A CONJECTURE ON OPTIMAL RATES OF CONVERGENCE IN PERIODIC HOMOGENIZATION OF NON-DIVERGENCE FORM LINEAR ELLIPTIC PDE IN TWO DIMENSIONS

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1. Introduction

We study the optimal rates of convergence in periodic homogenization of linear elliptic equations in non-divergence form. Let $U \subset \mathbb{R}^n$ be a given bounded domain with smooth boundary. The equation of our main interest is

$$\begin{cases} -a_{ij}\left(\frac{x}{\varepsilon}\right) u^{\varepsilon}_{x_i x_j} = f(x) & \text{in } U, \\ u^{\varepsilon} = g & \text{on } \partial U. \end{cases} \tag{1.1}$$

The matrix function $A(y) = (a_{ij})_{1 \le i,j \le n} \in C^2(\mathbb{R}^n, \mathbb{R}^{n^2})$ is always assumed to be symmetric, $\mathbb{Z}^n$-periodic, and positive definite for all $y \in \mathbb{R}^n$. Denote by $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the flat n-dimensional torus, and $\mathcal{S}^n_+$ the set of all real symmetric, positive definite matrices of size $n \times n$, then we can also write that $A \in C^2\left(\mathbb{T}^n, \mathcal{S}^n_+\right)$. Assume $f \in C^2\left(\overline{U}\right)$ and $g \in C^4(\partial U)$. Here, we always use the Einstein summation convention.

The homogenization problem (1.1) was discussed in the classical books of Bensoussan, Lions, Papanicolaou [1], Jikov, Kozlov, Oleinik [4]. It is well-known that, as $\varepsilon \to 0$, $u^{\varepsilon} \to u$ uniformly on $\overline{U}$, where u solves the following effective equation

$$\begin{cases} -\overline{a}_{ij}\, u_{x_i x_j} = f(x) & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{1.2}$$

Here, $\overline{A} = \{\overline{a}_{ij}\}_{1 \le i,j \le n}$ is the *effective matrix* with constant entries, which is determined as follows. For each fixed $(k,l) \in \{1, \ldots, n\}^2$, consider the solution v^{kl} of the (k,l)-th cell problem

$$-a_{ij}(y) v^{kl}_{y_i y_j}(y) - a_{kl}(y) = -\overline{a}_{kl}, \qquad y \in \mathbb{T}^n, \tag{1.3}$$

where $\overline{a}_{kl} \in \mathbb{R}$ is the unique constant such that (1.3) has a solution v^{kl}. In fact, v^{kl} is unique up to an additive constant by the strong maximum principle. Then, for a symmetric matrix M, the corresponding *corrector* is

$$v(y, M) = M_{kl} v^{kl}(y). \tag{1.4}$$

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It is clear that $v(y,M)$ solves

$$-a_{ij}(M_{ij}+v_{y_iy_j}(y,M)) = -\overline{a}_{ij}M_{ij} \qquad \text{in } \mathbb{T}^n.$$

On the other hand, $\overline{A}$ can also be determined through the corresponding invariant measure as follows. Let $r \in C(\mathbb{T}^n)$ be the unique solution to

$$\begin{cases} -(a_{ij}(y)r(y))_{y_iy_j} = 0 & \text{in } \mathbb{T}^n, \\ r > 0 \quad \text{and} & \int_{\mathbb{T}^n} r(y)\,dy = 1. \end{cases} \tag{1.5}$$

We say that r is the *invariant measure* of the matrix $A \in C^2(\mathbb{T}^n, \mathcal{S}^n_+)$. Multiply (1.3) by r and integrate to yield, for $1 \le k,l \le n$,

$$\overline{a}_{kl} = \int_{\mathbb{T}^n} a_{kl}(y)r(y)\,dy.$$

And thus,

$$\overline{A} = \int_{\mathbb{T}^n} A(y)r(y)\,dy.$$

The main question of interests is about optimal rate of convergence of u^ε to u in L^∞ norm. In what follows, by "*the rate of convergence*" we refer to the rate of convergence of $\|u_\varepsilon - u\|_{L^\infty}$ to 0 as $\varepsilon \to 0$.

Theorem 1.1 ([1, Theorem 5.1, page 230], [4, page 33])**.** *Assume that $f \in C^2\left(\overline{U}\right)$ and $g \in C^4(\partial U)$. Then, there exists $C > 0$ depending only on the ellipticity of A, f, g such that*

$$\|u^\varepsilon - u\|_{L^\infty(U)} \le C\varepsilon. \tag{1.6}$$

In fact, using the doubling variable method in the theory of viscosity solutions, the regularity of f and g can be relaxed to allow $f \in C^1\left(\overline{U}\right)$ and $g \in C^3(\partial U)$. In any case, the regularity of f and g is not the main concern here. Theorem 1.1 is well known in the literature. See the classical books of Bensoussan, Lions, Papanicolaou [1], Jikov, Kozlov, Oleinik [4], and the review paper of Engquist, Souganidis [2].

