# A note on weak convergence methods

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This is based on the summer course "Weak convergence methods for nonlinear PDEs" taught by Prof. Hung Tran (University of Wisconsin, Madison) in July 2016 at University of Science, Ho Chi Minh City, Vietnam and his following reading course.

## 1 Introduction

We want to solve the equation

$$F[u] = 0$$

where $F[\cdot]$ is a differential operator and maybe not nice. One simple way: approximate the nice $F^\varepsilon[\cdot] \to F[\cdot]$ as $\varepsilon \to 0$. Suppose $u^\varepsilon$ solves $F^\varepsilon[u^\varepsilon] = 0$. As $\varepsilon \to 0$, do we have that $u^\varepsilon \to \bar{u}$ in some sense of convergence ? And if so, do we have $F[\bar{u}] = 0$ ?

*Key points:*

1. We need to understand the convergence of $u^\varepsilon \to \bar{u}$ if possible.

2. $$\begin{cases} F[\cdot] \text{ is a nonlinear operator} \\ F^\varepsilon[\cdot] \text{ is nonlinear operators.} \end{cases}$$

   Does nonlinearity intervene in $F[\bar{u}] = 0$ ?

*Example* 1 ($1 = 0$ !!). Approximation equation: $1 + \varepsilon u^\varepsilon = 0$. If $u^\varepsilon$ is bounded as $\varepsilon \to 0$, then, we have to conclude that $1 = 0$ ! Luckily, $u^\varepsilon = -1/\varepsilon$ is not bounded.

In PDEs, if the approximation equation is physically relevant, then we can often find bounds for $u^\varepsilon$. This is often called "a priority estimates".

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## 1.1 The weak convergence

Let $U \subset \mathbb{R}^d$ be an bounded, open set with smooth boundary.

i. For $1 < p < \infty$, let $\{f_n\}$ be a sequence in $L^p(U)$ and $f \in L^p(U)$. We say that $f_n$ converges weakly to $f$ and write $f_n \rightharpoonup f$ if

$$\int_U f_n \varphi \to \int_U f \varphi \qquad \text{for every } \varphi \in L^q(U)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

ii. For $p = \infty$, let $\{f_n\}$ be a sequence in $L^\infty(U)$ and $f \in L^\infty(U)$. We write $f_n \overset{*}{\rightharpoonup} f$ if

$$\int_U f_n \varphi \to \int_U f \varphi \qquad \text{for every } \varphi \in L^1(U).$$

**Theorem 1.1.** $(1 < p < \infty)$ *Let $\{f_n\}$ be a bounded sequence in $L^p(U)$. Then there exists a subsequence $\{f_{n_k}\}$ which converges weakly to some $f \in L^p(U)$.*

*Proof.* We use the diagonal argument. Since $L^q(U)$ is separable, let $\{e_k\}$ be a dense sequence in $L^q(U)$. Suppose $\{f_n\} \subset L^p(U)$ such that $\|f_n\|_p \le C$ for every $n$, then $\{\langle f_n, e_1\rangle\}$ is a sequence bounded by $C\|e_1\|_q$. Thus, we can extract a subsequence $\{f_{1,n}\} \subset \{f_n\}$ such that $\{\langle f_{1,n}, e_1\rangle\}$ converges to a limit, called $L(e_1)$. Similarly, we can extract a further subsequence $\{f_{2,n}\} \subset \{f_{1,n}\}$ such that $\{\langle f_{2,n}, e_2\rangle\}$ converges to a limit, called $L(e_2)$. Continue this process to obtain a sequence $\{f_{k,n}\} \subset \{f_{k-1,n}\}$ and a real number $L(e_k)$ such that $\langle f_{k,n}, e_k\rangle \to L(e_k)$ as $n \to \infty$ for every $k \in \mathbb{N}$. Put $g_n = f_{n,n}$. Since $g_n$ belongs to $\{f_{k,n}\}$ for every $k < n$, we have $\langle g_n, e_k\rangle \to L(e_k)$ as $n \to \infty$ for every $k$. Pick $\varphi \in L^q(U)$ and $\varepsilon > 0$ arbitrarily, there is a number $k \in \mathbb{N}$ such that $\|e_k - \varphi\|_q < \varepsilon$. Whenever $m, n \in \mathbb{N}$ such that $|\langle g_n - g_m, e_k\rangle| < \varepsilon$, we have

$$\begin{aligned} |\langle g_n - g_m, \varphi\rangle| \le |\langle g_n - g_m, e_k\rangle| + |\langle g_n - g_m, e_k - \varphi\rangle| &< \varepsilon + \varepsilon\|g_n - g_m\|_p \\ &\le \varepsilon + 2C\varepsilon. \end{aligned}$$

Hence, $\{\langle g_n, \varphi\rangle\}$ is a Cauchy sequence, so it converges to a limit called $L(\varphi)$. Noting that $L(\varphi) \le C\|\varphi\|_q$, the linear functional $L : \varphi \mapsto L(\varphi)$ is bounded. By Riesz's Representation Theorem, there exists a function $f \in L^p(U)$ such that $L(\varphi) = \langle f, \varphi\rangle$ for every $\varphi \in L^q(U)$. We conclude that $g_n \rightharpoonup f$. □

Using similar argument, it is easy to prove the following result:

**Theorem 1.2.** $(p = \infty)$ *Let $\{f_n\}$ be a bounded sequence in $L^\infty(U)$. Then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \overset{*}{\rightharpoonup} f$ for some $f \in L^p(U)$.*

**Proposition 1.3.** *For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, let $\{f_n\}$ be a sequence in $L^p(U)$ which converges weakly to $f$.*

*(i)* $\liminf_{n\to\infty} \|f_n\|_{L^p} \geq \|f\|_{L^p}$.

*(ii) Suppose $\|f_n\|_{L^p} \to \|f\|_{L^p}$, then $f_n \to f$ strongly in $L^p(U)$*

*(iii) Suppose $g_n \to g$ strongly in $L^q(U)$, then*

$$\int_U f_n g_n \to \int_U fg.$$

*Proof.* In order to prove $(i)$, choose

$$g = \frac{1}{\|f\|_{L^p}^{p/q}} \mathrm{sgn}(f)|f|^{\frac{p}{q}},$$

where

$$\mathrm{sgn}(f) = \begin{cases} -1 & \text{if } f < 0, \\ 1 & \text{if } f \geq 0. \end{cases}$$

Then, it is clear that $g \in L^q(U)$, $\|g\|_{L^q} = 1$ and

$$\int_U f(x)g(x)dx = \|f\|_{L^p}.$$

From definition of weak convergence and Holder inequality, we have

$$\|f\|_{L^p} = \int_U f(x)g(x)dx = \lim_{n\to\infty} \int_U f_n(x)g(x)dx \leq \liminf_{n\to\infty} \|f_n\|_{L^p}.$$

For $(ii)$, let $g_n = f_n/\|f_n\|_{L^p}$ and $g = f/\|f\|_{L^p}$. Then $g_n \rightharpoonup g$ weakly and therefore $(g_n + g)/2 \rightharpoonup g$ weakly. By (i), we have

$$1 = \|g\|_{L^p} \leq \liminf_{n\to\infty} \left\| \frac{g_n + g}{2} \right\|_{L^p} \leq 1.$$

So $\|(g_n + g)/2\|_{L^p} \to 1$. Since $L^p(U)$ is a uniformly convex space, it follows that $\|g_n - g\|_{L^p} \to 0$, which also implies

$$\|f_n - f\|_{L^p} \to 0.$$

The assertion $(iii)$ is followed from

$$\begin{aligned} \left| \int_U f_n g_n - \int_U fg \right| &\leq \left| \int_U f_n g_n - \int_U f_n g \right| + \left| \int_U f_n g - \int_U fg \right| \\ &\leq \|f_n\|_{L^p(U)} \|g_n - g\|_{L^q(U)} + \left| \int_U f_n g - \int_U fg \right| \end{aligned}$$

which tends to 0 as $n \to \infty$ since $\|f_n\|_{L^p(U)}$ is bounded by Uniformly Bounded Principle. □

*Question.* Suppose $f_n \rightharpoonup f$ weakly in $L^p(U)$ and $g_n \rightharpoonup g$ weakly in $L^q(U)$. Is it generally true that

$$\int_U f_n(x)g_n(x)dx \to \int_U f(x)g(x)dx \quad ?$$

*Answer:* No. For a counterexample, let $L^p(U) = L^2(0,\pi)$ and $f_n(x) = g_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. Since, $(f_n)_{n\in\mathbb{N}}$ is an orthogonal basis of the Hilbert space $L^2(0,\pi)$, it converges weakly to zero. However,

$$\int_0^\pi f_n(x)g_n(x)dx = 1$$

for every $n$.

*Remark.* Most of the time, functional spaces are infinite dimensional, so they lose compactness.

