10/02/2023 . Introduction to optimal control theory _ Chapter 2

Two basic problem → Infinite horizon problem (time going on forever) } Static problem.

→ Finite problem (fixed given time) } Cauchy problem.

Infinite horizon problem.

V given compact metric space ← our control set

Ex: $V = \{1, 2, \dots, k\}$

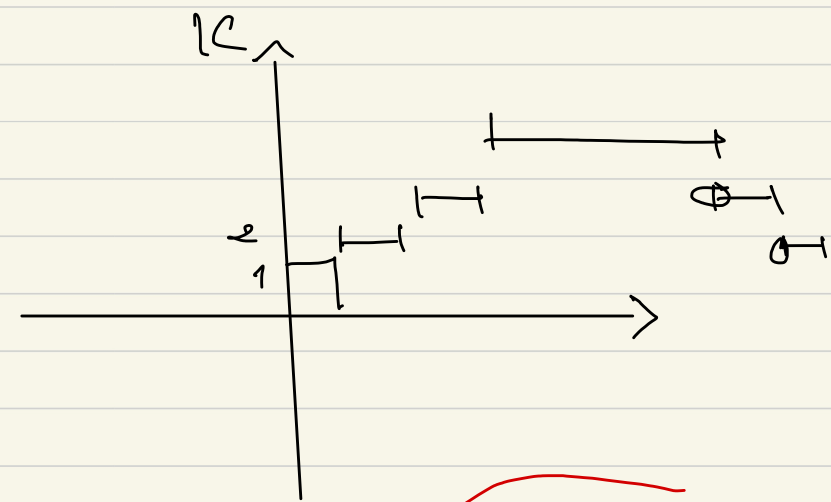
$V = \bar{B}(0,1) \subset \mathbb{R}^n$.

And $b : \mathbb{R}^n \times V \to \mathbb{R}^n$ is a given vetor field such that

$$\begin{cases} b \in C(\mathbb{R}^n \times V, \mathbb{R}^n), \\ |b(x,v)| \le C, \ \forall (x,v) \in \mathbb{R}^n \times V, \\ |b(x_1,v) - b(x_2,v)| \le c|x_1 - x_2|, \ \forall x_1, x_2 \in \mathbb{R}^n, v \in V. \end{cases}$$

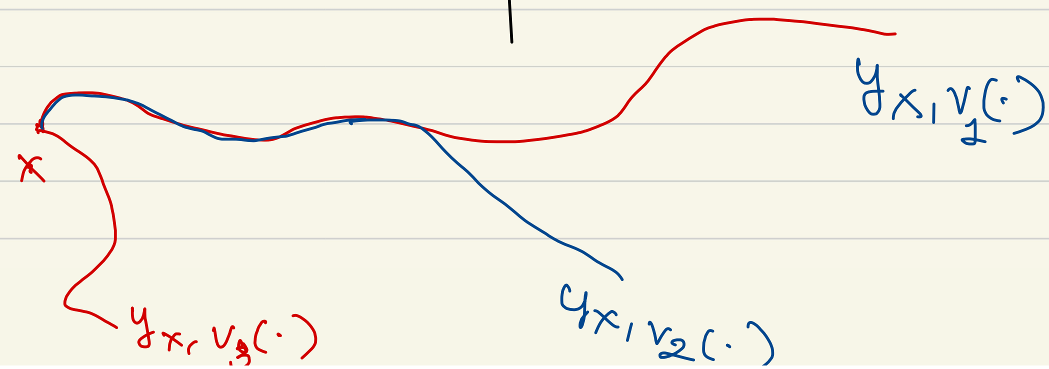
We can mange our control $v : [0,\infty) \longrightarrow V$,

$v(\cdot)$, v is measurable



For a given control $v(\cdot)$, we have an ODE

$$\begin{cases} y'(s) = b(y(s), v(s)), \ s > 0 \\ y(0) = x. \end{cases}$$



Name the solution to this ODE

$$y_{x,v(\cdot)}(t) \qquad \left(\text{No confusion : } y_x(t)\right)$$

Meaning of solution. $y_x(t)$: we mean

$$y_x(t) = x + \int_0^t b(y_x(s), v(s))\,ds, \; \forall t \geq 0.$$

Suppose y^1_x, y^2_x are 2 solutions.

$$|y^1_x - y^2_x| \leq \int_0^t |b(y^1_x(s), v(s)) - b(y^2_x(s), v(s))|\,ds$$

$$\leq c \underbrace{\int_0^t |y^1_x(s) - y^2_x(s)|\,ds}_{\varphi(t)}. \qquad \leq 0$$

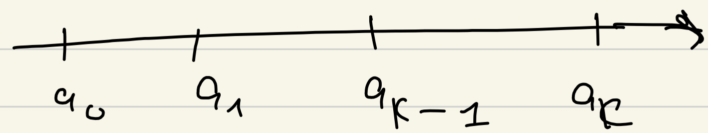
$$\varphi'(t) \leq c\,\varphi(t)$$

$$\Rightarrow \varphi(t) \le e^{ct}\varphi(0) = 0$$

$$\Rightarrow y^1_x = y^2_x .$$

<u>Ex</u>: ① $V = \{1,2,\dots,k\}$, then $b(x,i) = b_i(x)$

Then, if $v(\cdot) = \sum_{m=0}^{\infty} i_m \mathbb{1}_{[a_m, a_{m+1}]}$

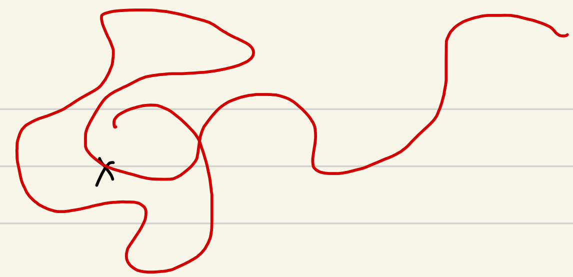


In (a_m, a_{m+1}) : $\gamma'(s) = b_{i_m}(\gamma(s))$

② $V = \overline{B}(0,1) \subset \mathbb{R}^n$. , ODE becomes

$$\begin{cases} y_x'(t) = v(t) \in \overline{B}(0,1) \\ y_x(0) = x . \end{cases}$$

$\Rightarrow \quad |y'_x(t)| \le 1 \quad , \forall t \in (0,\infty)$



Given cost function $f: \mathbb{R}^n \times V \to \mathbb{R}$ satisfying

$$\begin{cases} f \in C(\mathbb{R}^n \times V, \mathbb{R}) \\ |f(x, v)| \le C \\ |f(x_1, v) - f(x_2, v)| \le C|x_1 - x_2| \end{cases}$$

And $\lambda > 0$ is a given constant.

For each path $y_{x,v(\cdot)}$, are consider a cost functional

$$J(x, v(\cdot)) = \int_0^\infty e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\,ds$$

Main Goal. $\min_{v(\cdot)} J(x, v(\cdot))$ find minimum cost

find an optimal control $v^*(\cdot)$

Intuition 1, f bdd, to make the integral converges, need a discount term $e^{-\lambda s}$.

2 Discount term $e^{-\lambda s}$ indeed tell us that the cost decays (exponentially) in time.

Easy thing (Bellman 1950s) Look into the min cost function

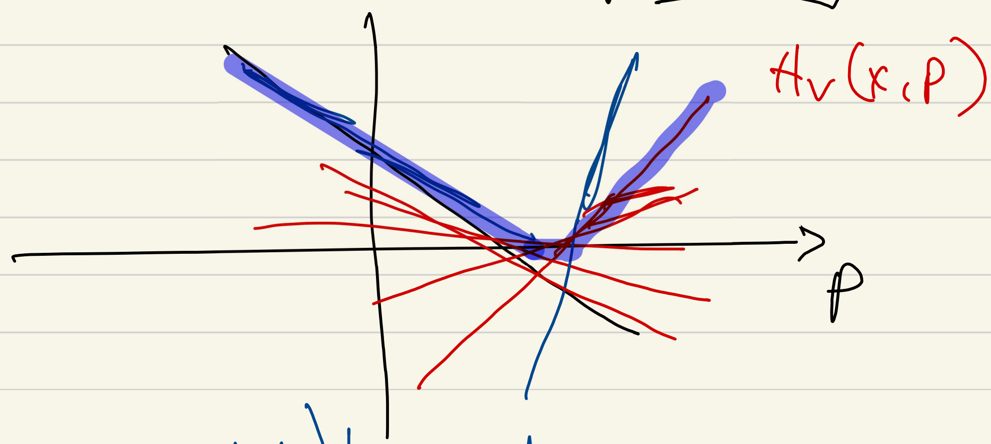
$$u(x) = \min_{v(\cdot)} J(x, v(\cdot)) = \min_{v(\cdot)} \int_0^\infty e^{-\lambda s} f(y_{x,u(\cdot)}(s), v(s))\, ds$$

Theorem: u is the unique viscosity solution to

$$\lambda u + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n.$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is $H(x,p) = \sup_{v \in V} \left[-b(x,v) \cdot p - f(x,v) \right]$.

Fix x : H is convex in p.



$$|H(x,p) - H(y,p)| \le C(1+|p|)|x-y|$$
$$|H(x,p) - H(x,q)| \le C|p-q|.$$

10/04/2023

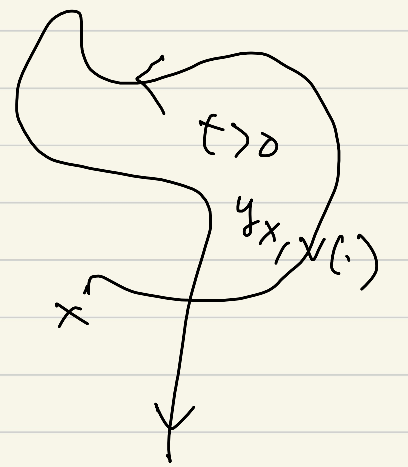
Recap - optimal control : V : compact metric space - control set

Vector field $b : \mathbb{R}^n \times V \to \mathbb{R}^n$ bounded, continuous, $|b(x_1, v) - b(x_2, v)| \le C|x_1 - x_2|$

Cost function $f : \mathbb{R}^n \times V \to \mathbb{R}^n$ $\longrightarrow$ $|f(x_1, v) - f(x_2, v)| \le C|x_1 - x_2|$

$v : [0, \infty) \to V$ measurable a control

(E.g.. V is finite $v(\cdot)$ . . .)



ODE : represents the local of agent

$$\begin{cases} y'_{x, v(\cdot)}(t) = b(y_{x, v(\cdot)}(t), v(t)) \\ y_{x, v(\cdot)}(0) = 0 \end{cases}$$

Cost functional: $J(x, v(\cdot)) = \int_0^\infty \underbrace{e^{-\lambda s}}_{\text{discount term}} f(y_{x, v(\cdot)}(s), v(s))\, ds$

(infinite horizon)

Min cost function

$$u(x) = \min_{v(\cdot)} J(x, v(\cdot))$$ only look at cost function $u(\cdot)$

ignore the underlying dynamics.

<u>Theorem</u>: u is the unique viscosity solution

$$\lambda u + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n.$$

$$H(x,p) = \sup_{v \in V} \left[-b(x,v) \cdot p - f(x,v) \right]$$

<u>No confusion</u>: $y_{x,v(\cdot)} = y_x$, $\quad y_x(t) = \underbrace{y_x(0)}_{= x} + \int_0^t b(y_x(s), v(s))\, ds$

<u>Lemma 1</u>: $|y_x(t) - y_x(s)| \leq C\,|t-s|$ (obvious from def)

$$|y_x(t) - y_x(s)| \leq e^{cT}\,|x-z|$$

proof:

$$|y_x(t) - y_x(s)| = \left| (x - z) + \int_0^t \Big(b(y_x(s), v(s)) - b(y_z(s), v(s)) \Big) ds \right|$$

$$\leq |z - x| + c \int_0^t |y_x(s) - y_z(s)| \, ds .$$

Let $\psi(t) = \int_0^t |y_x(s) - y_z(s)| \, ds$

$$\Rightarrow \psi'(t) = |y_x(t) - y_z(t)|$$

$$\psi'(t) \leq |z - x| + c\,\psi(t) .$$

$$\Rightarrow \psi'(t) - c\,\psi(t) \leq |x - z|$$

$$\left(e^{-ct} \psi(t) \right)' = e^{-ct} \left(\psi'(t) - c\,\psi(t) \right) \leq |x - z| \, e^{-ct}$$

$$e^{-ct}\varphi(t) - \varphi(0) \le |x-z| \cdot \int_0^t e^{-cs}\, ds$$

$$\Rightarrow \varphi(t) < C e^{ct} |x-z|.$$

H grows at most linearly in p

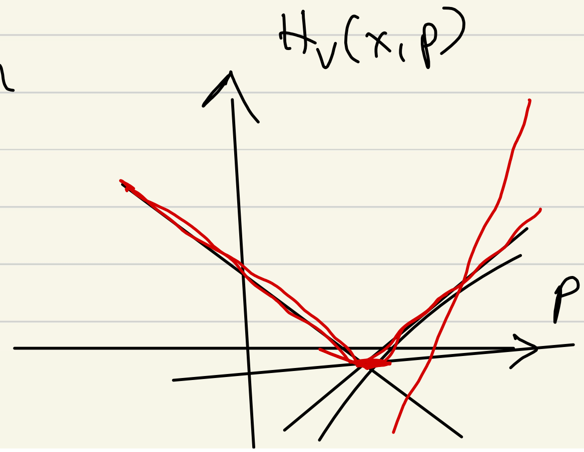
$|H(x,p)| \le C(1+|p|)$

Lemma 2: $$H(x,p) = \sup_{v \in V} \underbrace{\left[-b(x,v)\cdot p - f(x,v)\right]}_{H_v(x,p)}$$

$$= \sup_{v\in V} H_v(x,p)$$

Fix x, $p \to H_v(x,p)$ is just an affine function

$p \to H(x,p)$ is CONVEX.



$$\begin{cases} |H(x,p) - H(x,q)| \leq c|p-q| \\ |H(x,p) - H(y,p)| \leq c(1+|p|)\,|x-y| \end{cases}$$

Ex 1) $V = \overline{B}(0,1) \subset \mathbb{R}^n$, $f(x,v) = f(x)$: $b(x,v) = v$

$$H(x,p) = \sup_{|v| \leq 1} \big(-v \cdot p - f(x)\big) = |p| - f(x)$$

2) $V = \overline{B}(0,1) \subset \mathbb{R}^n$, $f(x,v) = 0$, $b(x,v) = a(x)\,v$

$a : \mathbb{R}^n \to \mathbb{R}$,

$$H(x,p) = \sup_{|v| \leq 1} \big(-a(x)\,v \cdot p\big) = a(x)\,|p|$$

3) $V = \{e_1, -e_1\} \subset \mathbb{R}^n$, $f(x,v) = f(x)$, $b(x,v) = v$.

(same as $V = \{s e_1 : -1 \leq s \leq 1\}$)

$$H(x,p) = \sup_{v \in V} \left(-v \cdot p - f(x) \right) = |p_1| - f(x)$$

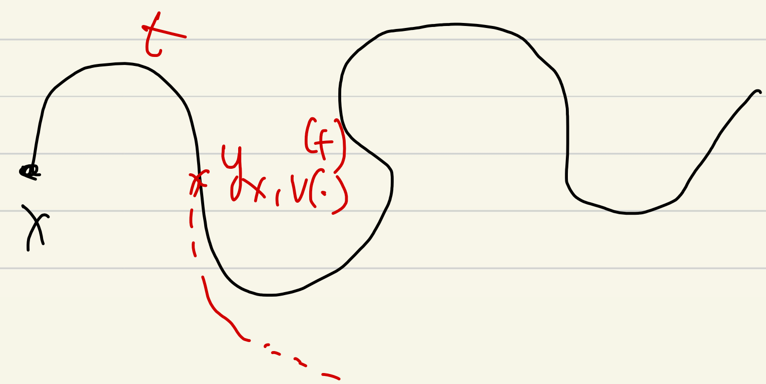
Proposition [DPP] – Dynamic Programming principle.

$$\boxed{u(x) = \inf_{v(\cdot)} \int_0^\infty e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\, ds}$$

$$\boxed{(DPP):\ u(x) = \inf_{v(\cdot)} \left[\int_0^t e^{-\lambda s} f(y_{x,v(\cdot)}, v(s))\, ds + e^{-\lambda t} u(y_{x,v(\cdot)}(t)) \right]}$$

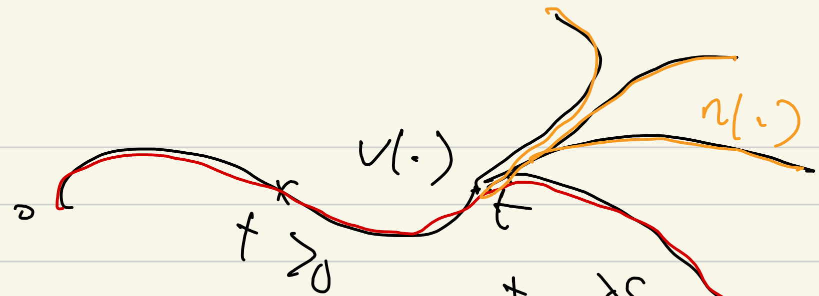
for any $t > 0$

Proof $\quad$ LHS $\leq$ RHS.



LHS $\geq$ RHS.

$\forall\, \varepsilon > 0$, $\exists\, v(\cdot)$ really optimal.



$$\int_0^t e^{-\lambda s} f(\cdot)\, ds.$$

$$u(x) + \varepsilon \geq \int_0^\infty e^{-\lambda s} f\big(y_{x,v(\cdot)}(s), v(s)\big)\, ds$$

$$u(x) + \varepsilon \geq \int_0^t e^{-\lambda s} f(\cdot)\, ds + \int_t^\infty e^{-\lambda s} f\big(y_{x,v(\cdot)}(s), v(s)\big)\, ds$$

$$= e^{-\lambda t} \Big(\underbrace{\int_0^\infty e^{-\lambda r} f\big(y_{\mathring{x}, v(\cdot+t)}(r), v(r+t)\big)\, dr}_{\geq\, u(y_{x,v(\cdot)}(t)).} \Big)$$

Idea: DPP $\to$ PDE

$t \to 0$ and use calculus.

Formal computations : $u \in C^\infty$

$v(\cdot) \equiv v$ constant control.

$$u(x) \leq \int_0^t e^{-\lambda s} f(y_x(s), v)\, ds + e^{-\lambda t} u(y_x(t))$$

$$\frac{u(x) - e^{-\lambda t} u(y_x(t))}{t} \leq \frac{1}{t} \int_0^t e^{-\lambda s} f(y_x(s), v)\, ds$$

$$\lim_{t \to 0^+} \frac{u(y_x(0)) - e^{-\lambda t} u(y_x(t))}{t} = -\frac{d}{dt}\left(e^{-\lambda t} u(y_x(t))\right)\Big|_{t=0}.$$

$$= \lambda u(x) - Du(x) \cdot \underbrace{y'_x(0)}_{b(x,v)} \leq f(x,v)$$

$$\lambda u(x) + [-b(x,v)\cdot Du(x) - f(x,v)] \leq 0, \forall v.$$

Take supremum.

$$H(x,p) = \sup_{v\in V} [-b(x,v)\cdot p - f(x,v)]$$

$$\lambda u(x) + H(x, Du(x)) \leq 0.$$

10/06/2023: DPP and Hamilton-Jacobi PDE.

Setting V: Control set – A compact metric space

$b: \mathbb{R}^n \times V \longrightarrow \mathbb{R}$ is given vector field; bounded, continuous, and Lipschitz in x:

$$|b(x_1,v) - b(x_2,v)| \leq C|x_1 - x_2|, \forall x_1, x_2, v.$$

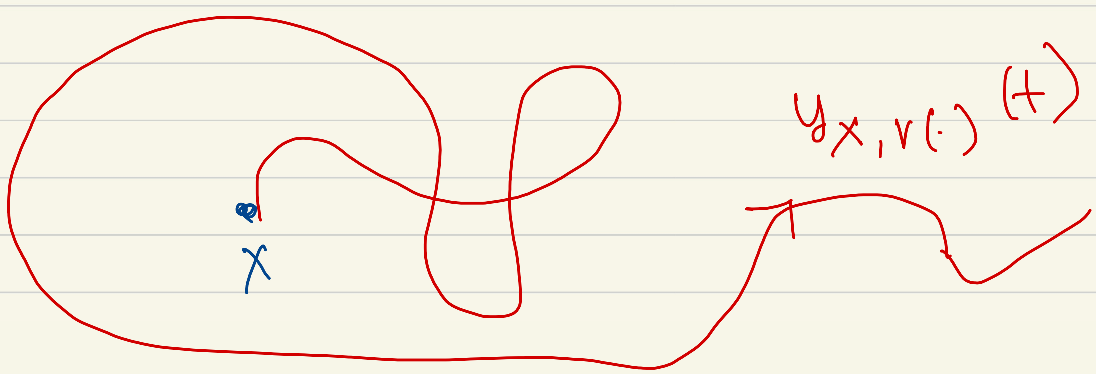
$f : \mathbb{R}^n \times V \to \mathbb{R}$ is a given cost function : bdd, continuous, and Lip in x

$$|f(x_1, v) - f(x_2, v)| \leq C|x_1 - x_2|.$$

$$\begin{cases} v(\cdot) : \text{a control} \longleftrightarrow v : [0, \infty) \longrightarrow V \text{ is a measurable function.} \\ \lambda : \text{a given positive constant.} \end{cases}$$

$y(\cdot) \longrightarrow$ a corresponding path of an agent :

$$\begin{cases} y'_{x, v(\cdot)}(t) = b(y_{x, v(\cdot)}(t), v(t)), \; t > 0 \\ y_{x, v(\cdot)}(0) = x \end{cases}$$



$$J[x, v(\cdot)] = \int_0^\infty e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\, ds.$$

(Minimum) Value function

$$\boxed{u(x) = \min_{v(\cdot)} J[x, v(\cdot)]}$$

For each x, dynamic are complex.

