ON THE LANGEVIN EQUATION WITH VARIABLE FRICTION

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Abstract. We study two asymptotic problems for the Langevin equation with variable friction coefficient. The first is the small mass asymptotic behavior, known as the Smoluchowski-Kramers approximation, of the Langevin equation with strictly positive variable friction. The second result is about the limiting behavior of the solution when the friction vanishes in regions of the domain. Previous works on this subject considered one dimensional settings with the conclusions based on explicit computations.

1. Introduction

The first topic in this paper is the study of the small mass asymptotic behavior, known as the Smoluchowski-Kramers approximation, of the generalized Langevin equation. The latter describes the motion, with variable strictly positive friction coefficient λ, of a particle of mass μ in a force field b which subject to random fluctuations modeled by a Brownian motion W with diffusivity σ, which represent random collisions of the given particle with other particles in the fluid.

More precisely, we consider the behavior as $\mu \to 0$ of the solution x^μ to

$$\mu \ddot{x}^\mu = b(x^\mu) - \lambda(x^\mu)\dot{x}^\mu + \sigma(x^\mu)\dot{W}, \quad x^\mu(0) = x \in \mathbb{R}^n, \quad \dot{x}^\mu(0) = p \in \mathbb{R}^n. \tag{1.1}$$

The result is that, under some regularity assumptions on b, σ and λ, and for every $T > 0$ and $\delta > 0$,

$$\lim_{\mu\to 0} \mathbb{P}(\max_{0\le t\le T} |x^\mu(t) - x(t)| > \delta) = 0, \tag{1.2}$$

with x evolving by the Itô stochastic differential equation

$$dx = \frac{1}{\lambda(x)} b(x)dt + \frac{1}{\lambda(x)}\sigma(x)dW, \quad x(0) = x \in \mathbb{R}^n. \tag{1.3}$$

We prove (1.2) by studying the pde governing the law of x^μ and showing that, as $\mu \to 0$, its solutions converge to solutions to the pde of the law of x.

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To this end, we rewrite (1.1) as

$$\begin{cases} \dot{x}^\mu = y^\mu, \\ \dot{y}^\mu = \dfrac{1}{\mu}\left(b(x^\mu) - \lambda(x^\mu)y^\alpha\right) + \dfrac{1}{\mu}\sigma(x^\mu)\dot{W}. \end{cases} \tag{1.4}$$

The generator L^μ of the law of the diffusion process (x^μ, y^μ) is, with $a = \sigma\sigma^t$,

$$L^\mu u(x,y) := \frac{1}{2\mu^2} a_{ij}(x) u_{y_i y_j} + \frac{1}{\mu}(b_i(x) - \lambda(x)y_i)u_{y_i} + y_i u_{x_i}. \tag{1.5}$$

The claim (see Theorem 2.1) is that solutions $u^\mu = u^\mu(x,y,t)$ to $u^\mu_t = L^\mu u^\mu$ converge, as $\mu \to 0$ and locally uniformly, to solutions $u = u(x,t)$ to $u_t = Lu$, where

$$Lu := \frac{1}{2\lambda(x)} a_{ij}(x) \left(\frac{u_{x_i}}{\lambda(x)}\right)_{x_j} + \frac{1}{\lambda(x)} b_i(x) u_{x_i}. \tag{1.6}$$

A result of this type was shown by Freidlin and Hu [4] under some simplifying assumptions, for example $a \equiv 1$ by exact computations.

The second topic of the paper is the study of the limiting generator at places where the friction vanishes. This question was raised by Freidlin, Hu and Wentzell [5], who considered that problem in one dimension with $a \equiv 1$ and found an explicit solution. Assuming that the nonnegative friction vanishes in some compact region, [5] approximates λ by $\lambda + \varepsilon$ and studies the behavior of the solutions as $\varepsilon \to 0$.

Motivated by [5], we consider the general boundary value problem

$$-a_{ij}(x)\left(\frac{u^\varepsilon_{x_i}}{\lambda + \varepsilon}\right)_{x_j} - 2b_i u^\varepsilon_{x_i} = 0 \text{ in } U, \quad u^\varepsilon = g \text{ on } \partial U, \tag{1.7}$$

in a domain $U \subset \mathbb{R}^n$ (all the precise assumptions are stated later in the paper) and $\lambda \equiv 0$ in $V \subset U$ and strictly positive in $\overline{U} \setminus \overline{V}$.

The result (see Theorem 2.3) is that, as $\varepsilon \to 0$ and uniformly in $\overline{U}$, $u^\varepsilon \to u$, the unique viscosity solution to

$$\begin{cases} -a_{ij}(x)\left(\dfrac{u_{x_i}}{\lambda}\right)_{x_j} - 2b_i u_{x_i} = 0 \text{ in } U \setminus \overline{V}, \quad u = g \text{ on } \partial U, \\ -a_{ij} u_{x_i x_j} = 0 \text{ in } V \quad \text{and} \quad a_{ij} u_{x_i} \nu_j = 0 \text{ on } \partial V, \\ \displaystyle\int_{\partial U} \frac{a_{ij} u_{x_i} \nu_j m}{\lambda} d\sigma = 0, \end{cases} \tag{1.8}$$

where $m \in C(\overline{U})$ is the unique solution of an appropriate adjoint problem and ν denotes the external normal vector to V and U.

Organization of the paper. In the next section we introduce the precise assumptions and state the main results. Section 3 is devoted to the proof of the small mass approximation. In Section 4, we prove the result about the degenerate friction. In Section 5, we study the adjoint problems that play an important role in the proofs in Section 4 and in identifying the limit. In Section 6, we give a brief explanation how to apply the standard theory of existence and uniqueness of solutions to the initial value problem for $u_t = L^\mu u$.

Terminology and Notation. Depending on the context throughout the paper solutions are either classical or in the viscosity sense. In particular the boundary value problem

$$-a_{ij}u_{x_ix_j} = 0 \ \text{ in } V \quad \text{ and } \quad a_{ij}u_{x_i}\nu_j = 0 \text{ on } \partial V,$$

is interpreted in the viscosity sense, that is, in the case of subsolution, for instance,

$$-a_{ij}u_{x_ix_j} \leq 0 \ \text{ in } V \quad \text{ and } \quad \min(-a_{ij}u_{x_ix_j},\, a_{ij}u_{x_i}\nu_j) \leq 0 \text{ on } \partial V.$$

Given $O \subset \mathbb{R}^k$ for some k, $\mathrm{BUC}\,(O)$ is the space of bounded uniformly continuous functions on O; its norm is denoted by $\|\cdot\|$. Moreover, $C_{\mathrm{b}}(O)$ and $C^m_{\mathrm{b}}(O)$ with $m \in \mathbb{N}$ denote respectively the spaces of the bounded continuous functions on O and the functions in $C_{\mathrm{b}}(O)$ with bounded continuous derivatives up to order m. Their respective norms are $\|\cdot\|_{C(O)}$ and $\|\cdot\|_{C^m(O)}$. When possible, the dependence on the domain of the spaces in the norms is omitted.

If $f_\varepsilon : A \to \mathbb{R}$, where $A \subset \mathbb{R}^k$, is such that $\sup_{\varepsilon\in(0,1)} \|f_\varepsilon\| < \infty$, the generalized (relaxed) upper and lower limits f^+ and f^- are given respectively by

$$f^+(x) := \limsup_{\varepsilon\to 0}{}^{*} f_\varepsilon(x) := \limsup_{\varepsilon\to 0, x'\to x} f_\varepsilon(x') \text{ and } f^-(x) := \liminf_{\varepsilon\to 0}{}_{*} f_\varepsilon(x) := \liminf_{\varepsilon\to 0, x'\to x} f_\varepsilon(x').$$

Finally, $O(r)$ denotes various functions of $r \geq 0$ such that $|O(r)| \leq Cr$ for all $r \geq 0$ for some constant $C > 0$ which is independent of the various parameters in the specific context.

Throughout the paper in writing equations we use the summation convention.

2. The assumptions and the results

Small mass approximation. In the first part of the paper we assume that

$$\sigma, b, \lambda, D\lambda \ \text{ are bounded and Lipschitz continuous on } \mathbb{R}^n, \tag{2.1}$$

and there exist $\Theta \geq \theta > 0$ such that, for all $x, \xi \in \mathbb{R}^n$,

$$\theta \leq \lambda(x) \leq \Theta \tag{2.2}$$

and

$$\theta|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Theta|\xi|^2. \tag{2.3}$$

We remark that we have not tried to optimize our assumptions and some of the results definitely hold with less regularity on the coefficients.

Fix $T > 0$ and given $u_0^\mu \in \mathrm{BUC}\,(\mathbb{R}^n \times \mathbb{R}^n)$, let $u^\mu \in C_{\mathrm{b}}(\mathbb{R}^n \times \mathbb{R}^n \times [0,T])$ be the unique (viscosity) solution to the initial value problem

$$\begin{cases} u^\mu_t = \dfrac{1}{2\mu^2}a_{ij}(x)u^\mu_{y_iy_j} + \dfrac{1}{\mu}(b_i(x) - \lambda(x)y_i)u^\mu_{y_i} + y_iu^\mu_{x_i} \text{ in } \mathbb{R}^n \times \mathbb{R}^n \times (0,T), \\ u^\mu(\cdot,\cdot,0) = u_0^\mu \text{ in } \mathbb{R}^n \times \mathbb{R}^n. \end{cases} \tag{2.4}$$

Because the coefficients $\lambda(x)y_i$ are not globally Lipschitz continuous on $\mathbb{R}^n \times \mathbb{R}^n$, (2.4) is a bit out of scope of the classical theory of viscosity solutions (see [1]). Nevertheless, in view of (2.1) and (2.2), there exists a unique viscosity solution of (2.4). We discuss this issue briefly (see Theorem 6.1) in Section 6.

The result about the small mass approximation is stated next.

Theorem 2.1. *Suppose* (2.1), (2.2) *and* (2.3). *Assume that* $u_0^\mu \in \mathrm{BUC}(\mathbb{R}^n \times \mathbb{R}^n)$ *is such that* $\lim_{\mu\to 0}\sup_{x,y\in\mathbb{R}^n}|u_0^\mu(x,y)-u_0(x)| = 0$. *Then, as* $\mu \to 0$ *and locally uniformly on* $\mathbb{R}^n \times \mathbb{R}^n \times [0,T)$, $u^\mu \to u \in \mathrm{BUC}(\mathbb{R}^n \times [0,T])$, *where* u *is the unique solution to*

$$\begin{cases} u_t = \dfrac{1}{2\lambda(x)} a_{ij}(x)\left(\dfrac{u_{x_i}}{\lambda(x)}\right)_{x_j} + \dfrac{1}{\lambda(x)} b_i(x) u_{x_i} \ \text{in } \mathbb{R}^n \times (0,T), \\ u(\cdot,0) = u_0 \ \text{in } \mathbb{R}^n. \end{cases} \tag{2.5}$$

As stated in the introduction, a special case of Theorem 2.1 for $n = 1$ was proved in [4].

Vanishing friction. We formulate next the result about the vanishing friction. We assume that for some $\alpha \in (0, 1)$,

$$\begin{cases} U \text{ is a } C^{2,\alpha}\text{-bounded, connected, open subset of } \mathbb{R}^n, \\ V \text{ is a } C^{2,\alpha}\text{-connected open subset of } U \text{ such that } \overline{V} \subset U, \end{cases} \tag{2.6}$$

$$a, b, \lambda \in C^{2,\alpha}(\bar{U}), \tag{2.7}$$

there exist $\Theta, \theta > 0$ such that, for all $x \in \overline{U}$ and $\xi \in \mathbb{R}^n$,

$$0 \le \lambda(x) \le \Theta, \tag{2.8}$$

and

$$\theta|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Theta|\xi|^2, \tag{2.9}$$

$$\lambda \equiv 0 \text{ on } \overline{V} \quad \text{and} \quad \lambda > 0 \text{ on } \overline{U} \setminus \overline{V}, \tag{2.10}$$

and, if d is the signed distance function of ∂V given by

$$d(x) := \begin{cases} \mathrm{dist}(x, \partial V) & \text{if } x \in \overline{U} \setminus V, \\ -\mathrm{dist}(x, \partial V) & \text{if } x \in V, \end{cases}$$

then

$$\begin{cases} \text{there exist } \lambda_0 \in C^2(\mathbb{R}) \text{ and } C_0 > 0 \text{ such that } \lambda_0 \equiv 0 \text{ on } (-\infty, 0], \text{ and} \\ \lambda_0(r) > 0,\ \lambda_0'(r) \ge 0 \text{ and } r\lambda_0'(r) \le C_0\lambda_0(r) \text{ for } r \in (0,\infty), \text{ and} \\ \qquad \lambda(x) = \lambda_0(d(x)) \quad \text{in a neighborhood of } \partial V \text{ in } U \setminus \overline{V}. \end{cases} \tag{2.11}$$

Assumption (2.11) is crucial in Lemmas 4.3 and 5.6 below. In what follows, one may replace d by a defining function $\rho \in C^2(\mathbb{R}^n)$ of V, that is, $\rho \in C^2$ such that $\rho < 0$ in V, $\rho > 0$ in $\mathbb{R}^n \setminus \overline{V}$, and $D\rho \neq 0$ on ∂V. Finally, as before, we remark that here we are not trying to optimize the assumptions.

