FIXED POINTS METHOD AND APPLICATIONS TO ODES AND PDES.

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Abstract

We study some basic results on the fixed points method and some interesting applications to ODEs and PDEs.

1 Introduction

These notes contain the essential materials and some further remarks of the summer course "Fixed points method and applications to ODEs and PDEs." taught by Prof. Hung V. Tran and Son Tu at University of Wisconsin - Madison from May 15 to June 6, 2017.

2 Lecture 1

2.1 Introduction.

The main goal: solve equations appears in various models in physics, economics, etc.
$F[u]=0$ where F is the operator and u is the unknown.
The first and most straightforward way is to solve the equations explicitly. However, in general, equations are not solvable explicitly.
So that leads us to the second way, which has two parts:

2.1.1 Look at equation in an abstract way and develop a solution theory

There are three parts for well-posedness:

$$\text{well-posedness}\left\{\begin{array}{l}\text{Existence}\\ \text{Uniqueness}\\ \text{Stability}\end{array}\right\}$$

Now we have a unique stable solution to an equation, and we could view the solution as a black box.

2.1.2 Study properties of the solution: beyond well-posedness theory

$$\left\{\begin{array}{l}\text{Regularity: If u solves an equation, then u is nice}\\ \text{Dynamical properties of solutions: If } u(x,t) \text{ solves an equation, then } u(x,t) \to \text{ what? if } t\to\infty \\ \text{Various aspect in terms of asymptotic theory}\end{array}\right\}$$

The main focus for this course is well-posedness theory. We are going to spend three lectures on: ODE theory, gradient flow , Deformation theory and Mountain-pass theorem.

2.2 ODE theory

If we have the following ODE:

$$u : \mathbb{R} \to \mathbb{R}$$
$$t \longmapsto u(t)$$

$$\begin{cases} u'(t) = F(t, u(t)) \\ u(0) = u_0 \in \mathbb{R} \end{cases}$$

Here $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Our goal is: $\left\{\begin{array}{l}\text{Well-posedness of solution in short time } t \in (-\delta,\delta), \text{ where } \delta > 0 \text{ is small}\\ \text{Well-posedness of solution for all time } t \geq 0\end{array}\right\}$

Example 2.1. Population Dynamic (logistic model)

$$\begin{cases} u'(t) = F(t, u) = u(M - u) \\ u(0) = u_0 \in (0, M) \end{cases}$$

If $u = u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$.

$$\text{Reaction-Diffusion equation} \left\{\begin{array}{l} u_t = F(t, u(x,t)) + u_{xx} \text{ in } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) \text{ on } \mathbb{R} \end{array}\right\}$$

In particular, if $F(t, u(t)) = u(1-u)$, the above PDE is called Fisher-KPP equation(1930s).
Example 2.2.

$$\begin{cases} u'(t) = F(t, u(t)) = u(t)^2 \\ u(0) = a \neq 0 \end{cases}$$

We know that $\frac{u'}{u^2} = 1$, then $\int \frac{u'}{u^2} dt = \int 1 dt$. So $-\frac{1}{u} = t + C$, then $u(t) = \frac{1}{C-t}$.
Note that well-posedness for short time is fine but Well-posedness for long time is not.
Example 2.3.

$$\begin{cases} u'(t) = \sqrt{|u(t)|} \\ u(0) = 0 \end{cases}$$

Roughly, $\frac{u'}{u^{1/2}} = 1$, then $\int \frac{u'}{u^{1/2}} dt = \int 1 dt$, so $2u^{1/2} = t + C$, $u = \frac{1}{4}(t+c)^2$. We know that uniqueness fails because there are too many solutions. But, we could use this idea to build all solutions.
Homework 1.1: Find rigorously all solutions of the above ODE.

Assumptions: Assume that $F \in C(\mathbb{R} \times \mathbb{R})$ and is Lipschitz in u, that is, there exist $c > 0$ such that:

$$|F(t,y) - F(t,z)| \leq c|y-z| \qquad \text{for all } t,y,z \in \mathbb{R}$$

In other words, F has linear growth in u.

Theorem 2.1. (Picard-Lindelof Theorem)
Let $R = \{(t,y) : |t| \leq a, |y-u_0| \leq b\}$ and $F : R \to \mathbb{R}$ such that $|F(t,y)| \leq M$ and F is Lipschitz in the second term. Then there is a unique solution $u : [-a^*, a^*] \to \mathbb{R}$ such that:

$$\begin{cases} u'(t) = F(t,u(t)) \text{ and } -a^* \leq t \leq a^* \\ u(0) = u_0 \end{cases}$$

Where $a^* = min(a, \frac{b}{M}, \frac{1}{c})$

Remark:

- This is a well-posedness result for a short time
- There are different view points to look at the ODE:
  - Normal form: $u'(t) = F(t,u(t))$
  - Integral form: $\int_0^t u'(s)ds = \int_0^t F(s,u(s))ds$, then $u(t) = u_0 + \int_0^t F(s,u(s))ds$

Proof. If we look at the above equation: $u(t) = u_0 + \int_0^t F(s,u(s))ds$. We can see that the left hand side is an identity map $u \to u$ and the right hand side is an operator $A : u \mapsto A[u](t)$ where $A(u) = u_0 + \int_0^t F(s,u(s))ds$. So this problem becomes: find u such that $u = A[u]$. This is a fixed point problem:

$$\begin{cases} A : C([-a^*, a^*], [u_0 - b, u_0 + b]) \to C([-a^*, a^*], [u_0 - b, u_0 + b]) \\ A \text{ is a contraction mapping} \end{cases}$$

Homework 1.2: Re-check this. □

Homework 1.3: Let $f : \mathbb{R} \to (0, \infty)$ be a continuous function, for the following ODE:

$$\begin{cases} u'(t) = f(u(t)) \\ u(0) = a \end{cases}$$

Show that blow up occurs if and only if when $\int_a^\infty \frac{1}{f(x)} dx \leq \infty$.
Here, blow up means exists $u : (0,T) \to \mathbb{R}$ such that:

$$\lim_{t \to T-} u(t) = \infty$$

Theorem 2.2. Global Uniqueness
If $F \in C(\mathbb{R} \times \mathbb{R})$ and F is Lipschitz in the second term. Then the uniqueness still holds for the above theorem.

Idea: keep gluing things together. But there is a problem: $|f(t,y)| \leq M$ and M can be large as $|y| \to \infty$, $\frac{b}{M} \to 0$ and $a_* \to 0$

Proof. Let $S = \{s > 0 :$ (ODE) has a solution $u(t)$ for the time $0 < t < s\}$. Then $s \neq \emptyset$ and by Picard-Lindelof Theorem. If $\sup S = +\infty$, we are done. So suppose $\sup S = T < \infty$.
Homework 1.4: Show that, if $s_1, s_2 \in S$ and u solves (ODE) for $0 < t < s_1$, and v solves (ODE) for $0 < t < s_2$, then $u \equiv v$ on $0 < t < \min(s_1, s_2)$.
Hint: use Picard-Lindelof correctly.
Now we have $u : [0, T) \to \mathbb{R}$ solves the following:

$$\begin{cases} u'(t) = F(t, u(t)) \text{ when } 0 < t < T \\ u(0) = u_0 \end{cases}$$

Claim: There exists $C_1 > 0$ such that $|u(t)| < C_1$ for all $0 \leq t < T$.
Proof of the Claim (Gronwall inequality):
Let $\phi(t) = |u(t)|^2$

$$\begin{aligned} \phi'(t) &= 2u(t)u'(t) = 2u(t)F(t, u(t)) \\ &= 2u(t)(F(t, u(t)) - F(t, 0) + F(t, 0)) \end{aligned}$$

Because F is Lipschitz, $F(t, u(t)) - F(t, 0) \leq c(u(t))$ and because F is continuous, $F(t, 0) \leq c$ for some constant c. And then we have:

$$\begin{aligned} \phi'(t) &\leq 2u(t)(c|u(t) + c|) \\ &\leq 2c|u(t)|^2 + c(u(t)^2 + 1) \\ &\leq C(\phi(t) + 1) \end{aligned}$$

Let $\psi(t) = \phi(t) + 1$, then $\psi(t) \leq C(\psi(t))$ and $\psi(0) = u_0^(2) + 1$. So $(e^{-Ct\psi(t)})' \leq 0$ and $\psi(t) \leq \psi(0)e^{-Ct}$.
Final Step: Because $|u(t)| \leq c$ for $0 \leq t < T$, $u'(t) = F(t, u(t))$ is bounded. So we could define $u(T) = lim_{t \to T-} u(t)$ and that contradicts with $\sup S = T$.
Homework 1.5: Complete the final step. □

3 Lecture 2

3.1 Another way to look at uniqueness

Proposition: Assume the requirements for Picard holds. Let u, v for $0 \leq t \leq T$ be solutions to the ODE, then $u \equiv v$.

