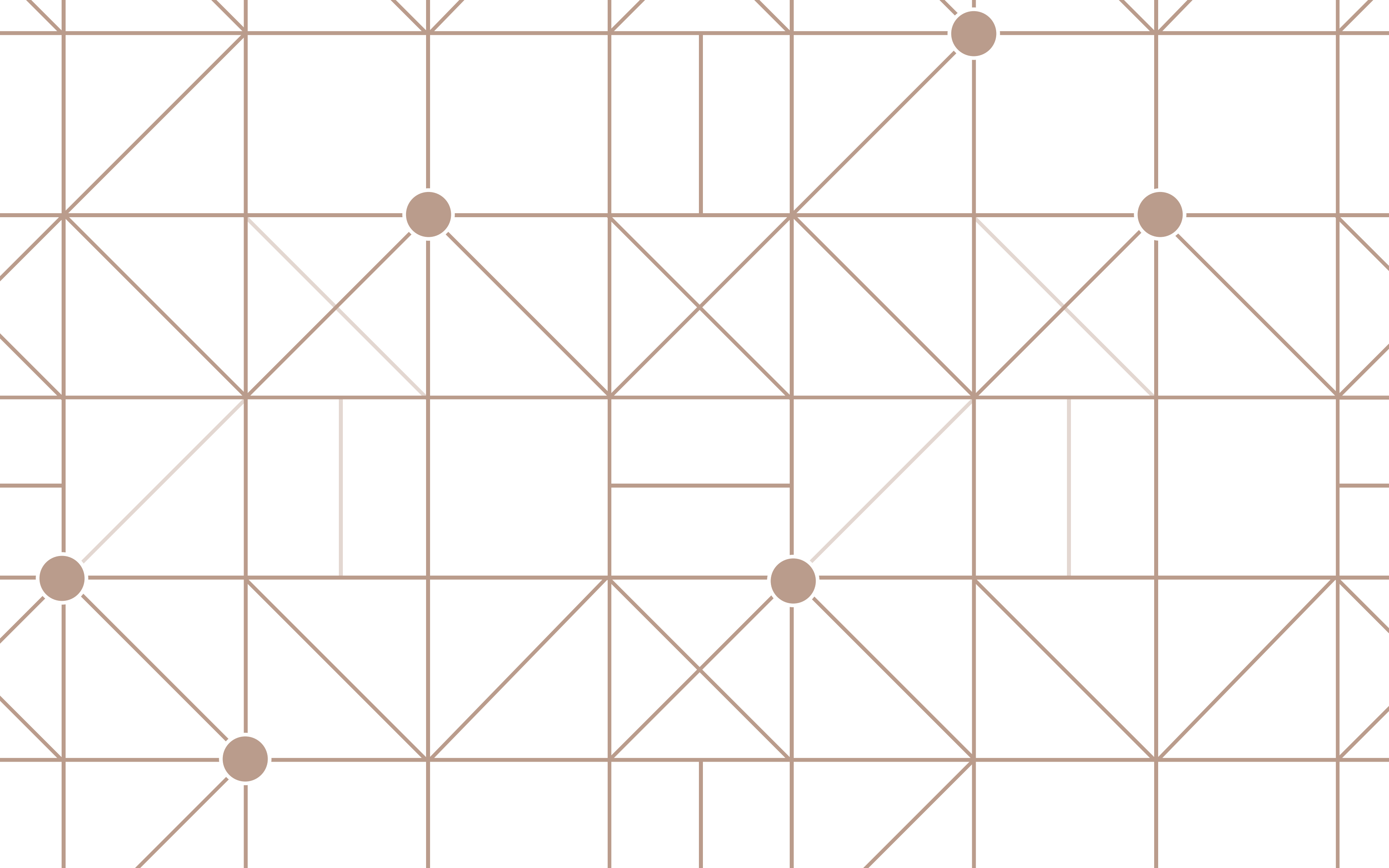


12/04/2023. Level-set mean Curvature Equation.

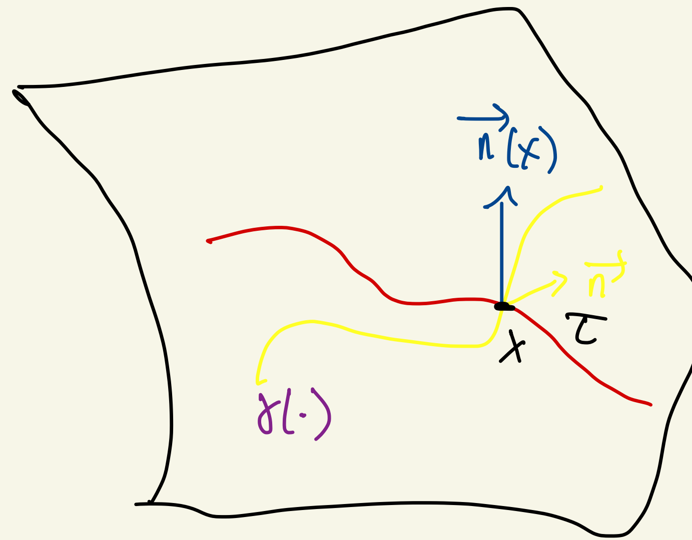
$$(\text{PDE}) \quad \begin{cases} u_t = |Du| \operatorname{div}\left(\dfrac{Du}{|Du|}\right) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \end{cases}$$

Basics on mean curvature.

$\Gamma \subset \mathbb{R}^n$: Smooth hypersurface.

$x \in \Gamma$, let $\vec{n}(x)$ be a unit normal to Γ_0 at x

We can extend $\vec{n}$ to be a smooth enough vector field in



a neighborhood of Γ.

$Dn(x) = \nabla n(x)$.

<u>Definition</u>: $A : T_x\Gamma \to \mathbb{R}^n$,

$$\tau \to A\tau = -D_\tau n(x) = -Dn(x)\tau.$$

<u>Bilinear form</u>: $B : T_x\Gamma \times T_x\Gamma \to \mathbb{R}$,

$$(\tau, \eta) \to B(\tau, \eta) = A\tau \cdot \eta = -D_\tau n(x) \cdot \eta$$

<u>Basic observations</u>:

① In fact, $A : T_x\Gamma \to T_x\Gamma$ as

$$A\tau \cdot \vec{n}(x) = -D_\tau \vec{n}(x) \cdot \vec{n}(x) = -\frac{1}{2} D_\tau\left(|\vec{n}(x)|^2\right) = 0$$

② For a function $\phi : \mathbb{R}^2 \to \Gamma \subset \mathbb{R}^n$ such that

$$\phi(0,0) = x, \ \phi_{x_1}(0,0) = \tau, \ \phi_{x_2}(0,0) = \eta.$$

Then, $\phi_{x_1 x_2}(0,0) \cdot \vec{n}(x) = B(\tau, \eta)$

proof: $\phi_{x_1}(x) \cdot \vec{n}(\phi(x)) = 0.$

$\frac{\partial}{\partial x_2}$: $\phi_{x_1 x_2}(x) \cdot \vec{n}(\phi(x)) + \phi_{x_1} \cdot (D\vec{n}\, \phi_{x_2}(x)) = 0.$

$\Rightarrow \phi_{x_1 x_2}(0) \cdot \vec{n}(x) = -\tau \cdot (D_\eta \vec{n}) = B(\tau, \eta).$

Definition. (Surface divergence)

For general vector field $\vec{F}$ defined in a neighborhood of Γ. For $x \in \Gamma$, let $\{\tau^{\ell} : 1 \leq \ell \leq n-1\}$

be a cononical orthonormal basis of $T_x \Gamma$

Then $$\operatorname{div}_\Gamma \vec{F}(x) = \sum_{l=1}^{n-1} \tau^l \cdot (D_{\tau^l} F)(x)$$

In particular, for the vector field $\vec{n}$,

$$\operatorname{div}_\Gamma \vec{n}(x) = \sum_{l=1}^{n-1} \tau^l \cdot (D_{\tau^l} \vec{n}) = -\sum_{l=1}^{n-1} B(\tau^l, \tau^l)$$

Mean curvature of Γ at x: Let's take an orthonormal basics.

