STOCHASTIC DIFFERENTIAL EQUATIONS WITH EXTRA PROPERTIES

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1. Introduction

The Loeb measure construction has been a powerful tool in proving existence theorems for stochastic differential equations. There are many strong existence theorems which depend on the richness of the adapted Loeb space and which cannot be proved by classical methods. See, for example, [1].

In these lectures we shall first use the method to show that solutions exist. We shall then exploit the method further to find solutions of stochastic differential equations with additional properties, such as solutions which are optimal in a variety of ways, and solutions which are Markov processes.

In most cases, a nonstandard existence proof shows more than mere existence of a solution—it also gives a characterization of the set of all solutions in terms of liftings. By the monad of a set C of stochastic processes we shall mean the set of all liftings of elements of C. A typical lifting theorem will show that the monad of the set of all solutions of the stochastic differential equation under consideration is a countable intersection of internal sets.

These lifting theorems draw their power from the fact that sets C whose monads are countable intersections of internal sets behave much like compact sets. For this reason, we call a set whose monad is a countable intersection of internal sets a **neocompact set**.

Some of the ideas developed here go back to the monograph [5], where several existence theorems for stochastic differential equations with extra properties were obtained. We are now taking another look at these ideas in the light of more recent developments. The notion of a neocompact set captures a common thread which appears in many proofs both in [5] and in the more recent literature.

In these lectures we use neocompact sets in the "conventional" nonstandard setting. In a recent series of papers (see [6] for a survey), the neocompact sets are instead taken as a primitive notion and used to prove existence theorems directly—avoiding the steps of lifting to the nonstandard universe and coming back down to the standard universe.

2. Spaces of Stochastic Processes

We begin by fixing notation and setting up a framework which is appropriate for studying liftings of stochastic processes. For simplicity we shall restrict time to the closed interval $[0,1]$. In the spirit of the previous lectures in this conference, we shall concentrate on square-integrable stochastic processes. We first look at liftings of random variables with values in a metric space $\mathcal{M}$, and then use the fact that a continuous or L^2 stochastic process with values in $\mathcal{M}$ is the same thing as a random variable with values in the metric space $C([0,1],\mathcal{M})$ or $L^2([0,1],\mathcal{M})$.

Let

$$T = \{0, \Delta t, 2\Delta t, \ldots, H\Delta t = 1\}$$

be a hyperfinite time line where H is an infinite hyperinteger and $\Delta t = 1/H$. Our sample space $\Omega = \Omega_0^T$ will be the set of all internal functions from T into Ω_0 where Ω_0 is a *finite set with at least two elements. Let P be the hyperfinite counting measure on Ω, so that every internal set A is P-measurable and $P(A) = |A|/|\Omega|$. P_L will denote the Loeb measure generated by P. For $\omega \in \Omega$ and $t \in T$ let

$$[\omega]_t = \{\alpha \in \Omega : \alpha(s) = \omega(s) \text{ for all } s < t\}.$$

Let $\mathcal{G}_t$ be the *-algebra composed of all internal sets A such that $[\omega]_t \subseteq A$ for all $\omega \in A$, and let $\sigma(\mathcal{G}_t)$ be the P_L-complete σ-algebra generated by $\mathcal{G}_t$. For $t \in [0,1)$ let

$$\mathcal{F}_t = \bigcap \{\sigma(\mathcal{G}_s) : {}^\circ s > t\},$$

and let $\mathcal{F}_1 = \sigma(\mathcal{G}_1)$.

We let $(\mathcal{M},\rho)$, $(\mathcal{N},\pi), \ldots$ be standard complete separable metric spaces. Let us pick out an element $m_0 \in \mathcal{M}$. The metric space $L^2(\Omega,\mathcal{M})$ is the space of all Loeb measurable random variables $x : \Omega \to \mathcal{M}$ such that $(\rho(x(\omega), m_0))^2$ is integrable, with the metric ρ_2 defined by

$$\rho_2(x,y) = \left[\int (\rho(x(\omega), y(\omega))^2 d\omega\right]^{1/2}.$$

We identify each $m \in \mathcal{M}$ with the constant function from Ω to m, so that $\mathcal{M} \subseteq L^2(\Omega, \mathcal{M})$. If $A \subseteq L^2(\Omega, \mathcal{M})$ and $r \in \mathbb{R}$, we let A^r be the set of all x such that $\rho_2(x, y) \leq r$ for some $y \in A$.

We also need an internal counterpart of $L^2(\Omega, \mathcal{M})$. To give us some flexibility, we first let $\mathcal{M}'$ be an internal subset of ${}^*\mathcal{M}$ which is S-dense, that is, every point of ${}^*\mathcal{M}$ is infinitely close to some point of $\mathcal{M}'$. We now define $SL^2(\Omega, \mathcal{M})$ as the internal set consisting of all internal functions $X : \Omega \to \mathcal{M}'$, with the internal metric

$$\bar{\rho}_2(X, Y) = \left[\sum ({}^*\rho(X(\omega), Y(\omega))^2 \Delta\omega\right]^{1/2}.$$

Let $X \in SL^2(\Omega, \mathcal{M})$ and $x : \Omega \to \mathcal{M}$. We shall say that X is S^2-**integrable** if $({}^*\rho(X(\omega), m_0))^2$ is S-integrable over Ω. We say that X **lifts** x, and that x is the **standard part** of X (in symbols $x = {}^\circ X$), if $X(\omega) \approx x(\omega)$ P_L-almost surely and X is S^2- integrable. If X has a standard part, we say that X is **near-standard** and write $X \in ns^2(\Omega, \mathcal{M})$.

The following proposition, which follows from the fundamental results in the earlier lectures, gives the connection between the standard part map and the spaces $L^2(\Omega, \mathcal{M})$.

2.1. Proposition. *(Loeb [8] and Anderson [2]).* $L^2(\Omega, \mathcal{M})$ *is the set of all standard parts of elements of* $ns^2(\Omega, \mathcal{M})$. □

We shall extend the standard part terminology to sets. For a set $A \subseteq SL^2(\Omega, \mathcal{M})$, the **standard part** of A is defined as the set

$${}^\circ A = \{{}^\circ X : X \in A \cap ns^2(\Omega, \mathcal{M})\}$$

of standard parts of near-standard elements of A. We say that A is near-standard if every element of A is near-standard. In the upward direction, the **monad** of a set $B \subseteq L^2(\Omega, \mathcal{M})$ is the set of all $X \in ns^2(\Omega, \mathcal{M})$ such that ${}^\circ X \in B$.