Proof. Directly by Gronwall:
Suppose both u(t) and v(t) are solutions to the ODE.
We can define $\phi(t) = |v(t) - u(t)|^2$ and $\phi(0) = 0$.

$$\begin{aligned} \phi\prime(t) &= 2|u(t) - v(t)|(u'(t) - v'(t)) \\ &= 2|u(t) - v(t)|(f(t, u(t)) - f(t, v(t))) \\ &\leq 2C\phi(t) \end{aligned}$$

So by Gronwall's inequality, $\phi(t) \leq e^{2Ct}\phi(0) = 0$.
So $\phi(t) \equiv 0$, which means $u(t) \equiv v(t)$. So we know that the solution is unique. □

Big Question: To prove the existence directly for time $0 < t < T$ for any given $T > 0$ for the following ODE:

$$\begin{cases} u'(t) = F(t, u(t)) \\ u(0) = u_0 \end{cases}$$

Idea: Use finite difference scheme.
For any $k \in \mathbb{N}$, we divide the whole interval into k equal-size subintervals and construct function u^k by the following: $u^k(0) = u_0$ and $u^k(t) = u^k(0) + f(0, u^k(0))t$. And need to prove that as $k \to \infty$, $u^k \to u$ on $[0, T]$ and u solves the ODE.
Homework 2.1: If we have the following:

$$\begin{cases} x'(t) = f(t, x(t)) + E_1 \\ y'(t) = f(t, y(t)) + E_2 \end{cases}$$

Where $|E_1| + |E_2| < \epsilon$ for some $\epsilon > 0$. Show that for $t > 0$,

$$|x(t) - y(t)| \leq c|x(0) - y(0)|e^{ct} + c\epsilon(e^{ct} - 1)$$

Remark: All results holds for vector-valued case: $u(t) : \mathbb{R} \to \mathbb{R}^n$, $f(t, u) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ continuous and f Lipschitz on the second term, then the same ODE still has a unique global solution.

3.2 Global flow

Gradient flow: Given $V : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function. Define $F(x) = -\nabla V(x) = V'(x) = -(V_{x1}, V_{x2}, ..., V_{xn})$. Then let's look at the ODE:

$$\begin{cases} u'(t) = F(t, u(t)) = -\nabla V(u(t)) \text{ when } t > 0 \\ u(0) = u_0 \end{cases}$$

Assume that there is $c > 0$ such that $|\frac{\partial^2 v}{\partial x_i \partial x_j}(x)| \leq c$ for $\forall x \in \mathbb{R}^n$ and $\forall i, j$. (Condition B)
Homework 2.2: Given the above assumption, show it implies that $F = -\nabla V$ is Lipschitz.
Idea of steepest descend method: Given $x_0 \in \mathbb{R}^n$ is the initial point, we want to find a direction $h \in \mathbb{R}^n$ such that $|h| = 1$, so we can decrease the value $v(x_0)$ fastest. So for $s \in \mathbb{R}$ small:

$$\begin{aligned} v(x_0 + sh) &= v(x_0 + v'(x_0)(sh) + v(sh)) \\ &= v(x_0) + s\nabla v(x_0)h + v(sh) \end{aligned}$$

Notice that in the above equation, the last term is just a remainder, so we want to make the second term as small as possible, so we get $h = -\frac{\nabla v(x_0)}{|\nabla v(x_0)|}$.
Lemma 3.1. If we assume condition B, then for the following equation:

$$\begin{cases} u'(t) = -\nabla V(u(t)) \\ u(0) = u_0 \end{cases}$$

We have $t \to v(u(t))$ is non-increasing.

Proof.

$$\frac{d}{dt}(v(u(t))) = \nabla v(u(t)) \cdot u'(t) = -|\nabla v(u(t))|^2 \leq 0$$

□

Homework 2.3: Assume that $V : \mathbb{R}^n \to \mathbb{R}$ is uniformly convex, which means:

$$\begin{cases} (n=1): 0 < c \leq V''(x) \leq C < \infty \\ (n \geq 2): cI \leq D^2 V \leq CI \end{cases}$$

And we have:

$$\begin{cases} u'(t) = -\nabla V(u(t)) \\ u(0) = u_0 \end{cases}$$

Find $\lim_{t\to\infty} u(t)$.

3.3 A bit on "Deformation"

This is about a first object transform to a second object.

$$\left\{ \begin{array}{l} \eta(t,x)\text{: } t \text{ is time, } x \text{ is location} \\ \eta \text{ is continuous} \\ \eta(0,x) \text{ is the first object, } \eta(1,x) \text{ is the second object} \end{array} \right\}$$

4 Lecture 3

4.1 Deformation Theorem

Setting $I : \mathbb{R}^n \to \mathbb{R}$ in C^2, $\lim_{|x|\to\infty} I(x) = +\infty$.
For $c \in \mathbb{R}$, define $A_c = \{x \in \mathbb{R}^n : I(x) \leq c\}$ and $K_c = \{x : I(x) = c \text{ and } I'(x) = 0\}$. So K_c contains all critical points of I at c-level set of I.
Remark: $I : \mathbb{R}^n \to \mathbb{R}$ so $I'(x) = \nabla I(x) = DI(x)$.

Theorem 4.1. Deformation Theorem
Assume $K_c = \emptyset$ for some $c \in \mathbb{R}$. Then for each $\epsilon > 0$ sufficiently small, $\exists \delta \in (0, \epsilon)$ such that there is a continuous function $\eta = \eta(t,x)$ $[0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ such that:

- $\eta(0,x) = \eta_0(x) = x$ for all $x \in \mathbb{R}^n$
- $\eta(1,x) = \eta_1(x) = x$ for all x not in $I^{-1}([c-\epsilon, c+\epsilon])$
- $t \to I[\eta(t,x)] = I[\eta_t(x)]$ is non-increasing.
- $\eta_1(A_{c+\delta}) \subset A_{c-\delta} \Leftarrow$ this is deformation.

Proof. **Claim:** Because $K_c = \emptyset$, there are $\epsilon > 0$ and $\delta > 0$ such that $|I'(x)| = |\nabla I(x)| > \sigma > 0$ for all $x \in I^{-1}([c-\epsilon, c+\epsilon])$.
Homework 3.1: Prove this claim.
Fix $0 < \delta < \epsilon$ such that $\delta < \frac{\sigma^2}{2}$. Note that for $a \in \mathbb{R}$, $A_a = I^{-1}((-\infty, a])$ is compact.
Step 1: To build the vector field $V : \mathbb{R}^n \to \mathbb{R}^n$.

$$\text{Let: } \left\{ \begin{array}{l} A = I^{-1}([c-\delta, c+\delta]) : \text{ compact.} \\ B = \mathbb{R}^n \backslash I^{-1}((c-\epsilon, c+\epsilon)) : \text{ closed set.} \\ 0 \end{array} \right\}$$

Set $\phi(x) : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\phi(x) = \frac{dist(x,B)}{dist(x,B) + dist(x,A)}$$

