HOMOGENIZATION OF INTERFACES MOVING IN SPATIALLY RANDOM TEMPORALLY PERIODIC ENVIRONMENT

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Abstract. We study the averaged behavior of interfaces moving with oscillatory normal velocity that is periodic in time and stationary ergodic in space. This problem can be interpreted as a homogenization problem of a Hamilton-Jacobi equation with a positively homogeneous and time dependent Hamiltonian. In an earlier work, we studied the setting when the environment is periodic in space and random in time, and established homogenization results through the averaging of reachable sets. In the present setting, we show that the minimal travel time between two spatial points, which now depends also on a starting time, has a deterministic averaging limit that is independent of the starting time. This averaged travel time function is then shown to determine the effective behavior of the moving interfaces.

Keywords: periodic and stochastic homogenization, minimal travel time, Hamilton-Jacobi equations, viscosity solutions, dynamic environment, level set method, front propagation.

AMS Classification: 35B27 70H20 49L25

1. Introduction

We study the homogenized (averaged) behavior of the solution $u^\varepsilon = u^\varepsilon(x,t,\omega)$ of the level-set equation

$$\begin{cases} \partial_t u^\varepsilon + a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right)|Du^\varepsilon| = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon = u_0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.1}$$

This equation describes fronts propagating with normal velocity $a(x,t,\omega)$, which is modeled as a random process in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is a Hamilton-Jacobi equation with Hamiltonian $H(x,t,p,\omega) := a(x,t,\omega)|p|$. The process $a(x,t,\omega)$ satisfies $\alpha \le \alpha(x,t,\omega) \le \beta$ for some positive real numbers α and β, for all realizations ω, and hence is $H(x,t,p,\omega)$ coercive and grows linearly in $|p|$. In this paper, $a(x,t,\omega)$ is assumed to be stationary ergodic in x, deterministic and periodic in t. The precise assumptions on a will be made clear later.

The goal of the homogenization problem is to find a deterministic Hamiltonian $\overline{H} : \mathbb{R}^n \to \mathbb{R}$ such that, almost surely in Ω, the unique solution u^ε of the problem above converges, locally uniformly in space and time, to the solution u of the effective equation

$$\begin{cases} \partial_t u + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = u_0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.2}$$

Date: March 15, 2016.

Moreover, in view of the positive homogeneity in p of the original Hamiltonian $H(x,t,p,\omega)$, we expect that the effective Hamiltonian is of the form

$$\overline{H}(p) = \overline{a}\left(\frac{p}{|p|}\right)|p| \quad \text{if } p \neq 0, \quad \text{and} \quad \overline{H}(0) = 0.$$

The main objective of this paper is to establish these results.

The stochastic homogenization for Hamilton-Jacobi equations with convex and coercive Hamiltonians was established independently by Souganidis [20] and Rezakhanlou and Tarver [17]. Results for viscous Hamilton-Jacobi equations were obtained by Lions and Souganidis [14] and Kosygina, Rezakhanlou and Varadhan [12], while problems with space-time oscillations were considered by Kosygina and Varadhan [13] and Schwab [19]. In [15] Lions and Souganidis gave a simpler proof for homogenization in probability using weak convergence techniques. Their program was extended by Armstrong and Souganidis in [1, 2], using the so-called the metric approach. The viscous case was later refined by Armstrong and Tran in [3]. Recently, Armstrong, Tran and Yu established stochastic homogenization for some specific non convex Hamiltonians in [4]. In those works, the Hamiltonian grows superlinearly in the momentum variable. The situation that H grows linearly in p appears in the G-equation which has an additional key difficulty of H being not coercive. The homogenization of the G-equation was established by Cardaliaguet, Nolen and Souganidis [7] in spatial periodic environments, and Xin and Yu [21] for space periodic incompressible flows independently. A specific random case in 2-dimensional space was obtained by Nolen and Novikov [16]. Cardaliaguet and Souganidis in [9] obtained the general homogenization result in random media, where highly nontrivial techniques were needed to develop certain reachability estimates. Linear growing and non-coercive Hamiltonian appears also if one allows a to change signs in (1.1). This was studied by Cardaliaguet, Lions and Souganidis [6] for periodic $a = a(x)$, and recently by Ciomaga, Souganidis and Tran [10] in the random setting, where they modified the metric approach using domain decompositions and interpolation operators.

In spite of its simple form, (1.1) falls outside of the scope of the available theory of homogenization of Hamilton-Jacobi equations listed above, due to the simultaneous occurrence of the temporal oscillations in the environment and the weak (not super-linear) growth of the Hamiltonian in $|p|$. On the one hand, if the environment is time independent, then uniform in ε and ω Lipschitz estimates for equations like (1.1) are available. On the other hand, if the Hamiltonian has super-linear growth in p, albeit having time oscillations, then uniform Hölder estimates are available [5, 8]. Such uniform in ε and ω moduli of continuity, which is very helpful for the homogenization theory [2, 19], is not available for the setting of (1.1). In [11] the authors established the first homogenization result in this setting, with the assumption that the environment is periodic in space and stationary ergodic in time. The main observation in that paper was that the homogenization result follows from the large time average of the reachable sets, that is the limit, as $t \to \infty$, of the set

$$\frac{\mathcal{R}_t(x,0)}{t}, \tag{1.3}$$

where $\mathcal{R}_t(x,s)$, $t \geq s$, is the set of points that can be reached from x, starting from time s, at time t by a path whose velocity satisfies a speed constraint along its trajectory. More precisely,

$$\begin{aligned} \mathcal{R}_t(x,s) := \{y \in \mathbb{R}^n \ : \text{there exists } \gamma \in C^{0,1}([s,t],\mathbb{R}^n) \text{ such that, } \gamma(s) = x,\ \gamma(t) = y \\ \text{and } |\gamma'(r)| \leq a(\gamma'(r), r, \omega) \text{ for all } r \in [s,t]\}. \end{aligned} \tag{1.4}$$

Thanks to the spatial periodicity, the reachable set from the unit square $\square = [0,1]^n$ is almost subadditive in the following sense

$$\mathcal{R}_m(\square,\omega) \subseteq \mathcal{R}_k(\square,\omega) + \mathcal{R}_{m-k}(\square,\tau_k\omega) + (-\square). \tag{1.5}$$

where the additions above are in the Minkowski sense, and $-\square$ denotes the reflection of the unit square with respect to the origin. A convex set valued subadditive ergodic theorem is then applied to show that the large time average of the convex hull of the reachable set converges to a deterministic convex compact set. The heart of the paper is then to show that this set is also the large time average of reachable set, without the convexification.

In this paper, we consider the somewhat complementary setting, where the environment is assumed to be periodic in time although its spatial dependence only needs be stationary ergodic. If the spatial dimension $n = 1$, the two settings coincide because, in view of $a > 0$ and the underlying optimal control problem (see e.g. [11]), the roles of space and time in (1.1) can be interchanged. When $n \geq 2$, the two settings are in general very different. Indeed, the subadditivity of the reachable set in (1.5) does not hold. Therefore, a different subadditive quantity, if it exists, should be devised for the purpose of homogenization. It turns out that such a quantity can be found by considering the minimal travel time function between two spatial points.

Let $x, y \in \mathbb{R}^n$, the minimal traveling time from x to y should be the smallest $T > 0$ so that y can be reached from x. In time dependent environment, this traveling time also depends on the starting time at x. We hence define

$$\begin{aligned} m(x,t,y) &:= \inf\{T \geq 0 \,:\, x \in \mathcal{R}_t(y, t-T)\} \\ &= \inf\{T \geq 0 \,:\, \exists \gamma \in C^{0,1}([0,T],\mathbb{R}^n), \gamma(0) = x, \gamma(T) = y, |\gamma'(r)| \leq a(\gamma(r), t-r,\omega)\}. \end{aligned} \tag{1.6}$$

The reversed direction of time in the above definition seems awkward, but it is well known to be natural if we start with an initial value problem like (1.1). The function m, as we see below, can also be interpreted as the time dependent analog of the metric problem in [2].

