SCATTERED SENTENCES HAVE FEW SEPARABLE RANDOMIZATIONS

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ABSTRACT. In the paper *Randomizations of Scattered Sentences*, Keisler showed that if Martin's axiom for aleph one holds, then every scattered sentence has few separable randomizations, and asked whether the conclusion could be proved in ZFC alone. We show here that the answer is "yes". It follows that the absolute Vaught conjecture holds if and only if every $L_{\omega_1\omega}$-sentence with few separable randomizations has countably many countable models.

1. INTRODUCTION

This note answers a question posed in the paper [K2], and grew out of a discussion following a lecture by Keisler at the Midwest Model Theory meeting in Chicago on April 5, 2016.

Fix a countable first order signature L. Countable structures with signature L can be coded in the natural way by subsets of $\mathbb{N}$. A sentence φ of the infinitary logic $L_{\omega_1\omega}$ is **scattered** if there is no countable fragment L_A of $L_{\omega_1\omega}$ such that there is a perfect set of codes of countable models of φ that are not L_A-equivalent. Scattered sentences were introduced by Morley [M], motivated by Vaught's conjecture. Morley showed that if φ is scattered then φ has at most $\aleph_1$ countable models.

The **Vaught conjecture** for an $L_{\omega_1\omega}$-sentence φ (Vaught [Va]) says: *Up to isomorphism, φ has either countably many*[1] *countable models or has $2^{\aleph_0}$ countable models.*

Following [BFKL], the **absolute Vaught conjecture for** φ says: *If φ is scattered, then φ has countably many countable models.*

The Vaught conjecture for φ is trivially true if the continuum hypothesis $2^{\aleph_0} = \aleph_1$ holds. The absolute Vaught conjecture for φ is equivalent to the Vaught conjecture for φ if the continuum hypothesis is false, and hence implies the Vaught conjecture for φ. The absolute Vaught conjecture for φ is absolute in the sense that it has the same truth value in all transitive models of ZFC that contain all ordinals and φ.

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[1] Here, countably many means of cardinality at most $\aleph_0$.

Given a signature L in continuous logic, the **pure randomization theory** P^R (from [BK]) is a theory whose signature L^R has a sort $\mathbb{K}$ for random elements and a sort $\mathbb{E}$ for events. For each formula $\theta(\cdot)$ of L with n free variables, L^R has a function symbol $[\![\theta(\cdot)]\!]$ of sort $\mathbb{K}^n \to \mathbb{E}$ for the event where $\theta(\cdot)$ is true. L^R also has Boolean operations $\sqcup, \sqcap, \neg$ in the event sort, a predicate μ from events to $[0,1]$, and distance predicates $d_{\mathbb{K}}, d_{\mathbb{E}}$ for each sort. The set of axioms for P^R is recursive in L. It insures that the functions $[\![\theta(\cdot)]\!]$ respect validity, connectives, and quantifiers, that each event is equal to the set where some pair of random elements agree, and that μ is an atomless probability measure on the set of events. There are also axioms that define $d_{\mathbb{K}}$ and $d_{\mathbb{E}}$ in the natural way:

$$d_{\mathbb{K}}(\mathbf{f}, \boldsymbol{g}) = \mu([\![\boldsymbol{f} \neq \boldsymbol{g}]\!]), \quad \mathbf{d}_{\mathbb{E}}(\mathsf{A}, \mathsf{B}) = \mu(\mathsf{A} \triangle \mathsf{B}).$$

Pre-models of P^R are called **randomizations**, and models of P^R are called **complete randomizations**. In Theorem 5.1 of [K2] (stated as Fact 2.7 below), it is shown that in a complete separable randomization, there is a unique mapping $[\![\cdot]\!]$ from $L_{\omega_1\omega}$-sentences to events that respects validity, countable connectives, and quantifiers. A **separable randomization of** an $L_{\omega_1\omega}$-sentence φ is a separable randomization whose completion satisfies $\mu([\![\varphi]\!]) = 1$. Intuitively, in a separable randomization of φ, a random element is obtained by randomly picking an element of a random countable model of φ, with respect to some underlying probability space.

An especially simple kind of separable randomization of φ, called a **basic randomization**, is built in the following way. Let $\mathcal{L}$ be the family of Borel subsets of $[0,1)$ and λ be the restriction of Lebesgue measure to $\mathcal{L}$. Take a countable sequence $\langle \mathcal{M}_n \rangle_{n \in \mathbb{N}}$ of countable models of φ, and a partition $[0,1) = \bigcup_m \mathsf{B}_n$ of $[0,1)$ into Borel sets of positive λ-measure. Then there is a unique separable randomization $\mathcal{P}$ of φ, called a basic randomization, such that:

- $\mathcal{P}$ has measure λ and event sort $\mathcal{L}$ with the usual Boolean operations.
- The random element sort $\mathcal{K}$ is the set of λ-measurable functions on $[0,1)$ that map each set B_n into $\mathcal{M}_n$.
- For each n, first order formula θ, and tuple $\vec{\boldsymbol{f}}$ in $\mathcal{K}$,

$$[\![\theta(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} \cap \mathsf{B}_n = \{t \in \mathsf{B}_n \colon \mathcal{M}_n \models \theta(\vec{\boldsymbol{f}}(t))\}.$$

One of the main results of [K2] is that if φ has only countably many countable models, the only separable randomizations of φ are those that are isomorphic to basic randomizations. In the special case that φ is a first order theory, this was previously proved in [AK]. φ is said to have **few separable randomizations** if every complete separable randomization of φ is isomorphic to a basic randomization. Thus if φ has only countably many countable models, then φ has few separable randomizations.

