# The Strength of Nonstandard Analysis

H. Jerome Keisler
University of Wisconsin, Madison

**Abstract**

A weak theory nonstandard analysis, with types at all finite levels over both the integers and hyperintegers, is developed as a possible framework for reverse mathematics. In this weak theory, we investigate the strength of standard part principles and saturation principles which are often used in practice along with first order reasoning about the hyperintegers to obtain second order conclusions about the integers.

## 1 Introduction

In this paper we revisit the work in [HKK] and [HK], where the strength of nonstandard analysis is studied. In those papers it was shown that weak fragments of set theory become stronger when one adds saturation principles commonly used in nonstandard analysis.

The purpose of this paper is to develop a framework for reverse mathematics in nonstandard analysis. We will introduce a base theory, "weak nonstandard analysis" ($WNA$), which is proof theoretically weak but has types at all finite levels over both the integers and the hyperintegers. In $WNA$ we study the strength of two principles that are prominent in nonstandard analysis, the standard part principle in Section 6, and the saturation principle in Section 9. These principles are often used in practice along with first order reasoning about the hyperintegers to obtain second order conclusions about the integers, and for this reason they can lead to the discovery of new results.

The standard part principle ($STP$) says that a function on the integers exists if and only if it is coded by a hyperinteger. Our main results show that in $WNA$, $STP$ implies the axiom of choice for quantifier-free formulas (Theorem 8.4), $STP+$ saturation for quantifier-free formulas implies choice for arithmetical formulas (Theorem 10.1), and $STP+$ saturation for formulas with first order quantifiers implies choice for formulas with second order quantifiers (Theorem 10.3). The last result might be used to identify theorems that are proved using nonstandard analysis but cannot be proved by the methods commonly used in classical mathematics.

The natural models of $WNA$ will have a superstructure over the standard integers $\mathbb{N}$, a superstructure over the hyperintegers $^*\mathbb{N}$, and an inclusion map $j : \mathbb{N} \to {}^*\mathbb{N}$. With the two superstructures, it makes sense to ask whether a higher order statement over the hyperintegers implies a higher order statement over the integers. As is commonly done in the standard literature on weak theories in higher types, we use functional superstructures with types of functions rather than sets. The base theory $WNA$ is neutral between the internal set theory approach and the superstructure approach to nonstandard analysis, and the standard part and saturation principles considered here arise in both approaches. For background in model theory, see [CK], Section 4.4.

The theory $WNA$ is related to the weak nonstandard theory $NPRA^\omega$ of Avigad [A], and the base theory $RCA_0^\omega$ for higher order reverse mathematics proposed by Kohlenbach [K]. The paper

[A] shows that the theory $NPRA^\omega$ is weak in the sense that it is conservative over primitive recursive arithmetic ($PRA$) for $\Pi_2$ sentences, but is still sufficient for the development of much of analysis. The theory $WNA$ is also conservative over $PRA$ for $\Pi_2$ sentences, but has more expressive power. In Sections 11 and 12 we will introduce a stronger, second order Standard Part Principle, and give some relationships between this principle and the theories $NPRA^\omega$ and $RCA_0^\omega$.

## 2 The theory $PRA^\omega$

Our starting point is the theory $PRA$ of primitive recursive arithmetic, introduced by Skolem. It is a first order theory which has function symbols for each primitive recursive function (in finitely many variables), and the equality relation $=$. The axioms are the rules defining each primitive recursive function, and induction for quantifier-free formulas. This theory is much weaker than Peano arithmetic, which has induction for all first order formulas.

An extension of $PRA$ with all finite types was introduced by Gödel [G], and several variations of this extension have been studied in the literature. Here we use the finite type theory $PRA^\omega$ as defined in Avigad [A].

There is a rich literature on constructive theories in intuitionistic logic that are very similar to $PRA^\omega$, such as the finite type theory $HA^\omega$ over Heyting arithmetic (See, for example, [TD]). However, in this paper we work exclusively in classical logic.

We first introduce a formal object $N$ and define a collection of formal objects called **types over** $N$.

(1) The **base type** over $N$ is $N$.

(2) If $\sigma, \tau$ are types over $N$, then $\sigma \to \tau$ is a type over $N$.

We now build the formal language $L(PRA^\omega)$. $L(PRA^\omega)$ is a many-sorted first order language with countably many variables of each type $\sigma$ over $N$, and the equality symbol $=$ at the base type $N$ only. It has the usual rules of many-sorted logic, including the rule $\exists f \forall u\, f(u) = t(u, \ldots)$ where $u, f$ are variables of type $\sigma, \sigma \to N$ and $t(u, \ldots)$ is a term of type $N$ in which $f$ does not occur.

We first describe the symbols and then the corresponding axioms. $L(PRA^\omega)$ has the following function symbols:

- A function symbol for each primitive recursive function.
- The primitive recursion operator which builds a term $R(m, f, n)$ of type $N$ from terms of type $N, N \to N$, and $N$.
- The definition by cases operator which builds a term $c(n, u, v)$ of type $\sigma$ from terms of type $N, \sigma$, and $\sigma$.
- The $\lambda$ operator which builds a term $\lambda v.t$ of type $\sigma \to \tau$ from a variable $v$ of type $\sigma$ and a term $t$ of type $\tau$.
- The application operator which builds a term $t(s)$ of type $\tau$ from terms $s$ of type $\sigma$ and $t$ of type $\sigma \to \tau$.

Given terms $r, t$ and a variable $v$ of the appropriate types, $r(t/v)$ denotes the result of substituting $t$ for $v$ in $r$. Given two terms $s, t$ of type $\sigma$, $s \equiv t$ will denote the infinite scheme of formulas $r(s/v) = r(t/v)$ where $v$ is a variable of type $\sigma$ and $r(v)$ is an arbitrary term of type $N$. $\equiv$ is a substitute for the missing equality relations at higher types.

The axioms for $PRA^\omega$ are as follows.

- Each axiom of $PRA$,
- The induction scheme for quantifier-free formulas of $L(PRA^\omega)$,
- Primitive recursion: $R(m, f, 0) = m$, $R(m, f, s(n)) = f(n, R(m, f, n))$,
- Cases: $c(0, u, v) \equiv u$, $c(s(m), u, v) \equiv v$,
- Lambda abstraction: $(\lambda u.t)(s) \equiv t(s/u)$.

The order relations for type $N$ can be defined in the usual way by quantifier-free formulas.

In [A] additional types $\sigma \times \tau$, and term-building operations for pairing and projections with corresponding axioms were also included in the language, but as explained in [A], these symbols are redundant and are often omitted in the literature.

On the other hand, in [A] the symbols for primitive recursive functions are not included in the language. These symbols are redundant because they can be defined from the primitive recursive operator $R$, but they are included here for convenience.

We state a conservative extension result from [A], which shows that $PRA^\omega$ is very weak.

**Proposition 2.1** *$PRA^\omega$ is a conservative extension of $PRA$, that is, $PRA^\omega$ and $PRA$ have the same consequences in $L(PRA)$.*

The natural model of $PRA^\omega$ is the full functional superstructure $V(\mathbb{N})$, which is defined as follows. $\mathbb{N}$ is the set of natural numbers. Define $V_N(\mathbb{N}) = \mathbb{N}$, and inductively define $V_{\sigma\to\tau}(\mathbb{N})$ to be the set of all mappings from $V_\sigma(\mathbb{N})$ into $V_\tau(\mathbb{N})$. Finally, $V(\mathbb{N}) = \bigcup_\sigma V_\sigma(\mathbb{N})$. The superstructure $V(\mathbb{N})$ becomes a model of $PRA^\omega$ when each of the symbols of $L(PRA^\omega)$ is interpreted in the obvious way indicated by the axioms. In fact, $V(\mathbb{N})$ is a model of much stronger theories than $PRA^\omega$, since it satisfies full induction and higher order choice and comprehension principles.

# 3 The Theory $NPRA^\omega$

In [A], Avigad introduced a weak nonstandard counterpart of $PRA^\omega$, called $NPRA^\omega$. $NPRA^\omega$ adds to $PRA^\omega$ a new predicate symbol $S(\cdot)$ for the standard integers (and $S$-relativized quantifiers $\forall^S, \exists^S$), and a constant $H$ for an infinite integer, axioms saying that $S(\cdot)$ is an initial segment not containing $H$ and is closed under each primitive recursive function, and a transfer axiom scheme for universal formulas. In the following sections we will use a weakening of $NPRA^\omega$ as a part of our base theory.

In order to make $NPRA^\omega$ fit better with the present paper, we will build the formal language $L(NPRA^\omega)$ with types over a new formal object ${}^*N$ instead of over $N$. The base type over ${}^*N$ is ${}^*N$, and if $\sigma, \tau$ are types over ${}^*N$ then $\sigma \to \tau$ is a type over ${}^*N$.

