A Metric on Probabilities, and Products of Loeb Spaces*

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Abstract

We first introduce two functions on finitely additive probability spaces that behave well under products: discrepancy, which measures how close one space comes to extending another, and bi-discrepancy, which is a pseudo-metric on the collection of all spaces on a given set, and a metric on the collection of complete spaces. We then apply these to show that the Loeb space of the internal product of two internal finitely additive probability spaces depends only on the Loeb spaces of the two original internal spaces. Thus the notion of a Loeb product of two Loeb spaces is well-defined. The Loeb operation induces an isometry from the nonstandard hull of the space of internal probability spaces on a given set to the space of Loeb spaces on that set, with the metric of bi-discrepancy.

1 Introduction

The most important construction in the application of model theory to probability is the Loeb measure construction from [10]. The Loeb operation converts an internal finitely additive probability space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ to a complete σ-additive probability space $L(\mathcal{M}) = (\Omega, L(\mathcal{F}), L(\mu))$. It provides a valuable tool for proving mathematical results using model-theoretic methods (see, for example, [1], [8], [9], and [11]). The purpose of this paper is to show that the "Loeb product" of two Loeb spaces is well-defined. That is, given two internal probability spaces, the Loeb space of the internal product depends only on the Loeb spaces of the original internal spaces. Thus, it is

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appropriate to call the Loeb spaces of the original spaces “factors”, and to call the Loeb space of the internal product the “Loeb product”.

For each $i = 1, 2$, let $\mathcal{M}_i = (\Omega_i, \mathcal{F}_i, \mu_i)$ be an internal probability space, where $\mathcal{F}_i$ is an internal algebra of subsets of a nonempty internal set Ω_i, and μ_i a finitely additive internal measure on $\mathcal{F}_i$. There are several ways to construct σ-additive product probability spaces based on the Loeb operations. First, we take the Loeb spaces $L(\mathcal{M}_i) = (\Omega_i, L(\mathcal{F}_i), L(\mu_i))$ for $i = 1, 2$ and take their usual σ-additive measure-theoretic product

$$L(\mathcal{M}_1) \otimes^\sigma L(\mathcal{M}_2) = (\Omega_1 \times \Omega_2, L(\mathcal{F}_1) \otimes^\sigma L(\mathcal{F}_2), \mu_1 \otimes^\sigma \mu_2).$$

Its completion is denoted by $(L(\mathcal{M}_1) \otimes^\sigma L(\mathcal{M}_2))^c$. Second, let

$$\mathcal{M}_1 \otimes \mathcal{M}_2 = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$$

be the internal product space, where $\mathcal{F}_1 \otimes \mathcal{F}_2$ is simply the internal algebra of all *finite disjoint unions of rectangles $A_1 \times A_2$ with $A_i \in \mathcal{F}_i$. The Loeb space $L(\mathcal{M}_1 \otimes \mathcal{M}_2)$ of $\mathcal{M}_1 \otimes \mathcal{M}_2$ is called the **Loeb product space**. Third, when the internal probability spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are ${}^*\sigma$-additive, let $\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2$ be the internal ${}^*\sigma$-additive probability space generated by the internal product $\mathcal{M}_1 \otimes \mathcal{M}_2$. The Loeb space $L(\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2)$ of $\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2$ is called the **Loeb σ-product space**.

As already noted in [2], $(L(\mathcal{M}_1) \otimes^\sigma L(\mathcal{M}_2))^c \subseteq L(\mathcal{M}_1 \otimes \mathcal{M}_2)$. It is shown in [13] that the inclusion is always proper when $L(\mathcal{M}_1)$ and $L(\mathcal{M}_2)$ are non-atomic (a specific example can be found in [1] p.74). In fact, Theorem 6.2 in [13] shows that the Loeb product space is very rich in the sense that it can be endowed with independent processes that are not measurable in $(L(\mathcal{M}_1) \otimes^\sigma L(\mathcal{M}_2))^c$ but have almost independent random variables with any variety of distributions. In addition, it is shown in [3] that there is a continuum of increasing Loeb product null sets with large gaps in the sense that their set differences have $L(\mu_1) \otimes^\sigma L(\mu_2)$-outer measure one. Thus, the Loeb product space is much richer than the usual product even on null sets. It is also clear that $L(\mathcal{M}_1 \otimes \mathcal{M}_2) \subseteq L(\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2)$ when both $\mathcal{M}_1$ and $\mathcal{M}_2$ are internally ${}^*\sigma$-additive.

Though the Loeb product (or Loeb σ-product) space is usually much richer than the completion of the usual product of the Loeb spaces, it still has the Fubini property ([9]). This property plays a key role in the discovery of some basic phenomena involving independence (see [13] and [14]). Loeb product spaces are also useful in solving stochastic differential equations ([1]), in chaos decompositions ([4] and Chapter 6 of[11] by Osswald), and in the model theory of stochastic processes ([8], [9]).

Recently, a number of special measure-theoretic properties of Loeb spaces were discovered and formulated in conventional terms; see, for example, [6], [8], [11], [13] and their references. For applications of these special properties, one can simply regard the Loeb space as a primitive object without going through internal operations at all. On the other hand, the definition of Loeb product (Loeb σ-product) relies on the internal product (the internal $^*\sigma$-additive product). Also, different internal probability spaces may give the same Loeb space. So the basic question is whether for any two given Loeb spaces, one can still define their Loeb product (or their Loeb σ-product) unambiguously. In this paper, we show that the Loeb product space $L(\mathcal{M}_1 \otimes \mathcal{M}_2)$ depends only on the factor Loeb spaces $L(\mathcal{M}_1)$ and $L(\mathcal{M}_2)$, and not on the internal spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ themselves. We also prove that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are $^*\sigma$-additive, then the Loeb σ-product $L(\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2)$ depends only on $L(\mathcal{M}_1)$ and $L(\mathcal{M}_2)$. Therefore both the Loeb product and the Loeb σ-product are well-defined functions of the factor Loeb spaces.

In Sections 2 and 3, we develop some general methods to study the collection of (finitely additive) standard probability spaces. A standard function $d(\mathcal{M}, \mathcal{N})$, which we call the discrepancy, is introduced. It measures how close one (finitely additive) probability space comes to extending another. The discrepancy function is of independent interest, and its properties are studied in detail in Section 2. In particular, given a nonempty set Ω, the bi-discrepancy $d_2(\mathcal{M}, \mathcal{N}) = d(\mathcal{M}, \mathcal{N}) + d(\mathcal{N}, \mathcal{M})$ is a pseudo-metric on the set of all probability spaces on Ω, and a metric on the set of all complete probability spaces on Ω. In Section 3 it is shown that the discrepancy from one product probability space to another product probability space is bounded by the sum of the discrepancies of the respective factor spaces.

In Section 4, we consider Loeb extensions and Loeb equivalence of internal probability spaces, which have useful characterizations in terms of the discrepancy. It shown that the space $\mathcal{L}_\Omega$ of all Loeb spaces on Ω is a complete metric space under bi-discrepancy, and that the Loeb operation induces an isometry from the nonstandard hull of the space of internal probability spaces to $\mathcal{L}_\Omega$. Section 5 presents the results on the uniqueness of the Loeb product and the Loeb σ-product. Finally, Section 6 deals with hyperfinite probability spaces. It is shown that when one of the factors is the Loeb space of a hyperfinite probability space, the Loeb product and the Loeb σ-product are the same. This leads to the study of spaces that are Loeb equivalent to a hyperfinite probability space.

A few questions are posed in Section 7. For background in nonstandard probability see, e.g., [1] or [11]. As usual, we work in an ω_1-saturated nonstandard universe.

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2 Discrepancy of probability spaces

A standard probability space $(\Omega, \mathcal{F}, \mu)$ is only understood to have a finitely additive measure μ on an algebra $\mathcal{F}$ of subsets of a nonempty set Ω with $\mu(\Omega) = 1$ unless we explicitly assume countable additivity (also termed σ-additivity). In this and the next section, we only work with standard probability spaces.

We first give a formal definition of the notion of discrepancy, which measures how close one probability space comes to extending another.