**Weak convergence of measures.** Let $\mathcal{R}(U)$ be the space of Radon measure on $U$. We have that $L^1(U) \subset \mathcal{R}(U)$ in following sense: For $f \in L^1(U)$, define $f \mapsto \mu_f$ by

$$\mu_f(B) = \int_U f(x)dx.$$

Then, for any $g \in L^\infty(U)$,

$$\int_U f(x)g(x)dx = \int_U g(x)d\mu_f(x).$$

*Compactness of* $\mathcal{R}(U)$: Let $\mu \in \mathcal{R}(U)$, define $|\mu|(U)$ as the total variation of $\mu$:

$$\mu(U) := \sup_{|h|\leq 1 \text{ a.e in } U} \int_U h d\mu.$$

For $f \in L^1(U)$,

$$|\mu_f|(U) = \int_U |f(x)|dx = \|f\|_{L^1(U)}.$$

For $\{\mu_n\} \subset \mathcal{R}(U)$ satisfying $|\mu_n|(U) \leq C$ for all $n$, then there exists a subsequence $\{\mu_{n_k}\}$ such that

$$\mu_{n_k} \rightharpoonup \mu \in \mathcal{R}(U) \qquad \text{in sense of measure,}$$

which means

$$\int_U g d\mu_{n_k} \to \int_U g d\mu \qquad \text{for all } g \in C_c(U).$$

**Lemma 1.4.** . *Assume that $\{\mu_n\}$ is a sequence of positive Radon measures that converges in measure to $\mu \in \mathcal{R}(U)$. Then*

*(i) For $K \subset U$ compact*

$$\limsup_{n\to\infty} \mu_n(K) \le \mu(K). \tag{1.1}$$

*(ii) For $V \subset U$ open*

$$\liminf_{n\to\infty} \mu_n(V) \ge \mu(V). \tag{1.2}$$

*Proof.* For *(i)*, let $K \subset U$ be compact. Choose an open set $U_\varepsilon \supset K$ such that $\mu(K) + \varepsilon > \mu(U_\varepsilon)$. By Urysohn lemma, there exists a function $f \in C_c(U_\varepsilon)$, $0 \le f \le 1$ and $f = 1$ on $K$. So

$$\mu(K) + \varepsilon > \mu(U_\varepsilon) \ge \int_{U_\varepsilon} f d\mu = \lim_{n\to\infty} \int_{U_\varepsilon} f d\mu_n \ge \limsup_{n\to\infty} \int_K f d\mu_n = \limsup_{n\to\infty} \mu_n(K).$$

Letting $\varepsilon \to 0$, we conclude that

$$\mu(K) \ge \limsup_{n\to\infty} \mu_n(K).$$

In order to verify *(ii)*, let $V \subset U$ be open. Choose a compact set $A_\varepsilon \subset V$ such that $\mu(A_\varepsilon) + \varepsilon > \mu(V)$. By Urysohn lemma again, there exists a function $f \in C_c(V)$, $0 \le f \le 1$ and $f = 1$ on $A_\varepsilon$. Thus

$$\mu(V) - \varepsilon < \mu(A_\varepsilon) \le \int_V f d\mu = \lim_{n\to\infty} \int_V f d\mu_n \le \liminf_{n\to\infty} \mu_n(V).$$

Letting $\varepsilon \to 0$, it follows that

$$\mu(V) \le \liminf_{n\to\infty} \mu_n(V).$$

□

*Question.* Show that in some cases, we don't have equalities in (1.1) and (1.2).

*Answer:* Suppose $\{\mu_n\}$ defined by

$$\mu_n(A) = \frac{n}{2}\left|A \cap \left[\frac{-1}{n}, \frac{1}{n}\right]\right| \quad \text{for Borel set } A,$$

$|\cdot|$ denotes the Lebesgue measure. Then, for any $f \in C_c(-1,1)$,

$$\int_{-1}^{1} f(x) d\mu_n(x) = \frac{n}{2} \int_{\frac{-1}{n}}^{\frac{1}{n}} f(x) dx \to f(0).$$

So $\mu_n$ converges in measure to the dirac mass $\delta_0$ at 0. Choose $K = \{0\}$, then $\mu_n(\{0\}) = 0$ while $\delta_0(\{0\}) = 1$. In this case, the equality in (1.1) doesn't hold. For the other, suppose $(\mu_n)$ defined by

$$\mu_n = \mu_{f_n}$$

where

$$f_n(x) = \begin{cases} -n^2 x + n & \text{for } 0 < x < \frac{1}{n}, \\ 0 & \text{for } \frac{1}{n} \leq x < 1. \end{cases}$$

Then $\mu_n \rightharpoonup 0$ weakly. However $\mu_n((0,1)) = \frac{1}{2}$ for all $n$. Thus, the equality in (1.2) doesn't hold in this case. □

## 1.2 More on functional spaces

Now, we turn to some important results in Sobolev spaces.

**Theorem 1.5** (Gagliardo-Nirenberg-Sobelev inequality). *Given $f \in C_c^1(\mathbb{R}^d)$, then*

$$\left(\int_{\mathbb{R}^d} |f(x)|^{p^*} dx\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^d} |Df(x)|^p dx\right)^{\frac{1}{p}},$$

*where $C$ is a constant depending only on $d$ and $p$. Here, for $d > p \geq 1$, $p^* = \frac{dp}{d-p}$.*

Why $p^*$ ? Suppose

$$\left(\int_{\mathbb{R}^d} |f(x)|^q dx\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^d} |Df(x)|^p dx\right)^{\frac{1}{p}},$$

for all $f \in C_c^1(\mathbb{R}^d)$. Scaling analysis: put $f_\lambda(x) = f(\lambda x)$ for $\lambda > 0$. Then, $Df_\lambda = \lambda Df$ and

$$\begin{aligned} \left(\int_{\mathbb{R}^d} |f(\lambda x)|^q dx\right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}^d} |f_\lambda(x)|^q dx\right)^{\frac{1}{q}} &\leq C \left(\int_{\mathbb{R}^d} |Df_\lambda(x)|^p dx\right)^{\frac{1}{p}} \\ &= C \left(\int_{\mathbb{R}^d} \lambda^p |Df(x)|^p dx\right)^{\frac{1}{p}}. \end{aligned}$$

Making some change in variables: $y = \lambda x$ and $dy = \lambda^d dx$, then

$$\left(\int_{\mathbb{R}^d} |f(y)|^q dx\right)^{\frac{1}{q}} \lambda^{-\frac{d}{q}} \leq C \left(\int_{\mathbb{R}^d} |Df_\lambda(y)|^p dx\right)^{\frac{1}{p}} \lambda^{1-\frac{d}{p}}.$$

Therefore, we must have

$$-\frac{d}{q} = 1 - \frac{d}{p} \Longrightarrow q = p^*.$$

Before proving the theorem, we need a technical result

**Lemma 1.6.** *For $f \in C_c^1(\mathbb{R}^d)$,*

$$\|f\|_{\frac{d}{d-1}} \leq \prod_{i=1}^{d} \|f_{x_i}\|_1^{1/d}.$$

*Proof.* For $d = 2$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^2(x_1, x_2)dx_1dx_2 \leq \int_{-\infty}^{\infty} \max_{x_2\in\mathbb{R}} |f(x_1, x_2)|dx_1 \int_{-\infty}^{\infty} \max_{x_1\in\mathbb{R}} |f(x_1, x_2)|dx_2.$$

Since

$$|f(x_1, x_2)| = \left|\int_{-\infty}^{x_2} f_{x_2}(x_1, t_2)dt_2\right| \leq \int_{-\infty}^{\infty} |f_{x_2}(x_1, t_2)|dt_2,$$

it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^2(x_1, x_2)dx_1dx_2 \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f_{x_1}(x)|dx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f_{x_2}(x)|dx.$$

Thus, the assertion holds for $d = 2$. Suppose it still holds for some $d > 2$. Put

$$\overline{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \quad \text{for} \quad x = (x_1, x_2, \ldots, x_d, x_{d+1}) \in \mathbb{R}^{d+1}.$$

Then, by induction and Holder inequality,

$$\begin{aligned}
\|f\|_{\frac{d+1}{d}}^{\frac{d+1}{d}} &= \int_{\mathbb{R}}\int_{\mathbb{R}^d} |f(\overline{x}, x_{d+1})|^{\frac{d+1}{d}} d\overline{x}dx_{d+1} \\
&\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}^d} |f(\overline{x}, x_{d+1})|d\overline{x}\right]^{\frac{1}{d}} \left[\int_{\mathbb{R}^d} |f(\overline{x}, x_{d+1})|^{\frac{d}{d-1}} d\overline{x}\right]^{\frac{d-1}{d}} dx_{d+1} \\
&\leq \|f_{x_{d+1}}\|_1^{1/d} \int_{\mathbb{R}} \left[\int_{\mathbb{R}^d} |f(\overline{x}, x_{d+1})|^{\frac{d}{d-1}} d\overline{x}\right]^{\frac{d-1}{d}} dx_{d+1} \\
&\leq \|f_{x_{d+1}}\|_1^{1/d} \int_{\mathbb{R}} \prod_{i=1}^{d} \left[\int_{\mathbb{R}^d} |f(\overline{x}, x_{d+1})|d\overline{x}\right]^{\frac{1}{d}} dx_{d+1} \\
&\leq \|f_{x_{d+1}}\|_1^{1/d} \prod_{i=1}^{d} \left[\int_{\mathbb{R}^{d+1}} |f_{x_i}(\overline{x}, x_{d+1})|d\overline{x}dx_{d+1}\right]^{\frac{1}{d}} \\
&= \prod_{i=1}^{d+1} \|f_{x_i}\|_1^{1/d}.
\end{aligned}$$

□

By the Lemma, we have

$$\|f\|_{\frac{d}{d-1}} \leq \|Df\|_1. \tag{1.3}$$

*Proof of Theorem 1.5.* Set

$$\lambda = \frac{p(d-1)}{d-p}.$$

Then

$$\frac{\lambda d}{d-1} = \frac{(\lambda-1)p}{p-1} = p^*.$$

Therefore, by (1.3), we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^d} |f(x)|^{p^*} dx\right)^{\frac{d-1}{d}} &= \left(\int_{\mathbb{R}^d} |f(x)|^{\frac{\lambda d}{d-1}} dx\right)^{\frac{d-1}{d}} \\
&\leq \lambda \int_{\mathbb{R}^d} |f(x)|^{\lambda-1}|Df(x)|dx \\
&\leq \lambda \left(\int_{\mathbb{R}^d} |f(x)|^{\frac{(\lambda-1)p}{p-1}} dx\right)^{\frac{p-1}{p}} \|Df\|_p \\
&= \lambda \left(\int_{\mathbb{R}^d} |f(x)|^{p^*} dx\right)^{\frac{p-1}{p}} \|Df\|_p.
\end{aligned}$$