Where we define $dist(x,A) = \inf\{|x-y| : y \in A\}$. Note that $0 \leq \phi(x) \leq 1$ for all x. For $x \in B$, $\phi(x) = 0$ and for $x \in A$, $\phi(x) = 1$.
Homework 3.2: Show that $x \to \phi(x)$ is lipschitz.
More generalized version: If A is compact and B is closed, show $x \to \phi(x)$ is lipschitz.
Define $V(x) = -\phi(x)I'(x) = -\phi(x)\nabla I(x)$.
Look at the following ODE, for $\forall x \in \mathbb{R}^n$

$$\begin{cases} \gamma'(t) = v(\gamma(t)) = -\phi(\gamma(t))\nabla I(\gamma(t)) \\ \gamma(0) = x \end{cases}$$

We can write $\eta(t,x) = \gamma(t)$ where t represents time and x represents the starting point.
Step 2: Show $\eta(t,x)$ fulfills all four conditions:

- Obvious as $\eta(0,x) = x$ starting point.
- For x not in $I^{-1}([c-\epsilon, c+\epsilon])$, then $xinB, \eta(t,x) = \gamma(t)$

$$\begin{cases} \gamma'(t) = v(\gamma(t)) = -\phi(\gamma(t))\nabla I(\gamma(t)) \\ \gamma(0) = x \in B \end{cases}$$

  $\phi(x) = 0$ if $x \in B$, so $\gamma(t) = x$ for all t.
-

$$\begin{aligned} \frac{d}{dt}[I(\eta(t,x))] &= I'(\eta(t,x)) \cdot \eta_t(t,x) \\ &= I'(\eta(t,x)) \cdot (-\phi(\gamma(t)))I'(\gamma(t)) \\ &= -\phi(\gamma(t)) \cdot |I'(\eta(t,x))|^2 \leq 0 \end{aligned}$$

- For $x \in A_{c+\delta}$, we need to show that $\eta(t,x) \in A_{c-\delta}$.
  If for some $t \in (0,1), \eta(t,x) \in A_{c-\delta}$ by the above proof we are done.
  So suppose that we always have $I(\eta(t,x)) > c - \delta$ for all $t \in [0,1]$. Then we have $\eta(t,x) \in A$ for all $t \in [0,1]$. So $\phi(\eta(t,x)) = 1$ for all $t \in [0,1]$.
  Look back, $\frac{d}{dt}[I(\eta(t,x))] \leq -|I'(\eta(t,x))|^2 \leq -\sigma_2$, so we get:

$$\begin{aligned} I(\eta(1,x)) &\leq I(\eta(0,x)) - \sigma_2 \\ &\leq c + \delta - \sigma_2 < c + \delta - 2\delta = c - \delta \end{aligned}$$

□

Key concern: To find critical points of I, that is, to solve $I'(x) = 0$.

4.2 Mountain-Pass Theorem

Theorem 4.2. Mountain-Pass Theorem.
Assume further:

$$\begin{cases} I(0) = 0 \\ I(x) = a > 0 \text{ if } |x| = r \\ \text{There is } y : |y| > r \text{ and } I(y) > 0 \end{cases}$$

Define $\Gamma = \{\gamma : [0,1] \to \mathbb{R}^n \text{ continuous}, \gamma(0) = 0, \gamma(1) = y\}$.
Let $c = \inf_{\gamma\in\Gamma}(\max_{t\in[0,1]} I(\gamma(t)))$.
Then $K_c \neq \emptyset$, that is, there exists $x \in \mathbb{R}^n$ such that $I(x) = c$ and $I'(x) = 0$.

Proof. Proof by contradiction: Assume that $K_c = \emptyset$, we now can apply the Deformation Theorem, given $0 < \delta < \epsilon$:

$$\begin{cases} \eta(0,x) = \eta_0(x) = x \text{ for all } x \in \mathbb{R}^n \\ \eta(1,x) = \eta_1(x) = x \text{ for all } x \text{ not in } I^{-1}([c-\epsilon, c+\epsilon]) \\ \eta(1,x) \in A_{c-\delta} \text{ if } x \in A_{c+\delta} \end{cases}$$

As $\delta > 0$, there is a path $\gamma \in \Gamma$ such that $\max_{t\in[0,1]} I(\gamma(t)) < c + \delta$.
Note: For $\forall\gamma \in \Gamma$, $\max_{t\in[0,1]} I(\overline{\gamma}(t)) \geq a > 0 \Rightarrow c \geq a$.
Then we can choose δ small such that $\delta < \frac{a}{2}$ and define $\alpha(t) = \eta(1, \overline{\gamma}(t))$ for $0 \leq t \leq 1$. So α is continuous and $\alpha(0) = \eta(1,0) = 0$ and $\alpha(1) = \eta(1,y) = y$. Then we can get that $\alpha \in \Gamma$.
By Deformation Theorem, $\alpha(t) \subset A_{c-\delta}$ for $\forall t$. Then $\max_{0<t\leq 1} I(\alpha(t)) \leq c - \delta$, and that is a contradiction. □

5 Lecture 4

5.1 Application of Mountain-Pass Theorem

An important remark: Many PDEs can be phrased as solving the equation $I'(z) = 0$.
This remark means that for PDEs can be phrased in terms of finding critical points of some functionals this is the area of calculus of variations.
Example 5.1. $U \subset \mathbb{R}^n$ is a given domain with smooth boundary, and

$$H = \{u : U \to \mathbb{R} \text{ such that } u(x) = 0 \text{ for all } x \in \partial u \text{ and } \int_U |Du|^2 dx < \infty\}$$

$$I : H \to \mathbb{R}$$

$$u \to I(u) = \int_U |Du|^2 dx \Rightarrow \text{ I is a functional.}$$

In H, we have a norm $||u||_H = (\int_U |Du|^2 dx)^{1/2}$, so $I[u] = ||u||^2$.
Question 1: What is $I'[u]$?

$$\begin{cases} I[u+v] = I[u] + I'[u]v + r(v) \\ \text{where } \lim_{||v||_H \to 0} \frac{|r(v)|}{||v||_H} = 0 \end{cases}$$

In this case, we can get:

$$\begin{aligned} I[u+v] &= \int_U |D(u+v)|^2 dx = \int_U (|Du|^2 + 2DuDv + |Dv|^2) dx \\ &= I[u] + 2\int_U Du \cdot Dv dx + ||v||_H^2 \end{aligned}$$

Then we can get $I'[u]v = 2\int_U Du \cdot Dv dx$.
Question 2: What does it mean by $I'[u] = 0$?
This means $2\int_U Du \cdot Dv dx = 0$ for $\forall v \in H$, then:

$$\int_U Du \cdot Dv dx = \int_U (-\Delta u) v dx + \int_{\partial U} \frac{\partial u}{\partial n} v dx$$

Because we have everything is 0 on the boundary, we can get:

$$\int_U (-\Delta u) v dx = 0 \text{ for any } v \in H$$

So we have $\Delta u = 0$ in U, an elliptic PDE.
Example 5.2. :
Homework 1: Prove the following statement.
H, U are both the same as the first example, and if we want to solve the following PDE:

$$\begin{cases} -\Delta u = f(x) \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases}$$

It is the same as finding critical points of the following functional:

$$\begin{aligned} &J : H \to \mathbb{R} \\ &\quad u \to J(u) = \int_U |Du|^2 - f(x)u(x) dx \end{aligned}$$

Example 5.3. We want to solve:

$$\begin{cases} -\Delta u = f(u) \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases}$$

It is the same as finding critical points of the following functional:

$$\begin{aligned} &K : H \to \mathbb{R} \\ &\quad u \to K(u) = \int_U |Du|^2 - F(u) dx \end{aligned}$$

Where $F(z) = \int_0^z f(s) ds$
Question: Do we know that this PDE has non-trivial solutions?

Theorem 5.1. Assume that $F(z) = |z|^s$ for $2 < s < \frac{2n}{n-2} = 2^*$, then the PDE has a non-trivial solution.