Definition 2.1 *Let $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and $\mathcal{N} = (\Omega, \mathcal{G}, \nu)$ be probability spaces. Define the* **outer measure** $\bar{\nu}$ *so that*

$$\bar{\nu}(B) = \inf\{\nu(C) : B \subseteq C \in \mathcal{G}\}$$

for any $B \subseteq \Omega$. Define the **discrepancy** $d(\mathcal{M}, \mathcal{N})$ *of $\mathcal{M}$ from $\mathcal{N}$ by*

$$d(\mathcal{M}, \mathcal{N}) = \sup\{\bar{\nu}(B) - \mu(B) : B \in \mathcal{F}\}.$$

Define the **bi-discrepancy** *between $\mathcal{M}$ and $\mathcal{N}$ as the sum*

$$d_2(\mathcal{M}, \mathcal{N}) = d(\mathcal{M}, \mathcal{N}) + d(\mathcal{N}, \mathcal{M}).$$

It is easy to see that if $\mathcal{N}$ extends $\mathcal{M}$ (denoted by $\mathcal{M} \subseteq \mathcal{N}$) then $d(\mathcal{M}, \mathcal{N}) = 0$, and in particular that $d(\mathcal{M}, \mathcal{M}) = d_2(\mathcal{M}, \mathcal{M}) = 0$. Note that the outer measure $\bar{\nu}$ as defined here may be different from the usual outer measure defined in a textbook on measure theory (see [5] or [12]), where a countable covering is used. The following lemma shows that the discrepancy function is non-negative and satisfies the triangle inequality.

Lemma 2.2 *Let $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$, $\mathcal{N} = (\Omega, \mathcal{G}, \nu)$ and $\mathcal{P} = (\Omega, \mathcal{H}, \rho)$ be probability spaces. Then*

(a) $d(\mathcal{M},\mathcal{N}) = \sup\{|\bar{\nu}(B) - \mu(B)| : B \in \mathcal{F}\}$*, and* $0 \le d(\mathcal{M},\mathcal{N}) \le 1$.
(b) The function $d(\cdot,\cdot)$ *satisfies the triangle inequality*

$$d(\mathcal{M},\mathcal{P}) \le d(\mathcal{M},\mathcal{N}) + d(\mathcal{N},\mathcal{P}).$$

Proof. (a) For any $B \in \mathcal{F}$, we have $\bar{\nu}(B) + \bar{\nu}(\Omega \setminus B) \ge 1$, and hence

$$\bar{\nu}(\Omega \setminus B) - \mu(\Omega \setminus B) \ge (1 - \bar{\nu}(B)) - (1 - \mu(B)) = -(\bar{\nu}(B) - \mu(B)).$$

This means that if $\bar{\nu}(B) - \mu(B) < 0$, then

$$|\bar{\nu}(B) - \mu(B)| = -(\bar{\nu}(B) - \mu(B)) \le \bar{\nu}(\Omega \setminus B) - \mu(\Omega \setminus B) \le d(\mathcal{M},\mathcal{N}).$$

When $\bar{\nu}(B) - \mu(B) \ge 0$, it is obvious that $|\bar{\nu}(B) - \mu(B)| \le d(\mathcal{M},\mathcal{N})$. By the arbitrary choice of B, we have $\sup\{|\bar{\nu}(B) - \mu(B)| : B \in \mathcal{F}\} \le d(\mathcal{M},\mathcal{N})$. The opposite inequality is clear. Hence $d(\mathcal{M},\mathcal{N}) = \sup\{|\bar{\nu}(B) - \mu(B)| : B \in \mathcal{F}\}$, which implies that $0 \le d(\mathcal{M},\mathcal{N}) \le 1$.

(b) For any $B \in \mathcal{F}$, we have $\bar{\nu}(B) - \mu(B) \le d(\mathcal{M},\mathcal{N})$. Let ϵ be any positive real number. There is a set $C \in \mathcal{G}$ such that $B \subseteq C$ and $\nu(C) - \epsilon/2 \le \bar{\nu}(B)$. Similarly, there is a set $D \in \mathcal{H}$ such that $C \subseteq D$ and $\rho(D) - \epsilon/2 \le \bar{\rho}(C)$. Add the three inequalities together and rearrange to obtain

$$\rho(D) - \mu(B) \le d(\mathcal{M},\mathcal{N}) + (\bar{\rho}(C) - \nu(C)) + \epsilon \le d(\mathcal{M},\mathcal{N}) + d(\mathcal{N},\mathcal{P}) + \epsilon.$$

Since $\bar{\rho}(B) \le \rho(D)$, we have $\bar{\rho}(B) - \mu(B) \le d(\mathcal{M},\mathcal{N}) + d(\mathcal{N},\mathcal{P}) + \epsilon$, which implies the desired triangle inequality by taking arbitrary choices of $B \in \mathcal{F}$ and $\epsilon > 0$. ■

Lemma 2.2(a) shows that in the special case that $\mathcal{M},\mathcal{N}$ are complete countably additive probability spaces with the same underlying σ-algebra $\mathcal{F} = \mathcal{G}$, the discrepancy $d(\mathcal{M},\mathcal{N})$ is equal to the total variation distance.

Theorem 2.3 *The bi-discrepancy* $d_2(\cdot,\cdot)$ *is a pseudo-metric on the family of all probability spaces on* Ω.

Proof. The bi-discrepancy is clearly symmetric, and the triangle inequality follows from Lemma 2.2 (b). ■

The following example shows that in general, the discrepancy d is not a pseudo-metric because $d(\mathcal{M},\mathcal{N}) \ne d(\mathcal{N},\mathcal{M})$.

Example 2.4 *(a) Let $\mathcal{N}$ be the probability space such that $\Omega = [0, 1)$, $\mathcal{G}$ be the set of all finite unions of disjoint sub-intervals $[a, b)$ in $[0, 1)$, and ν be the Lebesgue measure restricted to $\mathcal{G}$. Let $\mathcal{M}$ be the Borel probability space on $[0, 1)$ with the Lebesgue measure. Let Q be the set of rational numbers in $[0, 1)$. It is clear that $\bar{\nu}(Q) = 1$ and $\mu(Q) = 0$, and it follows that $d(\mathcal{M}, \mathcal{N}) = 1$ but $d(\mathcal{N}, \mathcal{M}) = 0$.*

(b) Here is a finite example. Let m be a positive integer greater than 3 and $\Omega = \{1, 2, \ldots, m\}$. Let $\mathcal{F}$ be the algebra generated by the partition $\{\{1, 2\}\} \cup \{\{i\} : 3 \le i \le m\}$ and let $\mu(\{1, 2\}) = 2/m, \mu(\{i\}) = 1/m$ for $3 \le i \le m$. Let $\mathcal{G}$ be the algebra of all subsets of Ω and let ν be the uniform counting probability measure. It is easy to check that $d(\mathcal{M}, \mathcal{N}) = 0$ and $d(\mathcal{N}, \mathcal{M}) = 1/m$.

The outer measure $\bar{\nu}$ associated with a finitely additive measure ν is used to define the discrepancy. We shall also consider the inner measure $\underline{\nu}$. The outer and inner measures are then used to extend ν to its completion ν^c.

Definition 2.5 *Let $\mathcal{N} = (\Omega, \mathcal{G}, \nu)$ be a probability space. Let $\underline{\nu}$ be the inner measure such that for any $B \subseteq \Omega$, $\underline{\nu}(B) = \sup\{\nu(C) : B \supseteq C \in \mathcal{G}\}$. The* **completion** *of $\mathcal{N}$ is defined as the probability space $\mathcal{N}^c = (\Omega, \mathcal{G}^c, \nu^c)$ where $\mathcal{G}^c = \{B \subseteq \Omega : \bar{\nu}(B) = \underline{\nu}(B)\}$, $\nu^c = \bar{\nu}|_{\mathcal{G}^c}$. $\mathcal{N}$ is said to be* **complete** *if $\mathcal{N}^c = \mathcal{N}$.*

Here is a collection of easy facts that will be needed.

Lemma 2.6 *(a) $\mathcal{N}^c$ is a finitely additive probability space extending $\mathcal{N}$.*

(b) If $\mathcal{N}$ is a countably additive probability space, then so is $\mathcal{N}^c$, which is the completion of $\mathcal{N}$ in the usual measure-theoretic sense.

(c) $\mathcal{N}^c$ is complete.

(d) If $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M}^c \subseteq \mathcal{N}^c$.

