SEPARABLE MODELS OF RANDOMIZATIONS

URI ANDREWS AND H. JEROME KEISLER

Abstract. Every complete first order theory has a corresponding complete theory in continuous logic, called the randomization theory. It has two sorts, a sort for random elements of models of the first order theory, and a sort for events. In this paper we establish connections between properties of countable models of a first order theory and corresponding properties of separable models of the randomization theory. We show that the randomization theory has a prime model if and only if the first order theory has a prime model. And the randomization theory has the same number of separable homogeneous models as the first order theory has countable homogeneous models. We also show that when T has at most countably many countable models, each separable model of T^R is uniquely characterized by a probability density function on the set of isomorphism types of countable models of T. This yields an analogue for randomizations of the results of Baldwin and Lachlan on countable models of ω_1-categorical first order theories.

1. Introduction

A randomization of a first order structure $\mathcal{M}$, as introduced by Keisler [Kei2] and formalized as a metric structure by Ben Yaacov and Keisler [BK], is a new structure $\mathcal{M}^R$ with two sorts, a sort for random elements of M, and a sort for events in an underlying probability space. Given a complete first order theory T, the theory T^R of randomizations of models of T forms a complete theory in continuous logic, which is called the randomization theory of T. One would expect that a first order theory and its randomization would be model theoretically similar. We continue the tradition in [BK] and [Ben] of examining which properties of a theory are preserved in its randomization. In particular, [BK] showed that ω-categoricity, having a countable or separable saturated model, ω-stability, and stability are preserved, and Ben Yaacov [Ben] showed that dependence is preserved.

This paper owes much to the paper Vaught [Va], "Denumerable Models of Complete Theories". We will cite [Va] for results on first order theories that were originally published there. The reader may also consult standard model theory texts, such as Section 2.3 in [CK].

In this paper we show that T^R has a prime model if and only if T has a prime model. T^R never has a prime model that is minimal, but T^R has a prime model that is minimal

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over its events (see Definition 9.6) if and only if T has a prime model that is minimal. We show that the number of separable homogeneous models of T^R is equal to the number of countable homogeneous models of T. We also show that every countable model of T is homogeneous if and only if every separable model of T^R is homogeneous over its events (see Definition 9.1). And unless T is ω-categorical, T^R has continuum many non-isomorphic separable models that are homogeneous over events but are not homogeneous. In the case where T has at most countably many countable models (up to isomorphism), we show that the separable models of T^R are exactly the completions of product randomizations (see Definition 7.1) of the countable models of T. This uniquely characterizes each separable model of T^R by a probability density function on the set of isomorphism types of countable models of T.

We were motivated by the theorem of Baldwin and Lachlan [BL] which shows that every ω_1-categorical theory in a countable language has countably many countable models, and all countable models are homogeneous. Moreover, these countable models are characterized by the dimension of the strongly minimal set, that takes values in $\mathbb{N} \cup \{\infty\}$. Our results yield a version of the Baldwin-Lachlan theorem for randomizations of ω_1-categorical theories. If T is ω_1-categorical, then every separable model $\mathcal{N}$ of T^R is characterized by a countable sequence of reals $s_0, s_1, \ldots, s_\omega \in [0,1]$ such that $\sum_{k \le \omega} s_k = 1$, and is homogeneous over events. The model $\mathcal{N}$ should be interpreted as the result of sampling from the k-dimensional model of T with probability s_k for each $k \le \omega$.

This paper is organized as follows. In Section 2 we review some notions we will need from the literature, including the key notion of the Borel randomization $(\mathcal{M}^{[0,1)}, \mathcal{L})$ of a first order structure $\mathcal{M}$. Section 3 contains results about prime models. In Sections 4 and 5 we prepare the way for our main results by examining in detail the strongly separable models— those that are embeddable in Borel randomizations of countable models. Our results about separable homogeneous models are in Section 6. In Section 7 we introduce product randomizations. Section 8 contains our characterization of the separable models of T^R when T has at most countably many countable models, and Section 9 contains results about models that are homogeneous over events, and about prime models that are minimal over events.

This paper is the result of merging two earlier submissions to the Journal of Symbolic Logic at the suggestion of the editors—a submission by the second author alone containing the results of Sections 3 on prime models and 6 on separable homogeneous models, and a submission by both authors containing the remaining results.

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2. Preliminaries

We refer to [BBHU] and [BU] for background in continuous model theory, and follow the notation of [BK]. We assume familiarity with the basic notions about continuous model theory as developed in [BBHU], including the notions of a theory, structure, pre-structure, formula, and model of a theory. In particular, the universe of a pre-structure is a pseudo-metric space, and the universe of a structure is a complete metric space. In continuous model theory, the analogue of a countable structure is a **separable structure**, that is, a structure whose universe is a separable metric space. A pre-structure is said to be separable if its universe is a separable pseudo-metric space. We remind the reader that formulas take truth values in $[0,1]$, and are built from atomic formulas using continuous connectives on $[0,1]$ and the quantifiers $\sup, \inf$.

We assume throughout that L is a finite or countable first order signature, and that T is a complete theory for L whose models have at least two elements. As in [BK], by a **countable model** of T we mean a model of T whose universe is either finite or of cardinality ω. A **tuple** is a finite sequence.

2.1. **The theory T^R.** A randomization of a model $\mathcal{M}$ of T is a two-sorted continuous structure with a sort $\mathbf{K}$ whose elements are random elements of $\mathcal{M}$, and a sort $\mathbf{B}$ whose elements are events in an underlying probability space. The probability of the event that a first order formula holds for a tuple of random elements will be expressible by a formula of continuous logic.

Formally, the **randomization signature** L^R is the two-sorted continuous signature with sorts $\mathbf{K}$ and $\mathbf{B}$, an n-ary function symbol $[\![\varphi(\cdot)]\!]$ of sort $\mathbf{K}^n \to \mathbf{B}$ for each first order formula φ of L with n free variables, a $[0,1]$-valued unary predicate symbol μ of sort $\mathbf{B}$ for probability, and the Boolean operations $\top, \bot, \sqcap, \sqcup, \neg$ of sort $\mathbf{B}$. The signature L^R also has distance predicates $d_{\mathbf{B}}$ of sort $\mathbf{B}$ and $d_{\mathbf{K}}$ of sort $\mathbf{K}$. In L^R, we use $\mathsf{B}, \mathsf{C}, \ldots$ for variables or parameters of sort $\mathbf{B}$, and $\mathsf{B} \doteq \mathsf{C}$ means $d_{\mathbf{B}}(\mathsf{B}, \mathsf{C}) = 0$.

