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Homogenization of Hamilton-Jacobi equations

Introduction.

We consider here various questions related to the behaviour, as ε goes to 0, of the solution u^ε of the following Hamilton-Jacobi equation

$$(1) \qquad \frac{\partial u^\varepsilon}{\partial t} + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 \qquad \text{in } \mathbb{R}^N \times]0,\infty[$$

together with the initial condition

$$(2) \qquad u^\varepsilon\big|_{t=0} = u_o(x) \qquad \text{in } \mathbb{R}^N .$$

Here and below u^ε, u_o are scalar, u_o is prescribed and the Hamiltonian $H(x,p) \in C(\mathbb{R}^N \times \mathbb{R}^N)$ is periodic in x (i.e. periodic in x_i for $1 \leqslant i \leqslant N$ of period 1). Finally, D or ∇ denotes the spatial gradient.

Such an asymptotic problem falls into the scope of homogeneization theory and we refer the reader to A. Bensoussan, J.L. Lions and G. Papanicolaou [2], E. De Giorgi [9], L. Tartar [18] (and their references) for similar problems. However, to our knowledge, our work is the first one concerning nonlinear first order equations (of hyperbolic type) and which is global in time.

Assuming only that $u_o \in BUC(\mathbb{R}^N)$ (★) and that

$$(3) \qquad H(x,p) \to +\infty \quad \text{as} \quad |p| \to \infty , \quad \text{uniformly for } x \in \mathbb{R}^N$$

then we will prove below that u^ε converges uniformly on $\mathbb{R}^N \times [0,T]$ (for all $T < \infty$) to the solution u of

$$(4) \qquad \frac{\partial u}{\partial t} + \overline{H}(Du) = 0 \qquad \text{in } \mathbb{R}^N \times]0,\infty[$$

satisfying (2), where $\overline{H}$ - the <u>effective Hamiltonian</u> - is given via the solution of a cell problem (ergodic stationary Hamilton-Jacobi equation) that we solve in details below.

(★) $BUC(X) = \{v \in C(X) , v$ bounded, uniformly continuous on $X\}$.

Up to know, we have been vague on the meaning of (1),(4). Let us only indicate for the moment that we will deal exclusively with viscosity solutions of Hamilton-Jacobi equations (which roughly speaking are the limits of the solutions of the approximated equations with vanishing viscosity). Viscosity solutions were introduced in M.G. Crandall and P.L. Lions [4] and we refer to M.G. Crandall, L.C. Evans and P.L. Lions [3], P.L. Lions [12], [13], G. Barles [1], H. Ishii [10], [11], M.G. Crandall and P.L. Lions [5], [6], [7], M.G. Crandall and P.E. Souganidis [8] for further works and references on viscosity solutions.

Let us explain now our motivations on two examples.

Example 1 : $H(x,p) = |p|^2 - V(x)$.

This is of course the standard Hamiltonian in classical mechanics. In this case, (1) is the Cauchy problem for the oscillating Hamiltonian $H(x,p) = |p|^2 - V(\frac{x}{\varepsilon})$ (recall that H and thus V are periodic in x). Recall - to see the physical motivation - that (1) in this case is known as the Eiconal equation and is obtained by considering solutions of

$$i\frac{\partial\phi}{\partial t} - \hbar^2\Delta\phi + \frac{1}{\hbar^2}V\left(\frac{x}{\varepsilon}\right)\phi = 0 \qquad \text{in } \mathbb{R}^N \times]0,\infty[$$

of the following form $\phi = e^{iu/\hbar} f$ and letting $\hbar \to 0$.

A case of particular interest is the case when $V = 0$ on $\bar{\omega}$, $V = +\infty$ on $\Pi-\omega$ where ω is an open set of Π and Π is the unit cube $(\Pi = [0,1]^N)$. Such a V is highly discontinuous and is not, strictly speaking, covered by our results. Nevertheless, this case may be treated by variants of the methods introduced here and we will come back on this problem in a future publication.

Example 2 : $H(x,p) = \sum_{i,j=1}^{N} a_{ij}(x)\, p_i\, p_j$.

The matrix (a_{ij}) is assumed to be symmetric and uniformly positive definite. The family of Hamiltonians $H_\varepsilon(x,p) = \sum_{i,j=1}^{N} a_{ij}\left(\frac{x}{\varepsilon}\right) p_i\, p_j$ then defines a sequence of Riemannian metrics on the torus $\mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N$. And, as we will recall, (1) is closely related to the determination of the associated distance functions $d_\varepsilon(x,y)$.

It is clear that the convergence theorem we mentioned above covers those two examples and therefore we explain below how to identify the effective Hamiltonian $\overline{H}$. In both cases, we also prove some qualitative properties of $\overline{H}$; in particular, we will observe a surprising phenomenon in Example 1 : for many potentials V (all except trivial ones in dimension 1) $\overline{H}$ is no more strictly convex in p and in fact $\overline{H}$ vanishes in a neighborhood of the origin.

Let us also mention that the Cauchy problem (1)-(2) is very much related to classical problems in the Calculus of Variations such as : find a path $\xi(t)$ in $\mathbb{R}^N$ minimizing

$$L_\varepsilon(x,y,t) = \text{Inf} \left\{ \int_0^t L\left(\frac{\xi(s)}{\varepsilon}, \dot{\xi}(s)\right) ds \ / \ \xi(0) = x \ , \ \xi(t) = y \right\}$$

where $x,y \in \mathbb{R}^N$, $t > 0$ and $L(x,p)$ (the Lagrangian) is convex in p, periodic in x. We will see that our main convergence theorem enables us to prove the convergence of L_ε as ε goes to 0 and to determine the limit.

Another relation that we wish to point out here is with the theory of scalar conservation laws : indeed if u^ε is the (viscosity) solution of (1),(2) when $N = 1$, then $v^\varepsilon = \frac{du^\varepsilon}{dx}$ is the (entropy) solution of

$$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} + \dfrac{\partial}{\partial x} \left[H\left(\dfrac{x}{\varepsilon}, v^\varepsilon\right) \right] = 0 & \text{in } \mathbb{R} \times]0,\infty[\\ v^\varepsilon \Big|_{t=0} = \dfrac{du_0}{dx} & \text{in } \mathbb{R}^N . \end{cases}$$

Thus, our convergence results yield results on the homogeneization of scalar conservation laws.

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I . Main convergence result.

I.1 Main results.

We consider, for any initial condition u_o in $BUC(\mathbb{R}^N)$, the unique viscosity solution u^ε of (1)-(2) in $BUC(\mathbb{R}^N \times [0,T])$ $(\forall\, T < \infty)$.

In order to introduce our results, it is natural to begin with the usual formal asymptotic expansions (see the book [2] for a systematic presentation of such ansatz)

(5) $$u^\varepsilon(x,t) = u^o(x,t) + \varepsilon\, u^1\left(\frac{x}{\varepsilon}, t\right) + \varepsilon^2 u^2 + \ldots$$

where $u^i(x,y,t)$ are periodic in y. Plugging (5) into (1) and identifying the terms in front of powers of ε, we find

(6) $$\frac{\partial u^o}{\partial t}(x,t) + H\left(y, D_x u^o(x,t) + D_y u^1(y,t)\right) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N \times]0,\infty[$$

Therefore, we are led to the following "cell problem" : for each $p \in \mathbb{R}^N$, find $\lambda \in \mathbb{R}$ such that there exists v viscosity solution of

(7) $$\boxed{H(y, p + D_y v) = \lambda \quad \text{in } \mathbb{R}^N, \quad v \text{ periodic in } y}$$

Of course, λ will depend on p and (provided we can solve (7) in a convenient way) we will denote $\lambda = \overline{H}(p)$. Then u^o "should satisfy" (4) and (2).

Our main result shows that all these formal guesses are correct.

Theorem 1. Let $H \in C(\mathbb{R}^N \times \mathbb{R}^N)$ be periodic in x and satisfy (3).