(e) $d_2(\mathcal{N}, \mathcal{N}^c) = 0$ and $d(\mathcal{M}, \mathcal{N}) = d(\mathcal{M}^c, \mathcal{N}^c)$.

(f) $d(\mathcal{M}, \mathcal{N}) = 0$ if and only if $\mathcal{M} \subseteq \mathcal{N}^c$.

(g) $d(\mathcal{M}^c, \mathcal{N}^c) = 0$ if and only if $\mathcal{M}^c \subseteq \mathcal{N}^c$.

(h) $d_2(\mathcal{M}^c, \mathcal{N}^c) = 0$ if and only if $\mathcal{M}^c = \mathcal{N}^c$.

Proof. The result (a) is essentially known. We include its proof for the sake of completeness.

We first show that $\mathcal{G}^c$ is an algebra. For any $E \in \mathcal{G}^c$, we have

$$\bar{\nu}(\Omega \setminus E) = 1 - \underline{\nu}(E) = 1 - \bar{\nu}(E) = \underline{\nu}(\Omega \setminus E),$$

and hence $(\Omega \setminus E) \in \mathcal{G}^c$.

Next, take any $B_i \in \mathcal{G}^c$ for $i = 1, 2$. For any given $\epsilon > 0$, there exist sets $C_i, D_i \in \mathcal{G}$ for $i = 1, 2$ such that $D_i \subseteq B_i \subseteq C_i$ and

$$\nu(C_i) - \epsilon/4 \leq \bar{\nu}(B_i) = \underline{\nu}(B_i) \leq \nu(D_i) + \epsilon/4.$$

It follows that

$$\nu(C_i \setminus D_i) = \nu(C_i) - \nu(D_i) \leq \epsilon/2.$$

Let $B = B_1 \cup B_2, C = C_1 \cup C_2$, and $D = D_1 \cup D_2$. Then $D \subseteq C$, and

$$C \setminus D = (C_1 \setminus D) \cup (C_2 \setminus D) \subseteq (C_1 \setminus D_1) \cup (C_2 \setminus D_2).$$

Therefore

$$0 \leq \bar{\nu}(B) - \underline{\nu}(B) \leq \nu(C) - \nu(D) = \nu(C \setminus D) \leq \nu(C_1 \setminus D_1) + \nu(C_2 \setminus D_2) \leq \epsilon.$$

Hence $\bar{\nu}(B) = \underline{\nu}(B)$, and $B \in \mathcal{G}^c$.

If $B_1 \cap B_2 = \emptyset$, then $D_1 \cap D_2 = \emptyset$, and

$$\bar{\nu}(B_1) + \bar{\nu}(B_2) \leq \nu(D_1) + \epsilon/4 + \nu(D_2) + \epsilon/4 = \nu(D) + \epsilon/2 \leq \bar{\nu}(B) + \epsilon/2,$$

which implies $\bar{\nu}(B_1) + \bar{\nu}(B_2) \leq \bar{\nu}(B)$ by arbitrarily choosing $\epsilon > 0$. Since it is easy to verify that $\bar{\nu}(B) \leq \bar{\nu}(B_1) + \bar{\nu}(B_2)$, we have $\bar{\nu}(B) = \bar{\nu}(B_1) + \bar{\nu}(B_2)$. Hence (a) is proven.

For (b), assume that $\mathcal{N}$ is countably additive. Then $B \in \mathcal{G}^c$ if and only if there exist $C, D \in \mathcal{G}$ such that $D \subseteq B \subseteq C$ and $\nu(C \setminus D) = 0$. Thus $\mathcal{N}^c$ is the completion of $\mathcal{N}$ in the usual measure-theoretic sense.

For (c), we first show that $\overline{\nu^c} = \bar{\nu}$. Take any $B \subseteq \Omega$. Since the set $\{\nu^c(C) : B \subseteq C \in \mathcal{G}^c\}$ contains the set $\{\nu(C) : B \subseteq C \in \mathcal{G}\}$, we have $\overline{\nu^c}(B) \leq \bar{\nu}(B)$. On the other hand, take any $E \in \mathcal{G}^c$ with $B \subseteq E$. For any given $\epsilon > 0$, there exists a set $D \in \mathcal{G}$ such that $D \supseteq E$ and $\nu(D) - \epsilon \leq \nu^c(E)$. Thus, $\bar{\nu}(B) - \epsilon \leq \nu^c(E)$. Then $\bar{\nu}(B) \leq \nu^c(E)$ follows from the arbitrary choice of ϵ. By taking the infimum on all the $E \in \mathcal{G}^c$ with $B \subseteq E$, we obtain that $\bar{\nu}(B) \leq \overline{\nu^c}(B)$. It follows that $\underline{\nu^c} = \underline{\nu}$. Therefore, $(\mathcal{G}^c)^c = \mathcal{G}^c$ and $(\nu^c)^c = \nu^c$, i.e., $(\mathcal{N}^c)^c = \mathcal{N}^c$.

For (d), note that for any $B \subseteq \Omega$, $\underline{\mu}(B) \leq \underline{\nu}(B) \leq \bar{\nu}(B) \leq \bar{\mu}(B)$. If $B \in \mathcal{F}^c$, then we must have $\underline{\nu}(B) = \bar{\nu}(B) = \bar{\mu}(B)$, and hence $B \in \mathcal{G}^c$ and $\nu^c(B) = \mu^c(B)$. (d) is shown.

For (e), note that $d(\mathcal{N}^c, \mathcal{N}) = \sup\{\bar{\nu}(B) - \nu^c(B) : B \in \mathcal{G}^c\}$. By the definition of $\mathcal{N}^c$, we know that $\nu^c(B) = \bar{\nu}(B)$ for $B \in \mathcal{G}^c$. Hence $d(\mathcal{N}^c, \mathcal{N}) =$

0. Since $\mathcal{N}^c$ is an extension of $\mathcal{N}$, $d(\mathcal{N}, \mathcal{N}^c) = 0$. Hence $d_2(\mathcal{N}, \mathcal{N}^c) = 0$, which together with the triangle inequality implies $d(\mathcal{M}, \mathcal{N}) = d(\mathcal{M}^c, \mathcal{N}^c)$.

Now we consider (f). First assume $d(\mathcal{M}, \mathcal{N}) = 0$. For any given $B \in \mathcal{F}$, Lemma 2.2(a) implies that $\bar{\nu}(B) = \mu(B)$. Thus

$$\underline{\nu}(B) = 1 - \bar{\nu}(\Omega \setminus B) = 1 - \mu(\Omega \setminus B) = \mu(B) = \bar{\nu}(B),$$

and hence $B \in \mathcal{G}^c$ and $\nu^c(B) = \mu(B)$. Therefore $\mathcal{M} \subseteq \mathcal{N}^c$. On the other hand, if $\mathcal{M} \subseteq \mathcal{N}^c$, then $d(\mathcal{M}, \mathcal{N}^c) = 0$, and $0 \leq d(\mathcal{M}, \mathcal{N}) \leq d(\mathcal{M}, \mathcal{N}^c) + d(\mathcal{N}^c, \mathcal{N}) = 0$. Hence $d(\mathcal{M}, \mathcal{N}) = 0$.

If $d(\mathcal{M}^c, \mathcal{N}^c) = 0$, then (e) and (f) imply that $d(\mathcal{M}, \mathcal{N}) = 0$ and $\mathcal{M} \subseteq \mathcal{N}^c$. By (c) and (d), $\mathcal{M}^c \subseteq (\mathcal{N}^c)^c = \mathcal{N}^c$. It is obvious that if $\mathcal{M}^c \subseteq \mathcal{N}^c$, then $d(\mathcal{M}^c, \mathcal{N}^c) = 0$. Hence (g) is shown.

(h) follows from (g) easily. ■

Theorem 2.7 *The bi-discrepancy $d_2(\cdot, \cdot)$ is a metric on the family of all complete probability spaces on Ω.*

Proof. By Lemmas 2.2 and 2.6. ■

The following proposition shows that the operation of countably additive extension is a contraction for the discrepancy function.