A pre-structure for T^R will be a pair $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ where $\mathcal{K}$ is the part of sort $\mathbf{K}$ and $\mathcal{B}$ is the part of sort $\mathbf{B}$. We call $\mathcal{B}$ the **event sort** of $\mathcal{N}$. In this paper we will only need to consider pre-structures of a special kind— the Borel randomizations and their (pre-)substructures. Borel randomizations are closely related to Boolean valued ultrapowers.

We let $\mathcal{L}$ be the family of Borel subsets of $[0,1)$, and let $([0,1), \mathcal{L}, \lambda)$ be the usual probability space[1] where λ is the restriction of Lebesgue measure to $\mathcal{L}$. The phrase "almost all t" will mean "for all t in a set $\mathsf{B} \in \mathcal{L}$ of λ-measure one". Given a model $\mathcal{M}$ of T, we let $\mathcal{M}^{[0,1)}$ be the set of $\mathcal{L}$-measurable functions with countable range from $[0,1)$ into $\mathcal{M}$. Intuitively, an element of $\mathcal{M}^{[0,1)}$ is an experiment in which an element of $\mathcal{M}$ is chosen at random. The elements of $\mathcal{M}^{[0,1)}$ are called **random elements of $\mathcal{M}$**.

Definition 2.1. The **Borel randomization of $\mathcal{M}$** is the pre-structure $(\mathcal{M}^{[0,1)}, \mathcal{L})$ for L^R whose universe of sort $\mathbf{K}$ is $\mathcal{M}^{[0,1)}$, whose universe of sort $\mathbf{B}$ is $\mathcal{L}$, whose measure μ is given by $\mu(\mathsf{B}) = \lambda(\mathsf{B})$ for each $\mathsf{B} \in \mathcal{L}$, and whose $[\![\psi(\cdot)]\!]$ functions are

$$[\![\psi(\vec{\mathbf{f}})]\!] = \{t \in [0,1) : \mathcal{M} \models \psi(\vec{\mathbf{f}}(t))\}.$$

[1] In [BK] the set $[0,1]$ is used instead of $[0,1)$. The set $[0,1)$ is more convenient here because it can be partitioned into intervals of the form $[r,s)$.

(So $[\![\psi(\vec{\mathbf{f}})]\!] \in \mathcal{L}$ for each first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{f}}$ in $\mathcal{M}^{[0,1)}$). Its distance predicates are defined by

$$d_{\mathbf{B}}(\mathsf{B},\mathsf{C}) = \mu(\mathsf{B}\triangle\mathsf{C}), \quad d_{\mathbf{K}}(\mathbf{f},\mathbf{g}) = \mu([\![\mathbf{f} \neq \mathbf{g}]\!]),$$

where $\triangle$ is the symmetric difference operation.

Fact 2.2. *([BK], Theorem 2.1) There is a unique complete theory T^R for L^R, called the **randomization theory** of T, such that for each model $\mathcal{M}$ of T, $(\mathcal{M}^{[0,1)},\mathcal{L})$ is a pre-model of T^R.*

It follows that for each first order sentence φ, if $T \models \varphi$ then $T^R \models [\![\varphi]\!] \doteq \top$. In both the first order and continuous settings, $\equiv$ denotes elementary equivalence. We will use $\mathcal{M},\mathcal{H}$ to denote models of the complete first order theory T with universes M and H, and use $\mathcal{N}$ and $\mathcal{P}$ to denote continuous structures or pre-structures with signature L^R. By a **pair in** a pre-structure $\mathcal{N} = (\mathcal{K},\mathcal{B})$ we mean a pair $(\mathbf{f},\mathsf{B})$ such that $\mathbf{f} \in \mathcal{K}$ and $\mathsf{B} \in \mathcal{B}$. We sometimes abuse notation by writing $(\mathbf{f},\mathsf{B}) \in \mathcal{N}$ instead of "$(\mathbf{f},\mathsf{B})$ is a pair in $\mathcal{N}$".

We extend the notions of embedding and elementary embedding to pre-structures in the natural way. Given pre-structures $\mathcal{P}$, $\mathcal{N}$, we write $h : \mathcal{P} \subseteq \mathcal{N}$ if h is a mapping from $\mathcal{P}$ into $\mathcal{N}$ which preserves the truth values of atomic formulas, and $h : \mathcal{P} \prec \mathcal{N}$ (h is an **elementary embedding**) if h preserves the truth values of all formulas. If $h\colon \mathcal{P} \subseteq \mathcal{N}$ where h is the inclusion mapping, we write $\mathcal{P} \subseteq \mathcal{N}$ and say that $\mathcal{P}$ is a **pre-substructure** of $\mathcal{N}$. If $h\colon \mathcal{P} \prec \mathcal{N}$ where h is the inclusion mapping, we write $\mathcal{P} \prec \mathcal{N}$ and say that $\mathcal{P}$ is an **elementary pre-substructure** of $\mathcal{N}$. If $h : \mathcal{P} \subseteq \mathcal{N}$, or even $h\colon \mathcal{P} \prec \mathcal{N}$, h preserves distance but is not necessarily one-to-one. Note that compositions of elementary embeddings are elementary embeddings.

Remark 2.3. It is easily seen that if $\mathcal{H} \prec \mathcal{M}$, then $(\mathcal{H}^{[0,1)},\mathcal{L}) \prec (\mathcal{M}^{[0,1)},\mathcal{L})$.

We write $h\colon \mathcal{P} \cong \mathcal{N}$ if $h : \mathcal{P} \prec \mathcal{N}$ and every element of $\mathcal{N}$ is at distance zero from some element of $h(\mathcal{P})$. We say that $\mathcal{P}$ and $\mathcal{N}$ are **isomorphic**, and write $\mathcal{P} \cong \mathcal{N}$, if $h\colon \mathcal{P} \cong \mathcal{N}$ for some h.

Remark 2.4. The isomorphism relation $\cong$ is an equivalence relation on pre-structures.