$\{\tau^l : 1 \leq l \leq n-1\}$ to be exactly coresponding with the principal curvatures at x. Sum of all principle curvatures at $x = H(x)$.

$$H(x) = \sum_{l=1}^{n-1} B(\tau^l, \tau^l) = -\mathrm{div}_\Gamma \vec{n}(x) = \tau^l_i n^i_{x_j} \tau^l_j$$

$$= \mathrm{tr}(Dn \, \tau^l \otimes \tau^l)$$

thus,

$$H(x) = \mathrm{tr}(\tau^l \otimes \tau^l D\vec{n}(x))$$

$$= \mathrm{tr}((I - \vec{n}(x) \otimes \vec{n}(x)) D\vec{n}(x))$$

$$= \mathrm{tr}(D\vec{n}(x) - n^j(x) \underbrace{n^i(x) n^i_{x_j}(x)}_{= \frac{1}{2}(|\vec{n}(x)|^2)_{x_j} = 0})$$

Identity: $H(x) = \mathrm{tr}\Big(\big(I - (\vec{n} \otimes \vec{n})\big) D\vec{n}(x)\Big) = \mathrm{tr}\big(D\vec{n}(x)\big).$

Remark: 1. $A : T_x\Gamma \to T_x\Gamma$ has $(n-1)$ eigenvalues $\lambda_1(x), \dots, \lambda_{n-1}(x)$, which are exactly $(n-1)$ principal curvature of Γ at x.

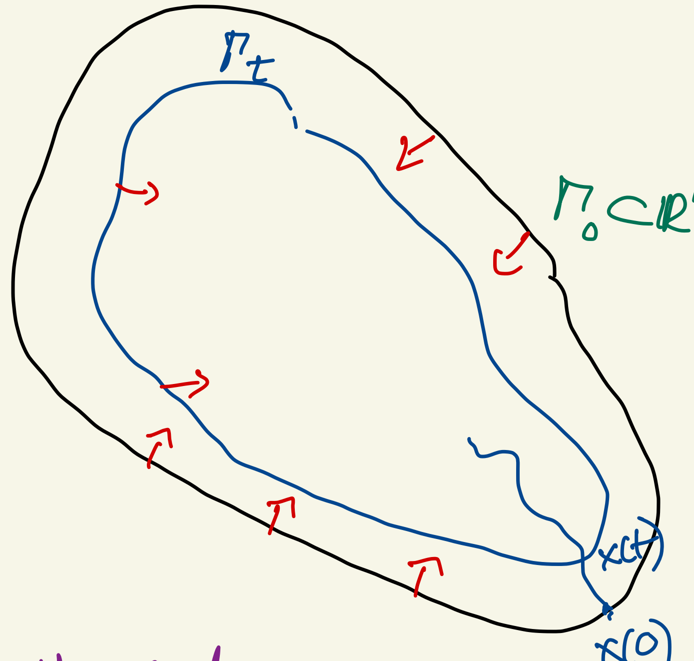
Choose $\{\tau^\ell : 1 \le \ell \le n-1\}$ such that

$$A\tau^\ell = -\lambda_\ell(x)\tau^\ell$$

$$\underbrace{\langle A\tau^\ell, \tau^\ell\rangle}_{\substack{\shortparallel \\ B(\tau^\ell, \tau^\ell)}} = -\lambda_\ell(x)$$

② Mean curvature $= \dfrac{\sum \lambda_i(x)}{n-1} = \dfrac{H(x)}{n-1}$.

By convention, we'll say that $H(x) = \mathrm{tr}(\vec{n}'(x))$ is the mean curvature.



2. PDEs

① Level-set mean curvature flow

$\Gamma_0 \subset \mathbb{R}^n$: given $(n-1)$ dim closed, compact, smooth manifold.

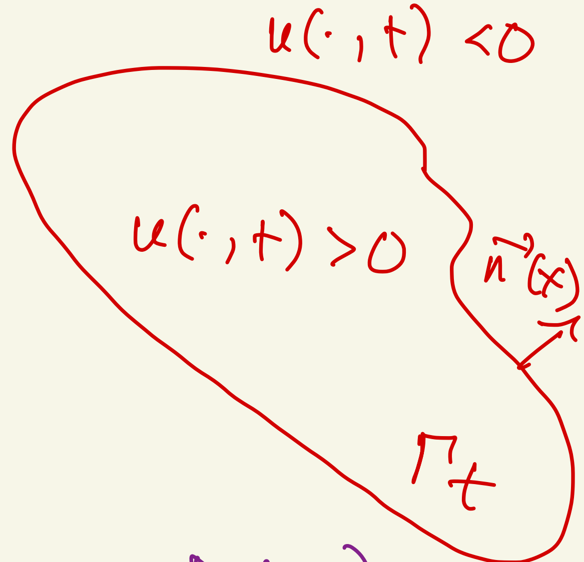
Law of motion:

$V(t) = H(x) . \vec{v}(x)$: $\vec{v}$ inner unit normal.

Define an unknown

$$u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$$

$$(x,t) \to u(x,t) \in \mathbb{R}.$$



$$\Gamma_t = \{ x : u(x,t) = 0 \}$$

$$\vec{n}(x) = \frac{-Du(x,t)}{|Du(x,t)|} \quad \text{and} \quad \vec{v}(x) = \frac{Du(x,t)}{|Du(x,t)|}$$

$$\dot{x}(t) = H(x(t))\,\vec{v}(x(t)) = \left(\operatorname{div} \vec{n}(x,t)\right) \frac{Du(x(t),t)}{|Du(x(t),t)|}$$

$$= -\operatorname{div}\left(\frac{Du}{|Du|}\right)\frac{Du}{|Du|}$$

$$u(x(t),t) = 0 \Rightarrow \frac{d}{dt}\Big(u(x(t),t)\Big) = 0$$

$$\Rightarrow u_t + Du \cdot \dot{x}(t) = 0$$

$\underline{\text{(PDE)}}$: $$u_t - |Du|\operatorname{div}\left(\frac{Du}{|Du|}\right) = 0$$