Proposition 2.8 *(a) If $\mathcal{N}$ can be extended to a countably additive probability space $\sigma(\mathcal{N}) = (\Omega, \sigma(\mathcal{G}), \nu^\sigma)$, then $d(\mathcal{M}, \sigma(\mathcal{N})) \leq d(\mathcal{M}, \mathcal{N})$.*

(b) Suppose that both $\mathcal{M}$ and $\mathcal{N}$ can be respectively extended to countably additive probability spaces $\sigma(\mathcal{M}) = (\Omega, \sigma(\mathcal{F}), \mu^\sigma)$ and $\sigma(\mathcal{N}) = (\Omega, \sigma(\mathcal{G}), \nu^\sigma)$. Then:
(1) $d(\sigma(\mathcal{M}), \sigma(\mathcal{N})) = d(\mathcal{M}, \sigma(\mathcal{N}))$, and
(2) $d(\sigma(\mathcal{M}), \sigma(\mathcal{N})) \leq d(\mathcal{M}, \mathcal{N})$.

Proof. (a) follows easily from the fact that for any $B \subseteq \Omega$,

$$\overline{\nu^\sigma}(B) = \inf\{\nu^\sigma(C) : B \subseteq C \in \sigma(\mathcal{G})\} \leq \inf\{\nu(C) : B \subseteq C \in \mathcal{G}\} = \bar{\nu}(B).$$

To prove the equality in (b)(1), we first observe that

$$\begin{aligned} d(\mathcal{M}, \sigma(\mathcal{N})) &= \sup\{\overline{\nu^\sigma}(B) - \mu(B) : B \in \mathcal{F}\} \\ &\leq \sup\{\overline{\nu^\sigma}(B) - \mu^\sigma(B) : B \in \sigma(\mathcal{F})\} \\ &= d(\sigma(\mathcal{M}), \sigma(\mathcal{N})). \end{aligned}$$

The proof of the other side of the inequality is more involved. Assume that $d(\mathcal{M}, \sigma(\mathcal{N})) < \delta$. Let $\mathcal{E}$ be the set of all $B \in \sigma(\mathcal{F})$ such that there exists a set $C \in \sigma(\mathcal{G})$ with $B \subseteq C$ and $\nu^{\sigma}(C) - \mu^{\sigma}(B) \leq \delta$. Thus, $\mathcal{E}$ contains the algebra $\mathcal{F}$. We show that $\mathcal{E}$ is a monotone class.

Let $A_0 \supseteq A_1 \supseteq \cdots$ be a decreasing chain of sets in $\mathcal{E}$ and $A = \bigcap_n A_n$. For each n there is a set $C_n \in \sigma(\mathcal{G})$ such that $A_n \subseteq C_n$ and $\nu^{\sigma}(C_n) - \mu^{\sigma}(A_n) \leq \delta$. We may take the C_n so that $C_0 \supseteq C_1 \supseteq \cdots$. Let $C = \bigcap_n C_n$. Then $A \in \sigma(\mathcal{F})$, $C \in \sigma(\mathcal{G})$, and

$$\nu^{\sigma}(C) - \mu^{\sigma}(A) = \lim_{n\to\infty} \nu^{\sigma}(C_n) - \lim_{n\to\infty} \mu^{\sigma}(A_n) = \lim_{n\to\infty} (\nu^{\sigma}(C_n) - \mu^{\sigma}(A_n)) \leq \delta,$$

so $A \in \mathcal{E}$.

Now let $B_0 \subseteq B_1 \subseteq \cdots$ be an increasing chain of sets in $\mathcal{E}$ and let $B = \bigcup_n B_n$. For each n choose a set $D_n \in \sigma(\mathcal{G})$ such that $B_n \subseteq D_n$ and $\nu^{\sigma}(D_n) - \mu^{\sigma}(B_n) \leq \delta$. Let $E_n = \bigcap_{m=n}^{\infty} D_m$. Then $E_n \in \sigma(\mathcal{G})$, $B_n \subseteq E_n \subseteq D_n$, and $E_0 \subseteq E_1 \subseteq \cdots$. It follows that $\nu^{\sigma}(E_n) - \mu^{\sigma}(B_n) \leq \delta$. Let $E = \bigcup_n E_n$. We have $B \in \sigma(\mathcal{F})$, $E \in \sigma(\mathcal{G}), B \subseteq E$, and

$$\nu^{\sigma}(E) - \mu^{\sigma}(B) = \lim_{n\to\infty} \nu^{\sigma}(E_n) - \lim_{n\to\infty} \mu^{\sigma}(B_n) = \lim_{n\to\infty} (\nu^{\sigma}(E_n) - \mu^{\sigma}(B_n)) \leq \delta,$$

and hence $B \in \mathcal{E}$.

We have shown that $\mathcal{E}$ is a monotone class. By the Monotone Class Theorem (e.g. see [5], p.15), $\mathcal{E}$ is the σ-algebra $\sigma(\mathcal{F})$ generated by $\mathcal{F}$. This proves that $d(\sigma(\mathcal{M}), \sigma(\mathcal{N})) \leq \delta$, and it follows that $d(\sigma(\mathcal{M}), \sigma(\mathcal{N})) = d(\mathcal{M}, \sigma(\mathcal{N}))$ as required.

(b)(2) follows easily from (a) and (b)(1). ■

Example 2.9 *Let $\mathcal{N}$ and $\mathcal{M}$ be the spaces from Example 2.4 (a). $\mathcal{N}$ is the restriction of Lebesgue measure to the algebra of finite unions of intervals in $[0, 1)$, $\mathcal{M}$ is the restriction of Lebesgue measure to the Borel algebra, and $d(\mathcal{M}, \mathcal{N}) = 1$. We note that $\mathcal{M} = \sigma(\mathcal{N})$, so*

$$0 = d(\mathcal{M}, \sigma(\mathcal{N})) = d(\sigma(\mathcal{M}), \sigma(\mathcal{N})) < d(\mathcal{M}, \mathcal{N}) = 1.$$

Thus both the inequalities in (a) and (b)(2) of Proposition 2.8 hold strictly in this case.

3 Discrepancy of product spaces

In this section we show that the discrepancy from one product probability space to another product probability space is dominated by the sum of the discrepancies of the respective factor spaces. The same result also holds for σ-product spaces.

Definition 3.1 *Given probability spaces* $\mathcal{M}_1 = (\Omega_1, \mathcal{F}_1, \mu_1)$ *and* $\mathcal{M}_2 = (\Omega_2, \mathcal{F}_2, \mu_2)$*, let*

$$\mathcal{M}_1 \otimes \mathcal{M}_2 = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$$

denote the product space. In this product, $\mathcal{F}_1 \otimes \mathcal{F}_2$ *is the set of all finite unions of rectangles* $A_1 \times A_2$ *where* $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2$*, and*

$$(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2).$$

The following result shows that the discrepancy is preserved for product spaces when one factor space is fixed. When both factor spaces are changed, the discrepancy of the product spaces are dominated by the sum of the factor discrepancies.

Theorem 3.2 *Let* $\mathcal{M}_1 = (\Omega_1, \mathcal{F}_1, \mu_1)$*,* $\mathcal{N}_1 = (\Omega_1, \mathcal{G}_1, \nu_1)$*,* $\mathcal{M}_2 = (\Omega_2, \mathcal{F}_2, \mu_2)$*, and* $\mathcal{N}_2 = (\Omega_2, \mathcal{G}_2, \nu_2)$ *be probability spaces. Then*
(a) $d(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{M}_2) = d(\mathcal{M}_1, \mathcal{N}_1)$*.*
(b) $d(\mathcal{N}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{N}_2) = d(\mathcal{M}_2, \mathcal{N}_2)$*.*
(c) $d(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{N}_2) \leq d(\mathcal{M}_1, \mathcal{N}_1) + d(\mathcal{M}_2, \mathcal{N}_2)$*.*

Proof. (a) We first prove $d(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{M}_2) \leq d(\mathcal{M}_1, \mathcal{N}_1)$. Let $\delta > d(\mathcal{M}_1, \mathcal{N}_1)$. Let $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$. Then B is the union of a finite family of rectangles $D_i \times E_i, 1 \leq i \leq K$ for some positive integer K, where each $D_i \in \mathcal{F}_1$ and $E_i \in \mathcal{F}_2$. One can arrange things so that the sets E_i are pairwise disjoint. For each i we have $\bar{\nu}_1(D_i) - \mu_1(D_i) < \delta$. We may therefore choose sets $C_i \in \mathcal{G}_1$ such that for each $1 \leq i \leq K$, $D_i \subseteq C_i$ and $\nu_1(C_i) - \mu_1(D_i) < \delta$. Let $C = \bigcup_{1 \leq i \leq K} C_i \otimes E_i$. Then $C \in \mathcal{G}_1 \otimes \mathcal{F}_2$, $B \subseteq C$, and $(\nu_1 \otimes \mu_2)(C) - (\mu_1 \otimes \mu_2)(B) < \delta$. This proves desired inequality.