We next apply our setup to spaces of L^2 stochastic processes and of continuous stochastic processes.

We first consider L^2 processes. Let $\mathcal{L}(\mathcal{M}) = L^2([0,1], \mathcal{M})$ be the space of L^2 paths in $\mathcal{M}$. Thus $L^2(\Omega, \mathcal{L}(\mathcal{M}))$ is the space of L^2 stochastic processes with values in $\mathcal{M}$. In this case we take the internal set $\mathcal{L}(\mathcal{M})'$ to be the set of all T-step functions induced by internal functions $X : T \to {}^*\mathcal{M}$. This set is S-dense in ${}^*(L^2([0,1], \mathcal{M}))$ as required. Then $ns^2(\Omega, \mathcal{L}(\mathcal{M}))$ turns out to be the set of all X such that $X(\omega, t)$ is S^2-integrable over $\Omega \times T$ and near-standard in ${}^*\mathcal{M}$ almost everywhere in $\Omega \times T$.

We now consider continuous processes. Let

$$\mathcal{C}(\mathcal{M}) = C([0,1], \mathcal{M})$$

be the space of continuous paths in $\mathcal{M}$ with the sup metric, and assume that $\mathcal{M}$ is a linear space. Then $L^2(\Omega, \mathcal{C}(\mathcal{M}))$ is the space of L^2 continuous stochastic processes with values in $\mathcal{M}$. This time we take the internal set $\mathcal{C}(\mathcal{M})'$ to be the set of all polygonal paths induced by internal functions $X : T \to {}^*\mathcal{M}$. This set is again S-dense. $ns^2(\Omega, \mathcal{C}(\mathcal{M}))$ is the set of all X such that $X(\omega)$ is S-continuous P_L-almost surely and is S^2-integrable over Ω.

Another space which is often used for the paths of a stochastic process is the space $D([0,1], \mathbb{R}^d)$ of right continuous functions with left limits and the Skorokhod metric. In the interest of simplicity, we shall avoid that space in these lectures.

We are now ready to study liftings of stochastic processes in a systematic way.

By an **adapted process** in $\mathcal{M}$ we shall mean a stochastic process $x \in L^2(\Omega, \mathcal{L}(\mathcal{M}))$ such that $x(\omega, t)$ is $\mathcal{F}_t$-measurable for each $t \in [0,1]$. A **continuous adapted process** in $\mathcal{M}$ is defined similarly but with $x \in L^2(\Omega, \mathcal{C}(\mathcal{M}))$. A **continuous martingale** in $\mathbb{R}^d$ is a continuous adapted process x in $\mathbb{R}^d$ such that $E[x(\bullet, t)|\mathcal{F}_s] = x(\omega, s)$ whenever $s \le t$.

An internal stochastic process $X \in SL^2(\Omega, \mathcal{L}(\mathcal{M}))$ or $X \in SL^2(\Omega, \mathcal{C}(\mathcal{M}))$ will be called **adapted after** r if $X(\omega, s)$ is $\mathcal{G}_t$-measurable whenever $s \le t \in T$ and $r \le t$, and called **adapted** if it is adapted after $1/n$ for each $n \in \mathbb{N}$. X is called a **martingale after** r if X is adapted after r and $E[X(\bullet, t)|\mathcal{G}_s] = X(\omega, s)$ whenever $r \le s \le t$, and a **martingale** if it is a martingale after each $1/n$.

We shall need the following lifting lemma which gives the connection between the standard notions of an adapted process and martingale and the nonstandard counterparts of these notions. We shall leave this lemma as an exercise for the reader, with a warning that the proof is not as easy as one would expect!

2.2. Lemma. *(i) A process $x \in L^2(\Omega, \mathcal{L}(\mathcal{M}))$ is adapted in $\mathcal{M}$ if and only if x has an adapted lifting $X \in SL^2(\Omega, \mathcal{L}(\mathcal{M}))$.*

(ii) A process $x \in L^2(\Omega, \mathcal{C}(\mathcal{M}))$ is continuous adapted in $\mathcal{M}$ if and only if x has an adapted lifting $X \in SL^2(\Omega, \mathcal{C}(\mathcal{M}))$.

(iii) A process $x \in L^2(\Omega, \mathcal{C}(\mathbb{R}^d))$ is a martingale in $\mathbb{R}^d$ if and only if x has an adapted lifting $X \in SL^2(\Omega, \mathcal{C}(\mathbb{R}^d))$ which is a martingale. □

Parts (i) and (ii) are proved in [5], and part (iii) is due to Hoover, Perkins, and Lindstrøm, (see [1]). Going up, the idea in the proof is to start with a lifting and modify it on a set of measure zero to a lifting which is adapted after $1/n$ for each n. Going down, the idea is to start with a standard part and modify it on a set of measure zero to an adapted process.

The following result is a lifting theorem for stochastic integrals. To avoid the complication of introducing $SL^2(w)$ liftings, we restrict our discussion to the case of uniformly bounded integrands.

2.3. Proposition. *(Anderson [2] for Brownian motions; Hoover, Perkins, and Lindstrøm in general). Suppose $f \in L^2(\Omega, \mathcal{L}(\mathbb{R}^{d\times d}))$ is uniformly bounded and adapted and $w \in L^2(\Omega, \mathcal{C}(\mathbb{R}^d))$ is a continuous martingale. Then for any uniformly bounded adapted lifting F of f and any martingale lifting W of w, the hyperfinite sum*

$$S(\omega,t) = \sum_{s<t} F(\omega,s)\Delta W(\omega,s)$$

is a lifting of the stochastic integral

$$I(\omega,t) = \int_0^t f(\omega,s)dw(\omega,s)$$

in the space $L^2(\Omega, \mathcal{C}(\mathbb{R}^d))$.

Sketch of Proof: The hyperfinite sum $S(\omega,t)$ is S-continuous by Lindstrøm Theorem 9.3. Since F is uniformly bounded, one can check that $S(\omega,t)$ is also SL^2, and hence near-standard. By Lindstrøm Theorem 12.2, $S(\omega,t)$ is a lifting of the stochastic integral $I(\omega,t)$.

