuzzo-Dolceta and Ishii [16] to obtain rigorously the rate of convergence result.

Theorem 4.1.3 (Rate of Convergence to Matched Solutions)**.** *For each $T > 0$, there exists $C := C(T) > 0$ such that*

$$\|u_i^\varepsilon - m_i^\varepsilon\|_{L^\infty(\mathbb{R}^n\times[0,T])} \leq C\varepsilon^{1/3} \text{ for } i = 1, 2,$$

where u is the solution of (4.1.3) *and*

$$m_i^\varepsilon(x,t) := u(x,t) + \frac{(f_i - f_j)(x)}{2} e^{-\frac{2t}{\varepsilon}} \tag{4.1.4}$$

with $j \in \{1, 2\}$ such that $\{i, j\} = \{1, 2\}$.

Finally, we study various properties of the effective Hamiltonian $\overline{H}$. It is always extremely hard to understand properties of the effective Hamiltonians even for single equations. Lions, Papanicolaou and Vadrahan [60] studied some preliminary properties of the effective Hamiltonians and pointed out a 1-dimensional example that $\overline{H}$ can be computed explicitly. After that, Concordel [19, 18] discovered some very interesting results related flat parts and non-flat parts of $\overline{H}$ for more general cases. Evans and Gomes [32] found some further properties on the strict convexity of $\overline{H}$ by using the weak KAM theory.

The properties of $\overline{H}$ for weakly coupled systems of Hamilton–Jacobi equations in this Chapter are even more complicated. In case $H_1 = H_2$ then the effective Hamiltonian for the weakly coupled systems and the single equations are obviously same. Therefore, we can view the cases of single equations as special cases of the weakly coupled systems. However, in general, we cannot expect the effective Hamiltonians for weakly coupled systems to have similar properties like single equations' cases.

The first few results on flat parts and non-flat parts of $\overline{H}$ are generalizations to the ones discovered by Concordel [19, 18], and are proved by using different techniques, namely the min-max formulas which are derived in Section 4.4 and the constructions of subsolutions. On the other hand, we investigate other cases which show that the properties of the effective Hamiltonians for weakly coupled systems are widely different from those of the effective Hamiltonians for single equations. Theorems 4.4.14, 4.4.17, 4.4.18 , 4.4.20, which are ones of our main results, describe some rather new results which do not appear in the context of single HJ equations. Since the theorems are technical, we refer the readers to Section 4.4 for details.

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This Chapter is organized as follows. In Section 2 we prove homogenization result, Theorem 4.1.2. Section 3 devotes to the study of initial layers and rate of convergence. We derive inner solutions, outer solutions, and matched asymptotic solutions in the heuristic mode and then prove Theorem 4.1.3. The properties of the effective Hamiltonian are studied in Section 4. We obtain its elementary properties in Section 4.1, the representation formulas in Section 4.2, and flat parts, non-flat parts near the origin in Section 4.3. In Section 5 we prove generalization results for systems of m equations for $m \geq 2$. We then prove also the homogenization result for Dirichlet problems and describe the differences of the effective data between Cauchy problems and Dirichlet problems in Section 4.6. Some lemmata concerning verifications of optimal control formulas for the Cauchy and Dirichlet problems are recorded in Section 4.7.

Notations. For $k \in \mathbb{N}$ and $A \subset \mathbb{R}^n$, we denote by $C(A)$, $C^{0,1}(A)$ and $C^k(A)$ the space of real-valued continuous, Lipschitz continuous and k-th continuous differentiable functions on A, respectively. We denote $L^\infty(A)$ by the set of bounded measurable functions and $\|\cdot\|_{L^\infty(A)}$ denotes the superemum norm. Let $\mathbb{T}^n$ denote the n-dimensional torus and we identify $\mathbb{T}^n$ with $[0,1]^n$. Define $\Pi : \mathbb{R}^n \to \mathbb{T}^n$ as the canonical projection. By abuse of notations, we denote the periodic extensions of any set $B \subset \mathbb{T}^n$ and any function $f \in C(\mathbb{T}^n)$ to the whole space $\mathbb{R}^n$ by B, and f themselves respectively. For $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. We call a function $m : [0, \infty) \to [0, \infty)$ a *modulus* if it is continuous, nondecreasing on $[0, \infty)$ and $m(0) = 0$.

4.2 Homogenization Results

Lemma 4.2.1 (Barrier Functions)**.** *We define the functions $\varphi_i^\pm : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ by*

$$\begin{cases} \varphi_1^\pm(x,t) = \dfrac{f_1(x) + f_2(x)}{2} + \dfrac{f_1(x) - f_2(x)}{2} e^{-\frac{2t}{\varepsilon}} \pm Ct \\ \varphi_2^\pm(x,t) = \dfrac{f_1(x) + f_2(x)}{2} + \dfrac{f_2(x) - f_1(x)}{2} e^{-\frac{2t}{\varepsilon}} \pm Ct. \end{cases} \tag{4.2.1}$$

If we choose $C \geq \max_{i=1,2} \max_{(\xi,p)\in\mathbb{R}^n\times B(0,r)} |H_i(\xi, p)|$, where $r = \|Df_1\|_{L^\infty(\mathbb{R}^n)} + \|Df_2\|_{L^\infty(\mathbb{R}^n)}$, then $(\varphi_1^-, \varphi_2^-)$ and $(\varphi_1^+, \varphi_2^+)$ are, respectively, a subsolution and a supersolution of (C_ε), and

$$(\varphi_1^-, \varphi_2^-)(\cdot, 0) = (\varphi_1^+, \varphi_2^+)(\cdot, 0) = (f_1, f_2) \text{ on } \mathbb{R}^n.$$

In particular, $\varphi_i^- \leq u_i^\varepsilon \leq \varphi_i^+$ on $\mathbb{R}^n \times [0,T]$ for $i = 1, 2$.

Proof. We calculate that

$$\begin{aligned} &(\varphi_1^-)_t + H_1(\frac{x}{\varepsilon}, D\varphi_1^-) + \frac{1}{\varepsilon}(\varphi_1^- - \varphi_2^-) \\ =& -\frac{f_1(x) - f_2(x)}{\varepsilon} e^{-\frac{2t}{\varepsilon}} - C + H_1(\frac{x}{\varepsilon}, D\varphi_1^-) + \frac{f_1(x) - f_2(x)}{\varepsilon} e^{-\frac{2t}{\varepsilon}} \\ =& -C + H_1(\frac{x}{\varepsilon}, D\varphi_1^-) \leq 0 \end{aligned}$$

for $C > 0$ large enough as chosen above. Similar calculations give us that $(\varphi_1^-, \varphi_2^-)$ and $(\varphi_1^+, \varphi_2^+)$ are, respectively, a subsolution and a supersolution of (C_ε). By the comparison principle for (C_ε) (see [26, 52]) we get $\varphi_i^- \leq u_i^\varepsilon \leq \varphi_i^+$ on $\mathbb{R}^n \times [0,T]$ for $i = 1, 2$. □

Proof of Theorem 4.1.2. By Lemma 4.2.1 we can take the following half-relaxed limits

$$\begin{cases} W(x,t) = \text{limsup}^*{}_{\varepsilon\to 0} \sup_{i=1,2}[u_i^\varepsilon](x,t) \\ w(x,t) = \text{liminf}_{*\varepsilon\to 0} \inf_{i=1,2}[u_i^\varepsilon](x,t). \end{cases}$$

We now show that W and w are, respectively, a subsolution and a supersolution of (4.1.3) in $\mathbb{R}^n \times (0,T)$ by employing the perturbed test functions method.

Since we can easily check $W(\cdot,0) = w(\cdot,0) = \overline{f}$ on $\mathbb{R}^n$ due to Lemma 4.2.1, it is enough to prove that w and W are a subsolution and a supersolution, respectively, of the equation in (4.1.3). We only prove that W is a subsolution since by symmetry we can prove that w is a supersolution. We take a test function $\phi \in C^1(\mathbb{R}^n \times (0,T))$ such that $W - \phi$ has a strict maximum at $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$. Let $P := D\phi(x_0,t_0)$. Choose a sequence $\varepsilon_m \to 0$ such that

$$W(x_0,t_0) = \text{limsup}^*{}_{m\to\infty} \max_{i=1,2} u_i^{\varepsilon_m}(x_0,t_0).$$

We define the perturbed test functions $\psi_i^{\varepsilon,\alpha}$ for $i = 1,2$ and $\alpha > 0$ by

$$\psi_i^{\varepsilon,\alpha}(x,y,t) := \phi(x,t) + \varepsilon v_i(\frac{y}{\varepsilon}) + \frac{|x-y|^2}{2\alpha^2},$$

where (v_1,v_2) is a solution of (E_P). By the usual argument in the theory of viscosity solutions, for every $m \in \mathbb{N}$, $\alpha > 0$, there exist $i_{m,\alpha} \in \{1,2\}$ and $(x_{m,\alpha}, y_{m,\alpha}, t_{m,\alpha}) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,T)$ such that

$$\max_{i=1,2} \max_{\mathbb{R}^n\times\mathbb{R}^n\times[0,T]} [u_i^{\varepsilon_m}(x,t) - \psi_i^{\varepsilon_m,\alpha}(x,y,t)] = u_{i_{m,\alpha}}^{\varepsilon_m}(x_{m,\alpha},t_{m,\alpha}) - \psi_{i_{m,\alpha}}^{\varepsilon_m,\alpha}(x_{m,\alpha},y_{m,\alpha},t_{m,\alpha}) \tag{4.2.2}$$

and up to passing some subsequences

$$\begin{aligned} &(x_{m,\alpha}, y_{m,\alpha}, t_{m,\alpha}) \to (x_m, x_m, t_m) \text{ as } \alpha \to 0, \\ &i_{m,\alpha} \to i_m \in \{1,2\} \text{ as } \alpha \to 0, \\ &(x_m,t_m) \to (x_0,t_0) \text{ as } m \to \infty, \\ &\lim_{m\to\infty}\lim_{\alpha\to 0} u_{i_{m,\alpha}}^{\varepsilon_m}(x_{m,\alpha},t_{m,\alpha}) = W(x_0,t_0). \end{aligned}$$

Choose $j_{m,\alpha}, j_m \in \{1,2\}$ such that $\{i_{m,\alpha}, j_{m,\alpha}\} = \{i_m, j_m\} = \{1,2\}$. By the definition of viscosity solutions, we have

$$\phi_t(x_{m,\alpha},t_{m,\alpha}) + H_{i_{m,\alpha}}(\frac{x_{m,\alpha}}{\varepsilon_m}, D\phi(x_{m,\alpha},t_{m,\alpha}) + \frac{x_{m,\alpha}-y_{m,\alpha}}{\alpha^2}) + \frac{1}{\varepsilon_m}(u_{i_{m,\alpha}}^{\varepsilon_m} - u_{j_{m,\alpha}}^{\varepsilon_m})(x_{m,\alpha},t_{m,\alpha}) \le 0. \tag{4.2.3}$$

Since (v_1,v_2) is a solution of (E_P) we have

$$H_{i_{m,\alpha}}(\frac{y_{m,\alpha}}{\varepsilon_m}, P + \frac{x_{m,\alpha}-y_{m,\alpha}}{\alpha^2}) + (v_{i_{m,\alpha}} - v_{j_{m,\alpha}})(\frac{y_{m,\alpha}}{\varepsilon_m}) \ge \overline{H}(P). \tag{4.2.4}$$

Let $\alpha \to 0$ in (4.2.3) and (4.2.4) to derive

$$\phi_t(x_m, t_m) + H_{i_m}(\frac{x_m}{\varepsilon_m}, D\phi(x_m, t_m) + Q_m) + \frac{1}{\varepsilon_m}(u^{\varepsilon_m}_{i_m} - u^{\varepsilon_m}_{j_m})(x_m, t_m) \le 0 \tag{4.2.5}$$

and

$$H_{i_m}(\frac{x_m}{\varepsilon_m}, P + Q_m) + (v_{i_m} - v_{j_m})(\frac{x_m}{\varepsilon_m}) \ge \overline{H}(P), \tag{4.2.6}$$

where $Q_m = \lim_{\alpha\to 0} \frac{x_{m,\alpha} - y_{m,\alpha}}{\alpha^2}$. Noting that the correctors v_i are Lipschitz continuous due to the coercivity of H_i, we see that $|Q_m| \le C$ for $C > 0$ which is independent of m. Combine (4.2.5) with (4.2.6) to get

$$\begin{aligned} \phi_t(x_m, t_m) + \overline{H}(P) \le\ & H_{i_m}(\frac{x_m}{\varepsilon_m}, P + Q_m) - H_{i_m}(\frac{x_m}{\varepsilon_m}, D\phi(x_m, t_m) + Q_m) \\ & + \frac{1}{\varepsilon_m}[u^{\varepsilon_m}_{j_m}(x_m, t_m) - (\phi(x_m, t_m) + \varepsilon_m v_{j_m}(\frac{x_m}{\varepsilon_m}))] \\ & - \frac{1}{\varepsilon_m}[u^{\varepsilon_m}_{i_m}(x_m, t_m) - (\phi(x_m, t_m) + \varepsilon_m v_{i_m}(\frac{x_m}{\varepsilon_m}))] \\ \le\ & \sigma(|P - D\phi(x_m, t_m)|) \end{aligned}$$

for some modulus σ. Letting $m \to \infty$, we get the result.

We finally prove that u is Lipschitz continuous. We can easily see that $\overline{f} \pm Mt$ are a supersolution and a subsolution of (4.1.3), respectively, for $M > 0$ large enough. By the comparison principle for (4.1.3) we have $|u(x,t) - \overline{f}(x)| \le Mt$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$. Moreover, the comparison principle for (4.1.3) also yields that

$$\sup_{x\in\mathbb{R}^n} |u(x, t+s) - u(x,t)| \le \sup_{x\in\mathbb{R}^n} |u(x,s) - \overline{f}(x)| \le Ms \text{ for all } t, s \ge 0, \text{ and}$$
$$\sup_{x\in\mathbb{R}^n} |u(x+z, t) - u(x,t)| \le \sup_{x\in\mathbb{R}^n} |\overline{f}(x+z) - \overline{f}(x)| \le r|z| \text{ for all } z \in \mathbb{R}^n,\ t \ge 0.$$

The proof is complete. □

4.3 Initial layers and rate of convergence

Inner solutions, Outer solutions, and Matched solutions

We first derive inner solutions, outer solutions and perform the matching asymptotic expansion method to find matched solutions in a heuristic mode.

As we already obtained in Section 4.2, outer solutions are same as the limit u give in Theorem 4.1.2. Now we need to find a right scaling for inner solutions. We let

$$w^\varepsilon_i(x,t) = u^\varepsilon_i(x, \frac{t}{\varepsilon}) \text{ for } i = 1, 2,$$

and plug into (C_ε) to obtain

$$(\mathrm{I}_\varepsilon) \qquad \begin{cases} (w_1^\varepsilon)_t + \varepsilon H_1(\frac{x}{\varepsilon}, Dw_1^\varepsilon) + (w_1^\varepsilon - w_2^\varepsilon) = 0 & \text{in } \mathbb{R}^n \times (0, T/\varepsilon), \\ (w_2^\varepsilon)_t + \varepsilon H_2(\frac{x}{\varepsilon}, Dw_2^\varepsilon) + (w_2^\varepsilon - w_1^\varepsilon) = 0 & \text{in } \mathbb{R}^n \times (T/\varepsilon), \\ w_i^\varepsilon(x,0) = f_i(x) & \text{on } \mathbb{R}^n \text{ for } i = 1,2. \end{cases}$$

We next assume that w_i^ε have the asymptotic expansions of the form

$$w_i^\varepsilon(x,t) = w_i(x,t) + \varepsilon w_{i1}(x,t) + \varepsilon^2 w_{i2}(x,t) \cdots, \text{ for } i = 1,2.$$

It is then relatively straightforward to see that (w_1, w_2) solves

$$(\mathrm{I}) \qquad \begin{cases} (w_1)_t + (w_1 - w_2) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ (w_2)_t + (w_2 - w_1) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ w_i(x,0) = f_i(x) & \text{on } \mathbb{R}^n \text{ for } i = 1,2. \end{cases}$$

Thus, we can compute the explicit formula for the inner solutions (w_1, w_2)

$$\begin{aligned} &(w_1(x,t), w_2(x,t)) \\ &= \left(\frac{f_1(x)+f_2(x)}{2} + \frac{f_1(x)-f_2(x)}{2} e^{-2t},\ \frac{f_1(x)+f_2(x)}{2} + \frac{f_2(x)-f_1(x)}{2} e^{-2t} \right). \end{aligned}$$

The final step is to obtain the matched solutions. We have in this particular situation

$$\lim_{t\to 0} u(x,t) = \lim_{t\to\infty} w_i(x,t) = \frac{f_1(x)+f_2(x)}{2},$$

which shows that the common part of the inner and outer solutions is $(f_1+f_2)(x)/2$. Hence, the matched solutions are

$$\begin{aligned} &\left(u(x,t) + w_1(x, \frac{t}{\varepsilon}) - \frac{f_1(x)+f_2(x)}{2}, u(x,t) + w_2(x, \frac{t}{\varepsilon}) - \frac{f_1(x)+f_2(x)}{2} \right) \\ &= \left(u(x,t) + \frac{f_1(x)-f_2(x)}{2} e^{-\frac{2t}{\varepsilon}}, u(x,t) + \frac{f_2(x)-f_1(x)}{2} e^{-\frac{2t}{\varepsilon}} \right) \\ &= (m_1^\varepsilon(x,t), m_2^\varepsilon(x,t)), \end{aligned}$$

where m_i^ε are the functions defined by (4.1.4).

As we can see, the matched solutions contain the layer parts which are essentially the same like the subsolutions and supersolutions that we build in Lemma 4.2.1. For any fixed $t > 0$, we can see that $(m_1^\varepsilon(x,t), m_2^\varepsilon(x,t))$ converges to $(u(x,t), u(x,t))$ exponentially fast. But for $t = O(\varepsilon)$ then we do not have such convergence. In particular, we have $(m_1^\varepsilon(x,\varepsilon), m_2^\varepsilon(x,\varepsilon))$ converges to $(u(x,t) + (f_1-f_2)(x)/(2e^2), u(x,t) + (f_2-f_1)(x)/(2e^2))$. On the other hand, the fact that $((m_1^\varepsilon)_t, (m_2^\varepsilon)_t)$ is not bounded also give us an intuition about the unboundedness of $((u_1^\varepsilon)_t, (u_2^\varepsilon)_t)$.

It is therefore interesting if we can study the behavior of the difference between the real solutions $(u_1^\varepsilon, u_2^\varepsilon)$ and the matched solutions $(m_1^\varepsilon, m_2^\varepsilon)$.

Rate of convergence

In this subsection, we assume further that

(A3) H_i are (uniformly) Lipschitz in the p-variable for $i = 1, 2$, i.e. there exists a constant $C_H > 0$ such that

$$|H_i(\xi, p) - H_i(\xi, q)| \le C_H|p - q| \text{ for all } \xi \in \mathbb{T}^n \text{ and } p, q \in \mathbb{R}^n.$$

We now prove Theorem 4.1.3 by splitting $\mathbb{R}^n \times [0, T]$ into two parts, which are $\mathbb{R}^n \times [0, \varepsilon|\log\varepsilon|]$ and $\mathbb{R}^n \times [\varepsilon|\log\varepsilon|, T]$. For the part of small time $\mathbb{R}^n \times [0, \varepsilon|\log\varepsilon|]$, we use the barrier functions in Lemma 4.2.1 and the effective equation to obtain the results. The L^∞-bounds of $|u_i^\varepsilon - m_i^\varepsilon|$ for $i = 1, 2$ on $\mathbb{R}^n \times [\varepsilon|\log\varepsilon|, T]$ can be obtained by using techniques similar to those of Capuzzo-Dolcetta and Ishii [16].

Proposition 4.3.1 (Initial Layer)**.** *There exists $C > 0$ such that*

$$|(u_i^\varepsilon - m_i^\varepsilon)(x, t)| \le C\varepsilon|\log\varepsilon| \textit{ for all } (x, t) \in \mathbb{R}^n \times [0, \varepsilon|\log\varepsilon|] \textit{ and } i = 1, 2.$$

Proof. We only prove the case $i = 1$. By symmetry we can prove the case $i = 2$. Let $C \ge \max_{i=1,2} \max_{(\xi,p)\in\mathbb{R}^n\times B(0,r)} |H_i(\xi, p)|$ be a constant, where $r = \|Df_1\|_{L^\infty(\mathbb{R}^n)} + \|Df_2\|_{L^\infty(\mathbb{R}^n)}$ and note that u is Lipschitz continuous with a Lipschitz constant $C_u := (1/2)(\|Df_1\|_\infty + \|Df_2\|_\infty)$. By Lemma 4.2.1 we have

$$\begin{aligned}
&\big|u_1^\varepsilon(x, t) - (u(x, t) + \frac{f_1(x) - f_2(x)}{2} e^{-\frac{2t}{\varepsilon}})\big| \\
&\le \big|u(x, t) - \frac{f_1(x) + f_2(x)}{2}\big| + Ct = |u(x, t) - u(x, 0)| + Ct \\
&\le (C + C_u)t \le (C + C_u)\varepsilon|\log\varepsilon|
\end{aligned}$$

for all $t \in [0, \varepsilon|\log\varepsilon|]$. ◻

Proposition 4.3.2. *Assume that (A3) holds. For $T > 0$ there exists $C = C(T) > 0$ such that*

$$|u_i^\varepsilon(x, t) - u(x, t)| \le C\varepsilon^{1/3} \quad \textit{for } (x, t) \in \mathbb{R}^n \times [\varepsilon|\log\varepsilon|, T] \textit{ and } i = 1, 2.$$

Lemma 4.3.3. *Assume that (A3) holds. For each $\delta > 0$ and $P \in \mathbb{R}^n$, there exists a unique solution $(v_1^\delta, v_2^\delta) \in C^{0,1}(\mathbb{T}^n)^2$ of*

$$(E_P^\delta) \qquad \begin{cases} H_1(\xi, P + Dv_1^\delta(\xi, P)) + (1 + \delta)v_1^\delta(\xi, P) - v_2^\delta(\xi, P) = 0 & \text{in } \mathbb{T}^n, \\ H_2(\xi, P + Dv_2^\delta(\xi, P)) + (1 + \delta)v_2^\delta(\xi, P) - v_1^\delta(\xi, P) = 0 & \text{in } \mathbb{T}^n. \end{cases}$$

Moreover,

(i) there exists a constant $C > 0$ independent of δ such that

$$\delta|v_i^\delta(\xi, P) - v_i^\delta(\xi, Q)| \le C|P - Q| \textit{ for } \xi \in \mathbb{T}^n, P, Q \in \mathbb{R}^n \textit{ and } i = 1, 2;$$

(ii) for each $R>0$, there exists a constant $C=C(R)>0$ independent of δ such that

$$|\delta v_i^\delta(\xi,P)+\overline{H}(P)|\le C(R)\delta \text{ for } \xi\in\mathbb{T}^n, P\in\overline{B}(0,R) \text{ and } i=1,2.$$

Proof. By the classical result (see [26, 52]) we can easily see that there exists a unique solution (v_1^δ,v_2^δ) of (E_P^δ) for any $P\in\mathbb{R}^n$. We see that $\big(v_1^\delta(\cdot,P)\pm C_H|P-Q|/\delta, v_2^\delta(\cdot,P)\pm C_H|P-Q|/\delta\big)$ are a supersolution and subsolution of (E_Q^δ), respectively, in view of (A3). By the comparison principle for (E_Q^δ) we have

$$v_i^\delta(\xi,P)-\frac{C_H|P-Q|}{\delta}\le v_i^\delta(\xi,Q)\le v_i^\delta(\xi,P)+\frac{C_H|P-Q|}{\delta} \text{ for } \xi\in\mathbb{T}^n,\ i=1,2,$$

which completes (i).