Existence and uniqueness of $\overline{H}$. For each $p \in \mathbb{R}^N$, there exists a unique $\lambda \in \mathbb{R}$ - that we denote by $\overline{H}(p)$ - such that there exists $v \in C(\mathbb{R}^N)$, periodic, viscosity solution of (7). And $\overline{H}$ is continuous in p.

Convergence of u^ε. For any $u_o \in BUC(\mathbb{R}^N)$, u^ε converges uniformly on

$\mathbb{R}^N \times [0,T]$ $(\forall\, T < \infty)$ to the viscosity solution u of (4)-(2) in $BUC(\mathbb{R}^N \times [0,T])$ $(\forall\, T < \infty)$.

Remarks : i) The existence and uniqueness of $\overline{H}$ is proved in section II.1 while we give in section II.2 various qualitative properties of $\overline{H}$.

ii) We give below explicit formula if $H(x,p) = |p|^2 - V(x)$, $N = 1$ and we will see that except in the trivial case when V is constant, solutions v of (7) are not unique (even up to the addition of constants). Recall that $v \equiv u^1$ is the corrector. Hence, we do not know to characterize u^1 and we do not know if the asymptotic expansion is valid globally in t .

iii) Except for very special initial conditions u_0 , we do not know the rate of convergence of u^ε to u .

We now treat explicitely (7) when $N = 1$ and $H(x,p) = |p|^2 - V(x)$. Without loss of generality, we may always assume that $\min_{\mathbb{R}} V = 0$. Denoting by $\langle\varphi\rangle$ the average of any periodic function φ on its period, we claim that $\overline{H}(p)$ is given by

$$(8)\qquad \begin{cases} \overline{H}(p) = 0 & \text{if } |p| \leq \langle V^{1/2}\rangle \\ \overline{H}(p) = \lambda \ \text{ solves } \ |p| = \langle (V+\lambda)^{1/2}\rangle , \ \lambda \geq 0 & \text{if } |p| \geq \langle V^{1/2}\rangle . \end{cases}$$

Indeed, we just have to exhibit $v \in C(\mathbb{R})$, viscosity solution of (7) for such a λ . If $|p| \leq \langle V^{1/2}\rangle$, one can find $x_0 \in [0,1]$, $\bar{x} \in [x_0, 1+x_0]$ such that

$$0 = V^{1/2}(x_0) \quad , \quad \int_{x_0}^{\bar{x}} V^{1/2} - p \, ds = \int_{\bar{x}}^{1+x_0} V^{1/2} + p \, ds$$

Then we set

$$v(x) = \int_{x_0}^{x} V^{1/2} - p \, ds \qquad \text{if } x_0 \leq x \leq \bar{x}$$

$$v(x) = \int_{x}^{1+x_0} V^{1/2} + p \, ds \qquad \text{if } \bar{x} \leq x \leq 1+x_0$$

and we extend v periodically.

Similarly, if $|p| \geq \langle V^{1/2} \rangle$, choosing λ as in (8), one can find $x_o \in [0,1]$ such that

$$(V+\lambda)^{1/2}(x_o) = p \qquad \text{if } p \geq 0$$

and one argues similarly if $p \leq 0$.

Then we set

$$v(x) = \int_{x_o}^{x} (V+\lambda)^{1/2} - p \, ds \qquad \text{if } x_o \leq x \leq 1+x_o$$

and we extend v periodically. □

At this stage, two important observations are to be made : first of all, even if H was strictly convex in p ($H(x,p) = |p|^2 - V(x)$!), the effective Hamiltonian $\overline{H}(p)$ satisfies $\overline{H} = 0$ for $|p| \leq \langle V^{1/2} \rangle$ and thus is not strictly convex in p (except if $V \equiv 0$). We will come back on this point in section II.2. Next, except if $V \equiv 0$, v is not unique in general even up to the addition of constants. Indeed, take for example $p = 0$ and assume V vanishes at x_o, x_1 satisfying $0 \leq x_o < x_1 \leq 1$. Then, we already gave one solution v of (7) venishing at x_o. A different one (if $V \not\equiv 0$) is given by

$$\tilde{v}(x) = \int_{x_o}^{x} V^{1/2} \, ds \text{ if } x_o \leq x \leq \bar{x}_1, \quad = \int_{x}^{x_1} V^{1/2} ds \text{ if } \bar{x}_1 \leq x \leq x_1$$

$$= \int_{x_1}^{x} V^{1/2} ds \text{ if } x_1 \leq x \leq \bar{x}_2, \quad = \int_{x}^{1+x_o} V^{1/2} ds \text{ if } \bar{x}_2 \leq x \leq 1+x_o$$

and one extends $\tilde{v}$ periodically, where $\bar{x}_1 \in [x_o, x_1]$, $\bar{x}_2 \in [x_1, 1+x_o]$ satisfy

$$\int_{x_o}^{\bar{x}_1} V^{1/2} \, ds = \int_{\bar{x}_1}^{x_1} V^{1/2} \, ds, \quad \int_{x_1}^{\bar{x}_2} V^{1/2} \, ds = \int_{\bar{x}_2}^{1+x_o} V^{1/2} \, ds .$$

[Concerning example 2, if $N = 1$ and $H(x,p) = a(x)p^2$ where $a > 0$, is continuous, periodic then $\overline{H}(p) = bp^2$ with $b = (\langle a^{-1/2} \rangle)^{-2}$.]

I.2 Affine data and exact solutions.

Our proof(s) of the convergence result in Theorem 1 relies on a simple observation we make in this section. It concerns affine initial conditions i.e.

(9) $$u_o(x) = \alpha + p\cdot x$$

for some $\alpha \in \mathbb{R}$, $p \in \mathbb{R}^N$.

For such an initial condition, the solution u of (4)-(2) is given by

(10) $$u(x,t) = \alpha + p\cdot x - t\,\overline{H}(p) \quad .$$

To be more precise, we recall from [11], [6] that if $u_o \in UC(\mathbb{R}^N)$, there exists a unique viscosity solution of (1)-(2) (or (4)-(2)) in $C(\mathbb{R}^N \times [0,\infty[)$, uniformly continuous in x uniformly in $t \in [0,T]$ ($\forall\ T < \infty$). Then, we choose one solution $v \in C(\mathbb{R}^N)$ of (7) (with $\lambda = \overline{H}(p)$) and we consider the formal asymptotic expansion of the preceding section

$$\tilde{u}^\varepsilon(x,t) = u(x,t) + \varepsilon v\left(\frac{x}{\varepsilon}\right) \quad .$$

One checks easily that $\tilde{u}^\varepsilon$ is a viscosity solution of (1) : formally, we have indeed

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t} + H\left(\frac{x}{\varepsilon}, D\tilde{u}^\varepsilon\right) = -\overline{H}(p) + H\left(\frac{x}{\varepsilon}, p+Dv\left(\frac{x}{\varepsilon}\right)\right) = 0$$

in view of (7).

The initial condition satisfies by $\tilde{u}^\varepsilon$ is

(11) $$\tilde{u}^\varepsilon\big|_{t=0} = \alpha + p\cdot x + \varepsilon v\left(\frac{x}{\varepsilon}\right) \qquad \text{in } \mathbb{R}^N$$

and thus if u^ε is the viscosity solution (in the appropriate class recalled above) of (1)-(2) corresponding to the choice (9) we have by the comparison results of [6], [11]

$$\sup_{\mathbb{R}^N \times [0,\infty[} |u^\varepsilon - \tilde{u}^\varepsilon| \leq \varepsilon \sup_{\mathbb{R}^N} |v| .$$

Hence, this yields

$$(12) \qquad \sup_{\mathbb{R}^N \times [0,\infty[} |u^\varepsilon - u| \leq 2\varepsilon \sup_{\mathbb{R}^N} |v| .$$

In particular, the convergence result in Theorem 1 is proved for affine initial conditions. And we will see in section III that this is enough to yield the convergence for arbitrary initial conditions. It is possible to explain the "sufficiency" of affine initial conditions as follows. Denoting by $S^\varepsilon(t)$ the semi-group on $BUC(\mathbb{R}^N)$ (or UC ...) induced by the Cauchy problem (1)-(2) and using properties of viscosity solutions [4], [6], we know that $S^\varepsilon(t)$ is order preserving, commutes with the addition of constants and is contractive in sup norm. We will also see that $S^\varepsilon(t)$ has a uniform (in ε) speed of propagation of the supports of initial conditions and that $S^\varepsilon(t)u_o$ is bounded in $W^{1,\infty}(\mathbb{R}^N \times (0,T))$ $(\forall\, T < \infty)$ if $u_o \in W^{1,\infty}(\mathbb{R}^N)$. Therefore, we may extract a subsequence $\varepsilon_n \to_n 0$, such that $S^{\varepsilon_n}(t)u_o$ converges uniformly on compact subsets of $\mathbb{R}^N \times [0,\infty[$ to $S(t)u_o$ for all $u_o \in UC(\mathbb{R}^N)$ and $S(t)$ is a semi-group enjoying all the properties listed above.