Next consider the other side of the inequality. Take any $A \subseteq \Omega_1$ and any $F \in \mathcal{G}_1 \otimes \mathcal{F}_2$ with $A \times \Omega_2 \subseteq F$. Then F is the union of a finite family of rectangles $G_i \times H_i, 1 \leq i \leq L$, where each $G_i \in \mathcal{G}_1$ and $H_i \in \mathcal{F}_2$. One can arrange things so that the sets H_i are pairwise disjoint. Since $A \times \Omega_2 \subseteq F$, we must have $\cup_{1 \leq i \leq L} H_i = \Omega_2$, and $A \subseteq G_i$ for each $1 \leq i \leq L$. Thus,

$$\begin{aligned}(\nu_1 \otimes \mu_2)(F) &= \sum_{1 \leq i \leq L} \nu_1(G_i)\mu_2(H_i) \geq \sum_{1 \leq i \leq L} \overline{\nu_1}(A)\mu_2(H_i) \\ &= \overline{\nu_1}(A) \sum_{1 \leq i \leq L} \mu_2(H_i) = \overline{\nu_1}(A).\end{aligned}$$

Hence, $\overline{\nu_1 \otimes \mu_2}(A \times \Omega_2) \geq \overline{\nu_1}(A)$, which implies that for any $A \in \mathcal{F}_1$,

$$\overline{\nu_1}(A) - \mu_1(A) \leq \overline{\nu_1 \otimes \mu_2}(A \times \Omega_2) - (\mu_1 \otimes \mu_2)(A \times \Omega_2).$$

Therefore, $d(\mathcal{M}_1, \mathcal{N}_1) \leq d(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{M}_2)$, and (a) follows.

The identity in (b) can be proven similarly.

By (a), (b) and the triangle inequality in Lemma 2.2,

$$\begin{aligned}d(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{N}_2) &\leq d(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{M}_2) + d(\mathcal{N}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{N}_2) \\ &= d(\mathcal{M}_1, \mathcal{N}_1) + d(\mathcal{M}_2, \mathcal{N}_2),\end{aligned}$$

and hence (c) is proven. ■

Corollary 3.3 *For any probability spaces $\mathcal{M}_1$ and $\mathcal{M}_2$,*

$$(\mathcal{M}_1)^c \otimes (\mathcal{M}_2)^c \subseteq (\mathcal{M}_1 \otimes \mathcal{M}_2)^c.$$

Proof. By Theorem 3.2 (c) and Lemma 2.6 (e),

$$d((\mathcal{M}_1)^c \otimes (\mathcal{M}_2)^c, \mathcal{M}_1 \otimes \mathcal{M}_2) \leq d((\mathcal{M}_1)^c, \mathcal{M}_1) + d((\mathcal{M}_2)^c, \mathcal{M}_2) = 0 + 0 = 0.$$

The result now follows by Lemma 2.6 (f). ■

Finally, we consider discrepancy under the operation of σ-product. We first give a formal definition for the σ-product.

Definition 3.4 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be countably additive (also called σ-additive) probability spaces.*

$$\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2 = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes^\sigma \mathcal{F}_2, \mu_1 \otimes^\sigma \mu_2)$$

denotes the σ-product space. This is the countably additive probability space generated by the finitely additive product space $\mathcal{M}_1 \otimes \mathcal{M}_2$.

Theorem 3.5 *Suppose $\mathcal{M}_1$, $\mathcal{N}_1$, $\mathcal{M}_2$, and $\mathcal{N}_2$ are countably additive probability spaces. Then*

$$d(\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2, \mathcal{N}_1 \otimes^\sigma \mathcal{N}_2) \leq d(\mathcal{M}_1, \mathcal{N}_1) + d(\mathcal{M}_2, \mathcal{N}_2).$$

Proof. Since $\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2 = \sigma(\mathcal{M}_1 \otimes \mathcal{M}_2)$ and $\mathcal{N}_1 \otimes^\sigma \mathcal{N}_2 = \sigma(\mathcal{N}_1 \otimes \mathcal{N}_2)$, Proposition 2.8 (b) and Theorem 3.2 (c) imply that

$$d(\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2, \mathcal{N}_1 \otimes^\sigma \mathcal{N}_2) \leq d(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{N}_2) \leq d(\mathcal{M}_1, \mathcal{N}_1) + d(\mathcal{M}_2, \mathcal{N}_2),$$

and the result follows. ■

4 Loeb extension and Loeb equivalence

From this section onwards, we work in an ω_1-saturated nonstandard universe. An internal probability space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ is understood to have a finitely additive (and hence *finitely additive by transfer) measure μ on an internal algebra $\mathcal{F}$ of subsets of a nonempty internal set Ω with $\mu(\Omega) = 1$, unless we explicitly assume $^*\sigma$-additivity. Note that the values of μ are in $^*[0, 1]$. We let $^o\mathcal{M} = (\Omega, \mathcal{F}, {^o\mu})$ be the standard probability space where $({^o\mu})(B)$ is the standard part of $\mu(B)$ for each $B \in \mathcal{F}$.

The **Loeb probability space** generated by an internal probability space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ is defined as the structure $L(\mathcal{M}) = (\Omega, L(\mathcal{F}), L(\mu))$, where $L(\mathcal{F})$ is the set of all $B \subseteq \Omega$ such that

$$\sup\{{^o\mu}(A) : B \supseteq A \in \mathcal{F}\} = \inf\{{^o\mu}(C) : B \subseteq C \in \mathcal{F}\},$$

and $L(\mu)(B)$ is defined as the above supremum.

Then Definition 2.5 implies that $L(\mathcal{M}) = ({^o\mathcal{M}})^c$, which is a finitely additive probability space by Lemma 2.6(a). However, Loeb in [10] showed that $L(\mathcal{M})$ is, in fact, a complete σ-additive probability space. Moreover, Lemma 2.6 (e) shows that $d_2({^o\mathcal{M}}, L(\mathcal{M})) = 0$.

The following definition gives a natural way of comparing two internal probability spaces on the same set.

Definition 4.1 *Given two internal probability spaces*

$$\mathcal{M} = (\Omega, \mathcal{F}, \mu)\,, \mathcal{N} = (\Omega, \mathcal{G}, \nu),$$

we say that $\mathcal{N}$ **Loeb extends** $\mathcal{M}$ *if* $L(\mathcal{G}) \supseteq L(\mathcal{F})$*, and* $L(\nu) \supseteq L(\mu)$ *(which simply means that* $L(\nu)$ *is an extension of* $L(\mu)$*) as a function.*

We say that $\mathcal{M}$ *is* **Loeb equivalent** *to* $\mathcal{N}$ *if they generate the same Loeb space,* $L(\mathcal{M}) = L(\mathcal{N})$*.*

Remark 4.2 *(a) If* $\mathcal{N}$ *is an internal extension of* $\mathcal{M}$*, i.e.,* $\mathcal{G} \supseteq \mathcal{F}$ *and* $\nu \supseteq \mu$*, then* $\mathcal{N}$ *Loeb extends* $\mathcal{M}$*.*

(b) $\mathcal{M}$ *is Loeb equivalent to* $\mathcal{N}$ *if and only if* $\mathcal{M}$ *Loeb extends* $\mathcal{N}$ *and* $\mathcal{N}$ *Loeb extends* $\mathcal{M}$*.*

(c) The Loeb extension relation is transitive.

By transfer, the function ${}^*d(\cdot, \cdot)$ can be defined on the family of internal probability spaces on a nonempty internal set Ω. When there is no ambiguity we will abuse notation by writing $d(\mathcal{M}, \mathcal{N})$ instead of ${}^*d(\mathcal{M}, \mathcal{N})$, dropping the star.

