A CONJECTURE ON OPTIMAL CONVERGENCE RATES FOR PERIODIC HOMOGENIZATION OF NONCONVEX HAMILTON-JACOBI EQUATIONS

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1. Introduction

Here is a brief description of the periodic homogenization of Hamilton-Jacobi equations. For each $\varepsilon > 0$, let $u^\varepsilon \in C(\mathbb{R}^n \times [0,\infty))$ be the viscosity solution to

$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.1}$$

We assume that the Hamiltonian $H = H(y,p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a given continuous function which is $\mathbb{Z}^n$-periodic in y and uniformly coercive in p. Assume also that $g \in \mathrm{BUC}(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$.

It is well-known that u^ε converges to u locally uniformly on $\mathbb{R}^n \times [0,\infty)$ as $\varepsilon \to 0$, and u solves the effective equation

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.2}$$

The effective Hamiltonian $\overline{H} \in C(\mathbb{R}^n)$ depends nonlinearly on H, and is determined by the cell problems. For each $p \in \mathbb{R}^n$, there exists a unique constant $\overline{H}(p) \in \mathbb{R}$ such that the following cell problem has a continuous viscosity solution

$$H(y, p + Dv) = \overline{H}(p) \qquad \text{in } \mathbb{T}^n. \tag{1.3}$$

The main goal here is to obtain rate of convergence of u^ε to u in L^∞, that is, an optimal bound for $\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n\times[0,\infty))}$ as $\varepsilon \to 0+$. See [1, 2] and the references therein for a complete account of the literature.

Capuzzo-Dolcetta and Ishii [1] proved that $\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n\times[0,T])} \le C\varepsilon^{1/3}$, and this is still the best known rate so far for the general nonconvex setting. Tran and Yu [2] showed that, if H is convex, then $\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \le C\varepsilon$, and this rate is optimal. Also, it was obtained in [2] that for a specific nonconvex case where $H(y,p) = \max\{|p|-1, 1-|p|\} + V(y)$ with $\max_{\mathbb{T}^n} V - \min_{\mathbb{T}^n} V \ge 1$, then one also has the optimal $O(\varepsilon)$ rate.

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2. A conjecture

The following is a conjecture that I made in Summer 2022.

Conjecture 1. *For the general nonconvex case, the optimal convergence rate should be $O(\varepsilon^{1/2})$, that is, for each $T > 0$, there exists $C = C(T) > 0$ such that*

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0,T])} \leq C\varepsilon^{1/2}.$$

A simpler question, which is probably more reasonable to study, is the following.

Question 1. *Give an example of a nonconvex H in which one has*

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0,1])} \geq c_0\varepsilon^\theta,$$

where $\theta \in (0,1)$ and $c_0 > 0$ are some fixed numbers.

References

[1] I. Capuzzo-Dolcetta, H. Ishii, *On the rate of convergence in homogenization of Hamilton–Jacobi equations*, Indiana Univ. Math. J. 50 (2001), no. 3, 1113–1129.

[2] H. V. Tran, Y. Yu, *Optimal convergence rate for periodic homogenization of convex Hamilton-Jacobi equations*, arXiv:2112.06896 [math.AP].

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