In addition, we claim that $S(t)$ commutes with translations. Indeed, let $z \in \mathbb{R}^N$ and let us denote by τ_z the translation by z i.e. $\tau_z \varphi(\cdot) = \varphi(\cdot + z)$. For all $n \geq 1$, we may find $z_n \in \varepsilon_n \mathbb{Z}^N$, $|r_n| \leq C\, \varepsilon_n$ such that $z = z_n + r_n$. Obviously, $S^{\varepsilon_n}(t)$ commutes with τ_{z_n} and passing to the limit we conclude that $S(t)$ commutes with τ_z.

But then, by the inverse result of P.L. Lions [15], [14], such a semi-group is automatically the semi-group of viscosity solutions of a certain Hamilton-Jacobi equation of the form

$$\frac{\partial u}{\partial t} + F(Du) = 0 \qquad \text{in } \mathbb{R}^N \times]0,\infty[$$

where $F \in C(\mathbb{R}^N)$. And we already know that if u_o is affine, $S(t)u_o =$

$= \alpha + p \cdot x - t \, \overline{H}(p)$ while the above characterization also yields

$$S(t)u_o = \alpha + p \cdot x - t \, F(p) \quad .$$

Therefore, $F \equiv \overline{H}$ and we conclude.

This scheme of proof may be justified, however we will prefer another one slightly different. Anyway, it indicates why affine initial conditions are enough here to determine the behaviour of the complete semi-group.

II . The cell problem.

II.1 Existence and uniqueness.

To prove the existence of v,λ satisfying (7), we consider the approximated equation

$$(13) \qquad H(y\,,\,p+D_y v_\alpha) + \alpha v_\alpha = 0 \qquad \text{in } \mathbb{R}^N\,,$$

where $\alpha > 0$.

Since $H \in BUC(\mathbb{R}^N \times \overline{B}_R)$, H satisfies (3), we know by [12], [13] that there exists a unique $v_\alpha \in W^{1,\infty}(\mathbb{R}^N)$, viscosity solution of (13). The uniqueness then implies that v_α is periodic. Furthermore, we have

$$(14) \qquad -\sup_y H(y,p) \leqslant \alpha v_\alpha \leqslant -\inf_y H(y,p) \quad .$$

And the combination of (13), (14) yields in view of (3)

$$\|Dv_\alpha\|_\infty \leqslant C$$

for some constant C independent of α .

We then set $\tilde{v}_\alpha = v_\alpha - \underset{\Pi}{\mathrm{Min}}\, v_\alpha$, recall that Π is the unit cube. Clearly, $\tilde{v}_\alpha$ is periodic, bounded in $W^{1,\infty}(\mathbb{R}^N)$ and (extracting a subsequence if necessary) we may assume that $(\tilde{v}_\alpha, -\alpha v_\alpha)$ converge uniformly on Π (or on $\mathbb{R}^N$) to some $(v,\lambda) \in W^{1,\infty}(\mathbb{R}^N) \times \mathbb{R}$ where v is periodic. By the properties of viscosity solutions v is a viscosity solution of (7). And the existence is proved. Notice that we also proved

$$(15) \qquad \inf_y H(y,p) \leqslant \overline{H}(p) \leqslant \sup_y H(y,p) \quad , \quad \forall\, p \in \mathbb{R}^N\,.$$

To prove the uniqueness of λ , suppose that (v,λ) , $(w,\mu) \in C(\mathbb{R}^N) \times \mathbb{R}$ satisfy (7). If $\lambda \neq \mu$, we may assume that $\lambda < \mu$ and remarking that we may add constants to v,w we may also assume that $v > w$ on $\mathbb{R}^N$. Then, for

α small enough we still have $\lambda + \alpha v \leqslant \mu + \alpha w$. On the other hand, v,w are the unique viscosity solutions in $BUC(\mathbb{R}^N)$ of

$$H(y,p+Dw) + \alpha w = (\mu+\alpha w) \quad , \quad H(y,p+Dv) + \alpha v = (\lambda+\alpha v) \quad \text{in } \mathbb{R}^N .$$

Then, by the comparizon results for viscosity solutions, we deduce

$$w \geqslant v \qquad \text{on } \mathbb{R}^N .$$

The contradiction proves the uniqueness of λ .

Observe that the uniqueness of λ implies easily the continuity of $\overline{H}(p)$. Observe also that we proved in fact that if $F(y,q) \in C(\mathbb{R}^N \times \mathbb{R}^N)$, is periodic in y and satisfies (3) then there exists a unique $\lambda = \lambda(F) \in \mathbb{R}$ such that there exists $v \in C(\mathbb{R}^N)$, viscosity solution of

(16) $$F(y,Dv) = \lambda \quad \text{in } \mathbb{R}^N \quad , \quad v \text{ periodic} .$$

Some properties of λ (as a function of F) are given in the next section.

II.2 Qualitative properties of the effective Hamiltonian.

Proposition 2 : Let $F_1, F_2 \in C(\mathbb{R}^N \times \mathbb{R}^N)$ be periodic in y and satisfy (3). Then, we have

(17) $$\lambda(tF_1) = t\lambda(F_1) \quad , \quad \lambda(F_1) = \lambda(F_1(\cdot, t\cdot)) \quad , \quad \forall t > 0 ,$$

(15) $$\inf_y F_1(y,0) \leqslant \lambda(F_1) \leqslant \sup_y F_1(y,0) \quad ,$$

(18) $$(\lambda(F_1)-\lambda(F_2))^+ \leqslant \sup \{(F_1(y,q)-F_2(y,q))^+ \,/\, y \in \mathbb{R}^N , \ |q| \leqslant R\}$$

where $R < \infty$ is such that $\inf \{F_i(y,q) \,/\, y \in \mathbb{R}^N , |q| \geqslant R\} \geqslant \sup_y F_i(y,0)$ for $i = 1,2$;

(19) $\lambda(F_1) = \lambda(\hat{F}_1)$ where $\hat{F}_1(y,q) = F_1(y,-z+q)$,

if $F_1(y,q) = F_1(y,-q-z)$ for some $z \in \mathbb{R}^N$;

(20) $\lambda(F) \leqslant \theta\lambda(F_2) + (1-\theta)\lambda(F_1)$

if $\theta \in]0,1[$, F_1 is convex in q , $F_2(y,q) = F_1(y,z+q)$ for some $z \in \mathbb{R}^N$ and $F(y,q) = F_1(y,\theta z+q)$.

Finally, if for some $\lambda \in \mathbb{R}$, one can find $w \in C(\mathbb{R}^N)$, periodic, viscosity subsolution (resp. supersolution) of (16) then $\lambda \geqslant \lambda(F)$ (resp. $\lambda \leqslant \lambda(F)$).

Remarks : i) We deduce in particular from (18) that if $F_1 \leqslant F_2$ then $\lambda(F_1) \leqslant \lambda(F_2)$.

ii) If H(x,p) is even in p , then $F_1(y,q) = H(y,p+q)$ satisfies $F_1(y,p) = F_1(y,-q-z)$ with $z = 2p$, and thus $\overline{H}(p)$ is even.

iii) If H(x,p) is convex in p , then (20) implies that $\overline{H}(p)$ is convex.