We study the behavior, as $\varepsilon \to 0$, of the solution u^ε to (1.7) with

$$g \in C^{2,\alpha}(\overline{U}). \tag{2.12}$$

An important ingredient of our analysis is the study of the asymptotic behavior of the solution m^ε of the "adjoint" problem

$$(\mathrm{Ad})_\varepsilon \qquad \begin{cases} -\left(\dfrac{(a_{ij}m^\varepsilon)_{x_i}}{\lambda+\varepsilon} - 2b_j m^\varepsilon\right)_{x_j} = 0 \text{ in } U \\ \left(\dfrac{(a_{ij}m^\varepsilon)_{x_i}}{\lambda+\varepsilon} - 2b_j m^\varepsilon\right)\nu_j = 0 \text{ on } \partial U \\ \displaystyle\int_U m^\varepsilon dx = 1 \text{ and } m^\varepsilon > 0 \text{ in } \overline{U}, \end{cases}$$

where ν denotes the outward unit normal vector to $\overline{U}$.

The limit problem of $(\mathrm{Ad})_\varepsilon$, as $\varepsilon \to 0$, is

$$(\mathrm{Ad1}) \qquad \begin{cases} -\left(\dfrac{(a_{ij}m)_{x_i}}{\lambda} - 2b_j m\right)_{x_j} = 0 \text{ in } U \setminus \overline{V}, \\ \left(\dfrac{(a_{ij}m)_{x_i}}{\lambda} - 2b_j m\right)\nu_j = 0 \text{ on } \partial U, \\ \displaystyle\int_U m dx = 1 \text{ and } m > 0 \text{ in } \bar{U}, \end{cases}$$

and

$$(\mathrm{Ad2}) \qquad -(a_{ij}m)_{x_i x_j} = 0 \text{ in } V \quad \text{and} \quad (a_{ij}m)_{x_i}\nu_j = 0 \text{ on } \partial V,$$

where, here, ν is the outward unit normal vector to $\overline{V}$.

To describe the limiting behavior of the u^ε's we need the following result which is a consequence of Theorem 4.1 below whose proof is provided in Section 5.

Theorem 2.2. *Assume* (2.6), (2.7), (2.8), (2.9), (2.10) *and* (2.11). *Then there exists a unique solution* $m \in C(\overline{U}) \cap C^2(\overline{U} \setminus \partial V)$ *of* (Ad1) *and* (Ad2).

The main result is:

Theorem 2.3. *Assume* (2.6), (2.7), (2.8), (2.9), (2.10), (2.11) *and* (2.12). *For each* $\varepsilon > 0$ *let* $u^\varepsilon \in C(\overline{U}) \cap C^2(U)$ *be the unique solution to* (1.7). *Then, as* $\varepsilon \to 0$ *and uniformly on* $\overline{U}$, $u^\varepsilon \to u$, *where* $u \in C(\overline{U}) \cap C^2(\overline{U} \setminus \partial V)$ *is the unique solution to*

$$-a_{ij}(x)\left(\frac{u_{x_i}}{\lambda}\right)_{x_j} + 2b_i u_{x_i} = 0 \text{ in } U \setminus \overline{V}, \tag{2.13}$$

$$u = g \text{ on } \partial U, \tag{2.14}$$

$$-a_{ij}u_{x_i x_j} = 0 \text{ in } V \quad \text{and} \quad a_{ij}u_{x_i}\nu_j = 0 \text{ on } \partial V, \tag{2.15}$$

and

$$\int_{\partial U} \frac{a_{ij}u_{x_i}\nu_j m}{\lambda} d\sigma = 0, \tag{2.16}$$

with $m \in C(\overline{U})$ *is given by Theorem 2.2.*

The meaning of (2.15) was discussed in the subsection about terminology and notation earlier in the paper.

3. The small mass approximation

The proof of Theorem 2.1 is based on a variant of the perturbed test function (see Evans [2,3]) and classical arguments from the theory of viscosity solutions.

Formal expansion. To identify the equation satisfied by the limit of the u^μ's we postulate the ansatz

$$u^\mu(x,y,t) = u(x,t) + \frac{\mu y_i}{\lambda(x)} v_i(x,t) + \frac{\mu^2 y_i y_j}{\lambda(x)^2} w_{ij}(x,t) + \frac{\mu^3 y_i y_j y_k}{\lambda(x)^3} z_{ijk}(x,t) + \cdots \tag{3.1}$$

where $u, v_i, w_{ij}, z_{ijk}, \dots$ are real-valued functions on $\mathbb{R}^n \times [0,\infty)$.

We assume that, for $1 \le i,j,k,\dots \le n$, $w_{ij} = w_{ji}$, $z_{ijk} = z_{jik} = \dots$, we insert (3.1) in (2.4), we organize in terms of powers of μ and we equate to 0 the coefficients of $O(1)$ and $O(\mu)$. From the former we get

$$u_t = \frac{1}{\lambda^2} a_{ij} w_{ij} + \frac{1}{\lambda}(b_i - \lambda y_i) v_i + y_i u_{x_i} = \frac{1}{\lambda^2} a_{ij} w_{ij} + \frac{1}{\lambda} b_i v_i + y_i(u_{x_i} - v_i), \tag{3.2}$$

while from the latter we find

$$\frac{y_i}{\lambda} v_{i,t} = \frac{3y_i}{\lambda^3} a_{jk} z_{ijk} + \frac{1}{\lambda^2}(b^i - \lambda y_i) 2 y_j w_{ij} + y_i y_j \left(\frac{v_i}{\lambda}\right)_{x_j}. \tag{3.3}$$

We deduce from (3.2) that $v_i = u_{x_i}$ for $1 \le i \le n$. Then (3.3) can be written , for $1 \le i \le n$, as

$$v_{i,t} = \frac{3}{\lambda^2} a_{jk} z_{ijk} + \frac{2b_j}{\lambda} w_{ij} + y_j \left(\lambda \left(\frac{u_{x_i}}{\lambda}\right)_{x_j} - 2 w_{ij} \right),$$

which yields that

$$w_{ij} = 2^{-1} \lambda \left(\lambda^{-1} u_{x_i}\right)_{x_j}.$$

Hence we obtain formally that $u = u(x,t)$ satisfies

$$u_t = \frac{1}{2\lambda(x)} a_{ij}(x) \left(\frac{u_{x_i}}{\lambda(x)}\right)_{x_j} + \frac{1}{\lambda(x)} b_i(x) u_{x_i}. \tag{3.4}$$

The rigorous convergence. We present here the rigorous proof of the asymptotics.

Proof of Theorem 2.1. Fix $T > 0$ and, without loss of generality, we only consider $\mu \in (0,1)$. In view of our assumptions, the u^μ's are bounded on $\mathbb{R}^n \times \mathbb{R}^n \times [0,T]$ uniformly in μ. To deal with the special (unbounded) dependence of (2.4) on y, we find it necessary to modify the definition of the relaxed upper and lower limits, which was introduced earlier.

In particular, taking into account the estimate (3.9) on y^μ below, we define generalized upper and lower limits u^+ and u^- on $\mathbb{R}^n \times [0,\, T]$ by

$$u^+(x,t) = \lim_{\delta \to 0+} \sup\{u^\mu(p,q,s) \,:\, 0 < \mu < \delta, |p-x| < \delta, |s-t| < \delta, |\mu q| < \delta\},$$

and

$$u^-(x,t) = \lim_{\delta \to 0+} \inf\{u^\mu(p,q,s) \,:\, 0 < \mu < \delta, |p-x| < \delta, |s-t| < \delta, |\mu q| < \delta\},$$

and prove that they are respectively sub- and super-solutions to (2.5). Since the arguments are almost identical, here we show the details only for the the generalized upper limit. Once the sub- and super-solution properties are established, we conclude, using that (2.5) has a comparison principle, that $u^+ = u^-$. This is a classical result in the theory of viscosity solutions, hence we omit the details.

We now show that u^+ is a viscosity subsolution to (2.5) on $\mathbb{R}^n \times [0, T)$, that is, including $t = 0$. To this end, we assume that, for some smooth test function ϕ, $u^+ - \phi$ has a strict global maximum at $(x_0, t_0) \in \mathbb{R}^n \times [0, T)$.

For the arguments below, it is convenient to assume that, there exists a compact neighborhood N of (x_0, t_0) such that

$$\begin{cases} \phi \text{ is constant for all } (x,t) \in (\mathbb{R}^n \times [0,T]) \setminus N, \\ \inf_{(\mathbb{R}^n \times [0,T)) \setminus N} \phi > 2 \sup_{0<\mu<1} \|u^\mu\| \text{ and } \ \phi(x_0, t_0) = 0. \end{cases} \tag{3.5}$$

We use a perturbed test function type argument to show that, at (x_0, t_0), if $t_0 > 0$ or if $t_0 = 0$ and $u^+(x_0, 0) > u_0(x_0)$, then

$$\phi_t \le \frac{1}{2\lambda} a_{ij} \left(\frac{\phi_{x_i}}{\lambda} \right)_{x_j} + \frac{1}{\lambda} b_i \phi_{x_i}. \tag{3.6}$$

First we consider the case $t_0 > 0$, in which case we choose N so that $N \subset \mathbb{R}^n \times (0, T)$.

We fix some $K > 0$ and replace ϕ by

$$\psi(x, y, t) = \psi^\mu(x, y, t) := \phi + \frac{\mu}{\lambda} y_i \phi_{x_i}(x, t) + \frac{\mu^2}{2\lambda} y_i y_j \left(\frac{\phi_{x_i}}{\lambda} \right)_{x_j} + K|\mu y|^3.$$

Straightforward computations together with (2.2) give

$$\psi_t(x, y, t) = \phi_t(x, t) + O(|\mu y| + |\mu y|^2),$$

$$\begin{aligned} \frac{a_{ij}\psi_{y_i y_j}}{2\mu^2} &= \frac{1}{2} a_{ij} \left(\frac{1}{\lambda} \left(\frac{\phi_{x_i}}{\lambda} \right)_{x_j} + 3K\mu(|y|^{-1} y_i y_j + |y|\delta_{ij}) \right) \\ &= \frac{1}{2\lambda} a_{ij} \left(\frac{\phi_{x_i}}{\lambda} \right)_{x_j} + O(K|\mu y|), \end{aligned}$$

$$\begin{aligned} \frac{b_i - \lambda y_i}{\mu} \cdot \psi_{y_i} &= \frac{b_i - \lambda y_i}{\mu} \cdot \left(\mu \frac{\phi_{x_i}}{\lambda} + \frac{\mu^2}{2\lambda} \left(y_j \frac{\phi_{x_i}}{\lambda} \right)_{x_j} + \frac{\mu^2}{2\lambda} \left(y_j \frac{\phi_{x_j}}{\lambda} \right)_{x_i} + 3\mu^3 K |y| y_i \right) \\ &= \frac{b_j \phi_{x_i}}{\lambda} - y_j \left(\phi_{x_j} + \mu \left(y_i \frac{\phi_{x_i}}{2\lambda} \right)_{x_j} \right) \\ &\quad - 3K\mu^2 \lambda |y|^3 + O\left(\mu |y| + K|\mu y|^2 \right), \end{aligned}$$

and

$$\begin{aligned} y_i \psi_{x_i} &= y_i \left(\phi_{x_i} + \mu \left(y_j \frac{\phi_{x_j}}{\lambda} \right)_{x_i} + \frac{\mu^2}{2\lambda} \left(y_k y_l \left(\frac{\phi_{x_k}}{\lambda} \right)_{x_l} \right)_{x_i} \right) \\ &= y_i \left(\phi_{x_i} + \mu \left(y_j \frac{\phi_{x_j}}{\lambda} \right)_{x_i} \right) + O(\mu^2 |y|^3); \end{aligned}$$

here, $O(r)$ is independent of y, μ and K.