The other two main results of [K2] are: If φ has few separable randomizations, then φ is scattered. If Martin's axiom for $\aleph_1$ holds and φ is scattered, then φ has few separable randomizations. [K2] asks whether the conclusion of this last result can be proved in ZFC. In this paper we show that the answer to that question is "yes". The idea is to use the Shoenfield absoluteness theorem to eliminate the use of Martin's axiom.

The results in the preceding paragraph show that being scattered is equivalent to having few separable randomizations. Thus the absolute Vaught conjecture is equivalent to the property that having few separable randomizations implies having countably many countable models.

2. Background

We refer to [BBHU] for background in continuous logic, [J] for background on absoluteness and Martin's axiom, and [K1] for background on $L_{\omega_1\omega}$.

2.1. Scott Sentences, Scattered Sentences, and Vaught's Conjecture.

$L_{\omega_1\omega}$ is the infinitary logic formed from the first order logic with signature L by adding countably infinite conjunctions and disjunctions. We assume throughout that φ is an $L_{\omega_1\omega}$-sentence that implies $(\exists x)(\exists y)x \neq y$.

By a **Scott sentence** for a countable first order structure $\mathcal{M}$ we mean an $L_{\omega_1\omega}$ sentence φ such that $\mathcal{M} \models \varphi$ and every countable model of φ is isomorphic to $\mathcal{M}$. We say that φ is a Scott sentence if φ is a Scott sentence for some countable first order structure $\mathcal{M}$.

Fact 2.1. *(Scott's Theorem; see [Sc], or Theorem 1 in [K1]). Every countable first order structure has a Scott sentence.*

Say that φ has **perfectly many countable models** if there is a perfect set of codes of pairwise non-isomorphic countable models of φ. By the number of countable models of φ we mean the greatest κ such that there is a set of κ pairwise non-isomorphic countable models of φ. It is easily seen that if φ has fewer than $2^{\aleph_0}$ countable models, then φ does not have perfectly many countable models, and if φ does not have perfectly many countable models, then φ is scattered.

Fact 2.2. *(Lemma 3.4 in [BFKL]) φ is scattered if and only if φ does not have perfectly many countable models* [2].

The Vaught conjecture for φ and the absolute Vaught conjecture for φ were stated in the introduction above.

To avoid the exceptional case where the continuum hypothesis holds and the Vaught conjecture is trivial, Steel [St] proposed the **strong Vaught conjecture for** φ: *If φ has uncountably many countable models, then φ has perfectly many countable models.*

Since perfect sets have cardinality $2^{\aleph_0}$, the strong Vaught conjecture for φ implies the Vaught conjecture for φ. If the continuum hypothesis fails, then the strong Vaught conjecture for φ is equivalent to the Vaught conjecture

[2] In [BFKL], being scattered is defined as not having perfectly many countable models, and Lemma 3.4 says that if φ is scattered as defined in Morley [M] (and here), then φ is scattered as defined in [BFKL].

for φ, by the result of Morley [M] that scattered sentences have at most $\aleph_1$ countable models.

Steel pointed out that the strong Vaught conjecture for φ is equivalent in ZFC to a Σ^1_2 formula, so by the Shoenfield absoluteness theorem, the strong Vaught conjecture for φ is absolute (i.e., has the same truth value) for all transitive models of ZFC that contain all ordinals and φ. The following corollary of Fact 2.2 shows that the absolute Vaught conjecture for φ is also absolute.

Corollary 2.3. *In ZFC, the absolute Vaught conjecture for φ is equivalent to the strong Vaught conjecture for φ.*

Proof. Assume the strong Vaught conjecture for φ. Suppose φ is scattered. By Fact 2.2, φ does not have perfectly many countable models. By the strong Vaught conjecture for φ, φ has only countably many countable models, so the absolute Vaught conjecture for φ holds.

Assume the absolute Vaught conjecture for φ. Suppose that φ has uncountably many countable models. By the absolute Vaught conjecture for φ, φ is not scattered. By Fact 2.2, φ has perfectly many countable models, so the strong Vaught conjecture holds for φ. □

2.2. **Randomizations.** We will not need the formal statement of the axioms of the pure randomization theory P^R, or the formal definition of $[\![\psi(\cdot)]\!]$ for $L_{\omega_1\omega}$-formulas $\psi(\cdot)$. In this section we state the definitions and results from [K2] that we need.