Proof of Proposition 2 : We already proved (15), while (17) is obvious, and (19) is easily deduced from the use of $\hat{v}(x) = v(-x)$. To prove (18), we observe that if v_1, v_2 are the viscosity solutions of

$$F_i(y,Dv_i^\alpha) + \alpha v_i^\alpha = 0 \qquad \text{in } \mathbb{R}^N$$

then $-\alpha v_i^\alpha \leqslant \sup_y F_i(y,0)$ and thus $\|Dv_i^\alpha\|_\infty \leqslant R$. Next, we know from [4], that

$$(v_1^\alpha - v_2^\alpha)^- \leqslant \frac{1}{\alpha} \sup \{(F_1(y,q)-F_2(y,q))^+ \ / \ y \in \mathbb{R}^N , |q| \leqslant R\} .$$

And we obtain (18) since $-\alpha v_i^\alpha$ converges to $\lambda_i(F)$ as α goes to 0 .

Next, the property of $\lambda(F)$ with viscosity subsolutions or supersolutions of (16) is proved exactly as we proved the uniqueness of λ . And (20) is deduced from this property since

$$\begin{aligned} F(y , \theta Dv_2 + (1-\theta)Dv_1) &= F_1(y , \theta z + \theta Dv_2 + (1-\theta)Dv_1) \\ &\leqslant \theta F_1(y , z+Dv_2) + (1-\theta)F_1(y , Dv_1) \\ &= \theta F_2(y , Dv_2) + (1-\theta)F_1(y , Dv_1) \\ &= \theta\lambda(F_2) + (1-\theta)\,\lambda(F_1) \end{aligned}$$

where v_1, v_2 solve (16) for F_1, F_2 .

Let us now review what we know on the effective Hamiltonian $\overline{H}(p)$ for the two examples mentionned in the Introduction.

Example 2: We consider $H(x,p) = \sum_{i,j=1}^{N} a_{ij}(x)\, p_i p_j$ where $a_{ij}(x) = a_{ji}(x)$ is continuous, periodic and

$$\mu\, I_N \geqslant (a_{ij}(x)) \geqslant \nu\, I_N \qquad \text{on } \mathbb{R}^N$$

for some $\mu, \nu > 0$. Then, by the remarks above, $\overline{H}$ is convex, even, homogeneous of degree 2 (use (17)) and

$$\mu|p|^2 \geqslant \overline{H}(p) \geqslant \nu|p|^2 \quad .$$

We do not know if this is a characterization of $\overline{H}$.

Example 1: We consider $H(x,p) = |p|^2 - V(x)$ where V is continuous periodic. Then, we know that $\overline{H}$ is convex, even and

$$|p|^2 - \max V \leqslant \overline{H}(p) \leqslant |p|^2 - \min V \quad , \quad \forall\, p \in \mathbb{R}^N \quad .$$

Recall that in section I.1, we gave explicit formula when $N = 1$ and we observed that

$$\overline{H}(p) = -\min V \quad \text{if} \quad |p| \leqslant \langle (V - \min V)^{1/2} \rangle \quad .$$

We want now to discuss a similar property of $\overline{H}$ if $N \geqslant 2$. Without loss of generality (adding constants), we may normalize V in such a way that $\min V = 0$.

Then, we claim that if V satisfies

(21) $$V_i(x_i) = \text{Min}\,\{V(x_1,\dots,x_N) \,/\, x_j \in [0,1],\ j\neq i\} \not\equiv 0 \quad \text{on } [0,1], \quad \forall\, 1 \leq i \leq N$$

then $\bar{H}$ vanishes in a neighborhood of 0. Indeed, we have $V \geq \sum_i V_i(x_i)$ and it is then easy to check that $\bar{H}(p) \leq \sum_i \bar{H}_i(p_i)$ (use (18)), where $\bar{H}_i$ is the effective Hamiltonian corresponding to $|p_i|^2 - V_i(x_i)$. Since $\bar{H}_i$ vanishes for $|p_i| \leq \langle V_i^{1/2}\rangle$, we deduce

(22) $$\bar{H}(p) = 0 \quad \text{if } |p_i| \leq \langle V_i^{1/2}\rangle \;, \quad \forall\, 1 \leq i \leq N .$$

On the other hand, if V satisfies

(23) $$\exists\, \xi \in BV(0,1;\mathbb{R}^N) \cap C([0,1];\mathbb{R}^N) \;, \quad \xi(1)-\xi(0) \in \mathbb{Z}^N - \{0\} \;, \quad V(\xi(t)) = 0 \quad \text{for } t \in [0,1]$$

then $\bar{H}(p)$ does not vanish in a neighborhood of the origin. We argue by contradiction and thus we assume that there exists $v \in W^{1,\infty}(\mathbb{R}^N)$ viscosity solution of

$$|p+Dv|^2 = V \quad \text{in } \mathbb{R}^N \;, \quad v \text{ periodic}$$

for some $p \in \mathbb{R}^N$ such that $(p\,,\xi(1)-\xi(0)) > 0$. Formally, we observe that on the path $\xi(t)$ we have

$$Dv(\xi(t)) = -p$$

and thus $0 = V(\xi(1)) - V(\xi(0)) = -(p\,,\xi(1)-\xi(0)) \neq 0$! Since v is not C^1, this computation has to be justified as in P.L. Lions [12] : by convolution, we may find $v_\varepsilon \in C^1(\mathbb{R}^N)$ such that

$$|p+Dv_\varepsilon|^2 \leq V + \varepsilon \quad \text{in } \mathbb{R}^N \;, \quad \|v_\varepsilon - v\|_\infty \leq \varepsilon .$$

Then, the argument above yields

$$(p\,,\,\xi(1)-\xi(0)) \leqslant \sqrt{\varepsilon}\int_0^1 |d\xi|$$

and letting ε go to 0 we conclude.

Let us conclude by pointing out the consequences of $\overline{H}$ vanishing say in a ball $\overline{B}_r$: let $u_o \in UC(\mathbb{R}^N)$ satisfy $\|Du_o\|_\infty \leqslant r$ then the solution $u^\varepsilon(x,t)$ of (1)-(2) converges as ε goes to $u_o(x)$ for all $t \geqslant 0$!

III . Proof of the convergence.

III.1 Convergence of the semi-groups.

In this section, we prove the convergence result in Theorem 1, assuming H smooth (at least locally Lipschitz in p, uniformly in x). We recall from [6], [10], [11] that for each $u_o \in UC(\mathbb{R}^N)$, there exists a unique viscosity solution u^ε of (1)-(2) in $C(\mathbb{R}^N \times [0,\infty[)$, uniformly continuous in x uniformly for $t \in [0,T]$ ($\forall\, T < \infty$). The unique solution thus yields a semi-group $S^\varepsilon(t)$ on $UC(\mathbb{R}^N)$ (which maps $BUC(\mathbb{R}^N)$ on $BUC(\mathbb{R}^N)$ ...) and $S^\varepsilon(t)$ is a contraction (in sup norm) semi-group, which is order preserving and commutes with the addition of constants (see [4], [3], [6]).

Next, if $u_o \in C(\mathbb{R}^N)$, $Du_o \in L^\infty(\mathbb{R}^N)$, one knows from [12] that

$$\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|_{L^\infty(\mathbb{R}^N \times (0,\infty))} \leqslant \left\|H\left(\frac{x}{\varepsilon}, Du_o\right)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant C_1 \tag{24}$$

for some constant C_1 independent of ε. Then, we deduce from (1) and (3)

$$\|Du^\varepsilon\|_{L^\infty(\mathbb{R}^N \times (0,\infty))} \leqslant C_2 \tag{25}$$

for some constant C_2 independent of ε.