Combining the above we get, for some $M>0$ depending only on ϕ, b and λ,

$$\begin{aligned}&-\psi_t+\frac{a_{ij}\psi_{y_iy_j}}{2\mu^2}+\frac{b_i-\lambda y_i}{\mu}\psi_{y_i}+y_i\psi_{x_i}\\&\qquad\le -\phi_t+\frac{1}{2\lambda}a_{ij}\left(\frac{\phi_{x_j}}{\lambda}\right)_{x_i}+\frac{b_i\phi_{x_i}}{\lambda}-3K\mu^2\lambda|y|^3+M((K+1)(|\mu y|+|\mu y|^2)+\mu^2|y|^2).\end{aligned}$$

Next we observe that $u^\mu-\psi$ has a global maximum on $\mathbb{R}^n\times\mathbb{R}^n\times(0,T)$. Indeed, note first that there exists a constant $R=R^\mu>0$ such that

$$\inf\{\psi(x,y,t):(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T),\ |y|>R\}\ge 1+2\|u^\mu\|, \tag{3.7}$$

consider the compact subset of $\mathbb{R}^n\times\mathbb{R}^n\times(0,T)$

$$N^\mu:=\{(x,y,t)\,:\,(x,t)\in N,\ y\in\overline{B}_R\},$$

note that

$$\psi(x,y,t)\le\phi(x,t)\quad\text{if}\quad (x,t)\in\mathbb{R}^n\times(0,T)\setminus N,$$

and, in view of (3.5) and (3.7), if $(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,T)$ and $(x,t)\notin N$, then

$$\begin{aligned}(u^\mu-\psi)(x,y,t)&\le u^\mu(x,y,t)-\phi(x,t)\le\|u^\mu\|-\inf_{(\mathbb{R}^n\times(0,T))\setminus N}\phi\\&-1-\|u^\mu\|\le -1+u^\mu(x_0,0,t_0)=-1+u^\mu(x_0,0,t_0)-\phi(x_0,t_0)\\&=-1+u^\mu(x_0,0,t_0)-\psi(x_0,0,t_0),\end{aligned}$$

and, if $|y|>R$, then

$$\begin{aligned}&(u^\mu-\psi)(x,y,t)\\&\qquad\le\|u^\mu\|-\inf\{\psi(p,q,s)\,:\,(p,q,s)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T),\ |q|>R\}\\&\qquad\le -1+\|u^\mu\|\le -1+u^\mu(x_0,0,t_0)-\psi(x_0,0,t_0).\end{aligned}$$

The two inequalities above yield

$$\sup_{(\mathbb{R}^n\times\mathbb{R}^n\times(0,T))\setminus N^\mu}(u^\mu-\psi)\le -1+(u^\mu-\psi)(x_0,0,t_0)<\max_{N^\mu}(u^\mu-\psi),$$

that is, $u^\mu-\psi$ has a global maximum at some point in N^μ.
Let (x^μ,y^μ,t^μ) be a global maximum point of $u^\mu-\psi$. Then, at (x^μ,y^μ,t^μ),

$$-\psi_t+\frac{a_{ij}\psi_{y_iy_j}}{2\mu^2}+\frac{b_i-\lambda y_i}{\mu}\psi_{y_i}+y_i\psi_{x_i}\ge 0,$$

and, hence, always at (x^μ,y^μ,t^μ),

$$-\phi_t+\frac{1}{2\lambda}a_{ij}\left(\frac{\phi_{x_j}}{\lambda}\right)_{x_i}+\frac{b_i\phi_{x_i}}{\lambda}-3K\mu^2\lambda|y|^3\ge M((K+1)(|\mu y|+|\mu y|^2)+\mu^2|y|^3).$$

Choosing $K=\frac{M+1}{3\theta}$ we obtain

$$-\phi_t+\frac{1}{2\lambda}a_{ij}\left(\frac{\phi_{x_j}}{\lambda}\right)_{x_i}+\frac{b_i\phi_{x_i}}{\lambda}\ge -M(K+1)(|\mu y|+|\mu y|^2)+\mu^2|y|^3. \tag{3.8}$$

In particular, for some $C>0$, we find

$$\mu^2|y^\mu|^3\le C(1+|\mu y^\mu|+|\mu y^\mu|^2).$$

Hence, $|\mu y^\mu| = O(\mu^{1/3})$, and, thus,

$$\lim_{\mu\to 0} \mu y^\mu = 0 \quad \text{and} \quad \lim_{\mu\to 0} (\psi(x^\mu, y^\mu, t^\mu) - \phi(x^\mu, t^\mu)) = 0. \tag{3.9}$$

Next we show that there is a sequence $\mu_j \to 0$ such that

$$\lim_{j\to\infty} (x^{\mu_j}, t^{\mu_j}) = (x_0, t_0).$$

In view of the definition of u^+, we may select a sequence $\{(\mu_j, p_j, q_j, s_j)\}_{j\in\mathbb{N}} \subset (0,\, 1) \times \mathbb{R}^n \times \mathbb{R}^n \times (0, T)$ such that

$$\lim_{j\to\infty} (\mu_j, p_j, s_j) = (0, x_0, t_0), \;\; \lim_{j\to\infty} \mu_j q_j = 0 \text{ and } \lim_{j\to\infty} u^{\mu_j}(p_j, q_j, s_j) = u^+(x_0, t_0). \tag{3.10}$$

Passing to a subsequence, we may assume that, for some $(\bar{x}, \bar{t}) \in N$,

$$\lim_{j\to\infty} (x^{\mu_j}, t^{\mu_j}) = (\bar{x}, \bar{t}).$$

Since (x^μ, y^μ, t^μ) is a global maximum of $u^\mu - \psi$, for any $\delta > 0$ and as soon as $|x^{\mu_j} - \bar{x}| < \delta$, $|t^{\mu_j} - \bar{t}| < \delta$ and $|\mu_j y^{\mu_j}| < \delta$, we have

$$(u^{\mu_j} - \psi)(p_j, q_j, s_j) \le (u^{\mu_j} - \psi)(x^{\mu_j}, y^{\mu_j}, t^{\mu_j}) \le v^\delta(\bar{x}, \bar{t}) - \psi(x^{\mu_j}, y^{\mu_j}, t^{\mu_j}) \tag{3.11}$$

where v^δ is defined by

$$v^\delta(x, t) := \sup\{u^\mu(p, q, s) \,:\, |p - x| < \delta, |s - t| < \delta, |\mu q| < \delta\}.$$

Now, since

$$\lim_{j\to\infty} \psi^{\mu_j}(p_j, q_j, s_j) = \phi(x_0, t_0) \quad \text{and} \quad \lim_{j\to\infty} \psi^{\mu_j}(x^{\mu_j}, y^{\mu_j}, t^{\mu_j}) = \phi(\bar{x}, \bar{t}),$$

we find from (3.11) that, for any $\delta > 0$,

$$(u^+ - \phi)(x_0, t_0) \le (v^\delta - \phi)(\bar{x}, \bar{t}),$$

which readily gives

$$(u^+ - \phi)(x_0, t_0) \le (u^+ - \phi)(\bar{x}, \bar{t}).$$

Since (x_0, t_0) is a strict global maximum point of $u^+ - \phi$, we see from the above that $(\bar{x}, \bar{t}) = (x_0, t_0)$, that is,

$$\lim_{j\to\infty} (x^{\mu_j}, t^{\mu_j}) = (x_0, t_0).$$

It then follows from (3.8) that, at (x_0, t_0),

$$\phi_t \le \frac{1}{2\lambda} a_{ij} \left(\frac{\phi_{x_i}}{\lambda}\right)_{x_j} + \frac{b_i \phi_{x_i}}{\lambda}.$$

Now we consider the case $t_0 = 0$ and $u^+(x_0, 0) > u_0(x_0)$, and show that (3.6) holds at $(x_0, 0)$. Let $\delta > 0$ be such that

$$(u^+ - \phi)(x_0, 0) > 3\delta + u_0(x_0) - \phi(x_0, 0) \tag{3.12}$$

and observe that there is a $\mu_0 \in (0,\, 1)$ such that

$$\sup_{0<\mu<\mu_0} \|u_0 - u_0^\mu\| < \delta. \tag{3.13}$$

Fix such a μ_0 and, henceforth, assume that $\mu \in (0, \mu_0)$. Moreover, since in the definition of ψ, we have, for some $C > 0$ independent of μ,

$$\psi^\mu(x, y, t) \ge \phi(x, t) - C(|\mu y| + |\mu y|^2) + K|\mu y|^3,$$

we may assume, choosing K large enough independently of μ, that

$$\psi^\mu(x,y,t) > \phi(x,t) - \delta \quad \text{for} \quad (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,T). \tag{3.14}$$

Then we select N to be a compact neighborhood of $(x_0,0)$ relative to $\mathbb{R}^n \times [0,\,T)$ as before, with the additional requirement, in view of (3.12), that

$$(u^+ - \phi)(x_0,0) > 3\delta + u_0(x) - \phi(x,0) \quad \text{if} \quad (x,0) \in N. \tag{3.15}$$

As before, we can select a global maximum point (x^μ, y^μ, t^μ) of $u^\mu - \psi$, where $(x^\mu, t^\mu) \in N$ for every $\mu \in (0,\mu_0)$, and a sequence $\{(\mu_j, p_j, q_j, s_j)\}_{j\in\mathbb{N}} \subset (0,\,\mu_0) \times \mathbb{R}^n \times \mathbb{R}^n \times [0,\,T)$ satisfying (3.10). Finally, may assume that $(p_j, s_j) \in N$ for $j \in \mathbb{N}$.

We claim that the sequence $\{t^{\mu_j}\}_{j\in\mathbb{N}}$ contains a subsequence, which we denote the same way as the sequence, such that $t^{\mu_j} > 0$.

Indeed arguing by contradiction, we suppose that, for $j \in \mathbb{N}$ large enough, $t^{\mu_j} = 0$. Fix such $j \in \mathbb{N}$ and observe that

$$(u^{\mu_j} - \psi)(x^{\mu_j}, y^{\mu_j}, 0) \geq (u^{\mu_j} - \psi)(p_j, q_j, 0),$$

and, in view of (3.13), (3.14) and (3.15),

$$\begin{aligned}(u^{\mu_j} - \psi)(x^{\mu_j}, y^{\mu_j}, 0) &< u_0^{\mu_j}(x^{\mu_j}, y^{\mu_j}, 0) - \phi(x^{\mu_j}, 0) + \delta < 2\delta + u_0(x^{\mu_j}) - \phi(x^{\mu_j}, 0)\\ &< -\delta + (u^+ - \phi)(x_0, 0).\end{aligned}$$

Hence, for such large j, we have

$$(u^{\mu_j} - \psi)(p_j, q_j, 0) < -\delta + (u^+ - \phi)(x_0, 0).$$

Letting $j \to \infty$ yields

$$(u^+ - \phi)(x_0, 0) \leq -\delta + (u^+ - \phi)(x_0, 0),$$

which is a contradiction, proving the claim.

We may now assume that $t^{\mu_j} > 0$, for all large j, and argue exactly as in the case $t_0 > 0$, to conclude that (3.6) holds.

This completes the proof of the subsolution property.

It is well-known that if u^+ (resp. u^-) is a subsolution (resp. supersolution) of (2.5) in the viscosity sense, as in the proof above, then $u^+(x,0) \leq u_0(x)$ (resp. $u^-(x,0) \geq u_0(x)$) for all $x \in \mathbb{R}^n$. □

4. Vanishing variable friction

The following two results are important for the proof of Theorem 2.3. The first asserts the existence of a uniques solution to adjoint problem. Its assertion (iii) is exactly Theorem 2.2. Its proof, which is rather long, is presented in Section 5.

Theorem 4.1. *Assume* (2.6), (2.7), (2.8), (2.9), (2.10) *and* (2.11). *Then:*
(i) *For any* $\varepsilon \in (0,\,1)$ *there exists a unique solution* $m^\varepsilon \in C^2(\overline{U})$ *of* $(\mathrm{Ad})_\varepsilon$.
(ii) *The family* $\{m^\varepsilon\}_{\varepsilon\in(0,\,1)}$ *converges, as* $\varepsilon \to 0$ *and uniformly on* $\overline{U}$, *to* $m \in C(\overline{U}) \cap C^2(\overline{U} \setminus \partial V)$.
(iii) *The function* m *is the unique solution to* (Ad1)–(Ad2).

Obviously, Theorem 2.2 is a direct consequence of Theorem 4.1 above.

The second preliminary result, which is proved at the end of this section, is about the behavior of the generalized upper-and lower limits u^+ and u^- of the family $\{u^\varepsilon\}_{\varepsilon\in(0,\,1)}$ in $\overline{U}$.

Lemma 4.2. *Suppose the assumptions of Theorem 2.3. Then*
(i) *the family* $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ *is uniformly bounded on* $\overline{U}$, *and*
(ii) u^+ *and* u^- *are respectively sub- and super-solution to* (2.15) *in* $\overline{V}$.