The idea of the proof of this last fact is as follows. For any sequence of adapted step functions f_n converging to f in $L^2(\Omega, \mathcal{L}(\mathbb{R}^d))$, the stochastic integrals

$$\int_0^t f_n(\omega,s)dw(\omega,s)$$

are defined in the natural way and can be shown to be convergent in $L^2(\Omega, \mathcal{C}(\mathbb{R}^d))$. The limit of this sequence is the standard definition of the stochastic integral

$$\int_0^t f(\omega,s)dw(\omega,s).$$

Taking F_n to be a step function lifting f_n, the hyperfinite sums

$$\sum_{s<t} F_n(\omega,s)\Delta W(\omega,s)$$

S-converge to $S(\omega,t)$, and it follows that $S(\omega,t)$ lifts $I(\omega,t)$. □

3. Solutions of Stochastic Differential Equations

To motivate our approach to solving stochastic differential equations, let us examine the simplest existence theorem for stochastic differential equations in [5]. Let $C(\mathbb{R}^d, \mathbb{R}^{d\times d})$ be the space of all continuous functions from $\mathbb{R}^d$ into $\mathbb{R}^{d\times d}$ with a metric for the topology of uniform convergence on compact sets.

3.1. Theorem. *Let $w(\omega, s)$ be a continuous martingale in $\mathbb{R}^d$ and let*

$$g \in L^2(\Omega, \mathcal{L}(C(\mathbb{R}^d, \mathbb{R}^{d\times d})))$$

be a uniformly bounded adapted process with values in the space $C(\mathbb{R}^d, \mathbb{R}^{d\times d})$. Then there exists a continuous martingale x in $\mathbb{R}^d$ such that

$$x(\omega, t) = \int_0^t g(\omega, s)(x(\omega, s))dw(\omega, s). \tag{1}$$

Proof: By Lemma 2.2, g has an adapted lifting G and w has a martingale lifting W. $W(\omega, \bullet)$ is S-continuous P_L-almost surely. It follows from Lindstrøm's lectures that the quadratic variation $[W]$ is S-continuous P_L-almost surely. By truncating we may take G to have the same finite bound as g. Define $X(\omega, t)$ as the unique solution of the hyperfinite difference equation

$$X(\omega, t) = \sum_{s<t} G(\omega, s)(X(\omega, s))\Delta W(\omega, s). \tag{2}$$

X is clearly an internal martingale. Since $[W]$ is S-continuous and G is bounded, $[X]$ is S-continuous, and therefore X is S-continuous. Similarly, since W is S^2-integrable, one can show that X is S^2-integrable. Therefore X is near-standard and has a standard part x which is a continuous martingale in $\mathbb{R}^d$. Furthermore,

$$^\circ G(\omega, t)(X(\omega, t)) = g(\omega, {}^\circ t)(x(\omega, {}^\circ t))$$

almost surely in $\Omega \times T$. By Proposition 2.3,

$$\sum_{s<t} G(\omega, s)(X(\omega, s))\Delta W(\omega, s)$$

lifts

$$\int_0^t g(\omega, s)(x(\omega, s))dw(\omega, s).$$

Taking standard parts we see that x is a solution of the original equation (1). □

This proof actually gives a characterization of the set C of all solutions of (1). Let $\hat{C}$ be the set of all $X \in SL^2(\Omega, \mathcal{C}(\mathbb{R}^d))$ such that

$$\bar{\rho}_2 \left(X(\omega, t), \sum_{s<t} G(\omega, s)(X(\omega, s))\Delta W(\omega, s) \right) \approx 0,$$

and

$$(\exists Y)[Y \text{ is adapted and } \bar{\rho}_2(X, Y) \approx 0].$$

The set $\hat{C}$ is the intersection of the decreasing chain of internal sets $\hat{C}_n$, where $\hat{C}_n$ is the set of all X such that

$$\bar{\rho}_2\left(X(\omega,t), \sum_{s<t} G(\omega,s)(X(\omega,s))\Delta W(\omega,s)\right) \le 1/n, \tag{3}$$

and

$$(\exists Y)[Y \text{ is adapted after } 1/n \text{ and } \bar{\rho}_2(X,Y) \le 1/n]. \tag{4}$$

If $x \in C$ and X lifts x, then $X \in \hat{C}$. Moreover, if $X \in \hat{C}$ then X is near-standard, and taking standard parts we see that $^\circ X \in C$. Therefore $\hat{C}$ is the monad of C. This shows that the set C of all solutions of equation (1) has the property that the monad of C is a countable intersection of internal sets. In the following definition, we shall call sets with this property neocompact sets. In these lectures we show how to exploit the fact that the set of solutions of a stochastic differential equation in an adapted Loeb space is neocompact

3.2. Definition. *By a Π^0_1 set we mean a countable intersection of internal sets. A set C of random variables or stochastic processes on Ω is* **neocompact** *if the monad of C is a Π^0_1 set. A neocompact relation, i.e. a neocompact set of n-tuples of random variables and/or stochastic processes, is defined similarly.*

3.3. Theorem. *(See [3]) For every neocompact set C of continuous martingales, the set D of all pairs (x,w) such that (x,w) solves equation (1) and $w \in C$ is neocompact.*

Proof: Let the monad of C be $\bigcap_n C_n$ where each set C_n is internal. Let $\hat{D}$ be the monad of D. Let D_n be the internal set consisting of all pairs (X,W) such that $W \in C_n$ and (X,W) satisfies 3 and 4. The proof of Theorem 3.1 shows that $\hat{D} = \bigcap_n D_n$, so $\hat{D}$ is a Π^0_1 set. Therefore D is neocompact. □

Here is an alternative proof of Theorem 3.1, the "delay" proof, which will be easier to generalize to other cases. Let us take $x(\omega,u)$ to be zero when $u<0$. Let h be the delayed stochastic integral function

$$h(x,u)(\omega,t) = \int_0^t g(\omega,s)(x(\omega,s-u))dw(\omega,s).$$

Using the liftings G and W as before, we may form the internal counterpart

$$H(X,U)(\omega,t) = \sum_{s<t} G(\omega,s)(X(\omega,s-U))\Delta W(\omega,s).$$

It follows as before that the set of all pairs (x,u) such that $x = h(x,u)$ is neocompact. For each $u>0$ we can easily build an x such that $x = h(x,u)$

by first building x on the time interval $[0,u]$, then building x on $[u,2u]$, and so on. This is done without using the lifting at all. From the lifting we see that the set D of all $u \in [0,1]$ such that $\exists x\; x = h(x,u)$ is also neocompact. We have $(0,1] \subseteq D$, so the monad of D contains all noninfinitesimals. By $\aleph_1$-saturation, the monad of D contains an infinitesimal. Therefore $0 \in D$, so there exists x such that $x = h(x,0)$. This shows that x is a solution of the equation (1). $\square$