For each type $\sigma$ over $N$, let ${}^*\sigma$ be the type over ${}^*N$ built in the same way. For each function symbol $u$ in $L(PRA^\omega)$ from types $\vec{\sigma}$ to type $\tau$, $L(NPRA^\omega)$ has a corresponding function symbol ${}^*u$ from types ${}^*\vec{\sigma}$ to type ${}^*\tau$. $L(NPRA^\omega)$ also has the equality relation $=$ for the base type ${}^*N$, and the extra constant symbol $H$ and the standardness predicate symbol $S$ of type ${}^*N$.

We will use the following conventions throughout this paper. When we write a formula $A(\vec{v})$, it is understood that $\vec{v}$ is a tuple of variables that contains all the free variables of $A$. If we want to allow additional free variables we write $A(\vec{v}, \ldots)$. We will always let:

- $m, n, \ldots$ be variables of type $N$,

- $x, y, \ldots$ be variables of type ${}^*N$,
- $f, g, \ldots$ be variables of type $N \to N$.

To describe the axioms of $NPRA^\omega$ we introduce the star of a formula of $L(PRA^\omega)$. Given a formula $A$ of $L(PRA^\omega)$, a **star of** $A$ is a formula ${}^*A$ of $L(NPRA^\omega)$ which is obtained from $A$ by replacing each variable of type $\sigma$ in $A$ by a variable of type ${}^*\sigma$ in a one to one fashion, and replacing each function symbol in $A$ by its star. The order relations on ${}^*N$ will be written $<, \leq$ without stars.

The axioms of $NPRA^\omega$ are as follows:

- The star of each axiom of $PRA^\omega$,
- $S$ is an initial segment: $\neg S(H) \wedge \forall x \forall y [S(x) \wedge y \leq x \to S(y)]$,
- $S$ is closed under primitive recursion,
- Transfer: $\forall^S \vec{x}\, {}^*A(\vec{x}) \to \forall \vec{x}\, {}^*A(\vec{x})$, $A(\vec{m})$ quantifier-free in $L(PRA^\omega)$.

It is shown in [A] that if $A(m, n)$ is quantifier-free in $L(PRA)$ and $NPRA^\omega$ proves $\forall^S x \exists y\, {}^*A(x, y)$, then $PRA$ proves $\forall m \exists n\, A(m, n)$. It follows that $NPRA^\omega$ is conservative over $PRA$ for $\Pi_2$ sentences.

The natural models of $NPRA^\omega$ are the internal structures ${}^*V(\mathbb{N})$, which are proper elementary extensions of $V(\mathbb{N})$ in the many-sorted sense, with additional symbols $S$ for $\mathbb{N}$ and $H$ for an element of ${}^*\mathbb{N} \setminus \mathbb{N}$.

## 4 The theory $WNA$

We now introduce our base theory $WNA$, weak nonstandard analysis. The idea is to combine the theory $PRA^\omega$ with types over $N$ with a weakening of the theory $NPRA^\omega$ with types over ${}^*N$, and form a link between the two by identifying the standardness predicate $S$ of $NPRA^\omega$ with the lowest type $N$ of $PRA^\omega$. In this setting, it will make sense to ask whether a formula with types over ${}^*N$ implies a formula with types over $N$.

The language $L(WNA)$ of $WNA$ has both types over $N$ and types over ${}^*N$. It has all of the symbols of $L(PRA^\omega)$, all the symbols of $L(NPRA^\omega)$ except the primitive recursion operator ${}^*R$, and has one more function symbol $j$ which goes from type $N$ to type ${}^*N$.

We make the axioms of $WNA$ as weak as we can so as to serve as a blank screen for viewing the relative strengths of additional statements which arise in nonstandard analysis.

The axioms of $WNA$ are as follows:

- The axioms of $PRA^\omega$,
- The star of each axiom of $PRA$,
- The stars of the Cases and Lambda abstraction axioms of $PRA^\omega$,
- $S$ is an initial segment: $\neg S(H) \wedge \forall x \forall y [S(x) \wedge y \leq x \to S(y)]$,
- $S$ is closed under primitive recursion,

- $j$ maps $S$ onto $\mathbb{N}$: $\forall x[S(x) \leftrightarrow \exists m\, x = j(m)]$,
- Lifting: $j(\alpha(\vec{m})) = {}^*\alpha(j(\vec{m}))$ for each primitive recursive function $\alpha$.

The star of a quantifier-free formula of $L(PRA)$, possibly with some variables replaced by $H$, will be called an **internal quantifier-free formula**. The stars of the axioms of $PRA$ include the star of the defining rule for each primitive recursive function, and the induction scheme for internal quantifier-free formulas (which we will call **internal induction**).

The axioms of $NPRA^\omega$ that are left out of $WNA$ are the star of the Primitive Recursion scheme, the star of the quantifier-free induction scheme of $PRA^\omega$, and Transfer. These axioms are statements about the hyperintegers which involve terms of higher type.

Note that $WNA$ is noncommittal on whether the characteristic function of $S$ exists in type ${}^*N \to {}^*N$, while the quantifier-free induction scheme of $NPRA^\omega$ precludes this possibility.

In practice, nonstandard analysis uses very strong transfer axioms, and extends the mapping $j$ to higher types. Strong axioms of this type will not be considered here.

**Theorem 4.1** *$WNA + NPRA^\omega$ is a conservative extension of $NPRA^\omega$, that is, $NPRA^\omega$ and $WNA + NPRA^\omega$ have the same consequences in $L(NPRA^\omega)$.*

**Proof.** Let $M$ be a model of $NPRA^\omega$, and let $M^S$ be the restriction of $M$ to the standardness predicate $S$. Then $M^S$ is a model of $PRA$. By Proposition 2.1, the complete theory of $M^S$ is consistent with $PRA^\omega$. Therefore $PRA^\omega$ has a model $K$ whose restriction $K^N$ to type $N$ is elementarily equivalent to $M^S$. By the compactness theorem for many-sorted logic, there is a model $M_1$ elementarily equivalent to $M$ and a model $K_1$ elementarily equivalent to $K$ with an isomorphism $j : M_1^S \cong K_1^N$. such that $\langle K_1, M_1, j\rangle$ is a model of $WNA + NPRA^\omega$. Thus every complete extension of $NPRA^\omega$ is consistent with $WNA + NPRA^\omega$, and the theorem follows. ■

**Corollary 4.2** *$WNA$ is a conservative extension of $PRA$ for $\Pi_2$ formulas. That is, if $A(m,n)$ is quantifier-free in $L(PRA)$ and $WNA \vdash \forall m \exists n\, A(m,n)$, then $PRA \vdash \forall m \exists n\, A(m,n)$.*

**Proof.** Suppose $WNA \vdash \forall m \exists n\, A(m,n)$. By the Lifting Axiom, $WNA \vdash \forall^S x \exists^S y\, {}^*A(x,y)$. By Theorem 4.1, $NPRA^\omega \vdash \forall^S x \exists^S y\, {}^*A(x,y)$. Then $PRA \vdash \forall m \exists n\, A(m,n)$ by Corollary 2.3 in [A]. ■

Each model of $WNA$ has a $V(\mathbb{N})$ part formed by restricting to the objects with types over $N$, and a $V({}^*\mathbb{N})$ part formed by restricting to the objects with types over ${}^*N$. Intuitively, the $V(\mathbb{N})$ and $V({}^*\mathbb{N})$ parts of $WNA$ are completely independent of each other, except for the inclusion map $j$ at the zeroth level. The standard part principles introduced later in this paper will provide links between types $N \to N$ and $(N \to N) \to N$ in the $V(\mathbb{N})$ part and types ${}^*N$ and ${}^*N \to {}^*N$ in the $V({}^*\mathbb{N})$ part.

$WNA$ has two natural models, the "internal model" $\langle V(\mathbb{N}), {}^*V(\mathbb{N}), j\rangle$ which contains the natural model ${}^*V(\mathbb{N})$ of $NPRA^\omega$, and the "full model" $\langle V(\mathbb{N}), V({}^*\mathbb{N}), j\rangle$ which contains the full superstructure $V({}^*\mathbb{N})$ over ${}^*\mathbb{N}$. In both models, $j$ is the inclusion map from $\mathbb{N}$ into ${}^*\mathbb{N}$. The full natural model $\langle V(\mathbb{N}), V({}^*\mathbb{N}), j\rangle$ of $WNA$ does not satisfy the axioms $NPRA^\omega$. In particular, the star of quantifier-free induction fails in this model, because the characteristic function of $S$ exists as an object of type ${}^*N \to {}^*N$.

# 5 Bounded Minima and Overspill

In this section we prove some useful consequences of the $WNA$ axioms.