The last property of S^ε we will be using is the finite speed of propagation of the support (cf. [4]). Let $u_o, v_o \in C(\mathbb{R}^N)$, $Du_o, Dv_o \in L^\infty(\mathbb{R}^N)$. We know from (25) that $Du^\varepsilon, Dv^\varepsilon$ are bounded in $L^\infty(\mathbb{R}^N \times (0,\infty))$ by a constant C_3 independent of ε. Then if we denote by $C_o = \sup\left\{\left|\frac{\partial H}{\partial p}\right| \;/\; y \in \mathbb{R}^N,\ |p| \leqslant C_3\right\}$ we have the following property :

$$\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^\infty(B(x_o, R - C_o t))} \leqslant \|u_o - v_o\|_{L^\infty(B(x_o,R))}$$

for all $t \leqslant R/C_o$, where x_o, R are arbitrary in $\mathbb{R}^N$, $(0,\infty)$.

Using these various properties of $S^{\varepsilon}(t)$, it is an easy exercise to extract a subsequence $\varepsilon_n \underset{n}{\to} 0$ such that $S^{\varepsilon_n}(t)u_o$ converges uniformly to $S(t)u_o$ on compact sets of $\mathbb{R}^N \times [0,\infty[$ to $S(t)u_o$, for all $u_o \in UC(\mathbb{R}^N)$ where $S(t)$ is a semi-group on $UC(\mathbb{R}^N)$ satisfying

$$(26)\qquad \begin{cases} \forall\, u_o \in UC(\mathbb{R}^N)\ , & S(t)u_o(x) \in C(\mathbb{R}^N \times [0,\infty[) \\ \forall\, u_o \in BUC(\mathbb{R}^N)\ , & S(t)u_o(x) \in BUC(\mathbb{R}^N \times [0,T]) \quad (\forall\, T < \infty) \end{cases}$$

$$(27)\qquad \|(S(t)u_o - S(t)v_o)^+\|_\infty \leq \|(u_o - v_o)^+\|_\infty \leq \infty\ , \quad \forall\, t \geq 0\ , \ \forall\, u_o, v_o \in UC(\mathbb{R}^N)$$

and in addition if $u_o, v_o \in C(\mathbb{R}^N)$, $Du_o, Dv_o \in L^\infty(\mathbb{R}^N)$ and if we denote by $u(x,t) = S(t)u_o(x)$, $v(x,t) = S(t)v_o(x)$, there exist constants C, C_o depending only on $\|Du_o\|_\infty$, $\|Dv_o\|_\infty$ such that

$$(28)\qquad \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(\mathbb{R}^N \times (0,\infty))} \leq C \ , \qquad \|Du\|_{L^\infty(\mathbb{R}^N \times (0,\infty))} \leq C$$

$$(29)\qquad \|u(t) - v(t)\|_{L^\infty(B(x_o, R - C_o t))} \leq \|u_o - v_o\|_{L^\infty(B(x_o,R))} \ , \quad \forall\, t \leq R/C_o$$

where $x^o \in \mathbb{R}^N$, $R < \infty$ are arbitrary.

Of course, we have to <u>identify $S(t)$</u> as the semi-group corresponding to the effective Hamiltonian i.e. equation (4). By the simple observation of section I.2, we already know that $S(t)$ is the right semi-group on affine initial conditions i.e.

$$(30)\qquad [S(t)u_o](x) = \alpha + p\cdot x - t\bar{H}(p) \quad \text{if} \quad u_o(x) = \alpha + p\cdot x \ .$$

We are going to show that this, combined with the above properties of $S(t)$, is enough to guarantee that $S(t)$ is indeed the right semi-group. In view of the verification result of P.L. Lions and M. Nisio [16], it is enough to show that for any $x_o \in \mathbb{R}^N$, $\varphi \in C_b^2(\mathbb{R}^N)$ we have

(31) $$\frac{1}{t}\{(S(t)u_o)(x_o) - u_o(x_o)\} \to -\overline{H}(Du_o(x_o)) \quad , \quad \text{as} \quad t \to 0_+$$

uniformly for u_o bounded in $C_b^2(\mathbb{R}^N)$.

To show (31), we introduce $\tilde{u}_o(x) = u_o(x_o) + Du_o(x_o)\cdot(x-x_o)$ and we use (29) to obtain

$$|(S(t)u_o)(x_o) - S(t)\tilde{u}_o(x_o)| \leq \|u_o-\tilde{u}_o\|_{L^\infty(B(x_o,C_ot))}$$

where C_o depends only on $\|Du_o\|_{L^\infty(\mathbb{R}^N)}$. We now use (30) to deduce

$$|S(t)u_o(x_o) - (u_o(x_o) - t\overline{H}(Du_o(x_o)))| \leq Ct^2$$

where C depends only on $\|u_o\|_{C_b^2}$. And this proves (31).

At this stage, we have proved that if $u_o \in UC(\mathbb{R}^N)$ then $u^\varepsilon(x,t)$ converges uniformly on compact sets of $\mathbb{R}^N \times [0,\infty[$ to $u(x,t)$ the viscosity solution of (4)-(2) as ε goes to 0 . In addition the uniform convergence on $B_R \times [0,T]$ $(\forall\, R,T < \infty)$ is uniform on bounded sets of initial conditions $u_o \in BUC(\mathbb{R}^N)$ which have a uniform modulus of continuity. By an easy translation argument, this yields the uniform convergence on $\mathbb{R}^N \times [0,T]$ $(\forall\, T < \infty)$. And Theorem 1 is proved for smooth H .

III.2 Approximation of Hamiltonians.

To conclude the proof of Theorem 1 we are going to deduce the convergence result for a general Hamiltonian from the particular case of a smooth one (that we treated above). To do so, we consider $H_n(x,p)$ converging uniformly to $H(x,p)$ on $\mathbb{R}^N \times \overline{B}_R$ $(\forall\, R < \infty)$, H_n periodic in x , H_n smooth and H_n satisfies (3) uniformly in n . We denote by S_n^ε, S_n the semi-groups corresponding to $H_n(\frac{x}{\varepsilon}, p)$, $\overline{H}_n(p)$ while we still denote by S^ε, S the semi-groups corresponding to $H(\frac{x}{\varepsilon}, p)$, $\overline{H}(p)$. We already know that for any

$u_o \in BUC(\mathbb{R}^N)$, $S_n^\varepsilon(t)u_o$ converges uniformly on $\mathbb{R}^N \times [0,T]$ $(\forall\, T < \infty)$ to $S_n(t)u_o$. Since S_n^ε, S_n, S^ε, S are contraction semi-groups, we only have to consider $u_o \in W^{1,\infty}(\mathbb{R}^N)$. For such an initial condition, we deduce from the fact that H_n satisfies (3) uniformly in n that

$$\| D\,[S_n^\varepsilon(t)u_o]\,(x)\|_{L^\infty(\mathbb{R}^N \times\,]0,\infty[)} \leqslant C_o$$

where C_o does not depend on n,ε . Then, we obtain from [4]

$$\| S_n^\varepsilon(t)u_o - S^\varepsilon(t)u_o \|_{L^\infty(\mathbb{R}^N)} \leqslant t \sup_{\substack{x \in \mathbb{R}^N \\ |p| \leqslant C_o}} |H_n(x,p) - H(x,p)|$$

for all $t \geqslant 0$. Using Proposition 2, we remark that $\overline{H}_n(p)$ converges uniformly on compact sets to $\overline{H}(p)$ as n goes to ∞ and thus a similar estimate holds for the difference between $S_n(t)$ and $S(t)$. We may now easily conclude.

IV . Calculus of Variations.

IV.1 Relations with Hamilton-Jacobi semi-groups.

Let $L(x,p) \in BUC(\mathbb{R}^N \times B_R)$ $(\forall\, R < \infty)$ be convex in p. We consider the following classical problem in the Calculus of Variations : let $x,y \in \mathbb{R}^N$, $t > 0$, we set

$$S(x,y,t) = \operatorname{Inf}\left\{\int_0^t L(\xi(s),\dot{\xi}(s))\,ds \;/\; \xi \in W^{1,\infty}(0,t;\mathbb{R}^N),\ \xi(0) = y\ ,\ \xi(t) = x\right\} \tag{32}$$

Clearly, $S(x,y,t)$ is finite if (for example)

$$L(x,p) \geq -C + \nabla W(x)\cdot p \tag{33}$$

for some $C > 0$, $W \in C^1(\mathbb{R}^N)$.