There are many other natural examples of neocompact sets. For instance, the set of all Brownian motions w in $L^2(\Omega, \mathcal{C}(\mathbb{R}^d))$ such that $w(\omega,0) = 0$ is neocompact. Its monad is the Π^0_1 set $\hat{B} = \bigcap_n \hat{B}_n$ where $\hat{B}_n$ is the internal set of all processes W such that $W(\omega,t)$ is within $1/n$ of a process which is adapted after $1/n$, and the law of W is within $1/n$ of the Wiener law (in the Prohorov metric on the set of measures on $C([0,1],\mathbb{R}^d)$).

Another important example is the set of all stopping times τ in the time interval $[0,1]$. A random variable $\tau \in \mathrm{L}^2(\Omega,[0,1])$ is a **stopping time** if the stochastic process $\min(t,\tau(\omega))$ is adapted. The corresponding notion of an internal stopping time was introduced in Lindstrøm's lectures. The set of all internal stopping times is itself internal. The monad of the set of stopping times is the Π^0_1 set of all X such that X is infinitely close some internal stopping time.

Lemma 2.2 shows that for every neocompact set C in either $L^2(\Omega,\mathcal{L}(\mathcal{M}))$ or $L^2(\Omega,\mathcal{C}(\mathcal{M}))$, the set of all adapted $x \in C$ is again neocompact. Similarly, for each neocompact set C in $L^2(\Omega,\mathcal{C}(\mathbb{R}^d))$, the set of all continuous martingales in C is neocompact.

The two proofs of Theorem 3.1 illustrate the usefulness of the following notion of a neocontinuous function.

3.4. Definition. *Let $B \subseteq L^2(\Omega,\mathcal{M})$ and $f : B \to L^2(\Omega,\mathcal{N})$. We say that a function $F : \hat{B} \to SL^2(\Omega,\mathcal{N})$ is a* **lifting** *of f if F is internal, $B \subseteq {}^\circ\hat{B}$, and whenever $X \in \hat{B}$ and ${}^\circ X = x \in B$ we have ${}^\circ(F(X)) = f(x)$. We say that f is* **neocontinuous** *if it has a lifting on each neocompact subset of B.*

It is clear that the composition of two neocontinuous functions is again neocontinuous.

Many examples of neocontinuous functions can be found in the earlier lectures. For example, the distance function

$$\rho : L^2(\Omega,\mathcal{M}) \times L^2(\Omega,\mathcal{M}) \to \mathbb{R}$$

is neocontinuous. The projection functions $(x,u) \mapsto x$ and $(x,u) \mapsto u$ are neocontinuous. For each bounded continuous function $\varphi : \mathcal{M} \to \mathbb{R}$, the function $x \mapsto E[\varphi(x(\bullet))]$ is a neocontinuous function from $L^2(\Omega,\mathcal{M})$ to $\mathbb{R}$.

Proposition 2.3 shows that the stochastic integral

$$(f, w) \mapsto \int_0^t f(\omega, s) dw(\omega, x)$$

is a neocontinuous function on the set of pairs (f, w) where f is uniformly bounded and adapted and w is a continuous martingale.

In the proof of Theorem 3.1 the application function

$$(g(\omega, t), x(\omega, t)) \mapsto g(\omega, t)(x(\omega, t))$$

is neocontinuous

$$L^2(\Omega, \mathcal{L}(C(\mathbb{R}^d, \mathbb{R}^{d\times d}))) \times L^2(\Omega, \mathcal{C}(\mathbb{R}^d)) \to L^2(\Omega, \mathcal{L}(C(\mathbb{R}^d, \mathbb{R}^{d\times d}))).$$

It follows that the function

$$(g, x, w) \mapsto \int_0^t g(\omega, s)(x(\omega, s)) dw(\omega, s)$$

is neocontinuous on the set of triples (g, x, w) where g is uniformly bounded and adapted, and x, w are continuous martingales. If $(x, y) \mapsto f(x, y)$ is a neocontinuous function of two variables, then $x \mapsto f(x, y_0)$ is neocontinuous in x for each y_0. Thus, for example,

$$\int_0^t g(\omega, s)(x(\omega, s)) dw(\omega, s)$$

is also neocontinuous as a function of x alone. The function $h(x, u)$ from the delay proof is also neocontinuous.

The following proposition about neocompact sets and neocontinuous functions contains the key facts needed in many of the applications. The proofs in a more general abstract setting are in [3] and [4].

3.5. Proposition. *Let C be a neocompact set and let f be a neocontinuous function on C.*

(i) C is closed and bounded.

(ii) f is continuous.

(iii) $f(C)$ is neocompact.

(iv) If D is neocompact then $C \cap f^{-1}(D)$ is neocompact.

(v) Every compact set is neocompact.

(vi) The intersection of any countable chain C_m of nonempty neocompact sets is nonempty.

Proof: (i) Let C be neocompact and let x be a limit of a sequence x_n of points in C. Let the monad of C be $\hat{C} = \bigcap_n \hat{C}_n$. Let X lift x. For each n there exists $Y_n \in C_n$ such that Y_n is within $1/n$ of X. By $\aleph_1$-saturation there exists $Y \in \hat{C}$ such that $Y \approx X$, and therefore $x \in C$. This proves that C is closed.

Suppose C is not bounded. Then for each n there is a pair of points X_n, Y_n in the monad of C such that $\bar{\rho}(X_n, Y_n) \geq n$. By $\aleph_1$-saturation there is a pair of points X, Y in the monad of C such that $\bar{\rho}(X, Y)$ is infinite, which is impossible.

(ii) By definition, f has an S-continuous lifting F.