The function $m(x,t,y,\omega)$ can be defined, of course, in the general setting. However, its behavior is much more complicated than its time independent version because, again, $m(\cdot,\cdot,y,\omega)$ satisfies a Hamilton-Jacobi equation in time dependent environment, and there is no uniform control of $|Dm|$ in general. However, in the time periodic setting, due to the compactness in t, much more can be said. We define

$$\theta(x,y,\omega) := \inf_{t\in[0,1]} m(x,t,y,\omega) = \inf_{t\in\mathbb{R}} m(x,t,y,\omega),$$

which is the absolute minimal travel time from x to y, since an additional minimizing in the starting time is considered. The compactness in time further implies uniform control between $m(x,\cdot,y) - \theta(x,y)$, almost subadditivity of $\theta(x,y,\omega)$ and uniform modulus of continuity of θ; see (2.8), (3.3) and (3.4) and below. In particular, subadditive ergodic theorem can be applied to show that, for some positively homogeneous function $\overline{\theta} : \mathbb{R}^n \to \mathbb{R}$, almost surely in Ω,

$$\lim_{t\to\infty} \frac{1}{t}\theta(ty,0,\omega) = \overline{\theta}(y). \tag{1.7}$$

The control between $m - \theta$ and the uniform modulus of continuity of θ further show

$$\lim_{t\to\infty} \frac{m(tx,s,ty,\omega)}{t} = \overline{m}(x-y) := \theta\left(\frac{x-y}{|x-y|}\right)|x-y| \text{ uniformly in } s, \text{ locally uniformly in } x \neq y.$$

The identification of the effective Hamiltonian, and the proof of the homogenization result for (1.1), then, follows essentially from the same argument as in the metric problem approach of [1, 2, 3].

Finally, we remark that the method described above does not apply to the setting of our earlier work [11] either, so the two settings are really complementary to each other. Indeed, when the environment is not periodic in t, we do not have a control of $\theta(x,y,\omega)-m(x,t,y,\omega)$ that is uniform (or at least sub-linear) in $|x-y|$, and the arguments above do not work.

The rest of this paper is organized as follows. In the next section, we specify the general assumptions on the velocity $a(x,t,\omega)$ and state the main theorems of this paper. In Section 3 we study some properties of the minimal travel time functions $m(x,t,y,\omega)$ and $\theta(x,y)$. In Section 4 we show that the large time average of the minimal time function $m(x,t,y,\omega)$ has a deterministic limit that does not depend on the initial time. In Section 4, viewing $m(x,t,y,\omega)$ as the time dependent metric problem, we prove that its large time average identifies the effective Hamiltonian $\overline{H}$ as the conjugate function of ωm.

Notations. We work in the n-dimensional Euclidean space $\mathbb{R}^n$. The open ball in $\mathbb{R}^n$ centered at x with radius $r>0$ is denoted by $B_r(x)$, and this notion is further simplified to B_r if the center is the origin. If $A\subset\mathbb{R}^n$ measurable set, we denote by $|A|$ its Lebesgue measure and call it the volume of A. The space-time cylinder with spatial radius $r>0$, height $h>0$ and center (x_0,t_0) on the top, is denoted by $Q_{r,h}(x_0,t_0)$ and is given by $B_r(x_0)\times(t_0-h,t_0]$; the center (x_0,t_0) is omitted if it is $(0,0)$; when $r=h$, we simplify the notation $Q_{r,h}$ to Q_r. Finally, if Ξ is a metric space, $\mathcal{B}(\Xi)$ denotes the Borel σ-algebra of Ξ

2. Preliminaries and main results

The setting. We consider a probability space $(\Omega,\mathcal{F},\mathbb{P})$ endowed with an ergodic group of measure preserving transformations $(\tau_x)_{x\in\mathbb{R}^n}$, that is, a family of measurable maps $\tau_x:\Omega\to\Omega$ satisfying, for all $x,x'\in\mathbb{R}^n$ and all $\mathcal{U}\in\mathcal{F}$,

$$\tau_{x+x'}=\tau_x\circ\tau_{x'}\quad\text{and}\quad\mathbb{P}[\tau_x\mathcal{U}]=\mathbb{P}[\mathcal{U}]$$

and

$$\text{if } \tau_x(\mathcal{U})=\mathcal{U}\text{ for every }x\in\mathbb{R}^n,\text{ then either }\mathbb{P}[\mathcal{U}]=1\text{ or }\mathbb{P}[\mathcal{U}]=0.$$

We assume that $a:\mathbb{R}^n\times\mathbb{R}\times\Omega\to\mathbb{R}$ is measurable with respect to the σ-algebra generated by $\mathcal{B}\times\mathcal{F}$, where $\mathcal{B}$ denotes the Borel σ-algebra of $\mathbb{R}^{n+1}$ and that

(A1) the function $a=a(x,t,\omega)$ is $\mathbb{Z}$-periodic in t, and stationary in x with respect to $(\tau_x)_{x\in\mathbb{R}^n}$, that is, for every $(x,t)\in\mathbb{R}^n\times\mathbb{R}$, $(z,k)\in\mathbb{R}^n\times\mathbb{Z}$, and $\omega\in\Omega$,

$$a(x+z,t+k,\omega)=a(x,t,\tau_z\omega),$$

(A2) $a(\cdot,\cdot,\omega)\in C^{0,1}(\mathbb{R}^{n+1})$ and there exist $\alpha,\beta>0$ such that for all $(x,t)\in\mathbb{R}^{n+1}$ and $\omega\in\Omega$,

$$\alpha\le a(x,t,\omega)\le\beta. \tag{2.1}$$

For simplicity, we combine all the assumptions into

(A) $a=a(x,t,\omega)$ satisfies (A1) and (A2).

The minimal travel time functions. As defined in the Introduction, given $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}^n$, the minimal travel time function $m(x,t,y,\omega)$ from x to y starting at time t, is given by

$$\inf\{T \geq 0 \,:\, \exists \gamma \in C^1([0,T]) \text{ such that } \gamma(0) = x, \gamma(T) = y, |\gamma'(r)| \leq a(\gamma(r), t-r, \omega)\}. \tag{2.2}$$

For notational simplicity in the sequel, we set $\mathcal{A}^{x,t}$ to be the set of admissible paths starting (going backward in time) from (x,t), that is

$$\mathcal{A}^{x,t} := \{\gamma \in C^1([0,\infty)) \,:\, \text{ such that } \gamma(0) = x, |\gamma'(r)| \leq a(\gamma(r), t-r, \omega)\}. \tag{2.3}$$

We will derived some properties of m below. In particular, $m(x,t,y,\omega)$ is periodic in t. As in the Introduction, minimizing this function in $t \in [0,1]$, we obtain the absolute minimal travel time

$$\theta(x,y,\omega) = \inf_{t\in[0,1]} m(x,t,y,\omega). \tag{2.4}$$

Next, we prove some properties of m and θ.

Proposition 2.1. *Assume (A). Then for every $\omega \in \Omega$, the following statements hold:*

(i) *for every $x, y, z \in \mathbb{R}^n$ and $k \in \mathbb{Z}$,*

$$m(x, t+k, y, \tau_z\omega) = m(x+z, t, y+z, \omega) \tag{2.5}$$

and if α and β denote the lower and upper bounds of a, then

$$\beta^{-1}|x-y| \leq m(x,t,y,\omega) \leq \alpha^{-1}|x-y|. \tag{2.6}$$

(ii) *the oscillations of $m(x,\cdot,y,\omega)$ can be controlled uniformly by*

$$\operatorname*{osc}_{t\in\mathbb{R}} m(x,t;0,\omega) = \sup_{t\in\mathbb{R}} m(x,t,0,\omega) - \inf_{s\in\mathbb{R}} m(x,s,0,\omega) \leq 1 \tag{2.7}$$

(iii) *the difference between m and θ can be controlled uniformly by*

$$\sup_{t\in\mathbb{R}} |m(x,t,y;\omega) - \theta(x,y;\omega)| \leq 1 \tag{2.8}$$

(iv) *the function $m(\cdot,\cdot,y;\omega)$ solves the Hamilton-Jacobi equation*

$$\begin{cases} \partial_t m(x,t;z,\omega) + a(x,t;\omega)|Dm| = 1 & \text{on } \mathbb{R}^n \setminus \{z\} \times \mathbb{R} \\ m(z,t;z) = 0 & \text{on } \{z\} \times \mathbb{R} \end{cases} \tag{2.9}$$

Proof. (i) The relation (2.5) is a direct consequence of the spatial stationarity and the temporal periodicity of the environment. To show (2.6), we observe that, on the one hand, the path $\zeta : s \mapsto x + \alpha s(y-x)/|y-x|$ satisfies $\zeta \in \mathcal{A}^{x,t}$ and $\zeta(\alpha^{-1}|x-y|) = y$. This shows, in view of the definition in (2.2), $m(x,t,y,\omega) \leq \frac{|y-x|}{\alpha}$. On the other hand, if $T = m(x,t,y,\omega)$ is achieved by some differentiable path $\xi \in \mathcal{A}^{x,t}$ satisfying $\xi(T) = y$. Then $|\xi'(r)| \in [a,b]$ and

$$|y-x| = |\xi(T) - \xi(0)| = \left|\int_0^t \xi'(s)ds\right| \leq \int_0^t |\xi'(s)|ds \leq \beta T$$

which shows $m(x,t,y,\omega) \geq \frac{|x-y|}{\beta}$, establishing the inequality (2.6).