Given pre-structures $\mathcal{N}, \mathcal{P}$ with signature L^R, we call $\mathcal{P}$ a **reduction of** $\mathcal{N}$ if $\mathcal{P}$ is obtained from $\mathcal{N}$ by identifying elements at distance zero, and call $\mathcal{P}$ a **completion of** $\mathcal{N}$ if $\mathcal{P}$ is a structure obtained from a reduction of $\mathcal{N}$ by completing the metrics.

We define an isomorphism between pre-structures as in [AK]. An **isomorphism** $h\colon \mathcal{N} \to \mathcal{P}$ is a mapping from $\mathcal{N}$ into $\mathcal{P}$ such that h preserves the truth values of all formulas of L^R, and every element of $\mathcal{P}$ is at distance zero from some element of $h(\mathcal{N})$. We say that $\mathcal{N}$ and $\mathcal{P}$ are **isomorphic**, in symbols $\mathcal{N} \cong \mathcal{P}$, if there is an isomorphism $h\colon \mathcal{N} \to \mathcal{P}$. By Remark 2.4 in [AK], $\cong$ is an equivalence relation on pre-structures.

Up to isomorphism, every pre-structure has a unique reduction and completion. The mapping that identifies elements at distance zero is called the **reduction mapping**, and is an isomorphism from a pre-structure onto its reduction. Note that if $\mathcal{N}$ and $\mathcal{P}$ are reduced structures, than every isomorphism $h\colon \mathcal{N} \to \mathcal{P}$ is an isomorphism in the usual sense—a bijection from N onto P that preserves all relations, functions, and constant symbols of L^R, and hence preserves the truth values of all formulas of L^R. Moreover, two pre-structures are isomorphic if and only if their reductions are isomorphic in the usual sense.

The axioms of P^R have the following consequence:

$$\mu([\![(\exists x)(\exists y)x \neq y]\!]) = 1.$$

So every separable randomization is a separable randomization of the sentence $(\exists x)(\exists y)x \neq y$. Since P^R has axioms saying that the functions $[\![\theta(\cdot)]\!]$ for first order θ respect connectives, and that every event is equal to $[\![\boldsymbol{a} = \boldsymbol{b}]\!]$ for some $\boldsymbol{a}, \boldsymbol{b}$, it follows that:

Fact 2.4. *Suppose $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ and $\mathcal{N}' = (\mathcal{K}', \mathcal{B}')$ are models of P^R, h maps $\mathcal{K}$ onto $\mathcal{K}'$, and*

$$\mathcal{N} \models \mu([\![\theta(\vec{\boldsymbol{a}})]\!]) \geq r \Leftrightarrow \mathcal{N}' \models \mu^{\mathcal{N}'}([\![\theta(h\vec{\boldsymbol{a}})]\!]) \geq r$$

for all first order θ, tuples $\vec{\boldsymbol{a}}$ in $\mathcal{K}$, and rational r. Then h can be extended to a unique isomorphism from $\mathcal{N}$ onto $\mathcal{N}'$.

The simplest examples of randomizations are the Borel randomizations, defined as follows. Let $\mathcal{L}$ be the family of Borel subsets of $[0, 1)$ and λ be the restriction of Lebesgue measure to $\mathcal{L}$.

Definition 2.5. *The **Borel randomization** of a model $\mathcal{M} \models (\exists x)(\exists y)x \neq y$ is the structure $(\mathcal{M}^{\mathcal{L}}, \mathcal{L})$ of sort L^R where $\mathcal{M}^{\mathcal{L}}$ is the set of all functions $\boldsymbol{f}\colon [0, 1) \to M$ with countable range such that $\{t \mid \boldsymbol{f}(t) = a\} \in \mathcal{L}$ for each $a \in M$, $\mathcal{L}$ has the usual Boolean operations, μ is interpreted by λ, and*

$$[\![\theta(\vec{\boldsymbol{f}})]\!] = \{t \mid \mathcal{M} \models \theta(\vec{\boldsymbol{f}}(t))\}.$$

A basic randomization of φ is formed by "gluing together" countably many Borel randomizations of countable models of φ. Basic randomizations were defined in the Introduction. We re-state the definition more carefully here.

Definition 2.6. *Suppose that*

- $[0, 1) = \bigcup_n \mathsf{B}_n$ *is a partition of* $[0, 1)$ *into countably many Borel sets of positive measure;*
- *for each n, $\mathcal{M}_n$ is a countable model of φ;*
- $\prod_n \mathcal{M}_n^{\mathsf{B}_n}$ *is the set of all functions* $\boldsymbol{f}\colon [0, 1) \to \bigcup_n M_n$ *such that for all n,*

  $(\forall t \in \mathsf{B}_n)\boldsymbol{f}(t) \in M_n$ *and* $(\forall a \in M_n)\{t \in \mathsf{B}_n \mid \boldsymbol{f}(t) = a\} \in \mathcal{L}$;
- $(\prod_n \mathcal{M}_n^{\mathsf{B}_n}, \mathcal{L})$ *has the usual Boolean operations, μ is interpreted by λ, and the $[\![\theta(\cdot)]\!]$ functions are*

$$[\![\theta(\vec{\boldsymbol{f}})]\!] = \bigcup_n \{t \in \mathsf{B}_n \mid \mathcal{M}_n \models \theta(\vec{\boldsymbol{f}}(t))\}.$$