So

$$\|f\|_{p^*} \leq \lambda \|Df\|_p.$$

□

Define

$$W^{1,p}(U) = \{f : U \to \mathbb{R} \text{ such that } f, Df \in L^p(U)\}.$$

This is a Banach space with the norm

$$\|f\|_{W^{1,p}(U)} = \|f\|_p + \|Df\|_p.$$

**Theorem 1.7** (Sobolev Compactness Embedding Theorem)**.** *For $1 \leq p < d$, assume $\{f_n\} \subset W^{1,p}(U)$ such that $\|f_n\|_{W^{1,p}(U)} \leq C$ for all $n$. Then there exists a subsequence $\{f_{n_k}\}$ and a function $f \in W^{1,p}(U)$ satisfying*

$$\begin{cases} f_{n_k} \to f & \text{in } L^q(U) \text{ for all } 1 \leq q \leq p^*, \\ Df_{n_k} \rightharpoonup Df & \text{weakly in } L^p(U). \end{cases}$$

**Difference quotient.** Assume $u : U \to \mathbb{R}$ is locally summable function and $V \subset\subset U$. The $i$-th difference quotient of size $h$ is

$$D_i^h u(x) := \frac{u(x+he_i) - u(x)}{h}, \qquad i = 1, \dots, d$$

for $x \in V$, $h \in \mathbb{R}$, $0 < |h| < \operatorname{dist}(V, \partial U)$ and $(e_i)_{1\le i\le d}$ is the standard basis of $\mathbb{R}^d$. We denote

$$D^h u := (D_1^h u, \dots, D_d^h u).$$

**Theorem 1.8.** *(i). Suppose* $1 \le p < \infty$ *and* $u \in W^{1,p}(U)$*. Then there exists a constant* $C > 0$ *such that*

$$\|D^h u\|_{L^p(V)} \le C\|Du\|_{L^p(U)}$$

*for every* $0 < |h| < \frac{1}{2}\operatorname{dist}(V, U)$*.*
*(ii). Assume* $1 < p < \infty$*,* $u \in L^p(V)$ *and there exists a constant* $C, \epsilon > 0$ *such that*

$$\|D^h u\|_{L^p(V)} \le C$$

*for every* $0 < |h| < \epsilon$*. Then* $u \in W^{1,p}(V)$ *and*

$$\|Du\|_{L^p(V)} \le C.$$

*Proof. (i).* Since $C^\infty(U)$ is dense in $W^{1,p}(U)$, it suffices to prove the assertion when $u$ is smooth. For $x \in V$ and $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$, we have

$$|u(x+he_i) - u(x)| \le |h| \int_0^1 |u_{x_i}(x+the_i)|\,dt \le |h| \int_0^1 |Du(x+the_i)|\,dt.$$

Thus,

$$\begin{aligned} \|D^h u\|_{L^p(V)}^p = \int_V |D^h u|^p &\le C \int_V \int_0^1 |Du(x+the_i)|^p \,dtdx \\ &\le C \int_0^1 \int_U |Du(x)|^p \,dxdt \\ &= C\|Du\|_{L^p(U)}^p. \end{aligned}$$

*(ii).* Since $\{D^h u\}$, $0 < |h| < \epsilon$, is bounded in $L^p(V; \mathbb{R}^d)$, we can take a sequence $h_n \to 0$ such that

$$D^{-h_n} u \rightharpoonup v = (v_1, \dots, v_d) \qquad \text{weakly in } L^p(V; \mathbb{R}^d). \tag{1.4}$$

Fix $\varphi \in C_c^\infty(V)$, when $|h|$ is small enough such that

$$\operatorname{supp}\varphi - he_i \subset V,$$

we get

$$\begin{aligned}
\int_V u(x)\frac{\varphi(x+he_i)-\varphi(x)}{h}\,dx &= \frac{1}{h}\left(\int_V u(x)\varphi(x+he_i)\,dx - \int_V u(x)\varphi(x)\,dx\right)\\
&= \frac{1}{h}\left(\int_V u(x-he_i)\varphi(x)\,dx - \int_V u(x)\varphi(x)\,dx\right)\\
&= -\int_V \varphi(x)D_i^{-h}u(x)\,dx.
\end{aligned}$$

Choose $h = h_n$ as $n \to \infty$, recall (1.4) and use Lebesgue's dominated convergence theorem to obtain

$$\int_V u(x)\varphi_{x_i}(x)\,dx = -\int_V v_i(x)\varphi(x)\,dx.$$

This holds for arbitrary $\varphi \in H_0^1(V)$; so, $u \in W^{1,p}(V)$ and $u_{x_i} = v_i$. Also, in view of (1.4), we have

$$\|Du\|_{L^p(V)} = \|v\|_{L^p(V)} \le \liminf_{n\to\infty} \|D^{-h_n}u\|_{L^p(V)} \le C.$$

□

# 2 Calculus of variations

Minimize an energy functional

$$E[u] = \int_U F(Du(x))dx \tag{2.1}$$

for $u \in \mathcal{A} = \{v \in W^{1,p}(U), v = g \text{ on } \partial U\}$.
*Assumption:*

(A1) *Growth condition:* There exists $C > 0$ such that

$$\frac{1}{C}|s|^p - C \le F(s) \le C|s|^p + C.$$

(A2) *Convexity:* $s \mapsto F(s)$ is convex.

**Theorem 2.1.** *The minimizers for energy functional (2.1) exist.*

*Proof.* Since

$$-C|U| \le \int\limits_U \left(\frac{1}{C}|Du|^p - C\right) \le \int\limits_U F(Du) = E[u] \le \int\limits_U C(|Du|^p + 1) < \infty,$$

$\inf_{u\in\mathcal{A}} E[u]$ exists. We want to show that the infimum is actually the minimum. There exists a sequence $\{u_k\}$ such that

$$E[u_k] \to \inf_{u\in\mathcal{A}} E[u].$$

Note that

$$C \ge E[u_k] = \int\limits_U F(Du_k) \ge \int\limits_U \left(\frac{1}{C}|Du|^p - C\right),$$

which implies $\|Du_k\|_p \le C$. Since $u_k = g$ on $\partial U$, by Sobolev Embedding Theorem, $\|u_k\|_p \le C$ and, thus, $\|u_k\|_{W^{1,p}(U)} \le C$. By Theorem 1.7, we can extract $\{u_{k_j}\}$ such that

$$\begin{cases} u_{k_j} \to \bar{u} & \text{in } L^p(U), \\ Du_{k_j} \rightharpoonup D\bar{u} & \text{weakly in } L^p(U). \end{cases}$$

Since $E[u_{k_j}] \to \inf E[u]$, it is sufficient to show

$$\liminf_{j\to\infty} E[Du_{k_j}] \ge E[D\bar{u}].$$

By the convexity assumption,

$$F(Du_{k_j}) \ge F(D\bar{u}) + DF(D\bar{u}) \cdot D(u_{k_j} - \bar{u}).$$

Therefore,

$$\begin{aligned} E[u_{k_j}] = \int\limits_U F(Du_{k_j}) &\ge \int\limits_U F(D\bar{u}) + \int\limits_U DF(D\bar{u}) \cdot D(u_{k_j} - \bar{u}) \\ &= E[\bar{u}] + \int\limits_U DF(D\bar{u}) \cdot D(u_{k_j} - \bar{u}). \end{aligned}$$

Since $|DF(D\bar{u})| \le C(|D\bar{u}|^{p-1} + 1)$, we have that $DF(D\bar{u}) \in L^q(U, \mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, as $Du_{k_j} \rightharpoonup D\bar{u}$,

$$\int\limits_U DF(D\bar{u}) \cdot D(u_{k_j} - \bar{u}) \to 0.$$

Consequently, at the limit,

$$\liminf_{j\to\infty} E[u_{k_j}] \ge E[\bar{u}].$$

□

*Remark.* 1. The same result holds for

$$E[u] = \int_U F(x, u(x), Du(x))dx.$$

2. This is only for real-valued function. For vector-valued function, there are still a lot of open problems.

*Question.* When the minimizers exist, do they satisfy some properties ?

Let $E[\bar{u}] = \min_{u \in \mathcal{A}} E[u]$. For any $v \in W_0^{1,p}(U)$, then $\bar{u} + tv \in \mathcal{A}$ for any $t \in \mathbb{R}$. Thus,

$$E[\bar{u}] = \min_{t \in \mathbb{R}} E[\bar{u} + tv] =: i(t).$$

The function $i(t)$ attains the minimum at $t = 0$, so

$$\begin{cases} i'(0) = 0 & \text{the first variation} \\ i''(0) \geq 0 & \text{the second variation.} \end{cases} \tag{2.2}$$

**The first variation.** We have

$$i'(t) = \frac{d}{dt} \int_U F(D\bar{u} + tDv) = \int_U DF(D\bar{u} + tDv) \cdot Dv.$$

By (2.2),

$$i'(0) = \int_U DF(D\bar{u}) \cdot Dv = 0.$$

$$\left( \text{so, } \int_U -\text{div}\,(DF(D\bar{u}))v = 0. \right)$$

Since $v$ is arbitrarily chosen from $W_0^{1,p}(U)$, we conclude that $\bar{u}$ is a weak solution of the Euler-Lagrange equation

$$-\text{div}\,(DF(D\bar{u})) = 0 \qquad \text{in } U.$$

*Question.* Suppose $f \in L^1(U)$ such that

$$\int_U f(x)\varphi(x)dx = 0$$

for all $\varphi \in C_c(U)$. Prove that $f = 0$ a.e in U.