Observe also that if $L(x,p) \equiv L(p)$ then

$$S(x,y,t) = t\,L\left(\frac{x-y}{t}\right) \tag{34}$$

and the infimum is achieved for $\xi(s) = y + s(x-y)/t$.

As in P.L. Lions [12], we need to investigate the continuity of S in (x,y,t).

Lemma 3 : We assume (33) and

$$\frac{\partial L}{\partial p}(x,p)\cdot p \leq C_1 L(x,p) + C_2 \tag{35}$$

for some positive constants $C_1, C_2 \geq 1$. Then, for any $\delta > 0$, $S(x,y,t) \in W^{1,\infty}(\Delta_\delta \times (\delta, 1/\delta))$ where $\Delta_\delta \in \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \;/\; |x-y| \leq 1/\delta\}$ and the $W^{1,\infty}$ bound depends only on δ, C, C_1, C_2 in (33),(35) and the bounds of $L(x,p)$ on $\mathbb{R}^N \times B_R$ for $R < \infty$.

Proof : For any path ξ admissible for (x,y,t), we set

$$\tilde{\xi}(s) = \xi(s) \quad \text{for} \quad 0 \leqslant s \leqslant t \quad , \quad = x + (s-t)\frac{h}{|h|} \quad \text{for} \quad t \leqslant s \leqslant t+|h|$$

where $h \in \mathbb{R}^N$. This choice yields

(36) $$S(x+h,y,t+|h|) \leqslant S(x,y,t) + C|h| \quad , \quad \forall\, x,y,h \in \mathbb{R}^N \, , \quad \forall\, t > 0 \, .$$

Similarly, considering for any path ξ admissible for $S(x,y,t)$ the path $\hat{\xi}(s) = \xi(s)$ for $0 \leqslant s \leqslant t$, $\hat{\xi}(s) = x$ for $t \leqslant s \leqslant \hat{t}$, we obtain

(37) $$S(x,y,\hat{t}) \leqslant S(x,y,t) + C(\hat{t}-t) \quad , \quad \forall\, x,y \in \mathbb{R}^N \, , \quad \forall\, \hat{t} \geqslant t > 0 \, .$$

Finally, let $t,h > 0$, we consider for any admissible path ξ for $S(x,y,t,h)$ the path $\bar{\xi}(s) = \xi\left(\frac{t+h}{t}\, s\right)$ for $0 \leqslant s \leqslant t$. And we have

$$S(x,y,t) \leqslant \int_0^t L(\bar{\xi}(s),\dot{\bar{\xi}}(s))ds = \frac{t}{t+h}\int_0^{t+h} L\left(\xi(s)\, , \frac{t+h}{t}\,\dot{\xi}(s)\right)ds$$

Observe next that (35) implies

(35') $$L(x,\lambda p) \leqslant \lambda^{C_1} L(x,p) + \frac{C_2}{C_1}\left(\lambda^{C_1} - 1\right) \quad , \quad \forall\, x,p \in \mathbb{R}^N \, , \forall \quad \lambda \geqslant 1 \, .$$

Therefore, we find

$$S(s,y,t) \leqslant \left(\frac{t+h}{t}\right)^{C_1 - 1} \int_0^{t+h} L(\xi,\dot{\xi})\; ds + C\left[\left(\frac{t+h}{t}\right)^{C_1} - 1\right]$$

or taking the infimum over all ξ

(38) $$S(x,y,t) \leqslant \left(\frac{t+h}{t}\right)^{C_1 - 1} S(x,y,t+h) + C\left(\frac{t+h}{t}\right)^{C_1} - 1$$

for all $x,y \in \mathbb{R}^N$, $t,h > 0$.

To conclude we observe that we have in view of (33)

(39) $$S(x,x,t) \geqslant -C\,t \quad , \quad S(x,x,t) \leqslant L(x,0)t$$

while clearly

(40) $$S(x,y,t) \leq C_R \qquad \text{if } |x-y| \leq R \ .$$

The combination of (36)-(40) yields Lemma 3. ∎

Let us also observe the following easy property (equivalent to the optimality principle of dynamic programming in Optimal Control theory)

(41) $$S(x,y,t+s) = \inf_{z \in \mathbb{R}^N} \{S(x,z,t) + S(z,y,s)\}$$

for all $x,y \in \mathbb{R}^N$, $t,s > 0$.

We may now recall the relations between $S(x,y,t)$ and Hamilton-Jacobi equations : <u>we will assume in all that follows</u> (35) <u>and</u>

(42) $$L(x,p)|p|^{-1} \to +\infty \quad \text{as} \quad |p| \to \infty \ , \quad \text{uniformly in} \quad x \in \mathbb{R}^N \ .$$

In particular, this implies (33) (with $W \equiv 0$). We denote by $H(x,p)$ the dual convex function (in p) of $L(x,\cdot)$ i.e.

(43) $$H(x,p) = \sup_{q \in \mathbb{R}^N} [\, p\cdot q - L(x,p)\,] \ ,$$

so that L is the dual convex function of H and $H \in BUC(\mathbb{R}^N \times B_R)$ for all $R < \infty$, H satisfies (42) and thus (3). We denote by $S(t)$ the viscosity semi-group (on $UC(\mathbb{R}^N)$, or on $BUC(\mathbb{R}^N)$) corresponding to the Hamiltonian-Jacobi equation

(44) $$\frac{\partial u}{\partial t} + H(x,Du) = 0 \qquad \text{in } \mathbb{R}^N \times (0,\infty) \ .$$

With these assumptions and notations, we deduce from Lemma 2 and from the results and methods of P.L. Lions [12] the following properties : for any fixed $y \in \mathbb{R}^N$, $S(x,y,t)$ is a viscosity solution of (44) which satisfies

(45) $$S(x,x,t) \to 0 \ \text{as} \ t \to 0_+ \ , \quad S(x,y,t) \to +\infty \ \text{as} \ t \to 0_+ \ ,$$
$$\text{uniformly for } |y-x| \geq \delta > 0$$

In addition, for any $u_o \in UC(\mathbb{R}^N)$, we have

(46) $$[S(t)u_o](x) = \inf_{y \in \mathbb{R}^N} \{u_o(y) + S(x,y,t)\} , \quad \forall\, t > 0 , \forall\, x \in \mathbb{R}^N .$$

Remarks : i) It is possible to relax (35), however since we need below the particular dependence on Lipschitz bounds of S we have in Lemma 3, we will skip such extensions.

ii) Let us mention that for fixed $x \in \mathbb{R}^N$, $S(x,y,t)$ is a viscosity solution of

(44') $$\frac{\partial u}{\partial t} + H(y,-Du) = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty) .$$ ■

IV.2 Convergence result.

We now turn to the homogeneization problems we are interested in : we thus assume that L (and thus H) is periodic in x and we introduce $S^\varepsilon(x,y,t)$ which corresponds (as in (32)) to the Lagrangian $L\left(\frac{x}{\varepsilon}, p\right)$. We thus deduce from (46), for any $u_o \in UC(\mathbb{R}^N)$

(47) $$[S^\varepsilon(t)u_o](x) = \inf_{y \in \mathbb{R}^N} \{u_o(y) + S^\varepsilon(x,y,t)\} , \quad \forall\, t > 0 , \forall\, x \in \mathbb{R}^N .$$

Furthermore, Lemma 3 (and its proof) implies that $S^\varepsilon(x,y,t)$ is bounded in $W^{1,\infty}(\Lambda_\delta \times (\delta,1/\delta))$ independently of ε (for all $\delta > 0$). Then, we deduce easily from Theorem 1 the :

Corollary 4 : Let $L(x,p) \in C(\mathbb{R}^N \times \mathbb{R}^N)$ be convex in p , periodic in x . We assume that L satisfies (35) and (42). Then, $S^\varepsilon(x,y,t)$ converges uniformly on $\bar{\Delta}_\delta \times [\delta, \frac{1}{\delta}]$ $(\forall\, \delta > 0)$ to $\bar{S}(x,y,t) = t\, \bar{L}\left(\frac{x-y}{t}\right)$ as ε goes to 0 , where $\bar{L}$ is the dual convex function of $\bar{H}$.