(ii) Assume, without loss of generality, that $0 \leq t_1 < t_2 \leq 1$ and $\delta := t_2 - t_1 \in (0,1)$. Set $T_j = m(x,t_j,y,\omega)$, $j = 1,2$. If T_1 is achieved by some $\gamma \in \mathcal{A}^{x,t_1}$ and $\gamma(T_1) = y$, then we define a path as follows

$$\zeta(r) = \begin{cases} x & \text{for } r \in [0,\delta), \\ \gamma(r-\delta) & \text{for } r \in [\delta, \delta + T_1]. \end{cases}$$

We check that $|\zeta'(r)| = 0 \le a(\zeta, t_2 - r, \omega)$ for $r \in [0, \delta)$ and, for $r \in [\delta, \delta + T_2]$,

$$|\zeta'(r)| = |\gamma'(r - \delta)| \le a(\zeta(r), t_1 - r + \delta, \omega) = a(\zeta(r), t_2 - r, \omega).$$

Hence, $\zeta \in \mathcal{A}^{x,t_2}$. Because $\zeta(\delta + T_1) = y$, we get $T_2 \le T_1 + \delta \le T_1 + 1$. To get the lower bound, we note $0 < 1 - \delta = (t_1 + 1) - t_2$, the same argument shows that $T_1 = m(x, t_1 + 1, y, \omega) \le T_2 + 1$.

(iii) The inequality (2.8) is a direct consequence of (ii).

(iv) This result follows from the observation that

$$m(x, t, y, \omega) = \inf_{\gamma \in \mathcal{A}^{x,t}} m(\gamma(s), t - s, y, \omega) + s \qquad \text{for all } s > 0.$$

In other words, m satisfies the dynamic programming principle. The verification of this fact, and the derivation of the Hamilton-Jacobi equation from this principle, are standard and hence omitted. □

Remark 1. The equation satisfied by $m(\cdot, \cdot, y, \omega)$ is the time dependent version of the metric problem in [1]. Since the equation is one homogeneous, the metric problem for m_μ with right hand side $\mu > 0$ is simply given by μm.

Main results. The first main result of this paper states that the large time average of the minimal travel time function has a deterministic limit which does not depend on the starting time.

Theorem 2.2. *Assume (A). There exists a deterministic Lipschitz continuous function* $\overline{\theta} : S^{n-1} \to [\beta^{-1}, \alpha^{-1}]$*, and an event* $\widetilde{\Omega}$ *with full probability measure, such that if* $\overline{m} : \mathbb{R}^n \to \mathbb{R}$ *is defined by*

$$\overline{m}(p) = \begin{cases} \overline{\theta}(\frac{p}{|p|})|p| & \text{if } p \neq 0, \\ 0 & \text{if } p = 0, \end{cases} \tag{2.10}$$

then for any $\omega \in \widetilde{\Omega}$ *and for any* $R > 0$,

$$\lim_{t \to \infty} \sup_{x, y \in B_R} \sup_{s \in [0,1]} \left| \frac{1}{t} m(tx, s, ty, \omega) - \overline{m}(x - y) \right| = 0. \tag{2.11}$$

Note that though the minimal travel time function $m(x, t, y, \omega)$ depends on the initial time, the effective (large scale average) travel time $\overline{m}(x - y)$ does not. As expected, $\overline{m}$ is one homogeneous. $\overline{m}(e) = \overline{\theta}(e)$ can be interpreted as the reciprocal value of the effective speed in direction e. The effective reachable set from the origin at time one is then

$$D := \bigcup_{e \in S^{n-1}} \{re \,:\, r \in [0, (\overline{\theta}(e))^{-1}]\}. \tag{2.12}$$

We will prove that D is a compact and convex set. The effective Lagrangian is then given by

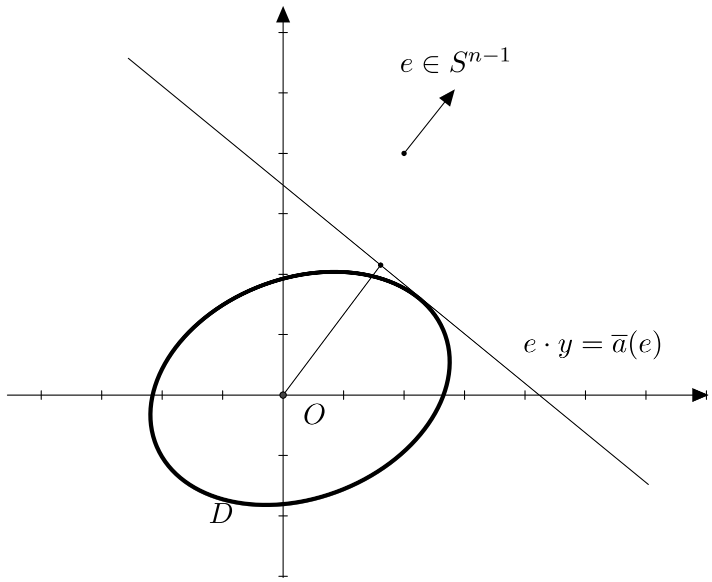
$$\overline{L}(v) = \begin{cases} 0 & \text{if } v \in D, \\ +\infty & \text{if } v \in D^c. \end{cases} \tag{2.13}$$

The effective Hamiltonian $\overline{H}$ is given by the Legendre transform (convex conjugate) of $\overline{L}$

$$\overline{H}(p) = \sup_{v \in D} p \cdot v = \begin{cases} \overline{a}(\frac{p}{|p|})|p| & \text{if } p \neq 0, \\ 0 & \text{if } p = 0. \end{cases} \tag{2.14}$$

The function $\overline{a}(e)$ is defined by $\sup_{v \in D} v \cdot e$; see Fig. 2 for illustration. We will prove in Proposition 3.4 below that, as in the framework of [1, 2, 3], for any $\mu > 0$, the function $\mu\overline{m}$ is the support

Figure 1. The effective unit reachable set and the determination of $\overline{a}(e)$



function of the convex sub-level set $\{\overline{H} \le \mu\}$. We could have defined $\overline{H}$ through this relation without the introduction of D and $\overline{L}$. Nevertheless, we choose to introduce those objects since they shine some light to the underlying picture. Finally, $\overline{H}(p)$ defined above is indeed the effective Hamiltonian. This is stated as follows.

Theorem 2.3. *Assume (A) and let $\widetilde{\Omega}$ and $\overline{H}$ be defined as above. Let $u \in C([0,\infty),\mathbb{R})$ be the unique viscosity solution of the homogenized level set equation (1.2). Then, for every $\omega \in \widetilde{\Omega}$ and for every $T \ge 0, R > 0$, we have*

$$\lim_{\varepsilon \to 0} \sup_{(x,t) \in B_R \times [0,T]} |u^{\varepsilon}(x,t,\omega) - u(x,t)| = 0. \tag{2.15}$$

In view of the Lax-Hoft formula for the first order Hamilton-Jacobi equation (1.2), its solution $u(x,t)$ is given by

$$u(x,t) = \inf \left\{ u_0(y) \,:\, \frac{x-y}{t} \in D \right\} = \inf_{e \in S^{n-1}} \inf_{0 < s \le t/\overline{\theta}(e)} u_0(x - se).$$

The proof of Theorem 2.3 is standard and follows, by the perturbed test function argument of Evans, if one can check that $\overline{H}(p)$ is the locally uniform limit of an approximate cell problem. This convergence is proved in detail in a later section.

3. The large time average of minimal travel time

In this section, we prove Theorem 2.2, and derive some property of the effective travel time function $\overline{\theta}$. To this end, we show that the absolute minimal function $\theta(x,y,\omega)$, defined in (2.4), is almost subadditive.

Proposition 3.1. *Assume (A). Then, for all $x, y, z \in \mathbb{R}^n$ and $\omega \in \Omega$,*

(i) *θ is stationary in the sense that*

$$\theta(x, y, \tau_z \omega) = \theta(x + z, y + z, \omega). \tag{3.1}$$

(ii) *θ satisfies the bounds*

$$\beta^{-1}|x - y| \leq \theta(x, y, \omega) \leq \alpha^{-1}|x - y|. \tag{3.2}$$

(iii) *$\theta + 1$ is subadditive*

$$\theta(x, y; \omega) \leq \theta(x, z; \omega) + \theta(z, y; \omega) + 1. \tag{3.3}$$

(iv) *θ also satisfies*

$$|\theta(x, y, \omega) - \theta(x, y, \tau_z \omega)| \leq C(|z| + 1) \tag{3.4}$$

Proof. For item (i) and (ii), we take the infimum in t on both sides of (2.5) to get (3.1); similarly, we get (3.2) by taking the infimum on (2.6).