Given a formula $A(x,\ldots)$ of $L(WNA)$, the **bounded minimum** operator is defined by

$$u = (\mu x < y)\, A(x,\ldots) \leftrightarrow [u \le y \wedge (\forall x < u)\neg A(x,\ldots) \wedge [A(u,\ldots) \vee u = y]],$$

where $u$ is a new variable. By this we mean that the expression to the left of the $\leftrightarrow$ symbol is an abbreviation for the formula to the right of the $\leftrightarrow$ symbol. In particular, if $z$ does not occur in $A$, $(\mu z < 1)\, A(\ldots)$ is the (inverted) characteristic function of $A$, which has the value 0 when $A$ is true and the value 1 when $A$ is false.

In $PRA$, the bounded minimum operator is defined similarly.

**Lemma 5.1** *Let $A(m,\vec{n})$ be a quantifier-free formula of $L(PRA)$ and let $\alpha(p,\vec{n})$ be the primitive recursive function such that in $PRA$, $\alpha(p,\vec{n}) = (\mu m < p)\, A(m,\vec{n})$. Then*
*(i) $WNA \vdash {}^*\alpha(y,\vec{z}) = (\mu x < y)\, {}^*A(x,\vec{z})$.*
*(ii) In $WNA$, there is a quantifier-free formula $B(p,\ldots)$ such that*

$$(\forall m < p)\, A(m,\ldots) \leftrightarrow B(p,\ldots), \qquad (\forall x < y)\, {}^*A(x,\ldots) \leftrightarrow {}^*B(y,\ldots).$$

*Similarly for $(\exists x < y)\, {}^*A(x,\ldots)$, and $u = (\mu x < y)\, {}^*A(x,\ldots)$.*

**Proof.** (i) By the axioms of $WNA$, the defining rule for ${}^*\alpha$ is the star of the defining rule for $\alpha$.

(ii) Apply (i) and observe that in $WNA$,

$$(\forall x < y)\, {}^*A(x,\ldots) \leftrightarrow y = (\mu x < y)\, \neg{}^*A(x,\ldots).$$

■

Let us write $\forall^\infty x\, A(x,\ldots)$ for $\forall x[\neg S(x) \to A(x,\ldots)]$ and $\exists^\infty x\, A(x,\ldots)$ for $\exists x\, [\neg S(x) \wedge A(x,\ldots)]$.

**Lemma 5.2** *(Overspill) Let $A(x,\ldots)$ be an internal quantifier-free formula. In $WNA$, $\forall^S x A(x,\ldots) \to \exists^\infty x\, A(x,\ldots)$, and $\forall^\infty x\, A(x,\ldots) \to \exists^S x A(x,\ldots)$.*

**Proof.** Work in $WNA$. If $A(H,\ldots)$ we may take $x = H$. Assume $\forall^S x A(x,\ldots)$ and $\neg A(H,\ldots)$. By Lemma 5.1 (ii) we may take $u = (\mu x < H)\, \neg A(x,\ldots)$. Then $\neg S(u)$. Let $x = u - 1$. We have $x < u$, so $A(x,\ldots)$. Since $S$ is closed under the successor function, $\neg S(x)$. ■

We now give a consequence of $WNA$ in the language of $PRA$ which is similar to Proposition 4.3 in [A] for $NPRA^\omega$. **$\Sigma_1$-collection** in $L(PRA)$ is the scheme

$$(\forall m < p)\exists n\, B(m,n,\vec{r}) \to \exists k(\forall m < p)(\exists n < k)B(m,n,\vec{r})$$

where $B$ is a formula of $L(PRA)$ of the form $\exists q\, C$, $C$ quantifier-free.

**Proposition 5.3** *$\Sigma_1$-collection in $L(PRA)$ is provable in $WNA$.*

**Proof.** We work in $WNA$. By pairing existential quantifiers, we may assume that $B(m,n,\vec{r})$ is quantifier-free. Assume $(\forall m < p)\exists n\, B(m,n,\vec{r})$. Let ${}^*B$ be the formula obtained by starring each function symbol in $B$ and replacing variables of type $N$ by variables of type ${}^*N$.

By the Lifting Axiom and the axiom that $S$ is an initial segment,

$$(\forall x < p)\exists^S y\, {}^*B(x,y,j(\vec{r})).$$

Then

$$\forall^\infty w(\forall x < p)(\exists y < w)\, {}^*B(x, y, j(\vec{r})).$$

By Lemma 5.1 and Overspill,

$$\exists^S w(\forall x < p)(\exists y < w)\, {}^*B(x, y, j(\vec{r})).$$

By the Lifting Axiom again,

$$\exists k(\forall m < p)(\exists n < k)B(m, n, \vec{r}).$$

■

# 6 Standard Parts

This section introduces a standard part notion which formalizes a construction commonly used in nonstandard analysis, and provides a link between the type $N \to N$ and the type ${}^*N$.

In type $N$ let $(n)_k$ be the power of the $k$-th prime in $n$, and in type ${}^*N$ let $(x)_y$ be the power of the $y$-th prime in $x$. The function $(n, k) \mapsto (n)_k$ is primitive recursive, and its star is the function $(x, y) \mapsto (x)_y$.

*Hereafter, when it is clear from the context, we will write $t$ instead of $j(t)$ in formulas of $L(WNA)$.*

Intuitively, we identify $j(t)$ with $t$, but officially, they are different because $t$ has type $N$ while $j(t)$ has type ${}^*N$. This will make formulas easier to read. When a term $t$ of type $N$ appears in a place of type ${}^*N$, it really is $j(t)$.

In the theory $WNA$, we say that $x$ is **near-standard**, in symbols $ns(x)$, if $\forall^S z\, S((x)_z)$. Note that this is equivalent to $\forall n S((x)_n)$. We employ the usual convention for relativized quantifiers, so that $\forall^{ns} x\, B$ means $\forall x[ns(x) \to B]$ and $\exists^{ns} x\, B$ means $\exists x[ns(x) \wedge B]$. We write

$$x \approx y \text{ if } ns(x) \wedge \forall^S z\, (x)_z = (y)_z.$$

This is equivalent to $ns(x) \wedge \forall n\, (x)_n = (y)_n$. We write $f = {}^o x$, and say $f$ is the **standard part of** $x$ and $x$ is a **lifting of** $f$, if

$$ns(x) \wedge \forall n f(n) = (x)_n.$$

Note that the operation $x \mapsto {}^o x$ goes from type ${}^*N$ to type $N \to N$. In nonstandard analysis, this often allows one to obtain results about functions of type $N \to N$ by reasoning about hyperintegers of type ${}^*N$.

**Lemma 6.1** *In $WNA$, suppose that $x$ is near-standard. Then*
*(i) If $x \approx y$ then $ns(y)$ and $y \approx x$.*
*(ii) $(\exists y < H)\, x \approx y$.*

**Proof.** (i) Suppose $x \approx y$. If $S(z)$ then $S((x)_z)$ and $(y)_z = (x)_z$, so $S((y)_z)$. Therefore $ns(y)$, and $y \approx x$ follows trivially.

(ii) Let $\beta$ be the primitive recursive function $\beta(m, n) = \Pi_{i<m}\, p_i^{(n)_i}$. By Lifting and defining rules for $\beta$ and ${}^*\beta$, $\forall x \forall u \forall z[z < u \to (x)_z = ({}^*\beta(u, x))_z]$. Therefore $\forall^\infty u\, \forall^S z\, (x)_z = ({}^*\beta(u, x))_z$, and hence $\forall^\infty u\, x \approx {}^*\beta(u, x)$. We have $\forall^S w\, w^w < H$, and by Overspill, there exists $w$ with $\neg S(w) \wedge w^w < H$. Since $x$ is near-standard, $\forall^S u[u \leq w \wedge (\forall z < u)\, p_z^{(x)_z} < w]$. By Overspill,

$$\exists^\infty u[u \leq w \wedge (\forall z < u)\, p_z^{(x)_z} < w].$$

Let $y = {}^*\beta(u, x)$. Then $x \approx y$. By internal induction,

$$\forall u[(\forall z < u)\, p_z^{(x)_z} < w \rightarrow {}^*\beta(u, x) < w^u].$$

Then $y \leq w^u \leq w^w < H$. ■

We now state the Standard Part Principle, which says that every near-standard $x$ has a standard part and every $f$ has a lifting.

**Standard Part Principle ($STP$):**

$$\forall^{ns} x \exists f\, f = {}^o x \wedge \forall f \exists x\, f = {}^o x.$$

The following corollary is an easy consequence of Lemma 6.1.

**Corollary 6.2** *In $WNA$, $STP$ is equivalent to*

$$(\forall^{ns} x < H)\exists f\, f = {}^o x \wedge \forall f (\exists x < H)\, f = {}^o x.$$

The **Weak Koenig Lemma** is the statement that every infinite binary tree has an infinite branch. The work in reverse mathematics shows that many classical mathematical statements are equivalent to the Weak Koenig Lemma.