*$(\prod_n \mathcal{M}_n^{\mathsf{B}_n}, \mathcal{L})$ is called a **basic randomization** of φ.*

Fact 2.7. *(Theorem 5.1 in [K2]) Let $\mathcal{P} = (\mathcal{K}, \mathcal{E})$ be a complete separable randomization, and let Ψ_n be the class of $L_{\omega_1\omega}$ formulas with n free variables. There is a unique family of functions $[\![\psi(\cdot)]\!]^{\mathcal{P}}$, $\psi \in \bigcup_n \Psi_n$, such that:*

(i) *When $\psi \in \Psi_n$, $[\![\psi(\cdot)]\!]^{\mathcal{P}}\colon \mathcal{K}^n \to \mathcal{E}$.*
(ii) *When ψ is a first order formula, $[\![\psi(\cdot)]\!]^{\mathcal{P}}$ is the usual event function for the structure $\mathcal{P}$.*

(iii) $[\![\neg\psi(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} = \neg[\![\psi(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}}$.
(iv) $[\![(\psi_1 \vee \psi_2)(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} = [\![\psi_1(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} \sqcup [\![\psi_2(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}}$.
(v) $[\![\bigvee_k \psi_k(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} = \sup_k [\![\psi_k(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}}$.
(vi) $[\![(\exists u)\theta(u, \vec{\boldsymbol{f}})]\!]^{\mathcal{P}} = \sup_{\boldsymbol{g} \in \mathcal{K}} [\![\theta(\boldsymbol{g}, \vec{\boldsymbol{f}})]\!]^{\mathcal{P}}$.

Moreover, for each $\psi \in \Psi_n$, the function $[\![\psi(\cdot)]\!]^{\mathcal{P}}$ is Lipschitz continuous with bound one, that is, for any pair of n-tuples $\vec{\boldsymbol{f}}, \vec{\boldsymbol{h}} \in \mathcal{K}^n$ we have

$$d_{\mathbb{E}}([\![\psi(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}}, [\![\psi(\vec{\boldsymbol{h}})]\!]^{\mathcal{P}}) \leq \sum_{m<n} d_{\mathbb{K}}(\boldsymbol{f}_m, \boldsymbol{h}_m).$$

Definition 2.8. *Let $\mathcal{N}$ be a separable randomization with completion $\mathcal{P}$, and φ be an $L_{\omega_1\omega}$-sentence. We write*

$$\mu^{\mathcal{N}}([\![\varphi]\!]) = \mu^{\mathcal{P}}([\![\varphi]\!]) = \mu([\![\varphi]\!]^{\mathcal{P}}).$$

If $\mu^{\mathcal{N}}([\![\varphi]\!]) = 1$, we say that $\mathcal{N}$ is a ***randomization of*** *φ.*

We say that φ has ***few separable randomizations*** *if every complete separable randomization of φ is isomorphic to a basic randomization of φ.*

Fact 2.9. *(Lemma 4.3 and Theorem 4.6 in [K2].) Every basic randomization of φ is isomorphic to its reduction, which is a complete separable randomization of φ (and thus a model of P^R).*

Fact 2.10. *(Lemma 9.4 in [K2]) Let $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ be a basic randomization. For each $j \in J$, let δ_j be a Scott sentence of $\mathcal{M}_j$. Then for each complete separable randomization $\mathcal{P}$ of φ, the following are equivalent.*

- *$\mathcal{P}$ is isomorphic to $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$.*
- *$\mu^{\mathcal{P}}([\![\delta_n]\!]) = \lambda(\mathsf{B}_j)$ for each $j \in J$.*

Fact 2.11. *(Lemma 9.5 in [K2]) φ has few separable randomizations if and only if for every complete separable randomization (or every countable randomization) $\mathcal{N}$ of φ there is a Scott sentence δ such that $\mu^{\mathcal{N}}([\![\delta]\!]) > 0$.*

Fact 2.12. *(Theorem 10.1 in [K2]). If φ has few separable randomizations, then φ is scattered.*

Fact 2.13. *(Theorem 10.3 in [K2]). Assume that Lebesgue measure is $\aleph_1$-additive (e.g. assume that MA($\aleph_1$) holds). Then every scattered sentence has few separable randomizations.*

Question 11.4 in [K2] asks whether or not the conclusion of Fact 2.13 can be proved in ZFC.

3. The Main Result

We prove the following theorem, which answers Question 11.4 in [K2] affirmatively.

Theorem 3.1. *Every scattered sentence has few separable randomizations.*

Fact 2.12 and Theorem 3.1 give us the following two corollaries.