Proof. It is clear that the relation $\cong$ is reflexive and transitive. To show that $\cong$ is symmetric, suppose $h : \mathcal{P} \cong \mathcal{N}$. Pick a random element $k(\mathbf{g})$ of $\mathcal{P}$ for each random element $\mathbf{g}$ of $\mathcal{N}$, and an event $k(\mathsf{B})$ of $\mathcal{P}$ for each event B of $\mathcal{N}$, such that $(h(k(\mathbf{g})) \doteq \mathbf{g}$ and $h(k(\mathsf{B})) \doteq \mathsf{B}$. Then $k\colon \mathcal{N} \cong \mathcal{P}$. $\blacksquare_{2.4}$

We say that $\mathcal{N}$ is a **reduction of** $\mathcal{P}$ if $\mathcal{P} \cong \mathcal{N}$ and the distance relations $d_{\mathbf{K}}$, $d_{\mathbf{B}}$ for $\mathcal{N}$ are metrics. For every pre-structure $\mathcal{P}$, one can obtain a reduction of $\mathcal{P}$ by identifying elements that are at distance zero from each other. Note that if $\mathcal{P}$ and $\mathcal{N}$ are reduced structures, then $\mathcal{P} \cong \mathcal{N}$ has the usual meaning, that there is a bijection from $\mathcal{P}$ to $\mathcal{N}$

that preserves truth values of (atomic) formulas. In general, $\mathcal{P}$ is isomorphic to $\mathcal{N}$ if and only if a reduction of $\mathcal{P}$ is isomorphic to a reduction of $\mathcal{N}$.

We call $\mathcal{N}$ a **completion of** $\mathcal{P}$ if $\mathcal{N}$ is a structure obtained from a reduction of $\mathcal{P}$ by completing the metrics. Every pre-structure $\mathcal{P}$ has a completion, and any two completions of $\mathcal{P}$ are isomorphic. Following [BK], we say that $\mathcal{P}$ is **pre-complete** if a reduction of $\mathcal{P}$ is already a structure (i.e., its metrics are already complete).

Fact 2.5. *([BK], Corollary 3.6) For each model $\mathcal{M}$ of T, the Borel randomization $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is pre-complete.*

Fact 2.6. *([BK], Theorem 2.7) Every model or pre-complete model $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ of T^R has perfect witnesses, i.e.,*

(i) for each first order formula $\varphi(x, \vec{y})$ and each $\vec{\mathbf{g}}$ in $\mathcal{K}^n$ there exists $\mathbf{f} \in \mathcal{K}$ such that

$$[\![\varphi(\mathbf{f}, \vec{\mathbf{g}})]\!] \doteq [\![(\exists x\varphi)(\vec{\mathbf{g}})]\!];$$

(ii) for each $\mathsf{B} \in \mathcal{B}$ there exist $\mathbf{f}, \mathbf{g} \in \mathcal{K}$ such that $\mathsf{B} \doteq [\![\mathbf{f} = \mathbf{g}]\!]$.

Corollary 2.7. *Let $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ be a pre-complete model of T^R and let $\mathbf{f}, \mathbf{g} \in \mathcal{K}$ and $\mathsf{B} \in \mathcal{B}$. Then there is an element $\mathbf{h} \in \mathcal{K}$ that agrees with $\mathbf{f}$ on B and agrees with $\mathbf{g}$ on $\neg\mathsf{B}$, that is, $[\![\mathbf{h} = \mathbf{f}]\!] \sqcap \mathsf{B} \doteq \mathsf{B}$ and $[\![\mathbf{h} = \mathbf{g}]\!] \sqcap \neg\mathsf{B} \doteq \neg\mathsf{B}$.*

Proof. By Fact 2.6 (ii), there exist $\mathbf{d}, \mathbf{e} \in \mathcal{K}$ such that $\mathsf{B} \doteq [\![\mathbf{d} = \mathbf{e}]\!]$. The first order sentence

$$(\forall u)(\forall v)(\forall x)(\forall y)(\exists z)[(x = y \to z = u) \wedge (x \neq y \to z = v)]$$

is logically valid, so we must have

$$[\![(\exists z)[(\mathbf{d} = \mathbf{e} \to z = \mathbf{f}) \wedge (\mathbf{d} \neq \mathbf{e} \to z = \mathbf{g})]]\!] \doteq \top.$$

By Fact 2.6 (i) there exists $\mathbf{h} \in \mathcal{K}$ such that

$$[\![\mathbf{d} = \mathbf{e} \to \mathbf{h} = \mathbf{f}]\!] \doteq \top, \quad [\![\mathbf{d} \neq \mathbf{e} \to \mathbf{h} = \mathbf{g}]\!] \doteq \top,$$

so $[\![d = e]\!] \sqsubseteq [\![\mathbf{h} = \mathbf{f}]\!]$ and $[\![\mathbf{d} \neq \mathbf{e}]\!] \sqsubseteq [\![\mathbf{h} = \mathbf{g}]\!]$. $\blacksquare_{2.7}$

Fact 2.8. *(Strong quantifier elimination, [Kei2]. See Theorem 2.9 in [BK]) Every formula Φ in the continuous language L^R is T^R-equivalent to a formula with the same free variables and no quantifiers of sort $\mathbf{K}$ or $\mathbf{B}$.*

It follows that if $\mathcal{P}, \mathcal{N}$ are pre-models of T^R and $\mathcal{P} \subseteq \mathcal{N}$, then $\mathcal{P} \prec \mathcal{N}$. We will also need the following result about pre-substructures of Borel randomizations.

Fact 2.9. *([Kei2]. See Proposition 2.2 in [BK]) If $(\mathcal{K}, \mathcal{B}) \subseteq (\mathcal{M}^{[0,1)}, \mathcal{L})$, $\mathcal{B}$ is atomless, and $(\mathcal{K}, \mathcal{B})$ has perfect witnesses, then $(\mathcal{K}, \mathcal{B})$ is a pre-model of T^R (so $(\mathcal{K}, \mathcal{B}) \prec (\mathcal{M}^{[0,1)}, \mathcal{L})$ by Fact 2.8).*

2.2. **First order types.** We assume familiarity with the first order notion of a type being realized in a structure. $S_n(T)$ is the set of all complete types for T in n variables. Given a first order structure $\mathcal{M}$ and a tuple $\vec{a}$ in M, the structure $\mathcal{M}$ with an added symbol for each element of $\vec{a}$ is denoted by $\mathcal{M}_{\vec{a}}$ or $(\mathcal{M}, \vec{a})$, and the complete theory of $\mathcal{M}_{\vec{a}}$ is denoted by $T_{\vec{a}}$. We say that $\mathcal{M}$ is **countable saturated** if M is countable and for each tuple $\vec{a}$ in M, every type in $S_n(T_{\vec{a}})$ is realized in $\mathcal{M}_{\vec{a}}$. $\mathcal{M}$ is **countable homogeneous** if M is countable, and for all n-tuples $\vec{a}, \vec{b}$ in M, if $\mathcal{M}_{\vec{a}} \equiv \mathcal{M}_{\vec{b}}$ then $(\forall c \in M)(\exists d \in M)\mathcal{M}_{\vec{a},c} \equiv \mathcal{M}_{\vec{b},d}$. Note that if $\mathcal{M}$ is countable saturated or homogeneous, then so is $\mathcal{M}_{\vec{a}}$ for each tuple $\vec{a}$ in M. We will use the following classical results for first order theories with countable signatures.