Let $C_1(P)=\max_{i=1,2}\max_{\xi\in\mathbb{T}^n}|H_i(\xi,P)|$. It is clear that $(-C_1(P)/\delta,-C_1(P)/\delta)$ and $(C_1(P)/\delta,C_1(P)/\delta)$ are a subsolution and a supersolution of (E_P^δ), respectively. Note that

$$|H_i(\xi,P)|\le|H_i(\xi,0)|+|H_i(\xi,0)-H_i(\xi,P)|\le C(1+|P|)$$

for $C\ge\max_{i=1,2,\,\xi\in\mathbb{T}^n}|H_i(\xi,0)|\vee C_H$. Therefore, by the comparison principle again we get

$$\delta\|v_i^\delta(\cdot,P)\|_{L^\infty(\mathbb{T}^n)}\le C_1(P)\le C(1+|P|). \tag{4.3.1}$$

Next, sum up the two equations of (E_P^δ) to get

$$H_1(\xi,P+Dv_1^\delta(\xi,P))+H_2(\xi,P+Dv_2^\delta(\xi,P))\le 2C(1+|P|).$$

Thus, for each $R>0$, there exists a constant $C=C(R)\ge0$ so that

$$\|Dv_i^\delta(\cdot,P)\|_{L^\infty(\mathbb{T}^n)}\le C(R) \text{ for } |P|\le R \text{ and } i=1,2. \tag{4.3.2}$$

We look back at (E_P^δ) and take the inequalities (4.3.1), (4.3.2) into account to deduce that

$$\|v_1^\delta(\cdot,P)-v_2^\delta(\cdot,P)\|_{L^\infty(\mathbb{T}^n)}\le C(R) \text{ for } |P|\le R. \tag{4.3.3}$$

Let $\mu^+:=\max_{i=1,2,\,\xi\in\mathbb{T}^n}\delta v_i^\delta(\xi,P)$ and $\mu^-:=\min_{i=1,2,\,\xi\in\mathbb{T}^n}\delta v_i^\delta(\xi,P)$. Then we have

$$\mu^-\le-\overline{H}(P)\le\mu^+. \tag{4.3.4}$$

Indeed, suppose that $\mu^+<-\overline{H}(P)$ and then by the comparison principle we have $v_i^\delta\ge w_i$ on $\mathbb{T}^n$ for any solution $(w_1,w_2,\overline{H}(P))$ of (E_P). This is a contradiction, since for any $C_2\in\mathbb{R}$ $(w_1+C_2,w_2+C_2,\overline{H}(P))$ is a solution of (E_P) too. Similarly we see that $\mu^-\le-\overline{H}(P)$. By (4.3.2)–(4.3.4) we can get the desired conclusion of (ii). ◻

Proof of Proposition 4.3.2. Let $(v_1^\delta(\cdot,P),v_2^\delta(\cdot,P))$ be the solution of (E_P^δ) for $P\in\mathbb{R}^n$. We consider the auxiliary functions

$$\Phi_i(x,y,t,s):=u_i^\varepsilon(x,t)-u(y,s)-\varepsilon v_i^\delta(\frac{x}{\varepsilon},\frac{x-y}{\varepsilon^\beta})-\frac{|x-y|^2+(t-s)^2}{2\varepsilon^\beta}-K(t+s)$$

for $i = 1, 2$, where $\delta = \varepsilon^\theta$ and $\beta, \theta \in (0, 1)$ and $K \geq 0$ to be fixed later.

For simplicity of explanation we assume that Φ_i takes a global maximum on $\mathbb{R}^{2n} \times [\varepsilon|\log\varepsilon|, T]^2$ and let $(\hat{x}, \hat{y}, \hat{t}, \hat{s})$ be a point such that

$$\max_{i=1,2} \max_{\mathbb{R}^n \times [\varepsilon|\log\varepsilon|, T]} \Phi_i(x, y, t, s) = \Phi_1(\hat{x}, \hat{y}, \hat{t}, \hat{s}). \tag{4.3.5}$$

For a more rigorous proof we need to add the term $-\gamma|x|^2$ to Ψ_i for $\gamma > 0$. See the proof of Theorem 1.1 in [16] for the detail. We first consider the case where $\hat{t}, \hat{s} > \varepsilon|\log\varepsilon|$.

Claim. If $0 < \theta < 1 - \beta$, then there exists $M > 0$ such that $(|\hat{x} - \hat{y}| + |\hat{t} - \hat{s}|)/\varepsilon^\beta \leq M$.

We use $\Phi_1(\hat{x}, \hat{y}, \hat{t}, \hat{s}) \geq \Phi_1(\hat{x}, \hat{x}, \hat{t}, \hat{t})$, Lemma 4.3.3 (i) and that u is Lipschitz continuous to deduce that

$$\begin{aligned}
\frac{|\hat{x} - \hat{y}|^2 + |\hat{t} - \hat{s}|^2}{2\varepsilon^\beta} &\leq |u(\hat{x}, \hat{t}) - u(\hat{y}, \hat{s})| + \varepsilon|v_1^\delta(\frac{\hat{x}}{\varepsilon}, \frac{\hat{x} - \hat{y}}{\varepsilon^\beta}) - v_1^\delta(\frac{\hat{x}}{\varepsilon}, 0)| + K|\hat{t} - \hat{s}| \\
&\leq C_u(|\hat{x} - \hat{y}| + |\hat{t} - \hat{s}|) + C\varepsilon\frac{1}{\varepsilon^\theta}\frac{|\hat{x} - \hat{y}|}{\varepsilon^\beta} + K|\hat{t} - \hat{s}| \\
&\leq C'(|\hat{x} - \hat{y}| + |\hat{t} - \hat{s}|)
\end{aligned}$$

for some $C, C' > 0$, which implies the result of Claim.

We fix $(y, s) = (\hat{y}, \hat{s})$ and notice that the function

$$(x, t) \mapsto u_1^\varepsilon(x, t) - \varepsilon v_i^\delta(\frac{x}{\varepsilon}, \frac{x - \hat{y}}{\varepsilon^\beta}) - \frac{|x - \hat{y}|^2 + (t - \hat{s})^2}{2\varepsilon^\beta} - Kt$$

attains the maximum at $(\hat{x}, \hat{t})$. For $\alpha > 0$, we define the function ψ by

$$\psi(x, \xi, z, t) := u_1^\varepsilon(x, t) - \varepsilon v_1^\delta(\xi, \frac{z - \hat{y}}{\varepsilon^\beta}) - \frac{|x - \hat{y}|^2 + |t - \hat{s}|^2}{2\varepsilon^\beta} - \frac{|x - \varepsilon\xi|^2 + |x - z|^2}{2\alpha} - Kt.$$

Let ψ attain the maximum at $(x_\alpha, \xi_\alpha, z_\alpha, t_\alpha)$ and then we may assume that $(x_\alpha, \xi_\alpha, z_\alpha, t_\alpha) \to (\hat{x}, \hat{x}/\varepsilon, \hat{x}, \hat{t})$ as $\alpha \to 0$ up to passing some subsequences if necessary. By the definition of viscosity solutions, we have

$$K + \frac{t_\alpha - \hat{s}}{\varepsilon^\beta} + H_1(\frac{x_\alpha}{\varepsilon}, \frac{x_\alpha - \hat{y}}{\varepsilon^\beta} + \frac{x_\alpha - \varepsilon\xi_\alpha}{\alpha} + \frac{x_\alpha - z_\alpha}{\alpha}) + \frac{1}{\varepsilon}(u_1^\varepsilon - u_2^\varepsilon)(x_\alpha, t_\alpha) \leq 0, \tag{4.3.6}$$

and

$$H_1(\xi_\alpha, \frac{z_\alpha - \hat{y}}{\varepsilon^\beta} + \frac{x_\alpha - \varepsilon\xi_\alpha}{\alpha}) + (1 + \delta)v_1^\delta(\xi_\alpha, \frac{z_\alpha - \hat{y}}{\varepsilon^\beta}) - v_2^\delta(\xi_\alpha, \frac{z_\alpha - \hat{y}}{\varepsilon^\beta}) \geq 0. \tag{4.3.7}$$

Next, since $\psi(x_\alpha, \xi_\alpha, z_\alpha, t_\alpha) \geq \psi(x_\alpha, \xi_\alpha, x_\alpha, t_\alpha)$ we get

$$\frac{|x_\alpha - z_\alpha|^2}{2\alpha} \leq \varepsilon(v_1^\delta(\xi_\alpha, \frac{x_\alpha - \hat{y}}{\varepsilon^\beta}) - v_1^\delta(\xi_\alpha, \frac{z_\alpha - \hat{y}}{\varepsilon^\beta})) \leq C\varepsilon^{1-\theta-\beta}|x_\alpha - z_\alpha|$$

by Lemma 4.3.3 (i). Thus, $|x_\alpha - z_\alpha|/\alpha \le C\varepsilon^{1-\theta-\beta}$. Send $\alpha \to 0$ and we get

$$K+\frac{\hat{t}-\hat{s}}{\varepsilon^\beta}+\overline{H}(\frac{\hat{x}-\hat{y}}{\varepsilon^\beta})+\frac{1}{\varepsilon}(u_1^\varepsilon-u_2^\varepsilon)(\hat{x},\hat{t})-v_1^\delta(\frac{\hat{x}}{\varepsilon},\frac{\hat{x}-\hat{y}}{\varepsilon^\beta})+v_2^\delta(\frac{\hat{x}}{\varepsilon},\frac{\hat{x}-\hat{y}}{\varepsilon^\beta})-C(\varepsilon^\theta+\varepsilon^{1-\theta-\beta}) \le 0. \tag{4.3.8}$$

Similarly we fix $(x,t)=(\hat{x},\hat{t})$ and do a similar procedure to the above to obtain

$$-K+\frac{\hat{t}-\hat{s}}{\varepsilon^\beta}+\overline{H}(\frac{\hat{x}-\hat{y}}{\varepsilon^\beta})+C(\varepsilon^\theta+\varepsilon^{1-\theta-\beta}) \ge 0. \tag{4.3.9}$$

Combining (4.3.8), (4.3.9), and (4.3.5), we get

$$2K \le C(\varepsilon^\theta+\varepsilon^{1-\theta-\beta}). \tag{4.3.10}$$

Now we choose $\theta=\beta=1/3$ and $K=K_1\varepsilon^{1/3}$ for K_1 large enough to get the contradiction in (4.3.10). Hence either $\hat{t}=-\varepsilon\log\varepsilon$ or $\hat{s}=-\varepsilon\log\varepsilon$ holds. The proof is complete immediately. □

Theorem 4.1.3 is a straightforward result of Propositions 4.3.1, 4.3.2.

4.4 Properties of effective Hamiltonians

Elementary properties

Proposition 4.4.1.
(i) (Coercivity) $\overline{H}(P) \to +\infty$ *as* $|P| \to \infty$.
(ii) (Convexity) If H_i *are convex in the p-variable for* $i=1,2$, *then* $\overline{H}$ *is convex.*

Proof. (i) For each $\delta>0$ and $P\in\mathbb{R}^n$, let (v_1^δ, v_2^δ) be a solution of (E_P^δ) and without loss of generality, we may assume that $v_1^\delta(\xi_0,P)=\max_{i=1,2,\,\xi\in\mathbb{T}^n} v_i^\delta(\xi,P)$ for some $\xi_0\in\mathbb{T}^n$. By the definition of viscosity solutions we have $H_1(\xi_0,P)\le H_1(\xi_0,P)+(v_1^\delta-v_2^\delta)(\xi_0,P)\le -\delta v_1^\delta(\xi_0,P)$. We let $\delta\to 0$ to derive that

$$\overline{H}(P) \ge \min_{i=1,2,\,\xi\in\mathbb{T}^n} H_i(\xi,P).$$

Since H_i are coercive for $i=1,2$, so is $\overline{H}$.

(ii) We argue by contradiction. Suppose that $\overline{H}$ is not convex and then there would exist $P,Q\in\mathbb{R}^n$ such that

$$2\varepsilon_0 := \overline{H}(\frac{P+Q}{2})-\frac{\overline{H}(P)+\overline{H}(Q)}{2} > 0. \tag{4.4.1}$$

We define the functions $w_i\in C(\mathbb{T}^n)$ so that $w_i(\xi):=(v_i(\xi,P)+v_i(\xi,Q))/2$ for $i=1,2$, where $(v_1(\cdot,P),v_2(\cdot,P))$ and $(v_1(\cdot,Q),v_2(\cdot,Q))$ are solutions of (E_P) and (E_Q), respectively.

Due to the convexity of H_i for $i = 1, 2$ we have

$$\begin{cases} H_1(\xi, \dfrac{P+Q}{2} + Dw_1(\xi)) + w_1(\xi) - w_2(\xi) \le \dfrac{\overline{H}(P) + \overline{H}(Q)}{2}, \\ H_2(x, \dfrac{P+Q}{2} + Dw_2(\xi)) + w_2(\xi) - w_1(\xi) \le \dfrac{\overline{H}(P) + \overline{H}(Q)}{2}. \end{cases}$$

By (4.4.1) there exists a small constant $\delta > 0$ such that

$$\begin{cases} H_1(\xi, \dfrac{P+Q}{2} + Dw_1(\xi)) + (1+\delta)w_1(\xi) - w_2(\xi) \le \overline{H}(\dfrac{P+Q}{2}) - \varepsilon_0, \\ H_2(x, \dfrac{P+Q}{2} + Dw_2(\xi)) + (1+\delta)w_2(\xi) - w_1(\xi) \le \overline{H}(\dfrac{P+Q}{2}) - \varepsilon_0, \end{cases}$$

and

$$\begin{cases} H_1(\xi, \dfrac{P+Q}{2} + Dv_1(\xi, \dfrac{P+Q}{2})) + (1+\delta)v_1(\xi, \dfrac{P+Q}{2}) - v_2(\xi, \dfrac{P+Q}{2}) \ge \overline{H}(\dfrac{P+Q}{2}) - \varepsilon_0, \\ H_2(\xi, \dfrac{P+Q}{2} + Dv_2(\xi, \dfrac{P+Q}{2})) + (1+\delta)v_2(\xi, \dfrac{P+Q}{2}) - v_1(\xi, \dfrac{P+Q}{2}) \ge \overline{H}(\dfrac{P+Q}{2}) - \varepsilon_0. \end{cases}$$

By the comparison principle we get

$$\frac{v_i(\xi, P) + v_i(\xi, Q)}{2} \le v_i(\xi, \frac{P+Q}{2}) \qquad \text{for } i = 1, 2. \tag{4.4.2}$$

Notice that (4.4.2) is still correct even if we replace $v_i(\xi, (P+Q)/2)$ by $v_i(\xi, (P+Q)/2) + C_1$ for $i = 1, 2$ and for any $C_1 \in \mathbb{R}$, which yields the contradiction. □

The uniqueness of the effective Hamiltonian for (E_P) and the cell problem for single Hamilton–Jacobi equations gives the following proposition.

Proposition 4.4.2. *If $H_1 = H_2 = K$, then*

$$\overline{H}(P) = \overline{K}(P) \text{ for all } P \in \mathbb{R}^n,$$

where $\overline{K}$ is the effective Hamiltonian corresponding to K.

Proposition 4.4.3. *If H_i are homogeneous with degree 1 in the p-variable for $i = 1, 2$, then $\overline{H}$ is positive homogeneous with degree 1.*

Proof. Let $(v_1, v_2, \overline{H}(P))$ be a solution of (E_P) for any $P \in \mathbb{R}^n$. If H_i is homogeneous with degree 1 in the p-variable, then $(rv_1, rv_2, r\overline{H}(P))$ is a solution of (E_{rP}) for any $r > 0$. Therefore by the uniqueness of the effective Hamiltonian we get the conclusion. □

Proposition 4.4.4. *We define the Hamiltonian K as*

$$K(\xi, p) := \max\{H_1(\xi, p), H_2(\xi, p)\}.$$

Let $\overline{K}$ be its corresponding effective Hamiltonian and then

$$\overline{H}(P) \le \overline{K}(P).$$

Proof. For each $P \in \mathbb{R}^n$, there exists $\varphi(\cdot, P) \in C^{0,1}(\mathbb{T}^n)$ such that

$$K(\xi, P + D\varphi(\xi, P)) = \overline{K}(P).$$

Thus $(\varphi(\cdot, P), \varphi(\cdot, P), \overline{K}(P))$ is a subsolution of (E_P). We hence get $\overline{K}(P) \geq \overline{H}(P)$ by Proposition 4.4.6. ☐

We give an example that we can calculate the effective Hamiltonian explicitly.

Example 4.4.5. Let $n = 1$ and $H_1(\xi, p) = |p|$, $H_2(\xi, p) = a(\xi)|p|$, where

$$a(\xi) := \frac{1 - (\frac{1}{8\pi^2}\cos(2\pi\xi) + \frac{1}{4\pi}\sin(2\pi\xi))}{1 + (\frac{1}{2} + \frac{1}{8\pi^2})\cos(2\pi\xi)} > 0.$$

By Proposition 4.4.3 we have that $\overline{H}(P) = \overline{H}(1)P$ for $P \geq 0$. Set

$$v_1(\xi, 1) := \frac{1}{16\pi^3}\sin(2\pi\xi) - \frac{1}{8\pi^2}\cos(2\pi\xi), \ v_2(\xi, 1) := (\frac{1}{4\pi} + \frac{1}{16\pi^3})\sin(2\pi\xi).$$

Then we can confirm that $(v_1(\cdot, 1), v_2(\cdot, 1), 1)$ is a solution of (E_1). Therefore $\overline{H}(1) = 1$ and thus, $\overline{H}(P) = P$ for $P \geq 0$.

For any $P < 0$ we have $\overline{H}(P) = \overline{H}(-1) \cdot (-P)$. Set

$$v_1(\xi, -1) := -\big(\frac{1}{16\pi^3}\sin(2\pi\xi) - \frac{1}{8\pi^2}\cos(2\pi\xi)\big), \ v_2(\xi, 1) := -(\frac{1}{4\pi} + \frac{1}{16\pi^3})\sin(2\pi\xi).$$

Then we can confirm that $(v_1(\cdot, -1), v_2(\cdot, -1), 1)$ is a solution of (E_{-1}). Therefore $\overline{H}(-1) = 1$ and thus, $\overline{H}(P) = -P$ for $P \leq 0$. Thus, we get $\overline{H}(P) = |P|$.

Representation formulas for the effective Hamiltonian

In this subsection we derive representation formulas for the effective Hamiltonian $\overline{H}(P)$. See [20, 45] for the min-max formulas for the effective Hamiltonian for single equations.

Proposition 4.4.6 (Representation formula 1)**.** *We have*

$$\overline{H}(P) = \inf\{c : \text{ there exists } (\phi_1, \phi_2) \in C(\mathbb{T}^n)^2 \text{ so that the triplet } (\phi_1, \phi_2, c) \text{ is a subsolution of } (E_P)\}. \quad (4.4.3)$$

Proof. Fix $P \in \mathbb{R}^n$ and we denote by $c(P)$ the right-hand side of (4.4.3). By the definition of $c(P)$ we can easily see that $\overline{H}(P) \geq c(P)$. We prove the other way around. Assume by contradiction that there exist a triplet $(\phi_1, \phi_2, c) \in C(\mathbb{T}^n)^2 \times \mathbb{R}$ which is a subsolution of (E_P) and $c < \overline{H}(P)$. Let $(v_1, v_2, \overline{H}(P))$ be a solution of (E_P) and take $C > 0$ so that $\phi_i > v_i - C =: \overline{v}_i$ on $\mathbb{T}^n$. Then since $\overline{v}_i$ and ϕ_i are bounded on $\mathbb{T}^n$, for $\varepsilon > 0$ small enough, we have

$$\begin{cases} H_1(\xi, P + D\overline{v}_1) + (1+\varepsilon)\overline{v}_1 - \overline{v}_2 \geq H_1(\xi, P + D\phi_1) + (1+\varepsilon)\phi_1 - \phi_2 \\ H_2(\xi, P + D\overline{v}_2) + (1+\varepsilon)\overline{v}_2 - \overline{v}_1 \geq H_2(\xi, P + D\phi_2) + (1+\varepsilon)\phi_2 - \phi_1. \end{cases}$$

By the comparison principle (see [26, 52]) we deduce $\overline{v}_i \geq \phi_i$ on $\mathbb{T}^n$ which yields the contradiction. ☐

If we assume the convexity on $H_i(\xi,\cdot)$ for any $\xi\in\mathbb{R}^n$, by the classical result on the representation formula for the effective Hamiltonian for single Hamilton–Jacobi equations we can easily see that

$$\begin{aligned}\overline{H}(P) &= \inf_{\varphi\in C^1(\mathbb{T}^n)}\max_{\xi\in\mathbb{T}^n}[H_1(\xi,P+D\varphi(\xi))+v_1(\xi,P)-v_2(\xi,P)] \\ &= \inf_{\psi\in C^1(\mathbb{T}^n)}\max_{\xi\in\mathbb{T}^n}[H_2(\xi,P+D\psi(\xi))+v_2(\xi,P)-v_1(\xi,P)]\end{aligned} \tag{4.4.4}$$

for any solution $(v_1(\cdot,P),v_2(\cdot,P))$ of (E_P), which is in a sense an implicit formula. For the weakly coupled system we have the following representation formula.

Proposition 4.4.7 (Representation formula 2)**.** *If H_i are convex in the p-variable for $i=1,2$, then*

$$\overline{H}(P)=\inf_{(\phi_1,\phi_2)\in C^1(\mathbb{T}^n)^2}\ \max_{i=1,2,\,\xi\in\mathbb{T}^n}[H_i(\xi,P+D\phi_i(\xi))+\phi_i(\xi)-\phi_j(\xi)], \tag{4.4.5}$$

where we take $j\in\{1,2\}$ so that $\{i,j\}=\{1,2\}$.

Lemma 4.4.8. *Assume that H_i are convex in the p-variable. Let $(v_1,v_2,\overline{H}(P))\in C(\mathbb{T}^n)^2$ be a subsolution of (E_P). Set $v_{i\delta}(x):=\rho_\delta * v_i(x)$, where $\rho_\varepsilon(x):=(1/\varepsilon^n)\rho(x/\varepsilon)$ and $\rho\in C^\infty(\mathbb{R}^n)$ be a standard mollification kernel, i.e., $\rho\ge 0$, $\operatorname{supp}\rho\subset B(0,1)$, and $\int_{\mathbb{R}^n}\rho(x)\,dx=1$. Then, $(v_{1\delta},v_{2\delta},\overline{H}(P)+\omega(\delta))$ is a subsolution of (E_P) for some modulus ω.*

Proof. Note that in view of the coercivity of H_i, v_i are Lipschitz continuous and $(v_1,v_2,\overline{H}(P))$ solves (E_P) almost everywhere. Fix any $\xi\in\mathbb{T}^n$. We calculate that

$$\begin{aligned}\overline{H}(P) &\ge \rho_\delta * \big(H_1(\cdot,Dv_1(\cdot))+(v_1-v_2)\big)(\xi) \\ &= \int_{B(\xi,\delta)}\rho_\delta(\xi-\eta)\big(H_1(\eta,Dv_1(\eta))+(v_1-v_2)(\eta)\big)\,d\eta \\ &\ge \int_{B(\xi,\delta)}\rho_\delta(\xi-\eta)\big(H_1(\xi,Dv_1(\eta))-\omega(\delta)\big)\,d\eta+(v_{1\delta}-v_{2\delta})(\xi) \\ &\ge H_1(\xi,\rho_\delta * Dv_1(\xi))+(v_{1\delta}-v_{2\delta})(\xi)-\omega(\delta) \\ &= H_1(\xi,Dv_{1\delta}(\xi))+(v_{1\delta}-v_{2\delta})(\xi)-\omega(\delta),\end{aligned}$$

where the third inequality follows by using Jensen's inequality. □

Proof of Proposition 4.4.7. Let $c(P)$ be the constant on the right-hand side of (4.4.5). Noting that for any $(\phi_1,\phi_2)\in C^1(\mathbb{T}^n)^2$

$$H_i(\xi,P+D\phi_i(\xi))+(\phi_i-\phi_j)(\xi)\le\max_{i=1,2,\,\xi\in\mathbb{T}^n}[H_i(\xi,P+D\phi_i(\xi))+(\phi_i-\phi_j)(\xi)]=:a_{\phi_1,\phi_2}$$

for every $\xi\in\mathbb{T}^n$. By Proposition 4.4.6 we see that $\overline{H}(P)\le a_{\phi_1,\phi_2}$ for all $(\phi_1,\phi_2)\in C^1(\mathbb{T}^n)^2$. Therefore we get $\overline{H}(P)\le c(P)$.

Conversely, we observe that by Proposition 5.2.1 $(v_{1\delta}(\cdot,P),v_{2\delta}(\cdot,P),\overline{H}(P)+\omega(\delta))\in C^1(\mathbb{T}^n)^2\times\mathbb{R}$ is a subsolution of (E_P). Therefore, by the definition of $c(P)$ we see that $c(P)\le\overline{H}(P)+\omega(\delta)$. Sending $\delta\to 0$ yields the conclusion. □

If H_i are convex in the p-variable, then there is a variational formula of solutions of the initial value problem and the cell problem as stated in Introduction. Therefore, naturally we have the following variational formula

$$\begin{aligned}\overline{H}(P) &= -\lim_{\delta\to 0}\inf_{\eta}\mathbb{E}_i\Big[\int_0^{+\infty} e^{-\delta s}\big(-P\cdot\dot\eta(s)+L_{\nu(s)}(\eta(s),-\dot\eta(s))\big)\,ds\Big]\\ &= -\lim_{t\to\infty}\frac{1}{t}\inf_{\eta}\mathbb{E}_i\Big[\int_0^{t}\big(-P\cdot\dot\eta(s)+L_{\nu(s)}(\eta(s),-\dot\eta(s))\big)\,ds\Big],\end{aligned}$$

where the infimum is taken over $\eta\in \mathrm{AC}\,([0,+\infty),\mathbb{R}^n)$ such that $\eta(0)=x$ and $\mathbb{E}_i$ denotes the expectation of a process with $\nu(0)=i$ given by (4.1.2).

Remark 4.4.9. When we consider the nonconvex Hamilton–Jacobi equations, in general we cannot expect the formula (4.4.5). Take the Hamiltonian

$$H_i(\xi,p):=(|p|^2-1)^2 \text{ for } i=1,2 \tag{4.4.6}$$

for instance. In this example if we calculate the right-hand side of (4.4.5) with $P=0$, then it is 0. But we can easily check that $\overline{H}(0)=1$, since in this case we have solutions (which are constants).