**Theorem 6.3** *The Weak Koenig Lemma is provable in $WNA + STP$.*

**Proof.** Work in $WNA + STP$. Let $B(n)$ be the formula

$$(\forall m < n)[(n)_m < 3 \wedge (\forall k < m)[(n)_k = 0 \rightarrow (n)_m = 0]].$$

$B(n)$ says that $n$ codes a finite sequence of 1's and 2's. Write $m \lhd n$ if

$$B(m) \wedge B(n) \wedge m < n \wedge (\forall k < m)[(m)_k > 0 \rightarrow (m)_k = (n)_k].$$

This says the sequence coded by $m$ is an initial segment of the sequence coded by $n$. Suppose that $\{n : f(n) = 0\}$ codes an infinite binary tree $T$, that is,

$$\forall m \exists n[m < n \wedge f(n) = 0] \wedge \forall n[f(n) = 0 \rightarrow B(n) \wedge \forall m[m \lhd n \rightarrow f(m) = 0]].$$

The formulas $B(n)$ and $m \lhd n$ are $PRA$-equivalent to quantifier-free formulas, which have stars ${}^*B(y)$ and $z \,{}^*\!\lhd\, y$. By $STP$, $f$ has a lifting $x$. By Lemma 5.1 and Overspill,

$$\exists^\infty y[{}^*B(y) \wedge (\forall z < y)[z \,{}^*\!\lhd\, y \rightarrow (x)_z = 0]].$$

Then $ns(y)$, and by the $STP$ there exists $g = {}^o y$. It follows that $g$ codes an infinite branch of $T$. ■

The next proposition gives a necessary and sufficient condition for $STP$ in $WNA + NPRA^\omega$. Let $\phi$ be a variable of type ${}^*N \rightarrow {}^*N$, and write $f \subset \phi$ for $\forall n f(n) = \phi(n)$.

**Proposition 6.4** *In $WNA + NPRA^\omega$, $STP$ is equivalent to*

$$\forall f \exists \phi f \subset \phi \wedge \forall \phi \exists f[\forall^S x\, S(\phi(x)) \rightarrow f \subset \phi].$$

**Proof.** Work in $WNA + NPRA^\omega$. Call the displayed sentence $STP'$.

Assume $STP$. Take any $f$. By $STP$, $f$ has a lifting $u$. Since $(u,y) \mapsto (u)_y$ is primitive recursive, $\exists\phi\forall y\phi(y) = (u)_y$. Then $\forall n\, f(n) = (u)_n = \phi(n)$, so $f \subset \phi$.

Now take any $\phi$ and assume that $\forall^S x\, S(\phi(x))$. Using the star of the primitive recursion scheme in $NPRA^\omega$, there exists $\psi$ such that $\forall x(\forall y < x)\,\phi(y) = (\psi(x))_y$. Let $u = \psi(H)$. We then have $(\forall y < H)\,\phi(y) = (u)_y$, so $\forall^S y\,\phi(y) = (u)_y$. It follows that $u$ is near-standard, and by $STP$ there exists $f$ with $f = {}^o u$ and hence $f \subset \phi$.

Now assume $STP'$. Take any $f$. By $STP'$ there exist $\phi$ with $f \subset \phi$. As before there exists $\psi$ such that $\forall x(\forall y < x)\,\phi(y) = (\psi(x))_y$. Let $u = \psi(H)$. Then $\forall n\,(u)_n = \phi(n) = f(n)$, so $u$ is a lifting of $f$.

Now let $u$ be near-standard. Since $(u,y) \mapsto (u)_y$ is primitive recursive, $\exists\phi\forall y\,\phi(y) = (u)_y$. Then $\forall^S x\, S(\phi(x))$, so by $STP'$ there exists $f$ with $f \subset \phi$. Then $\forall n\, f(n) = (u)_n = \phi(n)$, so $f = {}^o u$. ■

# 7 Liftings of Formulas

In this section we will define some hierarchies of formulas with variables of type $N$ and $N \to N$, and corresponding hierarchies of formulas with variables of type ${}^*N$. We will then define the lifting of a formula and show that liftings preserve the hierarchy levels and truth values of formulas.

In the following we restrict ourselves to formulas of $L(PRA^\omega)$ with variables of types $N$ and $N \to N$. We now introduce a restricted class of terms, the basic terms, which behave well with respect to liftings.

By a **basic term over** $N$ we mean a term of the form $\alpha(u_1, \dots, u_k)$ where $\alpha$ is a primitive recursive function of $k$ variables and each $u_i$ is either a variable $n$ of type $N$ or an expression of the form $f(n)$. These basic terms capture all primitive recursive functionals $\beta(\vec{m}, \vec{f})$ in sense that there is a basic term $t(\vec{m}, \vec{f}, n)$ over $N$ which gives the $n$th value in the computation of $\beta(\vec{m}, \vec{f})$ for each input $\vec{m}, \vec{f}, n$.

Let $QF$ be the set of Boolean combinations of equations between basic terms over $N$.

In $A \in QF$, then $(\forall m < n)\,A$, $(\exists m < n)\,A$, and $u = (\mu x < y)\,A$ are $PRA^\omega$-equivalent to formulas in $QF$.

The set $\Pi^1_0 = \Sigma^1_0$ of **arithmetical formulas** is the set of all formulas which are built from formulas in $QF$ using first order quantifiers $\forall m$, $\exists m$ and propositional connectives.

For each natural number $k$, $\Pi^1_{k+1}$ is the set of formulas of the form $\forall f A$ where $A \in \Sigma^1_k$, and $\Sigma^1_{k+1}$ is the set of formulas of the form $\exists f A$ where $A \in \Pi^1_k$.

We observe that up to $PRA^\omega$-equivalence, $\Pi^1_k \subseteq \Pi^1_{k+1} \cap \Sigma^1_{k+1}$, $\Pi^1_k$ is closed under finite conjunction and disjunction, and that negations of sentences in $\Pi^1_k$ belong to $\Sigma^1_k$ (and vice versa).

In the following we restrict our attention to formulas with variables of type ${}^*N$. We build a hierarchy of formulas of this kind.

By a **basic term over** ${}^*N$ we mean a term of the form ${}^*\alpha(u_1, \dots, u_k)$ where $\alpha$ is a primitive recursive function of $k$ variables and each $u_i$ is either a variable of type ${}^*N$ or the constant symbol $H$. $NQF$ is the set of finite Boolean combinations of equations $s = t$ and formulas $S(t)$ where $s, t$ are basic terms over ${}^*N$. Note that the constant symbol $H$ and the predicate symbol $S$ are allowed in formulas of $NQF$, but the symbol $j$ is not allowed.

The internal quantifier-free formulas are just the formulas $B \in NQF$ in which the symbol $S$ does not occur.

Let $N\Pi^0_0 = N\Sigma^0_0$ be the set of formulas which are built from formulas in $NQF$ using the relativized quantifiers $\forall^S$, $\exists^S$ and propositional connectives. Note that the relations $ns(x)$ and

$x \approx y$ are definable by $N\Pi^0_0$ formulas.

For each natural number $k$, $N\Pi^0_{k+1}$ is the set of formulas of the form $\forall^{ns} x\, A$ where $A \in N\Sigma^0_k$. $N\Sigma^0_{k+1}$ is the set of formulas of the form $\exists^{ns} x\, A$ where $A \in N\Pi^0_k$.

Up to $WNA$-equivalence, $N\Pi^0_k \subseteq N\Pi^0_{k+1} \cap N\Sigma^0_{k+1}$, $N\Pi^0_k$ is closed under finite conjunction and disjunction, and negations of sentences in $N\Pi^0_k$ belong to $N\Sigma^0_k$ (and vice versa).

We now define the lifting mapping on formulas, which sends $\Pi^1_k$ to $N\Pi^0_k$.