Proof. $K[u] = \int_U |Du|^2 - |u|^s dx$, we will apply the Mountain-Pass theorem:

- $K[0] = 0$
- Fix $w \in H, ||w||_H = 1$

Look at the ray $\{tw, t > 0\}$:

$$K[tw] = \int_U |D(tw)|^2 - |tw|^s dx = \int_U t^2|Dw|^2 - t^s|w|^s dx$$
$$= t^2 \int_U |Dw|^2 dx - t^s \int_U |w|^s dx$$

Homework 2: Show, for $||w||_H$, then $\int_U |Dw|^2 dx > 0$ and $\int_U |w|^s dx > 0$.
There is $t_0 > 0$ large enough such that $K[t_0 w] < 0$, let $y = t_0 w$.
Then by Pomcare's inequality, $(\int_U |u|^s dx)^{1/s} \leq c(\int_U |Du|^2 dx)^{1/2}$

$$K[u] = \int_U |Du|^2 - \int_U |u|^s dx = ||u||_H^2 - ||u||_H^s$$
$$\geq ||u||_H^2 - c||u||_H^s > c > 0 \text{ for } ||H||_H = r > 0$$

□

Remark: In infinite dimensional case, we need to be careful with compactness. In general, people need the Palais-Smale condition.

5.2 A simple reaction-diffusion PDE

$$u : [0,1] \times [0, \infty) \to \mathbb{R}$$
$$(x, t) \longmapsto u(x, t)$$

we have the following PDE

$$\begin{cases} u_t = u_{xx} + f(t, u(x, t)) \\ u(0, t) = u(1, t) = 0 \text{ for all } t \\ u(x, 0) = u_0(x) \text{ for } x \in (0, 1) \end{cases}$$

with $u_0 \in C^2[0,1]$ and f : $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitz on y.
Main goal: To show the above PDE has unique global solution, and there are two ways to show it:

- First way: To use the contradiction mapping theorem to prove existence and uniqueness for short time and glue pieces of solutions together to have global solution in time.
- Second way: Idea similarly to the solution of the ODE question.

Step 1: prove uniqueness of solutions in $(0,1)\times(0,T)$ for $\forall T>0$ fixed. Hint: use gronwall's inequality.
Step 2: Discretizing time to prove existence for $t\in[0,T]$. Fix $k\in\mathbb{N}$, chop $[0,T]$ into k equal size subintervals $[0,\tau],[\tau,2\tau],\cdots,[(k-1)\tau,T]$ where $\tau=\frac{T}{k}$, when time $t=0$, $u_0(x)\in C^2([0,1])$.
Need to have an approximate solution $U_k(x,t)$, and the set-up is:

$$\begin{cases} u_k(x,0)=u_k^0(x)=u_0(x) \text{ for } \forall x\in(0,1) \\ u_k(x,\frac{T}{k})=u_k^1(x) \\ \frac{u_k^1(x)-u_k^0(x)}{\tau}=(u_k^1(x))_{xx}+f(0,u_k^0(x)) \end{cases}$$

This is equivalent to:

$$u_k^1(x)-\tau[u_k^1(x)]_{xx}=u_k^0(x)+\tau f[0,u_k^0(x)]$$

Iteratively:

$$u_k(x,t)=u_k^{j-1}(x)+[t-(j-1)\tau]\frac{u_k^j(x)-u_k^{j-1}(x)}{\tau}$$

By iteration, the right hand side is known, and the unknown is $u_k^j(x)$. Now the next steps are to show:

- $u_k^j(x)$ is bounded independent of k,j. Hint: use maximal principle.
- Prove $(u_k^j(x))_x$ is bounded, and for simplicity, ignore the boundary case of $x=0$ and $x=1$.

6 Homework solutions

6.1 Lecture 1

6.1.1 Problem 1

Proof. Since $u'(t)=\sqrt{|u(t)|}$ and $\sqrt{|u(t)|}\geq 0$, we could get that u is a non-decreasing function.
If there are two 0 points that are not connected, but it is contradict to the fact that u is non-decreasing. So $\{x\in\mathbb{R}\mid u(x)=0\}$ is empty, a point, or an interval.
Suppose it is empty or a point, then because $\frac{u'}{|u|^{1/2}}=1$, $\int\frac{u'}{|u|^{1/2}}dt=\int 1dt$, $2|u|^{1/2}=t+c$, and because there is at most one zero point:

$$u=\left\{\begin{array}{l} \frac{1}{4}(t+c)^2 \text{ if } t\geq -c \\ -\frac{1}{4}(t+c)^2 \text{ if } t<-c \end{array}\right\}$$

Notice that $t=-c$ is a 0 point, so the set cannot be empty.
Similarly, if $\{x\in\mathbb{R}\mid u(x)=0\}$ is a interval $[a,b]$, where $a<b$:

$$u=\left\{\begin{array}{lr} \frac{1}{4}(t-a)^2 & \text{if } t\geq a \\ -\frac{1}{4}(t-b)^2 & \text{if } t\leq -b \\ 0 & \text{if } a<t<b \end{array}\right\}$$

□

6.1.2 Problem 2

Proof. First, we need to show that A maps a function to itself.
Since $|F| \leq M$ and $|t| \leq a^* \leq b/M$, $|\int_0^t F(s,u(s))ds| \leq b$.
So $A(u)(x) \in [u_0 + b, u_0 - b]$ for any $u \in C([-a^*, a^*], [u_0 - b, u_0 + b])$. Then $A(u) \in C([-a^*, a^*], [u_0 - b, u_0 + b])$ for any u, so A map to itself.
Then need to show that A is a contraction mapping. For any given u, v, and since F is Lipschitz:

$$
\begin{aligned}
|A(u) - A(v)| &= |u_0 - v_0 + \int_0^t F(s,u(s)) - F(s,v(s))ds| \\
&\leq |\int_0^t c|u(s) - v(s)|ds| \leq |t|c|u - v|
\end{aligned}
$$

Because we know that $t \in [-a^*, a^*]$, $|t| \leq 1/c$. We could change the interval to $[-a^* + \epsilon, a^* - \epsilon]$ and let $L = (a^* - \epsilon)c$, we can have $|A(u) - A(v)| \leq L|u - v|$ and $L < 1$ and that satisfy the condition of contraction mapping, so there is a unique u such that $A(u) = u$. But notice that ϵ can be any number that bigger than 0, so if we let $\epsilon \to 0$, for each value of ϵ, there is a unique u solve the ODE and different u have to agree with each other, so the solution is unique on $[-a^*, a^*]$. □

6.1.3 Problem 3

Proof. $\Rightarrow$: Suppose u blows up at point T, that is, $\lim_{t \to T-} u(t) = \infty$. Then because $\frac{u'(t)}{f(u)} = 1$, $\lim_{x \to T-} \int_0^x \frac{u'(t)}{f(u)} dt = T$. Note that $u(0) = a$ and $\lim_{x \to T-} u(x) = \infty$, we use change of variable to get $\int_a^\infty \frac{1}{f(u)} du = T < \infty$.
$\Leftarrow$: If we have $\int_a^\infty \frac{1}{f(u)} du = T$, suppose u does not blow up, then suppose $u(T) = n < \infty$, so $\int_0^T \frac{u'(t)}{f(u)} dt = T$, and we use change of variable to get $\int_a^n \frac{1}{f(u)} du = T$. So $\int_n^\infty \frac{1}{f(x)} dx = 0$, but $f > 0$, contradiction! So u must blow up. □

6.1.4 Problem 4

Proof. Note that in Picard-lindelof Theorem, we can change the condition of $|t| \leq a$ to $|t - t_0| \leq a$ and let $v(t) = u(t - t_0)$, then we will get the same condition.
So we can choose $a = 1/c$ and first let $b = 1$,suppose F is bounded by M in that interval, because of the Lipschitz condition, if we increase b by x, the bound will increase at most xc, so it is possible for us to find a value of b such that $b/M \geq 1/2c$. So $a^* \geq 1/2c$.
By Picard-lindelof Theorem and the above condition, we can get that in the interval $[1/2c - 1/2c, 1/2c + 1/2c] = [0, 1/c]$, the solution of the ODE is unique, so the two solutions must agree on this interval.
We could do this step iteratively, each time pick a different value for b such that $b/M \geq 1/2c$, and we can get the solution is unique on $[1/c, 2/c], [2/c, 3/c]$... all the way to $\min(s_1, s_2)$, so the two solutions must agree with each other on the interval. □