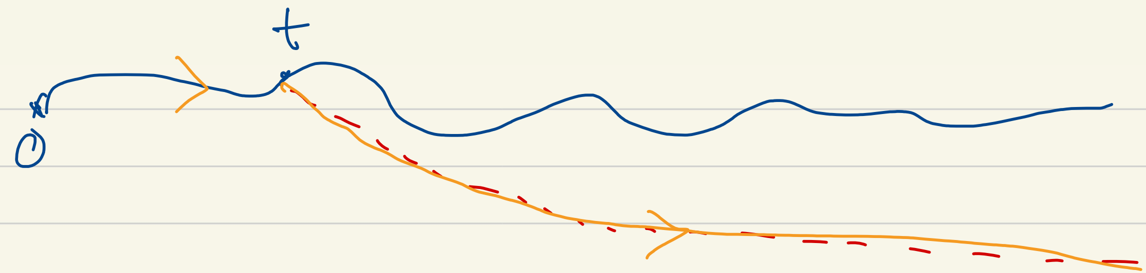
Theorem: u is the UNIQUE viscosity solution to

$$\boxed{\lambda u(x) + H(x, Du(x)) = 0 \quad \text{in } \mathbb{R}^n}$$

where $H(x,p) = \sup_{v \in V} \left(-b(x,v)\cdot p - f(x,v)\right).$

Proposition (DPP): For each fixed $t > 0$:

$$u(x) = \inf_{v(\cdot)} \left[\int_0^t e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\, ds + e^{-\lambda t} u(y_{x,v(\cdot)}(t)) \right]$$



Key idea: DPP $\Longrightarrow$ HJ

$t \to \delta t$

use calculus

Heuristic proof:

$$\boxed{u(x) = \inf_{v(\cdot)} \left[\int_0^t e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\, ds + e^{-\lambda t} u(y_{x,v(\cdot)}(t)) \right]}$$

Think of $v(\cdot) \equiv v \in V$, write $y_{x,v(\cdot)} = y_x$. We have

$$u(x) \leq \int_0^t e^{-\lambda s} f(y_x(s), v(s))\, ds + e^{-\lambda t} u(y_x(t))$$

$$\Rightarrow \quad \frac{u(x) - e^{-\lambda t} u(y_x(t))}{t} - \frac{1}{t}\int_0^t e^{-\lambda s} f(y_x(s), v(s))\,ds \leq 0$$

Take $\lim_{t \to 0^+}$ and ASSUME everything is SMOOTH

$$\lim_{t \to 0^+} - \frac{e^{-\lambda t} u(y_x(t)) - e^{-\lambda \cdot 0} u(y_x(0))}{t} = -\frac{d}{dt}\left(e^{-\lambda t} u(y_x(t))\right)\Big|_{t=0}$$

constant control

$$= \lambda u(x) - Du(x) \cdot y_x'(0) = \lambda u(x) - Du(x) \cdot b(x, v)$$

Then,

$$\lambda u(x) + \left[- b(x, v) \cdot Du(x) - f(x, v) \right] \leq 0 \quad , \forall\, v \in V$$

Recall :

$$H(x, p) = \sup_{v \in V} \left(- b(x, v) \cdot p - f(x, v) \right)$$

$$\lambda u(x) + H(x, Du(x)) \leq 0 \quad \Leftarrow \text{super property.}$$

Super solution : Assume $\exists\ v^*(\cdot) : [0,\infty) \longrightarrow V$ such that

$$\boxed{u(x) = \int_0^t e^{-\lambda s} f\big(y_{x,v^*(\cdot)}(s), v^*(s)\big)\,ds + e^{-\lambda t} u\big(y_{x,v^*(\cdot)}(t)\big)}$$

Write $y_{x,v^*(\cdot)} = y_x$. We compute

$$\underbrace{\left[e^{-\lambda\cdot 0} u(y_x(0)) - e^{-\lambda t} u(y_x(t))\right]}_{} - \int_0^t e^{-\lambda s} f(y_x(s), v^*(s))\,ds = 0$$

$$\Longleftrightarrow \quad -\int_0^t \frac{d}{ds}\left(e^{-\lambda s} u(y_x(s))\right) ds - \int_0^t e^{-\lambda s} f(y_x(s), v^*(s))\,ds = 0.$$

$$\Longleftrightarrow \quad \int_0^t \Big(\lambda u(y_x(s)) - Du(y_x(s))\cdot y_x'(s) - f(y_x(s), v^*(s))\Big) e^{-\lambda s}\,ds = 0$$

$$\Longleftrightarrow \quad \frac{1}{t}\int_0^t \Big(\lambda u(y_x(s)) \underbrace{- Du(y_x(s))\cdot b(y_x(s), v^*(s)) - f(y_x(s), v^*(s))}_{\leq H(y_x(s), Du(y_x(s)))}\Big) e^{-\lambda s}\,ds = 0$$

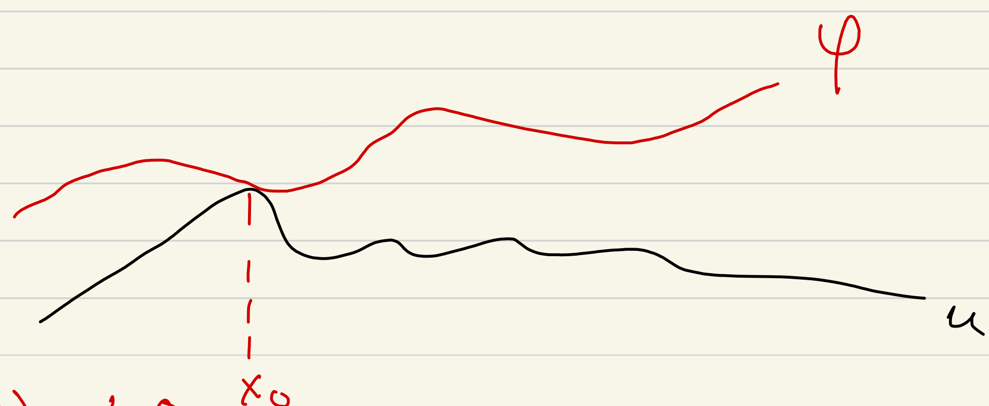
$$\left(\text{Fact: } H(x,p) = \sup_{v}\left(-b(x,v)\cdot p - f(x,v)\right) \geq -b(x,v)\cdot p - f(x,v)\right)$$

$$\Rightarrow \frac{1}{t}\int_0^t \left[\lambda u\left(y_x(s)\right) + H\left(y_x(s), Du\left(y_x(s)\right)\right)\right] ds \geq 0$$

let $t \to 0^+$: $\lambda u(x) + H\left(x, Du(x)\right) \geq 0$.

Rigorous proof:

(I) Subsolution proof:



Want to show:

$$\underset{\substack{\| \\ \varphi(x)}}{\lambda u(x)} + H\left(x, D\varphi(x)\right) \leq 0$$

Think of $v(\cdot) \equiv v \in V$, write $y_{x,v(\cdot)} = y_x$. We have $\longleftrightarrow \varphi(y_x(t))$

$$\varphi(x) = u(x) \leq \int_0^t e^{-\lambda s} f(y_x(s), v(s))\,ds + e^{-\lambda t} u(y_x(t))$$

$$\Rightarrow \frac{\varphi(x) - e^{-\lambda t}\varphi(y_x(t))}{t} - \frac{1}{t}\int_0^t e^{-\lambda s} f(y_x(s), v(s))\,ds \leq 0$$

Take $\lim_{t\to 0^+}$ and ASSUME everything is SMOOTH

$$\lim_{t\to 0^+} - \frac{e^{-\lambda t}\varphi(y_x(t)) - e^{-\lambda\cdot 0}\varphi(y_x(0))}{t} = -\frac{d}{dt}\left(e^{-\lambda t}\varphi(y_x(t))\right)\Big|_{t=0}$$

constant control

$$= \lambda\varphi(x) - D\varphi(x)\cdot y_x'(0) = \lambda\varphi(x) - D\varphi(x)\cdot b(x,v)$$

Then $\lambda\varphi(x) + [-b(x,v)\cdot D\varphi(x) - f(x,v)] \leq 0, \ \forall v \in V$

Sup over $v \in V$: $\lambda\varphi(x) + H(x, D\varphi(x)) \leq 0$

II/ Supersolution test.

Want to show : $\underset{\psi(x)}{\underset{\shortparallel}{\lambda u(x)}} + H(x, D\psi(x)) \geqslant 0$

DPP :

$$\underset{\psi(x)}{\underset{\shortparallel}{u(x)}} = \inf_{v(\cdot)} \Big[\int_0^t e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\,ds + e^{-\lambda t} \underbrace{u(y_{x,v(\cdot)}, t))}_{\geqslant\ \psi(y_{x,v(\cdot)}(t))} \Big]$$

$$\psi(x) - \inf_{v(\cdot)} \left[\int_0^t e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\,ds + e^{-\lambda t}\psi(y_{x,v(\cdot)}(t))\right] \overset{?}{\geq} 0$$

$$\iff \sup_{v(\cdot)} \left[\psi(x) - e^{-\lambda t}\psi(y_{x,v(\cdot)}(t)) - \int_0^t e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\,ds\right] \geq 0$$

$$\iff \sup_{v(\cdot)} \left[\overbrace{-\int_0^t \frac{d}{ds}\left(e^{-\lambda s}\psi(y_{x,v(\cdot)}(s))\right)ds} - \int_0^s e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\,ds\right] \geq 0$$

$$\iff \sup_{v(\cdot)} \frac{1}{t}\int_0^t e^{-\lambda s}\Big(\lambda\psi(y_{x,v(\cdot)}(s)) - D\psi(y_{x,v(\cdot)}(s))\cdot \underbrace{y'_{x,v(\cdot)}(s)}_{b(y_{x,v(\cdot)}(s),\, v(s))} - f(y_{x,v(\cdot)}(s), v(s))\Big)\,ds \geq 0$$

$$\Leftrightarrow \sup_{v(\cdot)} \frac{1}{t}\int_0^t e^{-\lambda s}\Big(\lambda\,\psi(y_{x,v(\cdot)}(s)) + H(y_{x,v(\cdot)}(s), D\psi(y_{x,v(\cdot)}(s))\Big)ds \geq 0$$

Recall: $y'_{x,v(s)}(s) = b(y_{x,v(\cdot)}(s), v(s))$

We proved: $|y_{x,v(\cdot)}(t) - y_{x,v(\cdot)}(s)| \leq C|t-s|$

$\Rightarrow$ for $t>0$ small: $|y_{x,v(\cdot)} - x| \leq Cs \leq Ct$ ← small.

$\Rightarrow$ Let $t \to 0^+$: $\lambda\psi + H(x, D\psi(x)) \geq 0$

Thus, we have shown that u (value function) is a viscosity solution to $\lambda u + H(x,Du) = 0$ in $\mathbb{R}^n$.

Recall $\begin{cases} p \to H(x,p) \text{ is convex: } H(x,p) = \sup\limits_{v \in V} \left(-b(x,v)\cdot p - f(x,v)\right) \\ |H(x,p) - H(x,q)| \le C|p-q| \\ |H(x,p) - H(y,q)| \le C(1+|p|)|x-y| \end{cases}$

Then we can conclude that u is the unique viscosity solution.

Note: $H(x,p) = \sup\limits_{v \in V} \left(-b(x,v)\cdot p - f(x,v)\right)$ is convex in p.

H may or may not be coercive in p $\left(\liminf\limits_{|x|\to+\infty} H(x,p) = +\infty\right)$

Ex: $V = \overline{B}(0,1)$, $f(x,v) = f(x)$, $b(x,v) = v$.

$$H(x,p) = \sup_{|v|\le 1} \left(-v\cdot p - f(x)\right) = |p| - f(x)$$

10/09/2023. Infinite Horizon Problem

$$\boxed{u(x) = \inf_{v(\cdot)} \int_0^\infty e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\, ds.}$$

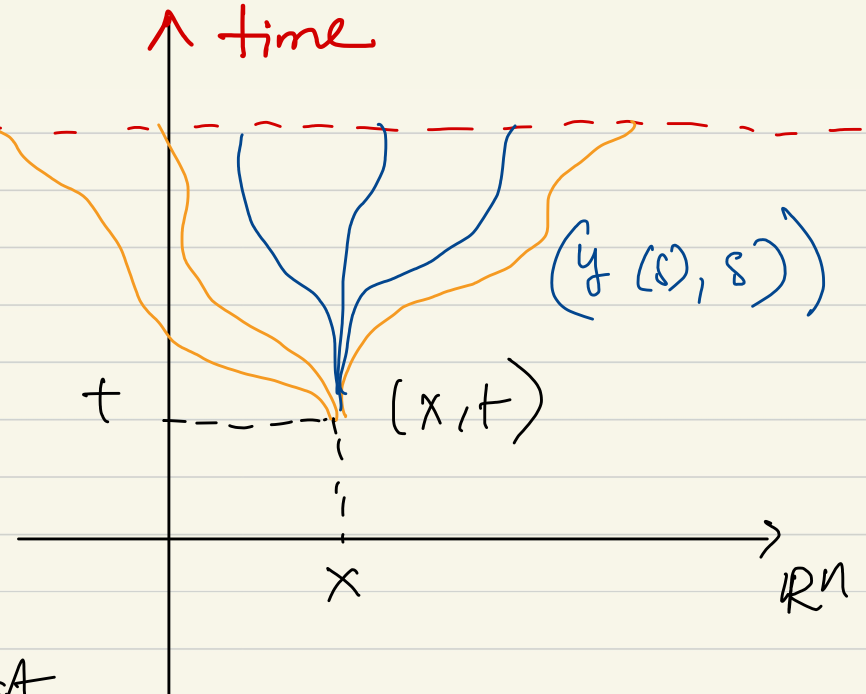
Then u is the unique solution of

$$\lambda u + H(x, Du) = 0 \text{ in } \mathbb{R}^n$$

where
$$H(x,p) = \max_{v \in V} \Big(\underbrace{-b(x,v)}_{\text{vector field}} \cdot p - \underbrace{f(x,v)}_{\text{cost function.}} \Big)$$

A pick description finite horizon problem

The dynamic stop in finite time (terminal time T)



$y(s)$: your position $t \le s \le T$

$y(x, t, v(\cdot)) = y$ for simplicity

mining cost + terminal cost = Total cost

V : compact metric space.

For each $t \in [0, T]$, $A_t = \{ v : [t, T] \to V : v \text{ is measurable} \}$

For $(x, t) \in \mathbb{R}^n \times (0, T]$ fixed & $v(\cdot) \in A_t$, consider.

$$\begin{cases} y'(s) = b(x(s), v(s)) & ; t \le s \le T, \\ y(t) = x \end{cases}$$

$y(t) = x$ starting point $\longrightarrow$ $y(T)$: ending point.

Running cost $= \int_t^T f(y(s), v(s))\, ds$ (works for $\int_t^T f(y(s), s, v(s))\, ds$

Terminal cost : ($g: \mathbb{R}^n \to \mathbb{R}$ is a given function) $\boxed{g(y(T))}$

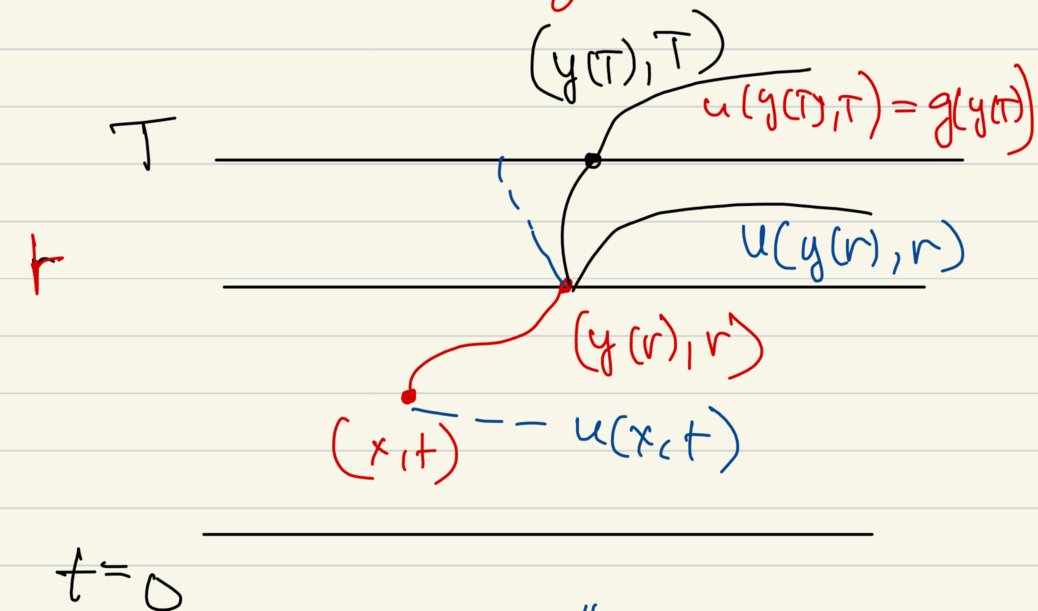
Cost functional : $J(x, t, v(\cdot)) = \int_t^T f(y(s), v(s))\, ds + g(y(T))$

Value function : $u(x,t) = \inf\limits_{v(\cdot) \in A_t} J(x, t, v(\cdot))$

$$= \inf_{v(\cdot) \in A_t} \left[\int_t^T f(y(s), v(s))\, ds + g(y(T)) \right]$$

DPP: For any fixed $t < r < T$

$$u(x,t) = \inf_{v(\cdot)\in \mathcal{A}_t} \left[\int_t^r f(y(s), r(s))\,ds + u(y(r), r) \right]$$



Theorem: u is the unique viscosity solution to the "terminal time" H-J equation

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Here $H(x,p) = \min_{v \in V} \left[b(x,v) \cdot p + f(x,v) \right] = -\max_{v \in V} \left[-b(x,v) \cdot p - f(x,v) \right]$

Remark. Min problem is harder than the traditional calculus of Variational problem.

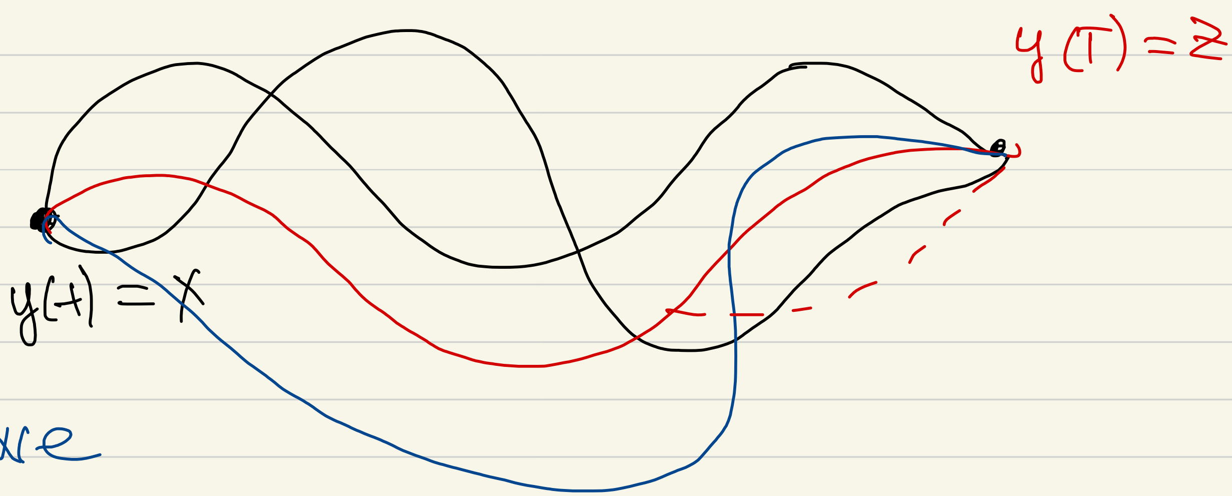
① Let's first fixed the end point $y(T) = z$.

$$\psi(z) = \inf_{\substack{v(\cdot) \\ y(t) = x \\ y(T) = z}} \left[\int_t^T f(y(s), v(s))\,ds + g(z) \right]$$

$$= g(z) + \inf_{\substack{v(\cdot) \\ y(t)=x \\ y(T)=z}} \left[\int_t^T f(y(s), v(s))\, ds \right].$$

② Then,

$$u(x,t) = \min_{z \in \mathbb{R}^n} \psi(z)$$



Remark: In practice, we want to be able to compute both $u(x,t)$ and the optimal control $v^*(\cdot)$ quickly.

How do we go from terminal time to initial time

$u(x,t) \longrightarrow$ changing of variable $\tilde{u}(x,t) = u(x, T-t)$.

Then $\tilde{u}(x,0) = u(x,T) = g(x)$, $\tilde{u}_t = -u_t(x, T-t)$

$$-\tilde{u}_t + H(x, D\tilde{u}) \Rightarrow \begin{cases} \tilde{u}_t + \tilde{H}(x, D\tilde{u}) = 0 \\ \tilde{u}(x,0) = g(x) \end{cases}$$

$$\tilde{H}(x,p) = \max_{v \in V} \left(-b(x,v)\cdot p - f(x,v)\right)$$

Regularity of u in infinite horizon problems

Claim: If H is coercive in p $\left(\lim_{|p|\to\infty} \inf_x H(x,p) = +\infty\right)$, then $u \in Lip(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$. (Theorem 2.9)

Claim 2: What happens if H is not coercive in p? Denote by

$$\lambda_0 = \|D_x b(x,v)\|_{L^\infty(\mathbb{R}^n \times V)}$$

i) If $\lambda > \lambda_0$, $u \in Lip(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$

ii) If $\lambda = \lambda_0$, $u \in C^{0,\alpha}(\mathbb{R}^n)$, $\forall\, 0 < \alpha < 1$.