**Definition 7.1** *Let $A(\vec{m}, \vec{f})$ be a formula in $\Pi^1_k$, where $\vec{m}, \vec{f}$ contain all the variables of $A$, both free and bound. The* ***lifting*** *$\overline{A}(\vec{z}, \vec{x})$ is defined as follows, where $\vec{z}$ and $\vec{x}$ are tuples of variables of type ${}^*N$ of the same length as $\vec{m}, \vec{f}$.*

- *Replace each primitive recursive function symbol in $A$ by its star.*
- *Replace each $m_i$ by $z_i$.*
- *Replace each $f_i(m_k)$ by $(x_i)_{z_k}$.*
- *Replace each quantifier $\forall m_i$ by $\forall^S z_i$, and similarly for $\exists$.*
- *Replace each quantifier $\forall f_i$ by $\forall^{ns} x_i$, and similarly for $\exists$.*

**Lemma 7.2** *(Zeroth Order Lifting) For each formula $A(\vec{m}, \vec{f}) \in \Pi^1_0$, we have $\overline{A}(\vec{z}, \vec{x}) \in N\Pi^0_0$, and*

$$WNA \vdash {}^o\vec{x} = \vec{f} \rightarrow [A(\vec{m}, \vec{f}) \leftrightarrow \overline{A}(\vec{m}, \vec{x})].$$

*Moreover, if $A \in QF$ then $\overline{A}(\vec{z}, \vec{x})$ is an internal quantifier-free formula.*

**Proof.** It is clear from the definition that $\overline{A}(\vec{z}, \vec{x}) \in N\Pi^0_0$, and if $A \in QF$ then $\overline{A}(\vec{z}, \vec{x})$ is an internal quantifier-free formula. In the case that $A$ is an equation between basic terms, the lemma follows from the Lifting Axiom. The general case is then proved by induction on the complexity of $A$, using the axiom that $j$ maps $\mathbb{N}$ onto $S$. ■

**Lemma 7.3** *(First Order Lifting) For each formula $A(\vec{m}, \vec{f}) \in \Pi^1_k$, we have $\overline{A}(\vec{z}, \vec{x}) \in N\Pi^0_k$ and*

$$WNA + STP \vdash {}^o\vec{x} = \vec{f} \rightarrow [A(\vec{m}, \vec{f}) \leftrightarrow \overline{A}(\vec{m}, \vec{x})].$$

**Proof.** Zeroth Order Lifting gives the result for $k = 0$. The general case follows by induction on $k$, using $STP$. ■

# 8 Choice Principles in $L(PRA^\omega)$

In this section we state two choice principles in the language $L(PRA^\omega)$, and show that for quantifier-free formulas they are consequences of the Standard Part Principle. Given a function $g$ of type $N \rightarrow N$, let $g^{(m)}$ be the function $g^{(m)}(n) = g(2^m 3^n)$.

In each principle, $\Gamma$ is a class of formulas with variables of types $N$ and $N \rightarrow N$, and $A(m, n, \ldots)$ denotes an arbitrary formula in $\Gamma$.

$(\Gamma, 0)$**-choice** $\forall m \exists n\, A(m, n, \ldots) \rightarrow \exists g \forall m\, A(m, g(m), \ldots)$.
$(\Gamma, 1)$**-choice** $\forall m \exists f\, A(m, f, \ldots) \rightarrow \exists g \forall m\, A(m, g^{(m)}, \ldots)$.

When $\Gamma$ is the set of all quantifier-free formulas of $PRA^\omega$, [K] calls these schemes $QF-AC^{0,0}$ and $QF-AC^{0,1}$ respectively. A related principle is

**Γ-comprehension** $\exists f \forall m\, f(m) = (\mu z < 1)\, A(m, \ldots)$.

$\Pi^1_0$-comprehension is called **Arithmetical Comprehension**. The following well-known fact is proved by pairing existential quantifiers.

**Proposition 8.1** *In $PRA^\omega$:*
$(\Pi^0_1, 0)$*-choice,* $(\Pi^1_0, 0)$*-choice, and Arithmetical Comprehension are equivalent.*
$(\Pi^0_k, 1)$*-choice is equivalent to* $(\Sigma^0_{k+1}, 1)$*-choice, and implies* $(\Sigma^0_{k+1}, 0)$*-choice;*
$(\Pi^1_k, 1)$*-choice is equivalent to* $(\Sigma^1_{k+1}, 1)$*-choice and implies* $(\Sigma^1_{k+1}, 0)$*-choice ;*
$(\Sigma^1_{k+1}, 0)$*-choice implies* $\Pi^1_k$*-comprehension.*

In $PRA^\omega$, one can define a subset of $\mathbb{N}$ to be a function $f$ such that $\forall n\, f(n) \leq 1$, and define $n \in f$ as $f(n) = 0$. With these definitions, $(\Pi^1_k, 1)$-choice implies $\Pi^1_k$-choice and $\Pi^1_k$-comprehension in the sense of second order number theory (see [S]).

**Lemma 8.2** *For each internal quantifier-free formula* $A(x, y, \vec{z})$*,*

$$WNA \vdash \forall^S x \exists^S y A(x, y, \vec{z}) \rightarrow (\exists^{ns} y < H) \forall^S x\, A(x, (y)_x, \vec{z}).$$

**Proof.** Work in $WNA$. Assume that $\forall^S x \exists^S y A(x, y, \vec{z})$. By Lemma 5.1, there is a primitive recursive function $\alpha$ such that ${}^*\alpha(u, \vec{z}, w) = (\mu v < w)\, A(u, v, \vec{z})$. By internal induction there exists $w$ such that $w^w < H \wedge \neg S(w)$. Then $\forall^S u\, S({}^*\alpha(u, \vec{z}, w))$ and

$$\forall^S x (\exists y < H)(\forall u < x)\, (y)_u = {}^*\alpha(u, \vec{z}, w).$$

By internal induction, there exists an $x$ such that $\neg S(x)$ and a $y < H$ such that

$$(\forall u < x)\, (y)_u = {}^*\alpha(u, \vec{z}, w).$$

It follows that $y$ is near-standard, and by the definition of $\alpha$, $(\forall u < x)\, A(u, (y)_u, \vec{z})$. Then

$$(\exists^{ns} y < H) \forall^S x\, A(x, (y)_x, \vec{z}).$$

■

**Theorem 8.3** $(QF, 0)$*-choice is provable in* $WNA + STP$*.*

**Proof.** We work in $WNA + STP$. Let $A(m, n, \vec{r}, \vec{h}) \in QF$ and assume $\forall m \exists n\, A(m, n, \vec{r}, \vec{h})$. Then $\overline{A}$ is an internal quantifier-free formula. By $STP$, $\vec{h}$ has a lifting $\vec{z}$. By First Order Lifting,

$$A(m, n, \vec{r}, \vec{h}) \leftrightarrow \overline{A}(m, n, \vec{r}, \vec{z}).$$

Then $\forall^S u \exists^S v\, \overline{A}(u, v, \vec{r}, \vec{z})$. By Lemma 8.2, there is a near-standard $y$ such that $\forall^S u\, \overline{A}(u, (y)_u, \vec{r}, \vec{z})$. By $STP$, $\exists g\, g = {}^o y$. Then by First Order Lifting, $\forall m\, A(m, g(m), \vec{r}, \vec{h})$. ■

**Theorem 8.4** $(QF, 1)$*-choice is provable in* $WNA + STP$*.*

**Proof.** We use $(QF,0)$-choice. Let $A(m,f,\vec{r},\vec{h}) \in QF$. Assume for simplicity that the tuple $\vec{r}$ is a single variable $r$. Suppose that $\forall m \exists f\, A(m,f,r,\vec{h})$. By the definition of $QF$ formulas, $f$ occurs in $A$ only in terms of the form $f(m)$ and $f(r)$. Then

$$A(m,f,r,\vec{h}) \leftrightarrow B(m,f(m),f(r),r,\vec{h})$$

where $B \in QF$. Hence $\forall m \exists k\, B(m,(k)_m,(k)_r,r,\vec{h})$. By $(QF,0)$-choice,

$$\exists f \forall m\, B(m,(f(m))_m,(f(m))_r,r,\vec{h}).$$

Applying $(QF,0)$-choice to the formula $\forall p\, \exists q\, q = (f((p)_0))_{(p)_1}$, we have $\exists g \forall p\, g(p) = (f((p)_0))_{(p)_1}$, and since $(2^m 3^n)_0 = m$ and $(2^m 3^n)_1 = n$,

$$\exists g \forall m \forall n\, g^{(m)}(n) = g(2^m 3^n) = (f(m))_n.$$

Then $\forall m\, B(m, g^{(m)}(m), g^{(m)}(r), r, \vec{h})$, and $\forall m\, A(m, g^{(m)}, r, \vec{h})$. ■

# 9 Saturation Principles

We state two saturation principles which formalize methods commonly used in nonstandard analysis. In each principle, $\Gamma$ is a class of formulas with variables of type ${}^*N$, and $A(x,y,\vec{u})$ denotes an arbitrary formula in the class $\Gamma$.

$(\Gamma,0)$**-saturation** $\quad \forall^{ns}\vec{u}\; [\forall^S x \exists^S y A(x,y,\vec{u}) \to \exists y \forall^S x\, A(x,(y)_x,\vec{u})]$.
$(\Gamma,1)$**-saturation** $\quad \forall^{ns}\vec{u}\; [\forall^S x \exists^{ns} y\, A(x,y,\vec{u}) \to \exists y \forall^S x A(x,(y)_x,\vec{u})]$.

Note that $(\Gamma,1)$-saturation implies $(\Gamma,0)$-saturation. $(N\Pi^0_k,1)$-saturation is weaker than the ${}^*\Pi_k$-saturation principle in the paper [HKK]. ${}^*\Pi_k$-saturation is the same as $(N\Pi^0_k,1)$-saturation except that the quantifiers $\forall^{ns}$, $\exists^{ns}$ are replaced by $\forall$, $\exists$.