<u>Compute</u>: $$|Du|\operatorname{div}\left(\frac{Du}{|Du|}\right) = |Du|\left(\frac{u_{x_i}}{|Du|}\right)_{x_i}$$

$$= |Du|\left(\frac{u_{x_i x_i}}{|Du|} - \frac{u_{x_i}}{|Du|^2}(|Du|)_{x_i}\right)$$

$$= |Du|\left(\frac{u_{x_i x_i}}{|Du|} - \frac{u_{x_i} u_{x_j} u_{x_i x_j}}{|Du|^3}\right)$$

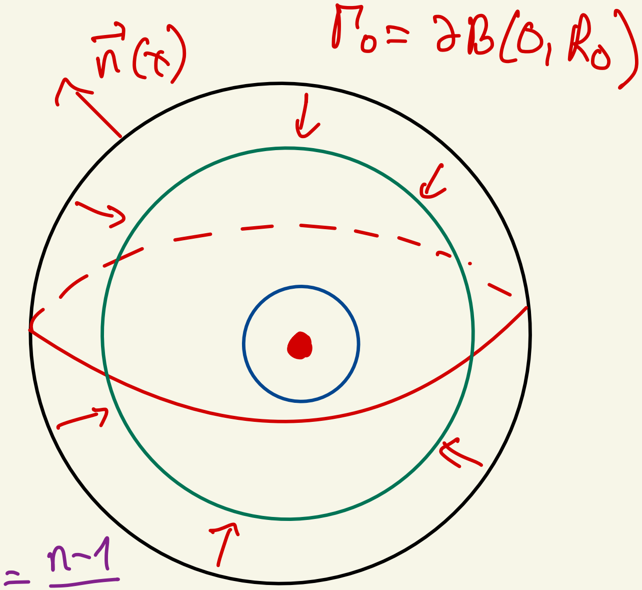
$$= u_{x_i x_j} - \frac{u_{x_i} u_{x_j} u_{x_i x_j}}{|Du|^2}$$

$$|Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = u_{x_i x_i} - \frac{u_{x_i} u_{x_j} u_{x_i x_j}}{|Du|^2}$$

$$= \Delta u - \operatorname{tr}\left(\left(\frac{Du}{|Du|} \otimes \frac{Du}{|Du|}\right) D^2 u\right)$$

$$= \operatorname{tr}\left(\left(I - \frac{Du}{|Du|} \otimes \frac{Du}{|Du|}\right) D^2 u\right)$$

$=$ Surface laplacian at x



Ex: $\vec{n}(x) = \dfrac{x}{|x|}$

$$\operatorname{div}\left(\frac{x}{|x|}\right) = \frac{n}{|x|} - \frac{x_i x_j}{|x|^3} = \frac{n-1}{|x|}$$

$$H(x) = \frac{n-1}{|x|}$$

Let $R(t)$ be such that $\Gamma_t = \partial B(0, R(t))$. Then

$$\begin{cases} \dot{R}(t) = -\dfrac{n-1}{R(t)} \\ R(0) = R_0 \end{cases}$$

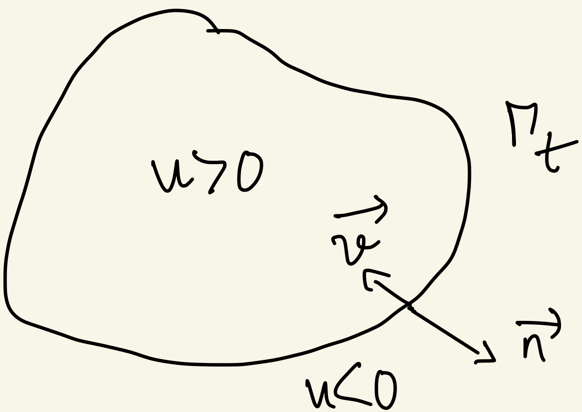
$$\frac{1}{2}\left(R(t)^2\right)' = -(n-1)$$

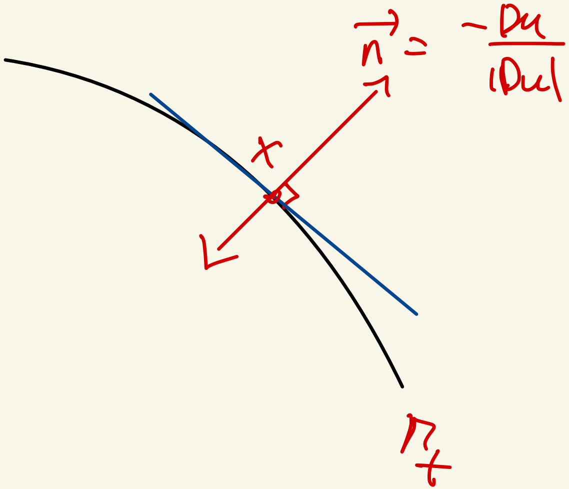
$$R(t)^2 = R(0)^2 - 2(n-1)$$

$$R(t) = \sqrt{R(0)^2 - 2(n-1)}\,, \qquad T = \frac{R(0)^2}{2(n-1)}$$

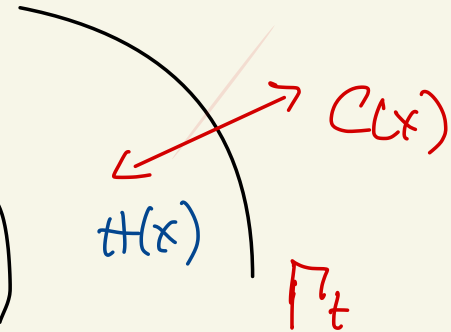
12/06/2023. Level-set MCF.

$$\begin{cases} u_t = |Du| \operatorname{div}\left(\frac{Du}{|Du|} \right) = \operatorname{tr}\left(\left(I - \underbrace{\frac{Du}{|Du|} \otimes \frac{Du}{|Du|}}_{\vec{n} \otimes \vec{n}} \right) D^2 u \right) \\ u(x,0) = g(x) \end{cases}$$





$$\vec{n} = \frac{-Du}{|Du|} \quad \text{and} \quad \vec{v} = -\vec{n} = \frac{Du}{|Du|}$$



Mean curvature

$$H = -\operatorname{div}_{\Gamma_t} \vec{n} = -\operatorname{div} \vec{n} = \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

<u>Forced level-set MCF</u>: $V(x) = (C(x) \pm H(x))\vec{n}$

$X(0) \in \Gamma_0$, $x(t) \in \Gamma_t$, and

$$\dot{x}(t) = (C(x(t)) + H(x(t))\vec{n}(x(t))).$$

$$= (C(x(t)) + H(x(t)) = \frac{-Du}{|Du|}$$

$$u(x(t),t) = 0 \Rightarrow u_t + Du \cdot \dot{x}(t) = 0$$

$$\Rightarrow u_t + Du \cdot \frac{-Du}{|Du|}\left(c(x(t)) + \operatorname{div}\left(\frac{Du}{|Du|}\right)\right) = 0$$

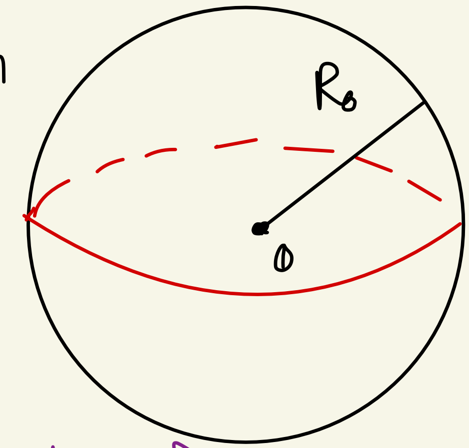
2 $\begin{cases} u_t = |Du|\left(c(x) + \operatorname{div}\left(\frac{Du}{|Du|}\right)\right) \\ u(x,0) = g(x). \end{cases}$

Remarks:

1 Let $c(x) \equiv 1$. Then $V = (1 + H(x))\vec{n}$ is at equilibirum if $1 + H(x) = 0$ on Γ_0

$H(x) = -1$ on Γ_0 : constant mean curvature.