It was shown recently in Guo, Tran, Yu [3], and Sprekeler, Tran [5] that $O(\varepsilon)$ is the optimal rate of convergence in general. Here is a very quick overview of some results in [3, 5]. For $1 \le j,k,l \le n$ fixed, denote by

$$c_j^{kl} = c_j^{kl}(A) = \int_{\mathbb{T}^n} a_{ij}(y)v^{kl}_{y_i}(y)r(y)\,dy. \tag{1.7}$$

Note that $c_j^{kl}(A)$ depends only on A but in a highly nonlinear way.

Set

$$h(x) = c_j^{kl}u_{x_jx_kx_l}(x) \qquad \text{for all } x \in U.$$

Let z be the solution to

$$\begin{cases} -\overline{a}_{ij}z_{x_ix_j} = -h(x) & \text{in } U, \\ z = 0 & \text{on } \partial U. \end{cases} \tag{1.8}$$

Theorem 1.2. [3, Theorem 1.2] *Assume that* $f \in C^3\left(\overline{U}\right)$ *and* $g \in C^5(\partial U)$*. Then, there exists* $C > 0$ *depending only on the ellipticity of* A, f, g *such that*

$$\|u^\varepsilon - u + 2\varepsilon z\|_{L^\infty(U)} \le C\varepsilon^2. \tag{1.9}$$

In particular, the following claims hold.

(i) *If* $h \equiv 0$*, then* $\|u^\varepsilon - u\|_{L^\infty(U)} \le C\varepsilon^2$*, and this rate of convergence* $O(\varepsilon^2)$ *is optimal.*

(ii) *If* $h \not\equiv 0$*, then* $\|u^\varepsilon - u\|_{L^\infty(U)} \le C\varepsilon$*, and this rate of convergence* $O(\varepsilon)$ *is optimal.*

Definition 1. *Let* $A \in C^2\left(\mathbb{T}^n, \mathcal{S}^n_+\right)$*. If, for all* $1 \le j, k, l \le n$,

$$C_{jkl}(A) = c_j^{kl}(A) + c_l^{jk}(A) + c_k^{jl}(A) = 0,$$

then we say that A *is a c-good matrix. Otherwise,* A *is a c-bad matrix.*

Clearly, c-good matrices give optimal rate of convergence $O(\varepsilon^2)$ as $h \equiv 0$. Moreover, for c-bad matrices, there are choices of f and g such that the optimal rate of convergence is only $O(\varepsilon)$.

Theorem 1.3. [3, Theorem 1.4] *Assume* $n \ge 2$*. The set of c-bad matrices is open and dense in* $C^{2,\alpha}\left(\mathbb{T}^n, \mathcal{S}^n_+\right)$.

Here is an explicit c-bad matrix pointed out in [5].

Theorem 1.4. [5, Theorem 1.10] *The matrix-valued function* $A : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ *given by*

$$A(y_1, y_2) := \frac{1}{r(y_1, y_2)} \begin{pmatrix} 1 - \frac{1}{2}\sin(2\pi y_1)\sin(2\pi y_2) & 0 \\ 0 & 1 + \frac{1}{2}\sin(2\pi y_1)\sin(2\pi y_2) \end{pmatrix}$$

with $r : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$r(y_1, y_2) := 1 + \frac{1}{4}(\cos(2\pi y_1) - 2\sin(2\pi y_1))\sin(2\pi y_2)$$

is c-bad. More precisely, there holds $c_1^{11} = c_1^{22} = -\frac{1}{128\pi}$ *and* $c_j^{kl} = 0$ *otherwise.*

For all c-bad matrices, the optimal rate of convergence of $\|u^\varepsilon - u\|_{L^\infty}$ is $O(\varepsilon)$, and we cannot expect a better rate $O(\varepsilon^2)$ here. However, there are some specific situations in which we can hope for a better rate. This leads to the following conjecture.

2. A conjecture

Here is a conjecture that we made in the summer of 2020.

Conjecture 1. *Assume that* $n = 2$, $A = \operatorname{diag}(a_1(y), a_2(y))$*, where*

$$a_1, a_2 \in C^2(\mathbb{T}^2, (0,1)) \quad \text{and} \quad a_1 + a_2 = 1.$$

Then, we conjecture that the optimal rate of convergence of $\|u^\varepsilon - u\|_{L^\infty}$ *to* 0 *is* $O(\varepsilon^2)$.

Similar claim should also hold in the discrete setting (where the differential equations are replaced by difference equations with periodic coefficients on the integer lattice $\mathbb{Z}^2$.)

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