Accepting Theorem 4.1 and Lemma 4.2, we complete the proof of Theorem 2.3.

Before presenting the proof we recall Green's formula that we will use in several occasions below. For any $\phi, \psi \in C^2(\overline{U})$ and $\varepsilon > 0$, we have:

$$\begin{aligned}\int_U \left(a_{ij}\left(\frac{\phi_{x_i}}{\lambda+\varepsilon}\right)_{x_j} + 2b_i\phi_{x_i}\right)\psi dx \\ = \int_{\partial U}\left\{\frac{a_{ij}\phi_{x_i}\nu_j\psi}{\lambda+\varepsilon} + \left(2b_i\psi - \frac{(a_{ij}\psi)_{x_j}}{\lambda+\varepsilon}\right)\nu_i\phi\right\}d\sigma \\ + \int_U\left(\left(\frac{(a_{ij}\psi)_{x_j}}{\lambda+\varepsilon}\right)_{x_i} - 2(b_j\psi)_{x_j}\right)\phi dx.\end{aligned} \tag{4.1}$$

Proof of Theorem 2.3. For $\varepsilon\in(0,\,1)$, let $m^\varepsilon \in C^2(\overline{U})$ be the unique solution of $(\mathrm{Ad})_\varepsilon$. Applying (4.1) to $\phi = u^\varepsilon$ and $\psi = m^\varepsilon$, we get, for all $\varepsilon \in (0,1)$,

$$\int_{\partial U}\frac{a_{ij}u^\varepsilon_{x_i}\nu_j m^\varepsilon}{\lambda+\varepsilon}d\sigma = 0. \tag{4.2}$$

Theorem 4.1 yields a unique function $m \in C(\overline{U})$ such that, as $\varepsilon\to 0$,

$$m^\varepsilon \to m \quad \text{uniformly on } \overline{U} \text{ and } m>0 \text{ in } \overline{U}. \tag{4.3}$$

Let K be a compact subset of $\overline{U}\setminus\partial V$. Since $a_{ij}u^\varepsilon_{x_ix_j} + 2\varepsilon b_i u^\varepsilon_{x_i} = 0$ in V, the assumption on λ implies that, for some $c_K>0$, $\lambda \geq c_K > 0$ in $K\setminus V$, the classical Schauder estimates ([6, Lemma 6.5]) yield a constant $C_K>0$ such that

$$\|u^\varepsilon\|_{C^{2,\alpha}(K)} \leq C_K. \tag{4.4}$$

It follows that there exist a sequence $\varepsilon_k\to 0$ and $u_0 \in C^2(\overline{U}\setminus\partial V)$ such that, as $k\to\infty$,

$$u^{\varepsilon_k}\to u_0 \quad \text{in } C^2(\overline{U}\setminus\partial V). \tag{4.5}$$

Let u^+ and u^- be the relaxed upper and lower limits of $\{u^{\varepsilon_k}\}_{k\in\mathbb{N}}$. It follows from (4.5) that

$$u^+ = u^- = u_0 \;\text{ on }\; \overline{U}\setminus\partial V. \tag{4.6}$$

Moreover, Lemma 4.2 yields that u^+ and u^- are respectively a viscosity sub- and super-solution to (2.15). Since any constant function is a solution to (2.15), combining the strong maximum principle as well as Hopf's lemma we get (see also Patrizi [8]) that

$$u^+ = \max_{\overline{V}} u^+ \;\text{ and }\; u^- = \min_{\overline{V}} u^- \;\text{ on }\; \overline{V}.$$

Then (4.6) gives that

$$u^+ = u^- \quad \text{on } \overline{V},$$

which proves that $u^+ = u^-$ on $\overline{U}$. If we write u for $u^+=u^-$, then

$$u^{\varepsilon_k}\to u \quad \text{in } C(\overline{U}).$$

Moreover, as observed already,

$$u\in C^2(\overline{U}\setminus\partial V) \quad\text{and}\quad \lim_{k\to\infty} u^{\varepsilon_k} = u \quad \text{in } C^2(\overline{U}\setminus\partial V).$$

It is clear from (4.2) and (4.3) that u satisfies (2.16) as well as (2.13) and (2.14).

To complete the proof, it suffices to show there is only one $u \in C(\overline{U}) \cap C^2(\overline{U} \setminus \partial V)$ satisfying (2.13)–(2.16).

Assume that $v, w \in C(\overline{U}) \cap C^2(\overline{U} \setminus \partial V)$ satisfy (2.13)–(2.16). Then, as already discussed above for u, the strong maximum principle yields that v and w are constant on $\overline{V}$.

Set $\phi = v - w$ on $\overline{U}$ and note that, for some $c \in \mathbb{R}$,

$$\phi = 0 \quad \text{on} \quad \partial U \quad \text{and} \quad \phi = c \quad \text{on } \overline{V};$$

interchanging v and w if needed, we may assume that $c \geq 0$.

It follows from (2.13) and (2.16) that

$$-a_{ij}\left(\frac{\phi_{x_i}}{\lambda}\right)_{x_j} - 2b_i\phi_{x_i} = 0 \quad \text{in } U \setminus \overline{V} \quad \text{ and } \quad \int_{\partial U} \frac{a_{ij}\phi_{x_i}\nu_j m}{\lambda} d\sigma = 0.$$

If $c = 0$, then the maximum principle gives $\phi = 0$ on $\overline{U} \setminus V$ and $v = w$ on $\overline{U}$.

If $c > 0$, then the strong maximum principle implies that $\phi > 0$ in $U \setminus \overline{V}$, moreover, Hopf's lemma yields

$$a_{ij}\phi_{x_i}\nu_j < 0 \quad \text{on } \partial U,$$

and, hence,

$$\int_{\partial U} \frac{a_{ij}\phi_{x_i}\nu_j m}{\lambda} d\sigma < 0,$$

which is a contradiction. We thus conclude that $v = w$ on $\overline{U}$. ∎

Next we turn to the proof of Lemma 4.2. For this we need an additional lemma. In preparation, for $\delta > 0$, we write

$$W_\delta := (\partial V)_\delta = \{x \in \overline{U} : \operatorname{dist}(x, \partial V) < \delta\}.$$

Lemma 4.3. *There exists $\delta \in (0,\, 1)$ and, for each $\varepsilon \in [0,\, 1)$, $\psi^\varepsilon \in C^2(\overline{W}_\delta)$ such that*

$$a_{ij}\left(\frac{\psi^\varepsilon_{x_i}}{\lambda + \varepsilon}\right)_{x_j} + 2b_i\psi^\varepsilon_{x_i} \leq 0 \quad \text{in} \quad \overline{W}_\delta, \tag{4.7}$$

and, as $\varepsilon \to 0$, $\psi^\varepsilon \to \psi^0$ in $C^2(\overline{W}_\delta)$ with $\psi^0 \equiv 0$ in $\overline{V} \cap \overline{W}_\delta$ and $\psi^0 > 0$ in $\overline{W}_\delta \setminus \overline{V}$.

Proof. Let $\delta, K > 0$ be such that $K\delta \leq \frac{1}{2}$ and $\overline{W}_\delta \subset U$, define, for $(\varepsilon, x) \in [0,\, 1] \times \overline{W}_\delta$,

$$\psi^\varepsilon(x) = \int_0^{d(x)} (\lambda_0(t) + \varepsilon)(1 - Kt)dt \quad \text{ for } (\varepsilon, x) \in [0,\, 1] \times \overline{W}_\delta,$$

and note that, as $\varepsilon \to 0$,

$$\psi^\varepsilon(x) = \psi^0(x) + \varepsilon \int_0^{d(x)} (1 - Kt)dt \to \psi^0(x) \quad \text{in} \quad C^2(\overline{W}_\delta).$$

Let $x \in \overline{W}_\delta$ and note that for $t \in [-|d(x)|,\, |d(x)|]$,

$$1 - Kt \geq 1 - |Kt| \geq 1 - K\delta \geq \frac{1}{2}.$$

Since $d > 0$ in $\overline{W}_\delta \setminus \overline{V}$, we find

$$\psi^0(x) = \int_0^{d(x)} \lambda_0(t)(1 - Kt)dt \geq \frac{1}{2}\int_0^{d(x)} \lambda_0(t)dt > 0,$$

while, since $d \le 0$ in $\overline{W}_\delta \cap \overline{V}$, thanks to (2.11),

$$\psi^0(x) = \int_0^{d(x)} \lambda_0(t)(1-Kt)dt \equiv 0.$$

To show that (4.7) holds, fix $\varepsilon \in (0,\,1)$ and compute

$$\psi^\varepsilon_{x_i} = (\lambda(x)+\varepsilon)(1-Kd(x))d_{x_i},$$

and

$$\left(\frac{\psi^\varepsilon_{x_i}}{\lambda+\varepsilon}\right)_{x_j} = [(1-Kd)d_{x_i}]_{x_j} = -Kd_{x_i}d_{x_j} + (1-Kd)d_{x_ix_j}.$$

Note that there exists $C_1>0$, which is independent of the choice of K, such that

$$|(1-Kd)| \le 1+\frac{1}{2} \le 2 \;\text{ and } |a_{ij}(1-Kd)d_{x_ix_j}| \le 2|a_{ij}d_{x_ix_j}| \le C_1 \text{ in } \overline{W}_\delta,$$

and

$$|b_i\psi^\varepsilon_{x_i}| = |b_i(\lambda+\varepsilon)(1-Kd)d_{x_i}| \le 2|b|(\lambda+\varepsilon) \le C_1 \text{ in } \overline{W}_\delta.$$

Thus

$$a_{ij}\left(\frac{\psi^\varepsilon_{x_i}}{\lambda+\varepsilon}\right)_{x_j} + 2b_i\psi^\varepsilon_{x_i} \le -K\theta|Dd|^2 + 3C_1 = 3C_1 - K\theta \quad \text{for } x \in \overline{W}_\delta.$$

We fix $K>0$ so that $3C_1 - K\theta \le 0$ and conclude that ψ^ε satisfies (4.7). □

Proof of Lemma 4.2. To prove (i) we apply the maximum principle to $u^\varepsilon - \|g\|_{C(\partial U)}$ and $-\|g\|_{C(\partial U)} - u^\varepsilon$ and get

$$\sup_{\varepsilon\in(0,1)} \|u^\varepsilon\|_{C(\overline{U})} \le \|g\|_{C(\partial U)}.$$

Next we show that u^+ is a viscosity subsolution of (2.15). Since

$$a_{ij}u^\varepsilon_{x_ix_j} + \varepsilon b_i u^\varepsilon_{x_i} = 0 \;\text{ in }\; V,$$

it is well-known that u^+ is a viscosity subsolution to $-a_{ij}w_{x_ix_j} = 0$ in V. Thus the only issue is to show that u^+ satisfies the boundary condition in the viscosity sense.
Let $\phi \in C^2(\overline{U})$ and assume that $x_0 \in \partial V$ is a strict maximum point of $u^+ - \phi$ on $\overline{V}$. Arguing by contradiction, we suppose that

$$-a_{ij}\phi_{x_ix_j}(x_0) > 0 \quad \text{and} \quad a_{ij}\phi_{x_i}\nu_j(x_0) > 0.$$

Let $\delta \in (0,\,1)$, W_δ and $\{\psi^\varepsilon\}_{\varepsilon\in[0,\,1)}$ be as in Lemma 4.3, select $\rho \in (0,\,\delta)$ so that

$$a_{ij}\phi_{x_ix_j} < 0 \quad \text{and} \quad a_{ij}\phi_{x_i}d_{x_j} > 0 \;\text{ in }\; B := B_\rho(x_0) \subset W_\delta.$$

Since

$$a_{ij}\left(\frac{\phi_{x_i}}{\lambda+\varepsilon}\right)_{x_j} + 2b_i\phi_{x_i} = \frac{1}{\lambda+\varepsilon}\left(a_{ij}\phi_{x_ix_j} - \frac{\lambda_0'(d)a_{ij}\phi_{x_i}d_{x_j}}{\lambda+\varepsilon} + 2(\lambda+\varepsilon)b_i\phi_{x_i}\right),$$

we may choose $\varepsilon_0 \in (0,\,1)$ and reselect $\rho > 0$ sufficiently small so that, for $\varepsilon \in (0,\,\varepsilon_0)$,

$$a_{ij}\left(\frac{\phi_{x_i}}{\lambda+\varepsilon}\right)_{x_j} + 2b_i\phi_{x_i} < 0 \;\text{ in }\; \overline{B}. \tag{4.8}$$