$R_0 = n-1 \to \partial B(0, n-1)$ is an equilibrium surface.



$\to$ Union of many disjoint spheres of radius $n-1$ also work.

② <u>Alexandrov's Theorem</u> (under some conditions)

If $H(x) = -1$ on Γ_0, then Γ_0 must be an union of many disjoint spheres of radius $n-1$.

Definition of Viscosity solution for (2).

(1) **Sub solution**: u is a Vis solution to (2) if $u(x,0) \le g(x)$ and for ANY smooth test function φ touching u from above at $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ then

a) If $D\varphi(x_0, t_0) \neq 0$,

$$\varphi_t(x_0, t_0) - \mathrm{tr}\left(\left(I - \frac{D\varphi}{|D\varphi|} \otimes \frac{D\varphi}{|D\varphi|}\right) D^2\varphi(x_0, t_0)\right)$$

$$- c(t_0) D\varphi(x_0, t_0) \le 0$$

b) If $D\varphi(x_0,t_0)=0$,

$$\varphi_t(x_0,t_0)-\mathrm{tr}\big((I-\xi\otimes\xi)D^2\varphi(x_0,t_0)\big)\leq 0,$$

for a vector $|\xi|\leq 1$.

② Supersolution:

Same as above: ψ smooth touching u from below at (x_1,t_1):

If $D\psi(x_1,t_1)=0$,

$$\psi_t(t_1,x_1)-\mathrm{tr}\big((I-\eta\otimes\eta)D^2\psi(t_1,x_1)\big)\geq 0$$ for

a vector $|\eta|\leq 1$.

Theorem [Comparison priciple for both (1) & (2)].

Let $u \in USC(\mathbb{R}^n \times [0,\infty))$, $v \in LSC(\mathbb{R}^n \times [0,\infty))$ be a sub, a super solution to (1), resp. Then $u \le v \Rightarrow$ UNIQUENESS.

Sketch of proof: Doubling Variables.

$$\boxed{\Psi^{\varepsilon,\lambda}(x,y,t,s) = u(x,t) - v(y,s) - \frac{|x-y|^2 + |t-s|^2}{2\varepsilon} - \lambda(t+s).}$$

for ε, λ sufficiently small.

Assume by contradiction that $u \not\le v$, $\sup_{\mathbb{R}^n}(u-v) = r > 0$.

$\Psi^{\varepsilon,\lambda}$ has a max at $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \xrightarrow{\varepsilon\to 0} (\hat{x}, \hat{x}, \hat{t}, \hat{x})$

$\in \mathbb{R}^{2n} \times (0,\infty)^2$.

$$\left(u(x_\varepsilon, t_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X^\varepsilon\right) \in \overline{J}^{2,+} u(x_\varepsilon, t_\varepsilon)$$

$$\left(v(y_\varepsilon, t_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y^\varepsilon\right) \in \overline{J}^{2,-} v(y_\varepsilon, t_\varepsilon)$$

and

$$-\frac{1}{k} I_{2n} \le \begin{pmatrix} X^\varepsilon & 0 \\ 0 & Y^\varepsilon \end{pmatrix} \le \frac{1}{\varepsilon - 2k} \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}$$

$$\Rightarrow X^\varepsilon \le Y^\varepsilon. \qquad \Leftrightarrow X^\varepsilon - Y^\varepsilon \le 0.$$

(1) $$u_t - \operatorname{tr}\left(\left(I - \frac{Du}{|Du|} \otimes \frac{Du}{|Du|}\right) D^2 u\right) = 0$$

$\boxed{\text{Sub solution test for } u.}$

$$\lambda + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} - \operatorname{tr}\left(\left(I - \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|} \otimes \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|}\right) X^\varepsilon\right) \leq 0$$

$\boxed{\text{Super-solution test for } v:}$

$$-\lambda + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} - \operatorname{tr}\left(\left(I - \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|} \otimes \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|}\right) Y^\varepsilon\right) \geq 0$$

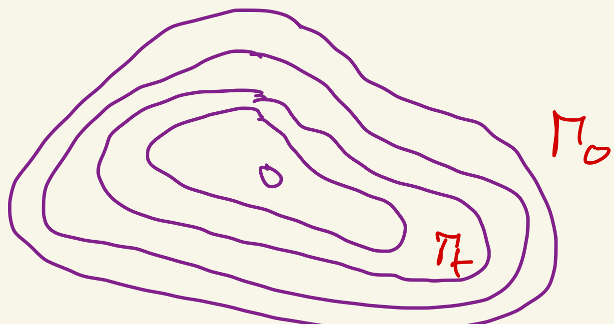
Subtracting the two

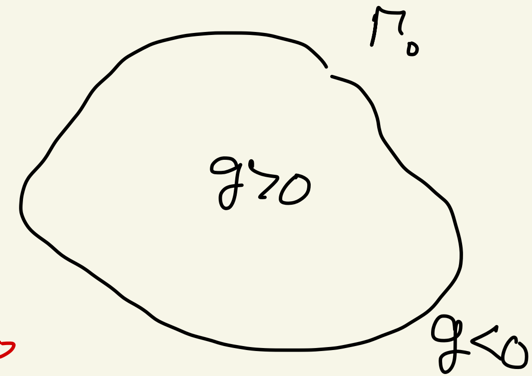
$$\underbrace{2\lambda}_{>0} + \mathrm{tr}\left(\underbrace{\left(I - \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|} \otimes \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|}\right)}_{\geqslant 0}\underbrace{\left(Y^\varepsilon - X^\varepsilon\right)}_{\geqslant 0}\right) \qquad (8)$$

Ref: Evans-Spruck, Chen-Giga-Goto (JDG)

Q: What about geometry?

Embed Γ_t to be the zero-level set of $u(\cdot, t)$





Theorem 2: Assume Γ_0 is a smooth, compact, no-body $(n-1)$-dim manifold. Construct initial condition g such that $\{g=0\} = \Gamma_0$, $g>0$ in the region enclosed by Γ_0, $g<0$ outside.

If the geometric evolution $V = \kappa \vec{n}$ is still classical $0<t<T$

Then, $\{u(\cdot,t)=0\} = \Gamma_t$ for $0 \le t \le T$.