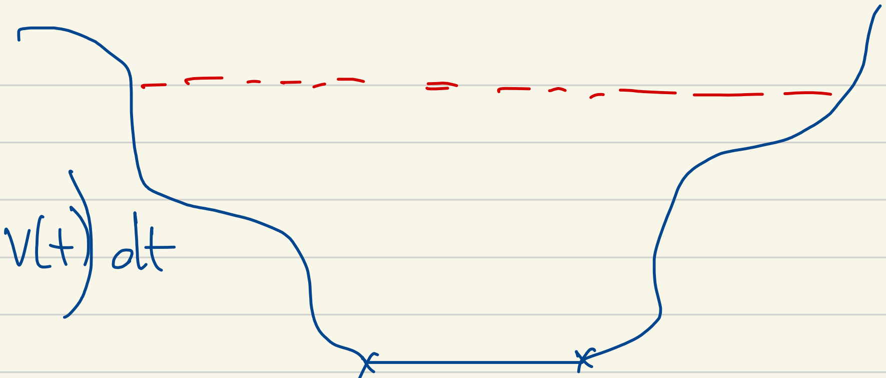
iii) If $0 < \lambda < \lambda_0$, $u \in C^{0,\lambda_0/\lambda}(\mathbb{R}^n)$

$$\begin{cases} y_x'(t) = b(y_x(t), v(t)) \\ y_x(0) = x \end{cases} \qquad \begin{cases} y_z'(t) = b(y_z(t), v(t)) \\ y_z(0) = z. \end{cases}$$

$$|y_x(t) - y_z(t)| \le C e^{\lambda_0 t} |x - z|$$

i) If $\lambda > \lambda_0$: $u(x) - u(z)$

$$\left| \int_0^\infty e^{-\lambda t} \left| f(y_x(t), v(t)) - f(y_z(t), v(t)) \right| dt \right.$$



$$\leq C \cdot \int_0^\infty e^{-\lambda t} \cdot e^{\lambda_0 t} |x - z| \, dt = \frac{C|x-z|}{\lambda - \lambda_0}$$

Also $|u(x)| = \left| \int_0^\infty e^{-\lambda s} f(y_x(s), v(s)) ds \right| \leq C \int_0^\infty e^{-\lambda s} ds = \frac{C}{\lambda}$

ii + iii

$$u(x) = \inf_{v(\cdot)} \left(\int_0^t e^{-\lambda s} f(y_x(s), v(s)) ds + e^{-\lambda t} u(y_x(t)) \right)$$

$$|u(x) - u(z)| \le C \int_0^t e^{-\lambda s} e^{\lambda_0 s} |x-z| \, ds + C e^{-\lambda t}$$

$(\lambda = \lambda_0)$ $$= C \underbrace{\left(t |x-z| + e^{-\lambda t} \right)}_{\varepsilon(t)}$$

Same for $\lambda < \lambda_0$.

<u>Example</u>: (G - equation : combustion literature)

$$H(x,p) = |p| + W(x) \cdot p = \max_{|v| \le 1} \left(p \cdot (v + W(x)) \right)$$

Here, $W(x)$ is a given vector field,

$\operatorname{div} W = 0$, and $\|W\|_{L^\infty}$ can be large (>1)

$\boxed{\underline{\text{Open 5}}: \quad \lambda u^\varepsilon + |Du^\varepsilon| + W(x)\cdot Du^\varepsilon = \varepsilon \Delta u^\varepsilon \text{ in } \mathbb{R}^n \qquad \begin{cases} u^\varepsilon \to u? \\ \text{Rate?} \end{cases}}$

<u>10/11/2023</u> Optimal Control Viewpoint

$b(x,v)$: driving vector field

$f(x,v)$: cost function

$$\boxed{H(x,p) = \max_{v \in V} \left[-b(x,v)\cdot p - f(x,v) \right]}$$

compact set

$\lambda u(x) + H(x, Du(x)) = 0$ in $\mathbb{R}^n$ | $\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \end{cases}$

Recall that in this situation, $p \to H(x,p)$ convex.

$\begin{cases} H \text{ grow at most linearly in } p. \\ \quad (|H(x,p)| \leq (1+|p|) \cdot C \\ H \text{ might not be coercive in } p \end{cases}$

(e.g., $H(x,p) = |p| + W(x) \cdot p$

$\operatorname{div} = 0$, $\|W\|_{L^\infty}$ can be large.

Question: But what if we go back to the most classical case.

$$H(x,p) = \underbrace{\tfrac{1}{2}|p|^2}_{\text{kinetic energy}} + \underbrace{V(x)}_{\text{potential energy}}$$

(grows quadratically in p)

Topic. Lagrangian framework.

x : position

v : its velocity

p : momentum

m : mass ($m=1$)

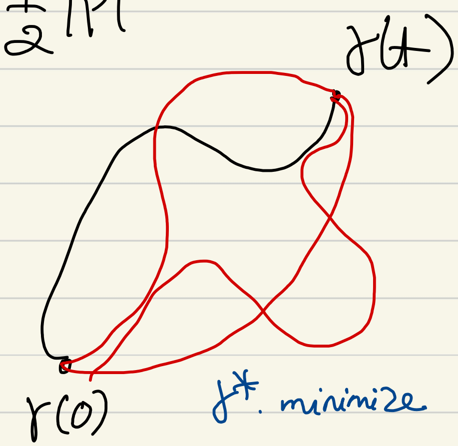
$p = m.v = v$

Kinetic energy $= \frac{1}{2} m |v|^2 = \frac{1}{2}|p|^2$

potential energy: $V(x)$

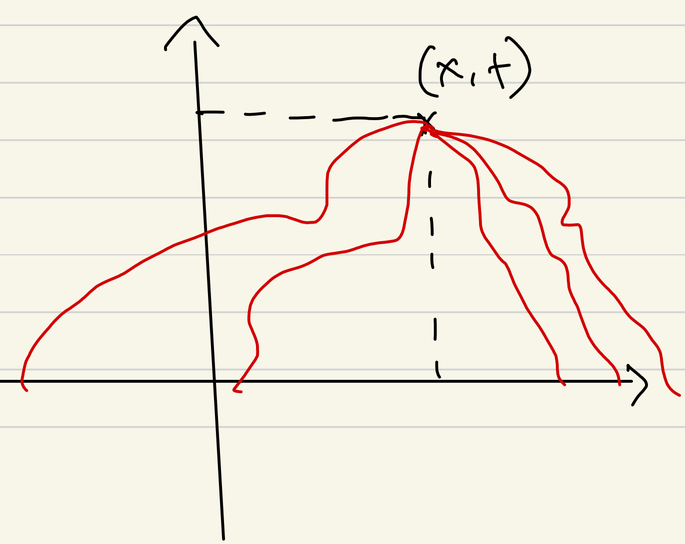
Action functional : minimize

$$\int_0^t \left(\left(\frac{1}{2}\right) |\dot{\gamma}(s)|^2 - V(\gamma(s)) \right) ds.$$



$L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$(x,v) \to L(x,v) = \frac{1}{2}|v|^2 - V(x)$



Prop. Assume $H(x,p) = \frac{1}{2}|p|^2 + V(x)$

Then, the viscosity solution to the Cauchy problem in

$$u(x,t) = \inf_{\gamma(t)=x} \left\{ \int_0^t \underbrace{\left[\frac{1}{2} |\dot\gamma(s)|^2 - V(\gamma(s)\right) ds}_{\text{Running cost}} + \underbrace{g(\gamma(0))}_{\text{terminal cost.}} \right\}$$

And the viscosity solution to the Static is $(\lambda w + H(x, Dw) = 0)$

$$W(x) = \inf_{\gamma(0)=x} \int_0^\infty e^{-\lambda s} \underbrace{\left(\frac{1}{2} |\dot\gamma(s)|^2 - V(\gamma(s))\right)}_{\text{Running cost}} ds$$

Let's look at

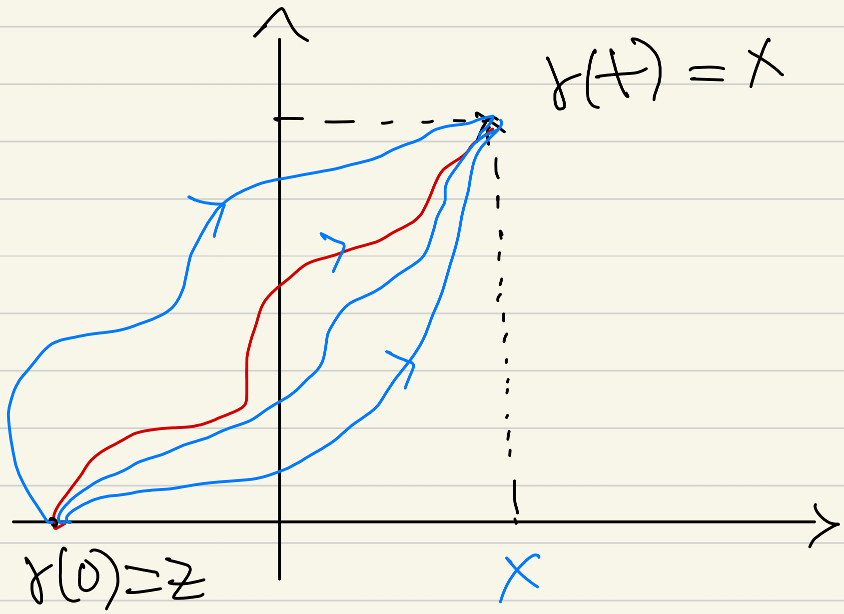
$$u(t,x) = \inf_{\gamma(t)=x} \left\{ \int_0^t \frac{1}{2} |\dot\gamma(s)|^2 - V(\gamma(s))) ds + g(\gamma(0)) \right\}$$

$$= \inf_{z \in \mathbb{R}^n} \left(\inf_{\substack{\gamma(0)=z \\ \gamma(t)=x}} \left(\int_0^t \frac{1}{2} |\dot{\gamma}(s)|^2 - V(\gamma(s))\, ds + g(z) \right) \right)$$

For the classical calculus of variation of

$$\inf_{\substack{\gamma(0)=z \\ \gamma(t)=x}} \left(\int_0^t \frac{1}{2} |\dot{\gamma}(s)|^2 - V(\gamma(s))\, ds \right)$$

$$= \int_0^t \frac{1}{2} |\dot{\xi}(s)|^2 - V(\xi(s))\, ds$$



ξ Satisfies the Euler–Lagrange equation

$$L(x,v) = \frac{1}{2}|v|^2 - V(x) \qquad , D_v L = v.$$

$$\frac{d}{ds}\left(D_v L(\xi(s), \dot{\xi}(s))\right) = D_x L(\xi(s), \dot{\xi}(s))$$

$$\frac{d}{ds}\left(\dot{\xi}(s)\right) = -D_x V(\xi(s))$$

$$\ddot{\xi}(s) = -D_x V(\xi(s))$$

$$ma = F.$$

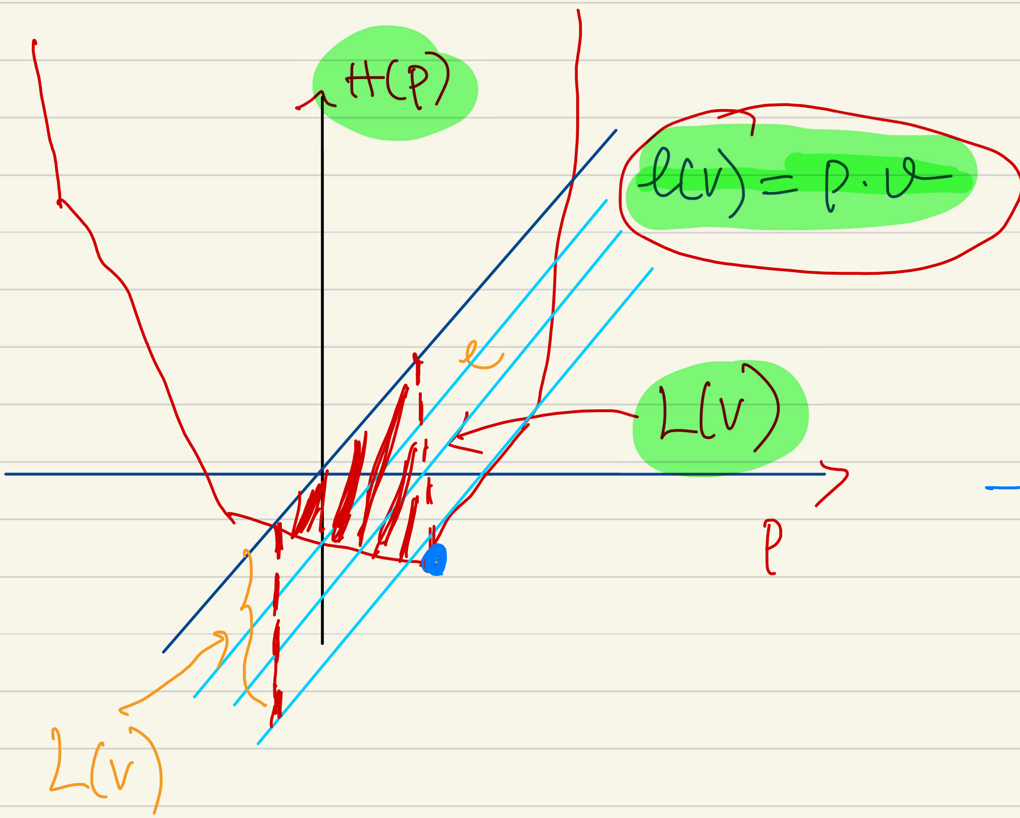
Legendre's transform.

$$H \text{ convex in } p \longrightarrow H^* = L \longrightarrow H^{**} = L^* = H$$

<u>Define</u>: Let's assume $H : \mathbb{R}^n \to \mathbb{R}$

$$\begin{cases} p \longrightarrow H(p) \text{ is convex in } p \\ \lim\limits_{|p| \to \infty} \dfrac{H(p)}{p} = +\infty \quad \text{superlinearity of } H \text{ in } p. \end{cases}$$

How to see $L(v)$ from here

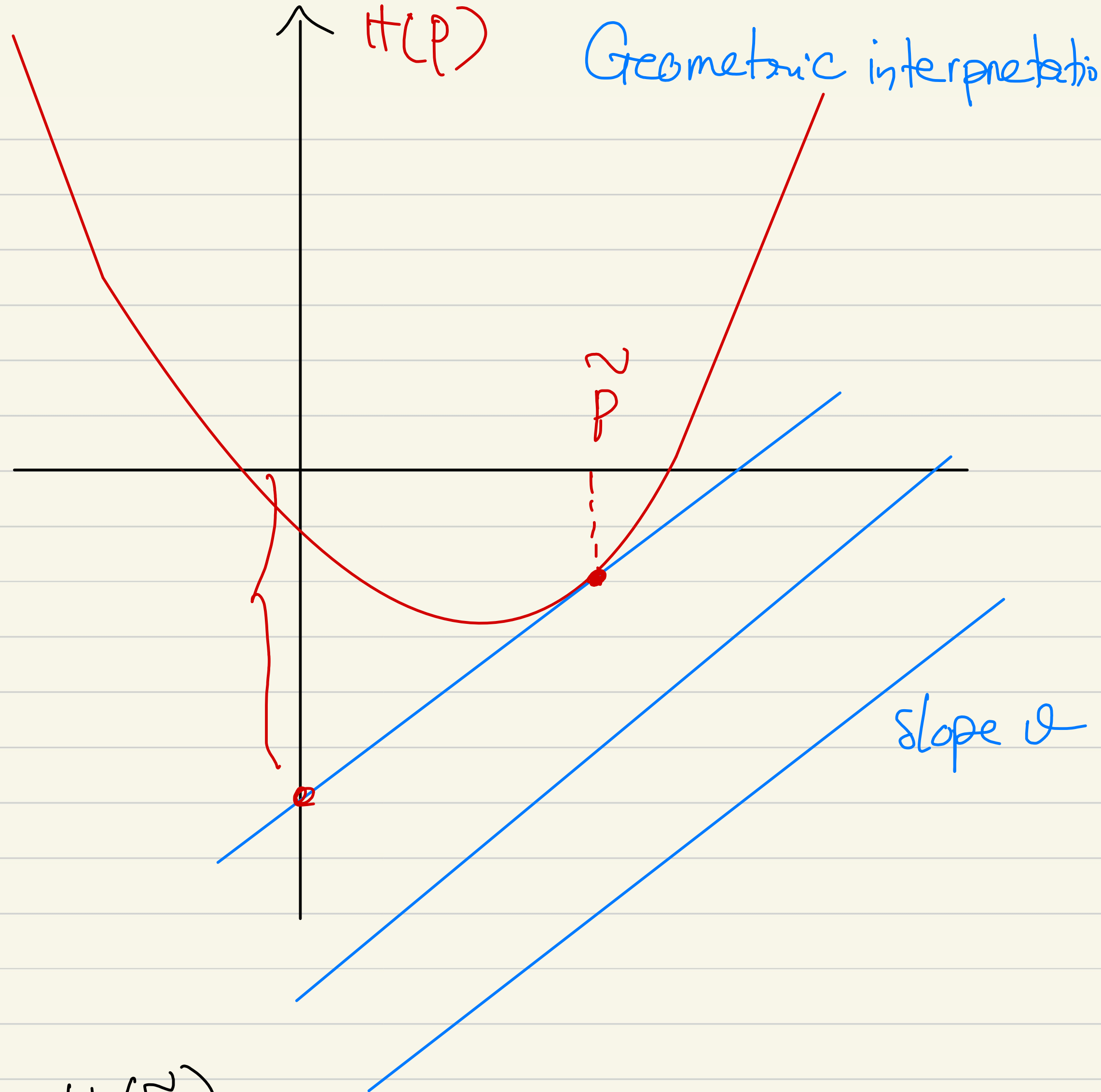


$H^* = L : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$H^*(v) = L(v)$$
$$= \sup_{p\in\mathbb{R}^n} \left(p\cdot v - H(p)\right)$$

Then $\begin{cases} L \text{ is convex in } v \\ L(v) \text{ is finite in each } v \end{cases}$

Geometric interpretatio



Take $c \in \mathbb{R}$ such that

$\ell(p) = p \cdot v - c$ touches H

from below at $\tilde{p}$

$$\ell(0) = -c$$

$$\ell(p) \leq H(p) \text{ and } \ell(\tilde{p}) = H(\tilde{p})$$

$$p\vartheta - c \le H(p)$$

$$\Leftrightarrow \; p\cdot\vartheta - H(p) \le C$$

$$\Leftrightarrow \; \sup_p \{ p\cdot\vartheta - H(p) \} = c = L(v) \quad (\text{equality at } \tilde{p})$$

$$L(v) = c = -l(0)$$

<u>Theorem</u> [Legendre's transform].

$$L(v) = H^*(v) = \sup_p (p\cdot v - H(p))$$

And $H^{**}(p) = L^*(p) = \sup_v (p\cdot v - L(v))$

Then, $\begin{cases} L \text{ is convex, finite, superlinear in } v. \\ \boxed{H^{**} = L^* = H.} \end{cases}$

$R > 0$

proof: L is superlinear in v

$$L(v) = \sup_{p} \left(p \cdot v - H(v) \right) = \sup_{|p| \leq R} \left(p \cdot v - H(p) \right)$$

$$\geq \sup_{|p| \leq R} p \cdot v - \max_{|p| \leq R} |H(p)|.$$

$$L(v) \geq \sup_{|p|\leq R} p\cdot v - \max_{|p|\leq R} |H(p)|$$

$$= R\cdot|v| - \max_{|p|\leq R} |H(p)|$$

$$\liminf_{|v|\to\infty} \frac{L(v)}{|v|} \geq R - \underbrace{\limsup_{|v|\to\infty} \frac{1}{|v|} \max_{|p|\leq R} H(p)}_{=0} \quad \forall R.$$

$$\Rightarrow \lim_{|v|\to\infty} \frac{L(v)}{|v|} = +\infty$$

Recall from optimal control

vector field

$$H(x,p) = \sup_{v \in V} \left[-b(x,v)\cdot p - \underset{\text{cost}}{\underset{\uparrow}{f(x,v)}} \right]$$

$\dot{\gamma}(t) = -v(t)$

From Legendre's tranforms.

$$\boxed{H(x,p) = \sup_{v \in V} \left[-b(x,v)\cdot p - f(x,v) \right]}$$

$$u(t,x) = \inf_{\gamma(t)=x} \left\{ \underbrace{\int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds}_{\text{running}} + \underbrace{g(\gamma(0))}_{\text{terminal}} \right\}$$

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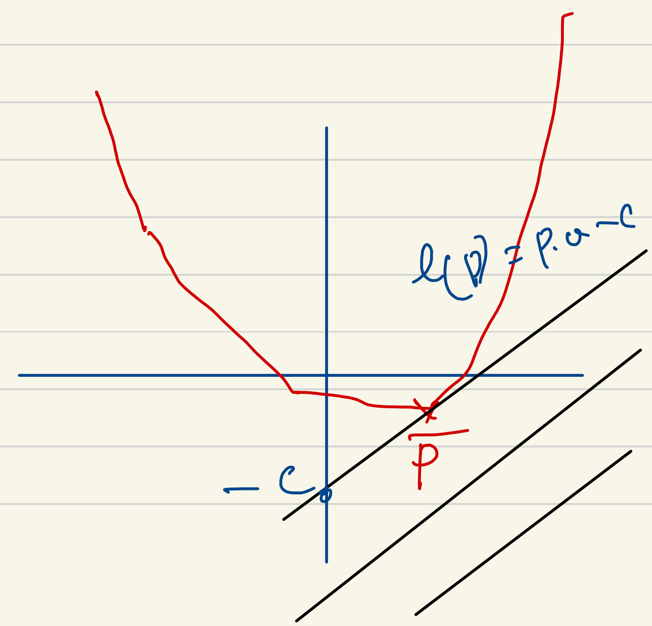
recap: $\begin{cases} p \to H(p) \text{ is convex} \\ \displaystyle\lim_{p\to\infty} \frac{H(p)}{p} = +\infty \text{, that is, } H \text{ is super linear.} \end{cases}$

Legendre's transform: $L = H^* : \mathbb{R}^n \to \mathbb{R}$

$$L(v) = H^*(v) = \sup_{p\in\mathbb{R}^n} [p\cdot v - H(p)]$$

Geometric interpresentation given v.



$\ell(p) = p\cdot v - c$ touches H from below at $\bar{p}$

Then, $h(v) = c = -l(0)$

$p \cdot v - c \leq H(p)$, $\forall p$ with equality at $p = \bar{p}$.

$\Leftrightarrow p \cdot v - H(p) \leq c$

$\Rightarrow L(v) = \sup_p \left(p \cdot v - H(p) \right) = c$.

<u>Theorem:</u> L satisfies

i) L is convex, finite, superlinear

finite $\uparrow$ superlinear of H

superlinear $\uparrow$ H is finite.

ii) $L^* = H^{**} = H$, where $L^*(p) = H^{**}(p) = \sup_v [p \cdot v - L(v)]$

Proof of the duality: $L(v) = \sup_p \left[p\cdot v - H(p) \right)$

$\Rightarrow$ Frechel's inequality $L(v) \geq p\cdot v - H(p)$

$$\Leftrightarrow L(v) + H(p) \geq p\cdot v \quad , \forall p, v.$$

$$\Rightarrow H(p) \geq p\cdot v - L(v)$$

Thus, $H(p) \geq \sup_v \left(p\cdot v - L(v) \right) = L^*(p)$

We now need to prove the converse: $L^*(p) \geq H(p)$.