Fact 2.10. *(Morley and Vaught [MV], and Keisler and Morley [KM]; see also [CK], Section 2.4)*

(i) *Every countable model of T has a countable homogeneous elementary extension.*
(ii) *Any two countable homogeneous models of T that realize the same types are isomorphic.*
(iii) *If $\mathcal{M}$ is countable homogeneous and $\mathcal{M}_{\vec{a}} \equiv \mathcal{M}_{\vec{b}}$ then $\mathcal{M}_{\vec{a}} \cong \mathcal{M}_{\vec{b}}$.*

Fact 2.11. *(Vaught [Va]) T has a countable saturated model if and only if $\bigcup_n S_n(T)$ is countable.*

$L_{\omega_1\omega}$ is the infinitary logic with signature L, countable conjunctions and disjunctions, negations, and the usual quantifiers $\exists x, \forall x$ (for background see [Kei1]). We say that an $L_{\omega_1\omega}$ sentence φ **defines** a countable structure $\mathcal{M}$ with signature L if for every countable structure $\mathcal{H}$ with signature L we have $\mathcal{H} \models \varphi$ if and only if $\mathcal{H} \cong \mathcal{M}$.

Fact 2.12. *(Scott [Sc]) For every countable structure $\mathcal{M}$, there is an $L_{\omega_1\omega}$ sentence φ that defines $\mathcal{M}$.*

2.3. **Continuous types.** For each n-tuple $\vec{\mathbf{f}}$ of elements in a continuous pre-structure $\mathcal{N}$, the type $tp(\vec{\mathbf{f}})$ of $\vec{\mathbf{f}}$ in $\mathcal{N}$ is the function p from formulas to $[0,1]$ such that for each formula $\Phi(\vec{x})$, we have $\Phi(\vec{x})^p = \Phi(\vec{\mathbf{f}})^{\mathcal{N}}$.

By quantifier elimination (Fact 2.8), the n-types in T^R of sort $\mathbf{B}$ do not depend on the theory T at all, and can be identified with the n-types in the continuous theory of atomless measure algebras $(\mathcal{B}, \top, \bot, \sqcap, \sqcup, \neg, \mu(\cdot))$. Formally, we have

Remark 2.13. Let $\mathcal{N}, \mathcal{P}$ be models of T^R and let $\vec{\mathsf{B}}, \vec{\mathsf{C}}$ be tuples of sort $\mathbf{B}$ in $\mathcal{N}$ and $\mathcal{P}$ respectively. Then $tp(\vec{\mathsf{B}}) = tp(\vec{\mathsf{C}})$ if and only if $\mu(\tau(\vec{\mathsf{B}}))^{\mathcal{N}} = \mu(\tau(\vec{\mathsf{C}}))^{\mathcal{P}}$ for every Boolean term τ.

By Fact 2.6 (ii), in a model of T^R we can always replace an element of sort $\mathbf{B}$ by a term $[\![\mathbf{f} = \mathbf{g}]\!]$. Thus every type in T^R of sort $\mathbf{B}$ can be obtained from a type in T^R of sort $\mathbf{K}$. The space of continuous n-types in T^R with variables of sort $\mathbf{K}$ will be denoted by $S_n(T^R)$. For each pre-model $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ of T^R and n-tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, the type $tp(\vec{\mathbf{f}})$ of

$\vec{\mathbf{f}}$ is the unique element $p \in S_n(T^R)$ such that for each first order formula $\varphi(\vec{v})$,

$$(\mu[\![\varphi(\vec{\mathbf{f}})]\!])^{\mathcal{N}} = (\mu[\![\varphi(\vec{v})]\!])^p.$$

We say that a type $p \in S_n(T^R)$ is **realized** in a pre-model $\mathcal{N}$ if we have $p = tp(\vec{\mathbf{f}})$ for some n-tuple $\vec{\mathbf{f}}$ in $\mathcal{N}$. By the Compactness Theorem, every type $p \in S_n(T^R)$ is realized in some separable model of T^R.

A continuous pre-structure $\mathcal{N}$ is ω-**homogeneous** if for every pair of n-tuples $\vec{\mathbf{f}}, \vec{\mathbf{g}}$ in $\mathcal{N}$ which realize the same type in $\mathcal{N}$, and every $\mathbf{h}$ in $\mathcal{N}$, there exists $\mathbf{k}$ in $\mathcal{N}$ such that $(\vec{\mathbf{f}}, \mathbf{h})$ and $(\vec{\mathbf{g}}, \mathbf{k})$ realize the same type in $\mathcal{N}$. We say that $\mathcal{N}$ is **separable homogeneous** if $\mathcal{N}$ is separable and ω-homogeneous.

The paper [BK] gave a useful connection between the types of T^R and the Borel probability measures on the space of types of T. Let $\mathfrak{R}(S_n(T))$ be the set of Borel probability measures on $S_n(T)$.

Fact 2.14. *Every measure $\nu \in \mathfrak{R}(S_n(T))$ is regular, that is, for each Borel set B, the measure of B is approximated above by the measures of open supersets of B, and below by the measures of compact subsets of B.*

Proof. $S_n(T)$ is a compact Polish space. Every Borel probability measure on a compact Polish space is regular (See, for example, [Bi]). $\blacksquare_{2.14}$

Fact 2.15. *([BK], Corollary 2.10) For every $p \in S_n(T^R)$ there is a unique measure $\nu_p \in \mathfrak{R}(S_n(T))$ such that for each formula $\varphi(\vec{v})$ of L,*

$$\nu_p(\{q \in S_n(T) : \varphi(\vec{v}) \in q\}) = (\mu[\![\varphi(\vec{v})]\!])^p.$$

Moreover, for each measure $\nu \in \mathfrak{R}(S_n(T))$ there is a unique $p \in S_n(T^R)$ such that $\nu = \nu_p$.