Idea: $g(x)=$ signed distance function to

$$\Gamma_0 = \begin{cases} d(x,\Gamma_0) & x \text{ inside} \\ -d(x,\Gamma_0) & x \text{ outside} \end{cases}$$

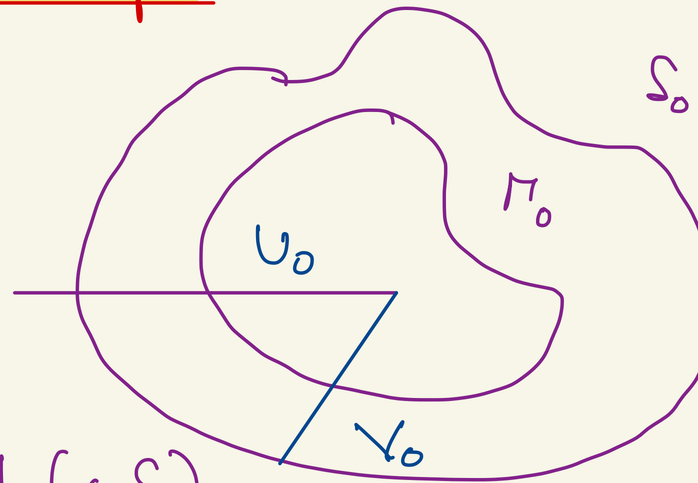
Keep track with this signed distance function.

$$|Dd(x, \Gamma_t)| = 1.$$

Some simple & puzzling examples

① Γ_t enclosed U_t

S_t enclosed V_t



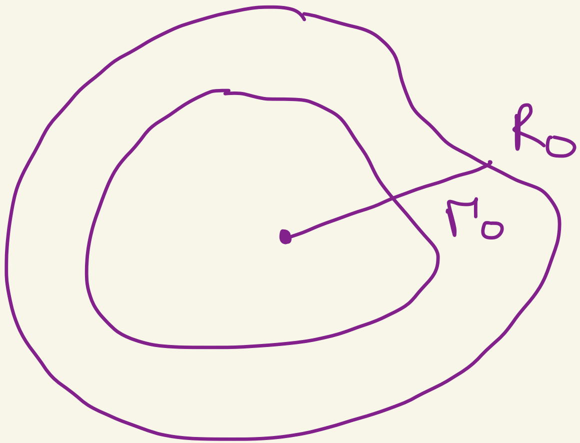
If $U_0 \subset V_0 \Rightarrow U_t \subset V_t$

$g_1 = sd(x, \Gamma_0) \leq g_2 = sd(x, S_0)$

① $u_1 \leq u_2$

$\Rightarrow U_t = \{u_1 \geq 0\} \subset V_t = \{u_2 \geq 0\}$

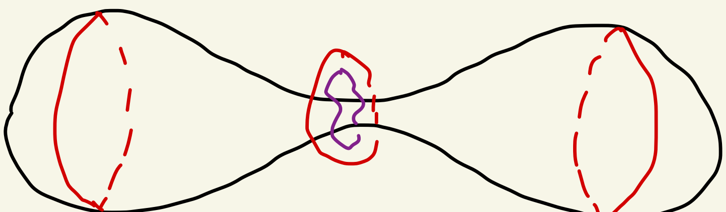
2

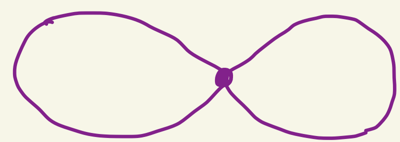


$\longrightarrow R(t) = \sqrt{R_0^2 - 2(n-1)t}$

$\Rightarrow \Gamma_t$ will develope singulties in time $\leq \dfrac{R_0^2}{2(n-1)}$

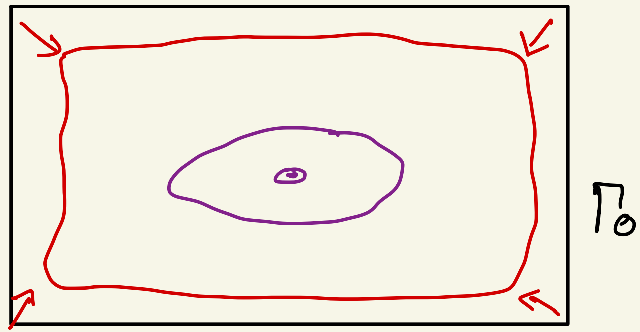
3





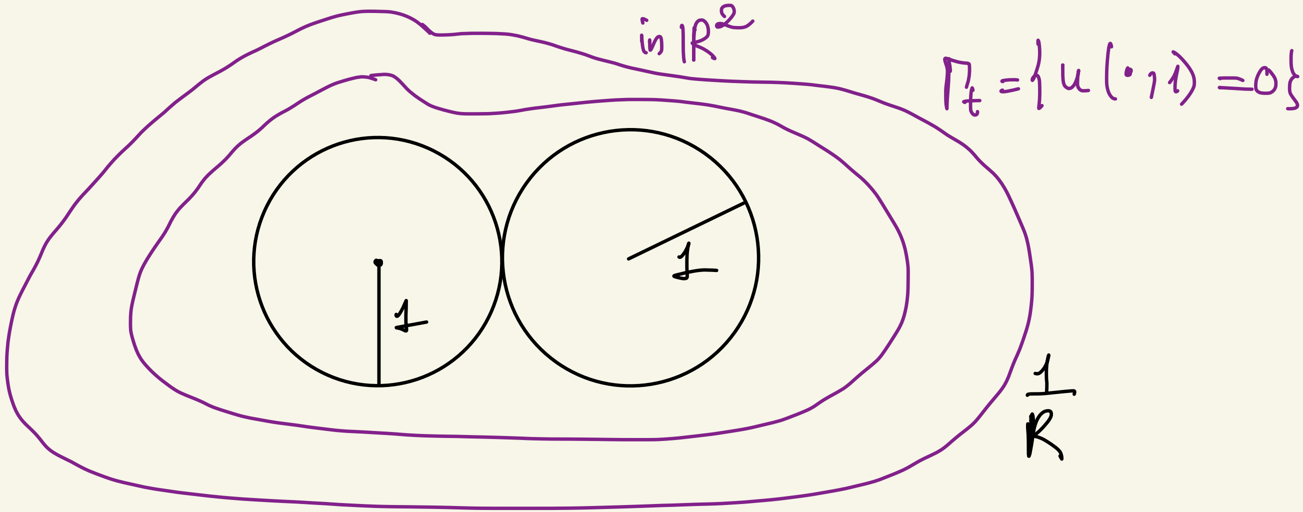
4

$V = x$



5

$V = x + 1$



12/08/2023

$$(1)\quad \begin{cases} u_t = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = \operatorname{tr}\left(\left(I - \frac{Du}{|Du|} \otimes \frac{Du}{|Du|}\right) D^2 u\right) \\ u(x,0) = g(x). \end{cases}$$

Goal: $\begin{cases} \text{Lipschitz regularity} \\ \text{Some observations.} \end{cases}$

Notations: $\widehat{Du} = \frac{Du}{|Du|}$; $\hat{p} = \frac{p}{|p|}$

$$(1) \iff \begin{cases} u_t - F(Du, D^2 u) = 0 \\ u(x,0) = g(x) \end{cases}$$

Hence, $F: (\mathbb{R}^n \setminus \{0\}) \times S^n \longrightarrow \mathbb{R}$.

$$(p, X) \longrightarrow F(p, X) = tr((I - \hat{p} \otimes \hat{p}) X)$$

* F is not defined at 0

$$F^*(0, X) = \limsup_{(p,Y) \to (0,X)} F(p, Y)$$

$$F_*(0, X) = \liminf_{(p,Y) \to (0,X)} F(p, Y)$$

* F is independent of x.