Note: $L^*(p) = \sup_v \left(p\cdot v - L(v) \right) = \sup_v \left[p\cdot v - \sup_r \left[r\cdot v - H(r) \right] \right]$

$$= \sup_v \inf_r \left[(p-r)\cdot v + H(r) \right] \overset{?!}{\geq} H(p)$$

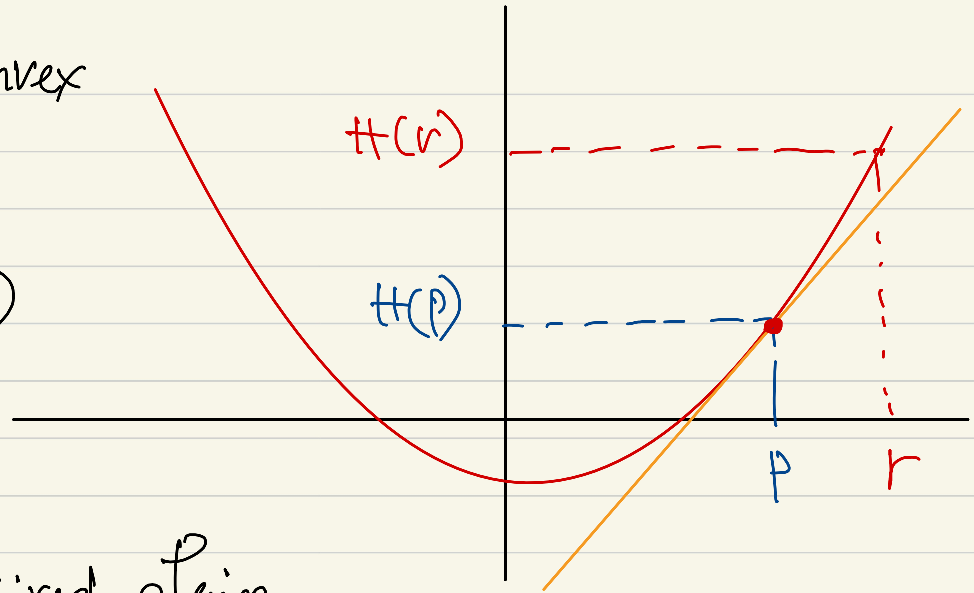
(can choose r, have inequality for all r)

Let's assume $H \in C^1(\mathbb{R}^n)$. As H convex

$$H(r) \geq H(p) + \underbrace{DH(p)} \cdot (r-p)$$

$$\Leftrightarrow H(r) + \underbrace{DH(p)} \cdot (p-r) \geq H(p)$$

$\forall r$.



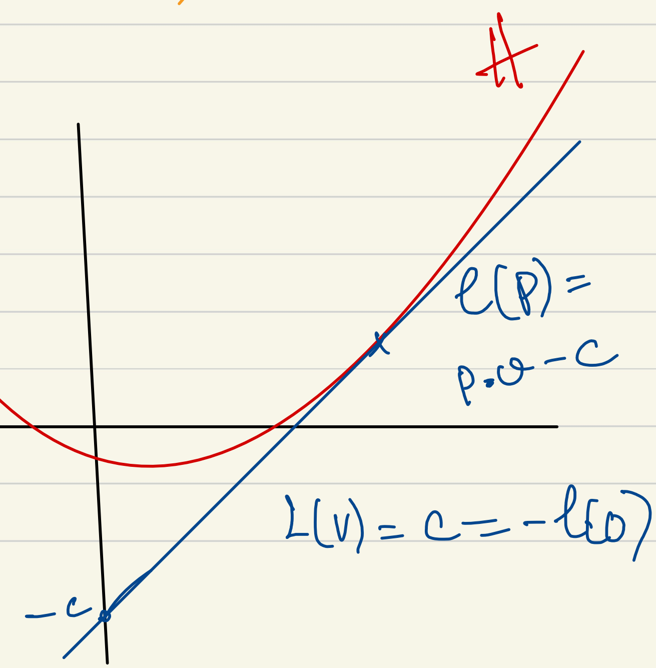
Choose $v = DH(p)$ to get the desired claim.

Remark

1) The proof holds in general: If H is not C^1,

$$v \in D^- H(p) = \partial H(p)$$

(convex analysis)



② Fenchel's equality.

$$H(p) + L(v) \geq p \cdot v$$

Equality happens $\Longleftrightarrow v \in D^- H(p) = \partial H(p)$

$\Longleftrightarrow p \in D^- L(v) = \partial L(v)$

③ If we only have $H : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex then

$$H^{**} = H.$$

④ Note Legendre's transform has 2 important properties acting on convex problem.

$$\begin{cases} H_1 \leq H_2 \Rightarrow H_1^* \geq H_2^* \\ H^{**} = H \end{cases}$$

Theorem [Characteristic theorem, 2009] Any transform acting on convex functions satisfying the above two properties has to be a Legendre's transform up to translations & dilations.

Examples: ① $H(x,p) = \frac{1}{2}|p|^{\alpha} + V(x)$

(Note that Legendre's transform work in exactly the same with $H(x,p)$.

$$L(x,v) = \sup_{p\in\mathbb{R}^n}\left[p\cdot v - H(x,p)\right]$$

$$= \sup_{p\in\mathbb{R}^n}\left(p\cdot v - \frac{1}{2}|p|^2 - V(x)\right)$$

$$= \sup_{p \in \mathbb{R}^n} \left(p \cdot v - \frac{1}{2} |p|^2 \right) - V(x)$$

$$\underline{\text{Cauchy Schwartz}} = \frac{1}{2} |v|^2 - V(x)$$

② $H(x,p) = \dfrac{|p|^m}{m} + V(x)$, $m > 1$.

$$L(x,v) = \sup_p \left[p \cdot v - H(x,p) \right] = \sup_p \left[p \cdot v - \frac{|p|^m}{m} - V(x) \right]$$

$$= \sup_p \left[p \cdot v - \frac{|p|^m}{m} \right] - V(x)$$

$$= \frac{|p|^{m'}}{m'} - V(x) \qquad \left(\frac{1}{m} + \frac{1}{m'} = 1. \right)$$

Optimal control formula via Lagrangian framework.

Theorem 2: Consider $\lambda u + H(x, Du) = 0$ in $\mathbb{R}^n$.

Assume H is convex & superlinear in p. Then, we have

$$\boxed{u(x) = \inf_{\gamma(0)=x} \int_0^\infty e^{-\lambda s} L(\gamma(s), -\dot{\gamma}(s))\, ds}$$

Proof: DPP $\longrightarrow$ v is solution.

Comparison between optimal control & Lagrangian.

Y: compact control set	$\mathbb{R}^n$ Not compact.
$b(x,v)$: vector field	$b(x,v) = -v$: vector field. $(v \in \mathbb{R}^n)$
$f(x,v)$: cost	$L(x,v)$: cost

$$H(x,p) = \max_{v \in V} \left[-b(x,v)\cdot p - f(x,v) \right]$$

$$u(x) = \inf_{v(\cdot)} \int_0^\infty e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\, ds$$

$$\begin{cases} y'_{x,v(\cdot)}(s) = b(y_{x,v(\cdot)}(s), v(s)) \\ y'_{x,v(\cdot)}(0) = x \end{cases}$$

$$H(x,p) = \max_{v \in \mathbb{R}^n} \left[v\cdot p - L(x,v) \right]$$

Vector field $- v = b(x,v)$

$$\begin{cases} v(s) = -\dot{\gamma}(s) \\ y_{x,v(\cdot)}(s) = \gamma(s) \end{cases}$$

$$u(x) = \inf_{\gamma(0)=x} \int_0^\infty e^{-\lambda s} L(\gamma, \dot{\gamma})\, ds.$$

First formula,

$$u(x) = \inf_{v(\cdot)} \int_0^\infty e^{-\lambda s} f(y_{x,v(\cdot)}(s), v(s))\, ds.$$

is used in given situation with given b, f.

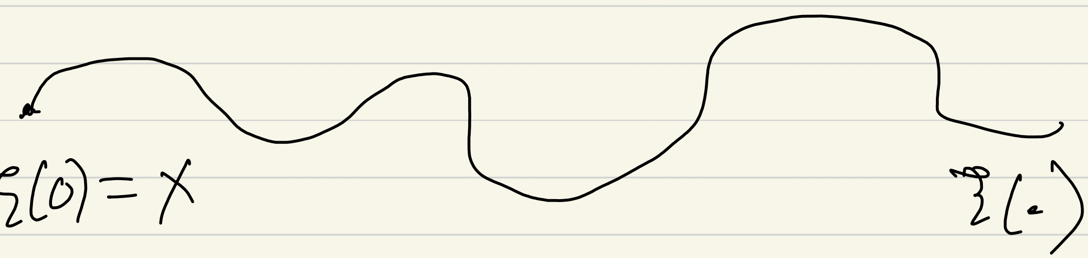
<u>Second formula</u>: If we're given PDE: $\lambda u(x) + H(x, Du(x)) = 0$ in $\mathbb{R}^n$,

Lagrangian's transform

$$u(x) = \inf_{\gamma(0)=x} \int_0^\infty e^{-\lambda s} L(\gamma(s), -\dot{\gamma}(s))\,ds.$$

Better in the sense that we can find a minimizer $\xi(\cdot)$.

$$u(x) = \int_0^\infty e^{-\lambda s} L(\xi(s), -\dot{\xi}(s))\,ds$$



$D^2_{vv} L > 0$

$\xi \in C^k$.

Careful: velocity of admissible curve & minimizer might not be bdd.

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Applications of the Lagrangian/optimal formulations.

$$H(x, p) \longleftrightarrow L(x, v)$$

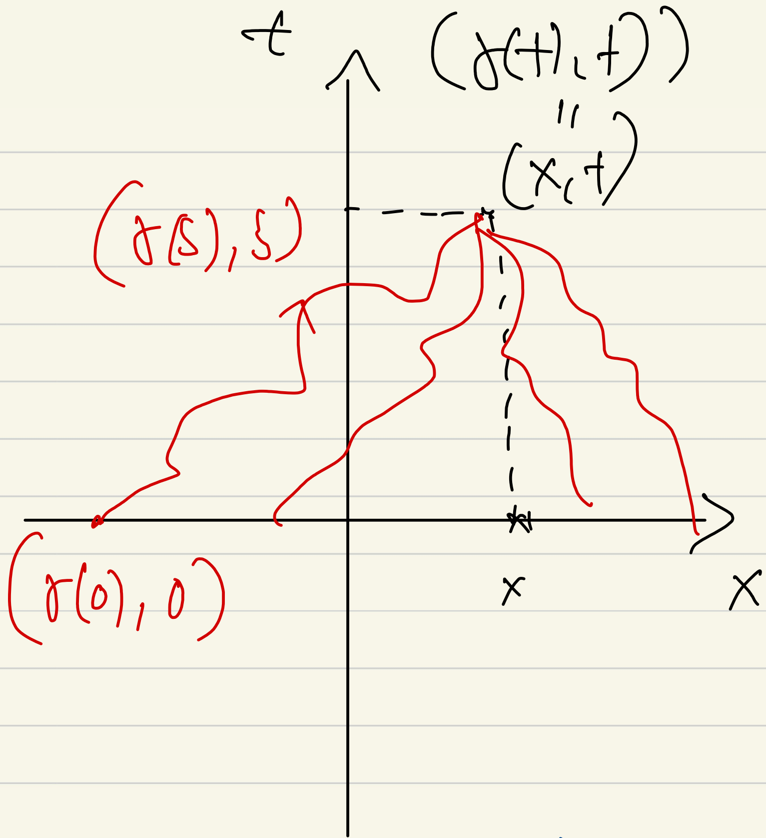
position momentum position velocity.

If H is convex in p, then L is convex in v,

$$L^* = H^{**} = H$$

[Lagrangian formulation]

(c) $$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$



$$u(x,t) = \inf_{\substack{\gamma(t) = x \\ \gamma \in AC([0,t], \mathbb{R}^n)}} \left[\underbrace{\int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds}_{\text{Running cost}} + \underbrace{g(\gamma(0))}_{\text{terminal cost}} \right]$$

$\gamma \in AC([0,T], \mathbb{R}^n)$ = set of all absolutely continuous curves $[0,T] \to \mathbb{R}^n$

I. Hopf - Lax's formula.

($\longleftrightarrow$ Lax - Oleinik formula for conversation laws).

Theorem 1: Assume $H(x,p) = H(p)$ (homogenous situation) is convex. Then

$$(C) \begin{cases} u_t + H(Du) = 0 \\ u(x,0) = g(x) \in BUC(\mathbb{R}^n) \cap Lip(\mathbb{R}^n). \end{cases}$$

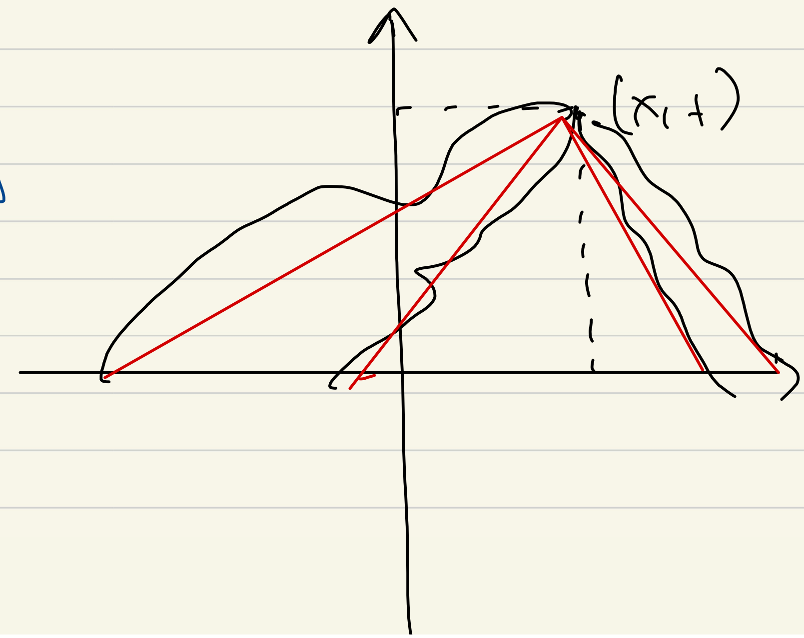
We have

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \left[tL\left(\frac{x-y}{t}\right) + g(y) \right]$$ is a v.s solution to (c).

And it's unique if either $\underbrace{H \text{ is Lipchitz globally}}_{\downarrow}$ OR $\underbrace{H \text{ is corecive in } p}_{\downarrow}$

Structed assumption for uniqueness

Solutions are Lp uniqueness.



Pf: Let's just use Jensen's inequality

$$\int_0^t L(\gamma(s), \dot\gamma(s))\,ds = \int_0^t L(\dot\gamma(s))\,ds$$

$$= t\left(\frac{1}{t}\int_0^t L(\dot\gamma(s))\,ds\right)$$

$$\geq t\, L\left(\frac{1}{t}\int_0^t \dot{\gamma}(s)\, ds\right) = t\, L\left(\frac{\gamma(t) - \gamma(0)}{t}\right)$$

$$= t\, L\left(\frac{x - \gamma(0)}{t}\right)$$

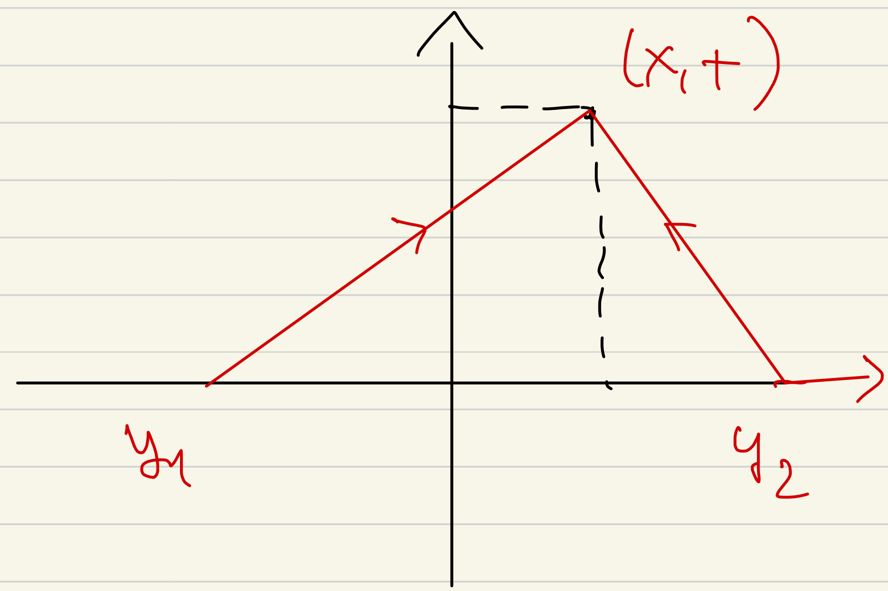
[use: $\sum_{i=1}^{K} \lambda_i L(v_i) \geq L\left(\sum_{i=1}^{K} \lambda_i v_i\right)$ $\quad \lambda_i \geq 0$, $\sum_{i=1}^{K} \lambda_i = 1$.

Equality happens if $\gamma(s) = \gamma(0) + s\,\frac{x - \gamma(0)}{t}$, $0 \leq s \leq t$.

Then the optimal control formula reduces to

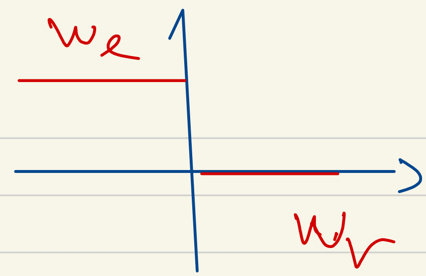
$$u(x,t) = \inf_{\gamma(t)=x} \left[t\, L\left(\frac{x - \gamma(0)}{t}\right) + g(\gamma(0)) \right]$$

$$= \inf_{y (= \gamma(0))} \left[t\, L\left(\frac{x-y}{t}\right) + g(y) \right].$$



Remark:

① If min problem has ≥ 2 solutions at y_1, y_2, u migh be not different at (x, t)



② Scalar conversation law in 1D

$$w_t + (F(w))_x = 0$$

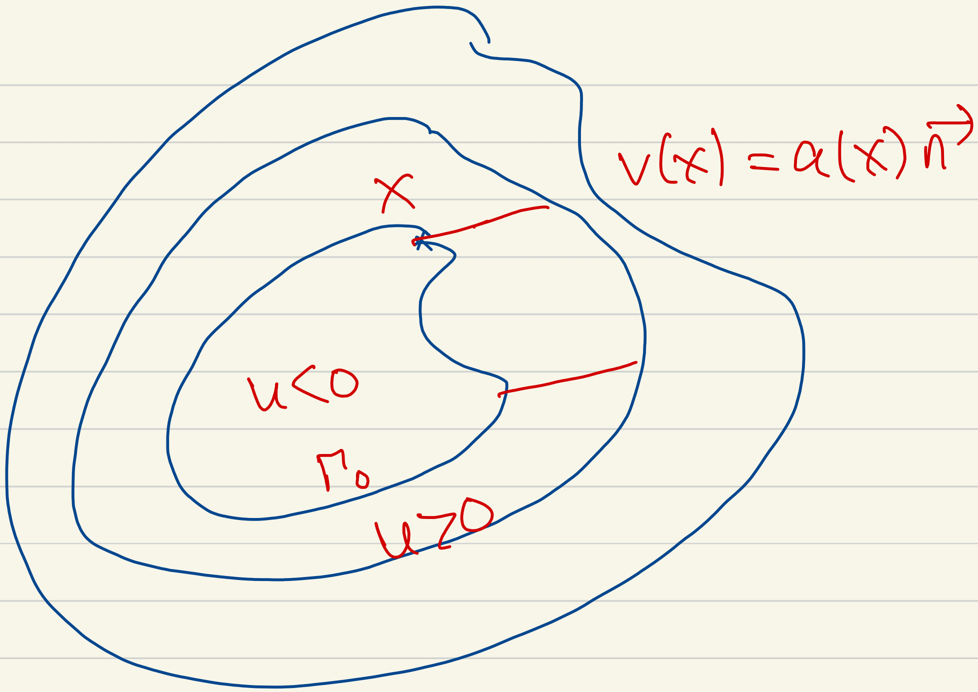
Ideal: Write: $w = u_x \Longrightarrow u_{xt} + (F(u_x))_x = 0$

$$\Longrightarrow u_t + F(u_x) = 0.$$

Riemman problems: $w(x,0) = \begin{cases} w_l, & x < 0 \\ w_r, & x > 0 \end{cases}$

II Front propagation problem. Simplest case first

$$(1) \quad \begin{cases} u_t + a(x)\,|Du| = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$



Assume $a \in C(\mathbb{R}^n, (0,\infty))$

$H(x,p) = a(x)|p|$ convex in p, but only have linear growth

$$L(x,v) = \sup_{p\in\mathbb{R}^n} \left[p\cdot v - H(x,p) \right] = \sup_{p\in\mathbb{R}^n} \left(p\cdot v - a(x)|p| \right)$$

$$= \begin{cases} 0 & \text{if } a(x) \geq |v| \\ +\infty & \text{if } a(x) < |v| \end{cases}$$

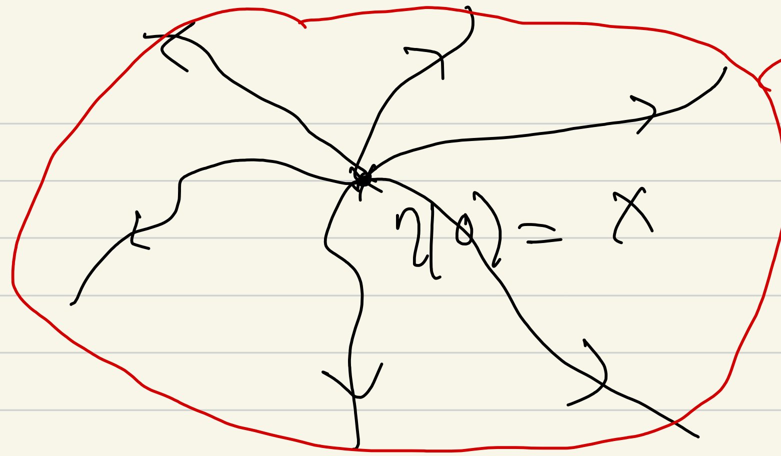
(take $p = s \cdot v$, $s \to +\infty$.)