For each $\varepsilon \in (0,\,\varepsilon_0)$, we select $x_\varepsilon \in \overline{B}$ so that

$$(u^\varepsilon - (\phi+\psi^\varepsilon))(x_\varepsilon) = \max_{\overline{B}}(u^\varepsilon - (\phi+\psi^\varepsilon)). \tag{4.9}$$

In view of the definition of u^+, we may choose $y_k \in B$ and $\varepsilon_k \in (0, \varepsilon_0)$ such that, as $k \to \infty$,

$$y_k \to x_0, \ \ \varepsilon_k \to 0 \text{ and } u^{\varepsilon_k}(y_k) \to u^+(x_0).$$

We may also assume that there is $\bar{x} \in \overline{B}$ such that $\lim_{k\to\infty} x_{\varepsilon_k} = \bar{x}$.
It follows from (4.9) that

$$(u^{\varepsilon_k} - (\phi + \psi^{\varepsilon_k}))(x_{\varepsilon_k}) \geq (u^{\varepsilon_k} - (\phi + \psi^{\varepsilon_k}))(y_k),$$

and thus

$$(u^+ - (\phi + \psi^0))(\bar{x}) \geq (u^+ - (\phi + \psi^0))(x_0) = (u^+ - \phi)(x_0), \tag{4.10}$$

which implies $(u^+ - \phi)(x_0) \leq (u^+ - \phi)(\bar{x})$, since $\psi^0 \geq 0$ in $\overline{B}$, and that $\bar{x} = x_0$ because x_0 is a unique maximum point of $u^+ - \phi$.
Selecting $k \in \mathbb{N}$ large enough so that $x_{\varepsilon_k} \in B$, we deduce using (4.7), (4.8) and the maximum principle that, at x_{ε_k},

$$\begin{aligned} 0 &\leq a_{ij}\left(\frac{u^{\varepsilon_k}_{x_i}}{\lambda + \varepsilon_k}\right)_{x_j} + 2b_i u^{\varepsilon_k}_{x_i} \\ &\leq a_{ij}\left(\frac{(\phi + \psi^{\varepsilon_k})_{x_i}}{\lambda + \varepsilon_k}\right)_{x_j} + 2b_i(\phi + \psi^{\varepsilon_k})_{x_i} \\ &\leq a_{ij}\left(\frac{\phi_{x_i}}{\lambda + \varepsilon_k}\right)_{x_j} + 2b_i \phi_{x_i} < 0. \end{aligned}$$

This is a contradiction and, thus, u^+ is a viscosity subsolution of (4.7).
The argument for the supersolution property is similar. □

5. The proof of Theorem 4.1

We remark that the existence of $m^\varepsilon \in C^2(\overline{U})$ that satisfies $(\mathrm{Ad})_\varepsilon$ follows from the following Fredholm alternative type of argument.
The adjoint problem to $(\mathrm{Ad})_\varepsilon$ is the Neumann boundary value problem

$$\begin{cases} a_{ij}\left(\dfrac{v_{x_i}}{\lambda + \varepsilon}\right)_{x_j} + 2b_i v_{x_i} = 0 \ \text{ in } \ U, \\ a_{ij} v_{x_i} \nu_j = 0 \ \text{ on } \ \partial U. \end{cases} \tag{5.1}$$

Since any constant function is a solution to (5.1), the eigenvalue problem

$$\begin{cases} a_{ij}\left(\dfrac{v_{x_i}}{\lambda + \varepsilon}\right)_{x_j} + 2b_i v_{x_i} + \rho v = 0 \ \text{ in } \ U, \\ a_{ij} v_{x_i} \nu_j = 0 \ \text{ on } \ \partial U, \end{cases} \tag{5.2}$$

has $\rho = 0$ as its principal eigenvalue. Consequently, in principle, the problem

$$\begin{cases} \left(\dfrac{(a_{ij}v)_{x_i}}{\lambda + \varepsilon} - 2b_j v\right)_{x_j} + \rho v = 0 \ \text{ in } \ U, \\ \left(\dfrac{(a_{ij}v)_{x_i}}{\lambda + \varepsilon} - 2b_j v\right)\nu_j = 0 \ \text{ on } \ \partial U, \end{cases} \tag{5.3}$$

should have $\rho = 0$ as its principal eigenvalue and there should be a positive function $m^\varepsilon \in C^2(\overline{U})$ that satisfies $(\mathrm{Ad})_\varepsilon$.

We organize the important parts of the proof of Theorem 4.1 in a series of lemmata.

Lemma 5.1. *There exists a unique solution $m^\varepsilon \in C^2(\overline{U})$ of* $(\mathrm{Ad})_\varepsilon$*. Moreover, If $\mu \in C^2(\overline{U})$ satisfies the first two equations of* $(\mathrm{Ad})_\varepsilon$*, then there exists $c \in \mathbb{R}$ such that $\mu = cm^\varepsilon$ on $\overline{U}$.*

We postpone the proof of the lemma above until the end of the proof of Theorem 4.1 and we continue with several other technical steps.

Lemma 5.2. *There exists a positive solution $\psi_0 \in C^2(\overline{V})$ to* (Ad2)*. Furthermore, if $\phi \in C^2(\overline{V})$ is a solution of* (Ad2)*, then $\phi = c\psi_0$ on $\overline{V}$ for some $c \in \mathbb{R}$.*

Proof. The claim is a consequence of Lemma 5.1, with U, $\lambda + \varepsilon$ and b_i replaced by V, 1 and 0, respectively. □

Lemma 5.3. *There exists at most one $m \in C(\overline{U}) \cap C^2(\overline{U} \setminus \partial V)$ that satisfies* (Ad1) *and* (Ad2)*.*

We prepare the next result which is needed for the proof of the lemma above. For $\gamma > 0$ we set

$$V_\gamma = \{x \in \mathbb{R}^n \,:\, \mathrm{dist}(x, V) < \gamma\},$$

which for sufficiently small γ is a $C^{2,\alpha}$-domain and $\overline{V}_\gamma \subset U$, and consider the Dirichlet problem

$$\begin{cases} a_{ij}\left(\dfrac{v_{x_i}}{\lambda}\right)_{x_j} + 2b_j v_{x_j} = 0 & \text{in } U \setminus \overline{V}_\gamma, \\ v = 0 \quad \text{on } \partial V_\gamma \quad \text{and} \quad v = 1 & \text{on } \partial U. \end{cases} \tag{5.4}$$

The classical Schauder theory (see [6, Theorem 6.14]) and the hypotheses of Theorem 4.1 yield that, for $\gamma > 0$ sufficiently small, (5.4) has a unique solution $v^\gamma \in C^{2,\alpha}(\overline{U} \setminus V_\gamma)$.

Lemma 5.4. *There exist constants $\gamma_0 \in (0,\, 1)$ and $C > 0$ such that, if $\gamma \in (0,\, \gamma_0)$, then the Dirichlet problem* (5.4) *has a unique solution $v^\gamma \in C^{2,\alpha}(\overline{U} \setminus V_\gamma)$ and it satisfies*

$$\begin{cases} |Dv^\gamma(x)| \le C\lambda(x) & \textit{for all } x \in \partial V_\gamma, \\ v^\gamma(x) \le C\Lambda_0(d(x)) & \textit{for all } x \in \overline{U} \setminus V_\gamma, \end{cases}$$

where Λ_0 denotes the primitive of λ_0 given by $\Lambda_0(r) := \int_0^r \lambda_0(t)dt$.

Proof. Let $\delta \in (0,\, 1)$ and $\psi^0 \in C^2(\overline{W}_\delta)$ be from Lemma 4.3. We may assume by replacing $\delta > 0$ by a smaller number that, if $0 < \gamma < \delta$, then $U \setminus \overline{V}_\gamma$ is a $C^{2,\alpha}$-domain. The Schauder theory guarantees that, if $\gamma \in (0,\, \delta)$, there is a unique solution $v^\gamma \in C^{2,\alpha}(\overline{U} \setminus V_\gamma)$ of (5.4). According to the proof of Lemma 4.3, the function ψ^0 has the form

$$\psi^0(x) = \Lambda(d(x)) \quad \text{in } \overline{V}_{\delta_0} \setminus V,$$

where $\Lambda \in C^3([0,\, \delta])$ satisfies the conditions that $\Lambda(0) = 0$, $\Lambda(r) > 0$ for $r \in (0, \delta]$, and Λ is nondecreasing on $[0,\, \delta]$. A careful review of the proof assures that $\Lambda'(r) \le 2\lambda_0(r)$ for $r \in [0,\, \delta]$ and, hence, $\Lambda(r) \le 2\Lambda_0(r)$ for $r \in [0,\, \delta]$. Also, the function ψ^0 is a supersolution of

$$a_{ij}\left(\frac{v_{x_i}}{\lambda}\right)_{x_j} + 2b_j v_{x_j} = 0 \quad \text{in } V_\delta \setminus \overline{V}.$$

Fix constants $\gamma_0 \in (0,\, \delta)$ and $M > 0$ so that $\Lambda(\gamma_0) < \Lambda(\delta)$ and $M(\Lambda(\delta) - \Lambda(\gamma_0)) \ge 1$.

Let $\gamma \in (0, \gamma_0)$, and consider the function

$$w(x) := M(\psi^0(x) - \Lambda(\gamma)) = M(\Lambda(d(x)) - \Lambda(\gamma)) \quad \text{on } \overline{V}_\delta \setminus V_\gamma.$$

Note that $w = 0$ on ∂V_γ and $w \geq 1$ on ∂V_δ. It is clear that w is a supersolution of

$$a_{ij}\left(\frac{w_{x_i}}{\lambda}\right)_{x_j} + 2b_j w_{x_j} = 0 \quad \text{in } V_\delta \setminus \overline{V}_\gamma.$$

Since the constant functions 0 and 1 are a sub- and super-solution of (5.4) including the boundary conditions, we see by the maximum principle that $0 \leq v^\gamma \leq 1$ on $\overline{U} \setminus V_\gamma$. Using again the maximum principle in the domain $V_\delta \setminus \overline{V}_\gamma$, we find that $v^\gamma \leq w$ on $\overline{V}_\delta \setminus V_\gamma$. Thus, we have $0 \leq v^\gamma \leq w$ on $\overline{V}_\delta \setminus V_\gamma$, which yields

$$\begin{cases} |Dv^\gamma| \leq M\Lambda'(\gamma) \leq 2M\lambda_0(\gamma) \text{ on } \partial V_\gamma, \\ v^\gamma(x) \leq M\Lambda(d(x)) \leq 2M\Lambda_0(d(x)) \quad \text{for } x \in V_\delta \setminus V_\gamma. \end{cases}$$

The last inequality is valid even for $x \in \overline{U} \setminus V_\delta$, since $2M\Lambda_0(r) \geq 2M\Lambda_0(\delta) \geq 1$ for $r \geq \delta$. Thus, the lemma is valid with $C = 2M$. □

Proof of Lemma 5.3. Let $m_1, m_2 \in C(\overline{U}) \cap C^2(\overline{U} \setminus \partial V)$ satisfy (Ad1)–(Ad2) and, as in Lemma 5.2, $\psi_0 \in C^2(\overline{V})$ a solution to (Ad2) which is positive on $\overline{V}$. Since $m_1, m_2 > 0$ on $\overline{U}$, we may choose a constant $c > 0$ so that

$$\min_{\overline{U}}(cm_1 - m_2) = 0,$$

and set $w = cm_1 - m_2$ on $\overline{U}$.
Lemma 5.2 yields $\alpha_1, \alpha_2 > 0$ such that

$$m_1 = \alpha_1\psi_0 \quad \text{and} \quad m_2 = \alpha_2\psi_0 \quad \text{on } \overline{V}.$$

Thus, $w = (c\alpha_1 - \alpha_2)\psi_0$ on $\overline{V}$, which implies that either $w \equiv 0$ on $\overline{V}$ or $w > 0$ on $\overline{V}$.
We show that $w \equiv 0$ on $\overline{U}$. Consider first the case when w has a minimum point at some point in $U \setminus \overline{V}$ and observe that, by the strong maximum principle, $w \equiv 0$ in $\overline{U} \setminus V$, which implies $w \equiv 0$ on $\overline{V}$ as well. Hence, $w \equiv 0$ on $\overline{U}$.
Next, we assume that $w > 0$ in $U \setminus \overline{V}$ and w attains a minimum value 0 at a point $x_0 \in \partial U$. Hopf's lemma then gives that, at x_0,

$$\left(\frac{(a_{ij}w)_{x_i}}{\lambda} - 2b_j w\right)\nu_j = \frac{a_{ij}w_{x_i}\nu_j}{\lambda} < 0,$$

which contradicts the second equality of (Ad1).
What remains is the possibility where $w > 0$ on $\overline{U} \setminus \overline{V}$ and $w \equiv 0$ on $\overline{V}$.
Now, let $\gamma_0 \in (0, 1)$ and $C > 0$ be the constants from Lemma 5.4. According to the lemma, (5.4) has a solution $v^\gamma \in C^{2,\alpha}(\overline{U} \setminus V_\gamma)$, $|Dv^\gamma| \leq C\lambda$ on ∂V_γ, and

$$0 \leq v^\gamma(x) \leq C\Lambda_0(d(x)) \quad \text{for} \quad x \in \overline{U} \setminus V_\gamma, \tag{5.5}$$

where the nonnegativity of v^γ is a consequence of the maximum principle and Λ_0 is the primitive of λ_0 chosen as in Lemma 5.4.
By the Schauder estimates, for any compact $K \subset \overline{U} \setminus \overline{V}$, there exists $C_K > 0$ such that, if $\gamma > 0$ is sufficiently small, then $\|v^\gamma\|_{C^{2,\alpha}(K)} \leq C_K$. Thus, we may choose a sequence