<u>Key observation</u>: u is solution, then $(x,t) \to u(x+y, t)$ or $(x,t) \to u(x, t+s)$, for all $y \in \mathbb{R}^n$, $s \geq 0$ is also a solution.

Theorem 1: Assume $g \in C^2(\mathbb{R}^n) \cap Lip(\mathbb{R}^n)$. Then there exists $c = c(g,n) > 0$ such that

$$\|u_t\|_{L^\infty(\mathbb{R}^n \times (0,\infty))} + \|Du\|_{L^\infty(\mathbb{R}^n \times (0,\infty))} \le C.$$

proof 1: $u(x,t)$ is a solution to (1) with initial data $g(x)$.

$(x,t) \to u(x+y,t) + \underbrace{\|Dg\|_{L^\infty} |y|}_{\text{constant}}$ is also to (1) with initial data $\underbrace{g(x+y) + \|Dg\|_{L^\infty} |y|}_{\geq g(x)}$.

By the comparison principle, $u(x+y,t) + \|Dg\|_{L^\infty} |y| \geq u(x,t)$

By symmetric, we get

$$|u(x+y,t) - u(x,t)| \leq \|Dg\|_{L^\infty} |y|.$$

By similar we get $\|u_t\|_{L^\infty} + \|Du\|_{L^\infty} \leq C.$

<u>proof</u> 2: [Vanishing process].

$$\begin{cases} u^\varepsilon_t = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}\, \mathrm{div}\left(\dfrac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} \right) = \mathrm{tr}\left(\left(I - \dfrac{Du^\varepsilon \otimes Du^\varepsilon}{\varepsilon^2 + |Du^\varepsilon|^2} \right) D^2u^\varepsilon \right) \\ u^\varepsilon(x,0) = g(x). \end{cases}$$

<u>Claim</u>. $\|u^\varepsilon_t\|_{L^\infty} + \|Du^\varepsilon\|_{L^\infty} \leq C.$

$$u^\varepsilon_t = \Delta u^\varepsilon - \frac{u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_i x_j}}{\varepsilon^2 + |Du^\varepsilon|^2}$$

Idea: u^ε_t, $u^\varepsilon_{x_k}$ solve linear parabolic PDEs.

$$u^\varepsilon_t = \Delta u^\varepsilon - \frac{u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_i x_j}}{\varepsilon^2 + |Du^\varepsilon|^2}$$

$$\boxed{\frac{\partial}{\partial t}} \qquad (u^\varepsilon_t)_t = \left[\Delta u^\varepsilon - \frac{u^\varepsilon_{x_i} u^\varepsilon_{x_j} (u^\varepsilon_t)_{x_i x_j}}{\varepsilon^2 + |Du^\varepsilon|^2}\right] - 2\,\frac{(u^\varepsilon_t)_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_i x_j}}{\varepsilon^2 + |Du^\varepsilon|^2}$$

$$+ \frac{u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_i x_j}}{(\varepsilon^2 + |Du^\varepsilon|^2)^2} \cdot 2Du^\varepsilon \cdot D(u^\varepsilon_t)$$

$\varphi = u^\varepsilon_t$ solves a linear parabolic PDE

$$\varphi_t = \mathrm{tr}\left(\left(I - \frac{u^\varepsilon_{x_i}\, u^\varepsilon_{x_j}}{\varepsilon^2 + |Du^\varepsilon|^2}\right) D^2\varphi\right) + b_i \cdot \varphi_{x_i}$$

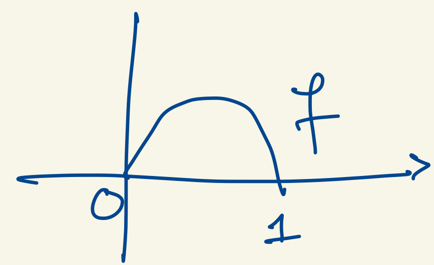
$$\max \varphi = \max \varphi(\cdot, 0). \Rightarrow \max_{\mathbb{R}^n \times [0,\infty)} u^\varepsilon_t = \max_{\mathbb{R}^n \times [0,\infty)} u^\varepsilon_t(\cdot, 0)$$

$$\Rightarrow g(x) - ct \le u^\varepsilon(x,t) \le g(x) + ct$$

$$\Rightarrow \quad |u^\varepsilon_t(\cdot, 0)| \le C.$$

<u>General Remark</u>: F-KPP equation

$$u_t - \Delta u = \underbrace{f(u)}_{g(x)}$$



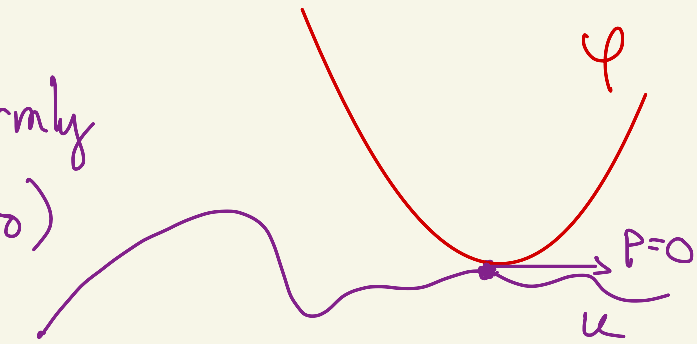
<u>Iterative method to solve PDE</u>: Take a first guess (u^1)

$$(u^2)_t + \Delta u^2 = \underbrace{f(u^1)}_{g(x)}$$

$$\vdots$$

$$(u^{k+1})_t + \Delta u^{k+1} = f(u^k)$$

$$u^k \to u.$$



Remark 2: $u^\varepsilon \to u$ local uniformly

Say $u-\varphi$ has a max at (x_0,t_0) and $D\varphi(x_0,t_0)=0$

$u^\varepsilon - \varphi$ has a nearby max at $(x_\varepsilon, t_\varepsilon)$

$$\varphi_t - \operatorname{tr}\left(\left(I - \underbrace{\frac{D\varphi}{\sqrt{\varepsilon^2+|D\varphi|^2}}}_{q_\varepsilon \otimes q_\varepsilon \text{ with } |q_\varepsilon| \le 1.} \otimes \frac{D\varphi}{\sqrt{\varepsilon^2+|D\varphi|^2}}\right) D^2\varphi(x_\varepsilon, t_\varepsilon)\right) \le 0$$

$q_\varepsilon \to \eta$ with $|\eta| \le 1$.