Lemma 4.3 *Let* $\mathcal{M}$*,* $\mathcal{N}$ *be internal probability spaces.*

(a) $d({}^o\mathcal{M}, {}^o\mathcal{N}) = {}^o(d(\mathcal{M}, \mathcal{N}))$*.*

(b) $d(L(\mathcal{M}), L(\mathcal{N})) = {}^o(d(\mathcal{M}, \mathcal{N}))$*.*

(c) $\mathcal{N}$ *Loeb extends* $\mathcal{M}$ *if and only if* $d(\mathcal{M}, \mathcal{N}) \approx 0$*.*

(d) $\mathcal{N}$ *Loeb extends* $\mathcal{M}$ *if and only if for every* $B \in \mathcal{F}$ *there exists* $C \in \mathcal{G}$ *such that* $B \subseteq C$ *and* $\nu(C) \approx \mu(B)$*.*

(e) $\mathcal{M}$ *is Loeb equivalent to* $\mathcal{N}$ *if and only if* $d_2(\mathcal{M}, \mathcal{N}) \approx 0$*.*

Proof. (a) It is obvious that for any internal subset B of Ω,

$$\inf\{{}^o\nu(C) : B \subseteq C \in \mathcal{G}\} = {}^o(\inf\{\nu(C) : B \subseteq C \in \mathcal{G}\}),$$

and thus ${}^o(\bar{\nu}(B)) = \overline{{}^o\nu}(B)$. Hence

$$\begin{aligned} d({}^o\mathcal{M}, {}^o\mathcal{N}) &= \sup\{\overline{{}^o\nu}(B) - {}^o\mu(B) : B \in \mathcal{F}\} = \sup\{{}^o(\overline{\nu}(B) - \mu(B)) : B \in \mathcal{F}\} \\ &= {}^o(\sup\{(\overline{\nu}(B) - \mu(B)) : B \in \mathcal{F}\}) = {}^o(d(\mathcal{M}, \mathcal{N})) \end{aligned}$$

(b) By the definition of Loeb spaces and by Lemma 2.6(e) and (a) of this lemma,

$$d(L(\mathcal{M}), L(\mathcal{N})) = d(({}^o\mathcal{M})^c, ({}^o\mathcal{N})^c) = d({}^o\mathcal{M}, {}^o\mathcal{N}) = {}^o(d(\mathcal{M}, \mathcal{N})).$$

(c) $\mathcal{N}$ Loeb extends $\mathcal{M}$ if and only if $L(\mathcal{M}) \subseteq L(\mathcal{N})$, i.e., $({}^o\mathcal{M})^c \subseteq ({}^o\mathcal{N})^c$. By Lemma 2.6(g) and (b) of this lemma, it is equivalent to $d(({}^o\mathcal{M})^c, ({}^o\mathcal{N})^c) = {}^o(d(\mathcal{M}, \mathcal{N})) = 0$.

(d) If $d(\mathcal{M},\mathcal{N}) \approx 0$, then Lemma 2.2 (a) implies that for each $B \in \mathcal{F}$, $\bar{\nu}(B) \approx \mu(B)$. Since $\bar{\nu}$ is an internal outer measure, there exists a set $C \in \mathcal{G}$ such that $B \subseteq C$ and $\nu(C) \approx \bar{\nu}(B)$, and hence $\nu(C) \approx \mu(B)$.

Next, assume that for each $B \in \mathcal{F}$ there exists $C \in \mathcal{G}$ such that $B \subseteq C$ and $\nu(C) \approx \mu(B)$. Take any standard positive real number ϵ. Then $\bar{\nu}(B) - \mu(B) \leq \nu(C) - \mu(B) < \epsilon$, which implies that $d(\mathcal{M},\mathcal{N}) \leq \epsilon$. Since $d(\mathcal{M},\mathcal{N})$ is a non-negative hyperreal number less than any standard positive real number, we have $d(\mathcal{M},\mathcal{N}) \approx 0$.

(e) follows easily from (c). ■

In the case that $\mathcal{F} \subseteq \mathcal{G}$, the Loeb extension relation behaves in a simple way.

Corollary 4.4 *Suppose $\mathcal{M},\mathcal{N}$ are such that $\mathcal{F} \subseteq \mathcal{G}$.*

(a) $\mathcal{N}$ Loeb extends $\mathcal{M}$ if and only if $\nu(A) \approx \mu(A)$ for all $A \in \mathcal{F}$.

(b) Let $\mathcal{P}$ be the subspace of $\mathcal{N}$ with the algebra $\mathcal{H} = \mathcal{F}$. $\mathcal{N}$ Loeb extends $\mathcal{M}$ if and only if $\mathcal{P}$ is Loeb equivalent to $\mathcal{M}$.

(c) If $\mathcal{F} = \mathcal{G}$, then $\mathcal{N}$ is Loeb equivalent to $\mathcal{M}$ if and only if $\mathcal{N}$ Loeb extends $\mathcal{M}$.

The Loeb extension relation between stars of standard spaces also takes a simple form.

Corollary 4.5 *Suppose $\mathcal{M} = {}^*\mathcal{M}_0$ and $\mathcal{N} = {}^*\mathcal{N}_0$ where $\mathcal{M}_0$ and $\mathcal{N}_0$ are standard probability spaces on the same set Ω_0.*

(a) $\mathcal{N}$ Loeb extends $\mathcal{M}$ if and only if $\mathcal{M}_0 \subseteq (\mathcal{N}_0)^c$.

(b) $\mathcal{M}$ and $\mathcal{N}$ are Loeb equivalent if and only if $(\mathcal{M}_0)^c = (\mathcal{N}_0)^c$.

Proof. Part (a) follows from Lemma 2.6 (f) and Lemma 4.3 (b) and (c). Part (b) then follows easily. ■

In this paper, by a **hyperfinite set** we will mean an internal set whose internal cardinality is a finite or infinite hyperinteger, that is, an element of ${}^*\mathbb{N}$. Using the next example, one can easily find different internal probability spaces $\mathcal{M},\mathcal{N}$ on a hyperfinite set Ω such that $\mathcal{M}$ and $\mathcal{N}$ are Loeb equivalent and have the same internal algebra $\mathcal{F} = \mathcal{G}$.

Example 4.6 *Let Ω be a hyperfinite set. Let $\mathcal{M} = (\Omega,\mathcal{F},\mu), \mathcal{N} = (\Omega,\mathcal{G},\nu)$ be internal probability spaces such that $\mathcal{F} = \mathcal{G} =$ the set of all internal subsets of Ω, and the sum $\Sigma_{\omega\in\Omega} |\mu(\{\omega\}) - \nu(\{\omega\})|$ is positive infinitesimal. Then $\mathcal{M}$ and $\mathcal{N}$ are Loeb equivalent.*

The Loeb extension or Loeb equivalence relation is much more difficult when it is not assumed that $\mathcal{F} \subseteq \mathcal{G}$; see the open questions at the end of this paper. The following example shows that one can indeed find different internal probability spaces $\mathcal{M}, \mathcal{N}$ that are Loeb equivalent, but (1) one is a proper extension of the other, or (2) neither internal algebra is an extension of the other.

Example 4.7 *Let Ω be a hyperfinite set. Suppose $\mathcal{N} = (\Omega, \mathcal{G}, \nu)$ is an internal probability space such that $\mathcal{G}$ is the set of all internal subsets of Ω, and S_1 and S_2 are disjoint internal subsets of Ω such that $\nu(S_i) \approx 0$ and $card(S_i) > 1$ for $i = 1, 2$. For a given $i = 1$ or 2, let $\mathcal{F}_i$ be the set of all internal sets $U \subseteq \Omega$ such that U either contains S_i or is disjoint from S_i, and let $\mathcal{M}_i = (\Omega, \mathcal{F}_i, \nu|_{\mathcal{F}_i})$. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are Loeb equivalent to each other and to their proper extension $\mathcal{N}$, but $\mathcal{F}_1$ and $\mathcal{F}_2$ are not extensions of each other.*

Let us now consider the space $(\mathcal{I}_\Omega, {}^*d_2)$ of all internal probability spaces on Ω, endowed with the internal pseudo-metric ${}^*d_2$. Let $\mathcal{L}_\Omega$ be the space of all Loeb spaces on Ω with the metric of the standard bi-discrepancy d_2. Let π be the Loeb operation from $\mathcal{I}_\Omega$ to $\mathcal{L}_\Omega$, where $\pi(\mathcal{M}) = L(\mathcal{M})$ for each internal probability space $\mathcal{M}$ on Ω.