(iii) Set $\theta_1 = \theta(x, z; \omega)$; assume it is achieved by $\gamma_1 \in \mathcal{A}^{x,s_1}$ for some $s_1 \in [0, 1]$, and $z = \gamma(\theta_1)$. Set $\theta_2 = \theta(z, y; \omega)$; assume it is achieved by $\gamma_2 \in \mathcal{A}^{z,s_2}$ for some $s_2 \in [0, 1]$, and $y = \gamma_2(\theta_2)$. Moreover, we can find a positive integer $k \in \mathbb{Z}$ so that $s_2 + k < s_1 - \theta_1 \leq s_2 + k + 1$. Note that $\delta := s_1 - \theta_1 - s_2 - k$ is a number in $(0, 1]$. Define the path

$$\gamma(r) = \begin{cases} \gamma_1(r) & r \in [0, \theta_1) \\ z & r \in [\theta_1, \theta_1 + \delta) \\ \gamma_2(r - \theta_1 - \delta) & r \in [\theta_1 + \delta, \theta_1 + \theta_2 + \delta) \end{cases}$$

We check that $\gamma \in \mathcal{A}^{x,s_1}$. Indeed, γ' clearly satisfies the speed constraint for $r \in [0, \theta_1 + \delta)$. On the other hand, if $r \geq \theta_1 + \delta$, we have

$$|\gamma'(r)| = |\gamma_2'(r - \theta_1 - \delta)| \leq a(\gamma(r), s_2 - r + \theta_1 + \delta, \omega) = a(\gamma(r), s_1 - r - k, \omega) = a(\gamma(r), s_1 - r, \omega).$$

Since $\gamma(\theta_1 + \theta_2 + \delta) = y$, we conclude that

$$\theta(x, y, \omega) \leq m(x, s_1, y, \omega) \leq \theta_1 + \theta_2 + \delta \leq \theta_1 + \theta_2 + 1.$$

This establishes (3.3).

(iv) This follows from (3.1), (3.3) and (3.2). We find

$$\begin{aligned} \theta(x, y, \tau_z \omega) - \theta(x, y, \omega) &= \theta(x + z, y + z, \omega) - \theta(x, y, \omega) \\ &\leq \theta(x + z, x, \omega) + \theta(x, y + z, \omega) + 1 - \theta(x, y, \omega) \\ &\leq \theta(x + z, x, \omega) + \theta(y, y + z, \omega) + 2 \leq 2(\alpha^{-1}|z| + 1). \end{aligned}$$

The other direction is obtained by the same calculation applied to $\theta(x, y, \tau_{-z}\omega') - \theta(x, y, \omega')$ where $\omega' = \tau_z \omega$. The proof is hence complete. □

Remark 2. By the proof of part (iv) above, we also verifies that

$$|\theta(x_1, y_1, \omega) - \theta(x_2, y_2, \omega)| \leq C(|x_1 - x_2| + |y_1 - y_2| + 1).$$

If we set $\theta^\varepsilon := \varepsilon\theta(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \omega)$, then we have the uniform in ε continuity in the limit, that is

$$\limsup_{\varepsilon \to 0} |\theta^\varepsilon(x_1, y_1, \omega) - \theta^\varepsilon(x_2, y_2, \omega)| \leq C(|x_1 - x_2| + |y_1 - y_2|). \tag{3.5}$$

Such uniform modulus of continuity in the limit (not necessarily Lipschitz) plays an important role in the proof of Theorem 2.2.

In view of the stationarity and subadditivity of $\theta(x,y,\omega)$, the large scale average $t^{-1}\theta(tx,ty,\omega)$ admits a limit as $t\to\infty$.

Lemma 3.2. *Assume (A). Then for any $e\in S^{n-1}$, there exists a real number $\overline{\theta}(e)\in\mathbb{R}$ and an event Ω_e with full probability measure, such that, for all $\omega\in\Omega_e$,*

$$\lim_{t\to\infty}\frac{1}{t}\theta(te,0;\omega)=\overline{\theta}(e). \tag{3.6}$$

In addition, $\overline{\theta}\in[\beta^{-1},\alpha^{-1}]$ on S^{n-1}.

Proof. For each e, the random mapping $[a,b)\mapsto\theta(be,ae;\omega)+1$, where $0\le a<b<\infty$, is stationary and subadditive with respect the translation group $\{\tau_{ce}\}_{c\in\mathbb{R}_+}$. Moreover, in view of (3.2), $|[a,b)|^{-1}(\theta(be,ae,\omega)+1)$ is bounded from below. Hence, the hypothesis of the subadditive ergodic theorem are verified, and we can find $\Omega_e\in\mathcal{F}$, $\mathbb{P}(\Omega_e)=1$ and for every $\omega\in\Omega_e$,

$$\frac{1}{t}\theta(te,0;\omega)=\overline{\theta}(e,\omega). \tag{3.7}$$

We may enlarge Ω_e so that it denote the set of realizations for which the left hand side above has a limit. Because $\{\tau_{ce}\}_{c\in\mathbb{R}_+}\}$ is not necessarily ergodic, it remains to show that the random variable $\overline{\theta}(e,\cdot)$ is deterministic.

To this end, take any $\omega\in\Omega_e$ and an arbitrary $z\in\mathbb{R}^n$, we check that (3.4) implies

$$\left|\frac{1}{t}\theta(te,0,\tau_z\omega)-\frac{1}{s}\theta(se,0,\tau_z\omega)\right|\le\left|\frac{1}{t}\theta(te,0,\omega)-\frac{1}{s}\theta(se,0,\omega)\right|+\frac{C(|z|+1)}{t\wedge s}$$

which shows that $\tau_z\omega\in\Omega_e$ and that Ω_e is $\{\tau_z\}_{z\in\mathbb{R}^n}$ invariant. Moreover, we have, by stationarity and by repeated usage of triangle inequality, that

$$\begin{aligned}\overline{\theta}(e,\tau_z\omega)&=\lim_{t\to\infty}\frac{1}{t}\theta(te,0,\tau_z\omega)=\lim_{t\to\infty}\frac{1}{t}\theta(te+z,z,\omega)\\&\le\liminf_{t\to\infty}\left(\frac{1}{t}\theta(te+z,te,\tau_z\omega)+\frac{1}{t}\theta(te,0,\omega)+\frac{1}{t}\theta(0,z,\omega)\right)=\overline{\theta}(e,\omega).\end{aligned}$$

Applying the same argument to $\overline{\theta}(e,\tau_{-z}\omega')$ with $\omega'=\tau_z\omega$, we get also $\overline{\theta}(e,\omega)\le\overline{\theta}(e,\tau_z\omega)$. It follows that $\overline{\theta}(e)$ must be deterministic.

Finally, the bounds on $\overline{\theta}$ is a direct consequence of (3.2). ☐

Now that the deterministic function $\overline{\theta}:S^{n-1}\to\mathbb{R}$ is defined, the function $\overline{m}:\mathbb{R}^n\to\mathbb{R}$ can be defined as in (2.10) through $\overline{\theta}$.

Lemma 3.3. *The function $\overline{m}$ is Lipschitz continuous, and there exists an event Ω_0 with $\mathbb{P}(\Omega_0)=1$ such that, for every $x\in\mathbb{R}^n$ and $\omega\in\Omega_0$,*

$$\lim_{t\to\infty}\frac{1}{t}\theta(tx,0,\omega)=\overline{m}(x). \tag{3.8}$$

In addition, $\overline{m}$ satisfies

$$\beta^{-1}|p|\le\overline{m}(p)\le\alpha^{-1}|p|. \tag{3.9}$$

Proof. For each $e\in\mathbb{R}\mathbb{Q}^{n-1}\cap S^{n-1}$, that is a rational direction, we let Ω_e denote the event given by Lemma 3.2. Define

$$\Omega_0=\bigcap_{e\in\mathbb{R}\mathbb{Q}^{n-1}\cap S^{n-1}}\Omega_e.$$

Note that Ω_0 is invariant with respect to the translations $\{\tau_z\}_{z\in\mathbb{R}^n}$ since each Ω_e is so.

If $x = 0$, (3.8) automatically holds for all $\omega \in \Omega$.