$$\varphi_t(x_0, t_0) - \operatorname{tr}\left(\left(I - \eta \otimes \eta\right) D^2\varphi(x_0, t_0)\right) \le 0 \text{ for some } |\eta| \le 1.$$

<u>Remark 3</u>.

$$u^\varepsilon_t = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}\, \operatorname{div}\left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}}\right)$$

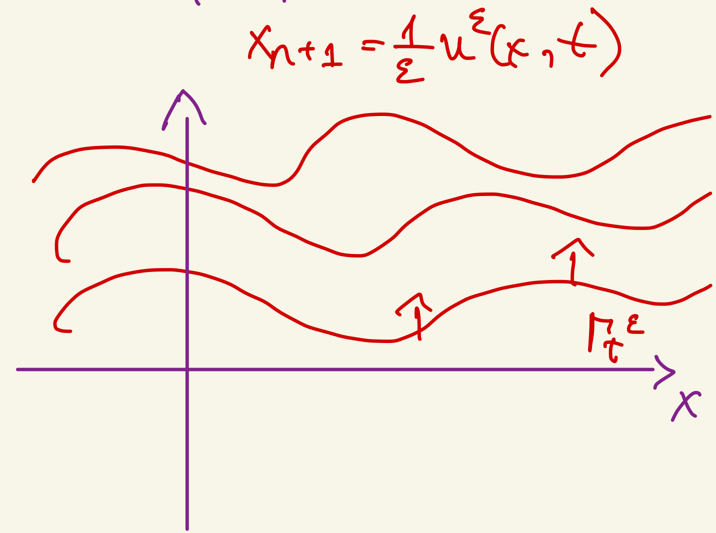
$x \in \mathbb{R}^n$, Consider $y = (x, x_{n+1}) \in \mathbb{R}^{n+1}$.

$$v^\varepsilon(y,t) = u^\varepsilon(x,t) - \varepsilon x_{n+1} \Rightarrow |Dv^\varepsilon|^2 = |Du^\varepsilon|^2 + \varepsilon^2$$

PDE because: $v_t^\varepsilon = |Dv^\varepsilon| \, \mathrm{div}_y \left(\frac{Dv^\varepsilon}{|Dv^\varepsilon|} \right)$

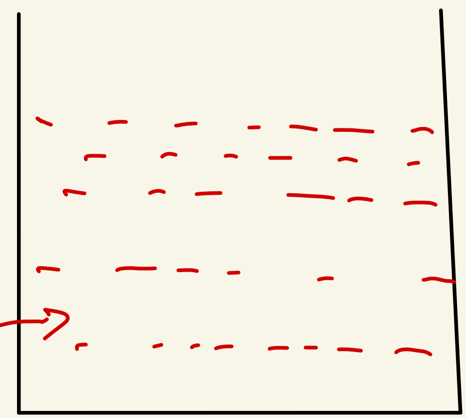
Key: $v^\varepsilon(y,t) = u^\varepsilon(x,t) - \varepsilon x_{n+1}$

$$\Gamma_t^\varepsilon = \{ y : v^\varepsilon(y,t) = 0 \}$$
$$= \{ y : x_{n+1} = \frac{1}{\varepsilon} u^\varepsilon(x,t) \}$$



12/11/2023. Crystal growths in supersaturated media / environment.

(See lecture note on Canvas)



supersaturated

(i.e, lost of crystal molecules)

12/13/2023 Birth and Spread model.

$$(1)\quad \begin{cases} u_t = |Du|\left(\operatorname{div}\left(\dfrac{Du}{|Du|}\right) + 1\right) - f(x) \\ u(x,0) = g(x) \end{cases}$$

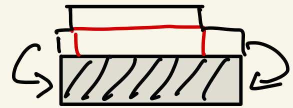
Birth: Vertical growth from the source term $f(x)$

Spread: Horizontal growth from adattoms to the surface.

$$V = \kappa + 1.$$

Hence, $u(x,t)$ represents the height of the crystal at (x,t).

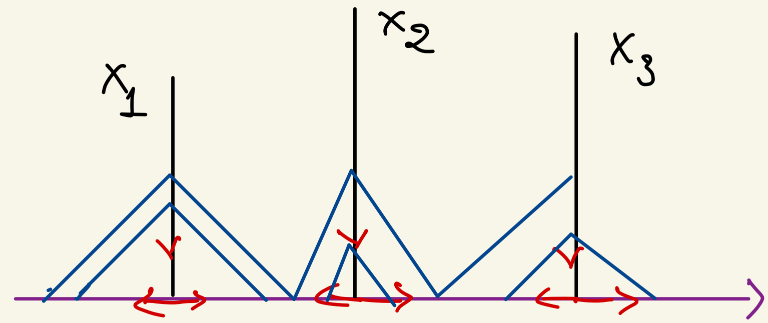
Level set of u evolve w.r.t geometric motions.



In principle, we can replace $|Du|\operatorname{div}\left(\frac{Du}{|Du|}+1\right)$ by any fully nonlinear $F(Du, D^2u)$.

Let's think of a simpler example (Kohn (NYU), Giga (Tokyo), ...)

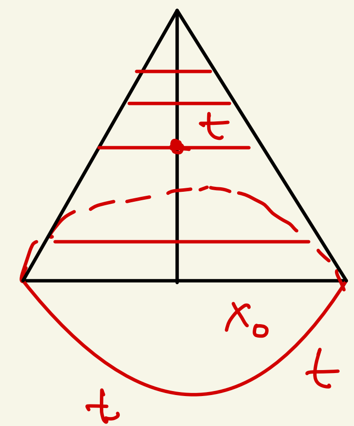
$$(2)\begin{cases} u_t = |Du| + 1\!\!1_E \\ u(x,0) = 0 \end{cases}$$



Let $E = \{x_1, \dots, x_k\}$

$$1\!\!1_E = 1\!\!1_{\{x_1\}} + \dots + 1\!\!1_{\{x_k\}}$$

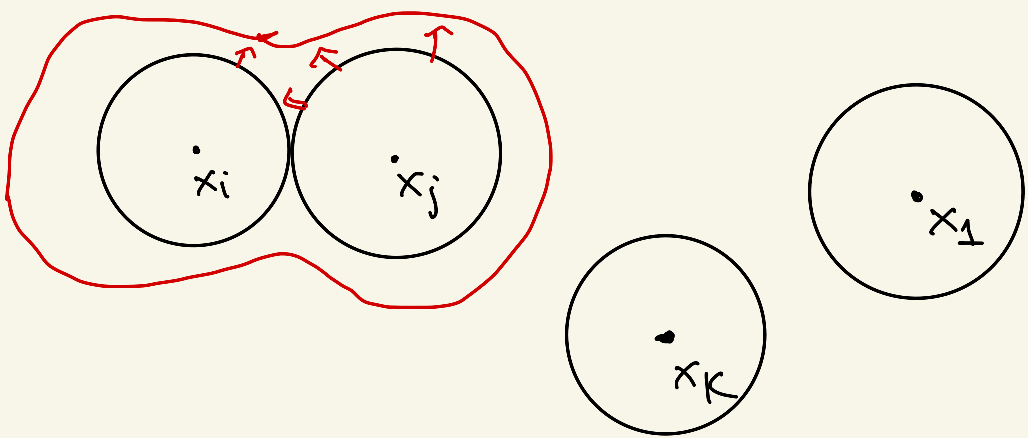
Locally, for each x_i, at time t, it look likes



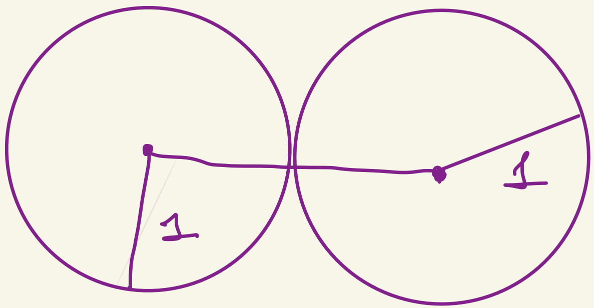
up to a time $T_1 > 0$:

2 sandpiles start overlap with each other

Top down picture



Back to (1), there will also cases in which level set of u exhibit singularities $\nu = \kappa + 1$.