We next conclude this section by giving another proof of the convergence result in Theorem 1 or above, using only the observation in section I.2 that is

(48) $$\inf_y \{p\cdot y + S^{\varepsilon}(x,y,t)\} \to p\cdot x - t\,\overline{H}(p) \quad , \quad \text{as} \quad \varepsilon \to 0$$

uniformly in $x \in \mathbb{R}^N$, $t \geq 0$. By the bounds proved in Lemma 3, we may find a sequence $\varepsilon_n \underset{n}{\to} 0$ such that

$$S^{\varepsilon_n}(x,y,t) \underset{n}{\to} \overline{S}(x,y,t)$$

uniformly on compact subsets of $\mathbb{R}^N \times \mathbb{R}^N \times]0,\infty[$.

First of all, considering the convex functions $\tilde{H}(p) = \max_{x \in \Pi} H(x,p)$, $\tilde{L}(p) = (\tilde{H})^{\star}(p)$, we remark that

(49) $$S^{\varepsilon}(x,y,t) \geq t\,\tilde{L}\left(\frac{x-y}{t}\right) \quad , \quad \forall\, x,y \in \mathbb{R}^N , \quad \forall\, t > 0 .$$

Since $\tilde{L}(p)|p|^{-1} \to +\infty$ as $|p| \to \infty$, we deduce from (48) and (41) the following relations

(50) $$p\cdot x - t\,\overline{H}(p) = \inf_y \{p\cdot y + \overline{S}(x,y,t)\} \quad , \quad \forall\, x,p \in \mathbb{R}^N , \forall\, t > 0$$

(51) $$\overline{S}(x,y,t+s) = \inf_z \{\overline{S}(x,z,t)+\overline{S}(z,y,s)\} \, , \quad \forall\, x,y \in \mathbb{R}^N , \forall\, t,s > 0 .$$

We now introduce the following quantities (closely related to Γ -limits in De Giorgi sense [9]) : let $t > 0$, $\xi \in C([0,t];\mathbb{R}^N)$, we set

$$E(t,\xi) = \operatorname{Inf}\left\{ \underline{\lim}_n \int_0^t L\left(\frac{\xi_n}{\varepsilon_n}, \dot{\xi}_n\right) ds \ / \ \xi_n \in W^{1,\infty}(0,t;\mathbb{R}^N) , \ \xi_n \underset{n}{\to} \xi \ \text{uniformly on } [0,t] , \ \varepsilon_n \underset{n}{\to} 0 \right\}$$

Because of (42), we see that if $\tilde{L}(\dot{\xi}) \notin L^1(0,t)$ (in particular if $\xi \notin W^{1,1}(0,t;\mathbb{R}^N)$) then $E(t,\xi) = +\infty$.

Again because of (42), one shows easily that

(52) $$\overline{S}(x,y,t) = \operatorname{Inf}\left\{E(t,\xi) \ / \ \xi(0) = y \ , \ \xi(t) = x \ , \ \xi \in C([0,t];\mathbb{R}^N)\right\}$$

for all $x,y \in \mathbb{R}^N$, $t > 0$. Notice that the infimum is in fact restricted to $\xi \in W^{1,1}(0,t;\mathbb{R}^N)$ and that $E(t,\cdot)$ is lower semicontinuous for the uniform convergence $(\forall\, t > 0)$.

Using easy translations arguments (similar to those we did several times) we observe that

(53) $$E(t,\tilde{\xi}) = E(t,\xi) \quad \text{if } \tilde{\xi}(\cdot) = \xi(\cdot)+z \text{ , for some } z \in \mathbb{R}^N$$

(54) $$E(t,\xi) = E(s,\xi_1) + E(t-s,\xi_2) \quad \text{if } 0<s<t \text{ , } \xi_1 = \xi|_{[0,s]}, \ \xi_2 = \xi|_{[s,t]}.$$

Notice that (53) implies obviously

(55) $$\bar{S}(x,y,t) = \bar{S}(x-y,0,t) \quad , \quad \forall\, x,y \in \mathbb{R}^N \text{ , } \forall\, t > 0 \text{ .}$$

And we denote by $\bar{S}(x,t) = \bar{S}(x,0,t)$.

We are going to prove now that the infimum in (52) is achieved when ξ is the straight line, that is

(56) $$\bar{S}(\bar{x},\bar{t}) = E(\bar{t},\xi_o) \quad \text{where } \xi_o(t) = \frac{t}{\bar{t}}\bar{x} \text{ , } \forall\, \bar{x} \in \mathbb{R}^N \text{ , } \forall\, \bar{t} > 0 \text{ .}$$

To prove this claim, we consider any minimizing ξ such that $E(\bar{t},\xi) < \infty$ and thus $\xi \in W^{1,1}(0,\bar{t}\,;\mathbb{R}^N)$. And we introduce a family of transformations τ_F on such ξ defined through sets F composed of an integer $m \geq 1$, a partition of $[0,\bar{t}\,]$, $0 < t_1 < \dots < t_m = \bar{t}$, and a permutation σ of $\{1,\dots,m\}$. Then, $\tilde{\xi} = \tau_F\xi$ is defined as follows (where $t_o = 0$)

$$\tilde{\xi}(s) = \xi(t_{\sigma(1)-1}+s) - \xi(t_{\sigma(1)-1}) \quad , \quad \text{if } 0 \leq s \leq t_{\sigma(1)}-t_{\sigma(1)-1} =: \tilde{t}_1$$

$$\tilde{\xi}(s) = \xi(t_{\sigma(2)-1}+s-\tilde{t}_1) - \xi(t_{\sigma(2)-1}) + \tilde{\xi}(\tilde{t}_1) \text{ , } \quad \text{if } \tilde{t}_1 \leq s \leq \tilde{t}_1+t_{\sigma(2)}-t_{\sigma(2)-1} =: \tilde{t}_2$$

.....

$$\tilde{\xi}(s) = \xi(t_{\sigma(m)-1}+s-\tilde{t}_{m-1}) - \xi(t_{\sigma(m)-1}) + \tilde{\xi}(\tilde{t}_{m-1}) \text{ , } \quad \text{if } \tilde{t}_{m-1} \leq s \leq \tilde{t}_m = \bar{t} \text{ .}$$

Observe that we still have $\tilde{\xi}(0) = 0$, $\tilde{\xi}(\bar{t}) = \bar{x}$ and that (53),(54) yield

$$E(\bar{t},\tau_F\xi) = E(\bar{t},\xi) \ .$$

We now leave to the reader (as an exercize !) to prove that it is possible to find a sequence of transformations τ_{F^n} such that the resulting path ξ_n converges uniformly to the straight line $\xi_o(t) = \frac{t}{\bar{t}}\bar{x}$. Using the lower semicontinuity, we conclude.

We next claim that we have

(57) $$\bar{S}(x,t) = \lambda\bar{S}\left(\frac{x}{\lambda},\frac{t}{\lambda}\right) \quad , \quad \forall\, x \in \mathbb{R}^N \ , \ \forall\, t,\lambda > 0 \ .$$

Indeed, we first remark that the combination of (53),(54),(56) yields

$$\bar{S}(x,t) = m\bar{S}\left(\frac{x}{m},\frac{t}{m}\right) \quad , \quad \forall\, x \in \mathbb{R}^N \ , \ \forall\, t > 0 \ , \ \forall\, m \geq 1$$

and thus (57) holds with $\lambda \in \mathbb{Q}$ and by density (57) is proved.