In the rest of this section we prove some consequences of $(NQF,0)$-saturation.

**Proposition 9.1** *Let us write $w = st(v)$ for*

$$S(v) \to w = v \wedge \neg S(v) \to w = 0.$$

*In $WNA$, $(NQF,0)$-saturation implies that*

$$\forall x \exists^{ns} y \forall^S z\, [(y)_z = st((x)_z)].$$

*In $WNA + STP$, $(NQF,0)$-saturation implies that*

$$\forall x \exists f \forall m\, [f(m) = st((x)_m)].$$

**Proof.** Work in $WNA$. Note that $w = st(v)$ stands for a formula in $NQF$. Take any $x$. We have $\forall^S z \exists^S w\, w = st((x)_z)$. Then by $(NQF,0)$-saturation, $\exists y \forall^S z\, (y)_z = st((x)_z)$, and it follows that $ns(y)$. The second assertion follows by taking $f = {}^o y$. ■

**Lemma 9.2** $WNA \vdash (\forall v > 1)(\exists w < v^{2vH})(\forall x < v)\,(w)_x = (\mu u < H)\,[(y)_{2^x 3^u} = 0]$.

**Proof.** Use internal induction on $v$. The result is clear for $v = 2$. Let $\alpha(x) = (\mu u < H)\,[(y)_{2^x 3^u} = 0]$. Assume the result holds for $v$, that is, $w < v^{2vH} \wedge (\forall x < v)\,(w)_x = \alpha(x)$. Let $z = w * p_v^{\alpha(v)}$. We have $p_v < v^2$ and $\alpha(v) < H$, so $z < v^{2vH} * v^{2H} \leq (v+1)^{2(v+1)H}$ and $(\forall x < v+1)\,(z)_x = \alpha(x)$. This proves the result for $v+1$ and completes the induction. ∎

**Lemma 9.3** *In $WNA$, $(NQF, 0)$-saturation implies that for every formula $A(x, \vec{u}) \in N\Pi^0_0$, $\forall^{ns}\vec{u}\ (\exists y < H)\forall^S x\,(y)_x = (\mu z < 1)A(x, \vec{u})$.*

**Proof.** Work in $WNA$ and assume $(NQF, 0)$-saturation. Let $\Phi$ be the set of formulas $A(x, \vec{u})$ such that $\forall^{ns}\vec{u}\,(\exists y < H)\forall^S x\,(y)_x = (\mu z < 1)A(x, \vec{u})$. We prove that $N\Pi^0_0 \subseteq \Phi$ by induction on quantifier rank. Suppose first that $A \in NQF$. Let $C(x, w, \vec{u})$ be the formula $w = (\mu z < 1)\,A(x, \vec{u})$. Then $C$ is a propositional combination of $A$, $w = 0$, and $w = 1$, so $C \in NQF$. We clearly have $\forall^{ns}\vec{u}\,\forall^S x\,\exists^S w\,C(x, w, \vec{u})$. By $(NQF, 0)$-saturation and Lemma 6.1 (ii),

$$\forall^{ns}\vec{u}\,(\exists y < H)\forall^S x\,C(x, (y)_x, \vec{u}),$$

so $A \in \Phi$.

It is clear that the set of formulas $\Phi$ is closed under propositional connectives. Suppose all formulas of $N\Pi^0_0$ of quantifier rank at most $n$ belong to $\Phi$, and $A(x, \vec{u}) = \exists^S w B(x, w, \vec{u})$ where $B(x, w, \vec{u}) \in N\Pi^0_0$ has quantifier rank at most $n$. There is a formula $D(v, \vec{u})$ with the same quantifier rank as $B$ such that in $WNA$, $D(2^x 3^w, \vec{u}) \leftrightarrow B(x, w, \vec{u})$. Then $D \in \Phi$, so

$$\forall^{ns}\vec{u}\,(\exists t < H)\forall^S v\,(t)_v = (\mu z < 1)\,D(v, \vec{u}).$$

Then

$$\forall^{ns}\vec{u}\,(\exists t < H)\forall^S x \forall^S w\,(t)_{2^x 3^w} = (\mu z < 1)\,B(x, w, \vec{u}).$$

Assume that $ns(\vec{u})$ and take $t$ as in the above formula. By Lemma 9.2 there exists $s$ such that

$$(\forall x < H)\,(s)_x = (\mu w < H)\,[(t)_{2^x 3^w} = 0].$$

It is trivial that $\forall^S x \exists^S y\, y = (\mu z < 1)\,S((s)_x)$. By $(NQF, 0)$-saturation and Lemma 6.1,

$$(\exists y < H)\forall^S x\,(y)_x = (\mu z < 1)\,S((s)_x).$$

Thus whenever $S(x)$, $(y)_x = 0$ iff $S((s)_x)$ iff $\exists^S w\,[(t)_{2^x 3^w} = 0]$ iff $\exists^S w\,B(x, w, \vec{u})$. It follows that

$$\forall^{ns}\vec{u}\,(\exists y < H)\forall^S x\,(y)_x = (\mu z < 1)\exists^S w\,B(x, w, \vec{u}),$$

so $A \in \Phi$. ∎

**Theorem 9.4** *In $WNA$, $(NQF, 0)$-saturation implies $(N\Pi^0_0, 0)$-saturation.*

**Proof.** We continue to work in $WNA$ and assume $(NQF, 0)$-saturation and $ns(\vec{u})$. Assume that $A(x, y, \vec{u}) \in N\Pi^0_0$ and $\forall^S x \exists^S y\,A(x, y, \vec{u})$. There is a formula $B(v, \vec{u}) \in N\Pi^0_0$ with $B(2^x 3^y, \vec{u})$ $WNA$-equivalent to $A(x, y, \vec{u})$. Applying Lemma 9.3 to $B$, we obtain $w$ such that

$$\forall^S v\,(w)_v = (\mu z < 1)\,B(v, \vec{u}),$$

so

$$\forall^S x \forall^S y\,(w)_{2^x 3^y} = (\mu z < 1)\,A(x, y, \vec{u}).$$

By Lemma 9.2 there exists $w'$ such that

$$(\forall x < H)\,(w')_x = (\mu y < H)\,[(w)_{2^x 3^y} = 0].$$

Then $\forall^S x\,A(x, (w')_x, \vec{u})$, and $w$ is near-standard because $\forall^S x \exists^S y\,A(x, y, \vec{u})$. ∎

**Theorem 9.5** *In $WNA$, $(NQF, 0)$-saturation implies that for every formula $A(\vec{u}) \in N\Pi^0_1$, there a formula $B \in NQF$ such that $\forall^{ns}\vec{u}\,[A(\vec{u}) \leftrightarrow \forall^{ns}x\exists^S y\, B(x, y, \vec{u})]$.*

**Proof.** Work in $WNA+(NQF, 0)$-saturation. Suppose $A \in N\Pi^0_1$. Then there is a least $k$ such that $A$ is equivalent to a formula $\forall^{ns}x\, C$ where $C$ is a prenex formula in $N\Pi^0_0$ of quantifier rank $k$. If $C$ has the form $\forall^S y\, D$, the quantifier $\forall^S y$ can be absorbed into the quantifier $\forall^{ns}x$, contradicting the assumption that $k$ is minimal. Suppose $C$ has the form $\exists^S y\,\forall^S z\, D(x, y, z, \vec{u})$ and assume that $ns(\vec{u})$. Then $\neg C$ is equivalent to $\forall^S y\,\exists^S z\,\neg D(x, y, z, \vec{u})$. By $(N\Pi^0_0, 0)$-saturation, $\neg C$ is equivalent to $\exists^{ns}z\forall^S y\,\neg D(x, y, (z)_y, \vec{u})$. Then $A$ is equivalent to $\forall^{ns}x\forall^{ns}z\exists^S y\, D(x, y, (z)_y, \vec{u})$. By combining the quantifiers $\forall^{ns}x\forall^{ns}z$, we contradict the assumption that $k$ is minimal. Therefore $C$ must have the form $\exists^S y\, B$ where $B \in NQF$, as required. ■

# 10 Saturation and Choice

In this section we prove results showing that in $WNA+STP$, saturation principles with quantifiers of type ${}^*N$ imply the corresponding choice principles with quantifiers of type $N \to N$.