$\{\gamma_k\}_{k\in\mathbb{N}} \subset (0, \gamma_0)$ converging to zero and a function $v^0 \in C^2(\overline{U} \setminus \overline{V})$ such that, for any compact $K \subset \overline{U} \setminus \overline{V}$, as $k \to \infty$,

$$v^{\gamma_k} \to v^0 \quad \text{in } C^2(K).$$

Moreover, in view of (5.5), we may assume that $v^0 \in C(\overline{U} \setminus V)$, $v^0 = 0$ on ∂V, and $v^0 = 1$ on ∂U

Applying Green's formula (4.1), with $(\phi, \psi, \lambda + \varepsilon, U)$ replaced by $(v^\gamma, w, \lambda, U \setminus \overline{V}_\gamma)$, we get

$$0 = \int_{\partial U} \frac{a_{ij} v^\gamma_{x_i} \nu_j w}{\lambda} d\sigma - \int_{\partial V_\gamma} \frac{a_{ij} v^\gamma_{x_i} \nu_j w}{\lambda} d\sigma, \tag{5.6}$$

where the unit normal vector ν on ∂V_γ is taken as being exterior normal to V_γ.

Note that, in view of by Lemma 5.4 and, for some independent of γ, $C_1 > 0$,

$$\Big| \int_{\partial V_\gamma} \frac{a_{ij} v^\gamma_{x_i} \nu_j m}{\lambda} d\sigma \Big| \le C_1 \|m\|_{C(\partial V_\gamma)} \|\lambda^{-1} D v^\gamma\|_{C(\partial V_\gamma)} \le C C_1 \|m\|_{C(\partial V_\gamma)}.$$

Setting $\gamma = \gamma_k$ and sending $k \to \infty$, we obtain from (5.6)

$$\int_{\partial U} \frac{a_{ij} v^0_{x_i} \nu_j m}{\lambda} d\sigma = 0. \tag{5.7}$$

It is obvious that $v^0 \in C(\overline{U} \setminus V) \cap C^2(\overline{U} \setminus \overline{V})$ solves (5.4), with V_γ replaced by V, and, for all $x \in \partial U$, $v_0(x) = 1 = \max_{\overline{U}\setminus V} v^0$. By the strong maximum principle and Hopf's lemma, we deduce that

$$a_{ij} v^0_{x_i} \nu_j > 0 \quad \text{on } \partial U.$$

In our current situation, we have $m > 0$ on ∂U, which together with the above inequalities gives a contradiction to (5.7), and thus we conclude that $w \equiv 0$ on $\overline{U}$.

The third identity of (Ad1) yields

$$0 = \int_U w dx = c \int_U m_1 dx - \int_U m_2 dx = c - 1,$$

from which we get $c = 1$, and, thus, $m_1 - m_2 = w = 0$ on $\overline{U}$. □

Lemma 5.5. *For each $\varepsilon \in (0, 1)$, let m^ε be the unique solution to* (Ad)$_\varepsilon$*. Assume that the family $\{m^{\varepsilon_j}\}_{j\in\mathbb{N}}$ is uniformly bounded on $\overline{U}$, and let $m^\pm$ on $\overline{U}$ be the relaxed upper and lower limit of the m^ε's. Then m^+ and m^- are respectively a viscosity sub- and super-solution to* (Ad2) *as functions on $\overline{V}$.*

The proof of Lemma 5.5 is very similar to the one of Lemma 4.2, with the role of Lemma 4.3 replaced by Lemma 5.6 below, hence we omit it.

Lemma 5.6. *There exist $\delta \in (0, \delta_0)$ and, for each $\varepsilon \in [0, 1)$, $\psi^\varepsilon \in C^2(\overline{W}_\delta)$ such that*

$$\left(\frac{(a_{ij}\psi^\varepsilon)_{x_i}}{\lambda + \varepsilon}\right)_{x_j} - 2(b_i \psi^\varepsilon)_{x_i} \le 0 \quad \text{in } \overline{W}_\delta \;\; \text{if} \;\; \varepsilon > 0, \tag{5.8}$$

and, as $\varepsilon \to 0$, $\psi^\varepsilon \to \psi^0$ in $C^2(\overline{W}_\delta)$ with $\psi^0 \equiv 0$ in $\overline{V} \cap \overline{W}_\delta$ and $\psi^0 > 0$ in $\overline{W}_\delta \setminus \overline{V}$.

The proof of the lemma above is similar to, but slightly more involved than that of Lemma 4.3, which needed the full strength of (2.11).

Proof. Let $\delta \in (0,\, \delta_0)$ and K be positive constants such that $K\delta \le \frac{1}{2}$ and $\overline{W}_\delta \subset U$. As in the proof of Lemma 4.3, we define, for $(\varepsilon, x) \in [0,\, 1] \times \overline{W}_\delta$, .

$$\psi^\varepsilon(x) = \int_0^{d(x)} (\lambda_0(t) + \varepsilon)(1 - Kt)dt.$$

and note that the functions ψ^ε have all the claimed properties except (5.8).
To show (5.8), we fix $\varepsilon \in (0,\, 1)$ and observe first that

$$\begin{aligned}\left(\frac{(a_{ij}\psi^\varepsilon)_{x_i}}{\lambda+\varepsilon}\right)_{x_j} &= \left(\frac{a_{ij}\psi^\varepsilon_{x_i} + a_{ij,x_i}\psi^\varepsilon}{\lambda+\varepsilon}\right)_{x_j} \\ &= \frac{a_{ij}\psi^\varepsilon_{x_ix_j} + 2a_{ij,x_i}\psi^\varepsilon_{x_j} + a_{ij,x_ix_j}\psi^\varepsilon}{\lambda+\varepsilon} - \frac{a_{ij}\psi^\varepsilon_{x_i}\lambda_{x_j} + a_{ij,x_i}\psi^\varepsilon\lambda_{x_j}}{(\lambda+\varepsilon)^2} \\ &= a_{ij}\left(\frac{\psi^\varepsilon_{x_i}}{\lambda+\varepsilon}\right)_{x_j} + \frac{2a_{ij,x_i}\psi^\varepsilon_{x_j} + a_{ij,x_ix_j}\psi^\varepsilon}{\lambda+\varepsilon} - \frac{a_{ij,x_i}\psi^\varepsilon\lambda_{x_j}}{(\lambda+\varepsilon)^2}.\end{aligned}$$

As seen in the proof of Lemma 4.3, we have

$$|D\psi^\varepsilon| = (\lambda+\varepsilon)(1 - Kd(x))|Dd| \le 2(\lambda+\varepsilon).$$

and, for some $C_1 > 0$,

$$a_{ij}\left(\frac{\psi^\varepsilon_{x_i}}{\lambda+\varepsilon}\right)_{x_j} \le C_1 - K\theta \quad \text{on} \quad \overline{W}_\delta.$$

Moreover, if C_0 is the constant from (2.11),

$$|\psi^\varepsilon| \le 2\left|\int_0^{d(x)} (\lambda_0(t)+\varepsilon)dt\right| \le 2(\lambda+\varepsilon)|d(x)| \le 2\delta(\lambda+\varepsilon),$$

and

$$|\lambda_{x_j}\psi^\varepsilon| \le \lambda_0'(d(x))|d_{x_j}||\psi^\varepsilon| \le 2\lambda_0'(d(x))|d(x)|(\lambda+\varepsilon) \le 2C_0(\lambda(x)+\varepsilon)^2.$$

Hence, we can choose a constant $C_2 > 0$, which is independent of K and ε, such that

$$\left|\frac{2a_{ij,x_i}\psi^\varepsilon_{x_j} + a_{ij,x_ix_j}\psi^\varepsilon}{\lambda+\varepsilon} - \frac{a_{ij,x_i}\psi^\varepsilon\lambda_{x_j}}{(\lambda+\varepsilon)^2}\right| \le C_2.$$

and

$$|(b_i\psi^\varepsilon)_{x_i}| \le |b_i||\psi^\varepsilon_{x_i}| + |b_{i,x_i}\psi^\varepsilon| \le C_2.$$

It follows that

$$\left(\frac{(a_{ij}\psi^\varepsilon)_{x_i}}{\lambda+\varepsilon}\right)_{x_j} - 2(b_i\psi^\varepsilon)_{x_i} \le C_1 + 2C_2 - \theta K \quad \text{on } \overline{W}_\delta.$$

Choosing $K \ge (C_1 + 2C_2)/\theta$, we obtain (5.8). □

Proof of Theorem 4.1. Assertion (i) is an immediate consequence of Lemma 5.1. According to [7, Lemma 3.1], there exists $C_1 > 0$, which is independent of $\varepsilon \in (0,\, 1)$, such that

$$\sup_{\varepsilon\in(0,\,1)} \|m^\varepsilon\|_{C(\overline{U})} \le C_1. \tag{5.9}$$

The interior Schauder estimates ([6, Corollary 6.3]) also imply that, for each compact $K \subset U \setminus \partial V$, there exists $C_K > 0$, again independent of ε, such that

$$\sup_{\varepsilon\in(0,\,1)} \|m^\varepsilon\|_{C^{2,\alpha}(K)} \le C_K. \tag{5.10}$$

We choose a smooth domain W such that $\overline{V} \subset W$ and $\overline{W} \subset U$ and set

$$h^\varepsilon = \left(\frac{(a_{ij} m^\varepsilon)_{x_i}}{\lambda + \varepsilon} - 2 b_j m^\varepsilon \right) \nu_j \ \text{ on } \ \partial W,$$

where ν denotes the inward unit normal vector of W.

Observe that $w = m^\varepsilon$ satisfies

$$\begin{cases} \left(\frac{(a_{ij} w)_{x_i}}{\lambda + \varepsilon} - 2 b_j w \right)_{x_j} = 0 \ \text{ in } \ U \setminus \overline{W}, \\ \left(\frac{(a_{ij} w)_{x_i}}{\lambda + \varepsilon} - 2 b_j w \right) \nu_j = 0 \ \text{ on } \ \partial U, \\ \left(\frac{(a_{ij} w)_{x_i}}{\lambda + \varepsilon} - 2 b_j w \right) \nu_j = h^\varepsilon \ \text{ on } \ \partial W. \end{cases}$$

We use the global Schauder estimates ([6, Theorem 6.30]) to find $C_W > 0$, independent of ε, such that

$$\sup_{\varepsilon \in (0,1)} \|m^\varepsilon\|_{C^{2,\alpha}(\overline{U} \setminus W)} \leq C_W. \tag{5.11}$$

Combining (5.10) and (5.11) shows that, for each compact $K \subset \overline{U} \setminus \partial V$, there exists $C_K > 0$, independent of ε, such that

$$\sup_{\varepsilon \in (0,1)} \|m^\varepsilon\|_{C^{2,\alpha}(K)} \leq C_K. \tag{5.12}$$

We may then select a sequence $\varepsilon_j \to 0$ such that, as $j \to \infty$ and for some $m \in C(\overline{U}) \cap C^2(\overline{U} \setminus \partial V)$,

$$m^{\varepsilon_j} \to m \ \text{ in } \ C^2(\overline{U} \setminus \partial V). \tag{5.13}$$

Let m^+ and m^- be the relaxed upper and lower limits of the m^{ε_j}'s, which exist in view of (5.9), and observe that $m = m^+ = m^-$, as function on $\overline{U} \setminus \overline{V}$, is a solution to

$$\begin{cases} \left(\frac{(a_{ij} m)_{x_i}}{\lambda} - 2 b_j m \right)_{x_j} = 0 \ \text{ in } \ U \setminus \overline{V}, \\ \left(\frac{(a_{ij} m)_{x_i}}{\lambda} - 2 b_j m \right) \nu_j = 0 \ \text{ on } \ \partial U, \end{cases} \tag{5.14}$$

while, in view of Lemma 5.5, m^+ and m^-, as functions on $\overline{V}$, are respectively a sub- and super-solution to

$$\begin{cases} (a_{ij} m)_{x_i x_j} = 0 \ \text{ in } \ V, \\ (a_{ij} m)_{x_i} \nu_j = 0 \ \text{ on } \ \partial V. \end{cases} \tag{5.15}$$

Let $\psi_0 \in C^2(\overline{V})$ be the positive solution to (Ad2) given by Lemma 5.2. Since $m^+ \geq m^- \geq 0$, there are exist constants $c^\pm \geq 0$ such that

$$\max_{\overline{V}} (m^+ - c^+ \psi_0) = 0 \quad \text{and} \quad \min_{\overline{V}} (m^- - c^- \psi_0) = 0.$$

Using the strong maximum principle, we find

$$m^+ = c^+ \psi_0 \quad \text{and} \quad m^- = c^- \psi_0 \ \text{ on } \ \overline{V}.$$

Since $m^+ = m^-$ in $\overline{U} \setminus \partial V$, we must have $c^+ = c^-$ and, accordingly, $m^+ = m^- = c^+\psi_0$ on $\overline{V}$ for some $c^+ \geq 0$. Thus,

$$m^+ = m^- \quad \text{on} \quad \overline{U},$$

and, therefore, if we set $m = m^+ = m^-$ on ∂V, then

$$\lim_{j\to\infty} m^{\varepsilon_j} = m \quad \text{in} \quad C(\overline{U}), \tag{5.16}$$

which completes the proof of assertion (ii).