Corollary 3.2. *A sentence of $L_{\omega_1\omega}$ is scattered if and only if it has few separable randomizations.*

Corollary 3.3. *For each $L_{\omega_1\omega}$-sentence φ, the following are equivalent.*

(i) *The absolute Vaught conjecture for φ holds.*
(ii) *If φ has few separable randomizations, then φ has countably many countable models.*

Note that each countable pre-structure $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ in the signature L^R can be coded in a natural way by a first order structure with universe $\mathbb{N}$ and a countable signature indexed by $\mathbb{N}$. In particular, the function $\mu\colon \mathcal{B} \to [0,1]$ can be coded by the set of $(e, m, n) \in \mathbb{N}^3$ such that e codes an event E and $m/n \leq \mu(\mathsf{E})$.

Let $\mathcal{A}$ be the set of subsets of $[0,1)$ that are finite unions of intervals with rational endpoints. Given a countable model $\mathcal{M}$ of $(\exists x)(\exists y)x \neq y$ with countable signature L, let $\mathcal{M}^{\mathcal{A}}$ be the set of functions $f\colon [0,1) \to \mathcal{M}$ with finite range such that for each $a \in \mathcal{M}$, $f^{-1}(a) \in \mathcal{A}$. Let $\widetilde{\mathcal{M}}$ be the completion of $(\mathcal{M}^{\mathcal{A}}, \mathcal{A})$. $\widetilde{\mathcal{M}}$ is isomorphic to the Borel randomization $(\mathcal{M}^{\mathcal{L}}, \mathcal{L})$ of $\mathcal{M}$. $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{M}^{\mathcal{A}}$ are countable and can be coded in the natural way by subsets of $\mathbb{N}$.

Lemma 3.4. *Let $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ be a countable randomization with a coding. Then the statement (S) below is equivalent (in ZFC) to a Σ^1_1 formula with parameter $\mathcal{N}$.*

(S) *There exists a Scott sentence δ such that $\mu^{\mathcal{N}}([\![\delta]\!]) > 0$.*

Proof. For each event C in the completion of $\mathcal{N}$ such that $\mu(\mathsf{C}) > 0$, let $\mu|\mathsf{C}$ be the conditional measure such that

$$(\mu|\mathsf{C})(\mathsf{E}) = \mu(\mathsf{E} \sqcap \mathsf{C})/\mu(\mathsf{C}),$$

and let $\mathcal{N}|\mathsf{C}$ be the completion of the pre-structure obtained from $\mathcal{N}$ by replacing μ by $\mu|\mathsf{C}$. We first show that (S) is equivalent to the following statement.

(S') There exists a countable model $\mathcal{M}$ of $(\exists x)(\exists y)x \neq y$ and an event C in the completion of $\mathcal{N}$ such that $\mu(\mathsf{C}) > 0$ and $\mathcal{N}|\mathsf{C} \cong \widetilde{\mathcal{M}}$.

Assume (S). Let δ be a Scott sentence δ such that $\mu^{\mathcal{N}}([\![\delta]\!]) > 0$. Let $\mathsf{C} = [\![\delta]\!]$, which is an event of positive measure in the completion of $\mathcal{N}$. Then $\mu^{\mathcal{N}|\mathsf{C}}([\![\delta]\!]) = 1$, so $\mathcal{N}|\mathsf{C}$ is a separable randomization of δ. Let $\mathcal{M}$ be a countable model of δ. By Fact 2.10, we have $\mathcal{N}|\mathsf{C} \cong \widetilde{\mathcal{M}}$, so (S') holds.

Now assume (S'). By Scott's theorem, $\mathcal{M}$ has a Scott sentence δ. Then by Fact 2.10, $\mu^{\widetilde{\mathcal{M}}}([\![\delta]\!]) = 1$, so

$$1 = \mu^{\mathcal{N}|\mathsf{C}}([\![\delta]\!]) = \mu^{\mathcal{N}}(\mathsf{C} \sqcap [\![\delta]\!])/\mu^{\mathcal{N}}(\mathsf{C}).$$

Hence

$$\mu^{\mathcal{N}}([\![\delta]\!]) \geq \mu^{\mathcal{N}}(C \sqcap [\![\delta]\!]) = \mu^{\mathcal{N}}(\mathsf{C}) > 0,$$

so (S) holds.

We now show that (S') is equivalent to the following statement.

(S") There exists a countable coded structure $\mathcal{M}$ with at least 2 elements, a sequence $\mathsf{B}\colon \mathbb{N} \to \mathcal{B}$, and double sequences $\alpha\colon \mathbb{N} \times \mathbb{N} \to \mathcal{M}^{\mathcal{A}}$, $\beta\colon \mathbb{N} \times \mathbb{N} \to \mathcal{K}$ such that

(a) B is Cauchy convergent in $d_{\mathbb{E}}$, and $\lim_{n\to\infty} \mu(\mathsf{B}_n) > 0$.

(b) For each $m \in \mathbb{N}$, $\langle \alpha_{m,n} \mid n \in \mathbb{N}\rangle$ and $\langle \beta_{m,n} \mid n \in \mathbb{N}\rangle$ are Cauchy convergent in $d_{\mathbb{K}}$.

(c) For each $x \in \mathcal{M}^{\mathcal{A}}$, there exists $m_x \in \mathbb{N}$ such that $\alpha_{m_x,n} = x$ for all $n \in \mathbb{N}$, and for each $y \in \mathcal{K}$, there exists $m_y \in \mathbb{N}$ such that $\beta_{m_y,n} = y$ for all $n \in \mathbb{N}$.