(iii) Let F be a lifting of f, and let the monad of C be $\hat{C} = \bigcap_n \hat{C}_n$. Let $\hat{B}$ be the domain of F. Then $\hat{B}$ is internal and $C \subseteq {}^\circ(\hat{B})$. Let $\hat{D} = \hat{B} \cap \hat{C}$, and $\hat{D}_n = \hat{B} \cap \hat{C}_n$. Then $C = {}^\circ\hat{D}$ and $f(C) = {}^\circ(F(\hat{D}))$. We have $F(\hat{D}) = \bigcap_n F(\hat{D}_n)$; the nontrivial inclusion follows from $\aleph_1$-saturation. By $\aleph_1$-saturation again,

$$^\circ(F(\hat{D})) = {}^\circ(\bigcap_n F(\hat{D}_n)) = \bigcap_n({}^\circ F(\hat{D}_n)) = \bigcap_n(({}^\circ F(\hat{D}_n))^{1/n}).$$

It follows that the monad of $f(C)$ is the Π^0_1 set $\bigcap_n(({}^\circ F(\hat{D}_n))^{1/n})$.

The proof of (iv) is similar.

(v) Let E be compact. For each n, there is a finite subset E_n such that $E \subseteq ((E_n)^{1/n})$. Then the monad of E is the Π^0_1 set $\bigcap_n \hat{E}_n$ where

$$\hat{E}_n = \{X : \bar{\rho}_2(X, E_n) \leq 1/n\}.$$

(vi) For each m we may represent the monad of C_m as an intersection $\bigcap_n \hat{C}_{m,n}$ of a decreasing chain of internal sets. Then the intersection of any finite number of the internal sets $\hat{C}_{m,n}$ is nonempty. By $\aleph_1$-saturation, the intersection $\bigcap_m \bigcap_n \hat{C}_{m,n}$ is nonempty. Let X belong to this intersection. Then X is near-standard and ${}^\circ X \in \bigcap_m C_m$. □

As a consequence, we see that if C is a nonempty neocompact set and $f : C \to \mathbb{R}$ is a neocontinuous function, then f has a minimum and a maximum. (Because the range $f(C)$ is a closed bounded set of reals). This allows us to prove that optimal solutions of various kinds exist.

Another consequence is that for any neocompact relation $C \subseteq \mathcal{M} \times \mathcal{N}$, the projection function $f(x, y) = x$ is neocontinuous and hence its range

$$\{x \in \mathcal{M} : (\exists y \in \mathcal{N})(x, y) \in C\}$$

is neocompact.

We can now very quickly get many applications of the result that the set of solutions of the stochastic differential equation (1) is neocompact. Here are several typical examples. In each case, we can conclude that optimal solutions exist and that the set of all optimal solutions is again neocompact.

3.6. Corollary. *(i) Let w be a continuous martingale, and let*

$$f : C([0,1], \mathbb{R}^d) \to \mathbb{R}$$

be a bounded continuous function. Then the set of solutions x of equation (1) such that $E[f(x(\omega))]$ is a minimum is nonempty and neocompact.

(ii) Let C be a nonempty neocompact set of continuous martingales, and let

$$f : C([0,1], \mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}$$

be a bounded continuous function. Then the set of pairs (x, w) such that $w \in C$, (x, w) solves equation (1), and $E[f(x(\omega), w(\omega))]$ is a minimum, is nonempty and neocompact.

(iii) For every pair of stochastic differential equations of the form (1), the set of pairs of solutions (x_1, x_2) such that $\rho_2(x_1, x_2)$ is a minimum is nonempty and neocompact.

(iv) Let w be a continuous martingale. For any nonempty neocompact set $C \subseteq L^2(\Omega, \mathcal{C}(\mathbb{R}^d))$ or $C \subseteq L^2(\Omega, \mathcal{L}(\mathbb{R}^d))$, the set of all $y \in C$, such that

$$\rho_2 \left(y, \int_0^t g(\omega, s)(y(\omega, s)) dw(\omega, s) \right)$$

is a minimum, is nonempty and neocompact.

3.7. Corollary. *Suppose that we have a sequence of equations*

$$x(\omega, t) = \int_0^t g_n(\omega, s)(x(\omega, s)) dw_n(\omega, s)$$

where each g_n is a bounded adapted process with values in the space $C(\mathbb{R}^d, \mathbb{R}^{d \times d})$), and w_n is a continuous martingale with values in $\mathbb{R}^d$. Assume that for each n there exists an x which is a solution of the first n equations. Then there exists an x which is a simultaneous solution of all the equations, and the set of all such x is again neocompact. □

3.8. Corollary. *(Stochastic differential equations with control)*

(i) Let $w(\omega, s)$ be a continuous martingale in $\mathbb{R}^d$, let $h(\omega, s)$ be a uniformly bounded adapted process with values in the space $C(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^{d \times d})$, and let x be an adapted process with values in $\mathbb{R}^d$. Then the set of all continuous martingales y in $\mathbb{R}^d$ such that

$$y(\omega, t) = \int_0^t h(\omega, s)(x(\omega, s), y(\omega, s)) dw(\omega, s) \tag{5}$$

is nonempty and neocompact.

(ii) For any neocompact set C of triples (h, x, w) of the appropriate kind, the set of quadruples (h, x, w, y) such that $(h, x, w) \in C$ and y is a continuous martingale which is a solution of the above equation is neocompact. □

3.9. Corollary. *Let y be a continuous martingale such that for some continuous martingale x, x solves equation (1) and (x, y) solves (5). Let*

$$f : C([0,1], \mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}$$

be bounded and continuous. Then the set of controls x such that x solves equation (1), (x, y) solves (5), and $E[f(x(\omega), y(\omega))]$ is a minimum, is nonempty and neocompact. □

4. Solutions which are Markov processes

In [5] it was shown that in the case that w is a Brownian motion and the coefficient g is deterministic, equation (1) has a solution with the strong Markov property. In this section we shall give a simpler argument and prove a weaker result—there is a solution with the ordinary Markov property. To find such a solution, we shall use a particular countable sequence of optimal solutions and a lifting theorem from [5] for Markov processes. A continuous stochastic process x in $\mathcal{M}$ is a **Markov process** (with respect to $\mathcal{F}_\bullet$) if it is adapted and for each pair of times $s < t$ in $[0, 1]$ and each bounded continuous function $\varphi : \mathcal{M} \to \mathbb{R}$,

$$E[\varphi(x(\bullet, t))|\mathcal{F}_s] = E[\varphi(x(\bullet, t))|x(\bullet, s)]. \tag{6}$$

That is, the value of x at time s gives all information available at time s about the value of x at time t. The strong Markov property is a stronger condition obtained by replacing the time s with a stopping time τ.