6.1.5 Problem 5

Proof. First, we need to show that $\lim_{t\to T}(u(t))$ exists. We could first build a sequence a where $a_i = u((i-1)T/i)$ Because we know that u is bounded, this sequence is also bounded, so there exists a subsequence that converges to some point x.
We want to show that $\lim_{t\to T}(u(t)) = x$. We know that $u'(t)$ is bounded, suppose $|u'(t)| < C$. For any give ϵ, we can find $\delta = \epsilon/2C$ such that for any $t_1, t_2 \in (T-\delta, T)$:

$$|u(t_1) - u(t_2)| \leq |t_1 - t_2|C \leq \delta C = \epsilon/2$$

Also, because we have the subsequence converges to x and the subsequence must have a point $t_s \in (T-\delta, T)$ such that $|u(t_s) - x| \leq \epsilon/2$, by triangle inequality, for $\forall t \in (T-\delta, T)$, $|u(t) - x| \leq \epsilon$. Since ϵ is arbitrary, $\lim_{t\to T}(u(t)) = x$.
Let $u(T) = x$, then we can find a such that u on $[t-a, t+a]$ is bounded and pick $b = 1$ so we could apply the Picard-lindelof Theorem to get that there is a unique solution of the ODE on $[T-a^*, T+a^*]$. Because of what we have done in question 4, we know that the solution agrees with others on $[T-a^*, T)$, so combine them we could get a solution on $(0, T+a^*)$, so $T+a^* \in S$. But $\sup S = T$, contradiction! So $\sup S = \infty$ □

6.2 Lecture 2

6.2.1 Problem 1

Proof. Let $\phi(t) = x(t) - y(t)$, then:

$$\begin{aligned}
|\phi'(t)| &= |x'(t) - y'(t)| = |f(t, x(t)) - f(t, y(t)) + E_1 - E_2| \\
&\leq c|\phi(t)| + \epsilon = c\left|\int_0^t \phi'(s)ds + \phi(0)\right| + \epsilon \\
&\leq c\int_0^t |\phi'(s)|ds + c|\phi(0)| + \epsilon
\end{aligned}$$

Then by Gronwall's inequality, $|\phi'(t)| \leq (c|\phi(0)| + \epsilon)e^{ct}$, so:

$$\begin{aligned}
|\phi(t)| &= \left|\int_0^t \phi'(s)ds + \phi(0)\right| \leq \left|\int_0^t |\phi'(s)|ds + \phi(0)\right| \\
&\leq \left|\int_0^t (c|\phi(0)| + \epsilon)e^{cs}ds + \phi(0)\right| = \left||\phi(0)|e^{ct} + \frac{\epsilon}{c}(e^{ct} - 1)\right| \\
&= |x(0) - y(0)|e^{ct} + \frac{\epsilon}{c}(e^{ct} - 1)
\end{aligned}$$

□

6.2.2 Problem 2

Proof. Given 2 points $a, b \in \mathbb{R}^n$, then we know that, because the derivative in every direction of ∇V is smaller or equal to c:

$$\begin{aligned}|F(a)-F(b)| &= \left|\int_0^{|b-a|} (\nabla(-\nabla V(a+\frac{t}{|b-a|}(b-a)))) \cdot (b-a)dt\right| \\ &\leq |b-a|c\end{aligned}$$

So we know F is Lipschitz. □

6.2.3 Problem 3

Proof. We can divide this into two cases.

- **Case 1**, when $n = 1$:
  Because V is uniformly convex, there is a unique point x such that $V'(x) = 0$. If $u(0) = x$, then $u'(0) = -V'(x) = 0$, so u would just be a constant function, so $\lim_{t\to\infty} = x$.
  Suppose $u(0) \neq x$, WLOG suppose $u(0) > x$. Then $u'(0) = -V'(u(0)) < 0$. So we know $u(t) > x$ and $u'(t) < 0$ for $\forall t > 0$. Because u is decreasing and bounded, u must converge. Suppose it converges to $n \neq x$. Then $u' = -V'(u)$ must converges to $V'(n)$ which is not 0, and that contradicts with the fact that u converges. So $\lim_{t\to\infty} u(t) = x$.

- **Case 2**, when $n > 1$:
  First, we need to show that there is a unique point x_0 such that $\nabla V(x_0) = 0$.
  We could assume that v's second derivative at any direction is between c and C.
  Then, I will show that it suffices to prove $\lim_{t\to\infty} v(u(t)) = v(x_0)$.
  Because of the uniqueness of x_0, $\nabla v(x_0) = 0$ and $\nabla v(x_0)$'s derivative at any direction is bigger than 0, we could know that x_0 is both the global and local minimal of the v function.
  Claim: if $x \in \mathbb{R}^n$, and $|x - x_0| = a$, then $\frac{1}{2}a^2c \leq v(x) - v(x_0) \leq \frac{1}{2}a^2C$, where c, C are constants of uniform convex.

  Proof.

  $$\begin{aligned}V(x)-V(x_0) &= \int_0^{|x-x_0|} (\nabla V(x_0+\frac{t}{|x-x_0|}(x-x_0))) \cdot (x-x_0)dt \\ &= \int_0^{|x-x_0|}\int_0^t \nabla(\nabla V(x_0+\frac{s}{|x-x_0|}(x-x_0))) \cdot (x-x_0)dsdt\end{aligned}$$

  Because the part inside integration is just second derivative at some point in some direction, that part is bounded by c and C, so we can get:

  $$\int_0^{|x-x_0|}\int_0^t cdsdt \leq V(x) - v(x_0) \leq \int_0^{|x-x_0|}\int_0^t Cdsdt$$

So we know $\frac{1}{2}a^2c \leq V(x) - V(x_0) \leq \frac{1}{2}a^2C$ if $|x - x_0| = a$. □

Given any $\epsilon > 0$, since $\lim_{t\to\infty} V(u(t)) = V(x_0)$, exist T such that when $t > T$, $|V(u(t)) - V(x_0)| \leq \frac{1}{2}\epsilon^2 c$, then by the above claim we know that $|x - x_0| \leq \epsilon$. So $\lim_{t\to\infty} u(t) = x_0$.
Now we just need to show that $\lim_{t\to\infty} V(u(t)) = V(x_0)$. Since we already know that x_0 is the unique minimal point and the only point whose derivative is 0 at all direction, and $\frac{d}{dt}(V(u(t))) = -|\nabla V(u(t))|^2 \leq 0$, similar to the proof in $n = 1$ case, we have $\lim_{t\to\infty} v(u(t)) = V(x_0)$.

□

6.3 Lecture 3

6.3.1 Problem 1

Proof. Suppose such ϵ, δ don't exist.
Then for any $\epsilon > 0$, there is sequence $x_n \in I^{-1}([c - \epsilon, c + \epsilon])$ such that $\lim_{i\to\infty} I'(x_i) = 0$. So we could let $\epsilon \to 0$ and could get a sequence x_n such that $\lim_{i\to\infty} I'(x_i) = 0$ and $\lim_{i\to\infty} I(x_i) = c$.
Because we have $\lim_{|x|\to\infty} I(x) = +\infty$, we know that all elements in x_n are in a bounded interval. So there is a subsequence y_n that converges to some point x_0. Now we are going to show that $I(x_0) = c$ and $I'(x_0) = 0$.
We have $\lim_{i\to\infty} I(y_i) = c$ and $\lim_{i\to\infty} y_i = x_0$, because of the continuity of I, $I(x_0) = c$ and similarly, $I'(x_0) = 0$. Then $x_0 \in K_c$ and that contradicts with the fact that $K_c = \emptyset$ □