The following formula is a revised min-max formula for the effective Hamiltonian for nonconvex Hamilton–Jacobi equations.

Proposition 4.4.10. *We have*

$$\overline{H}(P)=\inf_{(\phi_1,\phi_2)\in C^{0,1}(\mathbb{T}^n)^2}\ \max_{i=1,2,\,\xi\in\mathbb{T}^n}\ \sup_{p\in D^+\phi_i(\xi)}[H_i(\xi,P+p)+(\phi_i-\phi_j)(\xi)], \tag{4.4.7}$$

where if $D^+\phi_i(\xi)=\emptyset$, then we set $\sup_{p\in D^+\phi_i(\xi)}[H_i(\xi,P+p)+(\phi_i-\phi_j)(\xi)]=-\infty$ by convention.

We notice that if H_i are given by (4.4.6), then the right-hand side of (4.4.7) with $P=0$ is 1.

Proof. The proof is already in the proof of Proposition 4.4.7. We just need to be careful for the definition of viscosity subsolutions. Indeed, let c be the right-hand side of (4.4.7) and noting that for any $(\phi_1,\phi_2)\in C^{0,1}(\mathbb{T}^n)^2$, $\xi\in\mathbb{T}^n$, and $q\in D^+\phi_i(\xi)$,

$$H_i(\xi,P+q)+(\phi_i-\phi_j)(\xi)\le \max_{\xi\in\mathbb{T}^n,i=1,2}\ \sup_{p\in D^+\phi_i(\xi)}[H_i(\xi,P+p)-(\phi_i-\phi_j)(\xi)]=:a_{\phi_1,\phi_2}.$$

Thus, $\overline{H}(P)\le a_{\phi_1,\phi_2}$ for all $(\phi_1,\phi_2)\in C^{0,1}(\mathbb{T}^n)^2$ by Proposition 4.4.6. Therefore, $\overline{H}(P)\le c$.

Conversely, there exists a viscosity subsolution $(v_1(\cdot,P),v_2(\cdot,P),\overline{H}(P))\in C^{0,1}(\mathbb{T}^n)^2\times\mathbb{R}$ of (E_P). By the definition of viscosity subsolutions we have

$$H_i(\xi,P+p)+(v_i-v_j)(\xi)\le\overline{H}(P) \text{ for all } \xi\in\mathbb{T}^n \text{ and } p\in D^+v_i(\xi).$$

Thus,

$$\max_{\xi\in\mathbb{T}^n,i=1,2}\ \sup_{p\in D^+v_i(\xi)}[H_i(\xi,P+p)+(v_i-v_j)(\xi)]\le\overline{H}(P),$$

which implies $c\le\overline{H}(P)$. ∎

Flat parts and Non-flat parts near the origin

In this subsection, we study the results concerning flat parts and non-flat parts of the effective Hamiltonian $\overline{H}$ near the origin. We first point out that there are some cases in which we can obtain similar results to those of Concordel's results for single equations. We present different techniques to obtain these results , namely the min-max formulas, and the construction of subsolutions. In this subsection, we only deal with the Hamiltonians of the form $H_i(\xi,p)=|p|^2-V_i(\xi)$, where $V_i\in C(\mathbb{T}^n)$ for $i=1,2$ unless otherwise stated.

Theorem 4.4.11. *Assume that $V_i\ge 0$ in $\mathbb{T}^n$ and $\{V_i=0\}=:U_i\subset\mathbb{T}^n$ for $i=1,2$. We assume further that $U_1\cap U_2\ne\emptyset$ and there exist open sets W_1,W_2 in $\mathbb{T}^n$, and a vector $q\in\mathbb{R}^n$ such that $\Pi(q+W_2)\Subset(0,1)^n$ and*

$$U_1\cup U_2\subset W_1\subset W_2 \text{ and } dist(W_1,\partial W_2),\ dist(U_1\cup U_2,\partial W_1)>0, \tag{4.4.8}$$

then there exists $\gamma>0$ such that $\overline{H}(P)=0$ for $|P|\le\gamma$.

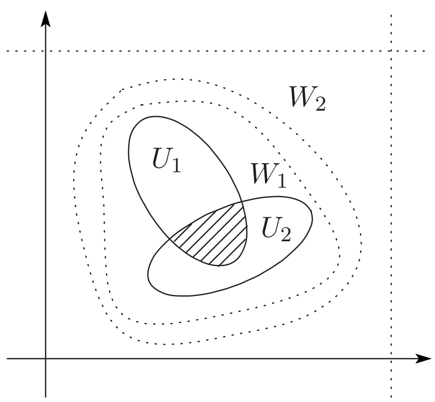

Fig. 4.1. The figure of U_i, W_i.

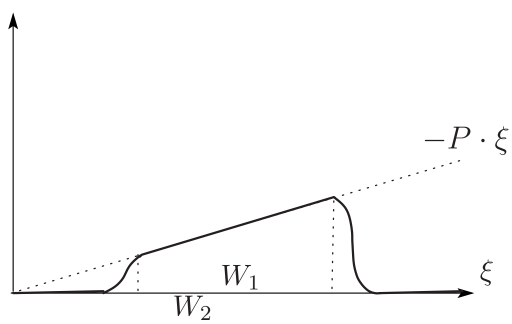

Fig. 4.2. The graph of φ in case $n=1$.

Proof. Without loss of generality, we may assume that $q=0$. Take $\xi_0\in U_1\cap U_2$. By Proposition 4.4.7 we have

$$\begin{aligned}\overline{H}(P)&\ge\inf_{(\varphi_1,\varphi_2)\in C^1(\mathbb{T}^n)}\max_{i=1,2}\left[|P+D\varphi_i(\xi_0)|^2-V_i(\xi_0)+\varphi_i(\xi_0,P)-\varphi_j(\xi_0,P)\right]\\&\ge\inf_{(\varphi_1,\varphi_2)\in C^1(\mathbb{T}^n)}\max_{i=1,2}\left[\varphi_i(\xi_0,P)-\varphi_j(\xi_0,P)\right]=0.\end{aligned}$$

Now, let $d:=\min\{\mathrm{dist}(W_1,\partial W_2),\ \mathrm{dist}(U_1\cup U_2,\partial W_1)\}>0$. There exists $\varepsilon_0>0$ such that

$$V_i(\xi)\ge\varepsilon_0 \quad \text{for } x\in\mathbb{T}^n\setminus W_1,\ i=1,2. \tag{4.4.9}$$

We define a smooth function φ on $\mathbb{T}^n$ such that

$$\begin{aligned}&\varphi(\xi)=-P\cdot\xi \text{ on } W_1,\ \varphi(\xi)=0 \text{ on } \mathbb{T}^n\setminus W_2,\\&|D\varphi|\le\frac{C|P|}{d} \text{ on } \mathbb{T}^n.\end{aligned}\tag{4.4.10}$$

Notice that

$$|P + D\varphi(\xi)|^2 - V_i(\xi) = \begin{cases} = -V_i(\xi) \le 0, & \text{on } W_1, \\ \le \dfrac{C|P|^2}{d^2} - \varepsilon_0, & \text{on } \mathbb{T}^n \setminus W_1. \end{cases}$$

Thus, $|P + D\varphi(\xi)|^2 - V_i(\xi) \le 0$ on $\mathbb{T}^n$ provided that $|P| \le d\sqrt{\varepsilon_0}/C =: \gamma$. We hence have that $(\varphi, \varphi, 0)$ is a subsolution of (E_P) for $|P| \le \gamma$. Therefore $\overline{H}(P) \le 0$ for $|P| \le \gamma$ by Proposition 4.4.6. □

Remark 4.4.12. (i) In fact, the result of Theorem 4.4.11 still holds for more general Hamiltonians

$$H_i(\xi, p) := F_i(\xi, p) - V_i(\xi),$$

where $F_i \in C(\mathbb{T}^n \times \mathbb{R}^n)$ and $V_i \in C(\mathbb{T}^n)$ are assumed to satisfy

(a) the functions $p \mapsto F_i(\xi, p)$ are convex and $F_i(\xi, p) \ge F_i(\xi, 0) = 0$ for all $(\xi, p) \in \mathbb{T}^n \times \mathbb{R}^n$,

(b) V_1, V_2 satisfy the conditions of Theorem 4.4.11.

(ii) Notice that the assumptions $V_i \ge 0$ and $\{V_i = 0\} \ne \emptyset$ for $i = 1, 2$ are just for simplicity. In general, we can normalize V_i by $V_i - \min_{\xi \in \mathbb{T}^n} V_i(\xi)$ to get back to such situation.
(iii) From the proof of Theorem 4.4.11 we have

$$v_1(\xi, P) = v_2(\xi, P) \quad \text{for all } \xi \in \{V_1 = 0\} \cap \{V_2 = 0\}$$

for any solution $(v_1(\cdot, P), v_2(\cdot, P))$ of (E_P).
(iv) By Proposition 4.4.4 we can give another proof to Theorem 4.4.11 as follows. In this case, we explicitly have

$$K(\xi, p) = \max\{|p|^2 - V_1(\xi), |p|^2 - V_2(\xi)\} = |p|^2 - V(\xi)$$

where $V(\xi) = \min\{V_1(\xi), V_2(\xi)\}$. Note that $V \ge 0$ and $\{V = 0\} = \{V_1 = 0\} \cup \{V_2 = 0\}$. Hence, we can either repeat the above proof for single equations to show that $\overline{H}(P) \le \overline{K}(P) = 0$ for $|P| \le \gamma$ or we can use Concordel's result directly.

Notice that condition (4.4.8) is crucial and plays an important role in the construction of the subsolution $(\varphi, \varphi, 0)$ of (E_P) and could not be removed in the proof of Theorem 4.4.11. We point out in the next Theorem that there are cases when (4.4.8) does not hold, then the flatness near the origin of $\overline{H}$ does not appear.

Theorem 4.4.13. *Assume that $V_i \ge 0$ in $\mathbb{T}^n$ and*

$$\{V_1 = 0\} = \{V_2 = 0\} = \{\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{T}^n : \ \xi_j = 1/2 \text{ for } j \ge 2\} =: K. \tag{4.4.11}$$

The followings hold.

(i) There exists $\gamma > 0$ such that $\overline{H}(P) = |P_1|^2$ provided that $|P'| \le \gamma$ for any $P = (P_1, P') \in \mathbb{R} \times \mathbb{R}^{n-1}$.

(ii) $\overline{H}(P) \geq |P_1|^2$ *for all* $P \in \mathbb{R}^n$*.*

Proof. Firstly, we prove that $\overline{H}(P) \leq |P_1|^2$ provided that $|P'| \leq \gamma$ for some $\gamma > 0$ small enough by using exactly the same idea in the proof of Theorem 4.4.11. We build a function $\varphi(\xi) = \varphi(\xi_2, \cdots, \xi_n) \in C^1(\mathbb{T}^n)$, which does not depend on ξ_1, so that

$$\sum_{j=2}^{n} |P_j + \varphi_{\xi_j}(\xi)|^2 - V_i(\xi) \leq 0 \text{ on } \mathbb{T}^n$$

for $i = 1, 2$ and for $|P'| \leq \gamma$ with $\gamma > 0$ small enough. Thus

$$|P + D\varphi(\xi)|^2 - V_i(\xi) \leq |P_1|^2 \text{ on } \mathbb{T}^n$$

for $i = 1, 2$. By Proposition 4.4.6 $\overline{H}(P) \leq |P_1|^2$.

We now prove that $\overline{H}(P) \geq |P_1|^2$. For each $\xi_0 \in K$, we have in view of (4.4.4)

$$\begin{aligned}\overline{H}(P) &= \inf_{\varphi \in C^1(\mathbb{T}^n)} \max_{\xi \in \mathbb{T}^n} [|P + D\varphi(\xi)|^2 - V_1(\xi) + v_1(\xi, P) - v_2(\xi, P)] \\ &\geq \inf_{\varphi \in C^1(\mathbb{T}^n)} [|P + D\varphi(\xi_0)|^2 + v_1(\xi_0, P) - v_2(\xi_0, P)],\end{aligned}$$

and similarly

$$\overline{H}(P) \geq \inf_{\psi \in C^1(\mathbb{T}^n)} [|P + D\psi(\xi_0)|^2 + v_2(\xi_0, P) - v_1(\xi_0, P)].$$

Take an arbitrary function $\varphi \in C^1(\mathbb{T}^n)$ and observe that

$$\begin{aligned}&\int_K |P + D\varphi(\xi)|^2 \, d\xi_1 \\ &\geq \int_K |P_1 + \varphi_{\xi_1}(\xi)|^2 \, d\xi_1 = \int_K |P_1|^2 + |\varphi_{\xi_1}(\xi)|^2 + 2P_1\varphi_{\xi_1}(\xi) \, d\xi_1 \\ &\geq \int_K |P_1|^2 + 2P_1\varphi_{\xi_1}(\xi) \, d\xi_1 = |P_1|^2.\end{aligned}$$

Thus, it is clear to see that $\overline{H}(P) \geq |P_1|^2$, which implies the result. □

The above two Theorems describe several examples that we can obtain similar results of the flat part or non-flat part of $\overline{H}$ to those of single Hamilton–Jacobi equations in [19, 18]. Indeed, the structures on the potentials V_i for $i = 1, 2$ are very related in such a way that we obtain the shape of $\overline{H}$ like for single equations. We rely on the idea of building the subsolutions $(\varphi, \psi, \overline{H}(P))$ of (E_P) where $\varphi = \psi$, which does not work in general cases.

Next, we start investigating the properties of $\overline{H}$ in some cases where the structures of the potentials V_i for $i = 1, 2$ are widely different and in general we cannot expect $\overline{H}$ to have simple properties. The next question is that: Can we read of information of the effective Hamiltonian in the case where $\{V_1 = \min_{\xi \in \mathbb{T}^n} V_1(\xi)\} \cap \{V_2 = \min_{\xi \in \mathbb{T}^n} V_2(\xi)\} = \emptyset$?

Theorem 4.4.14. *Let $n = 1$ and assume that for $\varepsilon_0 > 0$ small enough the following properties hold.*

(a) $\{V_1 = 0\} = [\frac{4}{16}, \frac{12}{16}]$, $\{V_1 = -\varepsilon_0\} = [0,1] \setminus (\frac{3}{16}, \frac{13}{16})$, *and* $-\varepsilon_0 \le V_1 \le 0$ *on* $\mathbb{T}$ *for some* $\varepsilon_0 > 0$.

(b) $\{V_2 = 0\} = [\frac{7}{16}, \frac{9}{16}]$, $\{V_2 = 2\} = [0,1] \setminus (\frac{6}{16}, \frac{10}{16})$, *and* $0 \le V_2 \le 2$ *on* $\mathbb{T}$.

There exists $\gamma > 0$ *such that* $\overline{H}(P) = 0$ *for* $|P| \le \gamma$.

Lemma 4.4.15. *We have*

$$\overline{H}(P) \ge -\frac{1}{2} \min_{\xi \in \mathbb{T}^n} (V_1 + V_2)(\xi).$$

Proof. Sum up the two equations in (E_P) to get

$$|P + Dv_1|^2 + |P + Dv_2|^2 - V_1 - V_2 = 2\overline{H}(P),$$

which implies $2\overline{H}(P) \ge -(V_1 + V_2)(\xi))$ for a.e. $\xi \in \mathbb{T}^n$, and the proof is complete. ☐

Proof of Theorem 4.4.14. Noting that $\min_{\xi \in \mathbb{T}} (V_1 + V_2)(\xi) = 0$, and $\{V_1 = -\varepsilon\} \cap \{V_2 = 0\} = \emptyset$, we have $\overline{H}(P) \ge 0$ by Lemma 4.4.15. We construct a subsolution $(\varphi, \psi, 0)$ of (E_P) for $|P|$ small enough. Let

$$W_1 = (\frac{6}{16}, \frac{10}{16}),\ W_2 = (\frac{5}{16}, \frac{11}{16}),\ W_3 = (\frac{4}{16}, \frac{12}{16}).$$

Let $P < 0$ for simplicity. We define the functions φ, ψ by

$$\varphi(\xi) := \begin{cases} -P \cdot \xi & \text{for } x \in W_2 \\ 0 & \text{for } \xi \in \mathbb{T} \setminus W_3 \end{cases}$$

and $|D\varphi| \le C_1 |P|$ for some $C_1 > 0$, $0 \le \varphi \le -P \cdot \xi$ on $[0, 1]$ and

$$\psi(\xi) = \begin{cases} -P \cdot \xi & \text{for } \xi \in W_1 \\ C_2 & \xi \in \mathbb{T} \setminus W_2 \end{cases}$$

for some $C_2 \in (1/64, 1)$, $|P + D\psi| \le 1$, and $\psi \ge -P \cdot \xi$ on $[0, 1]$.

We have

$$\begin{aligned} & |P + D\varphi(\xi)|^2 - V_1(\xi) + \varphi(\xi) - \psi(\xi) \\ \le & \begin{cases} \varphi(\xi) - \psi(\xi) \le 0 & \text{if } \xi \in W_2 \\ 2(C_1^2 + 1)|P|^2 + \varepsilon_0 + |P| - C_2 & \text{if } \xi \in \mathbb{T} \setminus W_2. \end{cases} \end{aligned}$$

If $|P|$ and ε_0 are small enough, then $|P+D\varphi(\xi)|^2-V_1(\xi)+\varphi(\xi)-\psi(\xi)\le 0$ on $\mathbb{T}$. Besides,

$$\begin{aligned}&|P+D\psi(\xi)|^2-V_2(\xi)+\psi(\xi)-\varphi(\xi)\\ \le&\begin{cases}0 & \text{if } \xi\in W_1\\ 1-2+C_2-0\le 0 & \text{if } \xi\in\mathbb{T}\setminus W_1.\end{cases}\end{aligned}$$

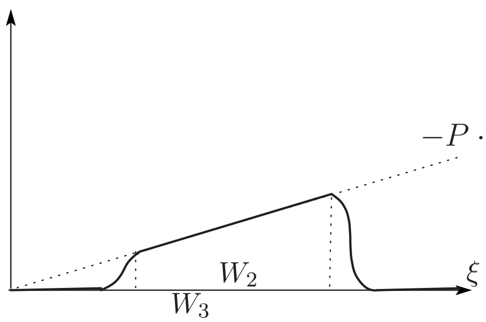

Fig. 4.3. The graph of φ.

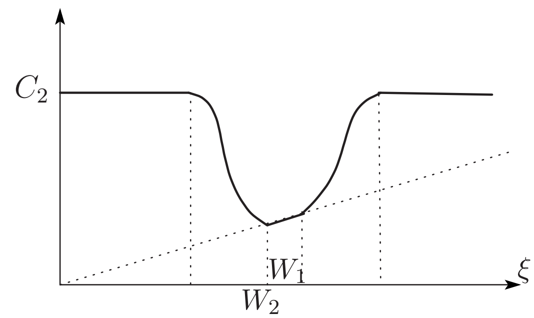

Fig. 4.4. The graph of ψ.

Thus $(\varphi,\psi,0)$ is a subsolution of (E_P), and the proof is complete. ☐

Remark 4.4.16. It is worth to notice that

$$\overline{H}(0)\neq -\frac{1}{2}\min_{x\in\mathbb{T}^n}(V_1+V_2)(x)$$

in general. Indeed, set

$$\begin{aligned}V_1(\xi)&=4\pi^2\sin^2(2\pi\xi)+\cos(2\pi\xi)-\sin(2\pi\xi),\\ V_2(\xi)&=4\pi^2\cos^2(2\pi\xi)+\sin(2\pi\xi)-\cos(2\pi\xi).\end{aligned}$$

If we check that $(\cos(2\pi\xi),\sin(2\pi\xi),0)$ is a solution of (E_0), then we realize $\overline{H}(0)=0$. In this case

$$\overline{H}(0)=0\neq -2\pi^2=-\frac{1}{2}(V_1+V_2)(\xi)\text{ for all }\xi\in\mathbb{T}.$$

In Theorem 4.4.11 the fact that $\Pi(\mathbb{R}^n\setminus(U_1\cup U_2))$ is connected, where $U_i=\{V_i=0\}$ plays an important role in the construction of subsolutions as stated just before Theorem 4.4.13. In the next couple of Theorems we make new observations that we can get the flat parts of effective Hamiltonians even though $\Pi(\mathbb{R}^n\setminus(U_1\cup U_2))$ is not connected.

Theorem 4.4.17. *Let $n=1$ and assume $V_1\equiv 0$, $V_2\ge 0$ on $[0,1]$ and $\{V_2=0\}=\{1/2\}$. Then there exists $\gamma>0$ such that $\overline{H}(P)=0$ for $|P|\le\gamma$.*

Sketch of Proof. The proof is almost the same as the proof of Theorem 4.4.14 but let us present it here for the sake of clarity. Since $\min_{\xi\in\mathbb{T}^n}(V_1+V_2)(\xi)=0$, we have $\overline{H}(P)\ge 0$ by Lemma 4.4.15. Let

$$W_1=(\frac{3}{8},\frac{5}{8}),\ W_2=(\frac{2}{8},\frac{6}{8}),\ W_3=(\frac{1}{8},\frac{7}{8}).$$

There exists $M \in (0,1)$ so that

$$V_2(\xi) \geq M \text{ for } \xi \notin W_1.$$

Assume $P < 0$ for simplicity. We now construct the functions φ, ψ so that $(\varphi, \psi, 0)$ is a subsolution of (E_P) for small $|P|$, which implies the conclusion. Take $|P| \leq M/4$ first. We define the functions φ, ψ by

$$\varphi(\xi) := \begin{cases} -P \cdot \xi & \text{for } x \in W_2 \\ 0 & \text{for } \xi \in \mathbb{T} \setminus W_3 \end{cases}$$

and $|D\varphi| \leq C_1|P|$ for some $C_1 > 0$, $0 \leq \varphi \leq -P \cdot \xi$ on $[0,1]$ and

$$\psi(\xi) = \begin{cases} -P \cdot \xi & \text{for } \xi \in W_1 \\ C_2 & \xi \in \mathbb{T} \setminus W_2 \end{cases}$$

for some $C_2 \in (M/128, M/2)$, $|P + D\psi| \leq M/2$, and $\psi \geq -P \cdot \xi$ on $[0,1]$.

We have

$$|P + D\varphi(\xi)|^2 + \varphi(\xi) - \psi(\xi) \leq \begin{cases} \varphi(\xi) - \psi(\xi) \leq 0 & \text{if } \xi \in W_2 \\ 2(C_1^2 + 1)|P|^2 + |P| - C_2 & \text{if } \xi \in \mathbb{T} \setminus W_2. \end{cases}$$

If $|P|$ is small enough, then $|P + D\varphi(\xi)|^2 + \varphi(\xi) - \psi(\xi) \leq 0$ on $\mathbb{T}$. Besides,

$$\begin{aligned} &|P + D\psi(\xi)|^2 - V_2(\xi) + \psi(\xi) - \varphi(\xi) \\ &\leq \begin{cases} 0 & \text{if } \xi \in W_1 \\ \dfrac{M^2}{4} - M + C_2 - 0 \leq \dfrac{M^2}{4} - M + \dfrac{M}{2} \leq 0 & \text{if } \xi \in \mathbb{T} \setminus W_1. \end{cases} \end{aligned}$$

Thus $(\varphi, \psi, 0)$ is a subsolution of (E_P), and the proof is complete. □

We can actually generalize Theorem 4.4.17 as following.

Theorem 4.4.18. *Assume that $V_1 \equiv 0$, $V_2 \geq 0$ and there exist an open set W in $\mathbb{T}^n$ and a vector $q \in \mathbb{R}^n$ such that $\Pi(q + W) \Subset (0,1)^n$ and $\emptyset \neq \{V_2 = 0\} \subset W$. Then there exists $\gamma > 0$ such that $\overline{H}(P) = 0$ for $|P| \leq \gamma$.*

The proof of this Theorem is basically the same as the proof of Theorem 4.4.17, hence omitted. The following Corollary is a direct consequence of Theorem 4.4.18

Corollary 4.4.19. *Assume that V_1, $V_2 \geq 0$ and there exist an open set W in $\mathbb{T}^n$ and a vector $q \in \mathbb{R}^n$ such that $\Pi(q + W) \Subset (0,1)^n$ and*

$$\emptyset \neq \{V_1 = 0\} \cap \{V_2 = 0\} \subset \{V_2 = 0\} \subset W.$$

Then there exists $\gamma > 0$ such that $\overline{H}(P) = 0$ for $|P| \leq \gamma$.

The result of Corollary 4.4.19 is pretty surprising in the sense that flat part around 0 of $\overline{H}$ occurs even though we do not know much information about V_1. More precisely, we only need to control well $\{V_2 = 0\}$ and do not need to care about $\{V_1 = 0\}$ except that $\{V_1 = 0\} \cap \{V_2 = 0\} \neq \emptyset$.

Finally, we consider a situation in which the requirements of Theorem 4.4.18 and Corollary 4.4.19 fail.