We need a lifting theorem for conditional expectations of random variables.

4.1. Lemma. *Let $x \in L^2(\Omega, \mathbb{R}^d)$ be a random variable, let X lift x, and let $\mathcal{A}$ be a countably generated sigma-algebra contained in $\mathcal{F}_t$. Then for all sufficiently large $s \approx t$ in T,*

(i) $E[X|\mathcal{G}_s]$ is a lifting of $E[x|\mathcal{F}_t]$.

(ii) There is an internal algebra $\mathcal{B} \subseteq \mathcal{G}_s$ such that $E[X|\mathcal{B}]$ is a lifting of $E[x|\mathcal{A}]$.

Proof: Part (i) is in [5] and is left as an exercise. (ii) Let $\mathcal{A}_n$ be an increasing chain of finite algebras whose union generates $\mathcal{A}$. Then $E[x|\mathcal{A}] = \lim_{n\to\infty} E[x|\mathcal{A}_n]$. Let $\mathcal{B}_n$ be a finite internal algebra which approximates $\mathcal{A}_n$

within a null set. For each n, $E[X|\mathcal{B}_n]$ lifts $E[x|\mathcal{A}_n]$. Since $\mathcal{A}_n \subseteq \mathcal{F}_t$, we may take $\mathcal{B}_n$ so that $\mathcal{B}_n \subseteq \mathcal{G}_s$ for some $s \approx t$. By $\aleph_1$-saturation we may extend the sequence $\mathcal{B}_n$ to an internal sequence $\mathcal{B}_J, J \in {}^*\mathbb{N}$. By overspill, for all sufficiently small infinite J we have $\mathcal{B}_J \subseteq \mathcal{G}_s$ and $E[X|\mathcal{B}_J]$ lifts $E[x|\mathcal{A}]$. $\square$

4.2. Theorem. *(See [5]) Let w be a Brownian motion with values in $\mathbb{R}^d$ on Ω, and let $g \in \mathcal{L}(C(\mathbb{R}^d, \mathbb{R}^{d\times d}))$ be uniformly bounded. Then the stochastic differential equation*

$$x(\omega, t) = \int_0^t g(s, x(\omega, s))dw(\omega, s) \tag{7}$$

has a solution which is a Markov process.

One cannot expect to have a Markov solution in the case that the coefficient g depends on ω, because the value of x at time t will then depend on ω through g. Similarly, one cannot expect a Markov solution in the case that w is an arbitrary continuous martingale. However, in the case that w is a continuous martingale with the Markov property, the theorem can be improved, with more work, to say that the equation has a solution x such that the joint process (x, w) has the Markov property.

Proof of Theorem 4.2: Let Φ be a countable set of bounded continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$ such that whenever $E[\varphi(x(\omega))] = E[\varphi(y(\omega))]$ for all $\varphi \in \Phi$, x and y have the same distribution. Then for x to be a Markov process it is sufficient that equation (6) hold for all $\varphi \in \Phi$. Since each side of equation 6 is continuous in t, it is even sufficient that (6) holds for all rational t and all $\varphi \in \Phi$. Let (φ_n, t_n), $n \in \mathbb{N}$ be an enumeration of the countable set $\Phi \times (\mathbb{Q} \cap [0, 1])$.

Let C_0 be the set of all solutions of equation (7). We inductively define C_{n+1} to be the set of all $x \in C_n$ such that $E[\varphi_n(x(\bullet, t_n))]$ is maximal among all members of C_n. The functions $x \mapsto E[\varphi_n(x(\bullet, t_n))]$ are neocontinuous. Using Corollary 3.6, it follows by induction that for each n, the set C_n is nonempty and neocompact. The sets C_n form a decreasing chain. Then by countable compactness, the intersection $x \in \bigcap_n C_n$ is nonempty and neocompact.

Let $x \in \bigcap_n C_n$. x is a solution of (7) because it belongs to C_0. We shall prove that x is a Markov process. To do this we prove by induction that for all n,

$$E[\varphi_n(x(\bullet, t_n))|\mathcal{F}_s] = E[\varphi_n(x(\bullet, t_n))|x(\bullet, s)] \tag{8}$$

for all $s \leq t_n$. Suppose this holds for all $n < m$, but fails for m and some $s \leq t_m$. Since $\mathbb{R}^d$ is separable, the σ-algebra determined by $x(\bullet, s)$ is countably generated.

We now go up to the hyperfinite world. Let G be a uniformly bounded lifting of g. Let X be a martingale lifting of x. Then for all ω in a set U_0 of Loeb probability one,

$$(\forall t)^\circ X(\omega, t) = x(\omega, {}^\circ t)$$

and

$$(\forall t) X(\omega, t) \approx \sum_{s<t} G(s, X(\omega, s)) \Delta W(\omega, s). \tag{9}$$

Moreover, any X which satisfies (9) is a lifting of an element of C_0.

By Lemma 4.1, there exists $u \approx s$ in T and an internal algebra $\mathcal{B} \subseteq \mathcal{G}_u$ such that $E[\varphi_m(X(\bullet, t_m)|\mathcal{G}_u]$ lifts $E[\varphi_m(x(\bullet, t_m)|\mathcal{F}_s]$ and $E[\varphi_m(X(\bullet, t_m)|\mathcal{B}]$ lifts $E[\varphi_m(x(\bullet, t_m)|x(\bullet, s)]$. Since equation (8) fails, there is a set $U \in \mathcal{F}_s$ of positive Loeb measure and a real $\varepsilon > 0$ such that for all $\omega \in U$,

$$E[\varphi_m(x(\bullet, t_m))|\mathcal{F}_s](\omega) + \varepsilon \le E[\varphi_m(x(\bullet, t_m))|x(\bullet, s)](\omega).$$

U has an internal subset $V \in \mathcal{G}_u$ of positive Loeb measure such that both conditional expectation liftings hold at all $\omega \in V$.