Now, in view of Lemma 5.3, it only remains to show that m is positive on $\overline{U}$. Note that $m \geq 0$ and $m \not\equiv 0$ on $\overline{U}$. Since m satisfies (5.14) and (5.15), we infer using again the strong maximum principle together with the Hopf's lemma that, if m vanishes at a point in $\overline{V}$, then $m = 0$ on $\overline{V}$, and that, if m vanishes at a point in $\overline{U}\setminus\overline{V}$, then $m = 0$ on $\overline{U}\setminus\overline{V}$. In particular, if m vanishes at a point in $\overline{U} \setminus \overline{V}$, then $m = 0$ on $\overline{U}$, which is impossible. That is, we must have $m > 0$ in $\overline{U} \setminus \overline{V}$. □

We conclude with the last remaining proof.

Proof of Lemma 5.1. Given $f \in C(\overline{U})$, consider the problems

$$\begin{cases} -a_{ij}\left(\dfrac{v_{x_i}}{\lambda+\varepsilon}\right)_{x_j} - 2b_i v_{x_i} = \rho v + f & \text{in} \quad U, \\ a_{ij} v_{x_i}\nu_j = 0 \quad \text{on} \quad \partial U, \end{cases} \tag{5.17}$$

and

$$\begin{cases} -\left(\dfrac{(a_{ij}v)_{x_i}}{\lambda+\varepsilon} - 2b_j v\right)_{x_j} = \rho v + f & \text{in} \quad U, \\ \left(\dfrac{(a_{ij}v)_{x_i}}{\lambda+\varepsilon} - 2b_j v\right)\nu_j = 0 \quad \text{on} \quad \partial U. \end{cases} \tag{5.18}$$

The Schauder theory ([6, Theorem 6.31]) guarantees that, if $f \in C^{0,\alpha}(\overline{U})$ and $\rho < 0$, then (5.17) has a unique classical solution $v \in C^{2,\alpha}(\overline{U})$ and any constant function is a solution of (5.17), for $\rho = 0$ and $f = 0$. It follows that $\rho = 0$ is the principal eigenvalue for the eigenvalue problem corresponding to (5.17).

For $r > 0$, let S_r be the solution operator to (5.17), that is, for $f \in C^{0,\alpha}(\overline{U})$, $v = S_r f \in C^2(\overline{U})$ is the unique solution to (5.17) with $\rho = -r$.

The maximum principle gives

$$\|S_r f\|_{C(\overline{U})} \leq r^{-1}\|f\|_{C(\overline{U})},$$

which extends the domain of the S_r to $C(\overline{U})$. Obviously, for $f \in C(\overline{U})$, $S_r f$ is the unique viscosity solution to (5.17).

The classical existence and uniqueness theory for elliptic equations does not immediately apply to (5.18). In order to have a good monotonicity with respect to the boundary conditions, we need to make a change of unknowns.

Let $\phi, v \in C^2(\overline{U})$ and set $w(x) = e^{-\phi(x)}v(x)$. Straightforward computations yield

$$\frac{(a_{ij}v)_{x_i}}{\lambda+\varepsilon} - 2b_j v = \frac{(a_{ij}e^{\phi}w)_{x_i}}{\lambda+\varepsilon} - 2b_j e^{\phi} w = e^{\phi}\left(\frac{a_{ij}w_{x_i}}{\lambda+\varepsilon} + \left(\frac{a_{ij,x_i} + a_{ij}\phi_{x_i}}{\lambda+\varepsilon} - 2b_j\right) w\right),$$

and

$$\left(\frac{(a_{ij}v)_{x_i}}{\lambda+\varepsilon}-2b_jv\right)_{x_j}=\left\{e^{\phi}\left(\frac{a_{ij}w_{x_i}}{\lambda+\varepsilon}+\left(\frac{a_{ij,x_i}+a_{ij}\phi_{x_i}}{\lambda+\varepsilon}-2b_j\right)w\right)\right\}_{x_j}$$
$$=e^{\phi}\Bigg\{\frac{a_{ij}u_{x_ix_j}}{\lambda+\varepsilon}+\left(\frac{2a_{ij}\phi_{x_j}+2a_{ij,x_j}}{\lambda+\varepsilon}-\frac{a_{ij}\lambda_{x_j}}{(\lambda+\varepsilon)^2}-2b_i\right)w_{x_i}$$
$$+\left(\phi_{x_j}\left(\frac{a_{ij,x_i}+a_{ij}\phi_{x_i}}{\lambda+\varepsilon}-2b_j\right)+\left(\frac{a_{ij,x_i}+a_{ij}\phi_{x_i}}{\lambda+\varepsilon}-2b_j\right)_{x_j}\right)w\Bigg\}.$$

Choosing $\phi=M\,\mathrm{dist}(\cdot,\partial U)$ near ∂U, with $M>0$ sufficiently large, so that

$$D\phi=M\nu \quad \text{on} \quad \partial U,$$

we may assume that

$$\left(\frac{a_{ij,x_i}+a_{ij}\phi_{x_i}}{\lambda+\varepsilon}-2b_j\right)\nu_j\geq 0 \quad \text{on} \quad \partial U.$$

Let $R>0$ be sufficiently large so that

$$R\geq 1+\phi_{x_j}\left(\frac{a_{ij,x_i}+a_{ij}\phi_{x_i}}{\lambda+\varepsilon}-2b_j\right)+\left(\frac{a_{ij,x_i}+a_{ij}\phi_{x_i}}{\lambda+\varepsilon}-2b_j\right)_{x_j} \quad \text{on} \quad \overline{U}.$$

If v is a solution to (5.18) with $\rho=-R$, then w satisfies

$$\begin{cases} -\dfrac{a_{ij}w_{x_ix_j}}{\lambda+\varepsilon}-\tilde{b}_iw_{x_i}+(R-\tilde{c})w=e^{-\phi}f \quad \text{in} \quad U,\\ \dfrac{a_{ij}w_{x_i}\nu_j}{\lambda+\varepsilon}+\tilde{d}w=0 \quad \text{on} \quad \partial U,\end{cases} \tag{5.19}$$

where

$$\tilde{b}_i(x)=\left(\frac{2a_{ij}\phi_{x_j}+2a_{ij,x_j}}{\lambda+\varepsilon}-\frac{a_{ij}\lambda_{x_j}}{(\lambda+\varepsilon)^2}-2b_i\right),$$
$$\tilde{c}(x)=\phi_{x_j}\left(\frac{a_{ij,x_i}+a_{ij}\phi_{x_i}}{\lambda+\varepsilon}-2b_j\right)+\left(\frac{a_{ij,x_i}+a_{ij}\phi_{x_i}}{\lambda+\varepsilon}-2b_j\right)_{x_j},$$
$$\tilde{d}(x)=\left(\frac{a_{ij,x_i}+a_{ij}\phi_{x_i}}{\lambda+\varepsilon}-2b_j\right)\nu_j(x).$$

Note that

$$\tilde{d}\geq 0 \quad \text{on} \quad \partial U \quad \text{and} \quad R-\tilde{c}\geq 1 \quad \text{on} \quad \overline{U}.$$

Applying again the maximum principle and the Schauder theory to (5.19), we infer that, if $f\in C^{0,\alpha}(\overline{U})$, then (5.18) has a unique classical solution $v\in C^{2,\alpha}(\overline{U})$ and satisfies the maximum principle.

Let T denote the solution operator for (5.18) with $\rho=-R$, that is, if v is a classical solution of (5.18), then $Tf=v$.

As before applying the maximum principle, applied to the function $e^{-\phi}Tf$, we get

$$\|e^{-\phi}Tf\|_{C(\overline{U})}\leq\|e^{-\phi}f\|_{C(\overline{U})},$$

and, thus,

$$\|Tf\|_{C(\overline{U})}\leq e^{2\|\phi\|_{C(\overline{U})}}\|f\|_{C(\overline{U})},$$

which allows us to extend the domain of definition of T to $C(\overline{U})$.

Fix $r > 0$ and observe that for any $\psi, f \in C(\overline{U})$,

$$\Big|\int_U \psi(x) S_r f(x) dx\Big| \leq |U|\, \|\psi\|_{C(\overline{U})} \|S_r f\|_{C(\overline{U})} \leq r^{-1}|U|\, \|\psi\|_{C(\overline{U})} \|f\|_{C(\overline{U})}, \tag{5.20}$$

where $|U|$ denotes the Lebesgue measure of U, and, hence, that for each $\psi \in C(\overline{U})$ the mapping

$$C(\overline{U}) \ni f \mapsto \int_U \psi(x) S_r f(x) dx \in \mathbb{R}$$

is linear and continuous. Accordingly, there exists a unique $S_r^* \psi \in C(\overline{U})^*$, the dual space of $C(\overline{U})$, such that, if $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $C(\overline{U})^*$ and $C(\overline{U})$, then, for all $\psi \in C(\overline{U})$,

$$\int_U \psi(x) S_r f(x) dx = \langle S_r^* \psi, f \rangle.$$

Using the Riesz representation theorem, we may identify $S_r^* \psi$ as a Radon measure on $\overline{U}$.
Since, by (5.20),

$$|\langle S_r^\star \psi, f \rangle| \leq r^{-1}|U|\, \|f\|_{C(\overline{U})} \|\psi\|_{C(\overline{U})} \quad \text{for } f, \psi \in C(\overline{U}),$$

it follows that $C(\overline{U}) \ni \psi \mapsto S_r^* \psi \in C(\overline{U})^*$ is a continuous and linear map.
Next, we fix $f \in C^{0,\alpha}(\overline{U})$ and $r \in (0, R)$, and solve (5.18) for $\rho = -r$.
Without loss of generality, we may assume that $f \geq 0$ in $\overline{U}$.
We use an iteration argument, and consider the sequence $\{v_n\}_{n \in \mathbb{N}}$ given by $v_1 \equiv 0$ and, for $n > 1$, by the solution $v_n \in C^2(\overline{U})$ of

$$\begin{cases} -\left(\dfrac{(a_{ij} v_n)_{x_i}}{\lambda + \varepsilon} - 2 b_j v_n\right)_{x_j} = -R v_n + (R - r) v_{n-1} + f \ \text{ in } \ U, \\ \left(\dfrac{(a_{ij} v_n)_{x_i}}{\lambda + \varepsilon} - 2 b_j v_n\right) \nu_j = 0 \ \text{ on } \ \partial U, \end{cases} \tag{5.21}$$

Using the operator T, (5.21) can be stated as

$$v_n = T((R - r) v_{n-1} + f). \tag{5.22}$$

It follows from the maximum principle that, for all $n \in \mathbb{N}$,

$$v_n \geq 0 \ \text{ and } \ v_{n+1} \geq v_n \ \text{ on } \ \overline{U}. \tag{5.23}$$

We show that, in the sense of measures on $\overline{U}$ and for all $n \in \mathbb{N}$,

$$v_n \leq S_r^* f.$$

Indeed, first observe that, in view of (4.1), for any $\phi, \psi \in C^2(\overline{U})$, if

$$\left(\frac{(a_{ij} \phi)_{x_i}}{\lambda + \varepsilon} - 2 b_j \phi\right) \nu_j = a_{ij} \psi_{x_i} \nu_j = 0 \ \text{ on } \ \partial U,$$

then

$$\int_U \phi L_r \psi dx = \int_U L_r^* \phi \psi dx,$$

where

$$L_r \psi = -a_{ij} \left(\frac{v_{x_i}}{\lambda + \varepsilon}\right)_{x_i} - 2 b_j \psi_{x_j} + r \psi \quad \text{and} \quad L_r^* \phi = -\left(\frac{(a_{ij} \phi)_{x_i}}{\lambda + \varepsilon} - 2 b_j \phi\right)_{x_j} + r \phi.$$

We rewrite the above formula as

$$\langle \phi, L_r\psi\rangle = \langle L_r^*\phi, \psi\rangle \tag{5.24}$$

and apply it to $(\phi,\psi)=(v_{n+1}, S_r w)$, with $n\in\mathbb{N}$ and $w\in C^{0,\alpha}(\overline{U})$, to get

$$\langle L_r^* v_{n+1}, S_r w\rangle = \langle v_{n+1}, L_r S_r w\rangle = \langle v_{n+1}, w\rangle,$$

where the first term can be calculated as follows:

$$\langle L_r^* v_{n+1}, S_r w\rangle = \langle L_R^* v_{n+1} + (r-R)v_{n+1}, S_r w\rangle = \langle (R-r)(v_n - v_{n+1}) + f, S_r w\rangle.$$

Assume now that $w\geq 0$ on $\overline{U}$, and observe that, by the maximum principle, $S_r w\geq 0$ on $\overline{U}$ and

$$\langle (R-r)(v_n - v_{n+1}) + f, S_r w\rangle \leq \langle f, S_r w\rangle = \langle S_r^* f, w\rangle.$$

Hence, if $w\geq 0$,

$$\langle v_{n+1} - S_r^* f, w\rangle \leq 0,$$

which proves that $v_{n+1}\leq S_r^* f$.