*Answer:* Since $C_c(U)$ is dense in $L^1(U)$, let $(\varphi_n) \subset C_c(U)$ be a sequence converging in $L^1(U)$ to $f$. Take a subsequence $(\varphi_{n_k})$ converging almost everywhere to $f$. Then, by Lebesgue Dominated Convergence theorem, we have

$$0 = \int_U f(x) \frac{\varphi_{n_k}(x)}{1 + \varphi_{n_k}^2(x)} dx \to \int_U \frac{f^2(x)}{1 + f^2(x)} dx.$$

So $f = 0$ a.e. □

*Example* 2. When $F(s) = |s|^2$, $DF(s) = 2s$, $E[u] = \int_U |Du|^2$, the Euler-Lagrange equation becomes the Laplace equation:

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

*Example* 3. When $F(s) = |s|^p$, $DF(s) = p|s|^{p-2}s$, $E[u] = \int_U |Du|^p$, the Euler-Lagrange equation becomes the p-Laplace equation:

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) = 0 & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

**The second variation.** We have

$$i''(t) = \int_U \sum_{i,j=1}^d F_{s_i s_j}(D\bar{u} + tDv) v_{x_i} v_{x_j}.$$

Thus, (2.2) implies

$$i''(0) = \int_U \sum_{i,j=1}^d F_{s_i s_j}(D\bar{u}) v_{x_i} v_{x_j} \geq 0. \tag{2.3}$$

Take

$$v(x) = \varepsilon\, \xi(x)\, v_0\left(\frac{r \cdot x}{\varepsilon}\right)$$

where $\varepsilon > 0$, $\xi \in C_c^\infty(U)$, $r = (r_1, r_2, \ldots, r_d) \in \mathbb{R}^d$ and $v_0$ is the 2-periodic saw-tooth function on $\mathbb{R}$ defined by

$$v_0(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1, \\ 2 - x & \text{for } 1 \leq x \leq 2. \end{cases}$$

Then,

$$v_{x_i} = \varepsilon\, \xi_{x_i}(x)\, v_0\left(\frac{r \cdot x}{\varepsilon}\right) + \xi(x)\, v_0'\left(\frac{r \cdot x}{\varepsilon}\right) s_i.$$

Plug $v$ into (2.3) and send $\varepsilon \to 0$ to obtain

$$\int_U \sum_{i,j=1}^d F_{s_i s_j}(D\bar{u}) \xi^2 r_i r_j \geq 0.$$

Since $\xi$ arbitrarily belongs to $C_c^\infty(U)$, we conclude that

$$\sum_{i,j=1}^{d} F_{s_i s_j}(D\bar{u}) r_i r_j \geq 0 \qquad \text{for all } s \in \mathbb{R}^d.$$

*Remark.* 1. For real-valued $u$, convexity is natural.

2. For vector-valued $u$, it is completely different.

**Theorem 2.2.** *$E[\cdot]$ is lower semi-continuous with respect to weak convergence in $W^{1,p}(U)$ if and only if the convexity assumption (A2) holds.*

*Proof.* The "$\Longleftarrow$" is clear by the proof of Theorem 2.1.
For the "$\Longrightarrow$", assume $E[\cdot]$ is lower semi-continuous w.r.t weak convergence in $W^{1,p}(U)$, that is, for any $\{v_i\}$ such that $Dv_j \rightharpoonup Dv$ in $L^p(U)$, we have

$$\liminf_{j\to\infty} E[v_j] \geq E[v].$$

First, we assume $U = (-1,1)^n$. Divide $U$ into $k^n$ subcubes. For each subcube $Q_l$, let $x_l$ be its center. Take

$$\xi \in C_c^\infty\left(\left(\frac{-1}{2}, \frac{1}{2}\right)^n\right) \qquad \Longrightarrow \qquad \int\limits_{\left(\frac{-1}{2},\frac{1}{2}\right)^n} D\xi(x)dx = 0.$$

Choose $s \in \mathbb{R}^d$ and put

$$v_k(x) = s \cdot x + \frac{1}{k} \sum_l \xi(k(x - x_l)).$$

Then,

$$Dv_k(x) = s + \sum_l D\xi(k(x - x_l)).$$

Because of the "crazy" oscillation of $\sum_l D\xi(k(x - x_l))$ around 0 as $k \to \infty$, we have $Dv_k \rightharpoonup s$ weakly in $L^p(U)$. Thus, by the assumption,

$$\liminf_{k\to\infty} \int\limits_U F(Dv_k) \geq \int\limits_U F(s) = F(s)|U|.$$

On the other hand, for any $k$,

$$\int\limits_U F(Dv_k(x))dx = \sum_l \int\limits_{Q_l} F(s + D\xi(k(x - x_l)))dx = \int\limits_{\left(\frac{-1}{2},\frac{1}{2}\right)^n} F(s + D\xi(x))dx.$$

Therefore, with $Q = \left(\frac{-1}{2}, \frac{1}{2}\right)^n$,

$$\int\limits_Q F(s + D\xi(x))dx \geq \int\limits_U F(s) \geq \int\limits_Q F(s) = F(s)|Q|.$$

Let $v_0$ be the 2-periodic saw-tooth function on $\mathbb{R}$ defined by

$$v_0(x) = \begin{cases} x & \text{for } 0 \le x \le 1, \\ 2 - x & \text{for } 1 \le x \le 2. \end{cases}$$

Take $\varepsilon > 0$, $r \in \mathbb{R}^d$, $\zeta \in C_c^\infty(Q)$ and choose

$$\xi(x) = \varepsilon \zeta(x) v_0 \left( \frac{r \cdot x}{\varepsilon} \right).$$

Also, put

$$i(t) = \int_Q F(s + tDv(x)) dx.$$

Since $t = 0$ is the minimizer for $i$, we must have $i''(0) \ge 0$. Sending $\varepsilon \to 0$, as in page 12, we get

$$\int_Q \sum_{i,j=1}^n F_{x_i x_j}(p) \zeta^2(x) r_i r_j \, dx \ge 0$$

which implies

$$\sum_{i,j=1}^n F_{x_i x_j}(s) r_i r_j \ge 0.$$

Hence, $F$ is convex. The theorem is done when $U$ is a cube. For a general open, bounded set $U$ with smooth boundary, we just need to pick an inside cube and apply the above argument to deduce the result. $\square$

We can use the more direct approaching: Since $U$ is bounded and has a smooth boundary, we can approximate $U$ by a countable family of disjoint inside cubes. Suppose $E_1, E_2, \dots \subset U$ are disjoint cubes such that

$$U = \bigcup_{i=1}^\infty E_i.$$

Fix $s \in \mathbb{R}^d$. For an integer $k$, divide $E_k$ to $m^d$ subcubes and let $x_l^{(k,m)}$ be the center of each subcube. Take $\xi \in C_c^\infty(U)$ and for $x \in E_k$ belonging to the subcube $l$, let

$$w_m^{(k)}(x) = \frac{1}{m} \sum_l \xi \left( m \left( x - x_l^{(k,m)} \right) \right).$$

Then, set

$$v_k(x) = s \cdot x + \sum_{i=1}^k \chi_{E_i}(x) w_k^{(k)}(x).$$

Then $Dv_k \rightharpoonup s$ weakly in $L^p(U)$. Moreover

$$\lim_{k\to\infty} E[v_k] = \lim_{k\to\infty} \int_U F(D(v_k)) = \lim_{k\to\infty} \int_{\bigcup_{i=1}^k E_i} F(p + D\xi) = \int_U F(p + D\xi).$$

Therefore, by the semicontinuity, we have

$$\int_U F(s + D\xi) \geq F(s)|U|.$$

# 3 Galerkin method in elliptic PDEs

From now on, $U$ is a bounded open subset of $\mathbb{R}^d$ with smooth boundary. Consider a differential operator

$$Lu = \sum_{i,j=1}^{d} a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu$$

where $a_{ij}, b_i, c$ belong to $L^\infty(U)$. We also assume $L$ satisfies ellipticity condition, that is

$$\sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j \geq \theta |\xi|^2$$

for all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and for some constant $\theta > 0$. We are interested in the elliptic boundary equation

$$\begin{cases} Lu + \lambda u = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{3.1}$$

where $f \in L^2(U)$ and $\lambda$ is a constant. A function $u \in H_0^1(U)$ is called *weak solution* of (3.1) if it satisfies

$$\int_U \sum_{i,j=1}^{d} a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} v + cuv \, dx + \lambda \int_U uv \, dx = \int_U fv \, dx \qquad \text{for all } v \in H_0^1(U).$$

For simplicity, define the bilinear form from $H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ as

$$B[u,v] = \int_U \sum_{i,j=1}^{d} a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{d} b_i u_{x_i} v + cuv \, dx.$$

In this section, we will use Galerkin method to show that, under the ellipticity condition and with appropriate constant $\lambda \geq 0$, equation (3.1) always has an unique weak solution. To do this, we need some estimates:

**Theorem 3.1** (Energy estimate). *There are $\alpha > 0$ and $\beta \geq 0$ such that*

$$\alpha \|u\|^2_{H_0^1(U)} \leq B[u,u] + \beta \|u\|_{L^2(U)} \tag{3.2}$$

*for all $u \in H_0^1(U)$.*

*Proof.* From the ellipticity condition,

$$\theta \int_U |Du|^2 \, dx \le B[u,u] - \left( \int_U \sum_{i=1}^d b_i u_{x_i} u + c|u|^2 \, dx \right)$$
$$\le B[u,u] + \sum_{i=1}^d \|b_i\|_{L^\infty(U)} \int_U |Du||u| \, dx + \|c\|_{L^\infty(U)} \int_U |u|^2 \, dx.$$

By Cauchy's inequality, we have

$$\int_U |Du||u| \, dx \le \frac{\theta}{2\sum \|b_i\|_{L^\infty(U)}} \int_U |Du|^2 \, dx + \frac{\sum \|b_i\|_{L^\infty(U)}}{2\theta} \int_U |u|^2 \, dx.$$

Consequently,

$$\frac{\theta}{2} \int_U |Du|^2 \, dx \le B[u,u] + \left( \|c\|_{L^\infty(U)} + \frac{1}{2\theta} \sum_{i=1}^d \|b_i\|_{L^\infty(U)} \right) \int_U |u|^2 \, dx.$$

□

Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $H_0^1(U)$ and an orthogonal basis of $L^2(U)$. The idea of Galerkin method is to find "projectional solutions" of (3.1) on finite dimensional subspaces spanned by $(e_k)$; then pass to limit to obtain the desire solution.