Theorem 4.8 *The Loeb operation induces an isometry from the nonstandard hull of the space $(\mathcal{I}_\Omega, {}^*d_2)$ of internal probability spaces to the space $(\mathcal{L}_\Omega, d_2)$ of Loeb spaces, and $(\mathcal{L}_\Omega, d_2)$ is a complete metric space.*

Proof. By Lemma 2.2, $(\mathcal{I}_\Omega, {}^*d_2)$ is an internal pseudo-metric space. Then ${}^*d_2(\mathcal{M}, \mathcal{N}) \approx 0$ defines an equivalence relation on $\mathcal{I}_\Omega$. Let $\widehat{\mathcal{I}}_\Omega$ be the set of equivalence classes, and let the equivalence class of $\mathcal{M}$ be denoted by $\widehat{\mathcal{M}}$. For $\widehat{\mathcal{M}}, \widehat{\mathcal{N}} \in \widehat{\mathcal{I}}_\Omega$, define $\widehat{d_2}(\widehat{\mathcal{M}}, \widehat{\mathcal{N}}) = {}^{o*}d_2(\mathcal{M}, \mathcal{N})$. Since $0 \leq {}^*d(\mathcal{M}, \mathcal{N}) \leq 1$, $\widehat{d_2}(\widehat{\mathcal{M}}, \widehat{\mathcal{N}})$ is finite for all $\mathcal{M}, \mathcal{N}$. It is well known (see, for example, [11]) that the nonstandard hull of a metric space is a complete metric space. The same proof shows that the nonstandard hull of an internal pseudo-metric space is also a complete metric space, and hence $(\widehat{\mathcal{I}}_\Omega, \widehat{d_2})$ is a complete metric space.

Define a mapping $\widehat{\pi}$ from $\widehat{\mathcal{I}}_\Omega$ to $\mathcal{L}_\Omega$ by letting $\widehat{\pi}(\widehat{\mathcal{M}}) = \pi(\mathcal{M})$. By Lemma 4.3(d), $\widehat{\pi}$ is well-defined and a bijection from $\widehat{\mathcal{I}}_\Omega$ to $\mathcal{L}_\Omega$. By Lemma 4.3(b), we have

$$\widehat{d_2}(\widehat{\mathcal{M}}, \widehat{\mathcal{N}}) = {}^{o*}d_2(\mathcal{M}, \mathcal{N}) = d_2(L(\mathcal{M}), L(\mathcal{N})) = d_2(\widehat{\pi}(\widehat{\mathcal{M}}), \widehat{\pi}(\widehat{\mathcal{N}})),$$

and hence $\widehat{\pi}$ is an isometry. Since $(\widehat{\mathcal{I}}_\Omega, \widehat{d}_2)$ is a complete metric space, so is $(\mathcal{L}_\Omega, d_2)$. ■

5 Uniqueness of the Loeb product

We show in this section that for any two given Loeb spaces, their Loeb product (or the Loeb σ-product) is uniquely defined. We begin with a formal definition of the internal product.

Definition 5.1 *Given internal probability spaces $\mathcal{M}_1$ and $\mathcal{M}_2$, $\mathcal{M}_1 \otimes \mathcal{M}_2$ will denote the internal product space, where $\mathcal{F}_1 \otimes \mathcal{F}_2$ is the set of all hyperfinite unions of rectangles $A_1 \times A_2$, and $(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$.*

The following theorem shows that the Loeb product is well defined in terms of factor Loeb spaces. That is, the Loeb product space $L(\mathcal{M}_1 \otimes \mathcal{M}_2)$ depends only on the Loeb spaces $L(\mathcal{M}_1)$ and $L(\mathcal{M}_2)$, and not on the internal probability spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ that generate the Loeb spaces $L(\mathcal{M}_1)$ and $L(\mathcal{M}_2)$.

Theorem 5.2 *(a) If $\mathcal{N}_1$ Loeb extends $\mathcal{M}_1$ and $\mathcal{N}_2$ Loeb extends $\mathcal{M}_2$, then $\mathcal{N}_1 \otimes \mathcal{N}_2$ Loeb extends $\mathcal{M}_1 \otimes \mathcal{M}_2$.*

(b) If $\mathcal{M}_1$ is Loeb equivalent to $\mathcal{N}_1$ and $\mathcal{M}_2$ is Loeb equivalent to $\mathcal{N}_2$, then $\mathcal{M}_1 \otimes \mathcal{M}_2$ is Loeb equivalent to $\mathcal{N}_1 \otimes \mathcal{N}_2$.

Proof. (a) By Lemma 4.3(c), we have $d(\mathcal{M}_1, \mathcal{N}_1) \approx 0$ and $d(\mathcal{M}_2, \mathcal{N}_2) \approx 0$. It follows from Lemma 2.2(a) and Theorem 3.2(c) that $d(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{N}_2) \approx 0$. By Lemma 4.3(c) again, $\mathcal{N}_1 \otimes \mathcal{N}_2$ Loeb extends $\mathcal{M}_1 \otimes \mathcal{M}_2$.

(b) follows from Remark 4.2. ■

We now turn to σ-additive probability spaces.

The following example shows that an internal ${}^*\sigma$-additive probability space $\mathcal{N}$ need not be Loeb equivalent to an internal subspace that ${}^*\sigma$-generates it.

Example 5.3 *Let $\mathcal{N}$ be the star of Lebesgue measure on $[0, 1]$, and let $\mathcal{M}$ be the star of the subspace of finite unions of half-open intervals in $[0, 1]$. Then $\mathcal{N}$ is not Loeb equivalent to $\mathcal{M}$.*

Proof. The set of hyperrational numbers in ${}^*[0,1]$ has measure 0 in $\mathcal{N}$ but any superset in $\mathcal{F}$ has measure 1 in $\mathcal{M}$. ■

For a pair of internal ${}^*\sigma$-additive probability spaces, there is a second notion of product to be considered.

Definition 5.4 *Given two internal ${}^*\sigma$-additive probability spaces $\mathcal{M}_1$ and $\mathcal{M}_2$, $\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2$ will denote the internal ${}^*\sigma$-additive probability space generated by $\mathcal{M}_1 \otimes \mathcal{M}_2$*

The next theorem shows that the Loeb σ-product is well-defined.

Theorem 5.5 *Suppose $\mathcal{M}_1$, $\mathcal{N}_1$, $\mathcal{M}_2$, and $\mathcal{N}_2$ are ${}^*\sigma$-additive internal probability spaces.*

(a) If $\mathcal{N}_1$ Loeb extends $\mathcal{M}_1$ and $\mathcal{N}_2$ Loeb extends $\mathcal{M}_2$, then $\mathcal{N}_1 \otimes^\sigma \mathcal{N}_2$ Loeb extends $\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2$.

(b) If $\mathcal{N}_1$ is Loeb equivalent to $\mathcal{M}_1$ and $\mathcal{N}_2$ is Loeb equivalent to $\mathcal{M}_2$, then $\mathcal{N}_1 \otimes^\sigma \mathcal{N}_2$ is Loeb equivalent to $\mathcal{M}_1 \otimes^\sigma \mathcal{M}_2$.

Proof. The procedure for proving this theorem is exactly the same as that of Theorem 5.2 except using Theorem 3.5 instead of Theorem 3.2(c). ■

6 Hyperfinite probability spaces

The Loeb σ-product is especially simple when one of the factor spaces is hyperfinite. A **hyperfinite probability space** is an internal probability space $\mathcal{M}$ such that $\mathcal{F}$ is a hyperfinite set. (Note that we do not require Ω to be hyperfinite.)

Remark 6.1 *If at least one of the internal ${}^*\sigma$-additive probability spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ is hyperfinite, then the internal product space is the same as the internal σ-product space.*

Corollary 6.2 *Suppose that $\mathcal{M}, \mathcal{N}$ are internal ${}^*\sigma$-additive probability spaces and either $\mathcal{M}$ or $\mathcal{N}$ is Loeb equivalent to a hyperfinite probability space. Then $\mathcal{M} \otimes^\sigma \mathcal{N}$ is Loeb equivalent to $\mathcal{M} \otimes \mathcal{N}$.*

Proof. By Theorems 5.2 and 5.5 and Remark 6.1. ■

In view of Corollary 6.2, it is natural to ask which internal spaces are Loeb equivalent to hyperfinite probability spaces. The rest of this section deals with that question.