We will sometimes use Fact 2.15 to build types of T^R.

Example 2.16. For each first order type $q \in S_n(T)$, there is a unique type $q^\star \in S_n(T^R)$ such that ν_{q^*} is the point mass at q, that is, $\nu_{q^\star}(\{q\}) = 1$.

Let $p_0, p_1, \ldots$ be a finite or countable sequence of first-order types in $S_n(T)$ and let $\alpha_0, \alpha_1, \ldots$ be elements of $[0, 1]$ such that $\sum_i \alpha_i = 1$. Then there is a unique type $p \in S_n(T^R)$ such that $\nu_p(\{p_i\}) = \alpha_i$ for each i. We denote this type by

$$p = \sum_i \alpha_i p_i^\star.$$

Remark 2.17. Let $p = \sum_i \alpha_i p_i^\star$ be as in Example 2.16, and let $\mathcal{M}$ be a model of T.

(i) Suppose $\mathcal{N}$ is a model of T^R. If each type $p_i^\star$ is realized in $\mathcal{N}$, then p is realized in $\mathcal{N}$.
(ii) If each type p_i is realized in $\mathcal{M}$, then the type $p = \sum_i \alpha_i p_i^\star$ is realized in $(\mathcal{M}^{[0,1)}, \mathcal{L})$.
(iii) If $\vec{\mathbf{f}}$ is a tuple in $\mathcal{M}^{[0,1)}$, then $tp(\vec{\mathbf{f}}) = \sum_i \alpha_i p_i^\star$ where $\{\vec{a}_0, \vec{a}_1, \ldots\}$ is the range of $\vec{\mathbf{f}}$, and for each i, $p_i = tp(\vec{a}_i)$ and $\alpha_i = \lambda(\{t \colon \vec{\mathbf{f}}(t) = \vec{a}_i\})$.

(iv) In particular, if $\vec{\mathbf{f}}(t)$ has the constant value $\vec{a}$ for all $t \in [0,1)$, then $tp(\vec{\mathbf{f}}) = (tp(\vec{a}))^\star$.

Proof. (i) For each i, let $\vec{\mathbf{f}}_i$ realize the type $p_i^\star$ in $\mathcal{N}$. Using Corollary 2.7 countably many times and taking a limit, we can obtain a family of pairwise disjoint events A_i in $\mathcal{N}$ such that $\mu(\mathsf{A}_i) = \alpha_i$ for each i, and a tuple $\vec{\mathbf{f}}$ in $\mathcal{N}$ such that for each i, $\vec{\mathbf{f}}$ agrees with $\vec{\mathbf{f}}_i$ on A_i. Then $\vec{\mathbf{f}}$ realizes $\sum_i \alpha_i p_i^\star$ in $\mathcal{N}$. (ii)–(iv) follow easily from the definitions involved. $\blacksquare_{2.17}$

3. Prime models

In this section we show that T has a prime model if and only if its randomization theory T^R has a prime model.

Let $\mathcal{N}$ be a first order or continuous structure with a countable signature and let U be the complete theory of $\mathcal{N}$. By definition, $\mathcal{N}$ is **prime** if $\mathcal{N}$ is elementarily embeddable in every model of U. We will call a pre-structure **prime** if its completion is prime. We define an n-type $p \in S_n(U)$ to be **principal** if p is realized in every model of U. For continuous theories, [BBHU] gave a different definition of principal type, but the above definition is equivalent to theirs by Theorem 13.4 in [BBHU]. We use the following results from the literature.

Fact 3.1. *(Vaught [Va]; see also [CK], Section 2.3)*

(i) *A model $\mathcal{M}$ of T is prime if and only if $\mathcal{M}$ is countable and every type which is realized in $\mathcal{M}$ is principal.*
(ii) *Any two prime models of T are isomorphic.*
(iii) *T has a prime model if and only if every formula $\varphi(\vec{v})$ which is consistent with T belongs to a principal type.*
(iv) *A type in $S_n(T)$ is principal if and only if it contains a maximal consistent formula.*

Fact 3.2. *([BBHU], Corollary 13.7.) Let U be a complete continuous theory with a countable signature.*

(i) *A model $\mathcal{N}$ of U is prime if and only if $\mathcal{N}$ is separable and every type which is realized in $\mathcal{N}$ is principal.*
(ii) *Any two prime models of U are isomorphic.*

Lemma 3.3. *Let $\mathcal{M}$ be a countable model of T. $\mathcal{M}$ is prime if and only if $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is prime.*

Proof. Suppose first that $\mathcal{M}$ is not prime. By Fact 3.1 there is a tuple $\vec{a}$ in $\mathcal{M}$ and a countable model $\mathcal{H}$ of T such that the type of $\vec{a}$ in $\mathcal{M}$ is not realized in $\mathcal{H}$. One can then check that the type of the constant function at $\vec{a}$ in $\mathcal{M}^{[0,1)}$ is not realized in $(\mathcal{H}^{[0,1)}, \mathcal{L})$, so by Fact 3.2, $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is not prime.

Now suppose that $\mathcal{M}$ is prime. By Fact 3.2, it is enough to show that for every tuple $\vec{\mathbf{f}}$ in $\mathcal{M}^{[0,1)}$, the type p of $\vec{\mathbf{f}}$ in $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is realized in every model of T^R. By Remark 2.17 (iii), $p = \sum_i \alpha_i p_i^\star$, where $\alpha_i = \lambda(\{t \colon \vec{\mathbf{f}}(t) = \vec{a}_i\})$ and $p_i = tp(\vec{a}_i)$ in $\mathcal{M}$. Since $\mathcal{M}$ is prime, each type p_i is realized in every model of T. By Fact 3.1, each type p_i contains a maximal consistent formula. It follows that $p_i^\star$ is realized in every model of T^R. By Remark 2.17 (i), p is realized in every model of T^R. $\blacksquare_{3.3}$

Theorem 3.4. *(i) T has a prime model if and only if T^R has a prime model.*

(ii) A model $\mathcal{N}$ of T^R is prime if and only if $\mathcal{N}$ is isomorphic to the Borel randomization of a prime model of T.