$$u(x,t) = \inf_{\gamma(t)=x} \left\{ \int_0^t \underbrace{L(\gamma(t), \dot{\gamma}(t))}_{=0 \text{ a.e}} ds + g(\gamma(0)) \right].$$

$$= \inf \left\{ g(\gamma(0)) : |\dot{\gamma}(s)| \leq a(\gamma(s)) \text{ a.e}, s \in (0,t), \; \gamma(t) = x \right\}$$

(reverse time)

$$u(x,t) = \inf \left\{ g(\eta(t)) : |\dot{\eta}(s)| \leq a(\eta(s)) \text{ a.e. } s \in (0,t), \eta(0) = x \right\}$$



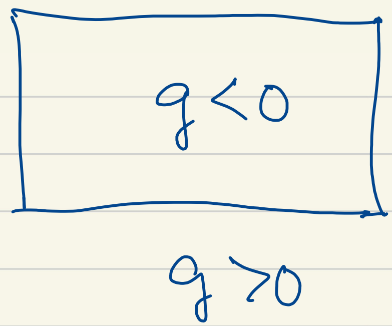
all possible endpoints $\eta(t)$ = Reachable set $R_t(x)$

$$u(x,t) = \min \{ g(y) : y \in R_t(x) \}$$

$$R_t(x) = \{ y \in \mathbb{R}^n : \exists\, \eta : [0,t] \to \mathbb{R}^n,\ \eta(0) = x,\ \eta(t) = y,\ |\dot\eta(s)| \le a(\eta(s)) \text{ a.e. } s \}$$

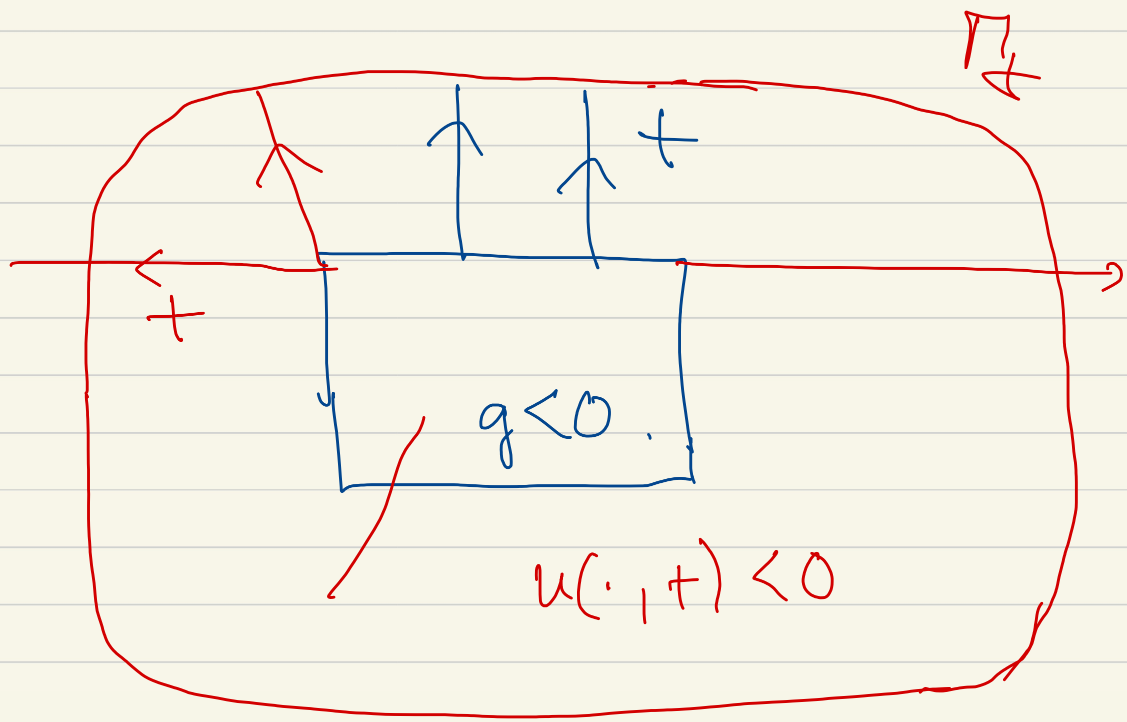
Back to the simplest example, $a \equiv 1$, $\begin{cases} u_t + |Du| = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$

$\Gamma_0 = \{ g = 0 \} =$



$|\mathring{n}| \le 1.$

Γ_t = 4 line segments + 4 quaters of circles radius t



i) Solutions (physical) pass singularities.

ii) viscosity solution is the one that select the maximal behavior among all possible choices. $u(\cdot,t) > 0$

Remark 1: $H(x,t,p) = a(x,t)\,|p|$,

$$u(x,t) = \inf \left\{ g(y) : \begin{array}{l} \exists\, \eta : [0,t] \to \mathbb{R}^n \\ \eta(0) = y,\ \eta(t) = x, \\ |\dot{\eta}(s)| \leq a(\eta(s), s) \end{array} \right\}$$

Also works for $H(x,t,p) = a(x,t)\,|p|$, where a can change signs

$$H(x,p) = \sup_{|v| \leq 1} [a(x)\, p \cdot v] = \sup [a(x) \cdot p \cdot (-v)]$$

$$= \sup_{|v|\le 1} \left(p \cdot (-a(x) v) \right),$$

$V = \overline{B_1}$, $b(x,v) = a(x) \cdot v$, $f \equiv 0$.

10/18/2023.

Convexity of H gives further hidden properties.

Equation : Static PDE with $\lambda \geq 0$

$$(S) : \lambda u + H(x, Du) = 0 \text{ in } \mathbb{R}^n.$$

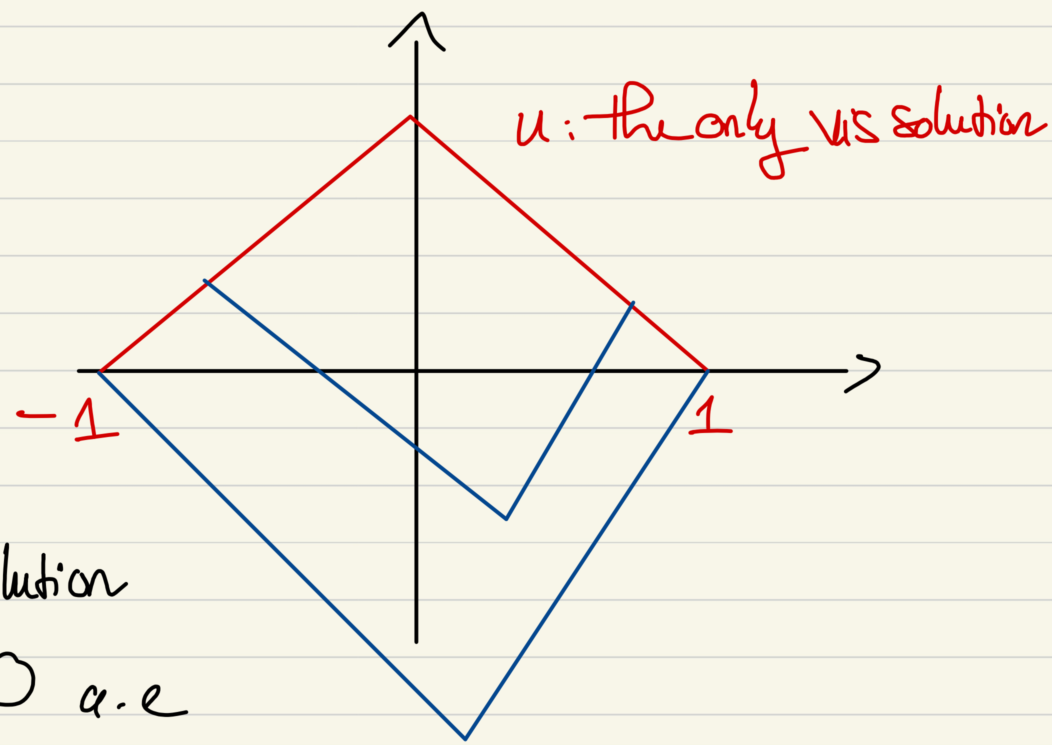
Assumption
$$\begin{cases} H \in BUC(\mathbb{R}^n \times B(0,R)), \ \forall R > 0, \\ \liminf\limits_{|p| \to \infty} \ \inf\limits_{x \in \mathbb{R}^n} H(x,p) = +\infty, \end{cases}$$

$\Big|\ p \to H(x,p)$ is convex.

Let's look at Lipschitz subsolution / supersolution to (S).

Results will explain much better about the Eikonal PDE

$$\begin{cases} |v'| = 1 & \text{in } (-1,0) \\ v(\pm 1) = 0. \end{cases}$$



<u>Theorem 1</u>: TFAE.

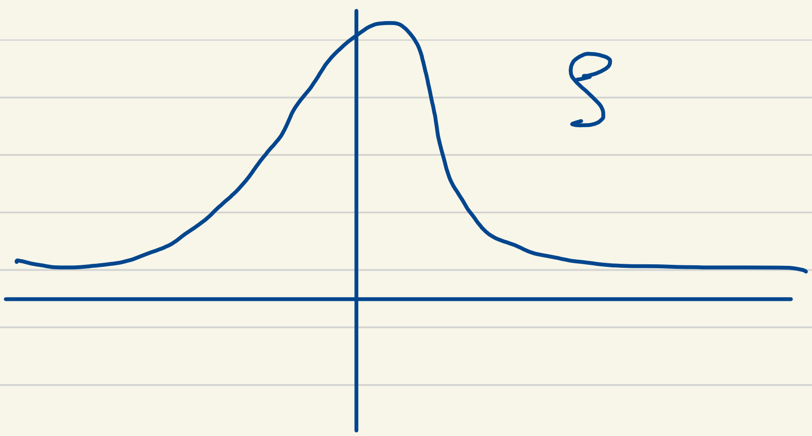
(i) $u \in$ Lipschitz is a viscosity solution to (S)

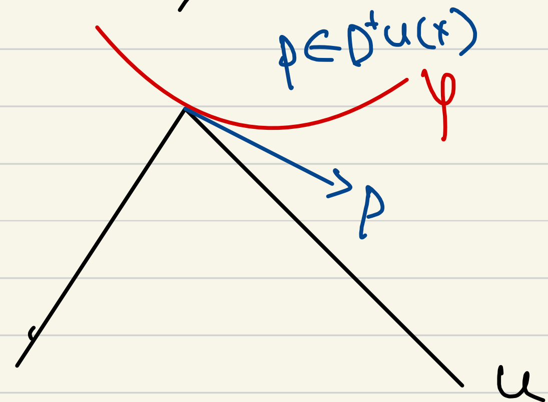
(ii) $u \in$ Lipschitz is an a.e subsolution to (S), i.e $\lambda u(x) + H(x, Du) \leq 0$ a.e proof: (i) $\Rightarrow$ (ii) obvious. At x – a differentiable point of u

$$\lambda u(x) + H(x, Du) \leq 0.$$

ii) $\Rightarrow$ i) : $u \in \mathrm{Lipschitz}(\mathbb{R}^n)$ solves $\lambda u(x) + H(x, Du) \leq 0$ a.e.

Smooth u up using standard mollifier.





$\rho \in C_c^\infty(\mathbb{R}^n, [0,\infty))$, $\mathrm{Supp}(\rho) \subset B_1$,

$$\int_{\mathbb{R}^n} \rho \, dx = 1$$

$$\varepsilon \in (0,1), \quad \rho^\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right), \quad \mathrm{Supp}(\rho^\varepsilon) \subset B_\varepsilon$$

Denote by: $u^{\varepsilon}(x) = (\rho^{\varepsilon} * u)(x) = \int_{\mathbb{R}^n} \rho^{\varepsilon}(x-y)\,u(y)\,dy.$

$$= \int_{B(0,\varepsilon)} \rho^{\varepsilon}(x-y)\,u(y)\,dy$$

$$\lambda u(x) + H(x, Du) \le 0 \quad \text{a.e.}$$

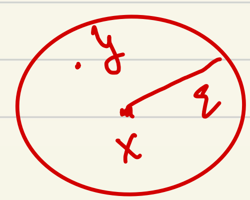
$$\int_{B(0,\varepsilon)} \rho^{\varepsilon}(x-y)\left[\lambda u(y) + H(y, Du(y))\right] dy \le 0$$

Convexity
Jensen's inequality

$$= \lambda u^{\varepsilon}(x) + \int_{B(x,\varepsilon)} \underbrace{\rho^{\varepsilon}(x-y)}\, H(y, Du(y))\,dy \le 0.$$

Replace y by x

Nonlinear



$$\lambda u^\varepsilon(x) + \int_{B(x,\varepsilon)} \underbrace{\rho^\varepsilon(x-y)}_{\geq} \left[H(x, Du(y)) - \omega(|x-y|) \right] dy \leq 0$$

$$\lambda u^\varepsilon(x) + H\left(x, \int_{B(x,\varepsilon)} \rho^\varepsilon(x-y)\, Du(y)\, dy\right) \leq \underbrace{\omega(\varepsilon)}_{\text{commutator error.}}$$

u^ε smooth solving
$$\boxed{\lambda u^\varepsilon(x) + H(x, Du^\varepsilon) \leq \omega(\varepsilon) \text{ in } \mathbb{R}^n}.$$

$$\left(\begin{array}{l} H(x, p_i) \leq 0, \forall i \\ c_i \geq 0, \sum c_i = 1. \end{array} \right.$$

$H(x, \sum c_i p_i \leq$

$$\sum_i c_i H(x, p_i) \leq 0$$

$u^\varepsilon \to u$ uniformly, $\omega(\varepsilon) \to 0$, and

by stability of viscosity solution

$\lambda u + H(x, Du) \leq 0$ in the vis solution.

Remark. (1) The proof holds for $\lambda \in \mathbb{R}$.

(2) Convolution trick & commutator estimates : really important

Ex: $u_t + b(x,t)\cdot Du - a_{ij}(x,t)\, u_{x_i x_j} = 0$
(≤ 0)

(u might be smooth)

$$u^\varepsilon = \rho^\varepsilon * u$$

Want: $u^\varepsilon_t + b(x,t)\cdot Du^\varepsilon - a_{ij}(x,t)\, u^\varepsilon_{x_i x_j} \leq \, ??$

Actually: $u^\varepsilon_t + \rho^\varepsilon * (b\cdot Du) - \rho^\varepsilon (a_{ij} u_{x_i x_j}) = 0$

Commutation Error $= \Big| \Big[\rho^\varepsilon * (b\cdot Du) - b\cdot(\rho^\varepsilon * Du) \Big] \Big|$
$+ \Big| \rho^\varepsilon * (a_{ij} u_{x_i x_j}) - a_{ij} \rho^\varepsilon * u_{x_i x_j} \Big] \Big|$

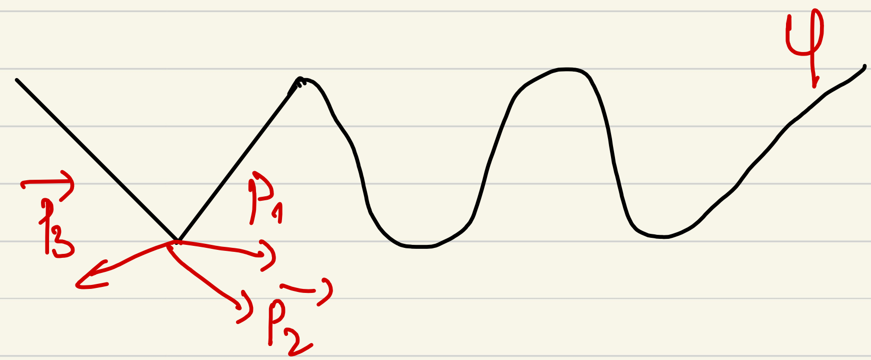
Theorem 2: TFAE $(\lambda \in \mathbb{R})$

(i) $u \in \mathrm{Lip}(\mathbb{R}^n)$ is a viscosity solution to (S)

ii) $u \in \mathrm{Lip}(\mathbb{R}^n)$ and $\forall\, x \in \mathbb{R}^n$, $\forall p \in D^- u(x)$,

$$\lambda u(x) + H(x, Du) = 0.$$

(Note: We only need to worry about corners from below)



proof: (ii) $\Rightarrow$ (i) is clear.

as $\lambda u(x) + H(x, Du(x)) = 0$

a.e at diff points $\Rightarrow$ u is a visc

subsolution and clearly (ii) gives u is a visc. supersolution.

(i) $\Rightarrow$ (ii): As u is a viscosity supersolution, $\forall x \in \mathbb{R}^n$, $\forall p \in D^- u(x)$

$$\lambda u(x) + H(x,p) \geq 0$$

Trick (only for 1^{st} order).

$$\lambda u(x) + H(x, Du(x)) = 0 \text{ a.e in } \mathbb{R}^n$$

$\boxed{V = -u}$

$$-\lambda v(x) + H(x, -Dv(x)) = 0 \text{ a.e in } \mathbb{R}^n$$

Let $G(x,p) = -\lambda v(x) + H(x, -p)$. Then G is convex in p.

$$G(x, Dv(x)) = 0 \text{ a.e}$$

$\Rightarrow v$ is viscosity subsolution to this

V is a viscosity subsolution to

$$G(x, Dv(x)) = 0$$

$-p \in D^+ v(x)$

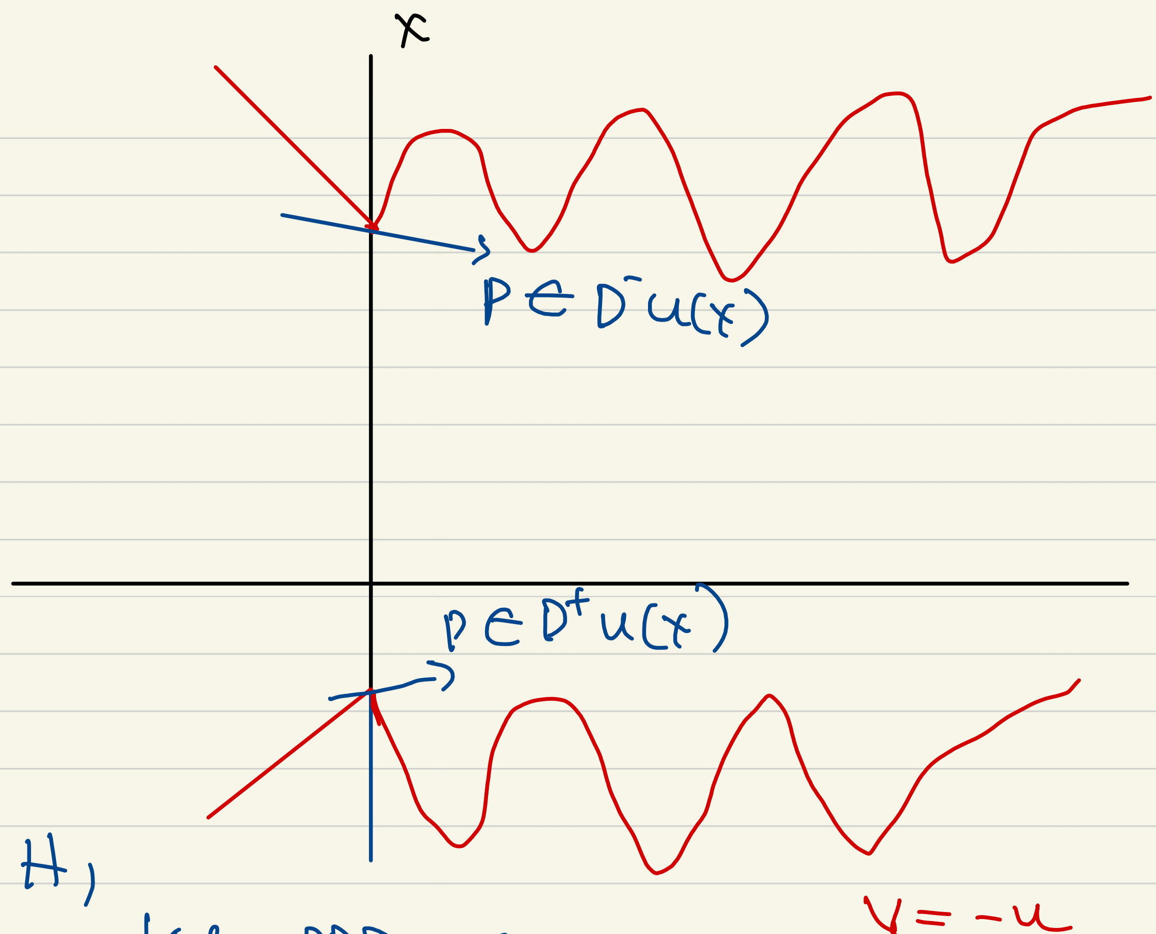
$$G(x, -p) \le 0$$

$$\|$$

$$\lambda u(x) + H(x, p) \le 0$$

Combining two inequalities

$$\lambda u(x) + H(x, p) = 0$$



Remarks: Theorems belongs to Barron–Jensen. For convex H, for u to be a vis solution

- satisfy PDE a.e
- wrong only about corners from below.