Proposition 6.3 *Suppose $\mathcal{M}$ and $\mathcal{N}$ are internal probability spaces, and $\mathcal{M}$ is hyperfinite.*

(a) If $\mathcal{N}$ Loeb extends $\mathcal{M}$, then $\mathcal{N}$ has a hyperfinite subspace $\mathcal{P}$ that Loeb extends $\mathcal{M}$.

(b) If $\mathcal{N}$ is Loeb equivalent to $\mathcal{M}$, then $\mathcal{N}$ has a hyperfinite subspace that is Loeb equivalent to $\mathcal{M}$.

Proof. We prove (a) first. By Lemma 4.3, for each $B \in \mathcal{F}$ there is a set $F(B) = C \in \mathcal{G}$ such that $B \subseteq C$ and $\nu(C) \approx \mu(B)$. The function F may be taken to be internal. Let $\mathcal{H}$ be the hyperfinite subalgebra of $\mathcal{N}$ generated by the range of F, $\rho = \nu|_{\mathcal{H}}$ and $\mathcal{P} = (\Omega, \mathcal{H}, \rho)$. Then by Lemma 4.3, $\mathcal{P}$ Loeb extends $\mathcal{M}$.

For (b), note that $L(\mathcal{M}) \subseteq L(\mathcal{P}) \subseteq L(\mathcal{N}) \subseteq L(\mathcal{M})$. Hence $L(\mathcal{M}) = L(\mathcal{P})$. ∎

An **atom** of $\mathcal{M}$ is a set $B \in \mathcal{F}$ such that $B \neq \emptyset$ but there is no $A \in \mathcal{F}$ such that $\emptyset \subset A \subset B$. Note that this is the notion of an atom in the Boolean algebra on $\mathcal{F}$, which is different from the notion of an atom in the measure algebra on $\mathcal{F}$ modulo the null sets.

Proposition 6.4 *Suppose $\mathcal{N}$ is an internal probability space. Then the following are equivalent.*

(a) $\mathcal{N}$ is Loeb equivalent to some hyperfinite probability space.

(b) There is a hyperfinite set $\mathcal{S}$ of atoms of $\mathcal{N}$ with $\nu(\bigcup_{S\in\mathcal{S}} S) \approx 1$.

Proof. Assume (a). By Proposition 6.3, $\mathcal{N}$ has a Loeb equivalent hyperfinite subspace $\mathcal{M}$. Let $\mathcal{S}$ be the set of all atoms of $\mathcal{M}$ that are also atoms of $\mathcal{N}$. We claim that $\nu(\bigcup_{S\in\mathcal{S}} S) \approx 1$, so (b) holds. Suppose not. Let $D = \Omega \setminus \bigcup_{S\in\mathcal{S}} S$. Since $\mathcal{M}$ is hyperfinite, D is the union of a hyperfinite set $\mathcal{U}$ of atoms of $\mathcal{M}$, so $D \in \mathcal{F}$. No element of $\mathcal{U}$ is an atom of $\mathcal{N}$, so there is a set $E \in \mathcal{G}$ such that $E \subseteq D$, and for each set $H \in \mathcal{U}$, $\emptyset \subset (E \cap H) \subset H$. Then whenever $A, C \in \mathcal{F}$ and $A \subseteq E \subseteq C$, we must have $A = \emptyset$ and $D \subseteq C$, so ${}^o\nu(C \setminus A) \geq {}^o\nu(D) > 0$. This means that $E \notin L(\mathcal{F})$, which contradicts the assumption that $\mathcal{M}$ is Loeb equivalent to $\mathcal{N}$. Hence it proves the claim.

Now assume (b). As in the last paragraph, let $D = \Omega \setminus \bigcup_{S\in\mathcal{S}} S$. Let $\mathcal{M}$ be the hyperfinite subspace generated by the partition $\mathcal{S} \cup \{D\}$. Then $\mathcal{F} \subseteq \mathcal{G}$. Let $B \in \mathcal{G}$. Then $B = (B \cap D) \cup \bigcup_{S\in\mathcal{S}}(B \cap S)$. For each atom E of $\mathcal{N}$, either $E \subseteq B$ or $E \cap B = \emptyset$. Hence $B = (B \cap D) \cup \bigcup_{E\in\mathcal{S}, E\subseteq B} E$. Let $A = \bigcup_{E\in\mathcal{S}, E\subseteq B} E$ and $C = A \cup D$. Then $B = A \cup (B \cap D) \subseteq C$ with

$\nu(C \setminus A) = \nu(D) \approx 0$. Hence $B \in L(\mathcal{F})$. By Lemma 4.3, $\mathcal{M}$ is Loeb equivalent to $\mathcal{N}$ and (a) holds. ■

Corollary 6.5 *If $\mathcal{N}$ is an internal probability space with no atoms of internally positive measure, then no hyperfinite probability space $\mathcal{M}$ is Loeb equivalent to $\mathcal{N}$.*

Example 6.6 *Let $\mathcal{N}$ be the star of Lebesgue measure on $[0,1]$. Then the above corollary implies that no hyperfinite probability space $\mathcal{M}$ is Loeb equivalent to $\mathcal{N}$.*

7 Some questions

We include a few open questions in this final section. In each question, $\mathcal{M}$ and $\mathcal{N}$ are assumed to be internal probability spaces.

Question 7.1 *Suppose $\mathcal{N}$ Loeb extends $\mathcal{M}$. Must $\mathcal{N}$ have an internal subspace that is Loeb equivalent to $\mathcal{M}$? What if $\mathcal{M}$ is assumed to be hyperfinite?*

Assuming a negative answer to Question 7.1, one can ask the following.

Question 7.2 *Suppose $\mathcal{N}$ Loeb extends $\mathcal{M}$ and $\mathcal{N}$ is hyperfinite. Must $\mathcal{M}$ be Loeb equivalent to a hyperfinite probability space?*

Question 7.3 *Suppose $\mathcal{M}$ is Loeb equivalent to $\mathcal{N}$, and let $\mathcal{H}$ be the internal set algebra generated by $\mathcal{F} \cup \mathcal{G}$. Must there be an internal probability measure λ on $\mathcal{H}$ such that $\mathcal{M}$ is Loeb equivalent to $(\Omega, \mathcal{H}, \lambda)$? What if $\mathcal{M}$ and $\mathcal{N}$ are assumed to be hyperfinite?*

We conclude with one more open question.

Question 7.4 *Let $\mathcal{M}$ and $\mathcal{N}$ be internal $^*\sigma$-additive probability spaces. Must the product $\mathcal{M} \otimes \mathcal{N}$ be Loeb equivalent to the σ-product $\mathcal{M} \otimes^\sigma \mathcal{N}$?*

In the case that at least one of the spaces $\mathcal{M}, \mathcal{N}$ is Loeb equivalent to a hyperfinite probability space, the answer is "yes" by Corollary 6.2.

In the case that the spaces $\mathcal{M}$ and $\mathcal{N}$ are stars of standard spaces, $\mathcal{M} = {}^*\mathcal{M}_0$ and $\mathcal{N} = {}^*\mathcal{N}_0$, by definition we have

$$\mathcal{M} \otimes \mathcal{N} = {}^*(\mathcal{M}_0 \otimes \mathcal{N}_0),\ \mathcal{M} \otimes^\sigma \mathcal{N} = {}^*(\mathcal{M}_0 \otimes^\sigma \mathcal{N}_0).$$

Therefore, in view of Corollary 4.5, Question 7.4 takes a standard form.

Let $\mathcal{M}_0$ and $\mathcal{N}_0$ be standard countably additive probability spaces. Must $(\mathcal{M}_0 \otimes \mathcal{N}_0)^c = (\mathcal{M}_0 \otimes^\sigma \mathcal{N}_0)^c$?

We have not been able to find an answer to this question in the literature, but we conjecture that the answer is "no" in the case that $\mathcal{M}_0$ and $\mathcal{N}_0$ are both equal to the unit interval $[0, 1]$ with Lebesgue measure.

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