Indeed, we look for appropriate real numbers $d_n^1, \dots, d_n^n$ such that the function of the form

$$u_n = d_n^1 e_1 + d_n^2 e_2 + \cdots + d_n^n e_n \in \operatorname{span}\{e_1, e_2, \dots, e_n\}$$

satisfies

$$B[u_n, e_k] + \lambda \int_U u_n e_k = \int_U f e_k \qquad \text{for } k = 1, \dots, n \tag{3.3}$$

or equivalently,

$$\sum_{l=1}^n \left( B[e_l, e_k] + \lambda \int_U e_l e_k \right) d_n^l = \int_U f e_k \qquad \text{for } k = 1, \dots, n. \tag{3.4}$$

In order to verify existence of such real numbers, put

$$\alpha_{ij} = B[e_j, e_i] + \lambda \int_U e_j e_i$$

and let $A = (\alpha_{ij})$ be a square matrix of size $n \times n$. Then (3.4) can be rewritten as

$$AD = P \tag{3.5}$$

where

$$D = \begin{pmatrix} d_n^1 \\ d_n^2 \\ \vdots \\ d_n^n \end{pmatrix} \qquad \text{and} \qquad P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix},$$

with

$$p_k = \int_U f e_k \qquad \text{for } k = 1, \dots, n.$$

Recall $\alpha$ and $\beta$ as in Theorem 3.1. If $\lambda \geq \beta$, then $A$ is invertible since

$$\begin{aligned} D \in \ker A \Longleftrightarrow AD = 0 &\Longrightarrow D^T AD = 0 \\ &\Longrightarrow B[u_n, u_n] + \lambda \int |u_n|^2 = 0 \\ &\Longrightarrow \alpha \|u_n\|^2_{H_0^1(U)} = 0 \qquad \text{(follows from (3.2))} \\ &\Longrightarrow u_n = 0 \\ &\Longrightarrow D = 0. \end{aligned}$$

Hence, when $\lambda \geq \beta$, $D = A^{-1}P$ satisfies (3.5) and therefore solves (3.3).

**Theorem 3.2** (Existence and uniqueness)**.** *Let $\alpha$ and $\beta$ be as in Theorem 3.1. Then for every $\lambda \geq \beta$, there exists unique weak solution for the boundary equation (3.1).*

*Proof. Existence.* Let $u_n \in \operatorname{span}\{e_1, e_2, \dots, e_n\}$ satisfying (3.3). It follows from (3.2), (3.3) and Poincare's inequality that

$$\alpha \|u_n\|^2_{H_0^1(U)} \leq B[u_n, u_n] + \lambda \int_U |u_n|^2 = \int_U f u_n \leq C \|u_n\|_{H_0^1(U)},$$

for some constant $C$, which implies the boundedness of $(u_n)$ in $H_0^1(U)$. Consequently, there exist a subsequence $(u_{n_l}) \subset (u_n)$ and a function $u \in H_0^1(U)$ such that

$$u_{n_l} \rightharpoonup u \quad \text{weakly in } H_0^1(U).$$

For any $k \in \mathbb{N}$ and $n_l \geq k$, from (3.3) we have

$$B[u_{n_l}, e_k] + \lambda \int_U u_{n_l} e_k = \int_U f e_k.$$

Passing to the limit as $l \to \infty$, the weak convergence of $(u_{n_l})$ to $u$ yields

$$B[u, e_k] + \lambda \int_U u e_k = \int_U f e_k \qquad \text{for all } k \in \mathbb{N}.$$

Since $(e_k)$ is dense in $H_0^1(U)$, we deduce that $u$ is a weak solution of (3.1).

*Uniqueness.* Suppose $u, v \in H_0^1(U)$ are two weak solutions of (3.1). Set $w = u - v$, then

$$B[w, \varphi] + \lambda \int_U w\varphi = 0 \qquad \text{for any } \varphi \in H_0^1(U).$$

In particular, choose $\varphi = w$ and recall (3.2) to conclude that $w = 0$. □

# 4 Monotonicity method in nonlinear PDEs

## 4.1 Quasilinear PDEs: existence, uniqueness & regularity of solution. Minty-Browder trick in $L^2$

In this section, we study the quasilinear equation

$$\begin{cases} -\operatorname{div} a(Du) = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{4.1}$$

where $f \in L^2(U)$ and $a : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector field. Also, $a$ has some properties:

**i.** *Monotonicity*:

$$(a(p) - a(q)) \cdot (p - q) \geq 0 \tag{4.2}$$

for all $p, q \in \mathbb{R}^d$.

**ii.** *Growth bound*:

$$|a(p)| \leq C(1 + |p|) \tag{4.3}$$

for all $p \in \mathbb{R}^d$ and a constant $C$.

**iii.** *Coercivity*:

$$a(p) \cdot p \geq \alpha |p|^2 - \beta \tag{4.4}$$

for all $p \in \mathbb{R}^d$ and constants $\alpha > 0$, $\beta \geq 0$.

If $a(p) = DF(p)$ for some smooth function $F : \mathbb{R}^d \to \mathbb{R}$, (4.1) becomes Euler-Lagrange equation

$$\begin{cases} -\operatorname{div} DF(Du) = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{4.5}$$

In section 2, the existence solution of (4.5) could be derived by minimizing the energy functional

$$u \mapsto \int_U F(Du) \tag{4.6}$$

in $H_0^1(U)$. Furthermore, the minimizer of (4.6) exists if and only if $F$ is convex. In this case,

$$(a(p) - a(q)) \cdot (p - q) = (DF(p) - DF(q)) \cdot (p - q) \geq 0.$$

Thus, the assumption (4.2) is naturally needed.
We will prove the existence of weak solution of (4.1), that is, $u \in H_0^1(U)$ and satisfies

$$\int_U a(Du) \cdot D\varphi = \int_U f\varphi \qquad \text{for every } \varphi \in H_0^1(U). \tag{4.7}$$

To do this, we again use the Galerkin approaching. The point is that our problem now is nonlinear. In section 3, the results heavily depend on linear structures. However, surprisingly, the Galerkin approaching can still be useful in this case, thanks to the monotonicity. To see this, we begin with a lemma

**Lemma 4.1.** *Suppose $v : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector-valued function satisfying*

$$v(x) \cdot x \geq 0 \qquad \text{whenever } |x| = r$$

*for a constant $r > 0$. Then there exists $x_0 \in B'(0, r)$ such that $v(x_0) = 0$.*

*Proof.* Assume, by contradiction, that $v(x) \neq 0$ for all $|x| \leq r$. Put

$$w(x) = -\frac{r}{|v(x)|} v(x).$$

Then $w : B'(0, r) \to \partial B'(0, r)$. By Brouwer's fixed point theorem (Theorem A.3), there exists $y \in B'(0, r)$ such that $w(y) = y$. But then, $|y| = r$ and

$$r^2 = w(y) \cdot y = -\frac{r}{|v(y)|} v(y) \cdot y \leq 0,$$

which is impossible. □

Let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal basis of $H_0^1(U)$. As an idea of Galerkin method, our aim is to find functions of the form

$$u_n = d_n^1 e_1 + d_n^2 e_2 + \cdots + d_n^n e_n \tag{4.8}$$

satisfying

$$\int_U a(Du_n) \cdot De_k = \int f e_k \qquad \text{for any } k = 1, \ldots, n. \tag{4.9}$$

In other words, $u_n$ is the "projectional solution" of (4.1) on the finite dimensional subspace spanned by $\{e_1, \ldots, e_n\}$.

**Theorem 4.2.** *For every $n \in \mathbb{N}$, there exists $u_n$ as in (4.8) satisfies (4.9).*

*Proof.* Put $v = (v^1, \ldots, v^n)$ where

$$v^k(d) = \int_U a\left(\sum_{i=1}^n d^i De_i\right) \cdot De_k \, dx - \int_U f e_k \, dx, \tag{4.10}$$

for $d = (d^1, \ldots, d^n) \in \mathbb{R}^n$. Then employ (4.4) to have

$$\begin{aligned} v(d) \cdot d &= \int_U a\left(\sum_{i=1}^n d^i De_i\right) \cdot \left(\sum_{i=1}^n d^i De_i\right) dx - \int_U f \sum_{i=1}^n d^i e_i \, dx \\ &\geq \int_U \alpha \left|\sum_{i=1}^n d^i De_i\right|^2 - \beta - f \sum_{i=1}^n d^i e_i \, dx \\ &= \alpha |d|^2 - \beta |U| - \sum_{i=1}^n d^i \int_U f e_i \, dx \\ &\geq \alpha |d|^2 - C(|d| + 1). \end{aligned}$$

Taking $r > 0$ large enough, above inequalities implies $v(d) \cdot d \geq 0$ for all $|d| = r$. According to Lemma 4.1, there exists $d_n = (d_n^1, \cdots, d_n^n)$ such that $v(d_n) = 0$. With such $d_n$, it follows from (4.10) that $u_n$ as in (4.8) satisfies (4.9). □

**Theorem 4.3** (Energy estimate)**.**

$$\|u_n\|_{H_0^1(U)} \leq C(1 + \|f\|_{L^2(U)})$$

*for every $n \in \mathbb{N}$ and a constant $C$.*

*Proof.* It follows from (4.9) that

$$\int_U a(Du_n) \cdot Du_n = \int_U f u_n.$$

Hence, the coercivity (4.4), Poincare's theorem and Cauchy's inequality yield

$$\begin{aligned} \alpha \|u_n\|_{H_0^1(U)}^2 = \alpha \int_U |Du_n|^2 \, dx &\leq \int_U f u_n \, dx + C \\ &\leq C + C\|f\|_{L^2(U)} \|u_n\|_{H_0^1(U)} \\ &\leq C + \frac{\alpha}{2}\|u_n\|_{H_0^1(U)}^2 + \frac{C^2}{2\alpha}\|f\|_{L^2(U)}^2, \end{aligned}$$

which is equivalent to

$$\frac{\alpha}{2}\|u_n\|_{H_0^1(U)}^2 \leq C + \frac{C^2}{2\alpha}\|f\|_{L^2(U)}^2.$$

□

Similar to section 3, we want to obtain the solution $u$ of (4.1) upon passing to the limit of $(u_n)$ as $n \to \infty$ in some sense of convergence. In this case, weak convergence is still helpful with assistance of monotonicity and Browder-Minty's trick.