Theorem 4.4.20. *We take two potentials $V^i : \mathbb{T} \to [0,\infty)$ such that V^i are continuous and $\{V^i = 0\} = \{y_{0i}\}$ for some $y_{0i} \in \mathbb{T}$ for $i = 1,2$. Assume that $V_1(\xi_1,\xi_2) = V^1(\xi_1)$ and $V_2(\xi_1,\xi_2) = V^2(\xi_2)$ for $(\xi_1,\xi_2) \in \mathbb{T}^2$. Then there exists $\gamma > 0$ such that $\overline{H}(P) = 0$ for $|P| \le \gamma$.*

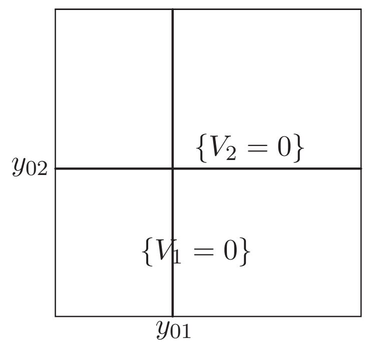

Fig. 4.5. The figures of $\{V_i = 0\}$

Proof. By using Theorem 4.4.17, for $P = (P_1, P_2)$ with $|P|$ small enough, there exist two pairs $(\varphi_i, \psi_i) \in C^{0,1}(\mathbb{T})^2$ for $i = 1,2$ such that

$$\begin{cases} |P_1 + \varphi_1'(\xi_1)|^2 - V^1(\xi_1) + \varphi_1(\xi_1) - \psi_1(\xi_1) = 0, \\ |P_1 + \psi_1'(\xi_1)|^2 + \psi_1(\xi_1) - \varphi_1(\xi_1) = 0 \end{cases}$$

and

$$\begin{cases} |P_2 + \varphi_2'(\xi_2)|^2 + \varphi_2(\xi_2) - \psi_2(\xi_2) = 0, \\ |P_2 + \psi_2'(\xi_2)|^2 - V^2(\xi_2) + \psi_2(\xi_2) - \varphi_2(\xi_2) = 0 \end{cases}$$

Now let $v_1(\xi_1,\xi_2) = \varphi_1(\xi_1) + \varphi_2(\xi_2)$, $v_2(\xi_1,\xi_2) = \psi_1(\xi_1) + \psi_2(\xi_2)$ for $(\xi_1,\xi_2) \in \mathbb{T}^2$. For $P = (P_1,P_2)$ with $|P| \le \gamma$, we easily get that $(v_1, v_2, 0)$ is a solution of (E_P), which means $\overline{H}(P) = 0$. □

4.5 Generalization

In this section we consider weakly coupled systems of m-equations for $m \ge 2$

$$(u_i^\varepsilon)_t + H_i(\frac{x}{\varepsilon}, Du_i^\varepsilon) + \frac{1}{\varepsilon}\sum_{j=1}^m c_{ij}(u_i^\varepsilon - u_j^\varepsilon) = 0 \text{ in } \mathbb{R}^n \times (0,T) \text{ for } i = 1,\dots,m,$$

with

$$u_i^\varepsilon(x,0) = f_i(x) \text{ on } \mathbb{R}^n \text{ for } i = 1, \ldots, m,$$

where c_{ij} are given nonnegative constants which are assumed to satisfy

$$\sum_{j=1}^{m} c_{ij} = 1 \text{ for all } i = 1, \ldots, m. \tag{4.5.1}$$

Set

$$K := \begin{pmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mm} \end{pmatrix}, \ \mathbf{u}^\varepsilon := \begin{pmatrix} u_1^\varepsilon \\ \vdots \\ u_m^\varepsilon \end{pmatrix}, \text{ and } \mathbf{f} := \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

Then the problem can be written as

$$\begin{cases} \mathbf{u}_t^\varepsilon + \begin{pmatrix} H_1(x/\varepsilon, Du_1^\varepsilon) \\ \vdots \\ H_m(x/\varepsilon, Du_m^\varepsilon) \end{pmatrix} + \dfrac{1}{\varepsilon}(I - K)\mathbf{u}^\varepsilon = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \mathbf{u}^\varepsilon(\cdot, 0) = \mathbf{f} & \text{on } \mathbb{R}^n, \end{cases} \tag{4.5.2}$$

where I is the identity matrix of size m. We obtain the following result.

Theorem 4.5.1. *The functions u_i^ε converge locally uniformly to the same limit u in $\mathbb{R}^n \times (0,T)$ as $\varepsilon \to 0$ for $i = 1, \ldots, m$ and u solves*

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(x,0) = \overline{f}(x) & \text{on } \mathbb{R}^n, \end{cases}$$

where $\overline{H}$ is the associated effective Hamiltonian and

$$\overline{f}(x) := \frac{1}{m}\sum_{i=1}^{m} f_i(x).$$

We only present barrier functions which are generalizations of the barrier function in case $m = 2$ defined by (4.2.1) in Lemma 4.2.1. Set

$$\mathbf{w}^\pm(\mathbf{x}, \mathbf{t}) := (\overline{f} \pm Ct)\mathbf{j} + \mathbf{g}^\varepsilon(x,t),$$

where C is a positive constant which will be fixed later, $\mathbf{j} := (1, \ldots, 1)^T$ and

$$\mathbf{g}^\varepsilon(x,t) := \left[e^{\frac{t}{\varepsilon}(K-I)}\mathbf{h}\right](x), \ \mathbf{h}(x) := \mathbf{f}(x) - \overline{f}(x)\mathbf{j}.$$

Since we assume (4.5.1), we can easily check that the Frobenius root of K, i.e., the maximum of the eigenvalues of K, is 1 and moreover $\mathbf{j}$ is an associated eigenvector. Moreover by the Perron–Frobenius theorem we have

Lemma 4.5.2. *There exists $\delta > 0$ such that $|e^{t(K-I)}\mathbf{h}| \le e^{-\delta t}|\mathbf{h}|$ provided that $\mathbf{h}\cdot\mathbf{j} = 0$.*

See [54, Lemma 5.2] for a more general result.

Proposition 4.5.3. *The functions $\mathbf{w}^{\pm}$ are a subsolution and a supersolution of (4.5.2) with $\mathbf{w}^{\pm}(\cdot, 0) = \mathbf{f}$ on $\mathbb{R}^n$, respectively, if $C > 0$ is large enough.*

Proof. It is easy to check $\mathbf{w}^{\pm}(\cdot, 0) = \mathbf{f}$ on $\mathbb{R}^n$. Note that

$$\frac{\partial \mathbf{g}^{\varepsilon}}{\partial t} = \frac{1}{\varepsilon}(K - I)\mathbf{g}^{\varepsilon} \text{ and } |D\mathbf{g}| \le Ce^{-\frac{\delta t}{\varepsilon}}.$$

Thus, we can check easily that $\mathbf{w}^{\pm}$ are a subsolution and a supersolution of (4.5.2), respectively, if $C > 0$ is large enough. □

By a rather standard argument by using the perturbed test functions we can get Theorem 4.5.1 as in the proof of Theorem 4.1.2.

4.6 Dirichlet Problems

In this section we consider the asymptotic behavior, as ε tends to 0, of the viscosity solutions $(u_1^{\varepsilon}, u_2^{\varepsilon})$ of Dirichlet boundary problems for weakly coupled systems of Hamilton–Jacobi equations

$$(\mathrm{D}_{\varepsilon}) \qquad \begin{cases} u_1^{\varepsilon} + H_1(\frac{x}{\varepsilon}, Du_1^{\varepsilon}) + \frac{1}{\varepsilon}(u_1^{\varepsilon} - u_2^{\varepsilon}) = 0 & \text{in } \Omega, \\ u_2^{\varepsilon} + H_2(\frac{x}{\varepsilon}, Du_2^{\varepsilon}) + \frac{1}{\varepsilon}(u_2^{\varepsilon} - u_1^{\varepsilon}) = 0 & \text{in } \Omega, \\ u_i^{\varepsilon}(x) = g_i(x) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with the Lipschitz boundary, the Hamiltonians $H_i \in C(\mathbb{R}^n \times \mathbb{R}^n)$ are assumed to satisfy (A1)-(A2) and $g_i \in C(\partial\Omega)$ are given functions.

Concerning the Dirichlet problem, most of the works required continuous solutions up to the boundary and prescribed data on the entire boundary. This can be achieved for special classes of equations by imposing compatibility conditions on the boundary data or by assuming the existence of appropriate super and subsolutions. However, in general, we do not expect that there exists a (viscosity) solution satisfying the boundary condition in the classical sense. After Soner studied the state constraints problems in terms of PDE, the viscosity formulation of Dirichlet conditions was introduced by Barles and Perthame [5] and Ishii [48]. In this Chapter we deal with solutions satisfying Dirichlet boundary conditions in the sense of viscosity solutions.

Theorem 4.6.1. *Let $(u_1^{\varepsilon}, u_2^{\varepsilon})$ be the solution of (D_{ε}). Then u_i^{ε} converge locally uniformly to the same limit u on Ω as $\varepsilon \to 0$ for $i = 1, 2$ and u solves*

$$\begin{cases} u + \overline{H}(Du) = 0 & \text{in } \Omega, \\ u(x, t) = \overline{g} & \text{on } \partial\Omega, \end{cases} \tag{4.6.1}$$

where $\overline{g} := \min\{g_1, g_2\}$ *on* $\partial\Omega$.

Lemma 4.6.2. *If* $(u_1^\varepsilon, u_2^\varepsilon) \in USC(\overline{\Omega})^2$ *is a bounded subsolution of* (D_ε), *then* $u_i^\varepsilon(x) \le g_i(x)$ *for all* $x \in \partial\Omega$ *and* $i = 1, 2$.

Proof. Fix $x_0 \in \partial\Omega$. Choose a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n \setminus \overline{\Omega}$ such that $|x_0 - x_k| = 1/k^2$. Define the functions $\phi_1 : \overline{\Omega} \to \mathbb{R}$ by $\phi_1(x) := u_1^\varepsilon(x) - k|x - x_k|$. Let $r > 0$ and $\xi_k \in B(x_0, r) \cap \overline{\Omega}$ be a maximum point of ϕ_1 on $B(x_0, r) \cap \overline{\Omega}$. Since $\phi_1(\xi_k) \ge \phi_1(x_0)$, we have $k|\xi_k - x_k| \le u_1^\varepsilon(\xi_k) - u_1^\varepsilon(x_0) + k|x_0 - x_k| \le C$, where $C > 0$ is a constant independent of k. Thus, $\xi_k \to x_0$ as $k \to \infty$. Moreover, noting that $u_1^\varepsilon(x_0) \le \liminf_{k\to\infty}(u_1^\varepsilon(\xi_k) + k|x_0 - x_k|) \le \limsup_{k\to\infty} u_1^\varepsilon(\xi_k) + \limsup_{k\to\infty} k|x_0 - x_k| \le u_1^\varepsilon(x_0)$, we get $u_1^\varepsilon(\xi_k) \to u_1^\varepsilon(x_0)$ as $k \to \infty$. By the viscosity property of u_1^ε, we have

$$\begin{aligned} &u_1^\varepsilon(\xi_k) + H_1(\frac{\xi_k}{\varepsilon}, p_k) + \frac{1}{\varepsilon}(u_1^\varepsilon(\xi_k) - u_2(\xi_k)) \le 0 \text{ or} \\ &u_1^\varepsilon(\xi_k) \le g_1(\xi_k), \end{aligned} \tag{4.6.2}$$

where $p_k = k(\xi_k - x_k)/|\xi_k - x_k|$. Noting that $|p_k| = k$, by (A1), we see that the left-hand side of (4.6.2) is positive for a sufficiently large $k \in \mathbb{N}$ and then we must have $u_1^\varepsilon(\xi_k) \le g_1(\xi_k)$. Sending $k \to \infty$, we get $u_1^\varepsilon(x_0) \le g_1(x_0)$. Similaly, we get $u_2^\varepsilon(x_0) \le g_2(x_0)$ on $\partial\Omega$. □

Lemma 4.6.3. *The families* $\{u_i^\varepsilon\}_{\varepsilon>0}$ *are equi-Lipschitz continuous for* $i = 1, 2$.

Proof. Set $M := \max_{i=1,2}(\|H_i(\cdot, 0)\|_{L^\infty(\mathbb{R}^n)} + \|g_i\|_{L^\infty(\partial\Omega)})$. Then $(-M, -M)$ and (M, M) are a subsolution and a supersolution of (D_ϵ), respectively. By the comparison principle for (D_ϵ) we have $|u_i^\varepsilon| \le M$. Adding two equations in (D_ϵ) we get

$$u_1^\varepsilon + u_2^\varepsilon + H_1(\frac{x}{\varepsilon}, Du_1^\varepsilon) + H_2(\frac{x}{\varepsilon}, Du_2^\varepsilon) = 0$$

for almost every $x \in \Omega$, which implies that $|Du_i^\varepsilon| \le M'$ in the sense of viscosity solutions for some $M' > 0$. □

Proof of Theorem 4.1.2. By Lemma 4.6.3 we can extract a subsequence $\{\varepsilon_j\}$ converging to 0 so that $u_i^{\varepsilon_j}$ converges locally uniformly to $u_i \in C(\overline{\Omega})$ for $i = 1, 2$. By usual observations, we get that $u_1 = u_2 =: u$. Since (4.6.1) has a unique solution, it is enough for us to prove that u is a solution of (4.6.1).

We only prove that u is a supersolution of (4.6.1), since in view of Lemma 4.6.2 we can easily see that u is a subsolution of (4.6.1).

Let $\phi \in C^1(\overline{\Omega})$ be a test function such that $u - \phi$ takes a strict minimum at $x_0 \in \overline{\Omega}$. We only consider the case where $x_0 \in \partial\Omega$, since we can prove by a similar way to the proof of Theorem 4.1.2 in the case where $x_0 \in \Omega$. It is enough for us to prove that $u(x_0) + \overline{H}(D\phi(x_0)) \ge 0$ provided that $(u - \overline{g})(x_0) < 0$.

Let (v_1, v_2) be a solution of (E_P) with $P := D\phi(x_0)$. We consider

$$m^\varepsilon := \min_{i\in\{1,2\}} \min_{x\in\overline{\Omega}} \big(u_i^\varepsilon(x) - \phi(x) - \varepsilon v_i(\frac{x}{\varepsilon})\big).$$

Pick $i^\varepsilon \in \{1,2\}$ and $x^\varepsilon \in \overline{\Omega}$ so that $m^\varepsilon = u^\varepsilon_{i^\varepsilon}(x^\varepsilon) - \phi(x^\varepsilon) - \varepsilon v_{i^\varepsilon}(x^\varepsilon/\varepsilon)$. Also choose $j^\varepsilon \in \{1,2\}$ such that $\{i^\varepsilon, j^\varepsilon\} = \{1,2\}$. We only consider the case where $x^\varepsilon \in \partial\Omega$ again. Since u_{i_ε} converges to u locally uniformly on $\overline{\Omega}$, $\varepsilon v(\cdot/\varepsilon)$ converges to 0 uniformly on $\overline{\Omega}$ as $\varepsilon \to 0$ and $u - \phi$ takes a strict maximum at x_0, we see that $x^\varepsilon \to x_0$ as $\varepsilon \to 0$. Thus, if ε is small enough, then we may assume that $(u^\varepsilon_{i^\varepsilon} - g_{i^\varepsilon})(x^\varepsilon) < 0$.

For $\alpha > 0$ we define the function $\Phi_\alpha : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\Phi_\alpha(x,y) := u^\varepsilon_{i^\varepsilon}(x) - \phi(x) - \varepsilon v_{i^\varepsilon}\big(\frac{y}{\varepsilon}\big) + \frac{1}{2\alpha^2}|x-y|^2 + \frac{1}{2}|x - x^\varepsilon|^2.$$

Let Φ_α achieve its minimum over $\overline{\Omega} \times \mathbb{R}^n$ at some $(x^\varepsilon_\alpha, y^\varepsilon_\alpha)$. Since we may assume by taking a subsequence if necessary that $x^\varepsilon_\alpha \to x^\varepsilon$ as $\alpha \to 0$, we have

$$(u_{i^\varepsilon} - g_{i^\varepsilon})(x^\varepsilon_\alpha) < 0 \text{ for small } \alpha > 0.$$

Therefore, by the definition of viscosity solutions, we have

$$u_{i^\varepsilon} + H_{i^\varepsilon}\big(\frac{x^\varepsilon_\alpha}{\varepsilon}, D\phi(x^\varepsilon_\alpha) - p^\varepsilon_\alpha - (x^\varepsilon_\alpha - x^\varepsilon)\big) + \frac{1}{\varepsilon}(u^\varepsilon_{i^\varepsilon} - u^\varepsilon_{j^\varepsilon})(x^\varepsilon_\alpha) \geq 0,$$

where $p^\varepsilon_\alpha := (x^\varepsilon_\alpha - y^\varepsilon_\alpha)/\alpha^2$. Also, we have

$$H_i\big(\frac{y^\varepsilon_\alpha}{\varepsilon}, P - p^\varepsilon_\alpha\big) + (v_{i^\varepsilon} - v_{j^\varepsilon})\big(\frac{y^\varepsilon_\alpha}{\varepsilon}\big) \leq \overline{H}(P),$$

since (v_1, v_2) is a solution of (E_P).

A priori Lipschitz estimate implies $|p^\varepsilon_\alpha| \leq C$ for some $C > 0$ which is independent of α and ε. Without loss of generality, we may assume that $p^\varepsilon_\alpha \to p^\varepsilon$ by taking a subsequence $\{\alpha_j\}$ converging to 0 if necessary. Send $\alpha \to 0$ in the above inequalities to obtain

$$u^\varepsilon_{i^\varepsilon}(x^\varepsilon) + H_{i^\varepsilon}\big(\frac{x^\varepsilon}{\varepsilon}, D\phi(x^\varepsilon) - p^\varepsilon\big) + \frac{1}{\varepsilon}(u^\varepsilon_{i^\varepsilon}(x^\varepsilon) - u^\varepsilon_{j^\varepsilon}(x^\varepsilon)) \geq 0,$$
$$H_{i^\varepsilon}\big(\frac{x^\varepsilon}{\varepsilon}, P - p^\varepsilon\big) + v_{i^\varepsilon}\big(\frac{x^\varepsilon}{\varepsilon}\big) - v_{j^\varepsilon}\big(\frac{x^\varepsilon}{\varepsilon}\big) \leq \overline{H}(P).$$

Noting that $u^\varepsilon_{i^\varepsilon}(x^\varepsilon) - \phi(x^\varepsilon) - \varepsilon v_{i^\varepsilon}(\frac{x^\varepsilon}{\varepsilon}) \leq u^\varepsilon_{j^\varepsilon}(x^\varepsilon) - \phi(x^\varepsilon) - \varepsilon v_{j^\varepsilon}(\frac{x^\varepsilon}{\varepsilon})$, we get that

$$u^\varepsilon_{i^\varepsilon}(x^\varepsilon) + \overline{H}(P) \geq H_{i^\varepsilon}\big(\frac{x^\varepsilon}{\varepsilon}, D\phi(x^\varepsilon) - p^\varepsilon\big) - H_{i^\varepsilon}\big(\frac{x^\varepsilon}{\varepsilon}, P - p^\varepsilon\big) \geq -\sigma(|D\phi(x^\varepsilon) - P|).$$

Sending $\varepsilon \to 0$ yields the conclusion. □

In order to explain the relation between (D_ε) and the exit-time problem in the optimal control theory, we assume that the Hamiltonians H_i are convex in the p-variable henceforth. We next define the associated value functions, which give us an intuition about the effective boundary datum $\overline{g}$ in Theorem 4.6.1.

For $\varepsilon > 0$ we define the functions $u_i^\varepsilon : \overline{\Omega} \to \mathbb{R}$ by

$$u_i^\varepsilon(x) := \inf \Big\{ \mathbb{E}_i \Big(\int_0^\tau e^{-s} L_{\nu^\varepsilon(s)}(\frac{\eta(s)}{\varepsilon}, -\dot{\eta}(s))\, ds + e^{-\tau} g_{\nu^\varepsilon(\tau)}(\eta(\tau)) \Big) \Big\}, \tag{4.6.3}$$

where the infimum is taken over $\eta \in \mathrm{AC}\,([0,\infty), \overline{\Omega})$ such that $\eta(0) = x$ and $\tau \in [0,\infty]$ such that $\eta(\tau) \in \partial\Omega$ and if $\tau = \infty$, then we set $e^{-\infty} := 0$. Here $\mathbb{E}_i$ denotes the expectation of a process with $\nu^\varepsilon(0) = i$, where ν^ε is a $\{1,2\}$-valued continuous-time Markov chain given by (4.1.2).

Theorem 4.6.4. *Assume that the functions u_i^ε given by* (4.6.3) *are continuous on $\overline{\Omega}$. Then the pair $(u_1^\varepsilon, u_2^\varepsilon)$ is a solution of (D_ε).*

The proof of Theorem 4.6.4 is given in Section 4.7. See [**I**, 5] for single equations. The value functions defined by (4.6.3) give us an intuitive explanation of the reason why the boundary datum $\overline{g}$ of the limit solution u is the minimum of g_i for $i = 1, 2$. If we send ε to 0, then the switching rate becomes very fast but it does not really affect the exit time as we can choose to stay in $\overline{\Omega}$ as long as we like. And hence, we can control the exit state in such a way that the exit cost is the minimum of two given exit costs g_i. On the other hand, when we consider the value function (4.1.1) associated with the initial value problem, we cannot control the terminal state and also the timing of jumps, which are only determined by a probabilistic way given by (4.1.2). This is the main difference between Dirichlet problems and initial value problems and the reason why the effective Dirichlet boundary value and the effective initial value are different.

4.7 Auxiliary Lemmata

We now prove Theorems 4.7.1, and 4.6.4 by basically using the dynamic programming principles, which are pretty standard in the theory of viscosity solutions. Throughout this section we always assume in addition to (A1), (A2) that $p \mapsto H_i(\xi, p)$ are convex for $i = 1, 2$.

Theorem 4.7.1 (Verification Theorem)**.** *Assume that the functions u_i^ε given by* (4.1.1) *are continuous on $\mathbb{R}^n \times [0,T]$. Then the pair $(u_1^\varepsilon, u_2^\varepsilon)$ is a solution of (C_ε).*

Let $\varepsilon = 1$ for simplicity in what follows. By abuse of notations we write (u_1, u_2) for (u_1^1, u_2^1).

Proposition 4.7.2 (Dynamic Programming Principle)**.** *For any $x \in \mathbb{R}^n$, $0 \le h \le t$ and $i = 1, 2$ we have*

$$u_i(x,t) = \inf \Big\{ \mathbb{E}_i \Big(\int_0^h L_{\nu(s)}(\eta(s), -\dot{\eta}(s))\, ds + u_{\nu(h)}(\eta(h), t-h) \Big) \Big\}, \tag{4.7.1}$$

where the infimum is taken over $\eta \in AC([0,h], \mathbb{R}^n)$ with $\eta(0) = x$.

Proof. We denote by $v_i(x,t;h)$ the right-hand side of (4.7.1). Let η be a trajectory in $\mathrm{AC}([0,t],\mathbb{R}^n)$ with $\eta(0)=x$ and ν be a process with $\nu(0)=i$ which satisfies (4.1.2). Set $\tilde\eta(s):=\eta(s+h)$ and $\tilde\nu(s):=\nu(s+h)$ for $s\in[0,t-h]$. We have

$$\begin{aligned}
&\mathbb{E}_i\Big(\int_0^t L_{\nu(s)}(\eta(s),-\dot\eta(s))\,ds+f_{\nu(t)}(\eta(t))\Big)\\
=&\,\mathbb{E}_i\Big(\int_0^h L_{\nu(s)}(\eta(s),-\dot\eta(s))\,ds+\int_h^t L_{\nu(s)}(\eta(s),-\dot\eta(s))\,ds+f_{\nu(t)}(\eta(t))\Big)\\
=&\,\mathbb{E}_i\Big(\int_0^h L_{\nu(s)}(\eta(s),-\dot\eta(s))\,ds\Big)+\mathbb{E}_i\Big(\int_0^{t-h} L_{\tilde\nu(s)}(\tilde\eta(s),-\dot{\tilde\eta}(s))\,ds+f_{\tilde\nu(t-h)}(\tilde\eta(t-h))\Big)\\
\geq&\,\mathbb{E}_i\Big(\int_0^h L_{\nu(s)}(\eta(s),-\dot\eta(s))\,ds+u_{\nu(h)}(\eta(h),t-h)\Big)\\
\geq&\,v_i(x,t;h),
\end{aligned}$$

which implies $u_i(x,t)\geq v_i(x,t;h)$.