We now form a new internal stochastic process Y as follows. For each equivalence class $[\omega]_u \subseteq V$, internally choose a new equivalence class $[\omega']_u$ such that ω, ω' belong to the same $\mathcal{B}$-equivalence class but

$$E[\varphi_m(X(\bullet, t_m))|\mathcal{G}_u](\omega) + \varepsilon/2 \le E[\varphi_m(X(\bullet, t_m))|\mathcal{G}_u](\omega').$$

Form the process Y from X by exchanging the set of paths in the class $[\omega]_u$ by a copy of the set of paths in the class $[\omega']_u$, for each $\omega \in V$. Then Y is an improvement on X for the function φ_m, because

$$E[\varphi_m(X(\bullet, t_m))] + \varepsilon \cdot P(V)/2 \le E[\varphi_m(Y(\bullet, t_m))].$$

Moreover, Y is near-standard, and we may take $y = {}^\circ Y$. Taking standard parts, the corresponding inequality also holds for x and y. We shall show that $y \in C_m$. This will contradict the fact that $x \in C_{m+1}$ and hence that $E[\varphi_m(x(\bullet, t_m))]$ is maximal.

Y still satisfies equation (9) and thus y belongs to the set C_0. By inductive hypothesis, x satisfies (8) for all $n < m$. The exchange procedure will not disturb this property, so y also satisfies (8) for all $n < m$. Therefore

$$E[\varphi_m(X(\bullet, t_m))] \approx E[\varphi_m(Y(\bullet, t_m))]$$

for all $n < m$. Then

$$E[\varphi_m(x(\bullet, t_m))] = E[\varphi_m(y(\bullet, t_m))]$$

for all $n < m$. This shows that $y \in C_m$ and completes the induction. □

The longer proof in [5] uses the same neocompact set $\bigcap_n C_n$ and shows that every $x \in \bigcap_n C_n$ is a strong Markov process.

4.3. Corollary. *Suppose that the solutions of the stochastic differential equation (7) in Theorem 4.2 are unique in distribution, that is, for any two solutions x and y, we have $E[\varphi(x(\bullet,t))] = E[\varphi(y(\bullet,t))]$ for each bounded continuous $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $t \in [0,1]$. Then every solution of (7) is a Markov process with respect to $\mathcal{F}_\bullet$.*

Proof: In the proof of Theorem 4.2, it was shown that every x in the neocompact set $\bigcap_n C_n$ is a Markov process. But in the case that the solutions of (7) are unique in distribution, every solution $x \in C_0$ maximizes $E[\varphi_n(x(\bullet,t_n))]$ for every n, so the sets C_n are all the same. Therefore the set $\bigcap_n C_n$ is equal to the set C_0 of all solutions of (7). □

In the above corollary, the weaker conclusion that every solution x is a Markov process with respect to the filtration generated by the process x itself is well known and easily proved by classical methods. The point of the above result is that all solutions are Markov processes with respect to the filtration $\mathcal{F}_\bullet$ which is given in advance and is rich enough so that the existence theorem holds.

5. A Fixed Point Theorem

We shall now prove a simple but quite general fixed point theorem which can be used to show that for many stochastic differential equations set of all solutions is both nonempty and neocompact.

Let $A^2(\Omega,\mathcal{C}(\mathcal{M}))$ be the set of all adapted processes in $L^2(\Omega,\mathcal{C}(\mathcal{M}))$.

Given a stochastic process $x \in L^2(\Omega,\mathcal{C}(\mathcal{M}))$ and a time $t \in [0,1]$, we let $x[0,t]$ be the restriction of x to the time interval $[0,t]$, that is, $(x[0,t])(\omega) = x(\omega) \cap ([0,t] \times \mathcal{M})$.

For $x \in L^2(\Omega,\mathcal{C}(\mathcal{M}))$ and $u \in [0,1]$, define the **delay function** dl by

$$dl(x,u)(\omega,t) = x(\omega, max(0,t-u)).$$

The delay function has the following properties:

$$dl(x,t+u) = dl(dl(x,t),u),$$

$$x[0,t] = y[0,t] \Rightarrow (dl(x,u))[0,t+u] = (dl(y,u))[0,t+u].$$

One can readily check that the delay function dl is neocontinuous from $L^2(\Omega,\mathcal{C}(\mathcal{M})) \times [0,1]$ to $L^2(\Omega,\mathcal{C}(\mathcal{M}))$, and also maps $A^2(\Omega,\mathcal{C}(\mathcal{M})) \times [0,1]$ to $A^2(\Omega,\mathcal{C}(\mathcal{M}))$.

5.1. Definition. *By an* **adapted function** *on $A^2(\Omega,\mathcal{C}(\mathcal{M}))$ we shall mean a function*

$$I : A^2(\Omega,\mathcal{C}(\mathcal{M})) \to A^2(\Omega,\mathcal{C}(\mathcal{M}))$$

such that for all x, y, t,

$$(I(x))(\omega, 0) = x(\omega, 0),$$

and

$$x[0,t] = y[0,t] \Rightarrow (I(x))[0,t] = (I(y))[0,t].$$

That is, $I(x)$ has initial value $x(\omega, 0)$ and for each t, $I(x)[0,t]$ depends only on $x[0,t]$.

For example, if

$$w \in A^2(\Omega, C(\mathbb{R}^{d\times d}))$$

is a continuous martingale and

$$g \in A^2(\Omega \times [0,1], L^2(\mathbb{R}^d, \mathbb{R}^{d\times d}))$$

is uniformly bounded then the stochastic integral

$$I(x)(\omega, t) = x(\omega, 0) + \int_0^t (g(\omega, s, x(\omega, s))dw(\omega, s)$$

is an adapted function on $A^2(\Omega, \mathcal{C}(\mathcal{M}))$.

5.2. Theorem. *(Fixed Point Theorem) Let $C \subseteq A^2(\Omega, \mathcal{C}(\mathcal{M}))$ be a nonempty neocompact set such that for each $x \in C$ and $t \in [0,1]$, $dl(x,t) \in C$. Let I be an adapted function on $A^2(\Omega, \mathcal{C}(\mathcal{M}))$ such that $I(C) \subseteq C$ and I is neocontinuous. Then there exists a point $x \in C$ such that $I(x) = x$ (a* **fixed point for** *I), and the set of all fixed points for I in C is neocompact.*

Proof: The function $j(x) = \rho(x, I(x))$ is a composition of neocontinuous functions and hence is itself neocontinuous on C. The set $\{0\}$ is neocompact, and therefore the inverse image

$$j^{-1}(\{0\}) = \{x \in C : x = I(x)\},$$

which is the set of all fixed points of I, is neocompact.