10/20/2023.

$$\begin{cases} H = H(x,p) \in BUC(\mathbb{R}^n \times B(0,R)), \forall R>0, \\ \lim\limits_{p\to\infty} \inf\limits_{x} H(x,p) = +\infty \\ p \longrightarrow H(x,p) \text{ convex.} \end{cases}$$

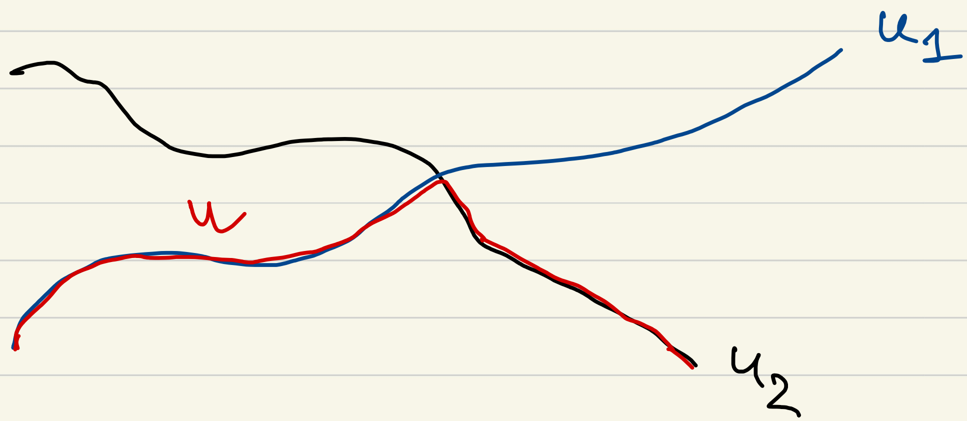
(S) $\lambda u(x) + H(x, Du) = 0$

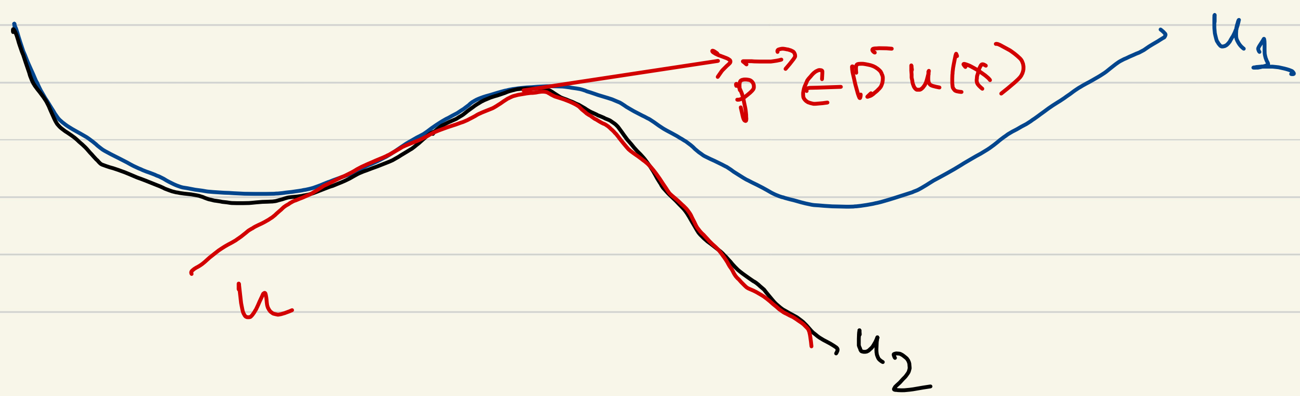
Theorem 1: $u \in Lip(\mathbb{R}^n)$ is a v.s subsolution $\Leftrightarrow$ $u \in Lip(\mathbb{R}^n)$ is an a.e solution

Theorem 2: $u \in Lip(\mathbb{R}^n)$ is a v.s solution $\Leftrightarrow$ $u \in Lip(\mathbb{R}^n)$, $\forall x \in \mathbb{R}^n$, $\forall p \in \bar{\partial} u(x)$, $\lambda u(x) + H(x,p) = 0$

Theorem 3: If u_1, u_2 are 2 Lipschitz solution to (S), then $\min\{u_1, u_2\}$ is also a viscosity solution of (S).

Proof:





Let $u = \min\{u_1, u_2\}$, then $u \in Lip(\mathbb{R}^n)$. Take any $x \in \mathbb{R}^n$ & $p \in D^- u(x)$. WOLG assume

$$u(x) = \min\{u_1, u_2\}(x) = u_1(x)$$

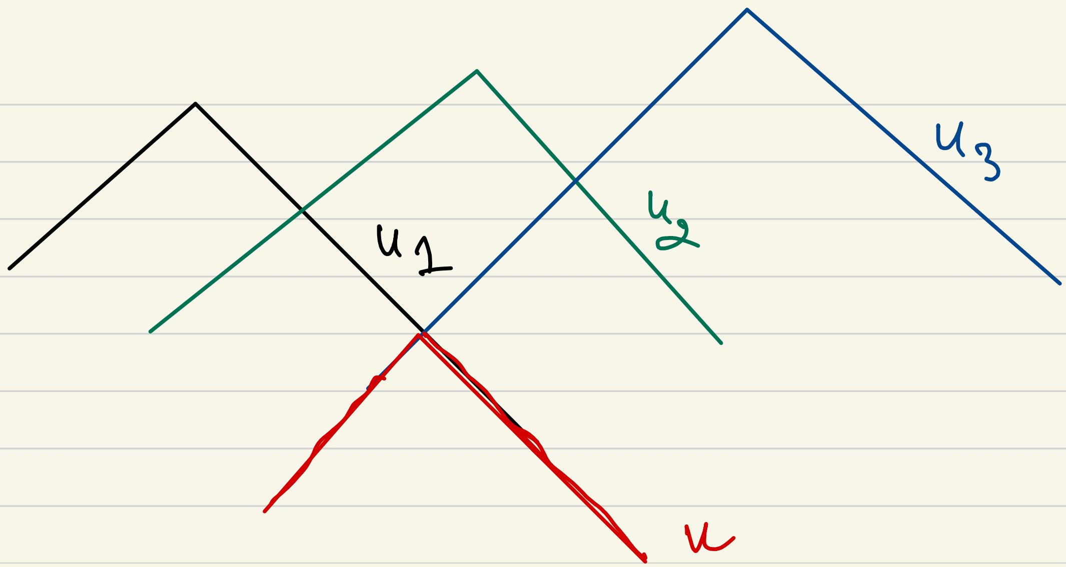
$$p \in D^- u(x) \Rightarrow p \in D^- u_1(x)$$

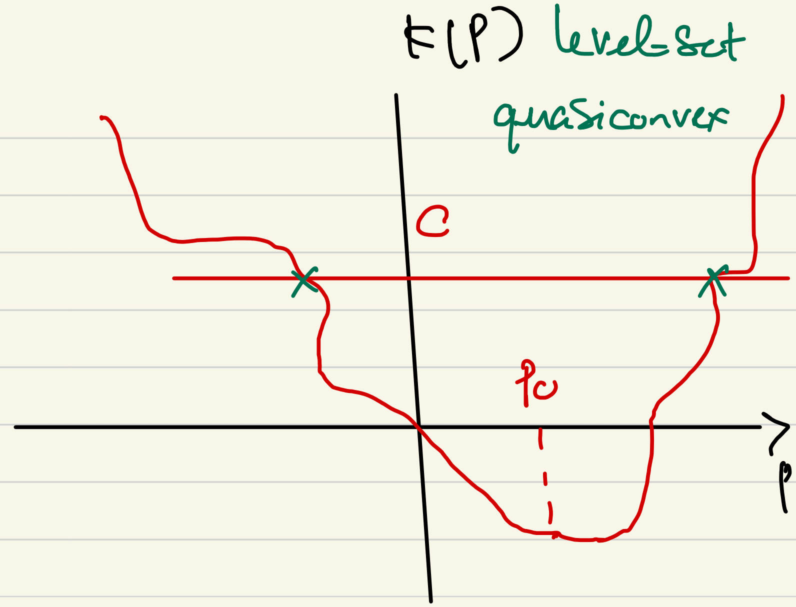
$$\Rightarrow \lambda u_1(x) + H(x, p) = 0$$

$$\parallel$$

$$\lambda u(x) + H(x, p) = 0$$

__Cor__: If $\{u_i\}_{i \in I}$ is a family of Lipschitz viscosity solution to (S) and $u = \inf_i u_i \in Lip(\mathbb{R}^n)$, then u is also a viscosity solution to (S)





In fact, Theorem 1 – 3 also hold for level-set quasiconvex Hamiltonians.

Def: $p \to H(x,p)$ is level-set quasiconvex iff, for each $x \in \mathbb{R}^n$ fixed, $\{p : H(x,p) \le s\}$ is convex for $s \in \mathbb{R}$.

A crucial tool we used for convex H is the Jensen inequality

$$\sum_{i=1}^{K} c_i H(x, p_i) \geq H\left(x, \sum_{i=1}^{K} c_i p_i\right) \qquad c_i \geq 0, \ \sum_{i=1}^{K} c_i = 1$$

$p_i \in \mathbb{R}^n$.

If $\rho \in C(\mathbb{R}^n, [0,\infty))$, $\int \rho \, dx = 1$, then

$$\int_{\mathbb{R}^n} H(x, p(y)\rho(y))\, dy \geq H\left(x, \int_{\mathbb{R}^n} p(y)\rho(y)\, dy\right)$$

An Analog of the above holds:

$$\max_{i=\overline{1,K}} H(x, p_i) \geq H\left(x, \sum_{i=1}^{K} c_i p_i\right)$$

Let's say $H(x, p_i) \leq c$

$p_i \in \{p_i : H(x,p) \leq c\}$

convex $\quad \ni$

$\sum c_i p_i$

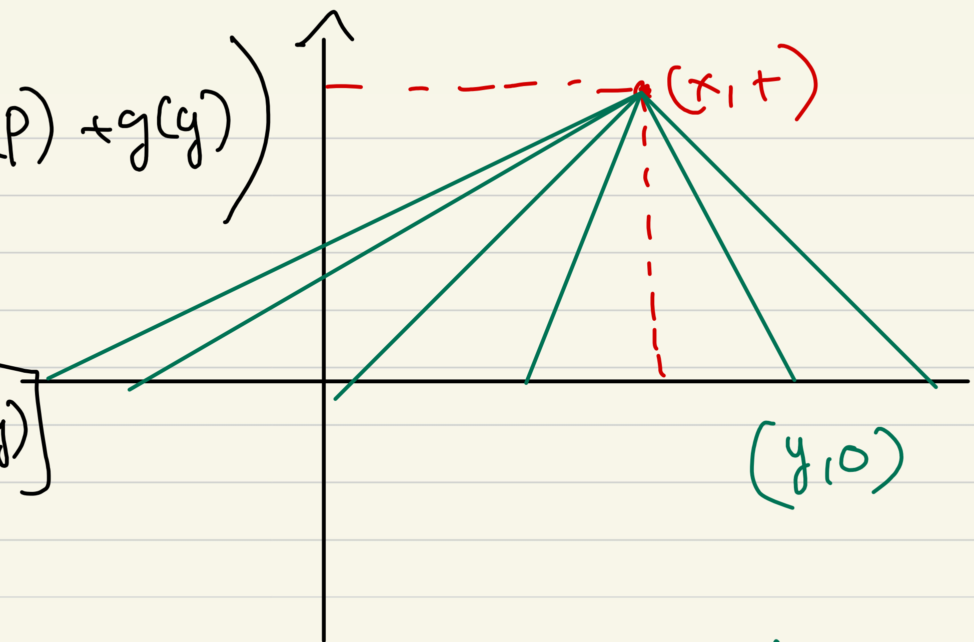
$$\sup_{y \in \mathbb{R}^n} H(x, p(y)) \geq H\left(x, \int_{\mathbb{R}^n} p(y)\,\rho(y)\,dy\right)$$

Hopf–Lax & optimal control formula revisited

Hopf–Lax formula
$$\begin{cases} u_t + H(Du) = 0, & \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \mathbb{R}^n. \end{cases}$$

H: convex: $$u(x,t) = \inf_{y \in \mathbb{R}^n} \Big[\underbrace{t\, L\left(\frac{y-x}{t}\right)}_{\text{Running cost}} + \underbrace{g(y)}_{\text{terminal cost.}} \Big]$$

$$u(x,t) = \inf_{y\in\mathbb{R}^n}\left[t \sup_p \left(p\cdot\frac{x-y}{t} - H(p) + g(y)\right)\right]$$



$$= \inf_y \sup_p \Big[\underbrace{p\cdot(x-y) - tH(p) + g(y)}_{\phi^{y,p}}\Big]$$

Let $\phi^{y,p}(x,t) = p\cdot(x-y) - tH(p) + g(y)$. is an affine function in (x,t)

$$\phi^{y,p}_t(x,t) = -H(p), \quad D\phi^{y,p} = p \Rightarrow \phi^{y,p}_t + H(D\phi^{y,p}) = 0$$

Family of affine (separable) solutions $\{\phi^{y,p}\}$ to our H-J PDE

Why not trying to find envelopes of these special solutions.

① $\inf_y \sup_p \phi^{y,p}$ $\longrightarrow$ Hopf_Lax formula (convex H case)

② $\sup_p \inf_y \phi^{y,p}$ $\longrightarrow$ Hopf formula (if g is convex
chapter 3)

Bardi _ Evans.

Back to Hopf_Lax formula

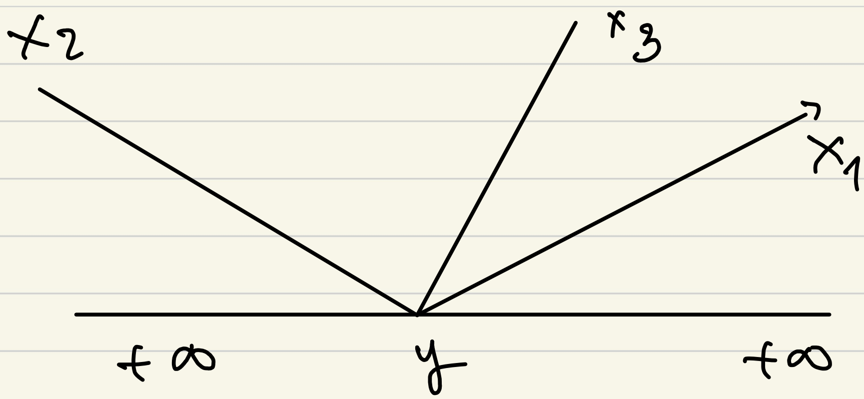
$$u(x,t) = \inf_y \sup_p \phi^{y,p}$$

$$\phi^{y,p}(x,t) = p\cdot(x-y) - tH(p) + g(y)$$

Initial data: $\phi^{y,p}(x,0) = p\cdot(x-y) + g(y) \longrightarrow \phi^{y,p}(y,0) = g(y)$

Let $\phi^{y}(x,t) = \sup_{p} \phi^{y,p}(x,t) = t\,L\left(\frac{x-y}{t}\right) + g(y)$

What is $\phi^{y}(x,0) = \sup_{p}\left(p\cdot(x-y) + g(y)\right) = \begin{cases} g(y) & \text{if } x=y \\ +\infty & \text{if } x\neq y \end{cases}$



We say that $\phi^{y}(x,t)$ is a fundamental solution to our H-J PDE.

Def: $\Phi^{y}(x,t) = t\,L\left(\frac{x-y}{t}\right)$ is a fundamental solution to H-J PDE

$$\Phi^y(x,0) = \begin{cases} 0 & \text{if } x = y \\ +\infty & \text{if } x \neq y. \end{cases}$$

Then, the Hopf–Lax formula reads

$$u(x,t) = \inf_y \left[\Phi^y(x,t) + g(y) \right]$$

(inf – convolution as opposed to usual convolution for heat kernel)

[heat PDE : $u(x,t) = \int_{\mathbb{R}^n} \Phi^y(x,t)\, g(y)\, dy$)

$$u(x,t) = \inf_y \left[\Phi^y(x,t) + g(y) \right]$$

(inf – convolution as opposed to usual convolution for heat kernel)

10/23/2023. Recap : Interpretation of Hopf–Lax & optimal control.

Hopf – Lax formula.

$$u(x,t) = \inf_{y\in\mathbb{R}^n}\Big[g(y) + \underbrace{tL\Big(\frac{x-y}{t}\Big)}_{=:\,\Phi^y(x,t)}\Big]$$

Here, $\Phi^y(x,t)$ is a fundamental solution to HJ equation

$$\begin{cases} \Phi^y_t + H(D\Phi^y) = 0 & \text{in } \mathbb{R}^n\times(0,\infty) \\ \Phi^y(x,0) = \begin{cases} 0, & x=y \\ +\infty, & x\neq y. \end{cases} \end{cases}$$

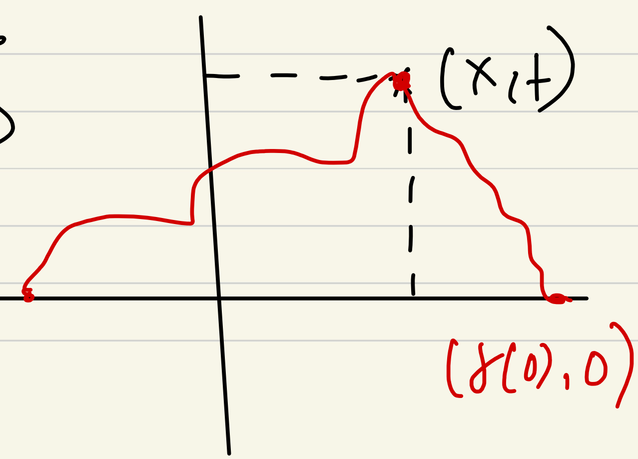
Note, $\Phi^y(x,t) = \Phi^0(x-y,t)$

From the fundamental solution, to find a general solution, perform inf convolution

[Key properties for convex / quasiconvex Hamiltonians: inf of a family of solution is a solution).

<u>More general solution</u>:
$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \end{cases}$$

$$u(x,t) = \inf_{\gamma(t)=x} \left\{ g(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\,ds \right\}$$

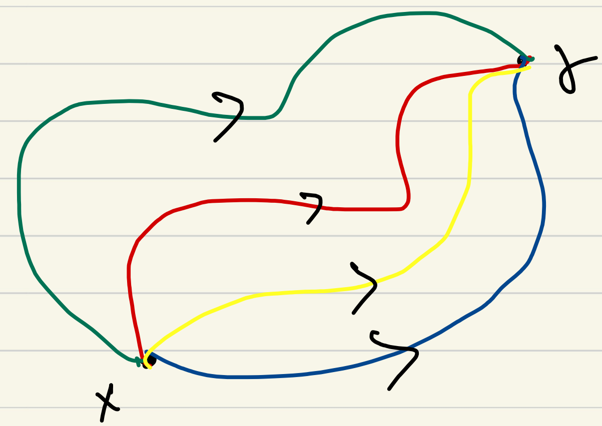


$$= \inf_{y\in\mathbb{R}^n} \inf_{\substack{\gamma(t)=x \\ \gamma(0)=y}} \left\{ g(y) + \int_0^t L(\gamma(s), \dot\gamma(s))\,ds \right\}$$

$$= \inf_{y \in \mathbb{R}^n} \left\{ g(y) + \underbrace{\inf_{\substack{\gamma(0)=y \\ \gamma(t)=x}} \int_0^t L(\gamma(s), \dot\gamma(s))\, ds}_{\Phi^y(x,t)} \right]$$

$\Phi^y(x,t)$: fundamental sln to our HJ

$$\begin{cases} \Phi^y_t + H(x, D\Phi^y) = 0 \\ \Phi^y(x,0) = \begin{cases} 0, & x = y \\ +\infty, & x \neq y. \end{cases} \end{cases}$$



$\mathbb{R}^n$ (spatial only)

Interpretation for $\Phi^y(x,t) = m(t,x,y)$: minimum cost going from x to y in time t.

Metric problems. _ Maximal subsolutions.

Assumption $\begin{cases} H = H(x,p) \in BUC(\mathbb{R}^n \times B(0,R)) \ \forall R>0 \\ \lim\limits_{|p|\to\infty} \inf\limits_{x} H(x,p) = +\infty \\ p \to H(x,p) \text{ is convex} \end{cases}$

Fix the origin, let's look at the PDE.

$$(\star) \quad \begin{cases} H(x,Du) \le \mu & \text{in } \mathbb{R}^n \setminus \{0\} \\ u(0) = 0 \end{cases}$$

(Consider $\mu \geq \mu_*$)

Here, $\mu \in \mathbb{R}$ is given real number. Let

$$\mu_* = \inf \left\{ c \in \mathbb{R} : \exists \text{ a subsolution to } (\star) \text{ with } c = \mu \right\}$$

<u>Ex 1</u>: $H(x,p) = |p|$ becomes

$$\begin{cases} |Du| \leq \mu & \text{in } \mathbb{R}^n \setminus \{0\} \\ u(0) = 0. \end{cases}$$

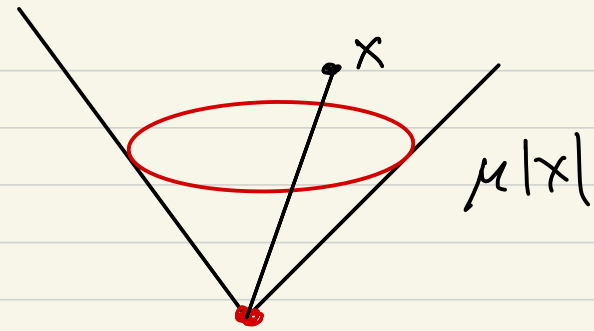
$$\mu_* = 0$$

If $\mu = \mu_* = 0$: $u \equiv 0$ is the only possible solution to $(\star)$

If $\mu > 0$: $u \equiv 0$ is a solution to $(\star)$, $u(x) = \mu |x|$.

$$Du(x) = \mu \frac{x}{|x|}, \quad x \neq 0$$

$|Du(x)| = \mu$ classically in $\mathbb{R}^n \setminus \{0\}$.



Cone-like structure

∃ many solutions.

Ex 2: $H(x,p) = a(x)|p|$ with $a \in C(\mathbb{R}^n, [0,\infty))$.

Definition [Maximal subsolutions].

For $\mu \geq \mu_*$, define

$$m_\mu(x) = \sup \{ u(x) : u \text{ is a vis subsolution to } (*) \}$$

Theorem: In fact, m_μ is a viscosity solution to ⊛,
meaning that,

$$\circledast' \quad \begin{cases} H(x, Dm_\mu(x)) = \mu & \text{in } \mathbb{R}^n \setminus \{0\}, \\ m_\mu(0) = 0 \end{cases}$$

Automatically, m_μ is also the maximal vis sln to ⊛.

proof: Def of m_μ reminds us of the perron method. But the Perron method required:

$$\left. \begin{aligned} W(x) = \sup\{u(x) :\ & u \text{ is a vis subsolution to } \circledast \ \& \\ & u \le \varphi(x), \text{ where } \varphi \text{ is a supersolution to } \circledast \end{aligned} \right\}$$

Missing: φ!