Proof. (i) Suppose T has a prime model $\mathcal{M}$. Then $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is prime by Lemma 3.3.

For the converse, suppose that T does not have a prime model, but T^R does have a prime model $\mathcal{N}$. We will arrive at a contradiction, completing the proof. By Fact 3.1, there is a formula $\varphi(\vec{v})$ which is consistent with T but does not belong to a principal type. Then $T \models (\exists \vec{v})\varphi(\vec{v})$, so $T^R \models [\![(\exists \vec{v})\varphi(\vec{v})]\!] \doteq \top$. By Fact 2.6, there is a tuple $\vec{\mathbf{f}}$ in $\mathcal{N}$ such that $\mathcal{N} \models \mu([\![\varphi(\vec{\mathbf{f}})]\!]) = 1$. Let $p = tp(\vec{\mathbf{f}})$. Then $\mu([\![\varphi(\vec{v})]\!])^p = 1$. By Fact 3.2, p is principal. Now consider an arbitrary countable model $\mathcal{H}$ of T. Since p is principal, p is realized by some tuple $\vec{\mathbf{g}}$ in $(\mathcal{H}^{[0,1)}, \mathcal{L})$. By Remark 2.17 (iii), $p = \sum_i \alpha_i p_i^\star$ for some sequence of types $p_i \in S_n(T)$ and some sequence of numbers $\alpha_i \in [0,1]$ such that $\sum_i \alpha_i = 1$ (the types p_i need not be distinct). Take an i such that $\alpha_i > 0$. We have $\lambda(\{t \colon tp((\vec{\mathbf{g}})(t)) = p_i\}) \geq \alpha_i > 0$, so p_i is realized in $\mathcal{H}$. Thus p_i is realized in every countable model of T, and hence is a principal type. But since $\mu([\![\varphi(\vec{v})]\!])^p = 1$, and $\vec{\mathbf{g}}$ realizes p in $(\mathcal{H}^{[0,1)}, \mathcal{L})$, $\mathcal{H} \models \varphi(\vec{\mathbf{g}}(t))$ for almost all t. Therefore $\varphi(\vec{v})$ belongs to a principal type p_i. This is a contradiction, and completes the proof of (i).

(ii) Let $\mathcal{N}$ be a model of T^R. By Lemma 3.3, if $\mathcal{N}$ is isomorphic to the Borel randomization of a prime model of T then $\mathcal{N}$ is prime. For the other direction, suppose $\mathcal{N}$ is prime. By (i), T has a prime model $\mathcal{M}$. By Lemma 3.3, $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is a prime model of T^R, and hence by Fact 3.2 (ii), $\mathcal{N} \cong (\mathcal{M}^{[0,1)}, \mathcal{L})$. $\blacksquare_{3.4}$

We will now show that the randomization theory T^R cannot have a minimal prime model. This is a place where the model theory of T^R differs from first order model theory.

Proposition 3.5. *T^R does not have a minimal prime model. In fact, for every prime model $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ of T^R, and any element $\mathsf{B} \in \mathcal{B}$ such that $\mathcal{N} \models 0 < \mu(\mathsf{B}) < 1$, $\mathcal{N}$ has an elementary substructure which does not contain B.*

Proof. By Theorem 3.4, T has a prime model $\mathcal{M}$. By Lemma 3.3 and Fact 3.2 (ii), $\mathcal{N} \cong (\mathcal{M}^{[0,1)}, \mathcal{L})$. It is well-known and easy to see that since $([0,1), \mathcal{L}, \lambda)$ is atomless, for each $\mathsf{B} \in \mathcal{L}$ such that $0 < \lambda(\mathsf{B}) < 1$ there is a λ-atomless σ-algebra $\mathcal{A} \subseteq \mathcal{L}$ such that $\mathcal{A}$ is independent of B with respect to λ. Let $(\mathcal{M}^{\mathcal{A}}, \mathcal{A})$ be the pre-structure where $\mathcal{M}^{\mathcal{A}}$ is the set of $\mathcal{A}$-measurable functions from $[0,1)$ into $\mathcal{M}$. By Example 3.4 in [BK], $(\mathcal{M}^{\mathcal{A}}, \mathcal{A})$ is a full pre-complete randomization of M, thus has perfect witnesses by Proposition 2.5 in [BK]. By Fact 2.9, $(\mathcal{M}^{\mathcal{A}}, \mathcal{A})$ is a pre-model of T^R and $(\mathcal{M}^{\mathcal{A}}, \mathcal{A}) \prec (\mathcal{M}^{[0,1)}, \mathcal{L})$. Since $\mathcal{A}$ is independent of B and $\mu(\mathsf{B}) > 0$, $(\mathcal{M}^{\mathcal{A}}, \mathcal{A})$ does not contain B. $\blacksquare_{3.5}$

4. Strongly Separable Models

Definition 4.1. A pre-model $\mathcal{N}$ of T^R is called **strongly separable** if $\mathcal{N}$ is elementarily embeddable in $(\mathcal{H}^{[0,1)}, \mathcal{L})$ for some countable model $\mathcal{H}$ of T.

From this point on, our main focus will be on strongly separable models of T^R.

By Corollary 3.8 in [BK], every strongly separable pre-model of T^R is separable. Example 3.10 of [BK] gives an example of a theory T and a separable model of T^R that is not strongly separable. More such examples can be found using Corollary 5.3 below.

Theorem 3.12 of [BK] shows that every separable model of T^R is strongly separable if and only if T has a countable saturated model. The results in Sections 3 and 4 of [BK] give some information about the separable pre-models of T^R in the case that T has a countable saturated model. In this section we will obtain results about the strongly separable pre-models of T^R in the general case that T is not assumed to have a countable saturated model. These results will be used in Sections 6 through 9.

Lemma 4.2. *A model $\mathcal{H}$ of T is countable if and only if the Borel randomization $(\mathcal{H}^{[0,1)}, \mathcal{L})$ is separable.*

Proof. Suppose first that $\mathcal{H}$ is countable. Let $\mathcal{A}$ be the algebra generated by the set of all subintervals of $[0, 1)$ with rational endpoints, and let F be the set of $\mathcal{A}$-measurable elements of $\mathcal{H}^{[0,1)}$ with finite range. $\mathcal{A}$ is countable and dense in $\mathcal{L}$. As shown in [BK], Lemma 3.7, F is countable and dense in $\mathcal{H}^{[0,1)}$. Therefore $(\mathcal{H}^{[0,1)}, \mathcal{L})$ is separable.