**Theorem 10.1** *In $WNA + STP$, $(NQF, 0)$-saturation implies Arithmetical Comprehension.*

**Proof.** Work in $WNA + STP$ and assume $(NQF, 0)$-saturation. By Proposition 8.1, Arithmetical Comprehension is equivalent to $(\Pi^1_0, 0)$-choice. By Theorem 9.4, $(N\Pi^0_0, 0)$-saturation holds. Let $A(m, n, \vec{r}, \vec{h})$ be an arithmetical formula such that $\forall m\exists n\, A(m, n, \vec{r}, \vec{h})$. By $STP$, $\vec{h}$ has a lifting $\vec{u}$. By First Order Lifting, we have $\forall^S x\exists^S y\,\overline{A}(x, y, \vec{r}, \vec{u})$, and $\overline{A} \in N\Pi^0_0$. By $(N\Pi^0_0, 0)$-saturation, there exists $y$ such that $\forall^S x[S((y)_x) \wedge \overline{A}(x, (y)_x, \vec{r}, \vec{u})]$. Then $y$ is near-standard, and by $STP$ there exists $g = {}^o y$. By First Order Lifting again, $\forall m A(m, g(m), \vec{r}, \vec{h})$. ■

We remark that the axioms of Peano Arithmetic are consequences of Arithmetical Comprehension, so $(NQF, 0)$-saturation implies Peano Arithmetic.

**Theorem 10.2** *In $WNA+STP$, $(N\Pi^0_k, 0)$-saturation implies $(\Pi^1_k, 0)$-choice, and $(N\Sigma^0_k, 0)$-saturation implies $(\Sigma^1_k, 0)$-choice.*

**Proof.** Work in $WNA+STP$. For the $\Pi^1_k$ case, assume $(N\Pi^0_k, 0)$-saturation. Let $A(m, n, \vec{r}, \vec{h}) \in \Pi^1_k$ and suppose that $\forall m\exists n\, A(m, n, \vec{r}, \vec{h})$. Now argue as in the proof of Theorem 10.1. The $\Sigma^1_k$ case is similar. ■

**Theorem 10.3** *In $WNA + STP$, $(N\Pi^0_k, 1)$-saturation implies $(\Sigma^1_{k+1}, 1)$-choice.*

**Proof.** Work in $WNA + STP$ and assume $(N\Pi^0_k, 1)$-saturation. It suffices to prove $(\Pi^1_k, 1)$-choice. Let $A(m, f, \vec{r}, \vec{h}) \in \Pi^1_k$ and suppose that $\forall m\exists f\, A(m, f, \vec{r}, \vec{h})$. By $STP$, $\vec{h}$ has a lifting $\vec{u}$. By First Order Lifting, $\forall^S x\exists^{ns}y\,\overline{A}(x, y, \vec{r}, \vec{u})$ and $\overline{A} \in N\Pi^0_k$. We may rewrite this as $\forall^S x\exists^{ns}y\,[ns(y) \wedge \overline{A}(x, y, \vec{r}, \vec{u})]$ and note that $ns(y) \wedge \overline{A} \in N\Pi^0_k$. By $(N\Pi^0_k, 1)$-saturation, there exists $y$ such that

$$\forall^S x\,[ns((y)_x) \wedge \overline{A}(x, (y)_x, \vec{r}, \vec{u})].$$

Applying $(N\Pi^0_k, 0)$-saturation to the formula $\forall^S x\exists^S z\, z = ((y)_{(x)_0})_{(x)_1}$, we get a near-standard $z$ such that $\forall^S x\,(z)_x = ((y)_{(x)_0})_{(x)_1}$. Then

$$\forall^S x\forall^S w\,(z)_{2^x 3^w} = ((y)_x)_w.$$

By $STP$, there exists $g = {}^o z$. Then for each $m, n$, $g^{(m)}(n) = g(2^m 3^n) = (z)_{2^m 3^n} = ((y)_m)_n$. Therefore $g^{(m)} = {}^o((y)_m)$ for each $m$. By First Order Lifting, we get the desired conclusion $\forall m\, A(m, g^{(m)}, \vec{r}, \vec{u})$. ∎

The literature in reverse mathematics (see [S]) shows that $\Pi^1_1$-comprehension is strong enough for almost all of classical mathematics. Let us work in $WNA{+}STP$ and aim for $\Pi^1_1$-comprehension. By Theorem 10.3, $(N\Pi^0_1, 1)$-saturation implies $\Pi^1_1$-comprehension. By Theorem 9.5, $(N\Pi^0_1, 1)$-saturation is equivalent to $(\Gamma, 1)$-saturation where $\Gamma$ is the set of formulas of the form $\forall^{ns} v \exists^S w\, B$ with $B \in NQF$, so $(\Gamma, 1)$-saturation also implies $\Pi^1_1$-comprehension. By Theorem 10.3 at the next level, $(N\Pi^0_2, 1)$-saturation implies $\Pi^1_2$-comprehension, which is stronger than the methods used in most of classical mathematics.

# 11 Second Order Standard Parts

In this section we introduce second order standard parts, which provide a link between the second level of $V(\mathbb{N})$ (type $(N \to N) \to N$), and the first level of $V({}^*\mathbb{N})$ (type ${}^*N \to {}^*N$). We will use $F, G, \ldots$ for variables of type $(N \to N) \to N$, and $\phi, \psi, \ldots$ for variables of type ${}^*N \to {}^*N$.

$\phi$ is called **near-standard**, in symbols $ns(\phi)$, if

$$\forall^{ns} x\, S(\phi(x)) \wedge \forall x \forall y [x \approx y \to \phi(x) = \phi(y)].$$

We write

$$\phi \approx \psi \text{ if } ns(\phi) \wedge \forall^{ns} x\, \phi(x) = \psi(x).$$

We write $G = {}^o\phi$, and say that $G$ is the **standard part of** $\phi$ and that $\phi$ is a **lifting of** $G$, if

$$ns(\phi) \wedge \forall^{ns} x \forall f\, [{}^o x = f \to \phi(x) = G(f)].$$

Note that the operation $\phi \mapsto {}^o\phi$ goes from type ${}^*N \to {}^*N$ to type $(N \to N) \to N$. The following lemma is straightforward.

**Lemma 11.1** *If $ns(\phi)$ and $\phi \approx \psi$ then $ns(\psi)$ and $\psi \approx \phi$.*

We now state the Second Order Standard Part Principle, which says that every near-standard $\phi$ has a standard part and every $F$ has a lifting.

**Second Order Standard Part Principle:**

$$\forall^{ns} \phi\, \exists F\, F = {}^o\phi \;\wedge \forall F \exists \phi\, F = {}^o\phi.$$

By $WNA + STP(2)$ we mean the theory $WNA$ plus both the first and second order standard part principles.

We now take a brief look at the consequences of $STP(2)$ in $WNA{+}NPRA^\omega$. Roughly speaking, in $WNA + NPRA^\omega$, the second order standard part principle imposes restrictions of the set of functionals which are reminiscent of constructive analysis. Besides the axioms of $WNA$, the only axiom of $NPRA^\omega$ that will be used in this section is the star of quantifier-free induction.

A functional $G$ is **continuous** if it is continuous in the Baire topology, that is,

$$\forall f \exists n \forall h\, [[(\forall m < n)\, h(m) = f(m)] \to G(h) = G(f)].$$

**Proposition 11.2** $WNA + NPRA^\omega + STP(2) \vdash \forall G\, G$ *is continuous.*

**Proof.** Work in $WNA + NPRA^\omega + STP(2)$. Suppose $G$ is not continuous at $f$. Then

$$\forall n \exists h\, [[(\forall m < n)\, h(m) = f(m)] \wedge G(f) \neq G(h)].$$

By $STP(2)$ there are liftings $\phi$ of $G$ and $x$ of $f$. By Lemma 6.1 and $STP$,

$$\forall n (\exists y < H)\, [[(\forall m < n)\, (y)_m = (x)_m] \wedge \phi(x) \neq \phi(y)].$$

By the star of $QF$ induction,

$$\exists^\infty w (\exists y < H)\, [[(\forall u < w)\, (x)_u = (y)_u] \wedge \phi(x) \neq \phi(y)].$$

But then $y \approx x$, contradicting the assumption that $\phi$ is near-standard. ■

This result is closely related to Proposition 5.2 in [A], which says that in $NPRA^\omega$, every function $f \in \mathbb{R} \to \mathbb{R}$ is continuous.

The sentence

$$(\exists^2) = \exists G \forall f [G(f) = 0 \leftrightarrow \exists n f(n) = 0]$$

played a central role in the paper [K], where many statements are shown to be equivalent to $(\exists^2)$ in $RCA_0^\omega$. Similar sentences are prominent in earlier papers, such as Feferman [F]. It is well-known that

$$PRA^\omega \vdash (\exists^2) \rightarrow \exists G\, G \text{ is not continouus.}$$

**Corollary 11.3** $WNA + NPRA^\omega + STP(2) \vdash \neg(\exists^2)$.

## 12 Functional Choice and $(\exists^2)$

In this section we obtain connections between $WNA$ and two statements which play a central role in the paper of Kohlenbach [K], the statement $(\exists^2)$ and the functional choice principle $QF - AC^{1,0}$.