As H is coercive in p, $\exists\, C_\mu > 0$ such that

$$H(x,p) \leq \mu \Rightarrow |p| \leq C_\mu$$

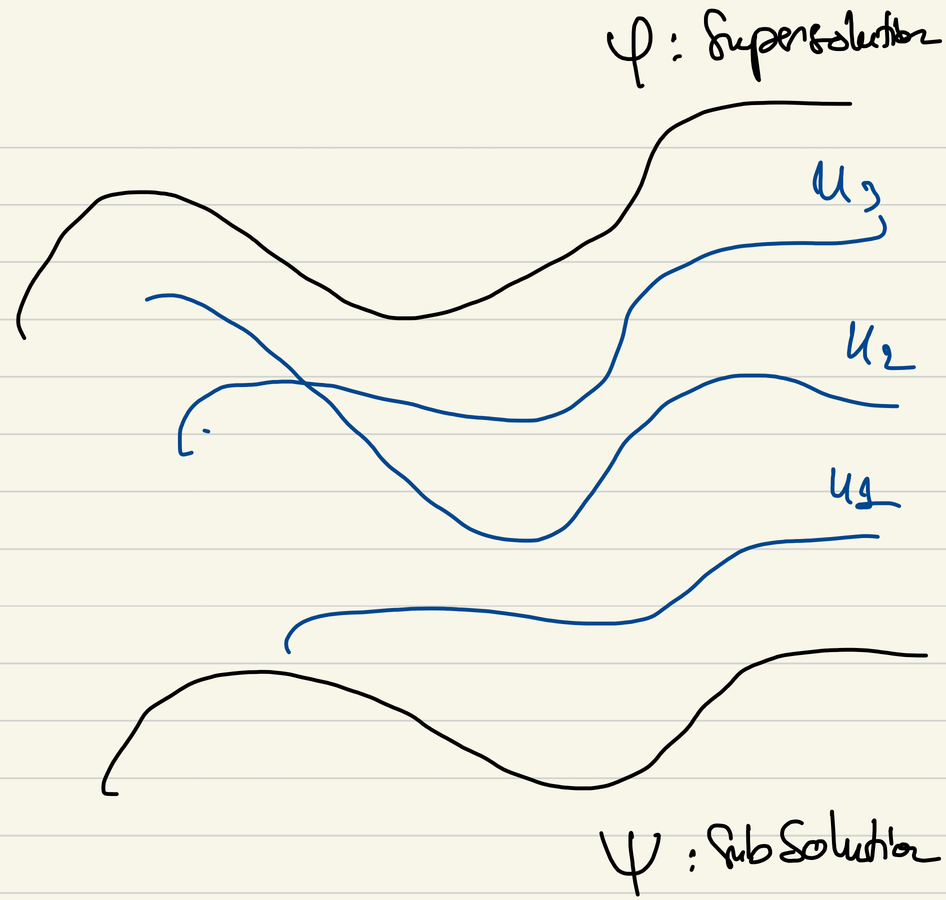
$\forall\, u$ subsolution to $\star$

$$\Rightarrow |Du| \leq C_\mu \Rightarrow u(x) \leq C_\mu |x|.$$

$$\Rightarrow m_\mu(x) \leq C_\mu |x| \quad , \forall x \in \mathbb{R}^n.$$

Pick $K_\mu > C_\mu$ such that

$$H(x, K_\mu e) > \mu \quad , \forall |e| = 1.$$



Then $\varphi(x) = K_\mu |x|$ is a supersolution to ⊛ in the classical sense $\left(D\varphi(x) = K_\mu \frac{x}{|x|}, x \neq 0 \right)$

$$m_\mu(x) = \sup \left\{ u(x) : \begin{array}{l} u \text{ is a vis subsolution to ⊛,} \\ u \leq \varphi(x) = K_\mu |x| \end{array} \right\}$$

$$= w(x)$$

By the perron method, m_μ is the maximal vis sln to ⊛.

back to Ex 1: For $\mu \geq 0$, it's clear

$$m_\mu(x) = \mu |x|, \forall x$$

Note, however that for $\mu > 0$, m_μ is NOT a vis sln to

$$H(x, Dm_\mu(x)) = |Dm_\mu(x)| = \mu \text{ in } \mathbb{R}^n.$$

It fails the supersolution test at $x = 0$.

As $0 \in D^- m_\mu(0)$ but $|0| - \mu = -\mu \not\geq 0$.

Cor: Another interesting interpresentation of m_μ is vertex is 0,

$$m_\mu(x) = \sup\Big\{ \underbrace{v(x) - v(0)}_{u(x)} : v \text{ is a vis subsolution to } H(x, Dv(x)) \leq \mu \text{ in } \mathbb{R}^n \Big\}$$

More the vertices around. For a given vertex $y \in \mathbb{R}^n$,

Write

$$m_\mu(x,y) = \sup\left\{ v(x)-v(y) : v \text{ is a v.s. subsolution to } H(x, Dv(x)) \leq \mu \text{ in } \mathbb{R}^n \right\}$$

variable $\quad$ vertex

Main properties: $\quad \mu \geq \mu_*$

(i) (Triangle inequality)

$$m_\mu(x,y) + m_\mu(y,z) \geq m_\mu(x,z)$$

ii) $m_\mu(x,y)$ respects exactly a 'Rieman' type distance

$$m_\mu(x,y) = \inf \left\{ \int_0^t (L(\gamma(s), \dot\gamma(s)) - \mu) ds : \begin{array}{l} \gamma(0) = y \\ \gamma(t) = x \end{array} \right.$$

where $t > 0$ is any possible

10/25/2023.

Today, finish Chapter 2.

Next 1-2 lectures: Section 6.1. New representation formula

Afterward. Second-order PDE : $\lambda u + H(x, Du, D^2u, u) = 0$.

Quickly recap:

$$\begin{cases} H(x,p) \in BUC(\mathbb{R}^n \times B_R), \ \forall R>0 \\ p \to H(x,p) \text{ is convex.} \\ \lim_{|p|\to\infty} \left(\inf_x H(x,p)\right) = +\infty. \end{cases}$$

Lowest thershold: $\mu_* = \inf\{ \mu \in \mathbb{R} : \exists$ an a.e subsolution u to $H(x,Du) \le \mu$ in $\mathbb{R}^n \}$

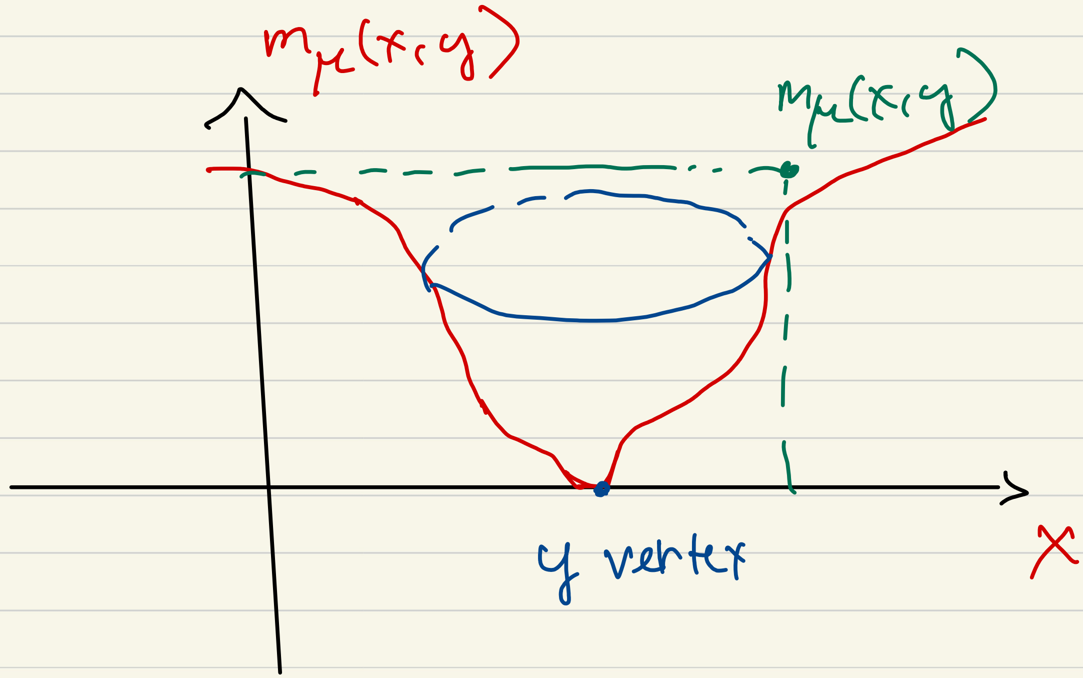
Remark: This is the same as asking for $\begin{cases} H(x,Du) \le \mu \text{ in } \mathbb{R}^n\setminus\{0\} \\ u(0) = 0. \end{cases}$

For $\mu \geq \mu_*$, define

$$m_\mu(\underset{\substack{\uparrow \\ \text{variable}}}{x}, \underset{\substack{\nwarrow \\ \text{vertex}}}{y}) = \sup\{ u(x) - u(y) : \forall u \text{ a.e subsolution to } H(x,Du) \le \mu \text{ in } \mathbb{R}^n \}$$

Theorem: $x \longmapsto m_\mu(x,y)$ is THE maximal vis solution

$$\begin{cases} H(x, Dm_\mu(x,y)) = \mu & \text{in } \mathbb{R}^n \setminus \{0\}, \\ m_\mu(y,y) = 0. \end{cases}$$



Lemma 1: m_μ enjoys the triangle inequality

$$m_\mu(x,y) + m_\mu(y,z) \geq m_\mu(x,z)$$

proof: Fix y, z. Think of x as a variable.

$$\iff m_\mu(x,y) \geq \underbrace{m_\mu(x,z) - m_\mu(y,z)}_{:= w(x)}$$

Then, $w(y) = m_\mu(y,z) - m_\mu(y,z) = 0$

For a.e $x \in \mathbb{R}^n$, $Dw(x) = Dm_\mu(x,z)$, and hence,

$$H(x, Dw(x)) = H(x, Dm_\mu(x,z)) \leq \mu.$$

By definition $m_\mu(x,y) : m_\mu(x,y) \geq w(x) - w(y) = w(x)$

$\blacksquare$.

Theorem 2: Assume H is superlinear, i.e $\lim_{|p| \to \infty} \frac{H(x,p)}{|p|} = +\infty$

Then we have the following rep formula

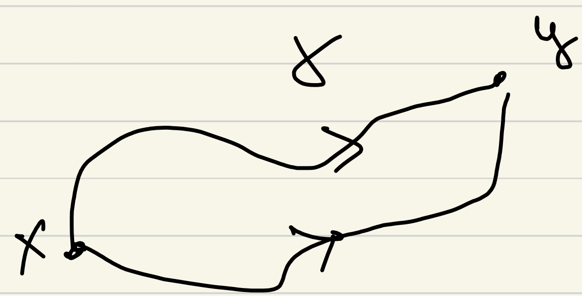
$$m_\mu(x,y) = \inf \left\{ \int_0^t \left(L(\gamma(s), \dot\gamma(s)) + \mu \right) ds : \begin{array}{l} \gamma(0) = y \\ \gamma(t) = x \end{array} \right\}$$

for any possible $t > 0$.

Difference:

① If discount term: $\lambda u + H(x, Du) = 0$

$$u(x) = \inf_{\gamma(0) = x} \int_0^\infty e^{-\lambda s} L(\gamma(s), -\dot\gamma(s))\, ds.$$



② If finite horizon

$$u_t + H(x, Du) = 0$$

$$u(x,t) = \inf_{\gamma(t)=x} \left\{ \int_0^t L(\gamma,\dot\gamma)\, ds + g(\gamma(0)) \right\}.$$

proof: For simplicity, let $g=0$, write $m_\mu(x,0) = m_\mu(x)$

WTS: $m_\mu(x) = w(x) = \inf \left\{ \int_0^t \left(L(\gamma(s), \dot\gamma(s)) + \mu \right) ds : \gamma(0) = 0,\ \gamma(t) = x \right\}$

① Show $m_\mu(x) \le w(x)$

Recall: $H(x, Dm_\mu(x)) \le \mu$ a.e in $\mathbb{R}^n$.

ρ : standard convolution kernel, $\rho^\varepsilon = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right)$, $0<\varepsilon<1$.

Denote by $u^\varepsilon(x) = (\rho^\varepsilon * m_\mu)(x)$

Then $H(x, Du^\varepsilon(x)) \leq \mu + w(\varepsilon)$, $\forall x \in \mathbb{R}^n$.

For any C^1 curve γ such that $\gamma(0) = 0$, $\gamma(t) = x$.

Recall $\boxed{H(x, p) + L(x, v) \geq p \cdot v.}$

$$\int_0^t \left(L(\gamma(s), \dot\gamma(s)) + \mu \right) ds \geq \int_0^t \left(L(\gamma(s), \dot\gamma(s)) + H(\gamma(s), Du^\varepsilon(\gamma(s))) \right) ds - t w(\varepsilon)$$

$$\geq \int_0^t Du^\varepsilon(\gamma(s)) \cdot \dot\gamma(s) ds - t w(\varepsilon).$$

$$= \int_0^t \frac{d}{ds} \left(u^\varepsilon(\gamma(s)) \right) - t w(\varepsilon) = u^\varepsilon(\gamma(t)) - u^\varepsilon(\gamma(0)) - t w(\varepsilon)$$

$$= u^{\varepsilon}(x) - u^{\varepsilon}(0) - t\omega(\varepsilon)$$

let $\varepsilon \to 0$: $\int_0^t \left(L(\gamma, \dot\gamma) + \mu\right) ds \geq m_\mu(x) - \underbrace{m_\mu(0)}_{=0}$

$$= m_\mu(x)$$

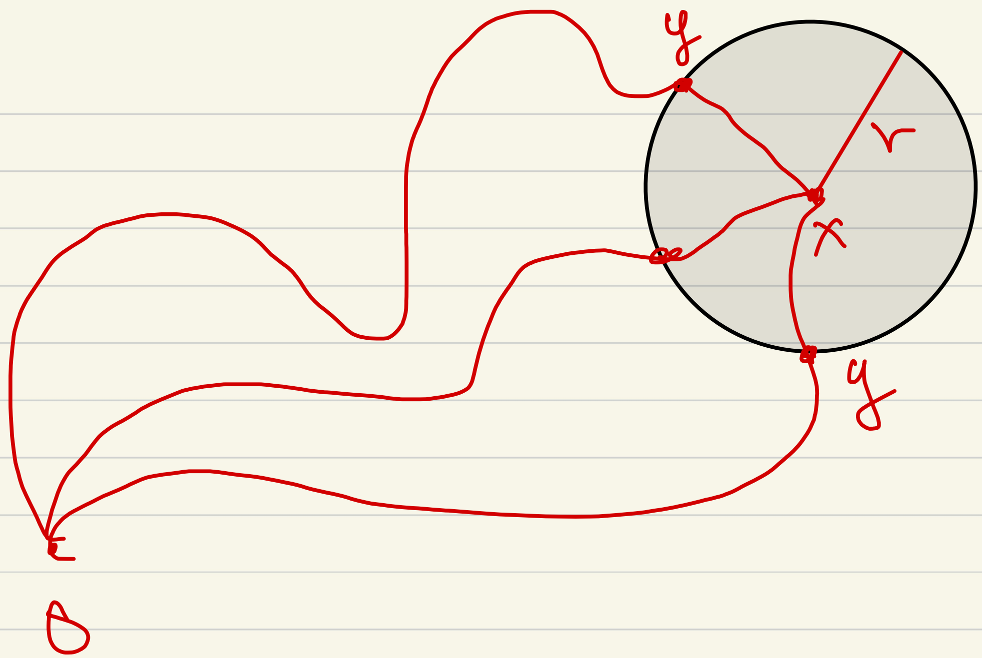
inf over all such possible curves

$w(x) \geq m_\mu(x)$

(2) Show w is a subsolution to $H(x, Dw) \leq \mu$ in $\mathbb{R}^n$ $\Big\}$

<u>Note</u>: $0 = m_\mu(0) \geq w(0) \leq 0 \Rightarrow w(0) = 0$

$\Rightarrow w(x) \leq m_\mu(x)$.



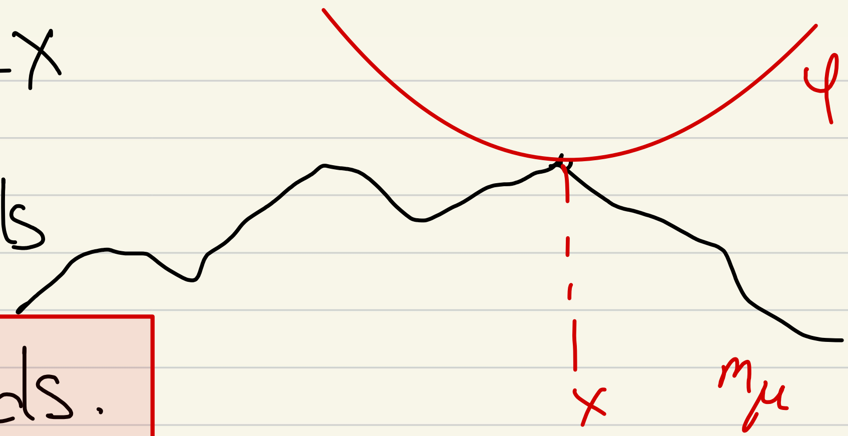
For $x \neq 0$, pick

$r \in \left(0, \frac{|x|}{2}\right)$

DPP: $$m_\mu(x) = \inf \left\{ m_\mu(y) + \int_0^t \left(L(\gamma, \dot\gamma) + \mu\right) ds : \begin{array}{l} \gamma(0) = y \in \partial B(x,r) \\ \gamma(t) = x \end{array} \right\}$$

$\forall y \in \partial B(x,r)$, $\gamma(0)=y$, $\gamma(t)=x$

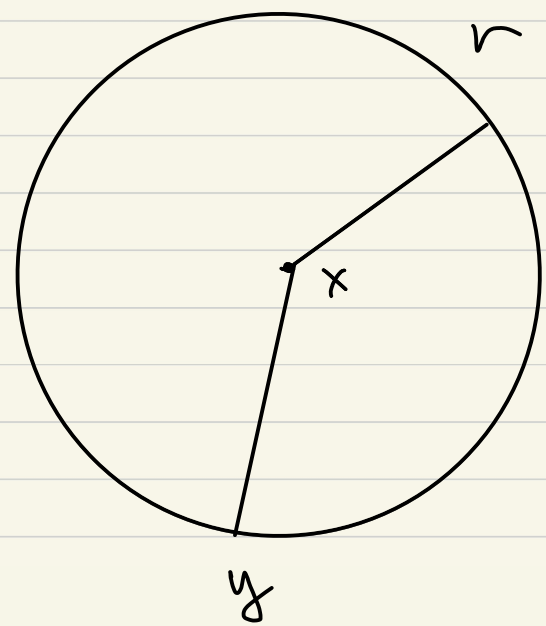
$$m_\mu(x) \le m_\mu(y) + \int_0^t \big(L(\gamma,\dot\gamma)+\mu\big)\,ds$$



$$\boxed{\varphi(x) \le \varphi(y) + \int_0^t \big(L(\gamma,\dot\gamma)+\mu\big)\,ds.}$$

Denote by

$$\gamma_e(s) = \underbrace{(x-te)}_{y} + se, \quad 0 \le s \le t.$$



$$\varphi(\gamma_e(t)) \leq \varphi(\gamma_e(0)) + \int_0^t \big(L(x-te+se, e)+\mu\big)\,ds$$

$$\frac{\varphi(\gamma_e(t)) - \varphi(\gamma_e(0))}{t} \leq \frac{1}{t}\int_0^t \big(L(x-te+se, e)+\mu\big)\,ds$$

Let $t \to 0^+$

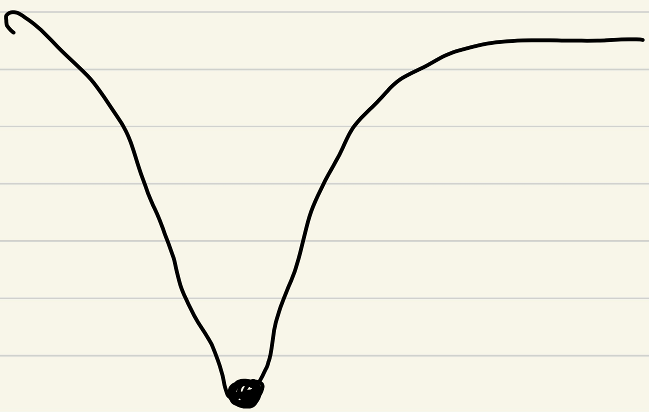
$$D\varphi(\gamma_e(0)) \cdot \gamma_e'(0) \leq L(x,e) + \mu$$

$$\Rightarrow \quad D\varphi(x)\cdot e - L(x,e) \leq \mu, \quad \forall\, e \in \mathbb{R}^n \setminus \{0\}$$

$$\sup_e \big(D\varphi(x)\cdot e - L(x,e)\big) = H(x, D\varphi(x)) \leq \mu.$$

Remark: 1 $m_\mu(x,0)$ is cone-like and representation a "metric" distance from $0 \to x$.

$$\begin{cases} H(x, Dm_\mu(x,0)) = \mu \text{ in } \mathbb{R}^n \setminus \{0\} \\ m_\mu(0,0) = 0 \end{cases}$$



study "large time averge"

$$\lim_{k\to\infty} \frac{1}{k}\, m_\mu(kx, 0) = ??$$

nicer metric?

② $m_\mu(x,0)$: comes from large deviation theory by Varadhan.

10/27/2023.

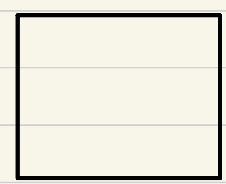
New representation formulas & applications, $\lambda > 0$.

$$(S) \quad \lambda u + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n.$$

Assumptions:

$$\begin{cases} H(x+k, p) = H(x,p), \ \forall k \in \mathbb{Z}^n \Leftrightarrow H \text{ is } \mathbb{Z}^n\text{-periodic in } x. \\ H \in C(\mathbb{R}^n \times \mathbb{R}^n) \\ \lim\limits_{|p|\to\infty} \left[\min\limits_{x \in \mathbb{R}} H(x,p)\right] = +\infty, \\ p \to H(x,p) \text{ is convex} \end{cases}$$

$\mathbb{T}^n = \mathbb{R}^n \setminus \mathbb{Z}^n$: flat n-dimensional torus



$Y = [0,1]^n$.