(d) For each L-formula $\psi(v_1, \dots, v_k)$,

$$\lim_{n\to\infty} \mu^{\widetilde{\mathcal{M}}}(\llbracket \psi(\alpha_{1,n}, \dots, \alpha_{k,n}) \rrbracket) = \lim_{n\to\infty} \mu^{\mathcal{N}}(\llbracket \psi(\beta_{1,n}, \dots, \beta_{k,n}) \rrbracket \sqcap \mathsf{B}_n)/\mu^{\mathcal{N}}(\mathsf{B}_n).$$

In (S"), $\mathcal{N}$ and $\mathcal{M}$ are coded structures, so (S") is clearly Σ^1_1 with parameter $\mathcal{N}$.

The functions $\llbracket \psi(\cdot) \rrbracket$ are uniformly continuous in each model of P^R. Whenever (a) and (b) hold, for each $m, n \in \mathbb{N}$ the reduction maps send $\alpha_{m,n}$ to an element $\alpha''_{m,n}$ of $\widetilde{\mathcal{M}}$, and $\beta_{m,n}$ to an element $\beta''_{m,n}$ of $\mathcal{N}|\mathsf{C}$, and the limits $\alpha'_m = \lim_{n\to\infty} \alpha''_{m,n}$ in $\widetilde{\mathcal{M}}$ and $\beta'_m = \lim_{n\to\infty} \beta''_{m,n}$ in $\mathcal{N}|\mathsf{C}$ exist. Therefore, (a) and (b) imply that for each L-formula $\psi(v_1, \dots, v_k)$,

$$\mu^{\widetilde{\mathcal{M}}}(\llbracket \psi(\alpha'_1, \dots, \alpha'_k) \rrbracket) = \lim_{n\to\infty} \mu^{\widetilde{\mathcal{M}}}(\llbracket \psi(\alpha_{1,n}, \dots, \alpha_{k,n}) \rrbracket) \tag{3.1}$$

and

$$\mu^{\mathcal{N}|\mathsf{C}}(\llbracket \psi(\beta'_1, \dots, \beta'_k) \rrbracket) = \lim_{n\to\infty} \mu^{\mathcal{N}}(\llbracket \psi(\beta_{1,n}, \dots, \beta_{k,n}) \rrbracket \sqcap \mathsf{B}_n)/\mu^{\mathcal{N}}(\mathsf{B}_n). \tag{3.2}$$

We next assume that (S') holds for some $\mathcal{M}$ and C, and prove (S"). We may take $\mathcal{M}$ to be a coded structure, and let h be an isomorphism from $\mathcal{N}|\mathsf{C}$ to $\widetilde{\mathcal{M}}$. We may choose mappings α' from $\mathbb{N}$ into $\widetilde{\mathcal{M}}$ and β' from $\mathbb{N}$ into $\mathcal{N}|\mathsf{C}$ such that range(α'), range(β') contain the images of $\mathcal{M}^{\mathcal{A}}$ and $\mathcal{K}$ under the reduction maps, and $\alpha'_n = h(\beta'_n)$ for each $n \in \mathbb{N}$. Then for each L-formula $\psi(v_1, \dots, v_k)$,

$$\mu^{\widetilde{\mathcal{M}}}(\llbracket \psi(\alpha'_1, \dots, \alpha'_k) \rrbracket) = \mu^{\mathcal{N}|\mathsf{C}}(\llbracket \psi(\beta'_1, \dots, \beta'_k) \rrbracket). \tag{3.3}$$

One can choose a sequence $\mathsf{B}\colon \mathbb{N} \to \mathcal{B}$, and double sequences $\alpha\colon \mathbb{N} \times \mathbb{N} \to \mathcal{M}^{\mathcal{A}}$, $\beta\colon \mathbb{N} \times \mathbb{N} \to \mathcal{K}$ such that (c) holds, the reduction of B_n converges to C, and for each $m \in \mathbb{N}$ the reductions of $\alpha_{m,n}$ and $\beta_{m,n}$ converge to α'_m and β'_m respectively. Then conditions (a) and (b) hold, so (3.1) and (3.2) hold for each L-formula $\psi(v_1, \dots, v_k)$, By (3.3), condition (d) holds, and hence (S") holds.