6.3.2 Problem 2

Proof. Let $dist(A,B) = \inf\{|x-y| : x \in A, y \in B\}$, then we are going to show that $dist(A,B) = c > 0$. Suppose the opposite, the there is a sequence $x_n \in A$ and $y_n \in B$ such that the sequence $|x_n - y_n| \to 0$. Then because A is compact, there is a subsequence of x_n x_{1n}that converges to a point a, because A is closed, $a \in A$. So the corresponding subsequence y_{1n} must converges to a as well, and because B is also closed, $a \in B$, contradiction.
Now we are going to show that $|\phi(x) - \phi(y)| \leq \frac{1}{dist(A,B)}|x - y|$.
For any given $x, y \in \mathbb{R}^n$, if both of them are in A or B, then $|\phi(x) - \phi(y)| = 0$, satisfy the inequality.
If $x \in A$ and $y \in B$, then $|\phi(x) - \phi(y)| = 1$ and $|x - y| \geq dist(A,B)$, so $|\phi(x) - \phi(y)| \leq \frac{1}{dist(A,B)}|x - y|$.
If $x \in A$ and y not in A or B. Then:

$$
\begin{aligned}
|\phi(x) - \phi(y)| = 1 - \phi(y) = 1 - \frac{dist(y,B)}{dist(y,B) + dist(y,A)} \\
= \frac{dist(y,A)}{dist(y,B) + dist(y,A)} \\
\leq \frac{|x - y|}{dist(A,B)}
\end{aligned}
$$

If $x \in A$ and y not in A or B, this is very similar to the above case.
If x and y not in A or B. Since $dist(x,A) = \inf\{|x-y| : y \in A\}$ and A is closed, there exists $a \in A$ such that $|x-a| = dist(x,A)$, similarly, exist $b \in B$ such that $|y-b| = dist(y,B)$, WLOG suppose $\phi(x) \geq \phi(y)$, so:

$$\begin{aligned}|\phi(x)-\phi(y)| &= \frac{dist(x,B)}{dist(x,B)+dist(x,A)} - \frac{dist(y,B)}{dist(y,B)+dist(y,A)} \\ &\leq \frac{|x-b|}{|x-b|+|x-a|} - \frac{|y-b|}{|y-b|+|y-a|}\end{aligned}$$

Now suppose $\psi(x) = \frac{|x-b|}{|x-b|+|x-a|}$, then:

$$\begin{aligned}\psi'(x) &= \lim_{|\Delta x|\to 0}\left(\frac{|x+\Delta x-b|}{|x+\Delta x-b|+|x+\Delta x-a|} - \frac{|x-b|}{|x-b|+|x-a|}\right)/\Delta x \\ &= \lim_{|\Delta x|\to 0}\left(\frac{|x+\Delta x-b|-|x-b|}{|x-b|+|x-a|}\right)/\Delta x \\ &= \frac{1}{|x-b|+|x-a|} \leq \frac{1}{|a-b|}\end{aligned}$$

So $\psi(x)$ is Lipschitz, and we can get:

$$\begin{aligned}|\phi(x)-\phi(y)| &\leq \frac{|x-b|}{|x-b|+|x-a|} - \frac{|y-b|}{|y-b|+|y-a|} \\ &= \psi(x)-\psi(y) \leq \frac{1}{|a-b|}|x-y| \\ &\leq \frac{1}{dist(A,B)}|x-y|\end{aligned}$$

□

6.4 Lecture 4

6.4.1 Problem 1

Proof.

$$\begin{aligned}J[u+v] &= \int_U (|D(u+v)|^2 - f(x)(u+v)(x))dx \\ &= \int_U |D(u+v)|^2 dx - \int_U f(x)(u+v)(x)dx \\ &= \int_U |D(u)|^2 dx + 2\int_U DuDvdx + \int_U |D(v)|^2 dx - \int_U f(x)u(x) - \int_U f(x)v(x)dx \\ &= J[u] + (2\int_U DuDvdx - \int_U f(x)v(x)dx) + \int_U |D(v)|^2 dx\end{aligned}$$

So we can have $J'[u]v = 2\int_U DuDvdx - \int_U f(x)v(x)dx$, and because

$$2\int_U Du\cdot Dvdx = \int_U (-\Delta u)vdx + \int_{\partial U}\frac{\partial u}{\partial n}vdx$$

Because everything is 0 on the boundary, $\int_U Du \cdot Dv dx = \int_U (-\Delta u) v dx$, so

$$J'[u]v = \int_U (-\Delta u) v dx - \int_U f(x)v(x)dx = \int_U (-\Delta u)v - f(x)v dx$$

So $J'[u] = 0 \Leftrightarrow -\Delta u = f(x)$. □

6.4.2 Problem 2

Proof. Because we have $||w||_H = 1$, so $(\int_U |Dw|^2 dx)^{1/2} = 1$, so $\int_U |Dw|^2 dx = 1 > 0$.
Because we have $|w| \geq 0$, so $|w|^s \geq 0$, so $\int_U |w|^s dx \geq 0$, now we can suppose $w \equiv 0$ in U, then $||w||_H = (\int_U |Dw|^2 dx)^{1/2} = (\int_U 0 dx)^{1/2} = 0 \neq 1$, contradiction.
So we have $\int_U |w|^s dx > 0$ □

7 Presentation projects: Fixed points method and application to ODEs and PDEs

The content below was done as a small independent study project and was presented by Hangyu Pi and my self on the presentation day, Friday, June 9, 2017 at UW-Madison. Each of us gave a short presentation in the usual seminar style for 25 minutes.

7.1 The ODE question

7.1.1 The question

Given a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that is continuous and there exist $C > 0$ such that $|f(t,y) - f(t,z)| \leq C|y - z|$ for all $t, y, z \in \mathbb{R}$ and for the following ODE:

$$\begin{cases} u'(t) = f(t, u(t)) \text{ for } 0 < t < T \\ u(0) = u_0 \end{cases}$$

Is there a solution? Is that solution unique?

7.1.2 Main goals

- First way: using Picard theorem to prove that there is a unique solution in a short time and glue the solutions together to get the global solution.
- Second way: Discretizing the problem and define the piece-wise linear functions, using Arzelà-Ascoli theorem to show the functions converges.
  - Uniqueness
  - Setting of the problem

- Boundness of u_k
- Lipschitz of u_k
- subsequence convergent and converges to the solution
- convergent of the whole sequence

7.1.3 Uniqueness Proof

Suppose both u(t) and v(t) are solutions to the ODE.
We can define $\phi(t) = |v(t) - u(t)|^2$ and $\phi(0) = 0$.

$$\begin{aligned}\phi\prime(t) &= 2|u(t) - v(t)|(u\prime(t) - v\prime(t)) \\ &= 2|u(t) - v(t)|(f(t,u(t)) - f(t,v(t))) \\ &\leq 2C\phi(t)\end{aligned}$$

So by Gronwall's inequality, $\phi(t) \leq e^{2Ct}\phi(0) = 0$.
So $\phi(t) \equiv 0$, which means $u(t) \equiv v(t)$. So we know that the solution is unique.

7.1.4 Setting of the problem

We define function u_k(t),divide the interval [0,T] into k intervals, with equal interval the length of T/k.
For t ∈ [0,T/k),

$$u_k(t) = u_0 + f(0,u_0)t$$

For t ∈ [T/k,2T/k)

$$\begin{aligned}&u_(T/k) = u_0 + f(0,u_0)T/k \\ &u_k(t) = u_k(T/k) + f(T/k,u_k(T/k))(t - T/k)\end{aligned}$$

Iteratively, For t ∈ [aT/k,(a+1)T/k),

$$\begin{aligned}&u_k(aT/k) = u_k[(a-1)T/k)] + f[(a-1)T/k,u_k((a-1)T/k)]T/K \\ &u_k(t) = u_k(aT/k) + f(aT/k,u_k(aT/k))(t - aT/k)\end{aligned}$$