Now suppose that $\mathcal{H}$ is uncountable. Then the set C of constant functions from $[0, 1)$ into H is an uncountable set of elements of $(\mathcal{H}^{[0,1)}, \mathcal{L})$ such that the distance between any two elements of C is one, so $(\mathcal{H}^{[0,1)}, \mathcal{L})$ is not separable. $\blacksquare_{4.2}$

Proposition 4.3. *A pre-model $\mathcal{N}$ of T^R is strongly separable if and only if $\mathcal{N}$ is separable and $\mathcal{N}$ is elementarily embeddable in the Borel randomization of some model of T.*

Proof. If $\mathcal{N}$ is strongly separable, then by definition, $\mathcal{N}$ is elementarily embeddable in $(\mathcal{H}^{[0,1)}, \mathcal{L})$ for some countable model $\mathcal{H}$ of T. By Corollary 3.8 in [BK], $\mathcal{N}$ is separable. Alternatively, by Lemma 4.2, $(\mathcal{H}^{[0,1)}, \mathcal{L})$ is separable, so $\mathcal{N}$ is separable.

For the other direction, suppose that $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ is separable and $h\colon \mathcal{N} \prec (\mathcal{M}^{[0,1)}, \mathcal{L})$ for some model $\mathcal{M}$ of T. Let $\mathcal{K}_0$ be a countable dense subset of $\mathcal{K}$. Since each $\mathbf{f} \in \mathcal{M}^{[0,1)}$ has countable range in M, there is an elementary submodel $\mathcal{H} \prec \mathcal{M}$ that contains the range of $h(\mathbf{g})$ for each $\mathbf{g} \in \mathcal{K}_0$. Then $h(\mathcal{K}_0) \subseteq \mathcal{H}^{[0,1)}$. It is clear that $\mathcal{H}^{[0,1)}$ is closed in $\mathcal{M}^{[0,1)}$, so $h(\mathcal{K}) \subseteq \mathcal{H}^{[0,1)}$. By Facts 2.2 and 2.8, we have $(\mathcal{H}^{[0,1)}, \mathcal{L}) \prec (\mathcal{M}^{[0,1)}, \mathcal{L})$, so $h\colon \mathcal{N} \prec (\mathcal{H}^{[0,1)}, \mathcal{L})$ and thus $\mathcal{N}$ is strongly separable. $\blacksquare_{4.3}$

To clarify the relationships between different strongly separable pre-models of T^R, we will fix once and for all a model $\mathcal{M}_\infty$ of T such that every countable model of T is elementarily embeddable in $\mathcal{M}_\infty$. The existence of such a model follows from the Compactness Theorem. We denote the Borel randomization of $\mathcal{M}_\infty$ by

$$\mathcal{N}_\infty = (\mathcal{M}_\infty^{[0,1)}, \mathcal{L}).$$

It follows from Proposition 4.3 that a pre-model $\mathcal{N}$ of T^R is strongly separable if and only if it is separable and elementarily embeddable in $\mathcal{N}_\infty$.

For each pre-substructure $\mathcal{N} \subseteq \mathcal{N}_\infty$, we will now construct three associated pre-structures, the reduction $\mathcal{N}^o$ of $\mathcal{N}$ in $\mathcal{N}_\infty$, the completion $\mathcal{N}^\wedge$ of $\mathcal{N}$, and the closure $\mathcal{N}^{cl}$ of $\mathcal{N}$ in $\mathcal{N}_\infty$. Up to isomorphism, these pre-structures will not depend on our choice of $\mathcal{M}_\infty$.

We first introduce the **reduction mapping** o on $\mathcal{N}_\infty$. For each random element $\mathbf{f} \in \mathcal{M}_\infty^{[0,1)}$ and event $\mathsf{A} \in \mathcal{L}$, let $\mathbf{f}^o$ and A^o be the equivalence classes

$$\mathbf{f}^o = \{\mathbf{g} \in \mathcal{M}_\infty^{[0,1)} \colon d_{\mathbf{K}}(\mathbf{f}, \mathbf{g}) = 0\}, \quad \mathsf{A}^o = \{\mathsf{B} \in \mathcal{L} \colon d_{\mathbf{B}}(\mathsf{A}, \mathsf{B}) = 0\}.$$

We let $(\mathcal{N}_\infty)^o$ be the pre-structure obtained from $\mathcal{N}_\infty$ by replacing each pair in $(\mathbf{f}, \mathsf{A}) \in \mathcal{N}_\infty$ by $(\mathbf{f}^o, \mathsf{A}^o)$. Note that $(\mathcal{N}_\infty)^o$ is a reduction of $\mathcal{N}_\infty$. By Fact 2.5, $\mathcal{N}_\infty$ is pre-complete, so $(\mathcal{N}_\infty)^o$ is also a completion of $\mathcal{N}_\infty$, and is thus a model of T^R.

When we apply the reduction mapping o to a pre-substructure $\mathcal{N} = (\mathcal{K}, \mathcal{B}) \subseteq \mathcal{N}_\infty$, we obtain a reduction[2] $\mathcal{N}^o = (\mathcal{K}^o, \mathcal{B}^o)$ of $\mathcal{N}$, which we call **the reduction of $\mathcal{N}$ in $\mathcal{N}_\infty$**. The advantage of working within the same pre-structure $\mathcal{N}_\infty$ is that we always have $\mathcal{P} \subseteq \mathcal{N} \subseteq \mathcal{N}_\infty$ implies $\mathcal{P}^o \subseteq \mathcal{N}^o$, and $\mathcal{P} \prec \mathcal{N} \prec \mathcal{N}_\infty$ implies $\mathcal{P}^o \prec \mathcal{N}^o$.

For each $\mathcal{N} = (\mathcal{K}, \mathcal{B}) \subseteq \mathcal{N}_\infty$, the **completion** of $\mathcal{N}$ in $(\mathcal{N}_\infty)^o$ is the continuous structure $\mathcal{N}^\wedge = (\mathcal{K}^\wedge, \mathcal{B}^\wedge)$ whose universe is the closure of the universe of $\mathcal{N}^o$ in $(\mathcal{N}_\infty)^o$. Thus $\mathcal{N}$ is pre-complete if and only if $\mathcal{N}^\wedge = \mathcal{N}^o$. In particular we have $(\mathcal{N}_\infty)^\wedge = (\mathcal{N}_\infty)^o$. We also let $(\mathcal{L}^o, \lambda^o)$ be the Lebesgue measure algebra, where λ^o is the unique measure on $\mathcal{L}^o$ such that $\lambda^o(\mathsf{B}^o) = \lambda(\mathsf{B})$ for each $\mathsf{B} \in \mathcal{L}$.