Finally, we assume (S") and prove (S'). Let $\mathsf{C} = \lim_{n\to\infty} \mathsf{B}_n$ in the completion of $\mathcal{B}$. Since (a) and (b) hold, (3.1) and (3.2) hold for each L-formula $\psi(v_1, \dots, v_k)$. Then by (d), (3.3) holds for every ψ. By (c), range$(\alpha') \supseteq \mathcal{M}^{\mathcal{A}}$ and range$(\beta') \supseteq \mathcal{K}$. Therefore range$(\alpha')$ is dense in the $\mathbb{K}$-sort of $\widetilde{\mathcal{M}}$, and range(β') is dense in the $\mathbb{K}$-sort of $\mathcal{N}|\mathsf{C}$. Hence every element of $\widetilde{\mathcal{M}}$ of sort $\mathbb{K}$ is equal to $\lim_{k\to\infty}\alpha'_{m_k}$ for some sequence $(m_0, m_1, \ldots) \in \mathbb{N}^{\mathbb{N}}$, and similarly for $\mathcal{N}|\mathsf{C}$ and β'. Since $d_{\mathbb{K}}(\boldsymbol{a}, \boldsymbol{b}) = \mu([\![\boldsymbol{a} \neq \boldsymbol{b}]\!])$ in any model of P^R, $\lim_{k\to\infty}\alpha'_{m_k}$ exists in $\widetilde{\mathcal{M}}$ if and only if $\lim_{k\to\infty}\beta'_{m_k}$ exists in $\mathcal{N}|\mathsf{C}$. Whenever $\lim_{k\to\infty}\alpha'_{m_k}$ exists in $\widetilde{\mathcal{M}}$, let $h(\lim_{k\to\infty}\alpha'_{m_k}) = \lim_{k\to\infty}\beta'_{m_k}$. Then h maps the $\mathbb{K}$-sort of $\widetilde{\mathcal{M}}$ onto the $\mathbb{K}$-sort of $\mathcal{N}|\mathsf{C}$. Since (3.3) holds and the functions $[\![\psi(\cdot)]\!]$ are uniformly continuous in $\widetilde{\mathcal{M}}$ and $\mathcal{N}|\mathsf{C}$,

$$\mu^{\widetilde{\mathcal{M}}}([\![\psi(\vec{\boldsymbol{a}})]\!]) = \mu^{\mathcal{N}|\mathsf{C}}([\![\psi(h\vec{\boldsymbol{a}})]\!])$$

for each L-formula ψ and tuple $\vec{\boldsymbol{a}}$ of sort $\mathbb{K}$ in $\widetilde{\mathcal{M}}$. Therefore by Fact 2.4, h can be extended to an isomorphism from $\widetilde{\mathcal{M}}$ onto $\mathcal{N}|\mathsf{C}$. This proves (S'). $\square$

By a **transitive model** of a set of sentences Z we mean a transitive set V such that $(V, \in) \models Z$. It is well known that there is a finite subset ZFC_0 of the set of axioms of ZFC such that the Shoenfield absoluteness theorem holds for all transitive models of ZFC_0. Assume hereafter that ZFC_0 is a finite subset of ZFC with that property, and also that ZFC_0 implies every result stated in Section 2, Lemma 3.4 above, and every consequence of ZFC that is used in the proofs of Lemmas 3.5 and 3.6 below.

Lemma 3.5. *Let V, V' be transitive models of ZFC_0 such that the signature L is in V, and $V \subseteq V'$. Suppose that in V it is true that φ is an $L_{\omega_1\omega}$-sentence and $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ is a countable randomization. Then in V' it is also true that φ is an $L_{\omega_1\omega}$-sentence and $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ is a countable randomization, and $\mu^{\mathcal{N}}([\![\varphi]\!])$ has the same value in V as in V'. Hence*

$$V \models \mathcal{N} \textit{ is a countable randomization of } \varphi$$

if and only if

$$V' \models \mathcal{N} \textit{ is a countable randomization of } \varphi.$$

Proof. It is easily proved using induction on the complexity of formulas that

$$V' \models \varphi \text{ is a an } L_{\omega_1\omega}\text{-sentence.}$$

Since the set of axioms of P^R is recursive in L, the property of being a countable randomization is Σ_1, and hence

$$V' \models \mathcal{N} \text{ is a countable randomization.}$$

Let $\mathcal{P}$ be the completion of $\mathcal{N}$ in V, and $\mathcal{Q}$ be the completion of $\mathcal{N}$ in V'. In V', $\mathcal{P}$ is a separable randomization that is not necessarily complete, and $\mathcal{Q}$ is the completion of $\mathcal{N}$ and also the completion of $\mathcal{P}$. For each $L_{\omega_1\omega}$-formula $\psi(\cdot)$ in V, let $[\![\psi(\cdot)]\!]^{\mathcal{P}}$ be the function obtained by applying Fact 2.7 to $\mathcal{P}$ in V, and let $[\![\psi(\cdot)]\!]^{\mathcal{Q}}$ be the function obtained by applying Fact 2.7 to $\mathcal{Q}$ in V'. Using Conditions (i)–(vi) of Fact 2.7, we show by induction on complexity that for every $L_{\omega_1\omega}$-formula $\psi(\cdot)$ in V and tuple $\vec{\boldsymbol{f}}$ in the reduction of $\mathcal{K}$, $[\![\psi(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} = [\![\psi(\vec{\boldsymbol{f}})]\!]^{\mathcal{Q}}$. The base step for first order formulas and the steps for negation and finite disjunction are easy.