If $x \in \mathbb{Q}^n$ and $x \neq 0$, set $e = \frac{x}{|x|}$, then since $\omega \in \Omega_0$ implies $\omega \in \Omega_e$, we have

$$\lim_{t\to\infty} \frac{1}{t}\theta(tx, 0, \omega) = |x| \lim_{t\to\infty} \frac{1}{t|x|}\theta(t|x|e, 0, \omega) = \overline{\theta}(e)|x| = \overline{m}(x).$$

Hence (3.8) holds for rational x. Moreover, for any Cauchy sequence $\{x_k\} \subseteq \mathbb{Q}^n$, for any $\omega \in \Omega_0$ and by (3.5), we have

$$\lim_{t\to\infty} \left|t^{-1}\theta(tx_k, 0, \omega) - t^{-1}\theta(tx_l, 0, \omega)\right| \leq C|x_k - x_l|. \tag{3.10}$$

It follows that $\overline{m} : \mathbb{Q}^n \to \mathbb{R}$ is Lipschitz continuous. By density of $\mathbb{Q}^n$ in $\mathbb{R}^n$, $\widetilde{m}$ extends to a Lipschitz continuous function on $\mathbb{R}^n$.

If $x \in \mathbb{R}^n \cap (\mathbb{Q}^n)^c$, then by density we can find $\{x_k\} \subseteq \mathbb{Q}^n$ so that $x_k \to x$. Then, an estimate similar to (3.10) shows that, for all $\omega \in \Omega_0$,

$$\lim_{t\to\infty} t^{-1}\theta(tx, 0, \omega) = \widetilde{m}(x).$$

Now that the left hand side above converges also to $\overline{m}(x)$ on an event of full probability, we must have $\widetilde{m}(x) = \overline{m}(x)$.

Finally, the bounds on $\overline{m}$ is a direct consequence of $\overline{\theta} \in [\beta^{-1}, \alpha^{-1}]$ on S^{n-1}. □

Remark 3. The pointwise convergence (with respect to x) in (3.8) can be upgraded to locally uniform convergence, for any $R > 0$ and $\omega \in \Omega_0$,

$$\lim_{t\to\infty} \sup_{x\in B_R} \left|\frac{1}{t}\theta(tx, 0, \omega) - \overline{m}(x)\right| = 0. \tag{3.11}$$

This can be proved, for instance, by a compactness argument. Indeed, the uniform continuity "in the limit" of the family $\{t^{-1}\theta(t\cdot, 0, \omega)\}_{t\geq 1}$, as described by (3.5), suffices for the argument.

Proof of Theorem 2.2. Step 1. We start with some reductions. Choosing a sequence of $R \to \infty$, and controlling the difference $m - \theta$ by (2.8), we only need to prove, for each large integer R, that

$$\mathbb{P}\left\{\limsup_{t\to\infty} \sup_{x,y\in B_R} \left|\frac{1}{t}\theta(tx, ty, \omega) - \overline{m}(x - y)\right| = 0\right\} = 1.$$

By (3.1) and some change of variables, this is equivalent to show that

$$\mathbb{P}\left\{\limsup_{t\to\infty} \sup_{x\in B_{2R}} \sup_{y\in B_R} \left|\frac{1}{t}\theta(tx, 0, \tau_{ty}\omega) - \overline{m}(x)\right| = 0\right\} = 1.$$

Step 2. Let Ω_0 be as defined in Lemma 3.3. For each $\omega \in \Omega_0$, set

$$Z_t(\omega) := \sup_{x\in B_{2R}} \left|\frac{1}{t}\theta(tx, 0, \omega) - \overline{m}(x)\right|.$$

Then Remark 3 shows that $t^{-1}Z_t \to 0$. It remains to upgrade this to

$$\mathbb{P}\left\{\limsup_{t\to 0} \sup_{y\in B_R} \frac{1}{t}Z_t(\tau_{ty}\omega) = 0\right\} = 1. \tag{3.12}$$

The techniques for this type of upgrade that allows the "vertex to flow" is somewhat standard. We present the details here for the sake of completeness. As we will see, the uniform continuity in the limit in (3.5), again, plays a key role.

By Egoroff's theorem, for any $0 < \varepsilon \le 1$ and, there exists $\Omega_\varepsilon \subset \Omega_0$ such that $\mathbb{P}(\Omega_\varepsilon) \ge 1 - \varepsilon^n$ and

$$\lim_{t\to\infty} \sup_{\omega\in\Omega_\varepsilon} t^{-1} Z_t(\omega) = 0.$$

In particular, there exists $T_\varepsilon > 0$ so that for all $t \ge T_\varepsilon$,

$$\sup_{\omega\in\Omega_\varepsilon} t^{-1} Z_t(\omega) < \varepsilon. \tag{3.13}$$

On the other hand, we apply the ergodic theorem to indicator function $\chi_{\Omega_\varepsilon}$ and find an $\widetilde{\Omega}_\varepsilon \in \mathcal{F}$ that is translation invariant, $\mathbb{P}(\widetilde{\Omega}_\varepsilon) = 1$ and, for every $\omega \in \widetilde{\Omega}_\varepsilon$,

$$\lim_{K\to\infty} \frac{1}{|B_K|} \int_{B_K} \chi_{\Omega_\varepsilon}(\tau_z\omega)dz = \mathbb{P}(\Omega_\varepsilon) \ge 1 - \varepsilon^n.$$

This shows in particular that, for each $\omega \in \widetilde{\Omega}_\varepsilon$, there exists $K_{\varepsilon,\omega}$, so that for $K \ge K_{\varepsilon,\omega}$ implies

$$|\{z \in B_K \,:\, \tau_z\omega \in \Omega_\varepsilon\}| \ge (1-\varepsilon^n)\,|B_K|. \tag{3.14}$$

Finally, we define $\widetilde{\Omega} = \cap_{k\in\mathbb{N}} \widetilde{\Omega}_{1/k}$. Clearly, $\mathbb{P}(\widetilde{\Omega}) = 1$. We prove below that $\widetilde{\Omega}$ is a subset of the event in (3.12).

Now fix an $\omega \in \widetilde{\Omega}$. For any $\varepsilon \in (0,1)$, let k be a large integer so that $k^{-1} < \varepsilon$. Let

$$M_{\varepsilon,\omega} := \max\{T_{k^{-1}}, (R+1)^{-1} K_{k^{-1},\omega}\}.$$

Then, if $t > M$, (3.13) and (3.14) are effective for t and $K = t(R+1)$. For any $y \in B_R$, we check that $B_{2\varepsilon(R+1)}(y) \subseteq B_{R+1}$ and their volume ratio is precisely $2^n\varepsilon^n$. Take $K = t(R+1)$ in (3.14), we see that there exists $\hat{y} \in B_{2\varepsilon(R+1)}(y)$ so that $\tau_{t\hat{y}}\omega \in \Omega_{1/k}$. Moreover,

$$\begin{aligned}
\frac{1}{t} Z_t(\tau_{ty}\omega) - \frac{1}{t} Z_t(\tau_{t\hat{y}}\omega) &= \sup_{x\in B_{2R}} \left| \frac{1}{t}\theta(tx,0,\tau_{ty}\omega) - \overline{m}(x) \right| - \sup_{z\in B_{2R}} \left| \frac{1}{t}\theta(tz,0,\tau_{t\hat{y}}\omega) - \overline{m}(z) \right| \\
&\le \sup_{x\in B_{2R}} \left| t^{-1}\theta(tx,0,\tau_{ty}\omega) - t^{-1}\theta(tx,0,\tau_{t\hat{y}}\omega) \right| \\
&\le C\left(|y-\hat{y}| + t^{-1}\right).
\end{aligned}$$

Since $\tau_{t\hat{y}}\omega \in \Omega_{1/k}$, the second item on the left hand is bounded by ε. We hence get

$$0 \le t^{-1} Z_t(\tau_{ty}\omega) \le C\left(2(R+1)\varepsilon + t^{-1}\right) + \varepsilon.$$

Note that this estimate is uniform in $y \in B_R$. Taking the supremum over $y \in B_R$, and send $t \to \infty$, we obtain (3.12) with the desired full measure event being $\widetilde{\Omega}$. The proof of the theorem is then complete. □

Some properties of the effective travel time function. We record some properties of the function $\overline{m}$ and their consequences.