Let $\delta_1\in\mathrm{AC}([0,h],\mathbb{R}^n)$ and $\delta_2\in\mathrm{AC}([0,t-h],\mathbb{R}^n)$ be trajectories with $\delta_1(h)=\delta_2(0)$ and $\delta_1(0)=x$. Set

$$\eta(s):=\begin{cases}\delta_1(s) & \text{for all } s\in[0,h],\\ \delta_2(s-h) & \text{for all } s\in[h,t].\end{cases}$$

Let ν be a process with $\nu(0)=i$ which satisfies (4.1.2). Note that

$$\begin{aligned}
&\int_0^h L_{\nu(s)}(\delta_1(s),-\dot\delta_1(s))\,ds+\int_0^{t-h} L_{\nu(s+h)}(\delta_2(s),-\dot\delta_2(s))\,ds+f_{\nu(t)}(\delta_2(t-h))\\
=&\int_0^t L_{\nu(s)}(\eta(s),-\dot\eta(s))\,ds+f_{\nu(t)}(\eta(t)).
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbb{E}_i\Big(\int_0^h L_{\nu(s)}(\delta_1(s),-\dot\delta_1(s))\,ds+\int_0^{t-h} L_{\nu(s+h)}(\delta_2(s),-\dot\delta_2(s))\,ds+f_{\nu(t)}(\delta_2(t-h))\Big)\\
=&\,\mathbb{E}_i\Big(\int_0^t L_{\nu(s)}(\eta(s),-\dot\eta(s))\,ds+f_{\nu(t)}(\eta(t))\Big)\\
\geq&\,u_i(x,t).
\end{aligned}$$

Take the infimum on all admissible δ_2 to obtain

$$\mathbb{E}_i\Big(\int_0^h L_{\nu(s)}(\delta_1(s),-\delta_1(s))\,ds+u_{\nu(h)}(\delta_1(h),t-h)\Big)\geq u_i(x,t),$$

which implies $v_i(x,t;h)\geq u_i(x,t)$. □

Proof of Theorem 4.7.1. It is obvious to see that $(u_1,u_2)(\cdot,0)=(f_1,f_2)$ on $\mathbb{R}^n$. We prove first that u_1 is a subsolution of (C_1). We choose a function $\phi\in C^1(\mathbb{R}^n\times(0,T))$ such that $u_1-\phi$ has a strict maximum at $(x_0,t_0)\in\mathbb{R}^n\times(0,T)$ and $(u_1-\phi)(x_0,t_0)=0$.

Let $h > 0$. By Proposition 4.7.1 we have

$$u_1(x_0,t_0) \le \mathbb{E}_i\Big(\int_0^h L_{\nu(s)}(\eta(s),-\dot\eta(s))\,ds + u_{\nu(h)}(\eta(h),t_0-h)\Big) \tag{4.7.2}$$

for any $\eta \in \mathrm{AC}\,([0,h],\mathbb{R}^n)$ with $\eta(0) = x_0 \in \mathbb{R}^n$ and $\dot\eta(0) = q \in \mathbb{R}^n$. Since ν is a continuous-time Markov chain which satisfies (4.1.2), the probability that $\nu(h) = 2$ is $c_1h + o(h)$ and the probability that $\nu(h) = 1$ is $1-(c_1h+o(h))$. By (4.7.2) we obtain

$$\begin{aligned}
&\phi(x_0,t_0) = u_1(x_0,t_0)\\
&\le (1-c_1h-o(h))\Big(\int_0^h L_1(\eta,-\dot\eta)\,ds + u_1(\eta(h),t_0-h)\Big)\\
&\quad + (c_1h+o(h))\Big(\int_0^h L_2(\eta,-\dot\eta)\,ds + u_2(\eta(h),t_0-h)\Big)\\
&\le \int_0^h L_1(\eta,-\dot\eta)\,ds + \phi(\eta(h),t_0-h)\\
&\quad + (c_1h+o(h))\Big(\int_0^h L_2(\eta,-\dot\eta)\,ds + u_2(\eta(h),t_0-h) - \int_0^h L_1(\eta,-\dot\eta)\,ds - u_1(\eta(h),t_0-h)\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\phi(\eta(0),t_0)-\phi(\eta(h),t_0-h)}{h}\\
&\le \frac{1}{h}\int_0^h L_1(\eta,-\dot\eta)\,ds + (c_1+\frac{o(h)}{h})(u_2(\eta(h),t_0-h)-u_1(\eta(h),t_0-h))\\
&\quad + (c_1+\frac{o(h)}{h})\Big(\int_0^h L_2(\eta,-\dot\eta)\,ds - \int_0^h L_1(\eta,-\dot\eta)\,ds\Big).
\end{aligned}$$

Sending $h \to 0$, we obtain

$$\phi_t(x_0,t_0) + D\phi(x_0,t_0)\cdot(-q) \le L_1(x_0,-q) + c_1(u_2-u_1)(x_0,t_0) \text{ for all } q \in \mathbb{R}^n,$$

which implies $\phi_t(x_0,t_0) + H(x_0,D\phi(x_0,t_0)) + c_1(u_2-u_1)(x_0,t_0) \le 0$.

Next we prove that u_1 is a supersolution of (C_1). We choose a function $\phi \in C^1(\mathbb{R}^n \times (0,T))$ such that $u_1-\phi$ has a strict minimum at $(x_0,t_0) \in \mathbb{R}^n\times(0,T)$ and $(u_1-\phi)(x_0,t_0) = 0$. Take $h,\delta > 0$. By Proposition 4.7.1 we have

$$u_1(x_0,t_0)+\delta > \mathbb{E}_i\Big(\int_0^h L_{\nu(s)}(\eta_\delta(s),-\dot\eta_\delta(s))\,ds + u_{\nu(h)}(\eta_\delta(h),t_0-h)\Big) \tag{4.7.3}$$

for some $\eta_\delta \in \mathrm{AC}\,([0,h],\mathbb{R}^n)$ with $\eta_\delta(0) = x_0$. Since ν is a continuous-time Markov chain which satisfies (4.1.2), by a similar calculation to the above we obtain

$$\begin{aligned}
&\phi(x_0,t_0)+\delta = u_1(x_0,t_0)+\delta\\
&> \int_0^h L_1(\eta,-\dot\eta)\,ds + \phi(\eta(h),t_0-h)\\
&\quad + (c_1h+o(h))\Big(\int_0^h L_2(\eta,-\dot\eta)\,ds + u_2(\eta(h),t_0-h) - \int_0^h L_1(\eta,-\dot\eta)\,ds - u_1(\eta(h),t_0-h)\Big).
\end{aligned}$$

Thus,

$$
\begin{aligned}
\frac{\delta}{h} >& \frac{1}{h}\int_0^h \frac{d\phi(\eta_\delta(s), t_0 - s)}{ds} + L_1(\eta_\delta, -\dot{\eta_\delta})\, ds \\
&+ (c_1 + \frac{o(h)}{h})(u_2(\eta_\delta(h), t_0 - h) - u_1(\eta_\delta(h), t_0 - h)) \\
&+ (c_1 + \frac{o(h)}{h})\Big(\int_0^h L_2(\eta_\delta, -\dot{\eta_\delta})\, ds - \int_0^h L_1(\eta_\delta, -\dot{\eta_\delta})\, ds\Big) \\
=& \frac{1}{h}\int_0^h -\phi_t(\eta_\delta(s), t_0 - s) - D\phi\cdot(-\dot{\eta_\delta}(s)) + L_1(\eta_\delta, -\dot{\eta_\delta})\, ds \\
&+ (c_1 + \frac{o(h)}{h})(u_2(\eta_\delta(h), t_0 - h) - u_1(\eta_\delta(h), t_0 - h)) + O(h) \\
\geq& \frac{1}{h}\int_0^h -\big(\phi_t(\eta_\delta(s), t_0 - s) + H_1(\eta_\delta(s), D\phi)\big)\, ds \\
&+ (c_1 + \frac{o(h)}{h})(u_2(\eta_\delta(h), t_0 - h) - u_1(\eta_\delta(h), t_0 - h)) + O(h).
\end{aligned}
$$

We finally set $\delta = h^2$ and let $h \to 0$ to yield the conclusion. □

By a similar argument to the proof of Proposition 4.7.2 we can prove

Proposition 4.7.3 (Dynamic Programming Principle for (4.6.3))**.** *For any $x \in \mathbb{R}^n$, $h \geq 0$ and $i = 1, 2$ we have*

$$
u_i(x) = \inf\Big\{\mathbb{E}_i\Big(\int_0^{h\wedge\tau} e^{-s} L_{\nu(s)}(\frac{\eta(s)}{\varepsilon}, -\dot{\eta}(s))\, ds + \mathbf{1}_{\{h<\tau\}} e^{-h} u_{\nu(h)}(\eta(h)) + \mathbf{1}_{\{h\geq\tau\}} e^{-\tau} g_{\nu(\tau)}(\eta(\tau))\Big)\Big\}, \quad (4.7.4)
$$

where ν with $\nu(0) = i$ is a $\{1, 2\}$-valued continuous-time Markov chain which satisfies (4.1.2) *and the infimum is taken over $\eta \in AC([0, h], \overline{\Omega})$ such that $\eta(0) = x$ and $\tau \in [0, h]$ such that $\eta(\tau) \in \partial\Omega$.*

Proof of Theorem 4.6.4. We only prove in what follows that u_i satisfy the Dirichlet boundary condition in the sense of viscosity solutions, as we can prove u_i satisfy the equations by an argument similar to the proof of Theorem 4.7.1. Since it is clear to see that $u_i \leq g_i$ on $\partial\Omega$ in the classical sense from the definition of u_i, we only need to prove that (u_1, u_2) is a supersolution of (D_ε) and particularly that u_1 satisfies the boundary condition in the viscosity solution sense. Take $x_0 \in \partial\Omega$ and suppose that

$$
(u_1 - g_1)(x_0) < 0. \quad (4.7.5)
$$

Let $\phi \in C^1(\overline{\Omega})$ satisfy $(u_1-\phi)(x_0) = \max_{\overline{\Omega}}(u_1-\phi) = 0$. By Proposition 4.7.3 we have

$$
\begin{aligned}
&u_1(x_0) + h^2 \\
&> \mathbb{E}_i\Big(\int_0^{h\wedge\tau_h} e^{-s} L_{\frac{\nu(s)}{\varepsilon}}(\eta_h(s), -\dot{\eta}_h(s))\,ds + \mathbf{1}_{\{h<\tau_h\}} e^{-h} u_{\nu(h)}(\eta_h(h)) + \mathbf{1}_{\{h\geq\tau_h\}} e^{-\tau_h} g_{\nu(\tau_h)}(\eta_h(\tau_h))\Big) \\
&\geq \mathbb{E}_i\Big(\int_0^{h\wedge\tau_h} e^{-s} L_{\nu(s)}(\frac{\eta_h(s)}{\varepsilon}, -\dot{\eta}_h(s))\,ds + e^{-(h\wedge\tau_h)} u_{\nu(h\wedge\tau_h)}(\eta_h(h\wedge\tau_h))\Big)
\end{aligned}
$$

for some $\eta_h \in \mathrm{AC}\,([0,h],\overline{\Omega})$ such that $\eta(0) = x$ and $\tau_h \in [0,h]$. In view of (4.7.5), we have $\tau_h > 0$ for small $h > 0$. Therefore by a similar calculation as in the proof of Theorem 4.7.1 we get

$$
u_1(x_0) + H_1(\frac{x_0}{\varepsilon}, D\phi(x_0)) + \frac{1}{\varepsilon}(u_1-u_2)(x_0) \geq 0. \qquad \square
$$

Chapter 5

Large time behavior of viscosity solutions of weakly coupled systems of Hamilton–Jacobi equations

5.1 Introduction

In this Chapter we study the large time behavior of the viscosity solutions of the following weakly coupled systems of Hamilton–Jacobi equations

$$\text{(C)} \qquad \begin{cases} (u_1)_t + H_1(x, Du_1) + c_1(u_1 - u_2) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ (u_2)_t + H_2(x, Du_2) + c_2(u_2 - u_1) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u_1(x,0) = u_{01}(x),\ u_2(x,0) = u_{02}(x) & \text{on } \mathbb{T}^n, \end{cases}$$

where the Hamiltonians $H_i \in C(\mathbb{T}^n \times \mathbb{R}^n)$ are given functions which are assumed to be *coercive*, i.e.,

$$\text{(A1)} \qquad \lim_{r\to\infty} \inf\{H_i(x,p) \mid x \in \mathbb{T}^n, |p| \geq r\} = \infty,$$

and u_{0i} are given real-valued continuous functions on $\mathbb{T}^n$, and $c_i > 0$ are given constants for $i = 1, 2$, respectively. Here u_i are the real-valued unknown functions on $\mathbb{R}^n \times [0,T]$ and $(u_i)_t := \partial u_i/\partial t, Du_i := (\partial u_i/\partial x_1, \ldots, \partial u_i/\partial x_n)$ for $i = 1, 2$, respectively. For the sake of simplicity, we focus on the system of two equations above but we can generalize it to general systems of m equations in Cases 1, 2 below.

Although it is already established well that existence and uniqueness results for weakly coupled systems of Hamilton–Jacobi equations hold (see [57, 26, 52] and the references therein for instance), there are not many studies on properties of solutions of (C). Shimano [76] and F. Camilli, O. Ley and P. Loreti [13] investigated homogenization problems for the system and obtained the convergence result, and the second author with F. Cagnetti and D. Gomes [9] very recently considered new nonlinear adjoint methods for weakly coupled systems of stationary Hamilton–Jacobi equations and obtained the speed of convergence by using usual regularized equations. As far as the authors know, there are few works on the large time behavior of solutions of weakly coupled systems of Hamilton–Jacobi equations.

Heuristic derivations and Main goal

First we heuristically derive the large time asymptotics for (C). For simplicity, from now on, we assume that $c_1 = c_2 = 1$. We consider the formal asymptotic expansions of the solutions u_1, u_2 of (C) of the form

$$\begin{aligned} u_1(x,t) &= a_{01}(x)t + a_{11}(x) + a_{21}(x)t^{-1} + \ldots, \\ u_2(x,t) &= a_{02}(x)t + a_{12}(x) + a_{22}(x)t^{-1} + \ldots \quad \text{as} \ \ t \to \infty. \end{aligned}$$

Plugging these expansions into (C), we get

$$\begin{aligned} &a_{01}(x) - a_{21}(x)t^{-2} + \ldots + H_1(x, Da_{01}(x)t + Da_{11}(x) + Da_{21}(x)t^{-1} + \ldots) \\ &+(a_{01}(x) - a_{02}(x))t + (a_{11}(x) - a_{12}(x)) + (a_{21}(x) - a_{22}(x))t^{-1} + \ldots = 0, \end{aligned} \tag{5.1.1}$$

and

$$\begin{aligned} &a_{02}(x) - a_{22}(x)t^{-2} + \ldots + H_2(x, Da_{02}(x)t + Da_{12}(x) + Da_{22}(x)t^{-1} + \ldots) \\ &+(a_{02}(x) - a_{01}(x))t + (a_{12}(x) - a_{11}(x)) + (a_{22}(x) - a_{21}(x))t^{-1} + \ldots = 0. \end{aligned} \tag{5.1.2}$$

Adding up the two equations above, we have

$$H_1(x, Da_{01}t + Da_{11} + O(1/t)) + H_2(x, Da_{02}t + Da_{12} + O(1/t)) + O(1) = 0$$

as $t \to \infty$. Therefore by the coercivity of H_1 and H_2 we formally get $Da_{01} = Da_{02} \equiv 0$. Then sending $t \to \infty$ in (5.1.1), (5.1.2), we derive

$$a_{01}(x) = a_{02}(x) \equiv a_0 \text{ for some constant } a_0,$$

and

$$\begin{cases} H_1(x, Da_{11}(x)) + a_{11}(x) - a_{12}(x) = -a_0 & \text{in } \mathbb{T}^n, \\ H_2(x, Da_{12}(x)) + a_{12}(x) - a_{11}(x) = -a_0 & \text{in } \mathbb{T}^n. \end{cases}$$

Therefore it is natural to investigate the existence of solutions of

$$\text{(E)} \qquad \begin{cases} H_1(x, Dv_1(x)) + v_1 - v_2 = c & \text{in } \mathbb{T}^n, \\ H_2(x, Dv_2(x)) + v_2 - v_1 = c & \text{in } \mathbb{T}^n. \end{cases}$$

Here one seeks for a triplet $(v_1, v_2, c) \in C(\mathbb{T}^n)^2 \times \mathbb{R}$ such that (v_1, v_2) is a solution of (E). If (v_1, v_2, c) is such a triplet, we call (v_1, v_2) a *pair of ergodic functions* and c an *ergodic constant*. By an analogous argument to that of the classical result of [60] we can see that there exists a solution of (E). Indeed the second author with F. Cagnetti, D. Gomes [9] recently proved that there exists a unique constant c such that the ergodic problem has continuous solutions (v_1, v_2).

Hence, our goal in this Chapter is to prove the following large time asymptotics for (C) under appropriate assumptions on H_i. For any $(u_{01}, u_{02}) \in C(\mathbb{T}^n)^2$ there exists a solution $(v_1, v_2, c) \in C(\mathbb{T}^n)^2 \times \mathbb{R}$ of (E) such that if $(u_1, u_2) \in C(\mathbb{R}^n \times [0, T])^2$ is the solution of (C), then, as $t \to \infty$,

$$u_i(x, t) + ct - v_i(x) \to 0 \;\text{ uniformly on } \mathbb{T}^n \tag{5.1.3}$$

for $i = 1, 2$. We call such a pair $(v_1(x) - ct, v_2(x) - ct)$ an *asymptotic solution* of (C).

It is worthwhile to emphasize here that for homogenization problems, the associated cell problems do not have the coupling terms. See [13] for the detail. Therefore it is relatively easy to get the convergence result by using the classical perturbed test function method introduced by L. C. Evans [30]. But when we consider the large time behavior of solutions of (C), we need to consider ergodic problems (E) with coupling terms. This fact seems to make convergence problems for large time asymptotics rather difficult. We are not yet able to justify rigorously convergence (5.1.3) for general Hamiltonians H_i for $i = 1, 2$ up to now. We are able to handle only three special cases which we describe below.

On the study of the large time behavior

In the last decade, a lot of works have been devoted to the study of large time behavior of solutions of Hamilton–Jacobi equations

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0, T), \tag{5.1.4}$$

where H is assumed to be coercive and general convergence results for solutions have been established. More precisely, the convergence

$$u(x, t) - (v(x) - ct) \to 0 \quad \text{uniformly on } x \in \mathbb{T}^n \text{ as } t \to \infty$$

holds, where $(v, c) \in C(\mathbb{T}^n) \times \mathbb{R}$ is a solution of the *ergodic* problem

$$H(x, Dv(x)) = c \quad \text{in } \mathbb{T}^n. \tag{5.1.5}$$

Here the ergodic eigenvalue problem for H is a problem of finding a pair of $v \in C(\mathbb{T}^n)$ and $c \in \mathbb{R}$ such that v is a solution of (5.1.5). G. Namah and J.-M. Roquejoffre in [70] were the first to get general results on this convergence under the following additional assumptions: $H(x, p) = F(x, p) - f(x)$, where F and f satisfy $p \mapsto F(x, p)$ is convex for $x \in \mathcal{M}$,

$$F(x, p) > 0 \text{ for all } (x, p) \in \mathcal{M} \times (\mathbb{R}^n \setminus \{0\}), F(x, 0) = 0 \text{ for all } x \in \mathcal{M}, \tag{5.1.6}$$

and

$$f(x) \geq 0 \text{ for all } x \in \mathcal{M} \text{ and } \{f = 0\} \neq \emptyset, \tag{5.1.7}$$

where $\mathcal{M}$ is a smooth compact n-dimensional manifold without boundary. Then A. Fathi [36] proved the same type of convergence result by using general dynamical approach and weak KAM theory. Contrary to [70], the results of [36] use strict convexity assumptions on $H(x,\cdot)$, i.e., $D_{pp}H(x,p) \geq \alpha I$ for all $(x,p) \in \mathcal{M} \times \mathbb{R}^n$ and $\alpha > 0$ (and also far more regularity) but do not need (5.1.6), (5.1.7). Afterwards J.-M. Roquejoffre [**R**] and A. Davini and A. Siconolfi [23] have refined the approach of A. Fathi and they studied the asymptotic problem for (5.1.4) on $\mathcal{M}$ or n-dimensional torus. By another approach based on the theory of partial differential equations and viscosity solutions, this type of results has been obtained by G. Barles and P. E. Souganidis in [7]. Moreover, we also refer to the literatures [6, 50, 46, 47] for the asymptotic problems without the periodic assumptions and the periodic boundary condition and the literatures [74, 65, 67, 66, 51, 4] for the asymptotic problems which treat Hamilton–Jacobi equations under various boundary conditions including three types of boundary conditions: state constraint boundary condition, Dirichlet boundary condition and Neumann boundary condition. We remark that results in [7, 6, 4] apply to nonconvex Hamilton–Jacobi equations. We refer to the literatures [81, 44, 43] for the asymptotic problems for noncoercive Hamilton–Jacobi equations.

Main results

The first case is an analogue of the study by G. Namah, J.-M. Roquejoffre [70]. We consider Hamiltonians H_i of the forms

$$H_i(x,p) = F_i(x,p) - f_i(x),$$

where the functions $F_i : \mathbb{T}^n \times \mathbb{R}^n \to [0,\infty)$ are coercive and $f_i : \mathbb{T}^n \to [0,\infty)$ are given continuous functions for $i = 1,2$, respectively. We use the following assumptions on F_i, f_i. For $i = 1,2$

(A2) $f_i(x) \geq 0$ for all $x \in \mathbb{T}^n$;

(A3) define $\mathcal{A}_1 := \{x \in \mathbb{T}^n \mid f_1(x) = 0\}$, $\mathcal{A}_2 := \{x \in \mathbb{T}^n \mid f_2(x) = 0\}$ and then $\mathcal{A} := \mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$;

(A4) $F_i(x,\lambda p) \leq \lambda F_i(x,p)$ for all $\lambda \in (0,1]$, $x \in \mathbb{T}^n \setminus \mathcal{A}$ and $p \in \mathbb{R}^n$;

(A5) $F_i(x,p) \geq 0$ on $\mathbb{T}^n \times \mathbb{R}^n$, and $F_i(x,0) = 0$ on $\mathbb{T}^n$.

With the above special forms of the Hamiltonians, we have

Theorem 5.1.1 (Convergence Result 1)**.** *Assume that the Hamiltonians H_i are of the forms*

$$H_i(x,p) = F_i(x,p) - f_i(x)$$

and H_i, F_i, f_i satisfy assumptions (A1)–(A5), then there exists a solution $(v_1,v_2) \in C(\mathbb{T}^n)^2$ of (E) with $c = 0$ such that convergence (5.1.3) *holds.*

Notice that the *directional convexity condition* with respect to the p variable on F_i, i.e.,

(A4') for any $p \in \mathbb{R}^n \setminus \{0\}$ and $x \in \mathbb{T}^n$, $t \mapsto F_i(x,tp)$ is convex, together with $F_i(x,0) = 0$ implies (A4). It is clear to see that assumption (A4) or (A4') does not require Hamiltonians to be convex. One explicit example of Hamiltonians in Theorem 5.1.1 is

$$H_i(x,p) = F_i(x,p) - f_i(x) = \begin{cases} a_i(x)|p|^{\alpha_i}\varphi_i(\frac{p}{|p|}) - f_i(x) & \text{for } p \neq 0, \\ -f_i(x) & \text{for } p = 0 \end{cases}$$

for some $\alpha_i \geq 1$, $a_i \in C(\mathbb{T}^n)$, $\varphi_i \in C(\mathbb{S}^{n-1})$ with $a_i, \varphi_i > 0$ and f_i satisfying (A2)–(A3) for $i = 1, 2$, where $\mathbb{S}^{n-1}$ denotes the $(n-1)$-dimensional unit sphere.

After this work has been completed, we learned of the interesting recent work of F. Camilli, O. Ley, P. Loreti and V. Nguyen [15], which announces results very similar to Theorem 5.1.1. Their result is somewhat more general along this direction. In fact they consider systems of m-equations which have coupling terms with variable coefficients instead of constant coefficients. Also, the control-theoretic interpretation of (C) is derived there.

In the second case, we consider the case where the Hamiltonians are independent of the x variable, i.e., $H_i(x,p) = H_i(p)$ for $i = 1, 2$. We assume that the Hamiltonians satisfy

(A6) H_i are uniformly convex, i.e.,

$$H_i(p) \geq H_i(q) + DH_i(q) \cdot (p-q) + \alpha|p-q|^2$$

for some $\alpha > 0$ and almost every $p, q \in \mathbb{R}^n$,

(A7) $H_i(0) = 0$

for $i = 1, 2$. Our main result is

Theorem 5.1.2 (Convergence Result 2)**.** *Assume that $H_i(x,p) = H_i(p)$ for $i = 1, 2$ and H_i satisfy assumptions (A1), (A6) and (A7), then there exists a constant M such that*

$$u_i(x,t) - M \to 0 \quad \textit{uniformly on } \mathbb{T}^n \textit{ for } i = 1, 2$$

as $t \to \infty$.

One explicit example of Hamiltonians in Theorem 5.1.2 is

$$H_i(p) = |p - b_i|^2 - |b_i|^2$$

for some constant vectors $b_i \in \mathbb{R}^n$ for $i = 1, 2$. Notice that the above Hamiltonians in general do not satisfy the conditions in the first case, particularly (A5). The idea for the proof of Theorem 5.1.2 can be applied to study more general forms of Hamiltonians, e.g.,

$$H_i(x,p) = |p - \mathbf{b}_i(x)|^2 - |\mathbf{b}_i(x)|^2$$

for $\mathbf{b}_i \in C^1(\mathbb{T}^n)$ with $\operatorname{div} \mathbf{b}_i = 0$ on $\mathbb{T}^n$ for $i = 1, 2$ as will be noted in Remark 5.4.4.