7.1.5 Boundedness

we notice that for any t ∈ [0,T], if t ∈ [aT/k,(a+1)T/k] then u_k(t)∈[u_k(aT/k),u_k(aT/k+T/k)] or [u_k(aT/k+T/k),u_k(aT/k)], so only have to prove u_k(aT/k) is bounded for all a and k

$$\begin{aligned}|u_k(aT/k + T/k) - u_k(aT/k)| &= f[aT/k,u_k(aT/k)]T/k \\ &\leq cT/k|u_k(aT/k)| + cT/k \\ |u_k(aT/k + T/k)| &\leq (cT/k + 1)|u_k(aT/k)| + cT/k\end{aligned}$$

$$|u_k(aT/k)| \le |u_0 - 1|(cT/k+1)^a + 1 \le |u_0 - 1|(cT/k+1)^k + 1$$

which is bounded,so $|u_k(ak)|$ is bounded for any a or k, thus $|u_k(t)|$is bounded for all k when t ∈ [0,T]

$$(cT/k+1)^k \le \lim_{k\to 0}(cT/k+1)^k = (e)^{cT}$$

7.1.6 Lipschitz

Since for any t ∈ [0,T],
$f[t,u_k(t)] \le c|u_k(t)|+c \le c(e)^{cT}+c$, so $f[t,u_k(t)]$ is bounded
For arbitrary x,y∈ [0,T] x≥y

$$\begin{aligned}
&x = pT/k + x_1 \quad x_1 < T/k \quad y = qT/k - y_1 \quad y_1 < T/k \\
&|u_k(x) - u_k(y)| = |x_1 f[pT/k, u_k(pT/k)] \\
&+ u_k(pT/k) - u_k(qT/k) + y_1 f[qT/k, u_k(qT/k)]| \\
&= |x_1 f[pT/k, u_k(pT/k)] \\
&+ \sum\nolimits_{a=q}^{p} f[aT/k, u_k(aT/k]T/k + y_1 f[qT/k, u_k(qT/k)]| \\
&\le |sup(f[t,u_k(t)])||x_1 + (p-q)T/k + y_1| \\
&= |sup(f[t,u_k(t)])||x - y|
\end{aligned}$$

We have $|sup(f[t,u_k(t)])| \le c(e)^{cT}+c$, which is bounded
so $u_k(t)$ is Lipschitz

7.1.7 u is the solution to the ODE

Since u_k is Lipschitz, for $\forall\epsilon$, we can just take $\delta = \epsilon/C_u$, then for any k,t, $|u_k(t) - u_k(t+\delta)| \le C_u\delta = \epsilon$. So U is equicontinuous.
According to the Arzelà-Ascoli theorem, we know that there is a subsequence of U that is uniformly convergent.
Let $u(t)$ be the function that the subsequence of $u_k(t)$ converges uniformly to.
Let $v_k(t) = u_0 + \int_0^t f(x, u_k(x))dx$. Then

$$\begin{aligned}
|u_k - v_k| &= |(u_0 + \int_0^t u_k\prime(x)dx) - (u_0 + \int_0^t f(x, u_k(x))dx)| \\
&= |\int_0^t u_k\prime(x) - f(x, u_k(x))dx|
\end{aligned}$$

Let T_n be the end of the nth interval, that is, $T_n = nT/k$.
We define $u_k\prime(t) = f(T_n, u_k(T_n))$ when $T_n \le t < T_{n+1}$. Since $u_k\prime(t)$ is a piece-wise constant function, the set of discontinuous points is countable, $u_k\prime(t)$ is integrable and its integral is just $u_k(t)$.
Suppose $T_n \le t < T_{n+1}$, notice that $u_k\prime(t) = f(T_n, u_k(T_n))$. So

$$\begin{aligned}
&|u_k\prime(t) - f(t, u_k(t))| \\
&= |f(T_n, u_k(T_k)) - f(t, u_k(t))| \\
&\le |f(T_n, u_k(T_n)) - f(t, u_k(T_n))| + |f(t, u_k(T_n)) - f(t, u_k(t))|
\end{aligned}$$

By the continuous of f, the first term will go to 0 as k goes to ∞.
On the other hand, because both f and u are Lipschitz

$$|f(t,u_k(T_n))-f(t,u_k(t))| \leq C|u_k(T_n)-u_k(t)| \leq CC_u|t-T_n| \leq CC_uT/k.$$

So the second term goes to 0 as well, so as $k \to \infty$, $|u_k - v_k| \to 0$. Also,

$$|v(t)-v_k(t)| \leq T|f(t,u(t))-f(t,u_k(t))| \leq TC|u(t)-u_k(t)|$$

Because T and C are constant, for the sub sequence of u_k that is uniformly convergent, as k goes to ∞, $|v(t)-v_k(t)|$ goes to 0, so v_k converges to v.
For the sub sequence of u_k that is uniformly convergent, u_k and v_k converge to the same thing and v_k converges to v, so the sub sequence of u_k converges to v.
Then we know that the subsequence of u_k that is uniformly convergent converges to $v(t) = u_0 + \int_0^t f(x,u(x))dx$ as well.
So $u(t) = u_0 + \int_0^t f(x,u(x))dx$, and $u\prime(t) = f(t,u(t))$ and $u(0) = u_0$. So $u(t)$ is a solution of the ODE. Then we know that there will always be a unique solution of the ODE.

7.1.8 Proof of the convergence

Lemma 7.1. For a sequence a_k in a normed vector space X and $a \in X$, if for every sub sequence of a_k, one of its sub sequences converges to a, then a_k converges to a.

Proof. Suppose a_k does not converges to a, then for $\forall \epsilon > 0$, there exists infinite number of $a_i \in a_k$ such that $|a_i - a| > \epsilon$. Those a_i form a sub sequence of a_k and it doesn't have a sub sequence that converges to a. Contradiction! □

For each of the sub sequence of U, we already proved that it is both bounded and equi-continuous, so by the Arzelà-Ascoli theorem, it has a sub sequence that converges. By the lemma, we know that U is convergent.

7.2 The PDE question

7.2.1 Problem - A Simple Reaction-Diffusion PDE

We want to study the well-posedness of the following PDE problem.

$$u : [0,1] \times [0,\infty) \to \mathbb{R}$$
$$(x,t) \longmapsto u(x,t)$$

we have the following PDE

$$\begin{cases} u_t = u_{xx} + f(t,u(x,t)) \\ u(0,t) = u(1,t) = 0 \text{ for all } t \\ u(x,0) = u_0(x) \text{ for } x \in (0,1) \end{cases}$$

with $u_0 \in C^2[0,1]$ and f : $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitz on y

- Is there a global solution?
- Is that solution unique?

7.2.2 Uniqueness Proof

Suppose both u(x,t) and v(x,t) are solutions to the PDE.
We can define $\phi(t) = \int_0^1 |v(x,t) - u(x,t)|^2 dx$. Clearly $\phi(0) = 0$.

$$\begin{aligned}
\phi_t &= \int_0^1 2(u-v)(u_t - v_t)dx \\
&= \int_0^1 2(u-v)(f(t,u) - f(t,v))dx + \int_0^1 2(u-v)(u_{xx} - v_{xx})dx \\
&\leq \int_0^1 2C(u-v)^2 dx + 2\int_0^1 (u-v)(u-v)_{xx} dx \\
&= 2C\phi(t) + (2(u-v)(u_x - v_x))|_0^1 - 2\int_0^1 (u_x - v_x)^2 dx \\
&\leq 2C\phi(t)
\end{aligned}$$

Then we know $\phi(t) \leq e^{2Ct}\phi(0) = 0$. So $\phi(t) \equiv 0$ for $\forall x$.
$(u-v)^2 = \frac{d\phi}{dx} = 0$ for $\forall x$, t. So $u \equiv v$, and the solution is unique.