Proposition 3.4. *Let $\overline{m} : \mathbb{R}^n \to \mathbb{R}$ be given by Lemma 3.3, D be defined as in (2.12), and $\overline{H}$ be defined as in (2.14). Then*

(i) *the mapping $x \mapsto \overline{m}(x)$ is convex; D is a compact and convex set.*

(ii) *the mapping* $p \mapsto \overline{H}(p)$ *is convex, and moreover, for each* $y \in \mathbb{R}^n$,

$$\overline{m}(y) = \sup\{p \cdot y \, : \, \overline{H}(p) \le 1\}. \tag{3.15}$$

Proof. (i) It is well known that the ergodic limit of a subadditive function is convex; see Proposition 4.1 of [1] for a proof. We check that

$$D = \{x \in \mathbb{R}^d \, : \, \overline{m}(x) \le 1\}.$$

If follows immediately from the convexity of $\overline{m}$ that D is convex and compact.

(ii) By definition (2.14), $\overline{H}(p)$ is the support function of the nonempty convex set D. It follows from basic convex analysis, see Theorem 13.2 of [18], that $\overline{H}$ is convex and positively homogeneous. Let $E := \{p' \in \mathbb{R}^n \, : \, \overline{H}(p') \le 1\}$ denote the unit sub-levelset of $\overline{H}$. The last statement of (ii) says $\overline{m}$ is also the support function of E. We may assume that $y \neq 0$. Since $y/\overline{m}(y) \in D$, we have $p \cdot y/\overline{m}(y) \le \overline{H}(p) \le 1$ for all $p \in E$. This shows $\overline{m}(y) \ge p \cdot y$ for all $p \in E$.

On the other hand, $y/\overline{m}(y) \in \partial D$, and hence there exists $p \in S^{n-1}$ so that $p \cdot y/\overline{m}(y) \ge p \cdot \xi$ for all $\xi \in D$. In other words, $\overline{H}(p) = p \cdot y/\overline{m}(y)$. Note that $\overline{H}(p) > 0$. Set $p_0 = p/\overline{H}(p)$. Then we check that $\overline{H}(p_0) = 1$ (hence $p_0 \in E$) and $p_0 \cdot y = \overline{m}(y)$. Therefore, (3.15) holds. □

In view of the positively homogeneity of $\overline{m}$ and $\overline{H}$, if we define $\overline{m}_\mu(y) = \mu\overline{m}(y)$ for $\mu > 0$, the equation (3.15) implies also

$$\overline{m}_\mu(y) = \sup\{p \cdot y \, : \, \overline{H}(p) \le \mu\}.$$

We can check that the reverse relation also holds:

$$\overline{H}(p) = \inf\{\mu > 0 \, : \, \mu\overline{m}(y) \ge p \cdot y \text{ for all } y \in \mathbb{R}^n\}.$$

which gives another characterization of the effective Hamiltonian, and which is standard in the metric problem approach of homogenization of Hamilton-Jacobi equations [2, 3]. The observations below then follows from basic convex analysis; see Lemma 3.2 of [3] for a detailed proof.

Lemma 3.5. *Let* $\mu > 0$ *and* $p \in \{\overline{H}(p') = \mu\}$, *then there exists a unit vector* $e \in S^{n-1}$ *such that*

$$\mu\overline{m}(e) - p \cdot e = 0 = \min_{e' \in S^{n-1}} (\mu\overline{m}(e') - p \cdot e'). \tag{3.16}$$

Moreover, if p *is an exposed point of the sub-levelset* $\{\overline{H}(p') \le \mu\}$, *then* e *can be chosen so that* $p = \mu D\overline{m}(e)$.

4. Proof of the homogenization result

In this section, we complete the proof of Theorem 2.3. This can be done by the method of perturbed test functions, provided we can show that $\varepsilon v^\varepsilon$, where v^ε is the solution to the approximate cell problem below, converges to $-\overline{H}$ uniformly on large balls of radius $\sim 1/\varepsilon$.

Given a vector $p \in \mathbb{R}^n$, the corresponding approximate cell problem is

$$\begin{cases} \varepsilon v^\varepsilon(x,t;p) + \partial_t v^\varepsilon + a(x,t)|p + Dv^\varepsilon| = 0 & \text{on } \mathbb{R}^{n+1} \\ v^\varepsilon \text{ is sublinear at infinity.} \end{cases}$$

We also define the scaled function $v_\varepsilon = \varepsilon v^\varepsilon(\cdot/\varepsilon, \cdot/\varepsilon)$. Then v_ε satisfies the equation

$$\begin{cases} v_\varepsilon(x,t;p) + \partial_t v_\varepsilon + a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)|p + Dv^\varepsilon| = 0 & \text{on } \mathbb{R}^{n+1} \\ v_\varepsilon \text{ is sublinear at infinity.} \end{cases} \tag{4.1}$$

By standard viscosity solution and assumption (A), there exists a unique viscosity solution v_ε to the equation above, and it satisfies $\|v_\varepsilon\|_{L^\infty} \leq \beta|p|$. We note that if $p = 0$, then v_ε is the constant zero function. For $p \neq 0$, due to the homogeneity of the Hamiltonian, we find that $v^\varepsilon(x,t;p) = |p| v^\varepsilon(x,t;\hat{p})$ with $\hat{p} = \frac{p}{|p|}$.

Convergence of the solution to the approximate problem. As mentioned above, the key step in the proof of Theorem 2.3 is to verify the following result. We only need to adapt the standard perturbed test function argument *in reverse* to the time dependent setting.

Theorem 4.1. *For every $\omega \in \widetilde{\Omega}$, $p \in \mathbb{R}^n$ and $R \geq 1$,*

$$\limsup_{\varepsilon \to 0} \sup_{(x,t) \in Q_R} \left| v_\varepsilon(x,t;p) + \overline{H}(p) \right| = 0. \tag{4.2}$$

Proof. If $p = 0$, then the equality $v_\varepsilon + \overline{H}(p) = 0$ holds and there is nothing to prove. We henceforth assume that $p \neq 0$ and set $\mu = \overline{H}(p) > 0$. We proceed in several standard steps.

Step 1. We establish that

$$\limsup_{\varepsilon \to 0} \sup_{(x,t) \in Q_R} v_\varepsilon(x,t;p,\omega) + \mu \leq 0. \tag{4.3}$$

By Lemma 3.5, there exists $e \in S^{n-1}$ so that p is a sub-differential of $\mu\overline{m}(y)$ at re for all $r > 0$:

$$\mu\overline{m}(y) - \mu\overline{m}(re) - p \cdot (y - re) \geq 0 \quad \text{for all } y \in \mathbb{R}^n. \tag{4.4}$$

To prove (4.3), we assume by contradiction that there exists $\delta > 0$, a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \searrow 0$, a sequence $\{(x_k, t_k)\} \in Q_R$, and

$$v_{\varepsilon_k}(x_k, t_k; p, \omega) + \mu \geq \delta > 0.$$

In the sequel, for notational simplicity, the subscript k in ε_k and the dependence of v_ε on p are suppressed, and (x_k, t_k) is denoted by (z, s).

For some small positive real numbers $c > 0$ to be determined, depending on δ, we define for all $x \in \mathbb{R}^n$ and $t \leq s$

$$w_\varepsilon(x,t) := v_\varepsilon(x,t,\omega) - v_\varepsilon(z,s,\omega) - c\delta\left(\sqrt{1 + |x - z|^2} - 1\right) - c\delta(s - t)$$

and set

$$U_\delta := \left\{ (x,t) \in \mathbb{R}^n \times (-\infty, s] \,:\, w_\varepsilon(x,t) \geq -\frac{\delta}{4} \right\}.$$

We claim that w_ε is a sub-solution to the following problem on U_δ:

$$\partial_t w_\varepsilon + a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)|p + Dw_\varepsilon(x,t)| \leq \mu - \frac{\delta}{4} \qquad \text{on } U_\delta \tag{4.5}$$

Let ∂U_δ denotes the space time boundary of U_δ. We note in particular that that (z,s) is in the interior of U_δ and not on the boundary. To prove that the above equation holds, let $\varphi(x,t)$ be a smooth function so that $w_\varepsilon - \varphi$ achieves its local maximum at (x_0,t_0), then

$$\begin{aligned}&(x,t)\mapsto v_\varepsilon(x,t) - (\varphi(x,t) + c\delta(s-t) + c\delta\psi(x;z) + v_\varepsilon(z,s))\\ &\text{achieves its local min at } (x_0,t_0).\end{aligned}$$

Here we denoted by $\psi(x;z) = \sqrt{1+|x-z|^2} - 1$ and we calculate that $D\psi(x;z) = \frac{x-z}{\sqrt{1+|x-z|^2}}$. In view of the equation satisfied by v_ε, we have

$$v_\varepsilon(x_0,t_0) + \partial_t\varphi(x_0,t_0) - c\delta + a\left(\frac{x_0}{\varepsilon},\frac{t_0}{\varepsilon}\right)|p + D\varphi(x_0,t_0) + c\delta D\psi(x_0;z)| \le 0.$$