In the third case, we generalize the result of G. Barles, P. E. Souganidis [7] for single equations to systems. We consider the case where the two Hamiltonians H_1, H_2 are same, i.e., $H := H_1 = H_2$. We normalize the ergodic constant c to be 0 by replacing H by $H - c$ and then we assume that H satisfies

(A8) either of the following assumption $(A8)^+$ or $(A8)^-$ holds:

$(A8)^+$ there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p+q) \geq \eta$ and $H(x, q) \leq 0$ for some $x \in \mathbb{T}^n$ and $p, q \in \mathbb{R}^n$, then for any $\mu \in (0, 1]$,

$$\mu H(x, \frac{p}{\mu} + q) \geq H(x, p+q) + \psi_\eta(1-\mu),$$

$(A8)^-$ there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p+q) \leq -\eta$ and $H(x, q) \geq 0$ for some $x \in \mathbb{T}^n$ and $p, q \in \mathbb{R}^n$, then for any $\mu \geq 1$,

$$\mu H(x, \frac{p}{\mu} + q) \leq H(x, p+q) - \frac{\psi_\eta(\mu-1)}{\mu}.$$

Assumption $(A8)^+$ was first introduced in [7] to replace the convexity assumption, and it mainly concerns the set $\{H \geq 0\}$ and the behavior of H in this set. Assumption $(A8)^-$ is a modified version of $(A8)^+$ which was introduced in [4] and on the contrary, it concerns the set $\{H \leq 0\}$. We can generalize them as in [7] but to simplify our arguments we only use the simplified version. See the end of Section 5.

Our third main result is

Theorem 5.1.3 (Convergence Result 3)**.** *If we assume that $H = H_1 = H_2$ and H satisfies (A1), (A8) and the ergodic constant c is equal to* 0*, then there exist a solution* $(v, v) \in C(\mathbb{T}^n)^2$ *of (E) with $c = 0$ such that convergence* (5.1.3) *holds.*

We notice that if H is smooth with respect to the p-variable, then (A8) is equivalent to a *one-sided directionally strict convexity* in a neighborhood of $\{p \in \mathbb{R}^n \mid H(x, p) = 0\}$ for all $x \in \mathbb{T}^n$, i.e.,

(A8') either of the following assumption $(A8')^+$ or $(A8')^-$ holds:

$(A8')^+$ there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p+q) \geq \eta$ and $H(x, q) \leq 0$ for some $x \in \mathbb{T}^n$ and $p, q \in \mathbb{R}^n$, then

$$D_pH(x, p+q) \cdot p - H(x, p+q) \geq \psi_\eta,$$

$(A8')^-$ there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0]$, there exists $\psi_\eta > 0$ such that if $H(x, p+q) \leq -\eta$ and $H(x, q) \geq 0$ for some $x \in \mathbb{T}^n$ and $p, q \in \mathbb{R}^n$, then

$$D_pH(x, p+q) \cdot p - H(x, p+q) \geq \psi_\eta.$$

We refer the readers to [7] for interesting examples of Hamiltonians in Theorem 5.1.3. Our conclusions in Cases 2, 3 seem to go beyond the recent work [15].

This Chapter is organized as follows: in Section 2 we give some preliminary results. Section 3, Section 4, and Section 5 are respectively devoted to the proofs of Theorems 5.1.1–5.1.3. In Section 5.6 we present the proof of the result on ergodic problems.

Notations. For $A \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $C(A)$, LSC$\,(A)$, USC$\,(A)$ and $C^k(A)$ the space of real-valued continuous, lower semicontinuous, upper semicontinuous and k-th continuous differentiable functions on A, respectively. We denote by $W^{1,\infty}(A)$ the set of bounded functions whose first weak derivatives are essentially bounded. We call a function $m : [0,\infty) \to [0,\infty)$ a modulus if it is continuous and nondecreasing on $[0,\infty)$ and vanishes at the origin.

5.2 Preliminaries

In this section we assume only (A1).

Proposition 5.2.1 (Ergodic Problem (E) (e.g., [9, Theorem 4.2]))**.** *There exists a solution* $(v_1, v_2, \overline{H}_1, \overline{H}_2) \in W^{1,\infty}(\mathbb{T}^n)^2 \times \mathbb{R}^2$ *of*

$$\begin{cases} H_1(x, Dv_1) + c_1(v_1 - v_2) = \overline{H}_1 \ \text{ in } \mathbb{T}^n, \\ H_2(x, Dv_2) + c_2(v_2 - v_1) = \overline{H}_2 \ \text{ in } \mathbb{T}^n. \end{cases} \tag{5.2.1}$$

Furthermore, $c_2\overline{H}_1 + c_1\overline{H}_2$ *is unique.*

We note that solutions v_1, v_2 of (5.2.1) are not unique in general even up to constants. Also it is easy to see that $\overline{H}_1, \overline{H}_2$ are not unique as well. Take $v_1' = v_1 + C_1, v_2' = v_2 + C_2$ for some constants C_1, C_2 then

$$\overline{H}_1' = \overline{H}_1 + c_1(C_1 - C_2),\ \overline{H}_2' = \overline{H}_2 + c_2(C_2 - C_1),$$

which shows that $\overline{H}_i$ can individually take any real value. But remarkably, we have

$$c_2\overline{H}_1 + c_1\overline{H}_2 = c_2\overline{H}_1' + c_1\overline{H}_2',$$

which is a unique constant. We can get the existence result by an argument similar to a classical result in [60] (see also the proof of Proposition 5.3.1 below). We give the sketch of the proof for the uniqueness of $c_2\overline{H}_1 + c_1\overline{H}_2$ in Section 5.6 for the reader's convenience.

We assume henceforth for simplicity that $c_1 = c_2 = 1$. Then the ergodic constant c is unique and is given by

$$c = \frac{\overline{H}_1 + \overline{H}_2}{2}.$$

The comparison principle for (C) is a classical result. See [57, 26, 52], [13, Proposition 3.1] for instance.

Proposition 5.2.2 (Comparison Principle for (C))**.** *Let* $(u_1, u_2) \in USC(\mathbb{R}^n \times [0,T])^2$, $(v_1, v_2) \in LSC(\mathbb{R}^n \times [0,T])^2$ *be a subsolution and a supersolution of (C), respectively. If* $u_i(\cdot, 0) \le v_i(\cdot, 0)$ *on* $\mathbb{T}^n$*, then* $u_i \le v_i$ *on* $\mathbb{R}^n \times [0,T]$ *for* $i = 1, 2$.

The following proposition is a straightforward application of Propositions 5.2.1, 5.2.2.

Proposition 5.2.3 (Boundedness of Solutions of (C))**.** *Let (u_1, u_2) be the solution of (C) and let c be the ergodic constant for (E). Then we have $|u_i(x,t) + ct| \leq C$ on $\mathbb{R}^n \times [0,T]$ for some $C > 0$ for $i = 1, 2$.*

In view of the coercivity assumption on H_i for $i = 1, 2$, we have the following Lipschitz regularity results.

Proposition 5.2.4 (Lipschitz Regularity of Solutions of (C))**.** *If $u_{0i} \in W^{1,\infty}(\mathbb{T}^n)$ for $i = 1, 2$, then $(u_1 + ct, u_2 + ct)$ is in $W^{1,\infty}(\mathbb{R}^n \times [0,T])^2$, where (u_1, u_2) is the solution of (C) and c is the ergodic constant.*

Proposition 5.2.5 (Lipschitz Regularity of Solutions of (E))**.** *Let $(v_1, v_2) \in USC(\mathbb{T}^n)^2$ be a subsolution of (E). Then $(v_1, v_2) \in W^{1,\infty}(\mathbb{T}^n)^2$.*

We *assume* henceforth that $u_{0i} \in W^{1,\infty}(\mathbb{T}^n)$ for $i = 1, 2$ in order to avoid technicalities but they are not necessary. We can easily remove these additional requirements on u_{0i}. See Remark 5.3.5 for details.

5.3 First Case

In this section we consider the case where Hamiltonians have the forms $H_i(x,p) = F_i(x,p) - f_i(x)$, and H_i, F_i, f_i satisfy assumptions (A1)–(A5). System (C) becomes

$$\text{(C1)} \quad \begin{cases} (u_1)_t + F_1(x, Du_1) + u_1 - u_2 = f_1(x) & \text{in } \mathbb{R}^n \times (0,T), \\ (u_2)_t + F_2(x, Du_2) + u_2 - u_1 = f_2(x) & \text{in } \mathbb{R}^n \times (0,T), \\ u_1(x,0) = u_{01}(x),\ u_2(x,0) = u_{02}(x) & \text{on } \mathbb{T}^n. \end{cases}$$

In order to prove Theorem 5.1.1, we need several following steps.

Stationary Problems

Proposition 5.3.1. *The ergodic constant c in (E) is equal to 0.*

Proof. For $\varepsilon > 0$ let us consider a usual approximate monotone system

$$\begin{cases} F_1(x, Dv_1^\varepsilon(x)) + (1+\varepsilon)v_1^\varepsilon - v_2^\varepsilon = f_1(x) & \text{in } \mathbb{T}^n, \\ F_2(x, Dv_2^\varepsilon(x)) + (1+\varepsilon)v_2^\varepsilon - v_1^\varepsilon = f_2(x) & \text{in } \mathbb{T}^n. \end{cases} \tag{5.3.1}$$

It is easy to see that $(0,0)$, $(C_1/\varepsilon, C_1/\varepsilon)$ are a subsolution and a supersolution of the above for $C_1 > 0$ large enough. By Perron's method and the comparison theorem for the monotone system, we have a unique solution $(v_1^\varepsilon, v_2^\varepsilon) \in C(\mathbb{T}^n)^2$ of (5.3.1). By the way of construction we have

$$0 \leq \varepsilon v_i^\varepsilon \leq C_1 \text{ on } \mathbb{T}^n \tag{5.3.2}$$

for $i = 1, 2$. Summing up both equations in (5.3.1), we have

$$F_1(x, Dv_1^\varepsilon) + F_2(x, Dv_2^\varepsilon) = -\varepsilon(v_1^\varepsilon + v_2^\varepsilon) + f_1(x) + f_2(x) \le C_2$$

for some $C_2 > 0$. By the coercivity of F_i we obtain

$$\|Dv_i^\varepsilon\|_{L^\infty(\mathbb{T}^n)} \le C_2$$

for $i = 1, 2$ by replacing C_2 by a larger constant if necessary. Therefore we see that $\{v_i^\varepsilon\}_{\varepsilon\in(0,1)}$ are equi-Lipschitz continuous.

We claim that there exists a constant $C_3 > 0$

$$|v_1^\varepsilon(x) - v_2^\varepsilon(y)| \le C_3 \text{ for all } x, y \in \mathbb{T}^n. \tag{5.3.3}$$

Indeed setting $m_i^\varepsilon := \max_{\mathbb{T}^n} v_i^\varepsilon = v_i^\varepsilon(z_i)$ for some $z_i \in \mathbb{T}^n$ for $i = 1, 2$. Take 0 as a test function in the first equation of (5.3.1) to derive

$$F_1(z_1, 0) + (1 + \varepsilon)v_1^\varepsilon(z_1) - v_2^\varepsilon(z_1) \le f_1(z_1),$$

which implies

$$v_1^\varepsilon(z_1) - v_2^\varepsilon(z_1) \le -F_1(z_1, 0) - \varepsilon v_1^\varepsilon(z_1) + f_1(z_1) \le C_3$$

for some $C_3 > 0$. Thus,

$$\begin{aligned} v_1^\varepsilon(x) - v_2^\varepsilon(y) &\le v_1^\varepsilon(z_1) - v_2^\varepsilon(y) \\ &= v_1^\varepsilon(z_1) - v_2^\varepsilon(z_1) + v_2^\varepsilon(z_1) - v_2^\varepsilon(y) \le C_3 \end{aligned}$$

by replacing C_3 by a larger constant if necessary. This implies (5.3.3). In particular, $|m_1^\varepsilon - m_2^\varepsilon| \le C_3$.

Let $w_i^\varepsilon(x) := v_i^\varepsilon(x) - m_i^\varepsilon$. Because of (5.3.2), $\{w_i^\varepsilon\}_{\varepsilon\in(0,1)}$ is a sequence of equi-Lipschitz continuous and uniformly bounded functions on $\mathbb{T}^n$. Moreover they satisfy

$$\begin{cases} F_1(x, Dw_1^\varepsilon(x)) + (1 + \varepsilon)w_1^\varepsilon - w_2^\varepsilon = f_1(x) - (1 + \varepsilon)m_1^\varepsilon + m_2^\varepsilon & \text{in } \mathbb{T}^n, \\ F_2(x, Dw_2^\varepsilon(x)) + (1 + \varepsilon)w_2^\varepsilon - w_1^\varepsilon = f_2(x) - (1 + \varepsilon)m_2^\varepsilon + m_1^\varepsilon & \text{in } \mathbb{T}^n \end{cases}$$

in the viscosity solution sense. By Ascoli-Arzela's theorem, there exists a sequence $\varepsilon_j \to 0$ so that

$$\begin{gathered} w_i^{\varepsilon_j} \to w_i, \\ -(1 + \varepsilon_j)m_1^{\varepsilon_j} + m_2^{\varepsilon_j} \to \overline{H}_1 \text{ and } -(1 + \varepsilon_j)m_2^{\varepsilon_j} + m_1^{\varepsilon_j} \to \overline{H}_2 \end{gathered}$$

uniformly on $\mathbb{T}^n$ as $j \to \infty$ for some $(w_1, w_2) \in W^{1,\infty}(\mathbb{T}^n)^2$ and $(\overline{H}_1, \overline{H}_2) \in \mathbb{R}^2$. By a standard stability result of viscosity solutions we see that $(w_1, w_2, \overline{H}_1, \overline{H}_2)$ is a solution of (5.2.1).

We now prove that $c := (\overline{H}_1 + \overline{H}_2)/2 = 0$. Noting that $m_i^{\varepsilon_j} \ge 0$ and

$$\frac{1}{2}\{(-(1+\varepsilon_j)m_1^{\varepsilon_j} + m_2^{\varepsilon_j}) + (-(1+\varepsilon_j)m_2^{\varepsilon_j} + m_1^{\varepsilon_j})\} = -\frac{1}{2}\varepsilon_j(m_1^{\varepsilon_j} + m_2^{\varepsilon_j}) \to c$$

as $j \to \infty$, we see that $c \le 0$. Furthermore, summing up the two equations in (5.2.1), we obtain

$$2c = \overline{H}_1 + \overline{H}_2 = F_1(x, Dw_1) + F_2(x, Dw_2) - f_1(x) - f_2(x) \ge -f_1(x) - f_2(x)$$

for almost every $x \in \mathbb{T}^n$. Since $\mathcal{A} \ne \emptyset$, we see that $c \ge 0$. Together with the above observation we get the conclusion. □

Theorem 5.3.2 (Comparison Principle for Stationary Problems)**.** *Let $(u_1, u_2) \in USC(\mathbb{T}^n)^2$, $(v_1, v_2) \in LSC(\mathbb{T}^n)^2$ be, respectively, a subsolution and a supersolution of*

$$\text{(S1)} \qquad \begin{cases} F_1(x, Dv_1(x)) + v_1 - v_2 = f_1(x) & \text{in } \mathbb{T}^n, \\ F_2(x, Dv_2(x)) + v_2 - v_1 = f_2(x) & \text{in } \mathbb{T}^n. \end{cases}$$

If $u_i \le v_i$ on $\mathcal{A}$, then $u_i \le v_i$ on $\mathbb{T}^n$ for $i = 1, 2$.

The idea of the proof below basically comes from the combination of those in [49] and [57, 26, 52]. It is worthwhile to mention that the set $\mathcal{A}$ plays the role of the boundary as in [39, 53]. See also [22] and [15, Theorem 3.3] for weakly coupled systems of Hamilton–Jacobi equations.

Proof. Fix any $\delta > 0$. We may choose an open neighborhood V of $\mathcal{A}$ and $\overline{\lambda} \in (0, 1)$ so that $\lambda u_i \le v_i + \delta$ on V for $\lambda \in [\overline{\lambda}, 1]$ and $i = 1, 2$. It is enough to show that $\lambda u_i \le v_i + \delta$ on $\mathbb{T}^n \setminus V$ for $\lambda \in [\overline{\lambda}, 1]$. Fix $\lambda \in [\overline{\lambda}, 1]$ and we set $u_i^\lambda := \lambda u_i$ and $v_i^\delta := v_i + \delta$. We prove the above statement by a contradiction argument. Suppose that $M := \max_{i=1,2, x \in \mathbb{T}^n \setminus V} (u_i^\lambda - v_i^\delta)(x) > 0$.

We take $i_0 \in \{1, 2\}$, $\xi \in \mathbb{T}^n \setminus V$ such that $M = (u_{i_0}^\lambda - v_{i_0}^\delta)(\xi)$. We may assume that $i_0 = 1$ by symmetry. We first consider the case where

$$M_\lambda = (u_1^\lambda - v_1^\delta)(\xi) = (u_2^\lambda - v_2^\delta)(\xi). \tag{5.3.4}$$

We define the function $\Psi : \mathbb{T}^{2n} \to \mathbb{R}$ by

$$\Psi(x, y) := u_1^\lambda(x) - v_1^\delta(y) - \frac{|x-y|^2}{2\varepsilon^2} - \frac{|x-\xi|^2}{2}.$$

Let Ψ achieve its maximum at some point $(x_\varepsilon, y_\varepsilon) \in \mathbb{T}^{2n}$. By the definition of viscosity solutions we have

$$\begin{aligned} &F_1(x_\varepsilon, \frac{1}{\lambda}(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} + x_\varepsilon - \xi)) + (u_1 - u_2)(x_\varepsilon) \le f_1(x_\varepsilon), \\ &F_1(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}) + (v_1 - v_2)(y_\varepsilon) \ge f_1(y_\varepsilon). \end{aligned}$$

By the usual argument we may assume that

$$x_\varepsilon, y_\varepsilon \to \xi,\ \frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} \to p \in \mathbb{R}^n$$

as $\varepsilon \to 0$ by taking a subsequence if necessary in view of the Lipschitz continuity of solutions. Therefore sending ε to 0 yields

$$F_1(\xi, \frac{p}{\lambda}) + (u_1 - u_2)(\xi) \le f_1(\xi), \tag{5.3.5}$$

$$F_1(\xi, p) + (v_1 - v_2)(\xi) \ge f_1(\xi). \tag{5.3.6}$$

In view of (A4), (5.3.5) transforms to read

$$F_1(\xi, p) + (u_1^\lambda - u_2^\lambda)(\xi) \le \lambda f_1(\xi) \text{ for all } \lambda \in [\overline{\lambda}, 1]. \tag{5.3.7}$$

Note that $(v_1 - v_2)(\xi) = (v_1^\delta - v_2^\delta)(\xi)$. By (5.3.4), (5.3.6) and (5.3.7) we get $f_1(\xi) \le \lambda f_1(\xi)$. Similarly, $f_2(\xi) \le \lambda f_2(\xi)$. Hence $f_1(\xi) + f_2(\xi) \le \lambda(f_1(\xi) + f_2(\xi))$ which is a contradiction since $f_1(\xi) + f_2(\xi) > 0$ and $\lambda \in (0, 1)$.

We next consider the case where

$$(u_1^\lambda - v_1^\delta)(\xi) \ne (u_2^\lambda - v_2^\delta)(\xi).$$

Then there exists $a > 0$ such that $(u_1^\lambda - v_1^\delta)(\xi) \ge (u_2^\lambda - v_2^\delta)(\xi) + a$ and therefore by (5.3.6), (5.3.7) we obtain

$$0 \ge (\lambda - 1) f_1(\xi) \ge (u_1^\lambda - v_1^\delta)(\xi) - (u_2^\lambda - v_2^\delta)(\xi) \ge a,$$

which is a contradiction. This finishes the proof. □

Convergence

Proposition 5.3.3 (Monotonicity Property 1)**.** *Set $U(x,t) := u_1(x,t) + u_2(x,t)$. Then the function $t \mapsto U(x,t)$ is nonincreasing for all $x \in \mathcal{A}$.*

Proof. It is easy to see that U satisfies $U_t \le 0$ on $\mathcal{A}$ in the viscosity sense and we get the conclusion. □

Proposition 5.3.4 (Monotonicity Property 2)**.** *Set*

$$V(x,t) := \max\{u_1(x,t), u_2(x,t)\} = \frac{1}{2}\big\{(u_1 + u_2)(x,t) + |(u_1 - u_2)(x,t)|\big\}.$$

Then the function $t \mapsto V(x,t)$ is nonincreasing for all $x \in \mathcal{A}$.

We notice that the result of Proposition 5.3.4 is included by the recent result of [15, Remark 5.7, (3)] but our proof seems to be more direct.

Proof. Fix $x \in \mathcal{A}$. For $\varepsilon, \delta > 0$ we set $K_\varepsilon(x) := x + [-\varepsilon, \varepsilon]^n$ and

$$V_\delta(x,t) := \frac{1}{2}\big((u_1+u_2)(x,t) + \langle (u_1-u_2)(x,t)\rangle_\delta\big),$$

where $\langle p\rangle_\delta := \sqrt{|p|^2+\delta^2}$. We note that V_δ converges uniformly to V as $\delta \to 0$.

We have for all $t, h \geq 0$

$$\int_{K_\varepsilon(x)} V_\delta(y,t+h) - V_\delta(y,t)\,dy = \int_{K_\varepsilon(x)\times[t,t+h]} (V_\delta)_t(y,s)\,dy\,ds.$$

Let (y,s) be a point at which u_1, u_2 are differentiable. We calculate that

$$\begin{aligned}
&(V_\delta)_t(y,s)\\
=&\frac{1}{2}\Big\{(u_1)_t + (u_2)_t + \frac{u_1-u_2}{\langle u_1-u_2\rangle_\delta}((u_1)_t-(u_2)_t)\Big\}\\
=&\frac{1}{2}\Big\{f_1+f_2+\frac{u_1-u_2}{\langle u_1-u_2\rangle_\delta}(f_1-f_2)\Big\}\\
&+\frac{1}{2}\Big\{-F_1-F_2+\frac{u_1-u_2}{\langle u_1-u_2\rangle_\delta}(F_2-F_1)\Big\} - \frac{1}{\langle u_1-u_2\rangle_\delta}(u_1-u_2)^2\\
\leq&\frac{1}{2}\Big\{f_1+f_2+\frac{u_1-u_2}{\langle u_1-u_2\rangle_\delta}(f_1-f_2)\Big\} + \frac{1}{2}\Big\{-F_1-F_2+\frac{u_1-u_2}{\langle u_1-u_2\rangle_\delta}(F_2-F_1)\Big\}.
\end{aligned}$$

In view of (A5) and (A3) sending $\delta \to 0$ yields

$$\begin{aligned}
&\int_{K_\varepsilon(x)} V(y,t+h)-V(y,t)\,dy\\
\leq&\int_{K_\varepsilon(x)\times[t,t+h]} \frac{1}{2}\Big\{f_1+f_2+\operatorname{sgn}(u_1-u_2)(f_1-f_2)\Big\}\\
&\quad+\frac{1}{2}\Big\{-F_1-F_2+\operatorname{sgn}(u_1-u_2)(F_2-F_1)\Big\}\,dyds\\
\leq&\int_{K_\varepsilon(x)\times[t,t+h]} \frac{1}{2}\Big\{f_1+f_2+\operatorname{sgn}(u_1-u_2)(f_1-f_2)\Big\}\,dyds\\
\leq&\int_{K_\varepsilon(x)\times[t,t+h]} \omega_{f_1}(|x-y|)+\omega_{f_2}(|x-y|)\,dyds\\
\leq&\,\varepsilon^n h(\omega_{f_1}(\sqrt{n}\varepsilon)+\omega_{f_2}(\sqrt{n}\varepsilon)),
\end{aligned}$$

where ω_{f_i} are the moduli of continuity of f_i for $i = 1, 2$. By dividing by ε^n and sending $\varepsilon \to 0$ we get the conclusion. □

Proof of Theorem 5.1.1. For any $x \in \mathcal{A}$ by Propositions 5.3.3, 5.3.4 we see that $(u_1 + u_2)(x,t) \to \alpha(x)$ and $|(u_1-u_2)(x,t)| \to \beta(x)$ as $t \to \infty$. If $\beta(x) > 0$, then $(u_1-u_2)(x,t)$ converges as $t \to \infty$ since $t \mapsto (u_1-u_2)(x,t)$ is continuous. The limit may be either $\beta(x)$ or $-\beta(x)$. Therefore $u_1(x,t), u_2(x,t)$ converge as $t \to \infty$. If $\beta(x) = 0$, then we have

$$(u_1+u_2)(x,t) - |(u_1-u_2)(x,t)| \leq 2u_1(x,t) \leq (u_1+u_2)(x,t) + |(u_1-u_2)(x,t)|,$$

which implies $u_1(x,t)$ and $u_2(x,t)$ converge to $(1/2)\alpha(x)$ as $t\to\infty$. Consequently, we see that $u_1(x,t)$, $u_2(x,t)$ converge for all $x\in\mathcal{A}$ as $t\to\infty$.