In particular, for all $n\in\mathbb{N}$, we have

$$\int_U v_{n+1}dx \leq \langle S_r^* f, 1\rangle \leq r^{-1}|U|\|f\|. \tag{5.25}$$

It follows from [?li83, Lemma 3.1] and [6, Theorem 6.30] that

$$\sup_{n\in\mathbb{N}} \|v_n\|_{C^{2,\alpha}(\overline{U})} < \infty,$$

and, hence, for some $v\in C^{2,\alpha}(\overline{U})$,

$$\lim_{n\to\infty} v_n = v \ \text{ in } \ C^2(\overline{U}).$$

Moreover, it is easily seen that v is a solution to (5.18) with $\rho=-r$.

Also, using (5.24), we deduce that, if $v\in C^2(\overline{U})$ is a solution to (5.18), with $\rho=-r$, then, for $w\in C^{0,\alpha}(\overline{U})$,

$$\langle S_r^* f, w\rangle = \langle f, S_r w\rangle = \langle L_r^* v, S_r w\rangle = \langle v, L_r S_r w\rangle = \langle v, w\rangle,$$

which shows that $v=S_r^* f$ and, in particular, the uniqueness of the solution to (5.18) for $\rho=-r$.

Fix a sequence $(0,R)\ni r_k\to 0$ and set $m_k=|U|^{-1}S_{r_k}^* r_k$, where the last r_k denotes the constant function $\overline{U}\ni x\mapsto r_k$.

Note that $m_k\in C^2(\overline{U})$ is a solution to (5.18) with $\rho=-r_k$ and $f=|U|^{-1}r_k$, $m_k\geq 0$ on $\overline{U}$, and

$$\langle m_k, 1\rangle = |U|^{-1}\langle r_k, S_{r_k}1\rangle = |U|^{-1}\langle r_k, r_k^{-1}\rangle = 1.$$

The Schauder theory (see [?li83, Lemma 3.1] and [6, Theorem 6.30]) imply that $\{m_k\}\subset C^{2,\alpha}(\overline{U})$ is bounded.

Hence, after passing to a subsequence, we may assume that, for some $m^\varepsilon\in C^{2,\alpha}(\overline{U})$,

$$\lim_{k\to\infty} m_k = m^\varepsilon \ \text{ in } \ C^2(\overline{U}).$$

It follows immediately that m^ε is a solution of $(\mathrm{Ad})_\varepsilon$, except the positivity of m^ε. It is clear that $m^\varepsilon\geq 0$ and $m^\varepsilon\not\equiv 0$. The strong maximum principle and Hopf's lemma yield that $m^\varepsilon>0$ on $\overline{U}$. Hence, m^ε is a solution of $(\mathrm{Ad})_\varepsilon$.

Let $\mu \in C^2(\overline{U})$ satisfy the first two equations of $(\mathrm{Ad})_\varepsilon$ and observe that, by the same reasoning as above, $\mu > 0$ on $\overline{U}$. Choose $c \in \mathbb{R}$ so that $\mu \le cm^\varepsilon$ on $\overline{U}$ and $\mu(x_0) = cm^\varepsilon(x_0)$ for some $x_0 \in \overline{U}$. Applying the strong maximum principle and Hopf's lemma to $cm^\varepsilon - \mu$, we find that, if $\mu \not\equiv cm^\varepsilon$ on $\overline{U}$, then $\mu < cm^\varepsilon$ in $\overline{U}$, which is a contradiction. It follows that $\mu = cm^\varepsilon$. This also implies the uniqueness of a solution of $(\mathrm{Ad})_\varepsilon$ and the proof is complete. □

6. The initial value problem (2.4)

In this section we briefly sketch the proof of the following theorem.

Theorem 6.1. *Let $\mu > 0$, $T > 0$ and assume* (2.1) *and* (2.2).
(i) *If v, w are bounded, respectively upper and lower semicontinuous on $\mathbb{R}^{2n} \times [0, T)$, viscosity sub- and super-solutions to $u_t = L^\mu u$ in $\mathbb{R}^{2n} \in (0, T)$ and $v(\cdot,\cdot,0) \le w(\cdot,\cdot,0)$ in $\mathbb{R}^{2n}$, then, $v \le w$ on $\mathbb{R}^{2n} \times [0, T)$.*
(ii) *Let $u_0^\mu \in \mathrm{BUC}\,(\mathbb{R}^{2n})$. There exists a unique viscosity solution $u^\mu \in C_\mathrm{b}(\mathbb{R}^{2n} \times [0, T))$ to* (2.4).

The uniform continuity assumption on u_0^μ can be relaxed and replaced by the continuity of u_0^μ in the above theorem. But this strong assumption makes it easy to prove assertion (ii).

Let $\langle x\rangle := (|x|^2 + 1)^{1/2}$ and set

$$p(x, y) := \langle x\rangle + \frac{1}{2}|y|^2 \quad \text{for } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{6.1}$$

Using the positivity of λ (see (2.2)) straightforward calculations imply that there exist exist $c > 0$ and $C > 0$ such that, for all $(x, y) \in \mathbb{R}^{2n}$,

$$L^\mu p(x, y) \le C - c|y|^2. \tag{6.2}$$

Proof of Theorem 6.1. Let p and C, c as in (6.1) and (6.2), set $q(x, y, t) = p(x, y) + Ct$ for $(x, y, t) \in \mathbb{R}^{2n} \times [0, T]$, and note that, for any $(x, y, t) \in \mathbb{R}^{2n} \times (0, T)$,

$$L^\mu q(x, y, t) = L^\mu p(x, y) \le C - c|y|^2 = q_t(x, y, t) - c|y|^2. \tag{6.3}$$

To prove (i), we fix $\delta > 0$ and observe that

$$v_\delta(x, y, t) = v(x, y, t) - \delta q(x, y, t),$$

is an upper semicontinuous subsolution to $u_t = L^\mu u$ in $\mathbb{R}^{2n} \times (0, T)$.

Since v and w are bounded and

$$p(x, y) \to \infty \quad \text{as} \quad |x| + |y| \to \infty,$$

we can choose a bounded open subset Ω of $\mathbb{R}^{2n}$ so that

$$v_\delta(x, y, t) \le w(x, y, t) \quad \text{if} \quad (x, y) \notin \Omega, \tag{6.4}$$

while

$$v_\delta(x, y, 0) \le v(x, y, 0) \le w(x, y, 0) \quad \text{for all} \quad (x, y) \in \mathbb{R}^{2n}.$$

Applying the standard comparison theorem (for instance, [1, Theorem 8.2] and its proof), we find that $v_\delta \le w$ on $\Omega \times [0, T)$, which together with (6.4) yields that $v_\delta \le w$ on $\mathbb{R}^{2n} \times [0, T)$. Letting $\delta \to 0$ implies the claim.

We turn to (ii). Since $u_0^\mu \in \mathrm{BUC}\,(\mathbb{R}^{2n})$, for each $\delta \in (0,\,1)$, we may choose $u_0^{\mu,\delta} \in C_{\mathrm{b}}^2(\mathbb{R}^{2n})$ so that $\|u_0^{\mu,\delta} - u_0^\mu\| < \delta$. Obviously, there exists a constant $C_\delta > 0$ so that

$$|L^\mu u_0^{\mu,\delta}| \leq C_\delta(1+|y|) \quad \text{on} \quad \mathbb{R}^{2n}.$$

Select $M_\delta > 0$ so that

$$|L^\mu u_0^{\mu,\delta}| + \delta L^\mu p \leq C_\delta(1+|y|) + \delta(C - c|y|^2) \leq M_\delta \quad \text{on} \quad \mathbb{R}^{2n},$$

set

$$v_\delta^\pm(x,y,t) = u_0^{\mu,\delta}(x,y) \pm (\delta + \delta p(x,y) + M_\delta t) \quad \text{on } R^{2n} \times [0,\,T],$$

observe that $v_\delta^-, v_\delta^+ \in C^2(\mathbb{R}^{2n} \times [0,\,T])$ are a sub- and super-solution of $u_t = L^\mu u$ in $\mathbb{R}^{2n} \times (0,\,T)$, and, finally, for $(x,y,t) \in \mathbb{R}^{2n} \times [0,\,T]$,

$$\tag{6.5} \begin{cases} v_\delta^\pm(x,y,0) = u_0^{\mu,\delta}(x,y) \pm \delta(1+p(x,y)), \\ v_\delta^-(x,y,t) \leq u_0^\mu(x,y) \leq v_\delta^+(x,y,t). \end{cases}$$

The stability property of viscosity solutions yields that, if, in $\mathbb{R}^{2n} \times [0,\,T]$,

$$v^+ = \inf_{0<\delta<1} v_\delta^+ \quad \text{and} \quad v^- = \sup_{0<\delta<1} v_\delta^+,$$

then the upper and lower semicontinuous envelopes w^- of v^- and w^+ of v^+ are respectively a viscosity sub- and super-solution of $u_t = L^\mu u$ in $\mathbb{R}^{2n} \times (0,\,T)$. Moreover, it follows from (6.5) that, for $(x,y,t) \in \mathbb{R}^{2n} \times [0,\,T]$,

$$\tag{6.6} \begin{cases} v^\pm(x,y,0) = u_0^\mu(x,y), \\ v^-(x,y,t) \leq w^-(x,y,t) \leq u_0^\mu(x,y) \leq w^-(x,y,t) \leq v^+(x,y,t). \end{cases}$$

According to Perron's method ([1]), if we set

$$u^\mu(x,y,t) = \sup\{u(x,y,t)\ : u \text{ is a subsolution of } \ u_t = L^\mu u \ \text{ in } \ \mathbb{R}^{2n} \times (0,\,T), \\ w^- \leq u \leq w^+ \ \text{ on } \mathbb{R}^{2n} \times [0,\,T)\},$$

then u^μ is a solution to $u_t = L^\mu u$ in $\mathbb{R}^{2n} \times (0,\,T)$ in the sense that the upper and lower semicontinuous envelopes $(u^\mu)^*$ and $(u^\mu)_*$ of u^μ are respectively a sub- and super-solution to $u_t = L^\mu u$ in $\mathbb{R}^{2n} \times (0,\,T)$.

Note that $v^-, -v^+$ are lower semicontinuous on $\mathbb{R}^{2n} \times [0,\,T]$, and hence, by (6.6), $(u^\mu)^*(x,y,0) = (u^\mu)_*(x,y,0) = u_0^\mu(x,y)$ for all $(x,y) \in \mathbb{R}^{2n}$.

Thus, by the comparison assertion (i), we obtain $(u^\mu)^* \leq (u^\mu)_*$ on $\mathbb{R}^{2n} \times [0,\,T)$, and we conclude that $u^\mu \in C_{\mathrm{b}}(\mathbb{R}^{2n} \times [0,\,T))$ and it is a solution of (2.4).

The uniqueness of u^μ is an immediate consequence of (i). □

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