We note that

$$|p + D\varphi(x_0,t_0)| \le |p + D\varphi(x_0,t_0) + c\delta D\psi(x_0)| + |-c\delta D\psi(x_0)|.$$

We get, in view of the bounds on w_ε in U_δ, the definition of w_ε and the bounds on v_ε,

$$\begin{aligned}\partial_t\varphi(x_0,t_0) + a\left(\frac{x_0}{\varepsilon},\frac{t_0}{\varepsilon}\right)|p + D\varphi(x_0,t_0)| &\le -v_\varepsilon(x_0,t_0) + c\delta + a\left(\frac{x_0}{\varepsilon},\frac{t_0}{\varepsilon}\right)|c\delta D\psi(x_0)|\\ &\le -v_\varepsilon(x_0,t_0) + c\delta(b+1)\\ &\le -w_\varepsilon(x_0,t_0) - v_\varepsilon(z,s) + c\delta(b+1)\\ &\le \frac{\delta}{4} - \delta + \mu + c\delta(b+1).\end{aligned}$$

We choose $c = c(b)$ small, so that $c(b+1) < \frac{1}{4}$. Then we have

$$\partial_t\varphi(x_0,t_0) + a\left(\frac{x_0}{\varepsilon},\frac{t_0}{\varepsilon}\right)|p + D\varphi(x_0,t_0)| \le \mu - \frac{\delta}{4} \qquad \text{in } U_\delta.$$

This verifies the equation (4.5).

We then compare w_δ with the function

$$\phi_\varepsilon(x,t) := \mu\varepsilon m\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon};\frac{z-re}{\varepsilon},\omega\right) - \mu\varepsilon m\left(\frac{z}{\varepsilon},\frac{t}{\varepsilon};\frac{z-re}{\varepsilon},\omega\right) - p\cdot(x-z), \tag{4.6}$$

where e is chosen as in (4.4). Using the bound $|v_\varepsilon| \le \beta|p|$, we can verify that

$$U_\delta \subset Q_{R_\delta}(z,s) \quad \text{where} \quad R_\delta := \frac{4\beta|p|}{c\delta} + \frac{1}{2c}.$$

As long as $r > R'$, the set U_δ is away from the pole $\{(z-re)/\varepsilon\}\times\mathbb{R}$, and ϕ_ε satisfies

$$\partial_t\phi_\varepsilon + a\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)|p + D\phi_\varepsilon| = \mu \quad \text{in } U_\delta. \tag{4.7}$$

We note that (z,s) is an interior point of U_δ and both w_ε and ϕ_ε vanishes there. By comparison principle,

$$0 \le \sup_{U_\delta}(w_\varepsilon - \phi_\varepsilon) = \sup_{\partial U_\delta}(w_\varepsilon - \phi_\varepsilon) = -\frac{\delta}{4} - \inf_{\partial U_\delta}\phi_\varepsilon.$$

Since $U_\delta \subset Q_{R_\delta}(z,s)$, the above implies

$$\inf_{Q_{R_\delta}(z,s)}\left(\mu\varepsilon m\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon};\frac{z-re}{\varepsilon},\omega\right) - \mu\varepsilon m\left(\frac{z}{\varepsilon},\frac{t}{\varepsilon};\frac{z-re}{\varepsilon},\omega\right) - p\cdot(x-z)\right) \le -\frac{\delta}{4}.$$

Note that R_δ does not depend on ε. Recall that the triple (ε, z, s) above denote a sequence $\{(\varepsilon_k, x_k, t_k)\}$ and we may assume $\{x_k\}$ converges. Sending $k \to \infty$ and using (2.11) we get

$$\inf_{B_{R_\delta}} (\mu\overline{m}(x + re) - \mu\overline{m}(re) - p \cdot x) \le -\frac{\delta}{4}.$$

This is a contradiction with (4.4).

Step 2. We establish that, if p is an exposed point of the sub-level set $\{\overline{H}(p') \le \mu\}$, then

$$\liminf_{\varepsilon \to 0} \inf_{(x,t) \in Q_R} v_\varepsilon(x, t; p, \omega) + \mu \ge 0. \tag{4.8}$$

By Lemma 3.5, there exists $e \in S^{n-1}$ so that $p = \mu D\overline{m}(e)$. In particular, for any $K > 0$,

$$\lim_{r \to \infty} \sup_{B_K} |\mu\overline{m}(x + re) - \mu\overline{m}(re) - p \cdot x| = 0. \tag{4.9}$$

We can then proceed in a similar way as in Step 2. Assume by contradiction that there exists $\delta > 0$, a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \searrow 0$, a sequence $\{(x_k, t_k)\} \in Q_R$, and

$$v_{\varepsilon_k}(x_k, t_k; p, \omega) + \mu \le -\delta < 0.$$

We then construct, for some small positive real numbers $c > 0$, the function

$$w^\varepsilon(x, t) := v_\varepsilon(x, t, \omega) - v_\varepsilon(z, s, \omega) + c\delta\left(\sqrt{1 + |x - z|^2} - 1\right) + c\delta(s - t).$$

It can be checked that w^ε is a super solution to

$$\partial_t w_\varepsilon + a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)|p + Dw_\varepsilon(x, t)| \ge \mu + \frac{\delta}{4} \qquad \text{on } U^\delta \tag{4.10}$$

where the domain U^δ is defined and bounded as follows:

$$U^\delta := \left\{(x, t) \in \mathbb{R}^n \times (-\infty, s] \,:\, w_\varepsilon(x, t) \le \frac{\delta}{4}\right\}, \quad U^\delta \subset Q_{R_\delta}(z, s)$$

where R_δ is defined as before. For $r > R_\delta$, equation (4.7) still holds on U^δ. By comparison principle, we get

$$\sup_{Q_{R_\delta}(z,s)} \left(\mu\varepsilon m\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}; \frac{z - re}{\varepsilon}, \omega\right) - \mu\varepsilon m\left(\frac{z}{\varepsilon}, \frac{t}{\varepsilon}; \frac{z - re}{\varepsilon}, \omega\right) - p \cdot (x - z)\right) \ge \frac{\delta}{4}.$$

Again, applying this result to the sequence $\{(\varepsilon_k, x_k, t_k)\}$ and assume $\{x_k\}$ converges, we get

$$\sup_{B_{R_\delta}} (\mu\overline{m}(x + re) - \mu\overline{m}(re) - p \cdot x) \le -\frac{\delta}{4}.$$

Let $r \to \infty$, then we get a contradiction with (4.9). Hence, (4.8) is true if p is an exposed point of the μ-level set of $\overline{H}$.

Step 3. We establish (4.8) for all $p \in E$ where $E := \partial\{\overline{H}(p') \le \mu\}$.

By Straszewicz's theorem [18, Theorem 18.6], if p is an extreme point of E, then there exists a sequence of exposed points $\{p_k\}$ of E and $p_k \to p$. Apply (4.8) to $\{p_k\}$ and then sending $k \to \infty$, by the stability of the Hamilton-Jacobi equation, we conclude that (4.8) holds for all extreme points of E. Finally, since any $p \in \partial E$ can be written as a convex combination of extreme points, we get (4.8) by the concavity of the map $p \mapsto v_\varepsilon(\cdot, \cdot; p, \omega)$. ☐

Homogenization of the level set equations. The homogenization result for the level set equation (1.1) follows from Theorem 4.1 by the standard perturbed test function argument. We present the details of the proof here for the sake of completeness.

Proof of Theorem 2.3. Define the upper and lower limits

$$u^*(x,t) = \lim_{\delta\searrow 0} \sup_{\substack{\varepsilon\le\delta, |x-z|\le\delta \\ |t-s|\le\delta}} u^\varepsilon(z,s), \qquad u_*(x,t) = \lim_{\delta\searrow 0} \inf_{\substack{\varepsilon\le\delta, |x-z|\le\delta \\ |t-s|\le\delta}} u^\varepsilon(z,s) \tag{4.11}$$

To establish (2.15), it suffices to prove that

$$u^*(x,t) \le u(x,t) \le u_*(x,t) \quad \text{for each } (x,t) \in B_R \times [0,T]. \tag{4.12}$$

We prove the inequality $u^*(x,t) \le u(x,t)$ in detail but omit the other inequality completely, since the proofs are essentially identical. Note that $u^*(x,t)$ is upper semi-continuous. The plan is to show that u^* is a sub-solution to the equation

$$\partial_t u^*(x,t) + \overline{H}(Du^*) \le 0. \tag{4.13}$$

Thanks to the uniform in ε and ω stability (in the locally uniform topology) of the Hamilton-Jacobi equation (1.1), we may assume that $u_0 \in C^{0,1}(\mathbb{R}^n)$. We then have $u^*(x,0) = u(x)$, and (4.13) follows from the comparison principle.