The proof that a solution exists is an abstract form of the delay argument.

Let D be the set of all pairs $(y, u) \in C \times [0,1]$ such that $y = I(dl(y, u))$. Then D is a neocompact set. Since the projection function $(y, u) \mapsto u$ is neocontinuous, the set E of all $u \in [0,1]$ such that $(\exists y \in C)(y, u) \in D$ is a neocompact subset of $[0,1]$. We show that $(0,1] \subseteq E$. Once this is done, the proof is completed as follows. Since E is neocompact it is closed, and therefore $0 \in E$. But this means that there exists $y \in C$ such that

$$y = I(dl(y, 0)) = I(y)$$

as required.

We let $u \in (0,1]$ and prove that $u \in E$. Choose an element $y_0 \in C$. Inductively define a sequence y_n by

$$y_{n+1} = I(dl(y_n, u)).$$

We see by induction that each y_n belongs to C.

We now claim that for each n,

$$y_{n+1}[0, nu] = y_n[0, nu].$$

We prove this claim by induction on n. For $n = 0$ we have

$$(y_1)(\omega, 0) = (I(dl(y_0, u)))(\omega, 0) = (dl(y_0, u))(\omega, 0) = (y_0)(\omega, 0),$$

so

$$y_1[0, 0] = y_0[0, 0].$$

Assume that the claim holds for n and let $t = nu$, so that

$$y_{n+1}[0, t] = y_n[0, t].$$

Then

$$(dl(y_{n+1}, u))[0, t+u] = (dl(y_n, u))[0, t+u],$$

and therefore

$$\begin{aligned} & y_{n+2}[0, t+u] \\ &= (I(dl(y_{n+1}, u))[0, t+u] \\ &= (I(dl(y_n, u)))[0, t+u] \\ &= y_{n+1}[0, t+u]. \end{aligned}$$

This completes the induction and proves the claim.

Now take k large enough so that $ku \geq 1$. Then by the claim,

$$y_{k+1} = y_k,$$

and therefore

$$y_k = I(dl(y_k, u)).$$

This shows that $(y_k, u) \in D$ and so $u \in E$ as required. $\square$

As a first illustration let us apply the Fixed Point Theorem to the case of equation (1). Let k be a uniform bound for the adapted continuous function g, and let C be the set of all stochastic integrals $\int_0^t h(\omega, s) dw(\omega, s)$ where w is a continuous martingale of dimension d and h is an adapted process in $L^2(\Omega, \mathbb{R}^{d \times d})$ with bound k. Then C is neocompact because its monad is the set of all X such that for each n, X is within $1/n$ of some hyperfinite sum

$$\sum_{s<t} H(\omega, s)\Delta W(\omega, s)$$

where H is adapted after $1/n$ and bounded by $k + 1/n$. Whenever $x \in C$ and $u \in [0, 1]$, we have $dl(x, u) \in C$. This can be seen by changing the coefficient h to be zero before u.

In this case,

$$I(x)(\omega, t) = \int_0^t g(\omega, s, x(\omega, s))dw(\omega, s).$$

Then I is an adapted function, $I : C \to C$, and I is neocontinuous. By the Fixed Point Theorem, the set of all fixed points $x \in C$ of I is a nonempty neocompact set, and this set is the set of all solutions of equation (1).

6. Stochastic Differential Equations with Nondegenerate Coefficients

In this section we apply the Fixed Point Theorem to give a short proof of a more difficult existence theorem. This is the case of stochastic differential equations where the coefficient is measurable rather than continuous in x, but the determinant of the coefficient is bounded away from zero. This result is from [5], and is an improvement of a weak existence theorem of Krylov [7]. The present proof uses some neocontinuity results from [3].

Let us choose a uniform bound $k > 0$ once and for all, and let J be the compact set of all $d \times d$ matrices A such that the entries of A are bounded by k and $det(AA^T) \geq 1/k$.

We collect the needed facts in a lemma which we state without proof.

6.1. Lemma. *([3]) Let w be a Brownian motion in $\mathbb{R}^d$. There is a neocompact set $C \subseteq L^2(\Omega, \mathcal{C}(\mathbb{R}^d))$ such that:*

(i) For each adapted process $y \in L^2(\Omega, \mathcal{L}(J))$, and $r \in [0, 1]$, the integral $\int_{min(r,t)}^t y(\omega, s)dw(\omega, s)$ belongs to C,

(ii) C is closed under delays,

(iii) For each function $g \in L^2([0, 1] \times \mathbb{R}^d, J)$ where $\mathbb{R}^d$ has the normal measure, the function

$$I(x)(\omega, t) = \int_0^t g(s, x(\omega, s))dw(\omega, s)$$

is neocontinuous on C. □

Here is the existence theorem.

6.2. Theorem. *([5]) Let w be a Brownian motion in $\mathbb{R}^d$. For each function $g \in L^2([0,1] \times \mathbb{R}^d, J)$ where $\mathbb{R}^d$ has the normal measure, the equation*

$$x(\omega, t) = \int_0^t g(s, x(\omega, s)) dw(\omega, s)$$

has a continuous martingale solution, and the set of all solutions is neocompact.

Proof: Let C be the neocompact set from Lemma 6.1 and let $I(x)$ be the stochastic integral function

$$I(x) = \int_0^t g(s, x(\omega, s)) dw(\omega, s).$$

I is neocontinuous on C by Lemma 6.1. By Lemma 2.2, we may take C to be included in the set of adapted processes and may also take C so that $x(\omega, 0) = 0$ for all $x \in C$. Then by part (i) of Lemma 6.1, $I(C) \subseteq C$, and $(I(x))(\omega, 0) = x(\omega, 0)$. Since $(I(x))(\omega, t)$ depends only on (ω, s) and the values of $x(\omega, s)$ for $s \le t$, I is an adapted function. The conclusion of the theorem now follows from the Fixed Point Theorem. □

It would be interesting to use the Fixed Point Theorem to find additional existence theorems. One candidate to be checked is the equation of Theorem 6.2 with the coefficient g being an adapted function rather than deterministic.

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