**Theorem 4.4** (Existence and uniqueness)**.** *There exists a weak solution for (4.1). In addition, if a is strictly monotone, that is, there exists $\gamma > 0$ such that*

$$(a(p) - a(q)) \cdot (p - q) \geq \gamma |p - q|^2 \tag{4.11}$$

*for every $p, q \in \mathbb{R}^d$, then the solution is unique.*

*Proof. Existence.* According to Theorem 4.3, since $(u_n)$ is bounded in $H_0^1(U)$, we can take a subsequence $(u_{n_l}) \subset (u_n)$ such that

$$u_{n_l} \rightharpoonup u \qquad \text{weakly in } H_0^1(U). \tag{4.12}$$

Also, in view of growth bound (4.3), $a(Du_n)$ is bounded in $L^2(U; R^d)$. Hence, by taking further subsequence if necessary, we assume

$$a(u_{n_l}) \rightharpoonup \zeta \qquad \text{weakly in } L^2(U; R^d). \tag{4.13}$$

By monotonicity (4.2), we have

$$\int_U (a(Du_{n_l}) - a(Dv)) \cdot (Du_{n_l} - Dv) \geq 0 \tag{4.14}$$

for any $l \in \mathbb{N}$ and $v \in H_0^1(U)$. From (4.9), we get

$$\int_U a(Du_{n_l}) \cdot Du_{n_l} = \int_U f u_{n_l}.$$

So, (4.14) is equivalent to

$$\int_U f u_{n_l} - a(Du_{n_l}) \cdot Dv - a(Dv) \cdot (Du_{n_l} - Dv) \geq 0.$$

Let $l \to \infty$ and invoke (4.12), (4.13) to obtain

$$\int_U f u - \zeta \cdot Dv - a(Dv) \cdot (Du - Dv) \geq 0. \tag{4.15}$$

Moreover, in (4.9), set $n = n_l$, send $l \to \infty$ and recall (4.13) to yield

$$\int_U \zeta \cdot De_k = \int_U f e_k \qquad \text{for all } k = 1, 2, \ldots$$

and, consequently,

$$\int_U \zeta \cdot Dv = \int_U fv \qquad \text{for every } v \in H_0^1(U). \tag{4.16}$$

Taking $v = u$ in (4.16) and substituting into (4.15):

$$\int_U (\zeta - a(Dv)) \cdot (Du - Dv) \geq 0 \qquad \text{for every } v \in H_0^1(U). \tag{4.17}$$

TRICK: Fix $w \in H_0^1(U)$ and put $v = u - \lambda w$ with $\lambda > 0$, in (4.17), we have

$$\int_U (\zeta - a(Du - \lambda Dw)) \cdot Dw \geq 0.$$

Sending $\lambda \to 0$, by Lebesgue's dominated convergence theorem,

$$\int_U (\zeta - a(Du)) \cdot Dw \geq 0. \tag{4.18}$$

Taking $-w$ in lieu of $w$, the inverse inequality of (4.18) also holds; thus, (4.18) is actually the equality. By (4.16), we obtain

$$\int_U a(Du) \cdot Dw = \int_U fw.$$

Since $w$ is arbitrary in $H_0^1(U)$, we conclude that $u$ is a weak solution for (4.1).
*Uniqueness.* Suppose $u$ and $\overline{u}$ are both solutions of (4.1). Then

$$\int_U (a(Du) - a(D\overline{u})) \cdot Dv = 0 \qquad \text{for every } v \in H_0^1(U).$$

In particular,

$$\int_U (a(Du) - a(D\overline{u})) \cdot (Du - D\overline{u}) = 0.$$

From (4.11), we deduce that $u = \overline{u}$. $\square$

The argument used in the proof of Theorem 4.4 yields the following:

**Theorem 4.5.** *Assume that $\{u_k\} \in H_0^1(U)$ and $f_k \in L^2(U)$ such that*

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in } H_0^1(U), \\ f_k \to f & \text{strongly in } L^2(U). \end{cases}$$

*Assume further more that $u_k$ solves*

$$\begin{cases} -\operatorname{div}(a(Du_k)) = f_k & \text{in } U, \\ u_k = 0 & \text{on } \partial U, \end{cases} \tag{4.19}$$

*then, $u$ solves (4.1)*

*Proof.* We have

$$\int_U a(Du_k) \cdot D\varphi = \int_U f_k \varphi \qquad \forall \varphi \in H_0^1(U).$$

For all $v \in H_0^1(U)$,

$$\begin{aligned} 0 &\le \int_U (a(Du_k) - a(Dv)) \cdot (Du_k - Dv) \\ &= \int_U a(Du_k) \cdot D(u_k - v) - \int_U a(Dv) \cdot D(u_k - v) \\ &= \int_U f_k(u_k - v) - \int_U a(Dv) \cdot D(u_k - v). \end{aligned}$$

Letting $k \to \infty$ and reminding that the convergence of $\{f_k\}$ is in the strong topology, we deduce

$$\int_U f(u-v) - \int_U a(Dv) \cdot D(u-v) \ge 0.$$

Now, apply the trick as in the proof of Theorem 4.4 to complete the proof. □

Under the strict monotonicity assumption (4.11), the unique solution $u$ obtained from Theorem 4.4 is actually in $H^2_{\text{loc}}(U)$ and therefore satisfies

$$-\operatorname{div} a(Du) = f \qquad \text{almost everywhere in } U.$$

**Theorem 4.6** ($H^2$ regularity)**.** *Suppose a is strictly monotone as in (4.11). Then the unique weak solution of (4.1) belongs to* $H^2_{\text{loc}}(U)$*.*

*Proof.* Fix an open set $V \subset\subset U$ and select other open set $W$ such that $V \subset\subset W \subset\subset U$. According to Urysohn's lemma, there is a smooth function $\zeta$ satisfying

$$\begin{cases} \zeta = 1 & \text{in } V \\ \zeta = 0 & \text{outside } W \\ 0 \le \zeta \le 1. \end{cases}$$

Set

$$\varphi = -D_k^{-h}(\zeta^2 D_k^h u) \in H_0^1(U)$$

for some sufficiently small $|h|$. Let $u$ be the weak solution of (4.1). Then, by definition,

$$\int_U a(Du) \cdot D\varphi = \int_U f\varphi. \tag{4.20}$$

Write

$$
\begin{aligned}
\int_U a(Du)\cdot D\varphi &= -\int_U a(Du)\cdot D_k^{-h} D(\zeta^2 D_k^h u) \\
&= \int_U D_k^h a(Du)\cdot D(\zeta^2 D_k^h u) \\
&= \int_U \zeta^2 \left(D_k^h a(Du)\cdot D_k^h Du\right) + \int_U D_k^h a(Du)\cdot \left(D_k^h u 2\zeta D\zeta\right).
\end{aligned}
\tag{4.21}
$$

From the strict monotonicity condition (4.11), we have

$$
\begin{aligned}
D_k^h a(Du(x))\cdot D_k^h Du(x) &= \frac{a(Du(x+he_k)) - a(Du(x))}{h}\cdot\frac{Du(x+he_k)-Du(x)}{h} \\
&\geq \frac{1}{h^2}\gamma |Du(x+he_k)-Du(x)|^2 \\
&= \gamma \left|D_k^h Du(x)\right|^2
\end{aligned}
$$

thereby obtaining

$$
\int_U \zeta^2 \left(D_k^h a(Du)\cdot D_k^h Du\right) \geq \gamma \int_U \zeta^2 \left|D_k^h Du\right|^2. \tag{4.22}
$$

Moreover, rewrite

$$
\int_U D_k^h a(Du)\cdot \left(D_k^h u 2\zeta D\zeta\right) = -\int_U a(Du)\cdot D_k^{-h}\left(D_k^h u 2\zeta D\zeta\right). \tag{4.23}
$$

By Cauchy's inequality,

$$
\left|\int_U a(Du)\cdot D_k^{-h}\left(D_k^h u 2\zeta D\zeta\right)\right| \leq \frac{1}{4\varepsilon}\int_U |a(Du)|^2 + \varepsilon \int_U \left|D_k^{-h}\left(D_k^h u 2\zeta D\zeta\right)\right|^2. \tag{4.24}
$$

According to Theorem 1.8,

$$
\begin{aligned}
\int_U \left|D_k^{-h}\left(D_k^h u 2\zeta D\zeta\right)\right|^2 &= \int_U \sum_{i=1}^d \left|D_k^{-h}(D_k^h u 2\zeta\zeta_{x_i})\right|^2 \\
&\leq C\int_U \sum_{i=1}^d \left|D(D_k^h u 2\zeta\zeta_{x_i})\right|^2 \\
&= C\int_U \sum_{i=1}^d \left|(D_k^h Du)2\zeta\zeta_{x_i} + (D_k^h u)D(2\zeta\zeta_{x_i})\right|^2 \\
&\leq C\int_U \zeta^2\left|D_k^h Du\right|^2 + C\int_W \left|D_k^h u\right|^2 \\
&\leq C\int_U \zeta^2\left|D_k^h Du\right|^2 + C\int_U |Du|^2
\end{aligned}
\tag{4.25}
$$

for some constant $C > 0$. Put (4.21)-(4.25) together, we have

$$\int_U a(Du)\cdot D\varphi \geq \gamma \int_U \zeta^2 \left|D_k^h Du\right|^2 - \frac{1}{4\varepsilon}\int_U |a(Du)|^2 - C\varepsilon \int_U \zeta^2\left|D_k^h Du\right|^2 - C\varepsilon \int_U |Du|^2.$$