The goal of the rest of this proof is, hence, to establish (4.13). Assume, by contradiction, that this fails. Then for some smooth function φ, $u^* - \varphi$ achieves its local strict maximum at (x_0, s_0), but

$$\partial_t \varphi(x_0,t_0) + H(D\varphi(x_0,t_0)) \ge \delta > 0 \tag{4.14}$$

for some $\delta > 0$. Set $p = D\varphi(x_0,t_0)$ and $\mu = \overline{H}(p)$. Let v_ε be the solution to the approximate cell problem (4.1) with vector p. Define

$$\varphi^\varepsilon(x,t) = \varphi(x,t) + v_\varepsilon(x,t;p).$$

We claim that, for ε sufficiently small and $r > 0$ small to be chosen,

$$\partial_t \varphi^\varepsilon(x,t) + a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right)|D\varphi^\varepsilon| \ge \frac{\eta}{2} > 0 \qquad \text{in } Q_r(x_0,t_0). \tag{4.15}$$

To this end, suppose ψ is a smooth function and $\varphi^\varepsilon - \psi$ achieves its local minimum at (y,s). Then

$$(x,t) \mapsto v_\varepsilon - (\psi(x,t) - \varphi(x,t)) \text{ achieves its local minimum at } (y,s).$$

By the equation satisfied by v_ε, we obtain

$$v_\varepsilon(y,s) + \partial_t\psi(y,s) - \partial_t\varphi(y,s) + a\left(\frac{y}{\varepsilon}, \frac{s}{\varepsilon}, \omega\right)|p + D\psi(y,s) - D\varphi(y,s)| \ge 0.$$

This can be rewritten as

$$\begin{aligned}\partial_t\psi(y,s) + a\left(\frac{y}{\varepsilon}, \frac{s}{\varepsilon}, \omega\right)|D\psi(y,s)| &\ge a\left(\frac{y}{\varepsilon}, \frac{s}{\varepsilon}, \omega\right)|p - D\varphi(y,s)| + \partial_t\varphi(y,s) - v_\varepsilon(y,s) \\ &\ge \alpha|p - D\varphi(y,s)| + \partial_t\varphi(y,s) + \mu - (v_\varepsilon(y,s) + \mu).\end{aligned}$$

For ε sufficiently small, $|v_\varepsilon(y,s) + \mu| < \frac{\eta}{4}$. In view of (4.14) and by choosing r sufficiently small, depending on φ but not on ε, we can make

$$\alpha|p - D\varphi(y,s)| < \frac{\eta}{4}, \quad |\partial_t\varphi(y,s) - \partial_t\varphi(x_0,t_0)| < \frac{\eta}{4} \quad \text{in } Q_r(x_0,t_0)$$

which lead to

$$\partial_t\psi(y,s) + a\left(\frac{y}{\varepsilon},\frac{s}{\varepsilon},\omega\right)|D\psi(y,s)| \geq \frac{\eta}{4} \quad \text{in } Q_r(x_0,y_0).$$

This establishes (4.15).

We then compare φ^ε with u^ε in $Q_r(x_0,t_0)$ and get

$$\max_{Q_r(x_0,t_0)} u^\varepsilon - \varphi^\varepsilon = \max_{\partial Q_r(x_0,t_0)} u^\varepsilon - \varphi^\varepsilon \leq \max_{\partial Q_r(x_0,t_0)} u^* - \varphi^\varepsilon.$$

Send $\varepsilon \to 0$, we get

$$(u^* - \varphi)(x_0,t_0) = \max_{\partial Q_r(x_0,t_0)} u^* - \varphi^\varepsilon$$

which is a contradiction to the assumption that $u^* - \varphi$ achieves its maximum at (x_0,t_0). This shows that (4.14) cannot be true. Therefore, (4.13) holds and we finished the proof of $u^* \leq u$. □

References

[1] Armstrong, S. N., and Souganidis, P. E. Stochastic homogenization of Hamilton-Jacobi and degenerate Bellman equations in unbounded environments. *J. Math. Pures Appl. (9) 97*, 5 (2012), 460–504.

[2] Armstrong, S. N., and Souganidis, P. E. Stochastic homogenization of level-set convex Hamilton-Jacobi equations. *Int. Math. Res. Not. 2013*, 15 (2013), 3420–3449.

[3] Armstrong, S. N., and Tran, H. V. Stochastic homogenization of viscous Hamilton-Jacobi equations and applications, preprint, arXiv:1310.1749 [math.AP].

[4] Armstrong, S. N., Tran, H. V., and Yu, Y. Stochastic homogenization of a nonconvex Hamilton-Jacobi equation, preprint, arXiv:1311.2029 [math.AP].

[5] Cannarsa, P., and Cardaliaguet, P. Hölder estimates in space-time for viscosity solutions of Hamilton-Jacobi equations. *Comm. Pure Appl. Math. 63*, 5 (2010), 590–629.

[6] Cardaliaguet, P., Lions, P.-L., and Souganidis, P. E. A discussion about the homogenization of moving interfaces. *J. Math. Pures Appl. (9) 91*, 4 (2009), 339–363.

[7] Cardaliaguet, P., Nolen, J., and Souganidis, P. E. Homogenization and enhancement for the G-equation. *Arch. Ration. Mech. Anal. 199*, 2 (2011), 527–561.

[8] Cardaliaguet, P., and Silvestre, L. Hölder continuity to Hamilton-Jacobi equations with superquadratic growth in the gradient and unbounded right-hand side. *Comm. Partial Differential Equations 37*, 9 (2012), 1668–1688.

[9] Cardaliaguet, P., and Souganidis, P. E. Homogenization and enhancement of the G-equation in random environments. *Comm. Pure Appl. Math. 66*, 10 (2013), 1582–1628.

[10] Ciomaga, A., Souganidis, P. E., and Tran, H. V. Stochastic homogenization of interfaces moving with changing sign velocity, 2015.

[11] Jing, W., Souganidis, P. E., and Tran, H. V. Large time average of reachable sets and applications to homogenization of interfaces moving with oscillatory spatio-temporal velocity.

[12] Kosygina, E., Rezakhanlou, F., and Varadhan, S. R. S. Stochastic homogenization of Hamilton-Jacobi-Bellman equations. *Comm. Pure Appl. Math. 59*, 10 (2006), 1489–1521.

[13] Kosygina, E., and Varadhan, S. R. S. Homogenization of Hamilton-Jacobi-Bellman equations with respect to time-space shifts in a stationary ergodic medium. *Comm. Pure Appl. Math. 61*, 6 (2008), 816–847.

[14] Lions, P.-L., and Souganidis, P. E. Homogenization of "viscous" Hamilton-Jacobi equations in stationary ergodic media. *Comm. Partial Differential Equations 30*, 1-3 (2005), 335–375.

[15] Lions, P.-L., and Souganidis, P. E. Stochastic homogenization of Hamilton-Jacobi and "viscous"-Hamilton-Jacobi equations with convex nonlinearities—revisited. *Commun. Math. Sci. 8*, 2 (2010), 627–637.

[16] Nolen, J., and Novikov, A. Homogenization of the G-equation with incompressible random drift in two dimensions. *Commun. Math. Sci. 9*, 2 (2011), 561–582.

[17] Rezakhanlou, F., and Tarver, J. E. Homogenization for stochastic Hamilton-Jacobi equations. *Arch. Ration. Mech. Anal. 151*, 4 (2000), 277–309.

[18] Rockafellar, R. T. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[19] Schwab, R. W. Stochastic homogenization of Hamilton-Jacobi equations in stationary ergodic spatio-temporal media. *Indiana Univ. Math. J. 58*, 2 (2009), 537–581.

[20] Souganidis, P. E. Stochastic homogenization of Hamilton-Jacobi equations and some applications. *Asymptot. Anal. 20*, 1 (1999), 1–11.

[21] Xin, J., and Yu, Y. Periodic homogenization of the inviscid G-equation for incompressible flows. *Commun. Math. Sci. 8*, 4 (2010), 1067–1078.

Acknowledgements

WJ is supported in part by the NSF grant DMS-1515150. PS is supported in part by the NSF grant DMS-1266383. HT is supported in part by the NSF grant DMS-1361236.

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