7.2.3 Existence Proof - Set up of the Proof

Fix $k \in \mathbb{N}$, define $u_k(x,0) = u_k^0(x) = u_0(x)$ for all $x \in [0,1]$ let τ = T/k
$u_k(x, T/k) = u_k^1(x)$ and $u_k^1(.)$ satisfies

$$[u_k^1(x) - u_k^0(x)]/\tau = [u_k^1(x)]_{xx} + f[0, u_k^0(x)]$$

Then:

$$u_k^1(x) - \tau[u_k^1(x)]_{xx} = u_k^0(x) + \tau f[0, u_k^0(x)]$$

Iteratively, u_k^j is the solution of the following ODE:

$$\begin{cases} u_k^j(x) - \tau[u_k^j(x)]_{xx} = u_k^{j-1}(x) + \tau f[(j-1)\tau, u_k^{j-1}(x)] \\ u_k^j(0) = u_k^j(0) = 0 \end{cases}$$

For $t \in ((j-1)\tau, j\tau)$, u_k is a linear combination of u_k^{j-1} and u_k^j:

$$u_k(x,t) = u_k^{j-1}(x) + [t - (j-1)\tau]\frac{u_k^j(x) - u_k^{j-1}(x)}{\tau}$$

7.2.4 An important Lemma

Lemma 7.2. Let $w \in C^2([0,1])$ with $w(0) = w(1) = 0$ be a given function. Let $g : \mathbb{R} \to \mathbb{R}$ be Lipschitz. Assume v is the solution to the following:

$$\begin{cases} v - \tau v_{xx} = w + \tau g(w) \\ v(0) = v(1) = 0 \end{cases}$$

Then there exists a constant $C > 0$ such that:

$$||v||_{L^\infty} \leq (1 + C\tau)||w||_{L^\infty}$$

Proof. $w + \tau g(w) \leq w + \tau(g(0) + cw) = (1 + \tau c)w + \tau g(0)$
Noticed that $g(0)$ is just a constant, so we can get $|v - \tau v_{xx}| \leq |(1 + \tau C)w|$ for some C. Suppose v reaches its maximum at x_1, minimum at x_2, if x_1 or x_2 is at boundary, we are done. Otherwise $v(x_1)_{xx} \leq 0$ and $v(x_2)_{xx} \geq 0$. Then we know $-|v - \tau v_{xx}| \leq v \leq |v - \tau v_{xx}|$. The lemma is proved. □

7.2.5 Existence Proof - Boundedness

According to our set up:

$$u_k^j(x) - \tau(u_k^j(x)_{xx} = (u_k^{j-1}(x)) + \tau f((j-1)\tau, u_k^{j-1}(x))$$

So by our lemma,

$$|u_k^j(x)| \leq |(1 + \tau c)u_k^{j-1}(x)|$$

for some $c > 0$. So

$$|u_k^j(x)| \leq |u_k^0(x)|(c\tau + 1)^j \leq |u_k^0(x)|e^{cT}$$

So u_k is bounded for any j, x.
For simplicity, we assume f is smooth. Because we know u_k^j and u_k^{j-1} are differentiable, $u_k^j(x)_{xx}$ is differentiable. Because f is Lipschitz and by the chain rule:

$$|f((j-1)\tau), u_k^{j-1}(x))_x| = |f((j-1)\tau, x)_x u_k^{j-1}(x)_x| \leq |cu_k^{j-1}(x)_x|$$

And by our set up of u:

$$|u_k^j(x)_x - \tau(u_k^j(x)_{xxx}| \leq |(u_k^{j-1}(x))_x + \tau c u_k^{j-1}(x)_x| = (1 + \tau c)|u_k^{j-1}(x)_x|$$

For simplicity, we avoid complications on the boundary and assume that $(u_k^j(0))$ and $(u_k^j(1))$ are always bounded. With the similar idea of the lemma, we can get:

$$(\tau c + 1)|u_k^{j-1}(x)_x| \geq (u_k^j(x))_x \geq -(\tau c + 1)|u_k^{j-1}(x)_x|$$

So we know $(u_k^j)_x$ is bounded by $(\tau c + 1)|u_k^{j-1}(x)_x|$. So for $\forall j$, $(u_k^j)_x \leq e^{cT} u_0(x)_x$ and since $u_0(x)_x$ is bounded, $(u_k^j)_x$ is bounded.
We first prove that $(u_k)_{xx}$ is bounded. Let $g(x) = f((j-1)\tau, x)$, by the chain rule, we can get:

$$\begin{aligned} f((j-1)\tau, u_k^{j-1}(x))_{xx} &= g''(u_k^{j-1}(x)) \\ &= g'' u_k^{j-1}(x)_x + g' u_k^{j-1}(x)_{xx} \end{aligned}$$

Because we assumed that f is smooth, so g is smooth, so g'' is continuous. And because u_k^j is bounded, the domain of g'' is bounded. So in the above equation, g'' is bounded. So

$$|u_k^j(x)_{xx} - \tau(u_k^j(x)_{xxxx}| \leq (1+\tau c)u_k^j(x)_{xx} + \tau C$$

So we can get $(u_k^j)_{xx} \leq e^{cT}u_0(x)_{xx}$, so $(u_k)_{xx}$ is bounded. Then:

$$\frac{u_k^j(x) - u_k^{j-1}(x)}{\tau} = g(u_k^{j-1}(x)) + (u_k^j(x))_{xx}$$

So u_k is Lipschitz in t. Doing this recursively, we can get all of u_k's derivatives in x of any order is bounded.

7.2.6 Existence Proof - Convergence

Because we proved earlier that u_k's derivative to any order is bounded, we know that $(u_k)_{xx}$ is bounded and Lipschitz, so by AA, we know that a subsequence of $(u_k)_{xx}$ converges, suppose it converges to u_2. Then we let $u_1 = \int u_2 + C$ with C such that $(u_k)_x$ converges to u_1, and let $u = \int u_1 + C$ with C such that (u_k) converges to u. We define:

$$v(x,t) = u_0(x) + \int_0^t f[s, u(x,s)] + u_{xx}(x,s)ds$$

$$v_k(x,t) = u_0(x) + \int_0^t f[s, u_k(x,s)] + [u_k]_{xx}(x,s)ds$$

First we prove that $|u_k - v_k| \to 0$. Although $(u_k)_s(s)$ is not continuous, it has finite discontinuous points, so we can take its integral and will get u_k. So

$$|u_k - v_k| = \int_0^t |(u_k)_s(s) - f[s, u_k(x,s)] - (u_k)_{xx}(x,s)|ds$$

For $t \in ((j-1)\tau, j\tau]$, we can cut $(u_k)_s$ into j parts and get:

$$\begin{aligned}|u_k - v_k| = |\sum_{i=0}^{j-2}\int_{i\tau}^{(i+1)\tau} & f(i\tau, u_k^i) - f[s, u_k(x,s)] + (u_k^{i+1})_{xx} - (u_k)_{xx}ds \\ + \int_{(j-1)}^{t} & f((j-1)\tau, u_k^{j-1}) - f[s, u_k(x,s)] + (u_k^j)_{xx} - (u_k)_{xx}ds|\end{aligned}$$

Because we have f is smooth and we have it on a bounded interval, we have f is Lipschitz on for all variables. Also, because u_k is Lipschitz, we get (suppose the Lipschitz constant for f in x, t and u are c, C, c'):

$$\begin{aligned}&|f[(j-1)\tau, u_k^{j-1}] - f[s, u_k(x,s)]| \\ &\leq |f[(j-1)\tau, u_k^{j-1}(x)] - f[(j-1)\tau, u_k(x,s)]| \\ &+ |f[(j-1)\tau, u_k(x,s)] - f[s, u(x,s)]| \\ &\leq c|u_k^{j-1}(x) - u_k(x,s)| + C(s-(j-1)\tau) \leq (cc'+C)\tau\end{aligned}$$

Also, since $(u_k)_{xx}$ is a linear combination of $(u_k^{j-1})_{xx}$ and $(u_k^j)_{xx}$, $|(u_k^j)_{xx}-(u_k)_{xx}| \leq |(u_k^j)_{xx}-(u_k^{j-1})_{xx}| \leq c'\tau$. So we can get:

$$|u_k - v_k| \leq \int_0^t (cc' + C)\tau + c'\tau ds = (cc' + C + c')t\tau$$

This clearly goes to 0 as τ goes to 0, so $|u_k - v_k| \to 0$. Then:

$$\begin{aligned}|v_k - v| &= \int_0^t |f(s,u_k) + u_{kxx} - f(s,u) - u_{xx}|ds \\ &\leq |\int_0^t |f(s,u_k) - f(s,u)|ds + \int_0^t |(u_k)_{xx} - u_{xx}|ds \\ &\leq c|u_k - u| + |(u_k)_{xx} - u_{xx}|\end{aligned}$$

Since u_k and $(u_k)_{xx}$ converges to u and u_{xx} uniformly, this goes to 0. Then we have $v_k \to v$. Put these results together we have u_k uniformly converges to v, however we already have that u_k uniformly converges to u. So $u = v$, u satisfies the PDE. □

References

[1] L. C. Evans, Partial Differential Equations, Graduate studies in Mathematics, AMS, Providence, RI, 1998.