Choose $\varepsilon = \frac{\gamma}{2C}$ and recall growth bound (4.3) to obtain

$$\begin{aligned}\left|\int_U a(Du)\cdot D\varphi\right| &\geq \frac{\gamma}{2}\int_U \zeta^2\left|D_k^h Du\right|^2 - C\int_U |a(Du)|^2 - C\int_U |Du|^2 \\ &\geq \frac{\gamma}{2}\int_U \zeta^2\left|D_k^h Du\right|^2 - C\int_U |Du|^2 - C.\end{aligned} \tag{4.26}$$

In order to estimate the right side of (4.20), apply Theorem 1.8 again to have

$$\begin{aligned}\int_U |\varphi|^2 \leq C\int_U \left|D(\zeta^2 D_k^h u)\right|^2 &\leq C\int_W \left|D_k^h u\right|^2 + C\int_W \zeta^2\left|D_k^h Du\right|^2 \\ &\leq C\int_U |Du|^2 + C\int_U \zeta^2\left|D_k^h Du\right|^2.\end{aligned}$$

Therefore, by Cauchy's inequality,

$$\left|\int_U f\varphi\right| \leq \varepsilon\int_U \zeta^2\left|D_k^h Du\right|^2 + \frac{C}{\varepsilon}\int_U f^2. \tag{4.27}$$

Choosing $\varepsilon = \frac{\gamma}{4}$ in (4.27) and combining (4.20), (4.26) yields

$$\int_V \left|D_k^h Du\right|^2 \leq \int_U \zeta^2\left|D_k^h Du\right|^2 \leq C\int_U |Du|^2 + C\int_U f^2 + C,$$

for $k = 1, 2, \ldots, d$. Hence, by Theorem 1.8, we deduce that $u \in H^2(V)$. Since $V \subset\subset U$ is arbitrary, we conclude $u \in H^2_{\text{loc}}(U)$. □

## 4.2 Minty-Browder trick in $L^\infty$

We study the fully nonlinear PDE:

$$\begin{cases} F(D^2u) = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{4.28}$$

Here, $D^2u$ is the Hessan matrix. If $u \in C^2(U)$, then $D^2u \in \mathcal{S}^d$ - the set of all symmetric matrices of size $d$. Also, $F : \mathcal{S}^d \to \mathbb{R}$ is a function. We assume that $F$ is (degenerate) elliptic, that is, for $S, R \in \mathcal{S}^d$, then

$$S \geq R \qquad \Longrightarrow \qquad F(S) \leq F(R).$$

By saying $S \geq R$, we mean $P^T(S-R)P \geq 0$ for all column vector $P \in \mathbb{R}^d$.

*Example* 4. Set $F(S) = -\operatorname{trace} S$, then (4.28) becomes

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases}$$

*Example* 5. Set $F(S) = \max\{-\operatorname{trace} S; -\frac{1}{2}\operatorname{trace} S\}$, then (4.28) becomes

$$\begin{cases} \max\{-\Delta u; -\frac{1}{2}\Delta u\} = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases}$$

*Example* 6. Assume $u$ is convex in $U$ and $F(S) = -\det S$, (4.28) becomes the Monge-Ampere equation:

$$\begin{cases} -\det(D^2 u) = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases}$$

**Theorem 4.7.** *Assume that $u_k$ solves*

$$\begin{cases} F(D^2 u_k) = f_k & \text{in } U \\ u_k = 0 & \text{on } \partial U. \end{cases}$$

*and that*

$$\begin{cases} \|u_k\|_{L^\infty(\bar{U})} + \|Du_k\|_{L^\infty(\bar{U})} + \|D^2 u_k\|_{L^\infty(\bar{U})} \le C. \\ u_k \to u, \quad Du_k \rightharpoonup Du \quad \text{in } C(\bar{U}) \\ D^2 u_k \overset{*}{\rightharpoonup} D^2 u \quad \text{in } L^\infty(U) \\ f_k \to f \quad \text{in } C(\bar{U}). \end{cases}$$

*Then, $u$ solves (4.28).*

*Note:* We do not have the variational structure as $F$ is not linear. We also do not have the $L^2$ structure.

**Pairings in Banach spaces.** Let $X$ be a real Banach space, how do we define $[f, g]$ for $f, g \in X$ ? When $X$ is a Hilbert space, $[f, g] = \langle f, g \rangle$. For a real Banach space, we can choose

$$[f, g] = \lim_{\lambda \to 0^+} \frac{\|f + \lambda g\|^2 - \|f\|^2}{2\lambda}.$$

To see the existence of the limit, write

$$\frac{\|f + \lambda g\|^2 - \|f\|^2}{\lambda} = (\|f + \lambda g\| + \|f\|)(\|\lambda^{-1} f + g\| - \|\lambda^{-1} f\|).$$

Take a sequence $\lambda_n \to 0^+$. Clearly,

$$\|f + \lambda_n g\| + \|f\| \to 2\|f\|,$$

and

$$\|\lambda_n^{-1}f+g\|-\|\lambda_n^{-1}f\|\le\|g\|.$$

So it suffices to show that the sequence $\|\lambda_n^{-1}f+g\|-\|\lambda_n^{-1}f\|$ is monotone. This is true since for every $\lambda_m>\lambda_n$,

$$\begin{aligned}&\|\lambda_n^{-1}f+g\|-\|\lambda_n^{-1}f\|-\|\lambda_m^{-1}f+g\|-\|\lambda_m^{-1}f\|\le 0\\ \iff\ &\|\lambda_n^{-1}f+g\|-\|\lambda_m^{-1}f+g\|\le\|(\lambda_n^{-1}-\lambda_n^{-1})f\|\end{aligned}$$

which is the triangle inequality.

**Proposition 4.8.** *For $X=C(\overline{U})$, then*

$$\begin{aligned}[f,g]&=\lim_{\lambda\to 0^+}\frac{\max_{x\in\overline{U}}|f(x)+\lambda g(x)|^2-\max_{x\in\overline{U}}|f(x)|^2}{2\lambda}\\&=\max\left\{f(x_0)g(x_0)\quad \textit{for } x_0\in\overline{U}\ \textit{s.t}\ |f(x_0)|=\max_{x\in\overline{U}}|f(x)|\right\}.\end{aligned}$$

*Proof.* Put

$$M=\max\left\{f(x_0)g(x_0)\quad\text{for } x_0\in\overline{U}\text{ s.t }|f(x_0)|=\max_{x\in\overline{U}}|f(x)|\right\}.$$

If $|f(x_0)|=\max_{x\in\overline{U}}|f(x)|$, then

$$[f,g]\ge\lim_{\lambda\to 0^+}\frac{(f(x_0)+\lambda g(x_0))^2-f^2(x_0)}{2\lambda}=f(x_0)g(x_0).$$

So $[f,g]\ge M$. To prove the inverse inequality, let $\lambda_n\to 0^+$ and $x_n\in\overline{U}$ such that

$$|f(x_n)+\lambda_n g(x_n)|=\max_{x\in\overline{U}}|f(x)+\lambda_n g(x)|. \tag{4.29}$$

Since $\overline{U}$ is compact, we can assume further that $x_n\to y$ for some $y\in\overline{U}$. Then

$$[f,g]\le\lim_{n\to\infty}\frac{(f(x_n)+\lambda g(x_n))^2-f^2(x_n)}{2\lambda}=f(y)g(y). \tag{4.30}$$

Moreover, letting $n\to\infty$ in (4.29), we have $|f(y)|=\max_{x\in\overline{U}}|f(x)|$. So, in (4.30), we conclude that $[f,g]\le M$. ☐

**Proposition 4.9.** *Define $A[u]=F(D^2u)$ for $u\in C^2(\bar{U})$ such that $u=0$ on $\partial U$. Then*

$$[u-v,A[u]-A[v]]\ge 0.$$

*Proof.* WLOG, assume

$$(u-v)(x_0)=\max|u-v|\ge 0.$$

Then, $D^2u(x_0)\le D^2v(x_0)$ which implies $F(D^2u(x_0))\ge F(D^2v(x_0))$. Hence,

$$[u-v,A[u]-A[v]]\ge(u-v)(x_0)(F(D^2u(x_0))-F(D^2v(x_0))\ge 0.$$

☐

**Corollary 4.10** (Viscosity solution)**.** *Suppose that $u \in C^2(\bar{U})$ and solves*

$$\begin{cases} F(D^2u) = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

*Let $\varphi \in C^2(U)$.*

*i. If $u - \varphi$ has a strict maximum at $x_0 \in U$, then*

$$F(D^2\varphi(x_0)) \leq f(x_0).$$

*ii. If $u - \varphi$ has a strict minimum at $x_0 \in U$, then*

$$F(D^2\varphi(x_0)) \geq f(x_0).$$

**Lemma 4.11.** *Let $u_k$ and $u$ be as in Theorem 4.7, then*

$$[u - \varphi, f - F(D^2\varphi)] \geq 0,$$

*for all $\varphi \in C_0^2(\overline{U})$.*

*Proof.* By Proposition 4.9,

$$[u_k - \varphi, f_k - F(D^2\varphi)] \geq 0.$$

Let $x_k \in \overline{U}$ such that

$$|u_k(x_k) - \varphi(x_k)| = \max_{x \in \overline{U}} |u_k(x) - \varphi(x)|.$$

Taking subsequence if necessary, assume $x_k \to y$ for some $y \in \overline{U}$. Then

$$|u(y) - \varphi(y)| = \max_{x \in \overline{U}} |u(x) - \varphi(x)|.$$

Hence, by Proposition 4.8,

$$\begin{aligned} [u - \varphi, f - F(D^2\varphi)] &= (u(y) - \varphi(y)) \left(f(y) - F(D^2\varphi(y))\right) \\ &= \lim_{k \to \infty} (u_k(x_k) - \varphi(x_k)) \left(f_k(x_k) - F(D^2\varphi(x_k))\right) \\ &= \lim_{k \to \infty} [u_k - \varphi, f_k - F(D^2\varphi)] \geq 0. \end{aligned}$$