Simplications:

$$\begin{cases} \lambda \|u\|_{L^\infty(\mathbb{R}^n)} \leq \|H(\cdot, 0)\|_{L^\infty(\mathbb{R}^n)} \\ \|Du\|_{L^\infty(\mathbb{R}^n)} \leq C \quad \text{b/c of corecive of } H \text{ in } p. \end{cases}$$

① Claim: u is also $\mathbb{Z}^n$-periodic $\Longleftrightarrow$ $u \in \mathrm{Lip}(\mathbb{T}^n)$.

$V(x) = u(x+K)$ for K fixed in $\mathbb{Z}^n$, is also a vis solution to (S).

By uniquess, $u(x) = V(x) = u(x+K) \Rightarrow u$ is $\mathbb{Z}^n$-periodic.

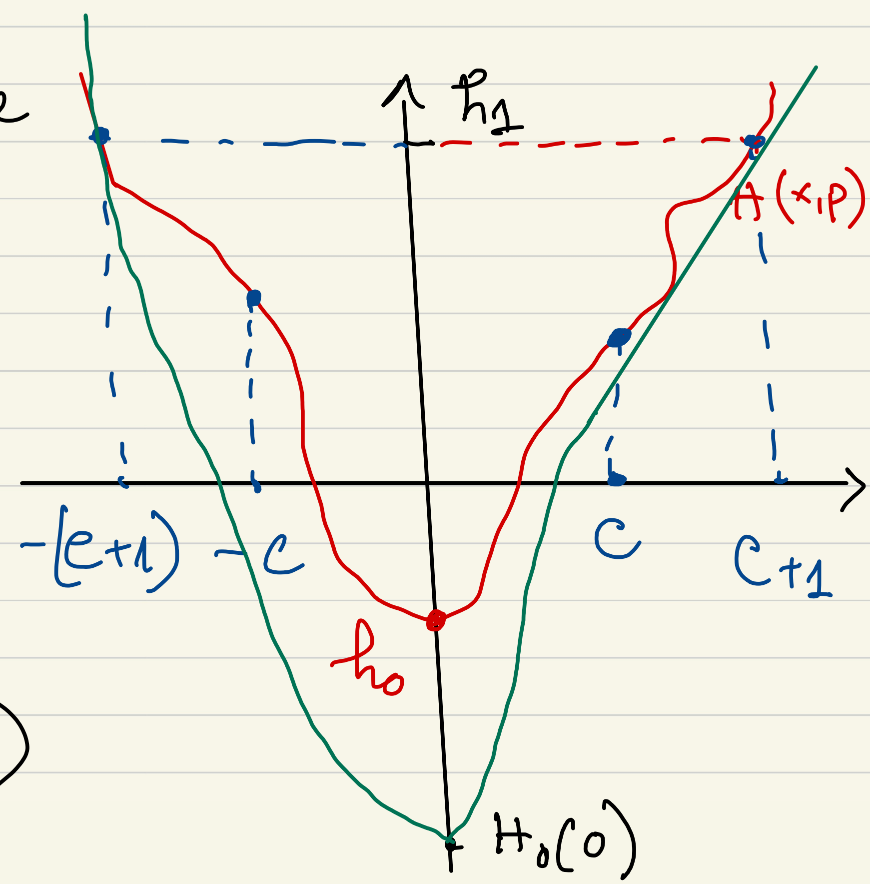
② $\|Du\|_{L^\infty} \leq C$. $\Longrightarrow$ the values of $H(x,p)$ for $|p| \geq C+1$ don't play any role in our PDE, we're free to modify H in a maner that we want.

Let $h_0 = \min\limits_{(x,p)} H(x,p)$, and $h_1 > h_0$ such that

$$H(x,p) \leq h_1 \text{ for } (x,p) \in \mathbb{T}^n \times \overline{B}(0,c+1)$$

Denote by

$H_0(p) = h_0 + (h_1 - h_0)(|p| - c)$ convex cone



$$H_0(p) = h_0 + (h_1 - h_0)(-c) < h_0$$

And for $|p| \leq c$

$$H_0(p) \leq h_0 \leq H(x,p)$$

For $|p| = c+1$.

$$H_0(p) = h_0 + h_1 - h_0 = h_1 > H(x,p)$$

Denote $\tilde{H}(x,p) = \begin{cases} \max\{ H(x,p), H_0(p) \}, & (x,p) \in \mathbb{T}^n \times \bar{B}(0,c) \\ H_0(p) & (x,p) \in \mathbb{T}^n \times \bar{B}(0,c)^c. \end{cases}$

$\tilde{H}(x,p)$ is convex in p, agrees with $H(x,p)$ for $(x,p) \in \mathbb{T}^n \times \bar{B}(0,c)$

Thus, (S) with

$$(S') : \lambda \tilde{u} + \tilde{H}(x, D\tilde{u}) = 0 \text{ in } \mathbb{R}^n$$

will yield $\tilde{u} = u$. We then just need to deal with $\tilde{H}$

Let $h = h_1 - h_0 > 0$ then we have

$$\tilde{H}(x,p) = H_0(p) = h_0 + h|p| \text{ for } (x,p) \in \mathbb{T}^n \times \bar{B}(0,c+1)^c.$$

Then $\tilde{L}(x,v) = \sup_p [p \cdot v - \tilde{H}(x,p)] = +\infty$ if $|v| > h$

$\Rightarrow \tilde{H}(x,p) = \sup_{v \in \mathbb{R}^n} [p \cdot v - \tilde{L}(x,v)] = \sup_{|v| \le h} [p \cdot v - \tilde{L}(x,v)]$

By the abuse of notation, write $H = \tilde{H}$

$$H(x,p) = \sup_{|v| \le h} [p \cdot v - L(x,v)]$$

$\mathbb{T}^n$, $\bar{B}_h$ compact

II/- Formulation of our new viewpoint [duality framework $\Longleftrightarrow$ optimal transport]

$\forall \phi \in C(\mathbb{T}^n \times \bar{B}_h)$, define

$$H_\phi(x,p) = \sup_{|v| \le h} [p \cdot v - \phi(x,v)]$$

Define $\mathcal{F}_\lambda = \{(\phi, u) \in C(\mathbb{T}^n \times \bar{B}_R) \times C(\mathbb{T}^n) : u$ is a Lipschitz solution to $\lambda u + H_\phi(x, Du) \leq 0$, in $\mathbb{T}^n\}$

Ex: $(0,0) \in \mathcal{F}_\lambda$ as $\phi \equiv 0$, $H_\phi(x,p) = h|p|$

$\Longrightarrow$ 0 is subsolution $(h, u) \in \mathcal{F}_\lambda$

Lemma 1: $\mathcal{F}_\lambda \subset C(\mathbb{T}^n \times \bar{B}_R) \times C(\mathbb{T}^n)$ is a CONVEX set.

Define: (Evalution cone). For $z \in \mathbb{T}^n$ fixed,

$$\mathcal{G}_{z,\lambda} = \{\phi(\cdot,\cdot) - \lambda u(z) : (\phi, u) \in \mathcal{F}_\lambda\}$$

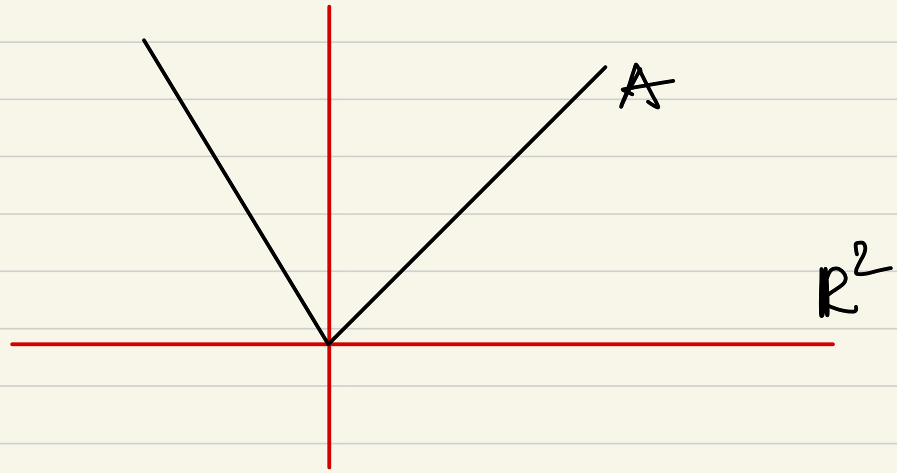
Lemma 2: $\mathcal{G}_{z,\lambda} \subset C(\mathbb{T}^n \times \bar{B}_R)$ is a convex cone.

$$C(\mathbb{T}^n \times \overline{B}_R) \xleftrightarrow{\text{dual space}} \mathcal{R} = \mathcal{R}(\mathbb{T}^n \times \overline{B}_R) : \text{Space of Radon measures on } \mathbb{T}^n \times \overline{B}_R$$

$$\mathcal{P} = \mathcal{P}(\mathbb{T}^n \times \overline{B}_R) : \text{Space of probability Radon measures on } \mathbb{T}^n \times \overline{B}_R$$

<u>Define</u> (convex dual cone of $\mathcal{G}_{z,\lambda}$)

$$\boxed{\mathcal{G}'_{z,\lambda} = \left\{ \mu \in \mathcal{R} : \langle \mu, f \rangle = \int_{\mathbb{T}^n \times \overline{B}_R} f(x,v)\, d\mu(x,v) \geq 0, \text{ for all } f \in \mathcal{G}_{z,\lambda} \right\}}$$



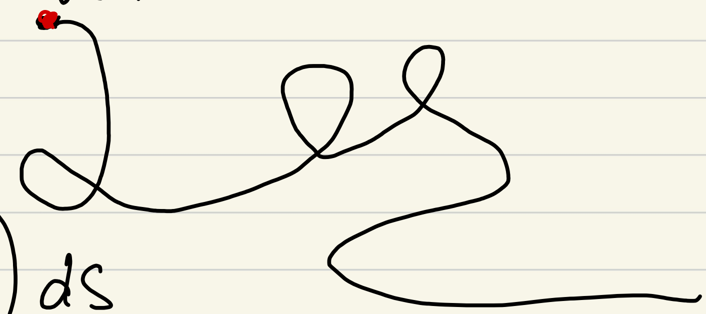
Theorem: $z \in \mathbb{T}^n$

$$\lambda u(z) = \inf_{\mu \in \mathcal{P} \cap \mathcal{C}'_{z,\lambda}} \int_{\mathbb{T}^n \times \bar{B}_R} L(x,v)\, d\mu(x,v)$$

$$= \min_{\mu \in \mathcal{P} \cap \mathcal{C}'_{z,\lambda}} \int_{\mathbb{T}^n \times \bar{B}_R} L(x,v)\, d\mu(x,v)$$

Remark: Ealier, Langrane formulations gives

$\gamma(0) = z$



$$u(z) = \inf_{\gamma(0)=z} \int_0^\infty e^{-\lambda s} L(\gamma(s), -\dot{\gamma}(s))\, ds$$

Relaxation Idea: Curves being replaced by measures

$$\lambda u(z) = \inf_{\gamma(0)=z} \int_0^\infty \left(\lambda e^{-\lambda s}\right) L(\cdot, \cdot)\, ds.$$

$$\left(\lambda \frac{u_1 + u_2}{2} + \sup_{\|N\| \le 1} \left(Du \cdot \nu - \frac{\phi_1 + \phi_2}{2} \right) \le 0 \right)$$

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$$(S) \quad \lambda v^\lambda + H(x, Dv^\lambda) = 0 \quad \text{in } \mathbb{R}^n \quad (\lambda > 0)$$

Lagraigian formulation :

$$\boxed{v^\lambda(x) = \inf_{\gamma(0)=x} \int_0^\infty e^{-\lambda s} \cdot L(\gamma(s), -\dot{\gamma}(s))\, ds}$$

<u>New rep. formulation</u>: Under assumption that H is $\mathbb{Z}^n$-periodic in x, convex & coercive in p, we can do some simplification & assume

$$\boxed{H(x,p) = \sup_{|v|\le R} \Big(p\cdot v - \underbrace{L(x,v)}_{\text{cost function}}\Big) \quad \left(\text{compact control setting}\right)}$$

<u>Defn</u>: $\phi \in C(\mathbb{T}^n \times \overline{B}_R)$, define $H_\phi(x,p) = \sup_{|v|\le R} \big(p\cdot v - \phi(x,v)\big)$

Note $H = H_L$ then)

Convex cones: $\mathcal{F}_\lambda = \{ (\phi, u) : \lambda u + H_\phi(x, Du) \leq 0 \text{ a.e in } \mathbb{T}^n \}$

(Evaluation cone) $\mathcal{G}_{z,\lambda} = \{ \phi - \lambda u(z) : (\phi, u) \in \mathcal{F}_\lambda \} \subset C(\mathbb{T}^n \times \bar{B}_R)$

Lemma 1: $\mathcal{F}_\lambda \subset C(\mathbb{T}^n \times \bar{B}_R) \times C(\mathbb{T}^n)$ convex.

Lemma 2: $\mathcal{G}_{z,\lambda} \subset C(\mathbb{T}^n \times \bar{B}_R)$ is a convex cone $\longleftrightarrow$ $\mathcal{G}'_{z,\lambda} \subset \mathcal{R}(\mathbb{T}^n \times \bar{B}_R)$

Theorem:
$$\lambda v^\lambda(z) = \inf_{\mu \in \mathcal{G}'_{z,\lambda} \cap \mathcal{P}} \int_{\mathbb{T}^n \times \bar{B}_R} L(x,v)\, d\mu(x,v)$$
$$= \min_{\mu \in \mathcal{G}'_{z,\lambda} \cap \mathcal{P}} \int_{\mathbb{T}^n \times \bar{B}_R} L(x,v)\, d\mu(x,v)$$

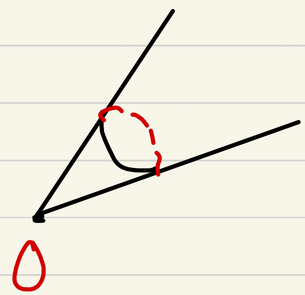
Relaxation idea. curves $\left(\lambda e^{-\lambda s}\, ds\right)$

$$\bigcap$$

$$\mathcal{D} \cap \mathcal{G}^{1}_{z,\lambda}$$

Sketch of proof of Lemma 2

For $s > 0$, $(\phi, u) \in \mathcal{F}_\lambda$



$f \in C,\ sf \in C,\ \forall s \geq 0.$

$$\lambda u + \sup_{|v| \leq h} \left[v \cdot Du - \phi(x, v) \right] \leq 0 \text{ a.e}$$

$$\iff \lambda u + \sup_{|v| \leq h} \left[v \cdot D(su) - s\phi(x, v) \right] \leq 0 \text{ a.e}$$

$$\iff (s\phi, su) \in \mathcal{F}_\lambda \Rightarrow \text{Both } \mathcal{F}_\lambda, \mathcal{G}_{z,\lambda} \text{ are convex cones.}$$

proof of theorem:

$$\lambda v^\lambda + H(x, Dv^\lambda) = 0 \quad (H = H_L)$$

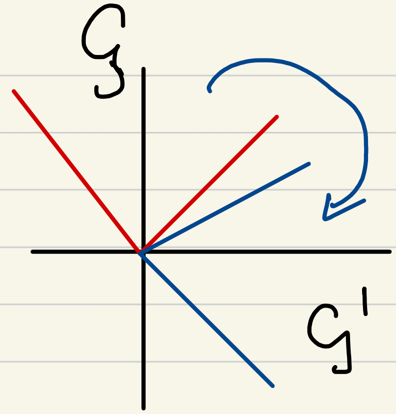
$$\Rightarrow (L, v^\lambda) \in \mathcal{F}_\lambda \Rightarrow L - \lambda v^\lambda(z) \in \mathcal{G}_{z,\lambda}.$$

$$\forall \mu \in \mathcal{G}'_{z,\lambda} \cap P : \langle \mu, L - \lambda v^\lambda(z) \rangle = \int_{\mathbb{T}^n \times \bar{B}_R} (L - \lambda v^\lambda(z))\, d\mu \geq 0$$

$$\Leftrightarrow \int_{\mathbb{T}^n \times \bar{B}_R} L\, d\mu \geq \int_{\mathbb{T}^n \times \bar{B}_R} \lambda v^\lambda(z)\, d\mu = \lambda v^\lambda(z).$$

$$\Rightarrow \lambda v^\lambda(z) \leq \min_{\mu \in \mathcal{G}'_{z,\lambda} \cap P} \int_{\mathbb{T}^n \times \bar{B}_R} L\, d\mu.$$

Converse otherwise that $\exists\, \varepsilon > 0$ such that



$$\lambda v^\lambda(z) + \varepsilon < \min_{\mu \in G'_{z,\lambda} \cap P} \langle \mu, L \rangle$$

<u>Duality trick</u>:

$$\inf_{f \in G_{\lambda,z}} \langle \mu, f \rangle = \begin{cases} 0 & \text{if } \mu \in G'_{z,\lambda} \\ -\infty & \text{if } \mu \notin G'_{z,\lambda} \end{cases}$$

$\exists f_1 , \langle \mu, f_1 \rangle < 0 \to \langle \mu, f_1 k \rangle \xrightarrow{k \to \infty} -\infty$

$$\lambda v^\lambda(z) + \varepsilon < \min_{\mu \in P} \min_{\mu \in G'_{z,\lambda}} \langle \mu, L \rangle = \min_{\mu \in P} \Big(\langle \mu, L \rangle - \inf_{f \in G_{\lambda,z}} \langle \mu, f \rangle$$

$$\boxed{\lambda v^\lambda(z)+\varepsilon < \min_{\mu\in P}\ \sup_{f\in \mathcal{G}_{z,\lambda}} \langle \mu, L-f\rangle}$$

forces $\mu \in \mathcal{G}'_{z,\lambda}$

(penalization idea)

Sion's maximin theorem (Appendix B): P compact, convex, $\mathcal{G}_{\lambda,z}$ convex $\lambda v^\lambda(z)+\varepsilon < \sup_{f\in\mathcal{G}_{z,\lambda}} \min_{\mu\in P} \langle \mu, L-f\rangle$.

$\exists\, f\in \mathcal{G}_{z,\lambda}$, $f=\phi-\lambda u(z)$ with $(\phi,u)\in \mathcal{F}_\lambda$ such that

$$\lambda v^\lambda(z)+\varepsilon < \min_{\mu\in P}\langle \mu, L-f\rangle = \min_{\mu\in P}\langle \mu, L-\phi+\lambda u(z)\rangle$$

$$\Rightarrow \lambda v^\lambda(z)+\varepsilon < L(x,v)-\phi(x,v)+\lambda u(z)$$

$$\forall \mu = \delta_{(x,v)}, \ (x,v) \in \mathbb{T}^n \times \overline{B}_h.$$

$$H(x,p) = \sup_{|v| \le h} \big(p \cdot v - L(x,v)\big) \le \sup_{|v| \le h} \Big(p \cdot v - \phi(x,v)$$

$$+ \lambda(u - v^\lambda)(z) - \varepsilon\Big)$$

$$\Rightarrow H(x,p) \le H_\phi(x,p) + \lambda(u - v^\lambda)(z) - \varepsilon$$

$$\lambda v^\lambda + H(x, Dv^\lambda) = 0$$

$$\Rightarrow \lambda v^\lambda + H_\phi(x, Dv^\lambda) + \lambda(u - v^\lambda)(z) - \varepsilon \geq 0$$

Thus, $\lambda v^\lambda + H_\phi(x, Dv^\lambda) + \underbrace{\lambda(u - v^\lambda)(z)}_{\text{constant.}} - \varepsilon \geq 0$

$$\begin{cases} \lambda\left(v^\lambda + (u - v^\lambda)(z) - \frac{\varepsilon}{\lambda}\right) + H_\phi(x, Dv^\lambda) \geq 0 \\ \lambda u + H_\phi(x, Du) \leq 0 \end{cases}$$

Comparison priciple $\Rightarrow$ $v^\lambda(x) + (u - v^\lambda)(z) - \frac{\varepsilon}{\lambda} \geq u(x)$

$x \in \mathbb{T}^n$.

For $x = z$, $-\frac{\varepsilon}{\lambda} \geq 0$ $\otimes$.

Remarks:

$$\lambda v^\lambda(z) = \min_{\mu \in \mathcal{P} \cap \mathcal{G}'_{\lambda, z}} \int_{\mathbb{T}^n \times \bar{B}_R} L(x, v)\, d\mu(x, v)$$

1. A weakness: $\mathcal{G}'_{z, \lambda}$ is a bit abstract and dependent on z.

(2) "practical / open question": Same formulation in the discrete setting. Can one then do numeric to find $g'_{z,\lambda}$ & λv^λ ?

(3) Major open question (both PDE & spectral theory)

$\varepsilon > 0$, consider

$$H(x, Dv^\varepsilon) = \varepsilon \Delta v^\varepsilon + \underbrace{\overline{H}^\varepsilon(0)}_{\text{a cost.}} \quad \text{in } \mathbb{T}^n$$

Formula of v^ε ? ($\hookrightarrow$ principal eigenvalue & eigenfunctions).

A <u>minimalistic description of applications</u>

$$\lambda v^\lambda + H(x, Dv^\lambda) = 0 \text{ in } \mathbb{T}^n.$$

① [Lions - papanicolaous - varadhan]

$$\lambda v^\lambda \rightrightarrows -\overline{H}(0) \in \mathbb{R} : \text{unique constant}$$

$$\underset{-c.}{\shortparallel} \qquad \text{many solutions.}$$

PDE becomes : $\boxed{H(x, Du) = \underset{\overset{\shortparallel}{c}}{\overline{H}(0)} \text{ in } \mathbb{T}^n}$

$$-c = -\overline{H}(0) = \min_{\mu \in \mathcal{P} \cap \mathcal{G}_0'} \int_{\mathbb{T}^n \times \overline{B}_R} L \, d\mu$$

H: original Hamiltonian

$\overline{H}$: effective Hamiltonian

If μ is a mimizer, we say that μ is a Mather measure.

(2) [Theorem] $v^\lambda + \frac{\overline{H}(0)}{\lambda} \xrightarrow{\lambda \to 0^+} v^0$ uniquely and

$$v^0 = \sup_{v \in \mathcal{E}} \quad \text{when} \quad \mathcal{E} = \left\{ v : \begin{array}{l} H(x, Dv) \leq \overline{H}(0) \\ \langle \mu, v \rangle \leq 0, \ \forall \mu \text{ Mather} \end{array} \right\}$$

So called Selection problem

μ_λ $\qquad\qquad$ x^λ

$\downarrow$

μ weakly

Convexity always has hidden duality.