Denoting by $\bar{S}(x) = \bar{S}(x,1)$, we deduce from (57)

(57') $$\bar{S}(x,y;t) = t\ \bar{S}((x-y)/t) \quad , \quad \forall\, x,y \in \mathbb{R}^N \ , \ \forall\, t > 0 \ .$$

In order to conclude, we just need to prove that $\bar{S}(x)$ is convex. Indeed, (50) implies that $(\bar{S})^* = \bar{H}$ and thus if $\bar{S}$ is convex then $\bar{S} = (\bar{H})^* = \bar{L}$ and we conclude. The convexity of $\bar{S}$ follows from (51) since (51) yields for $x,y \in \mathbb{R}^N$, $\theta \in (0,1)$

$$\begin{aligned}\bar{S}(\theta x+(1-\theta)y) &= \bar{S}(\theta x+(1-\theta)y,0,1)\\ &\leq \bar{S}(\theta x,(1-\theta)y,\theta) + \bar{S}((1-\theta)y,0,1-\theta)\\ &= \theta\bar{S}(x) + (1-\theta)\bar{S}(y) \ .\end{aligned}$$

V . Extensions to general Hamiltonians.

We now consider a more general Hamiltonian $H(x,y,p) \in BUC(\mathbb{R}^N \times \mathbb{R}^N \times B_R)$ $(\forall \ R < \infty)$, periodic in y , satisfying (as an exercise !)

$$(3') \qquad H(x,y,p) \to +\infty \quad \text{as} \quad |p| \to +\infty \quad , \quad \text{uniformly in} \quad x,y \in \mathbb{R}^N \quad .$$

For any $u_o \in BUC(\mathbb{R}^N)$, we denote by u^ε the unique viscosity solution in $BUC(\mathbb{R}^N \times [0,T])$ $(\forall \ T < \infty)$ of

$$(58) \qquad \frac{\partial u^\varepsilon}{\partial t} + H\left(x, \frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 \qquad \text{in} \ \mathbb{R}^N \times (0,\infty)$$

$$(59) \qquad u^\varepsilon|_{t=0} = u_o \qquad \text{in} \ \mathbb{R}^N \quad .$$

Then, with the notations of section II, we consider $\overline{H}(x,p) =$ $= \lambda(H(x,y,p+q))$. In view of Proposition 2, $\overline{H}(x,p) \in BUC(\mathbb{R}^N \times B_R)$ $(\forall \ R < \infty)$ and $\overline{H}$ satisfies

$$(59) \qquad \overline{H}(x,p) \to +\infty \quad \text{as} \quad |p| \to \infty \ , \quad \text{uniformly in} \ \ x \in \mathbb{R}^N \ .$$

Hence, there exists a unique viscosity solution $\bar{u}$ in $BUC(\mathbb{R}^N \times [0,T])$ $(\forall \ T < \infty)$ of

$$(60) \qquad \frac{\partial \bar{u}}{\partial t} + \overline{H}(x,D\bar{u}) = 0 \qquad \text{in} \ \mathbb{R}^N \times (0,\infty)$$

satisfying the initial condition (2). We then have

Theorem 5 : Let $H(x,y,p) \in BUC(\mathbb{R}^N \times B_R)$ $(\forall \ R < \infty)$, periodic in y , satisfy (3') and let $u_o \in BUC(\mathbb{R}^N)$. Then u^ε converges uniformly on $\mathbb{R}^N \times [0,T]$ $(\forall \ T < \infty)$ to $\bar{u}$.

Remark : In fact, by the methods which follow, we can treat even more general equations of the form

$$(58') \qquad \frac{\partial u^{\varepsilon}}{\partial t} + H\left(x, \frac{x}{\varepsilon}, t, u^{\varepsilon}, Du^{\varepsilon}\right) = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty)$$

where $H(x,y,t,s,p) \in BUC(\mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [-R,R] \times B_R)$ $(\forall R < \infty)$ is periodic in y and satisfies

$$\begin{cases} H(x,y,t,s,p) \to +\infty \text{ as } |p| \to \infty, \text{ uniformly in } x,y \in \mathbb{R}^N, \\ \qquad t \in [0,T], \ s \text{ bounded} \\ \exists \gamma > -\infty, \quad H(x,y,t,s_1,p) - H(x,y,t,s_2,p) \geq \gamma(s_1-s_2) \text{ if } s_1 \geq s_2 \\ \forall R < \infty, \exists C_R \geq 0, \quad H(x,y,t_1,s,p) - H(x,y,t_2,s,p) \leq C_R(t_1-t_2) \\ \qquad \text{if } t_1 \geq t_2, \ |s| \leq R. \end{cases}$$

In this case, the same result as above holds with the effective Hamiltonian $\overline{H}(x,t,s,p)$ given by

$$\overline{H}(x,t,s,p) = \lambda(H(x,y,t,s,p+q)), \quad \forall x,p \in \mathbb{R}^N, \forall t \in [0,T], \forall s \in \mathbb{R}.$$

<u>Proof of Theorem 5</u> : By the same argument as in section III.2, it is enough to prove Theorem 5 when H is smooth i.e. at least locally Lipschitz in p, uniformly in $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Next, estimates like (24),(25) still hold and as in section III.1, we may find a sequence $\varepsilon_n \to_n 0$ such that $S^{\varepsilon_n}(t)u_o$ converges uniformly on compact sets of $\mathbb{R}^N \times [0,\infty[$ to $S(t)u_o$ for all $u_o \in UC(\mathbb{R}^N)$ and $S(t)$ is a semi-group on $UC(\mathbb{R}^N)$, mapping $BUC(\mathbb{R}^N)$ into $BUC(\mathbb{R}^N)$ such that $[S(t)u_o](x) \in C(\mathbb{R}^N \times [0,\infty[)$, $S(t)u_o$ has a uniform modulus of continuity for $t \in [0,T]$ $(\forall T < \infty)$ and $[S(t)u_o](x) \in BUC(\mathbb{R}^N \times [0,T])$ $(\forall T < \infty)$ if $u_o \in BUC(\mathbb{R}^N)$. Furthermore, $S(t)$ preserves order and the finite speed of propagation property (29) holds. Hence, in view of [16], we just have to prove that if $u_o \in C_b^2(\mathbb{R}^N)$ then for all $x_o \in \mathbb{R}^N$

$$\frac{1}{t}\{[S(t)u_o](x_o) - u_o(x_o)\} \to -H(x_o, Du_o(x_o)) \quad \text{as } t \to 0_+$$

uniformly if u_o belongs to a bounded set of $C_b^2(\mathbb{R}^N)$.

To this end, we consider the Hamiltonians $H(x_o,y,p)$ and $\overline{H}(x_o,p)$.

We denote by $S_o^\varepsilon(t)$, $S_o(t)$ the viscosity semi-groups corresponding to $H\left(x_o, \frac{x}{\varepsilon}, p\right)$, $\overline{H}(x_o,p)$. By Theorem 1, we know that $S_o^\varepsilon(t)u_o$ converges uniformly on $\mathbb{R}^N \times [0,T]$ $(\forall\, T < \infty)$ to $S_o(t)u_o$. Furthermore, we also deduce from the finite speed of propagation of supports (see [4]) that if $u_o \in C(\mathbb{R}^N)$, $Du_o \in L^\infty(\mathbb{R}^N)$, then

$$|[S^\varepsilon(t)v_o](x_o) - [S_o^\varepsilon(t)v_o](x_o)| \leqslant$$

$$\leqslant t \sup \{|H(x,y,p) + H(x_o,y,p)| \,/\, |x-x_o| \leqslant Ct \,,\; y \in \mathbb{R}^N ,\; |p| \leqslant C\}$$

for some C depending only on $\|Dv_o\|_\infty$ and H.

Sending ε to 0, we deduce

$$\left|\frac{1}{t}\{[S(t)u_o](x_o) - u_o(x_o)\} - \frac{1}{t}\{[S_o(t)u_o](x_o) - u_o(x_o)\}\right|$$

$$\leqslant \varepsilon(t) \to 0 \qquad \text{as} \quad t \to 0_+$$

where $\varepsilon(\cdot)$ depends only on $\|Du_o\|_\infty$. But now, we do have for $S_o(t)$ the property we wish to prove and this enables us to conclude.