□

*Proof of Theorem 4.7.* Since $\|D^2u\|_{L^\infty} \leq C$, by Rademacher theorem, $u$ is twice differential almost everywhere. Let $x_0$ be the point at which $u$ is twice differential. By Taylor expansion, we have

$$\begin{aligned} u(x) &= u(x_0) + Du(x_0)\dot{(}x - x_0) + \frac{1}{2} \left\langle D^2u(x_0)(x - x_0), (x - x_0) \right\rangle + o(|x - x_0|^2) \\ &= Q(x) + o(|x - x_0|^2). \end{aligned}$$

For $\varepsilon > 0$, let $V$ be a small open set containing $x_0$ and $\varphi \in C_0^2(\overline{U})$ such that

$$\begin{cases} \varphi(x) = Q(x) + \varepsilon|x - x_0|^2 - 1 & \text{in } V, \\ |\varphi(x) - u(x)| < \frac{1}{2} & \text{outside } V. \end{cases}$$

Since $|u - \varphi|$ attains its maximum at the unique point $x_0$, by Lemma 4.11, we have

$$0 \leq [u - \varphi, f - F(D^2\varphi)] = f(x_0) - F(D^2\varphi(x_0))$$

which is

$$F(D^2u(x_0) + 2\varepsilon I(x_0)) \leq f(x_0).$$

Send $\varepsilon \to 0$ to get $F(D^2u(x_0)) \leq f(x_0)$. Similarly, we also get $F(D^2u(x_0)) \geq f(x_0)$. Since $x_0$ is an arbitrary point at which $D^2u$ exists, we conclude the proof.

□

# A Brouwer's fixed point theorem

We discuss an important theorem which plays a crucial role in the proof of Lemma 4.1. First, we need some technical results.

**Lemma A.1.** *Let $P = (p_{ij})_{1\leq i,j\leq n}$ be a square matrix of size $n \times n$. Then*

$$\frac{\partial}{\partial p_{ij}} \det P = (\operatorname{cof} P)_{ij} := (-1)^{i+j} \det \widetilde{P}_{ij},$$

*where $\widetilde{P}_{ij}$ is the square matrix of size $(n-1) \times (n-1)$ obtained by deleting the $i$-th row and $j$-th column from $P$.*

*Proof.* Using the expansion of determinant according to the $i$-th row to have

$$\det P = \sum_{j=1}^{n} (\operatorname{cof} P)_{ij} p_{ij}.$$

Since $(\operatorname{cof} P)_{ij}$ is independent of $p_{ik}$ for any $1 \leq j, k \leq n$, we get

$$\frac{\partial}{\partial p_{ij}} \det P = (\operatorname{cof} P)_{ij}.$$

□

**Lemma A.2.** *Suppose $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $f = (f^1, \ldots, f^n)$, is a smooth function. Define square matrices*

$$A_i = \begin{pmatrix} f^1_{x_1} & \cdots & f^1_{x_{i-1}} & f^1_{x_{i+1}} & \cdots & f^1_{x_{n+1}} \\ f^2_{x_1} & \cdots & f^2_{x_{i-1}} & f^2_{x_{i+1}} & \cdots & f^2_{x_{n+1}} \\ . & \cdots & . & . & \cdots & . \\ f^n_{x_1} & \cdots & f^n_{x_{i-1}} & f^n_{x_{i+1}} & \cdots & f^n_{x_{n+1}} \end{pmatrix} \qquad \textit{for } i = 1, 2, \ldots, n+1.$$

*Then*

$$\sum_{i=1}^{n+1}(-1)^i \frac{\partial}{\partial x_i} \det A_i = 0.$$

*Proof.* Let $B_{ij}$, $i \neq j$, be the square matrix obtained from $A_i$ by replacing column

$$\begin{pmatrix} f^1_{x_j} \\ f^2_{x_j} \\ \vdots \\ f^n_{x_j} \end{pmatrix} \quad \text{with column} \quad \begin{pmatrix} f^1_{x_j x_i} \\ f^2_{x_j x_i} \\ \vdots \\ f^n_{x_j x_i} \end{pmatrix}.$$

It is easy to observe that $\det B_{ij} = (-1)^{i-j+1} \det B_{ji}$ for $i \neq j$. We denote $\det B_{ii} := 0$. Then, by Lemma A.1, we have

$$\begin{aligned} \frac{\partial}{\partial x_i} \det A_i &= \sum_{\substack{j \le n+1 \\ j \neq i}} \sum_{k \le n} f^k_{x_j x_i} (\operatorname{cof} A_i)_{kj} \\ &= \sum_{\substack{j \le n+1 \\ j \neq i}} \sum_{k \le n} f^k_{x_j x_i} (\operatorname{cof} B_{ij})_{kj} \\ &= \sum_{\substack{j \le n+1 \\ j \neq i}} \det B_{ij} \\ &= \sum_{j \le n+1} \det B_{ij}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{i=1}^{n+1}(-1)^i \frac{\partial}{\partial x_i} \det A_i &= \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (-1)^i \det B_{ij} \\ &= \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (-1)^{j+1} \det B_{ji} \\ &= -\sum_{i=j}^{n+1} (-1)^j \frac{\partial}{\partial x_j} \det A_j. \end{aligned}$$

Therefore,

$$\sum_{i=1}^{n+1}(-1)^i \frac{\partial}{\partial x_i} \det A_i = 0.$$

□

**Theorem A.3** (Brouwer's fixed point)**.** *Let $B = \{x \in \mathbb{R}^n, |x| \le 1\}$ and $f : B \to B$ be a continuous function. Then there exists $x_0 \in B$ such that $f(x_0) = x_0$.*

*Proof.* We assume temporarily that $f$ is smooth and $f(x) \neq x$ for all $x \in B$. Set

$$g(x) = f(x) + k(x)\frac{x - f(x)}{|x - f(x)|}$$

where $k(x)$ is the larger root of the equation

$$\left| f(x) + k(x)\frac{x - f(x)}{|x - f(x)|} \right|^2 = 1,$$

that is

$$k(x) = -\frac{(x - f(x))f(x)}{|x - f(x)|} + \sqrt{\frac{|(x - f(x))f(x)|^2}{|x - f(x)|^2} + 1 - |f(x)|^2}.$$

Here are some properties of $g$ that can be easily checked:

- $g$ is smooth
- $|g(x)| = 1$ for all $x \in B$
- $g(x) = x$ for all $|x| = 1$.

Now, put

$$h(t, x) = h^t(x) = tg(x) + (1 - t)x \qquad \text{for } 0 \leq t \leq 1 \text{ and } x \in B.$$

We claim that

$$\frac{d}{dt}\int_B \det Dh^t(x)dx = 0. \tag{A.1}$$

To see this, consider $h(t, x)$ as a $\mathbb{R}^n$-valued function of $(n + 1)$ variables where $t$ is the $(n + 1)$-th variable. Let $A_i$ be as in Lemma A.2 with $h$ in lieu of $f$. Then it suffices to prove

$$\sum_{i=1}^{n}(-1)^i \int_B \frac{\partial}{\partial x_i} \det A_i = 0. \tag{A.2}$$

Observe that the last column of $A_i$ is

$$\begin{pmatrix} g^1(x) - x^1 \\ g^2(x) - x^2 \\ \vdots \\ g^n(x) - x^n \end{pmatrix}$$

which is zero on $\partial B$. Therefore, $\det A_i = 0$ on $\partial B$ and, consequently, by Green's theorem,

$$\int_B \frac{\partial}{\partial x_i} \det A_i = 0 \qquad \text{for all } i = 1, \ldots, n.$$

Thus, (A.2) holds and so does (A.1). It follows that

$$t \mapsto \int_B \det Dh^t(x)dx$$

is a constant function. Choosing $t = 0$ and $t = 1$ to have

$$\int_B \det Dg(x)dx = \int_B \det Dx dx = \int_B \det I_n dx = |B|. \tag{A.3}$$

On the other hand, that $|g(x)| = 1$ for all $x \in B$ implies

$$(Dg(x))g(x) = 0.$$

So $\ker Dg(x) \neq 0$ for all $x \in B$. Hence

$$\int_B \det Dg(x)dx = \int_B 0dx = 0,$$

which contradicts (A.3). Therefore, $f$ must have a fixed point. The theorem is done when assuming $f$ is smooth.
Now, for a general continuous function $f$, choose a sequence of smooth functions $(\varphi_n)$ converging to $f$ in $C(B, B)$. Let $x_n$ be the fixed point of $\varphi_n$ and assume further, since $B$ is compact, that $x_n \to x$ for some $x \in B$. Then, for every $\varepsilon > 0$ and $n$ large enough, we have

$$|f(x) - x| \leq |f(x) - f(x_n)| + |f(x_n) - \varphi_n(x_n)| + |\varphi_n(x_n) - x_n| + |x_n - x| \leq 3\varepsilon.$$

So, $x$ is a fixed point of $f$. The theorem is done. □

# References


[1] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, AMS 1990.

[2] L. C. Evans, *Partial Differential Equations*, 2nd edition, AMS 2010.