In [K], Kohlenbach proposed a base theory $RCA_0^\omega$ for higher order reverse mathematics which is somewhat stronger than $PRA^\omega$, and is a conservative extension of the second order base theory $RCA_0$. Its main axioms are the axioms of $PRA^\omega$ and the scheme

$$QF - AC^{1,0}: \qquad \forall f \exists n\, A(f, n, \ldots) \rightarrow \exists G \forall f\, A(f, G(f), \ldots)$$

where $A(f, n, \ldots)$ is quantifier-free.

In [K], the formula $A$ in the $QF - AC^{1,0}$ scheme is allowed to be an arbitrary quantifier-free formula in the language $L(PRA^\omega)$. Here we will make the additional restriction that $A(f, n, \ldots)$ is in the class $QF$ as defined in Section 7, that is, $A(f, n, \ldots)$ is a Boolean combination of equations and inequalities between basic terms. These formulas only have variables of type $N$ and $N \rightarrow N$, and do not have functional variables.

We show now that $QF - AC^{1,0}$ restricted in this way follows from $WNA$ plus the standard part principles.

**Theorem 12.1** $WNA + STP(2) \vdash QF - AC^{1,0}$.

**Proof.** Work in $WNA + STP(2)$. Assume $\forall f \exists n\, A(f, n, \vec{m}, \vec{h})$. By Zeroth Order Lifting, $\overline{A}(x, u, \vec{v}, \vec{z})$ is an internal quantifier-free formula, and

$$ {}^o x = f \wedge {}^o \vec{z} = \vec{h} \rightarrow [A(f, n, \vec{m}, \vec{h}) \leftrightarrow \overline{A}(x, n, \vec{m}, \vec{z})]. $$

By Lemma 5.1 there is a primitive recursive function $\alpha$ such that

$$ {}^*\alpha(x, w, \vec{v}, \vec{z}) = (\mu u < w)\, \overline{A}(x, u, \vec{v}, \vec{z}). $$

By $STP$, there exists $\vec{z}$ such that $\vec{h} = {}^o\vec{z}$. By the Lambda Abstraction axiom,

$$ \exists \phi \forall x\, \phi(x) = {}^*\alpha(x, H, \vec{m}, \vec{z}). $$

Then

$$ \forall^{ns} x\, [S(\phi(x)) \wedge \overline{A}(x, \phi(x), \vec{m}, \vec{z})]. $$

It follows that $\phi$ is near-standard. By $STP(2)$, there exists $G$ such that $G = {}^o\phi$. Therefore $\forall f\, A(f, G(f), \vec{m}, \vec{h})$. ∎

One of the advantages of $WNA$ over $NPRA^\omega$ is that one can add hypotheses which produce external functions and still keep the standard part principles. The simplest hypothesis of this kind is the following statement, which says that the characteristic function of $S$ exists:

$$ (1_S \text{ exists}): \qquad \exists \phi \forall y\, \phi(y) = (\mu z < 1)\, S(y). $$

It is clear that

$$ NPRA^\omega \vdash \neg(1_S \text{ exists}) $$

because by the star of quantifier-free induction, $\forall^S y\, \phi(y) = 0$ implies $\exists y\, [\neg S(y) \wedge \phi(y) = 0]$. However, ($1_S$ exists) is true in the full natural model $\langle V(\mathbb{N}), V({}^*\mathbb{N}), j\rangle$ of $WNA$. We now connect this principle with the statement $(\exists^2)$.

**Theorem 12.2** *$WNA + STP(2) \vdash (1_S$ exists$) \rightarrow (\exists^2)$.*

**Proof.** Work in $WNA$+$STP(2)$. Let $\alpha$ be the primitive recursive function such that ${}^*\alpha(x, w) = (\mu u < w)\, (x)_u = 0$. Let $\phi$ be the function $1_S$, so that $\forall y\, \phi(y) = (\mu z < 1)\, S(y)$. Then there exists $\psi$ such that $\forall x\, \psi(x) = \phi({}^*\alpha(x, H))$. Observe that

$$ \phi({}^*\alpha(x, H)) = 0 \leftrightarrow \exists^S u\, (x)_u = 0, $$

so $\psi(x) = 0 \leftrightarrow \exists^S u\ (x)_u = 0$. Moreover, $\forall x \psi(x) < 2$. We show that $\psi$ is near-standard.

Suppose $ns(x)$ and $x \approx y$. We always have $S(\phi(x))$ since $\phi(x) < 2$. If $\psi(x) = 0$ then there exists $u$ such that $S(u)$ and $(x)_u = 0$, so $(y)_u = 0$ and hence $\psi(y) = 0$. This shows that $ns(\psi)$. By $STP(2)$ there exists $G$ such that $G = {}^o\psi$. Consider any $f$. By $STP$, $f$ has a lifting $x$. Then $G(f) = 0$ iff $\psi(x) = 0$ iff $\exists^S u\, (x)_u = 0$ iff $\exists n f(n) = 0$, and thus $(\exists^2)$ holds. ∎

Let us now go back to Section 7 and redefine the set $QF$ of formulas by allowing basic terms of the form $G_i(f_k)$ in addition to the previous basic terms, and redefining the hierarchy $\Pi^1_k$ by starting with the new $QF$. Also redefine the set $NQF$ and the hierarchy $N\Pi^0_k$ by allowing additional basic terms of the form $\phi_i(x_k)$. When $STP(2)$ is assumed, the lifting lemmas from Section 7 and the results of Section 9 can be extended to the larger classes of formulas just defined. The hierarchies $\Pi^2_k$ and $N\Pi^1_k$ at the next level can now be defined in the natural way. One can then obtain the following result, with a proof similar to the proofs in Section 9.

**Theorem 12.3** *In $WNA + STP(2)$, $(N\Pi^1_k, 0)$-saturation implies $(\Pi^2_k, 0)$-choice, and $(N\Pi^1_k, 1)$-saturation implies $(\Pi^2_k, 1)$-choice.*

## 13 Conclusion

We have proposed weak nonstandard analysis, $WNA$, as a base theory for reverse mathematics in nonstandard analysis. In $WNA + STP$, one can prove:

The Weak Koenig Lemma,
$(QF, 0)$-choice and $(QF, 1)$-choice,
$(NQF, 0)$-saturation implies $(\Pi^1_0, 0)$-choice.
$(N\Pi^0_k, i)$-saturation implies $(\Pi^1_k, i)$-choice, $i = 0, 1$.

In $WNA + STP(2)$ one can prove:

$QF - AC^{1,0}$,
$NPRA^\omega$ implies $\forall G\, G$ is continuous,
$1_S$ exists implies $(\exists^2)$,
$(N\Pi^1_k, i)$-saturation implies $(\Pi^2_k, i)$-choice, $i = 0, 1$.

We envision the use of these results to calibrate the strength of particular theorems proved using nonstandard analysis. At the higher levels, this could give a way to show that a theorem cannot be proved with methods commonly used in classical mathematics.

Look again at the natural models of $WNA$ discussed at the end of Section 4. Let ${}^*V(\mathbb{N})$ be an $\aleph_1$-saturated elementary extension of $V(\mathbb{N})$ in the model-theoretic sense, and consider the internal natural model $\langle V(\mathbb{N}), {}^*V(\mathbb{N}), j\rangle$ and the full natural model $\langle V(\mathbb{N}), V({}^*\mathbb{N}), j\rangle$. Both of these models satisfy the axioms of $WNA$, the $STP$, the statement $(\exists^2)$, and $(N\Pi^1_k, 1)$-saturation. In view of Corollary 11.3, in the internal natural model the axioms of $NPRA^\omega$ hold and $STP(2)$ fails, while in the full natural model $STP(2)$ holds and the axioms of $NPRA^\omega$ fail.

## References


[A] J. Avigad. Weak theories of nonstandard arithmetic and analysis. To appear in Reverse Mathematics, ed. by S.G. Simpson.

[CK] C.C. Chang and H.J. Keisler. Model Theory, Third Edition. Elsevier 1990.

[F] S. Feferman, Theories of finite type related to mathematical practice, in Handbook of Mathematical Logic, ed. by J. Barwise, 1977.

[G] K. Gödel. Uber eine bisher noch nicht benutste Erweiterung des finiten Standpunktes. Dialectica 12, pp. 280-287 (1958).

[HKK] C.W. Henson, M. Kaufmann, and H.J. Keisler. The strength of proof theoretic methods in arithmetic. J. Symb. Logic 49 (1984), pp. 1039-1058.

[HK] C.W. Henson and H.J. Keisler. The strength of nonstandard analysis. J. Symb. Logic 51 (1986), pp. 377-386.

[K] U. Kohlenbach. Higher order reverse mathematics. To Appear in Reverse Mathematics, ed. by S.G. Simpson.

[S] S.G. Simpson. Subsystems of Second Order Arithmetic. Perspectives in Mathematical Logic. Springer-Verlag 1999.

[TD] A.S. Troelstra and D. van Dalen. Constructivism in Mathematics. An Introduction. Volume II. North-Holland 1988.