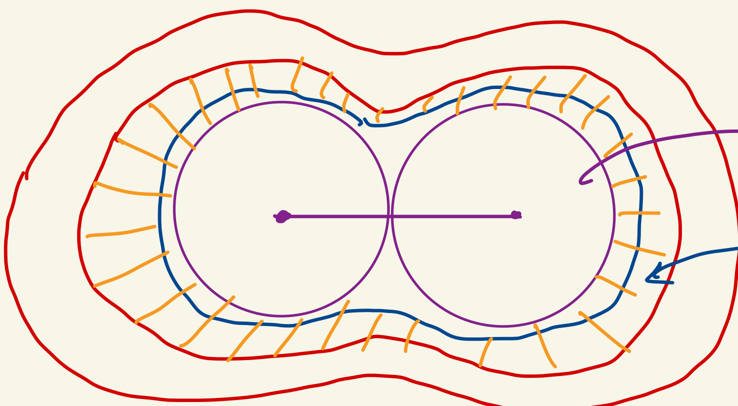


$\Gamma_0 = \partial B(x_1, 1) \cup \partial B(x_2, 1)$

<u>solution 1</u>: Nothing moves.

<u>solution 2</u>: (Maximal vis sln)

it goes to the whole $\mathbb{R}^2$ as $t \to \infty$.



$\kappa = \frac{1}{R} < 1$

Γ_t has interior.

Theorem: If $f \in C_c^1(\mathbb{R}^n, [0,\infty))$, then $\exists !\ c = c_f \geq 0$ such that
$$\lim_{t\to\infty} \frac{u(x,t)}{t} = c_f \quad \text{loc. uni. in } x \in \mathbb{R}^n.$$

Open problem: $u(x,t) - c_f t \xrightarrow{t\to\infty}$?? $v(x)$?

Sketch of proof:

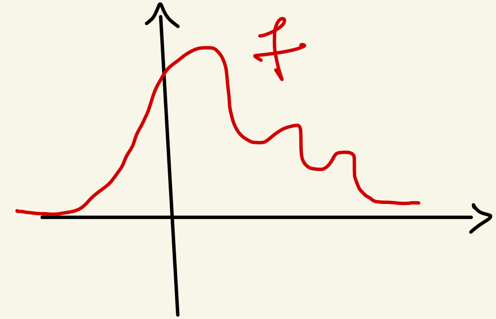
① u is Lipschitz (Berstein's method for $\varphi = |Du|^2$ or $\psi = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}$).

② $\operatorname{supp}(f) \subset B(0,R)$, $\max\limits_{x\in\mathbb{R}^n} u(x,t) = \max\limits_{x\in B(0,R)} u(x,t)$

Heuristic:

$$\max_x u(x,t) = u(x_0,t)$$

$$\Rightarrow u_t(x_0,t) \le f(x_0) \quad (MP).$$

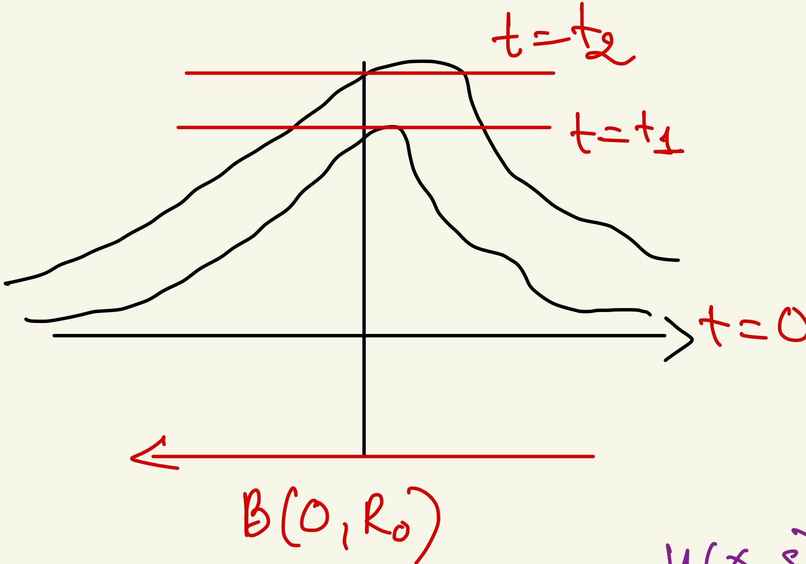


$$\max_{(x,t)\in\mathbb{R}^n\times[0,T]} u(x,t) = u(x_0,t_0)$$

$$\Rightarrow \underset{\substack{\text{\rotatebox{90}{$\le$}}\\ 0}}{u_t(x_0,t_0)} \le f(x_0) \Rightarrow x_0 \in \operatorname{supp}(f) \subset B(0,R_0)$$

Rigorous proof: $u(x,t) - \delta t - \gamma\sqrt{1+|x|^2}$)

③ Snapshots (sideway pictures of u).



$$M(t) = \max_x u(x,t)$$

Keep track tip of the crystal

M is subadditive:

$$M(t+s) \le M(t) + M(s), \quad \forall t, s \geq 0$$

$$u(x,s) \le M(s)$$

$\downarrow$ run for time t

$$u(x,t+s) \le M(s) + u(x,t)$$

$$\Longrightarrow M(t+s) \le M(s) + M(t).$$

Then by Fekete's lemma

$$\boxed{\lim_{t\to\infty} \frac{M(t)}{t} = C = \inf_{s>0} \frac{M(s)}{s}}$$

$$M(2s) \le 2M(s) \quad . \; . \; . \quad \frac{M(s_0)}{s_0} \le \frac{M(ks_0)}{ks_0}$$

Say $M(t) = u(x_t, t)$ for some $x_t \in B(0,R)$

For any x : $|u(x,t) - u(x_t,t)| \le C|x - x_t|$

$$\le C(|x| + R)$$

$$\left| \frac{u(x,t)}{t} - \frac{M(t)}{t} \right| \le C \, \frac{(|x|+R)}{t} \longrightarrow 0 \text{ loc. uni.}$$

OPEN problem 2: $c_f = c(f)$: dependent on what way?