Countable disjunction step: Let $\psi = \bigvee_k \psi_k$, and suppose $\vec{\boldsymbol{f}}$ is in the reduction of $\mathcal{K}$ and that $[\![\psi_k(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} = [\![\psi_k(\vec{\boldsymbol{f}})]\!]^{\mathcal{Q}}$ holds for each $k \in \mathbb{N}$. Let $\psi'_k = \bigvee_{n\le k} \psi_n$. Then $[\![\psi'_k(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} = [\![\psi'_k(\vec{\boldsymbol{f}})]\!]^{\mathcal{Q}}$ for each $k \in \mathbb{N}$, and

$$[\![\psi(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} = \lim_{k\to\infty} [\![\psi'_k(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} = \lim_{k\to\infty} [\![\psi'_k(\vec{\boldsymbol{f}})]\!]^{\mathcal{Q}} = [\![\psi(\vec{\boldsymbol{f}})]\!]^{\mathcal{Q}}.$$

Existential quantifier step: Let $\psi(\vec{u}) = (\exists v)\theta(\vec{u}, v)$ and suppose that $[\![\theta(\vec{\boldsymbol{f}}, \boldsymbol{g})]\!]^{\mathcal{P}} = [\![\theta(\vec{\boldsymbol{f}}, \boldsymbol{g})]\!]^{\mathcal{Q}}$ for all $\vec{\boldsymbol{f}}, \boldsymbol{g}$ in the reduction of $\mathcal{K}$. Since the reduction of $\mathcal{K}$ is dense in the sort $\mathbb{K}$ parts of both $\mathcal{P}$ and $\mathcal{Q}$, and the functions $[\![\theta(\cdot)]\!]^{\mathcal{P}}$ and $[\![\theta(\cdot)]\!]^{\mathcal{Q}}$ are both Lipschitz continuous with bound 1 by Fact 2.7, it follows that $[\![\psi(\vec{\boldsymbol{f}})]\!]^{\mathcal{P}} = [\![\psi(\vec{\boldsymbol{f}})]\!]^{\mathcal{Q}}$. This completes the induction.

Every event in $\mathcal{P}$ has the same measure in V as in V'. In particular, for the sentence φ, the measure of $[\![\varphi]\!]^{\mathcal{P}}$ is the same in V as in V'. We have

$$V \models \mu^{\mathcal{N}}([\![\varphi]\!]) = \mu([\![\varphi]\!]^{\mathcal{P}})$$

and

$$V' \models \mu^{\mathcal{N}}([\![\varphi]\!]) = \mu([\![\varphi]\!]^{\mathcal{Q}}) = \mu([\![\varphi]\!]^{\mathcal{P}}).$$

Therefore $\mu^{\mathcal{N}}([\![\varphi]\!])$ has the same value in V as in V'. □

Lemma 3.5 can also be proved by using the continuous analogue of the infinitary logic $L_{\omega_1\omega}$. Lemma 5.18 in the paper [EV] shows that for any metric structure $\mathcal{P}$ and continuous infinitary sentence Θ in V, the value of Θ in $\mathcal{P}$ computed in V is the same as the value computed in V'. Using Fact 2.7, one can find a continuous infinitary sentence Θ that has the same value as $\mu([\![\theta]\!]^{\mathcal{P}})$ in any complete separable randomization $\mathcal{P}$, and then use Lemma 5.18 in [EV] to get Lemma 3.5.

Lemma 3.6. *In any countable transitive model V of ZFC$_0$, it is true that every scattered sentence has few separable randomizations.*

Proof. By the result of Solovay and Tennenbaum, there is a countable transitive model V' of ZFC$_0$ with the same ordinals as V such that $V \subseteq V'$ and Martin's Axiom for $\aleph_1$ holds in V'. Suppose that in V it is true that φ is a scattered sentence, $\mathcal{N}$ is a countable randomization with a coding, and $\mu^{\mathcal{N}}([\![\varphi]\!]) = 1$.

We now work in V', and prove the statement (S) of Lemma 3.4. The property of being a scattered sentence is Π^1_2, so by the Shoenfield absoluteness theorem, φ is still a scattered sentence. By Lemma 3.5, $\mathcal{N}$ is still a countable randomization with $\mu^{\mathcal{N}}([\![\varphi]\!]) = 1$. So the completion of $\mathcal{N}$ is a complete separable randomization of φ. By Fact 2.13 and Martin's axiom, φ has few separable randomizations. By Fact 2.11, there exists a Scott sentence δ such that $\mu^{\mathcal{N}}([\![\delta]\!]) > 0$, so (S) holds.

By Lemma 3.4 and the Shoenfield absoluteness theorem (or even the weaker Mostowski absoluteness theorem), (S) also holds in V. So by Fact 2.11, it is true in V that φ has few separable randomizations. □

Proof. (Proof of Theorem 3.1) The following argument is well-known, and is included for completeness. Let η be the sentence in the vocabulary of ZFC that says that every scattered sentence has few separable randomizations. Assume $\neg\eta$. By the reflection theorem, $\mathrm{ZFC}_0 \cup \{\neg\eta\}$ has a transitive model. By the downward Löwenheim-Skolem theorem and the Mostowski collapsing lemma, $\mathrm{ZFC}_0 \cup \{\neg\eta\}$ has a countable transitive model. This contradicts Lemma 3.6, so η holds. □

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