Now, let us define the following half-relaxed semilimits

$$\overline{u}_i(x)=\limsup_{t\to\infty}{}^*[u_i](x,t) \text{ and } \underline{u}_i(x)=\liminf_{t\to\infty}{}_*[u_i](x,t)$$

for $x\in\mathbb{T}^n$ and $i=1,2$. By standard stability results of the theory of viscosity solutions, $(\overline{u}_1,\overline{u}_2)$, $(\underline{u}_1,\underline{u}_2)$ are a subsolution and a supersolution of (E), respectively. Moreover, $(\overline{u}_1,\overline{u}_2)=(\underline{u}_1,\underline{u}_2)$ on $\mathcal{A}$, since u_1,u_2 converge on $\mathcal{A}$ as $t\to\infty$. By the comparison principle, Theorem 5.3.2, we obtain $(\overline{u}_1,\overline{u}_2)=(\underline{u}_1,\underline{u}_2)$ in $\mathbb{T}^n$ and the proof is complete. □

Remark 5.3.5. (i) The Lipschitz regularity assumption on u_{0i} for $i=1,2$ is convenient to avoid technicalities but it is not necessary. We can remove it as follows. For each i, we may choose a sequence $\{u_{0i}^k\}_{k\in\mathbb{N}}\subset W^{1,\infty}(\mathbb{T}^n)$ so that $\|u_{0i}^k-u_{0i}\|_{L^\infty(\mathbb{T}^n)}\le 1/k$ for all $k\in\mathbb{N}$. By the maximum principle, we have

$$\|u_i-u_i^k\|_{L^\infty(\mathbb{R}^n\times(0,T))}\le\|u_{0i}-u_{0i}^k\|_{L^\infty(\mathbb{T}^n)}\le 1/k,$$

and therefore

$$u_i^k(x,t)-1/k\le u_i(x,t)\le u_i^k(x,t)+1/k \text{ for all } (x,t)\in\mathbb{R}^n\times[0,T],$$

where (u_1,u_2) is the solution of (C) and (u_1^k,u_2^k) are the solutions of (C) with $u_{0i}=u_{0i}^k$ for $i=1,2$. Therefore we have

$$u_{\infty i}^k(x)-1/k\le \operatorname{liminf}_{*t\to\infty}u_i(x,t)\le\operatorname{limsup}^*_{t\to\infty}u_i(x,t)\le u_{\infty i}^k(x)+1/k$$

for all $x\in\mathbb{T}^n$, where $u_{\infty i}^k(x):=\lim_{t\to\infty}u_i^k(x,t)$. This implies that

$$\operatorname{liminf}_{*t\to\infty}u_i(x,t)=\operatorname{limsup}^*_{t\to\infty}u_i(x,t)$$

for all $x\in\mathbb{T}^n$ and $i=1,2$.
(ii) Notice that if $\mathcal{A}=\emptyset$ then the comparison principle for (S1) holds, i.e., for any subsolution (v_1,v_2) and any supersolution (w_1,w_2) we have $v_i\le w_i$ on $\mathbb{T}^n$ for $i=1,2$ (e.g., [22, Theorem 3.3]). This fact implies that the ergodic constant c is negative (not 0!). Indeed, by the argument same as in the proof of Proposition 5.3.1 we easily see that $c\le 0$. Suppose that $c=0$ and then the comparison principle implies that (E) has a unique solution (v_1,v_2). However, that is obviously not correct since for any solution (v_1,v_2) of (E), (v_1+C,v_2+C) is also a solution for any constant C. In this case we do not know whether convergence (5.1.3) holds or not.

Systems of m-equations

This section was added after we had received the draft [15] in order for the readers to see the different ideas used in our work and [15].

In this subsection we consider weakly coupled systems of m-equations for $m \geq 2$

$$(u_i)_t + F_i(x, Du_i) + \sum_{j=1}^{m} c_{ij}u_j = f_i \text{ in } \mathbb{R}^n \times (0,T) \text{ for } i = 1, \ldots, m,$$

where F_i satisfy (A1), (A5) and the convexity with respect to the p-variable,

$$c_{ii} \geq 0,\ c_{ij} \leq 0 \text{ if } i \neq j \text{ and } \sum_{i=1}^{m} c_{ij} = \sum_{j=1}^{m} c_{ij} = 0 \tag{5.3.8}$$

for $i, j \in \{1, \ldots, m\}$ and f_i satisfy (A2) and

$$\mathcal{A} := \bigcap_{i=1}^{m} \{x \in \mathbb{T}^n \mid f_i(x) = 0\} \neq \emptyset$$

then the result of Theorem 5.1.1 still holds. In [15] the authors first found the importance of irreducibility of coupling term. Although it is not essential in our argument, we also somehow use it below. Let us first assume for simplicity that the coefficient matrix (c_{ij}) is irreducible, i.e.,

(M) For any $I \subsetneq \{1, \ldots, m\}$, there exist $i \in I$ and $j \in \{1, \ldots, m\} \setminus I$ such that $c_{ij} \neq 0$.

Condition (M) will be removed in Remark 5.3.6 at the end of this subsection.

We just give a sketch of the formal proof for the convergence. By a standard regularization argument we can prove it rigorously in the viscosity solution sense.

We only need to prove the convergence of u_i on $\mathcal{A}$ for each $i \in \{1, \ldots, m\}$, since we have an analogous comparison principle to Theorem 5.3.2 when (M) holds. For $(x,t) \in \mathbb{R}^n \times [0,T]$, we can choose $\{i_{x,t}\}_{i=1}^{m}$ such that $\{1_{x,t}, \ldots, m_{x,t}\} = \{1, \ldots, m\}$ and

$$u_{1_{x,t}}(x,t) \geq u_{2_{x,t}}(x,t) \geq \ldots \geq u_{m_{x,t}}(x,t)$$

and set $v_i(x,t) := u_{i_{x,t}}(x,t)$.

Fix $(x_0, t_0) \in \mathcal{A} \times (0, \infty)$ and we may assume without loss of generality that

$$1_{x_0,t_0} = 1 \text{ and } 2_{x_0,t_0} = 2.$$

Noting that $c_{1j} \leq 0$, $u_1 \geq u_j$ for all $j = 2, \ldots, m$, and $F_1 \geq 0$, we have

$$(v_1)_t = (u_1)_t \leq (u_1)_t + \sum_{j=1}^{m} c_{1j}u_1 \leq (u_1)_t + F_1(x_0, Du_1) + \sum_{j=1}^{m} c_{1j}u_j = 0$$

at the point (x_0, t_0), which implies that $v_1(x_0, \cdot)$ is nonincreasing for $x_0 \in \mathcal{A}$ and therefore $v_1(x_0, \cdot)$ converges as $t \to \infty$.

Noting that $u_2 \geq u_j$ and $c_{ij} \leq 0$ for all $i = 1, 2$, $j = 3, \ldots, m$, $\sum_{j=1}^{m} c_{2j} = 0$, and $F_i \geq 0$, we have

$$\begin{aligned}
(v_1 + v_2)_t = (u_1 + u_2)_t &\leq (u_1 + u_2)_t + \sum_{i=1}^{2} \sum_{j=3}^{m} c_{ij}(u_j - u_2) \\
&= (u_1)_t + (u_2)_t + (c_{11} + c_{12} + c_{21} + c_{22})u_2 + \sum_{i=1}^{2} \sum_{j=3}^{m} c_{ij} u_j \\
&\leq (u_1)_t + (u_2)_t + (c_{11} + c_{21})u_1 + (c_{12} + c_{22})u_2 + \sum_{i=1}^{2} \sum_{j=3}^{m} c_{ij} u_j \\
&\leq (u_1)_t + (u_2)_t + F_1(x_0, Du_1) + F_2(x_0, Du_2) + \sum_{i=1}^{2} \sum_{j=1}^{m} c_{ij} u_j = 0
\end{aligned}$$

at the point (x_0, t_0). Thus,

$$(v_1 + v_2)_t(x_0, t_0) \leq 0.$$

Therefore $(v_1 + v_2)(x_0, \cdot)$ is nonincreasing for $x_0 \in \mathcal{A}$. Since we have already known that $v_1(x_0, \cdot)$ converges, we see that $v_2(x_0, \cdot)$ converges as $t \to \infty$.

By the induction argument, we can prove that $(v_1 + \ldots + v_k)(x_0, \cdot)$ is nonincreasing for all $x_0 \in \mathcal{A}$ and $k \in \{1, \ldots, m\}$, which is a geralization of Proposition 5.3.4. Thus, we see that

$$v_i(x_0, t) \to w_i(x_0) \text{ as } t \to \infty \text{ for } i \in \{1, \ldots, m\},$$

which concludes that each $u_i(x_0, t)$ converges as $t \to \infty$ for $x_0 \in \mathcal{A}$.

Remark 5.3.6. (i) In general, condition (M) can be removed as follows. By possible row and column permutations, $\mathcal{C} := (c_{ij})$ can be written in the block triangular form

$$\mathcal{C} = (\mathcal{C}_{pq})_{p,q=1}^{l}$$

where $\mathcal{C}_{pq}$ are $s_p \times s_q$ matrices for $p, q \in \{1, \ldots, l\}$, $\sum_{k=1}^{l} s_k = m$, $\mathcal{C}_{kk}$ are irreducible for $k \in \{1, \ldots, l\}$ and $\mathcal{C}_{pq} = 0$ for $p > q$ as in [8]. By (5.3.8), we can easily see that $\mathcal{C}_{pq} = 0$ for $p < q$ as well. Therefore the convergence result above can be applied to each irreducible matrix $\mathcal{C}_{kk}$ to yield the result.
(ii) Our approach in this general case is slightly different from the one in [15]. The convergence of each $u_i(x, t)$ as $t \to \infty$ for $i \in \{1, \ldots, m\}$, for $x \in \mathcal{A}$ plays the key role here, while Lemma 5.6 plays the key role in [15]. See Lemma 5.6 in [15] for more details.

5.4 Second case

In this section we study the case where Hamiltonians are independent of the x-variable and then (C) reduces to

$$\text{(C2)}\qquad \begin{cases} (u_1)_t + H_1(Du_1) + u_1 - u_2 = 0 & \text{in } \mathbb{R}^n \times (0,T), \quad (5.4.1)\\ (u_2)_t + H_2(Du_2) + u_2 - u_1 = 0 & \text{in } \mathbb{R}^n \times (0,T), \quad (5.4.2)\\ u_1(x,0) = u_{01}(x),\ u_2(x,0) = u_{02}(x) & \text{on } \mathbb{T}^n. \end{cases}$$

Proposition 5.4.1. *The ergodic constant c is equal to* 0*, and problem (E) has only constant Lipschitz subsolutions* (a,a) *for* $a \in \mathbb{R}$*.*

Proof. Since we can easily see that the ergodic constant is 0, we only prove the second statement. To simplify the presentation, we argue as if H_i and v_i were smooth for $i = 1,2$ and rigorous proof can be made by a standard regularization argument. Summing up the two equations in (E) and using (A6), we obtain

$$\begin{aligned} 0 &\geq H_1(Dv_1) + H_2(Dv_2)\\ &\geq H_1(0) + DH_1(0)\cdot Dv_1 + \alpha|Dv_1|^2 + H_2(0) + DH_2(0)\cdot Dv_2 + \alpha|Dv_2|^2\\ &= DH_1(0)\cdot Dv_1 + \alpha|Dv_1|^2 + DH_2(0)\cdot Dv_2 + \alpha|Dv_2|^2. \end{aligned}$$

Integrate the above inequality over $\mathbb{T}^n$ to get

$$0 \geq \int_{\mathbb{T}^n} [DH_1(0)\cdot Dv_1 + \alpha|Dv_1|^2 + DH_2(0)\cdot Dv_2 + \alpha|Dv_2|^2]\,dx = \int_{\mathbb{T}^n} \alpha(|Dv_1|^2 + |Dv_2|^2)\,dx$$

which implies the conclusion. □

Lemma 5.4.2 (Monotonicity Property)**.** *Define*

$$M(t) := \max_{i=1,2}\max_{x\in\mathbb{T}^n} u_i(x,t) \text{ and } m(t) := \min_{i=1,2}\min_{x\in\mathbb{T}^n} u_i(x,t).$$

Then $t \mapsto M(t)$ *is nonincreasing and* $t \mapsto m(t)$ *is nondecreasing.*

Proof. Fix $s \in [0,\infty)$ and let $a = M(s)$. We have (a,a) is a solution of (C2) and $a \geq u_i(x,s)$ for all $x \in \mathbb{T}^n$ and $i = 1,2$. By the comparison principle for (C2), we have $a \geq u_i(x,t)$ for $x \in \mathbb{T}^n$, $t \geq s$ and $i = 1,2$. Thus $t \mapsto M(t)$ is nonincreasing. Similarly, $t \mapsto m(t)$ is nondecreasing. □

By Lemma 5.4.2, we can define

$$\overline{M} := \lim_{t\to\infty} M(t) \text{ and } \underline{m} := \lim_{t\to\infty} m(t).$$

Proof of Theorem 5.1.2. If $\overline{M} = \underline{m}$ then we immediately get the conclusion and therefore we suppose by contradiction that $\overline{M} > \underline{m}$ and show the contradiction.

Since $\{u_i(\cdot, t)\}_{t>0}$ is compact in $W^{1,\infty}(\mathbb{T}^n)$ for $i = 1, 2$, there exists a sequence $T_n \to \infty$ so that $\{u_i(\cdot, T_n)\}$ converges uniformly as $n \to \infty$ for $i = 1, 2$. By the maximum principle,

$$\|u_i(\cdot, T_n + \cdot) - u_i(\cdot, T_m + \cdot)\|_{L^\infty(\mathbb{T}^n\times(0,\infty))} \le \|u_i(\cdot, T_n) - u_i(\cdot, T_m)\|_{L^\infty(\mathbb{T}^n)}$$

for $i = 1, 2$ and $m, n \in \mathbb{N}$. Hence $\{u_i(\cdot, T_n + \cdot)\}$ is a Cauchy sequence in $\mathrm{BUC}\,(\mathbb{T}^n \times [0, \infty))$ and therefore they converge to $u_i^\infty \in \mathrm{BUC}\,(\mathbb{T}^n \times [0, \infty))$ for $i = 1, 2$.

By a standard stability result of the theory of viscosity solutions, (u_1^∞, u_2^∞) is a solution of (5.4.1), (5.4.2). Moreover for $t > 0$

$$\max_{i=1,2} \max_{x\in\mathbb{T}^n} u_i^\infty(x, t) = \lim_{n\to\infty} \max_{i=1,2} \max_{x\in\mathbb{T}^n} u_i(x, T_n + t) = \lim_{n\to\infty} M(T_n + t) = \overline{M},$$

and similarly

$$\min_{i=1,2} \min_{x\in\mathbb{T}^n} u_i^\infty(x, t) = \underline{m}.$$

Let (x_1, t_1) and (x_2, t_2) satisfy $\max_{i=1,2} u_i^\infty(x_1, t_1) = \overline{M}$ and $\min_{i=1,2} u_i^\infty(x_2, t_2) = \underline{m}$. Without loss of generality, we assume that $u_1^\infty(x_1, t_1) = \max_{i=1,2} u_i^\infty(x_1, t_1) = \overline{M}$. By taking 0 as a test function from above of u_1^∞ at (x_1, t_1) we have

$$u_1^\infty(x_1, t_1) - u_2^\infty(x_1, t_1) \le 0$$

and therefore we obtain $u_1^\infty(x_1, t_1) = u_2^\infty(x_1, t_1) = \overline{M}$. Similarly we obtain $u_1^\infty(x_2, t_2) = u_2^\infty(x_2, t_2) = \underline{m}$. In particular,

$$\max_{x\in\mathbb{T}^n} u_i^\infty(x, t) = \overline{M},\ \min_{x\in\mathbb{T}^n} u_i^\infty(x, t) = \underline{m} \tag{5.4.3}$$

for $t > 0$ and $i = 1, 2$.

On the other hand, we have

$$(u_1^\infty + u_2^\infty)_t + H_1(Du_1^\infty) + H_2(Du_2^\infty) = 0. \tag{5.4.4}$$

Integrate (5.4.4) over $\mathbb{T}^n$, use (A6), and do the same way as in the proof of Proposition 5.4.1 to get

$$\begin{aligned}
0 &= \frac{d}{dt}\int_{\mathbb{T}^n} (u_1^\infty + u_2^\infty)(x, t)\, dx + \int_{\mathbb{T}^n} [H_1(Du_1^\infty) + H_2(Du_2^\infty)]\, dx \\
&\ge \frac{d}{dt}\int_{\mathbb{T}^n} (u_1^\infty + u_2^\infty)(x, t)\, dx + \alpha \int_{\mathbb{T}^n} (|Du_1^\infty|^2 + |Du_2^\infty|^2)\, dx \\
&\ge \frac{d}{dt}\int_{\mathbb{T}^n} (u_1^\infty + u_2^\infty)(x, t)\, dx + C,
\end{aligned}$$

where the last inequality follows from Lemma 5.4.3 below. Thus

$$\frac{d}{dt}\int_{\mathbb{T}^n} (u_1^\infty + u_2^\infty)(x, t)\, dx \le -C,$$

which implies

$$\lim_{t\to\infty}\int_{\mathbb{T}^n}(u_1^\infty+u_2^\infty)(x,t)\,dx=-\infty.$$

This contradicts (5.4.3) and the proof is complete. □

Lemma 5.4.3. *There exists a constant $\beta>0$ depending only on n, C such that*

$$\int_{\mathbb{T}^n}|Df|^2\,dx\ge\beta$$

for all $f\in W^{1,\infty}(\mathbb{T}^n)$ such that $\|f\|_{W^{1,\infty}(\mathbb{T}^n)}\le C$, $\max_{\mathbb{T}^n}f=1$, and $\min_{\mathbb{T}^n}f=0$.

Proof. We argue by contradiction. Were the stated estimate false, there would exist a sequence $\{f_m\}\subset W^{1,\infty}(\mathbb{T}^n)$ such that $\|f_m\|_{W^{1,\infty}(\mathbb{T}^n)}\le C$, $\max_{\mathbb{T}^n}f_m=1$, $\min_{\mathbb{T}^n}f_m=0$, and

$$\int_{\mathbb{T}^n}|Df_m|^2\,dx\le\frac{1}{m}. \tag{5.4.5}$$

By Ascoli-Arzela's theorem, we may assume there exists $f_0\in W^{1,\infty}(\mathbb{T}^n)$ so that

$$f_m\to f_0\quad\text{uniformly on }\mathbb{T}^n$$

by taking a subsequence if necessary. It is clear that $\max_{\mathbb{T}^n}f_0=1$, $\min_{\mathbb{T}^n}f_0=0$.

Besides, $\|f_m\|_{H^1(\mathbb{T}^n)}\le C$ for all $m\in\mathbb{N}$. By the Rellich-Kondrachov theorem,

$$f_m\rightharpoonup f_0\quad\text{in }H^1(\mathbb{T}^n)$$

by taking a subsequence if necessary. By (5.4.5), we obtain $Df_0=0$ a.e. Thus f_0 is constant, which contradicts the fact that $\max_{\mathbb{T}^n}f_0=1$, $\min_{\mathbb{T}^n}f_0=0$. □

Remark 5.4.4. (i) Assumption (A7) is just for simplicity. Indeed we can always normalize the Hamiltonians so that they satisfy (A7) by substituting (u_1,u_2) with $(\overline{u}_1,\overline{u}_2)$, where

$$\begin{cases}\overline{u}_1(x,t):=u_1(x,t)+\dfrac{H_1(0)+H_2(0)}{2}t+\dfrac{H_1(0)-H_2(0)}{2}\\ \overline{u}_2(x,t):=u_2(x,t)+\dfrac{H_1(0)+H_2(0)}{2}t\end{cases}\quad\text{for }(x,t)\in\mathbb{R}^n\times[0,T].$$

(ii) It is clear to see that we can get a similar result for systems with m-equations.
(iii) The same procedure works for the following more general Hamiltonians

$$H_i(x,p)=|p-\mathbf{b}_i(x)|^2-|\mathbf{b}_i(x)|^2$$

for $\mathbf{b}_i\in C^1(\mathbb{T}^n)$ with $\operatorname{div}\mathbf{b}_i=0$ on $\mathbb{T}^n$ for $i=1,2$. This type of Hamiltonians is related to the ones in some recent works on periodic homogenization of G-equation. See [17, 80] for details. The new key observation comes from the fact that

$$\int_{\mathbb{T}^n}\mathbf{b}_i(x)\cdot D\phi(x)\,dx=-\int_{\mathbb{T}^n}(\operatorname{div}\mathbf{b}_i)\phi\,dx=0$$

for any $\phi \in W^{1,\infty}(\mathbb{T}^n)$. This identity was also used in [80] to study the existence of approximate correctors of the cell (corrector) problem of G-equation. The divergence free requirement on the vector fields $\mathbf{b}_i$ for $i = 1, 2$ is critical in our argument. In particular, it forces (E) to only have constant solutions (a, a) for $a \in \mathbb{R}$. We do not know how to remove this requirement up to now.

5.5 Third case

In this section we consider the third case pointed out in Introduction, i.e., we assume that $H = H_1 = H_2$ and H satisfies (A1) and (A8). Then (C) reduces to

$$\text{(C3)} \quad \begin{cases} (u_1)_t + H(x, Du_1) + u_1 - u_2 = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ (u_2)_t + H(x, Du_2) + u_2 - u_1 = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u_1(x, 0) = u_{01}(x), \ u_2(x, 0) = u_{02}(x) & \text{on } \mathbb{T}^n. \end{cases}$$

Let (u_1, u_2) be the solution of (C3).

Proposition 5.5.1. *The function $(u_1 - u_2)(x, t)$ converges uniformly to 0 on $\mathbb{T}^n$ as $t \to \infty$.*

Lemma 5.5.2. *Set $\gamma(t) := \max_{x \in \mathbb{T}^n} (u_1 - u_2)(x, t)$. Then γ is a subsolution of*

$$\dot{\gamma}(t) + 2\gamma(t) = 0 \text{ in } (0, \infty). \tag{5.5.1}$$

Proof of Lemma 5.5.2. Let $\phi \in C^1((0, \infty))$ and $\tau > 0$ be a maximum of $\gamma - \phi$. Choose $\xi \in \mathbb{T}^n$ such that $\gamma(\tau) = u_1(\xi, \tau) - u_2(\xi, \tau)$. We define the function Ψ by

$$\Psi(x, y, t, s) := u_1(x, t) - u_2(y, s) - \frac{1}{2\varepsilon^2}(|x - y|^2 + (t - s)^2) - |x - \xi|^2 - (t - \tau)^2 - \phi(t).$$

Let Ψ achieve its maximum at some $(\overline{x}, \overline{y}, \overline{t}, \overline{s})$. By the definition of viscosity solutions we have

$$\begin{aligned} &\frac{\overline{t} - \overline{s}}{\varepsilon^2} + 2(\overline{t} - \tau) + \dot{\phi}(\overline{t}) + H(\overline{x}, \frac{\overline{x} - \overline{y}}{\varepsilon^2} + 2(\overline{x} - \xi)) + u_1(\overline{x}, \overline{t}) - u_2(\overline{x}, \overline{t}) \le 0, \\ &\frac{\overline{t} - \overline{s}}{\varepsilon^2} + H(\overline{y}, \frac{\overline{x} - \overline{y}}{\varepsilon^2}) + u_2(\overline{y}, \overline{s}) - u_1(\overline{y}, \overline{s}) \ge 0. \end{aligned}$$

Subtracting the two inequalities above, we obtain

$$\begin{aligned} 2(\overline{t} - \tau) + \dot{\phi}(\overline{t}) + H(\overline{x}, \frac{\overline{x} - \overline{y}}{\varepsilon^2} + 2(\overline{x} - \xi)) - H(\overline{y}, \frac{\overline{x} - \overline{y}}{\varepsilon^2}) \\ + u_1(\overline{x}, \overline{t}) - u_2(\overline{x}, \overline{t}) - (u_2(\overline{y}, \overline{s}) - u_1(\overline{y}, \overline{s})) \le 0. \end{aligned} \tag{5.5.2}$$

By the usual argument we may assume that

$$\overline{x}, \overline{y} \to \xi, \ \overline{t}, \overline{s} \to \tau, \ \frac{\overline{x} - \overline{y}}{\varepsilon^2} \to p \tag{5.5.3}$$

as $\varepsilon \to 0$ by taking a subsequence if necessary. Sending $\varepsilon \to 0$ in (5.5.2), we get

$$\dot{\phi}(\tau) + 2\gamma(\tau) \le 0,$$

which is the conclusion. □

Proof of Proposition 5.5.1. Let γ be the function defined in Lemma 5.5.2 and set $C := \|u_{01} - u_{02}\|_{L^\infty(\mathbb{T}^n)}$ and $\beta(t) := Ce^{-2t}$ for $t \in (0, \infty)$. Then

$$\dot{\beta}(t) + 2\beta(t) = 0,$$

and $\beta(0) \ge \gamma(0)$. By the comparison principle we get $\gamma(t) \le \beta(t) = Ce^{-2t}$. Hence $u_1(x,t) - u_2(x,t) \le Ce^{-2t}$ for all $x \in \mathbb{T}^n$, $t \in (0,\infty)$. By symmetry, we get $u_2(x,t) - u_1(x,t) \le Ce^{-2t}$, which proves the proposition. □

In view of Proposition 5.5.1 we see that associated with the Cauchy problem (C3) is the ergodic problem:

$$H(x, Dv(x)) = c \quad \text{in } \mathbb{T}^n. \tag{5.5.4}$$

By the classical result on ergodic problems in [60], there exists a pair $(v, c) \in W^{1,\infty}(\mathbb{T}^n) \times \mathbb{R}$ such that v is a solution of (5.5.4). Then (v, v, c) is a solution of (E). As in Introduction we normalize the ergodic constant c to be 0 by replacing H by $H - c$.

We notice that $(v + M, v + M, 0)$ is still a viscosity solution of (E) for any $M \in \mathbb{R}$. Therefore subtracting a positive constant from v if necessary, we may assume that

$$1 \le u_i(x,t) - v(x) \le C \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,T],\ i = 1,2 \text{ and some } C > 0 \tag{5.5.5}$$

and we fix such a constant C.

We define the functions $\alpha_\eta^\pm, \beta_\eta^\pm : [0,\infty) \to \mathbb{R}$ by

$$\alpha_\eta^+(s) := \min_{x \in \mathbb{T}^n, t \ge s} \Big(\frac{u_1(x,t) - v(x) + \eta(t-s)}{u_1(x,s) - v(x)} \Big), \tag{5.5.6}$$

$$\beta_\eta^+(s) := \min_{x \in \mathbb{T}^n, t \ge s} \Big(\frac{u_2(x,t) - v(x) + \eta(t-s)}{u_2(x,s) - v(x)} \Big), \tag{5.5.7}$$

$$\alpha_\eta^-(s) := \max_{x \in \mathbb{T}^n, t \ge s} \Big(\frac{u_1(x,t) - v(x) - \eta(t-s)}{u_1(x,s) - v(x)} \Big),$$

$$\beta_\eta^-(s) := \max_{x \in \mathbb{T}^n, t \ge s} \Big(\frac{u_2(x,t) - v(x) - \eta(t-s)}{u_2(x,s) - v(x)} \Big)$$

for $\eta \in (0, \eta_0]$. By the uniform continuity of u_i and v, we have $\alpha_\eta^\pm, \beta_\eta^\pm \in C([0,\infty))$. It is easy to see that $0 \le \alpha_\eta^+(s), \beta_\eta^+(s) \le 1$ and $\alpha_\eta^-(s), \beta_\eta^-(s) \ge 1$ for all $s \in [0,\infty)$ and $\eta \in (0, \eta_0]$.

Lemma 5.5.3 (Key Lemma)**.** *Let C be the constant fixed in* (5.5.5).
(i) Assume that (A8)$^+$ holds. For any $\eta \in (0, \eta_0]$ there exists $s_\eta > 0$ such that the pair of the functions $(\alpha_\eta^+, \beta_\eta^+)$ is a supersolution of

$$\begin{cases} \max\{(\alpha_\eta^+)'(s) + \dfrac{\psi_\eta}{C}(\alpha_\eta^+(s) - 1) + F(\alpha_\eta^+(s) - \beta_\eta^+(s)), \\ \qquad\qquad\qquad\qquad \alpha_\eta^+(s) - 1\} = 0 \qquad \textit{in } (s_\eta, \infty), & (5.5.8) \\ \max\{(\beta_\eta^+)'(s) + \dfrac{\psi_\eta}{C}(\beta_\eta^+(s) - 1) + F(\beta_\eta^+(s) - \alpha_\eta^+(s)), \\ \qquad\qquad\qquad\qquad \beta_\eta^+(s) - 1\} = 0 \qquad \textit{in } (s_\eta, \infty), & (5.5.9) \end{cases}$$

where

$$F(r) := \begin{cases} Cr & \textit{if } r \geq 0, \\ \dfrac{r}{C} & \textit{if } r < 0. \end{cases}$$

(ii) Assume that (A8)$^-$ holds. For any $\eta \in (0, \eta_0]$ there exists $s_\eta > 0$ such that the pair of the functions $(\alpha_\eta^-, \beta_\eta^-)$ is a subsolution of

$$\begin{cases} \min\{(\alpha_\eta^-)'(s) + \dfrac{\psi_\eta}{C} \cdot \dfrac{\alpha_\eta^-(s) - 1}{\alpha_\eta^-(s)} + F(\alpha_\eta^-(s) - \beta_\eta^-(s)), \\ \qquad\qquad\qquad\qquad \alpha_\eta^-(s) - 1\} = 0 \qquad \textit{in } (s_\eta, \infty), & (5.5.10) \\ \min\{(\beta_\eta^-)'(s) + \dfrac{\psi_\eta}{C} \cdot \dfrac{\beta_\eta^-(s) - 1}{\beta_\eta^-(s)} + F(\beta_\eta^-(s) - \alpha_\eta^-(s)), \\ \qquad\qquad\qquad\qquad \beta_\eta^-(s) - 1\} = 0 \qquad \textit{in } (s_\eta, \infty). & (5.5.11) \end{cases}$$

Proof. We only prove (i), since we can prove (ii) similarly. Fix $\mu \in (0, \eta_0]$. By abuse of notations we write α, β for $\alpha_\eta^+, \beta_\eta^+$. Recall that $\alpha(s), \beta(s) \leq 1$ for any $s \geq 0$. By Proposition 5.5.1, there exists $s_\eta > 0$ such that $|u_1(x,t) - u_2(x,t)| \leq \eta/2$ for all $x \in \mathbb{T}^n$ and $t \geq s_\eta$.

We only consider the case where $(\alpha - \phi)(s) > (\alpha - \phi)(\sigma)$ for some $\phi \in C^1((0,\infty))$, $\sigma > s_\eta$, $\delta > 0$ and all $s \in [\sigma - \delta, \sigma + \delta] \setminus \{\sigma\}$, since a similar argument holds for β. Since there is nothing to check in the case where $\alpha(\sigma) = 1$, we assume that $\alpha(\sigma) < 1$. We choose $\xi \in \mathbb{T}^n$ and $\tau \geq \sigma$ such that

$$\alpha(\sigma) = \frac{u_1(\xi, \tau) - v(\xi) + \eta(\tau - \sigma)}{u_1(\xi, \sigma) - v(\xi)} =: \frac{\alpha_2}{\alpha_1}.$$

We write α for $\alpha(\sigma)$ henceforth.

Set $K := \mathbb{T}^{3n} \times \{(t, s) \mid t \geq s, s \in [\sigma - \delta, \sigma + \delta]\}$. For $\varepsilon \in (0,1)$, we define the function $\Psi : K \to \mathbb{R}$ by

$$\begin{aligned} &\Psi(x, y, z, t, s) \\ :=&\frac{u_1(x,t) - v(z) + \eta(t - s)}{u_1(y,s) - v(z)} - \phi(s) + \frac{1}{2\varepsilon^2}(|x-y|^2 + |x-z|^2) + |x - \xi|^2 + (t - \tau)^2. \end{aligned}$$

Let Ψ achieve its minimum over K at some $(\overline{x},\overline{y},\overline{z},\overline{t},\overline{s})$. Set

$$\begin{aligned}&\overline{\alpha}_1 := u_1(\overline{y},\overline{s}) - v(\overline{z}),\ \overline{\alpha}_2 = u_1(\overline{x},\overline{t}) - v(\overline{z}) + \eta(\overline{t}-\overline{s}),\ \overline{\alpha} := \frac{\overline{\alpha}_2}{\overline{\alpha}_1},\\ &\overline{p} := \frac{\overline{y}-\overline{x}}{\varepsilon^2} \text{ and } \overline{q} := \frac{\overline{z}-\overline{x}}{\varepsilon^2}.\end{aligned}$$

We observe the followings. Firstly, set

$$f_1(y,s) := \phi(s) - \frac{1}{2\varepsilon^2}(|\overline{x}-y|^2 + |\overline{x}-\overline{z}|^2) - |\overline{x}-\xi|^2 - (\overline{t}-\tau)^2.$$

Noting that $u_1(y,s)-v(\overline{z})>0$, we see that $u_1(y,s) - \big(u_1(\overline{x},\overline{t}) - v(\overline{z}) + \eta(\overline{t}-s)\big)\big(f_1(y,s) + \min\Psi\big)^{-1}$ takes its maximum at $(\overline{y},\overline{s})$. Secondly, set

$$f_2(z) := \phi(\overline{s}) - \frac{1}{2\varepsilon^2}(|\overline{x}-\overline{y}|^2 + |\overline{x}-z|^2) - |\overline{x}-\xi|^2 - (\overline{t}-\tau)^2.$$

Noting that for $\varepsilon>0$ small enough, then $\overline{\alpha}<1$, which implies $-a := u_1(\overline{x},\overline{t}) - u_1(\overline{y},\overline{s}) + \eta(\overline{t}-\overline{s}) < 0$. Then we see that $v(z) - a\big(f_2(z) + \min\Psi - 1\big)^{-1}$ takes its maximum at $\overline{z}$.

Thus, we have by the definition of viscosity solutions

$$\begin{cases} -\eta - 2\overline{\alpha}_1(\overline{t}-\tau) + H(\overline{x}, D_x u_1(\overline{x},\overline{t})) + (u_1-u_2)(\overline{x},\overline{t}) \ge 0,\\ -\dfrac{1}{\overline{\alpha}}(\eta + \overline{\alpha}_1\phi'(\overline{s})) + H(\overline{y}, D_y u_1(\overline{y},\overline{s})) + (u_1-u_2)(\overline{y},\overline{s}) \le 0,\\ H(\overline{z}, D_z v(\overline{z})) \le 0, \end{cases} \tag{5.5.12}$$

where

$$\begin{aligned} D_x u_1(\overline{x},\overline{t}) &= \overline{\alpha}_1\big\{\overline{p} + \overline{q} + 2(\xi - \overline{x})\big\},\\ D_y u_1(\overline{y},\overline{s}) &= \frac{\overline{\alpha}_1}{\overline{\alpha}}\overline{p},\\ D_z v(\overline{z}) &= \frac{\overline{\alpha}_1}{1-\overline{\alpha}}\overline{q}. \end{aligned}$$

By taking a subsequence if necessary, we may assume that

$$\overline{x},\overline{y},\overline{z} \to \xi \text{ and } \overline{t}\to\tau,\ \overline{s}\to\sigma \text{ as } \varepsilon\to 0.$$

Since u_i, v are Lipschitz continuous, we have

$$\frac{|\overline{x}-\overline{y}|}{\varepsilon^2} + \frac{|\overline{x}-\overline{z}|}{\varepsilon^2} \le M$$

for some $M>0$ and all $\varepsilon\in(0,1)$. We may assume that

$$\overline{p} := \frac{\overline{y}-\overline{x}}{\varepsilon^2} \to p,\ \overline{q} := \frac{\overline{z}-\overline{x}}{\varepsilon^2} \to q$$

as $\varepsilon \to 0$ for some $p, q \in B(0, M)$.

Sending $\varepsilon \to 0$ in (5.5.12) yields

$$\begin{aligned} -\eta + H(\xi, \tilde{P} + Q) + (u_1 - u_2)(\xi, \tau) &\geq 0, \\ -\frac{1}{\alpha(\sigma)}(\eta + \alpha_1 \phi'(\sigma)) + H(\xi, P) + (u_1 - u_2)(\xi, \sigma) &\leq 0, \\ H(\xi, Q) &\leq 0, \end{aligned} \tag{5.5.13}$$

where

$$P := \frac{\alpha_1}{\alpha} p, \ Q := \frac{\alpha_1}{1-\alpha} q, \ \tilde{P} = \alpha(P - Q).$$

Recalling that $(u_1 - u_2)(\xi, \tau) \leq \eta/2$, we have

$$H(\xi, \tilde{P} + Q) \geq \eta/2.$$

Therefore, by using $(A8)^+$, we obtain

$$H(\xi, \tilde{P} + Q) \leq \alpha H(\xi, P) - \psi_\eta (1 - \alpha) \tag{5.5.14}$$

for some $\psi_\eta > 0$.

Noting that

$$\beta(\sigma) \leq \frac{u_2(\xi, \tau) - v(\xi) + \eta(\tau - \sigma)}{u_2(\xi, \sigma) - v(\xi)} =: \frac{\beta_2}{\beta_1},$$

we calculate that

$$\begin{aligned} &(u_1 - u_2)(\xi, \tau) - \alpha(u_1 - u_2)(\xi, \sigma) \\ = &-(u_2(\xi, \tau) - v(\xi) + \eta(\tau - \sigma)) + \alpha(u_2(\xi, \sigma) - v(\xi)) \\ = &-\beta_1\big(\frac{\beta_2}{\beta_1} - \alpha\big) \\ \leq &-\beta_1(\beta(\sigma) - \alpha(\sigma)). \end{aligned}$$

Therefore by (5.5.14) and (5.5.13),

$$\begin{aligned} \eta \leq & H(\xi, \tilde{P} + Q) + (u_1 - u_2)(\xi, \tau) \\ \leq & \alpha\Big(\frac{1}{\alpha}(\eta + \alpha_1 \phi'(\sigma)) - (u_1 - u_2)(\xi, \sigma)\Big) - \psi_\eta(1 - \alpha) + (u_1 - u_2)(\xi, \tau) \\ \leq & \eta + \alpha_1 \phi'(\sigma) - \psi_\eta(1 - \alpha) + \beta_1(\alpha(\sigma) - \beta(\sigma)), \end{aligned}$$

which implies

$$\phi'(\sigma) + \frac{\psi_\eta}{C}(\alpha(\sigma) - 1) + \frac{\beta_1}{\alpha_1}(\alpha(\sigma) - \beta(\sigma)) \geq 0.$$

Combining the above inequality with the fact that $1/C \leq \beta_1/\alpha_1 \leq C$, we have

$$\phi'(\sigma) + \frac{\psi_\eta}{C}(\alpha(\sigma) - 1) + F(\alpha(\sigma) - \beta(\sigma)) \geq 0. \qquad \square$$

Lemma 5.5.4.
(i) Assume that (A8)$^+$ holds. The functions $\alpha_\eta^+(s)$ and $\beta_\eta^+(s)$ converge to 1 as $s \to \infty$ for each $\eta \in (0, \eta_0]$.
(ii) Assume that (A8)$^-$ holds. The functions $\alpha_\eta^-(s)$ and $\beta_\eta^-(s)$ converge to 1 as $s \to \infty$ for each $\eta \in (0, \eta_0]$.

Proof. Fix $\eta \in (0, \eta_0]$. We first recall that, by definition,

$$\alpha_\eta^+(s) \le 1 \le \alpha_\eta^-(s),\ \beta_\eta^+(s) \le 1 \le \beta_\eta^-(s)$$

for any $s \ge 0$. On the other hand, one checks easily that the pairs

$$\big(1 + (\gamma_1 - 1)\exp(-\frac{\psi_\eta}{C}t), 1 + (\gamma_1 - 1)\exp(-\frac{\psi_\eta}{C}t)\big)$$

and

$$\big(1 + (\gamma_2 - 1)\exp(-\frac{\psi_\eta}{C\gamma_2}t), 1 + (\gamma_2 - 1)\exp(-\frac{\psi_\eta}{C\gamma_2}t)\big)$$

are, respectively, a subsolution and a supersolution of (5.5.8)-(5.5.9) and (5.5.10)-(5.5.11) for $\gamma_1 = \min\{\alpha_\eta^+(0), \beta_\eta^+(0)\}$, and $\gamma_2 = \max\{\alpha_\eta^-(0), \beta_\eta^-(0)\}$. Therefore, by the comparison principle in [57, 26, 52] , we get

$$\alpha_\eta^+(s), \beta_\eta^+(s) \ge 1 + (\gamma_1 - 1)\exp(-\frac{\psi_\eta}{C}t)$$

and

$$\alpha_\eta^-(s), \beta_\eta^-(s) \le 1 + (\gamma_2 - 1)\exp(-\frac{\psi_\eta}{C\gamma_2}t),$$

which give us the conclusion. □

By Lemma 5.5.4, we immediately get the following proposition.

Proposition 5.5.5 (Asymptotically Monotone Property)**.**
(i) ***(Asymptotically Increasing Property)***
Assume that (A8)$^+$ holds. For $\eta \in (0, \eta_0]$, there exists a function $\delta_\eta : [0, \infty) \to [0, 1]$ such that

$$\lim_{s\to\infty} \delta_\eta(s) = 0$$

and

$$u_i(x, s) - u_i(x, t) - \eta(t - s) \le \delta_\eta(s)$$

for all $x \in \mathbb{T}^n$, $t \ge s \ge 0$ and $i = 1, 2$.
(ii) ***(Asymptotically Decreasing Property)***
Assume that (A8)$^-$ holds. For $\eta \in (0, \eta_0]$, there exists a function $\delta_\eta : [0, \infty) \to [0, 1]$ such that

$$\lim_{s\to\infty} \delta_\eta(s) = 0$$

and

$$u_i(x, t) - u_i(x, s) - \eta(t - s) \le \delta_\eta(s),$$

for all $x \in \mathbb{T}^n$, $t \ge s \ge 0$ and $i = 1, 2$.

Theorem 5.1.3 is a straightforward result of the above proposition and the proof follows as in [7, Section 4] or [4, Section 5]. We reproduce these arguments here for the reader's convenience.

Since $\{u_i(\cdot,t)\}$ is compact in $W^{1,\infty}(\mathbb{T}^n)$ for $i=1,2$, there exists a sequence $T_n \to \infty$ so that $\{u_i(\cdot,T_n)\}$ converges uniformly as $n\to\infty$ for $i=1,2$. By the maximum principle

$$\|u_i(\cdot,T_n+\cdot)-u_i(\cdot,T_m+\cdot)\|_{L^\infty(\mathbb{T}^n\times(0,\infty))} \le \|u_i(\cdot,T_n)-u_i(\cdot,T_m)\|_{L^\infty(\mathbb{T}^n)}$$

for $i=1,2$ and $m,n\in\mathbb{N}$. Hence $\{u_i(\cdot,T_n+\cdot)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{BUC}\,(\mathbb{T}^n\times[0,\infty))$ and therefore it converges to $u_i^\infty \in \mathrm{BUC}\,(\mathbb{T}^n\times[0,\infty))$ for $i=1,2$. For any $x\in\mathbb{T}^n$ and $t\ge s\ge 0$, by Proposition 5.5.5 we get

$$u_i(x,s+T_n)-u_i(x,t+T_n)+\eta(s-t)\le \delta_\eta(s+T_n).$$

Sending $n\to\infty$ then $\eta\to 0$, we get for any $t\ge s\ge 0$

$$u_i^\infty(x,s)\le u_i^\infty(x,t).$$

This implies $u_i^\infty(x,t)$ converges uniformly to $v_i(x)$ on $\mathbb{T}^n$ as $t\to\infty$ for some $v_i\in W^{1,\infty}(\mathbb{T}^n)$. Then (v_1,v_2) is a solution of (E) by a standard stability result of the theory of viscosity solutions.

Since $\{u_i(\cdot,T_n+\cdot)\}_{n\in\mathbb{N}}$ converges to u_i^∞ uniformly on $\mathbb{T}^n$,

$$-o_n(1)+u_i^\infty(x,t)\le u_i(x,t+T_n)\le o_n(1)+u_i^\infty(x,t),$$

where $\lim_{n\to\infty} o_n(1)=0$. Therefore,

$$-o_n(1)+v_i(x)\le \liminf_{t\to\infty}{}_*[u_i](x,t)\le \limsup_{t\to\infty}{}^*[u_i](x,t)\le o_n(1)+v_i(x).$$

Eventually, letting $n\to\infty$, we get the result.

Finally we remark that if we want to deal, at the same time, with the Hamiltonians of the form

$$H(x,p):=|p|-f(x),$$

we can generalize Theorem 5.1.3 as in [7]. We replace (A8) by

(A9) Either of the following assumption $(A9)^+$ or $(A9)^-$ holds:

$(A9)^+$ There exists a closed set $K\subset\mathbb{T}^n$ (K is possibly empty) having the properties

(i) $\min_{p\in\mathbb{R}^n} H(x,p)=0$ for all $x\in K$,

(ii) for each $\varepsilon>0$ there exists a modulus $\psi_\varepsilon(r)>0$ for all $r>0$ and $\eta_0^\varepsilon>0$ such that for all $\eta\in(0,\eta_0^\varepsilon]$ if $\operatorname{dist}(x,K)\ge\varepsilon$, $H(x,p+q)\ge\eta$ and $H(x,q)\le 0$ for some $x\in\mathbb{T}^n$ and $p,q\in\mathbb{R}^n$, then for any $\mu\in(0,1]$,

$$\mu H(x,\frac{p}{\mu}+q)\ge H(x,p+q)+\psi_\varepsilon(\eta)(1-\mu).$$

(A9)$^-$ There exists a closed set $K \subset \mathbb{T}^n$ (K is possibly empty) having the properties

(i) $\min_{p\in\mathbb{R}^n} H(x,p) = 0$ for all $x \in K$,

(ii) for each $\varepsilon > 0$ there exists a modulus $\psi_\varepsilon(r) > 0$ for all $r > 0$ and $\eta_0^\varepsilon > 0$ such that for all $\eta \in (0, \eta_0^\varepsilon]$ if $\operatorname{dist}(x,K) \geq \varepsilon$, $H(x, p+q) \leq -\eta$ and $H(x,q) \geq 0$ for some $x \in \mathbb{T}^n$ and $p, q \in \mathbb{R}^n$, then for any $\mu \in (0,1]$,

$$\mu H(x, \frac{p}{\mu} + q) \leq H(x, p+q) - \frac{\psi_\varepsilon(\eta)(\mu - 1)}{\mu}.$$

Theorem 5.5.6. *The result of Theorem 5.1.3 still holds if we replace (A8) by (A9).*

Sketch of Proof. By the argument same as in the proof of Propositions 5.3.3, 5.3.4 we can see $(u_1+u_2)|_K$ and $\max\{u_1,u_2\}|_K$ are nonincreasing and therefore we see that u_i converge uniformly on K as $t \to \infty$ for $i = 1, 2$.

Setting $K_\varepsilon := \{x \in \mathbb{T}^n \mid d(x,K) \geq \varepsilon\}$, we see that u_i are asymptotically monotone on $\mathbb{R}^n \setminus K_\varepsilon$ for every $\varepsilon > 0$, which implies that u_i converges uniformly on $\mathbb{R}^n \setminus K$ as $t \to \infty$ for $i = 1, 2$ as in [7]. □

5.6 Auxiliary Lemmata

We present a sketch of the proof based on Proposition 5.2.1 from [9] for the reader's convenience.

Sketch of the proof of Proposition 5.2.1. Without loss of generality, we may assume $c_1 = c_2 = 1$. The existence of $(v_1, v_2, \overline{H}_1, \overline{H}_2)$ can be proved by repeating the argument same as in the first part of Proposition 5.3.1. We here only prove that $\overline{H}_1 + \overline{H}_2$ is unique.

Suppose by contradiction that there exist two pairs $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $(\mu_1, \mu_2) \in \mathbb{R}^2$ such that $\lambda_1 + \lambda_2 < \mu_1 + \mu_2$ and two pair of continuous functions (v_1, v_2), $(\overline{v}_1, \overline{v}_2)$ such that (v_1, v_2), $(\overline{v}_1, \overline{v}_2)$ are viscosity solutions of the following systems

$$\begin{cases} H_1(x, Dv_1) + v_1 - v_2 = \lambda_1 \\ H_2(x, Dv_2) + v_2 - v_1 = \lambda_2 \end{cases} \text{in } \mathbb{T}^n,$$

and

$$\begin{cases} H_1(x, D\overline{v}_1) + \overline{v}_1 - \overline{v}_2 = \mu_1 \\ H_2(x, D\overline{v}_2) + \overline{v}_2 - \overline{v}_1 = \mu_2 \end{cases} \text{in } \mathbb{T}^n,$$

respectively.

For a suitably large constant $C > 0$, $(v_1 + \frac{\lambda_2 - \lambda_1}{2} - \frac{\lambda_1 + \lambda_2}{2}t - C, v_2 - \frac{\lambda_1 + \lambda_2}{2}t - C)$ and $(\overline{v}_1 + \frac{\mu_2 - \mu_1}{2} - \frac{\mu_1 + \mu_2}{2}t + C, \overline{v}_2 - \frac{\mu_1 + \mu_2}{2}t + C)$ are respectively a subsolution and a supersolution of (C). By the comparison principle for (C), Proposition 5.2.2, we obtain particularly

$$v_1 + \frac{\lambda_2 - \lambda_1}{2} - \frac{\lambda_1 + \lambda_2}{2} t - C \leq \overline{v}_1 + \frac{\mu_2 - \mu_1}{2} - \frac{\mu_1 + \mu_2}{2} t + C, \quad \text{in } \mathbb{R}^n \times [0, T]$$

which contradicts the fact that $\lambda_1 + \lambda_2 < \mu_1 + \mu_2$. □

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