For each $\mathcal{N} \subseteq \mathcal{N}_\infty$, the **closure** of $\mathcal{N}$ in $\mathcal{N}_\infty$ is the pre-structure $\mathcal{N}^{cl} \subseteq \mathcal{N}_\infty$ with universe sets

$$\mathcal{K}^{cl} = \{\mathbf{f} \in \mathcal{M}_\infty^{[0,1)} \colon \mathbf{f}^o \in \mathcal{K}^\wedge\}, \quad \mathcal{B}^{cl} = \{\mathsf{B} \in \mathcal{L} \colon \mathsf{B}^o \in \mathcal{B}^\wedge\}.$$

We say that $\mathcal{P}$ is **dense in** $\mathcal{N}$ if $\mathcal{P} \subseteq \mathcal{N} \subseteq \mathcal{P}^{cl}$. In particular, $\mathcal{N}$ is dense in $\mathcal{N}^{cl}$.

In the next remark we collect some easy observations about $\mathcal{N}$, $\mathcal{N}^o$, $\mathcal{N}^{cl}$, and $\mathcal{N}^\wedge$.

Remark 4.4. For all pre-substructures $\mathcal{P}$, $\mathcal{N} \subseteq \mathcal{N}_\infty$ we have

(i) $o : \mathcal{N} \cong \mathcal{N}^o$ and $\mathcal{N}^o \prec \mathcal{N}^\wedge$.
(ii) If $\mathcal{P}$ is dense in $\mathcal{N}$, then $\mathcal{P}^{cl} = \mathcal{N}^{cl}$, $\mathcal{P}^\wedge = \mathcal{N}^\wedge$, and $\mathcal{P} \prec \mathcal{N}$. In particular, $\mathcal{P} \prec \mathcal{P}^{cl}$.
(iii) If $\mathcal{P} \subseteq \mathcal{N}$, then the following are equivalent:

$$\mathcal{P} \prec \mathcal{N}, \qquad \mathcal{P}^o \prec \mathcal{N}^o, \qquad \mathcal{P}^\wedge \prec \mathcal{N}^\wedge, \qquad \mathcal{P}^{cl} \prec \mathcal{N}^{cl}.$$

(iv) $\mathcal{N}^{cl}$ is pre-complete. If $\mathcal{N}$ is pre-complete, then $\mathcal{N} \cong \mathcal{N}^{cl}$.
(v) If $\mathcal{N} \prec \mathcal{N}_\infty$, then $\mathcal{N}^\wedge = (\mathcal{N}^{cl})^\wedge$ and $\mathcal{N}^\wedge$ is a model of T^R.
(vi) If $\mathcal{N} \prec \mathcal{N}_\infty$ and $\mathcal{N}$ is separable, then $\mathcal{N}$, $\mathcal{N}^{cl}$, $\mathcal{N}^o$, and $\mathcal{N}^\wedge$ are strongly separable.

[2] The paper [BK] used the notation $\overline{\mathcal{N}}$ for the reduced pre-structure of $\mathcal{N}$ in $\mathcal{N}_\infty$.

Proof. Part (i) follows from Theorem 3.7 in [BBHU].

(ii) First check that $\mathcal{K}^{cl}$ is the topological closure of $\mathcal{K}$ in $\mathcal{M}^{[0,1)}_{\infty}$, and $\mathcal{B}^{cl}$ is the topological closure of $\mathcal{B}$ in $\mathcal{L}$. It follows that $\mathcal{P}^{cl} = \mathcal{N}^{cl}$ and $\mathcal{P}^{\wedge} = \mathcal{N}^{\wedge}$. Now apply (i) to get $\mathcal{P} \prec \mathcal{N}$.

(iii) Observe that by (ii) we have $\mathcal{P} \prec \mathcal{P}^{cl}$ and $\mathcal{N} \prec \mathcal{N}^{cl}$. The equivalences follow from that observation and (i).

(iv) Suppose $(\mathbf{g}, \mathsf{B}) \in (\mathcal{N}^{cl})^{\wedge}$. Then $(\mathbf{g}, \mathsf{B}) \in (\mathcal{N}_{\infty})^{\wedge}$. By Fact 2.5, $\mathcal{N}_{\infty}$ is pre-complete, so $(\mathbf{g}, \mathsf{B}) \in (\mathcal{N}_{\infty})^{o}$, and there exists $(\mathbf{f}, \mathsf{A}) \in \mathcal{N}_{\infty}$ such that $(\mathbf{g}, \mathsf{B}) = (\mathbf{f}, \mathsf{A})^{o}$. Hence $(\mathbf{f}, \mathsf{A}) \in \mathcal{N}^{cl}$ and $(\mathbf{g}, \mathsf{B}) \in (\mathcal{N}^{cl})^{o}$, so $\mathcal{N}^{cl}$ is pre-complete.

If $\mathcal{N}$ is pre-complete, then $(\mathcal{N}^{cl})^{o} = \mathcal{N}^{\wedge} = \mathcal{N}^{o}$, so $\mathcal{N} \cong \mathcal{N}^{cl}$.

(v) Since $\mathcal{N}$ is dense in $\mathcal{N}^{cl}$, (ii) gives $\mathcal{N}^{\wedge} = (\mathcal{N}^{cl})^{\wedge}$. (iii) gives $\mathcal{N}^{\wedge} \prec (\mathcal{N}_{\infty})^{\wedge}$, and since $\mathcal{N}^{\wedge}$ is complete it is a model of T^{R}.

(vi) Suppose $\mathcal{N} \prec \mathcal{N}_{\infty}$. Then by (iii), $\mathcal{N}^{cl} \prec (\mathcal{N}_{\infty})^{cl} = \mathcal{N}_{\infty}$. By (i) and (iii), $\mathcal{N}^{o} \prec \mathcal{N}^{\wedge} \prec (\mathcal{N}_{\infty})^{\wedge}$. By (i) and Fact 2.5, $\mathcal{N}_{\infty} \cong (\mathcal{N}_{\infty})^{o} = (\mathcal{N}_{\infty})^{\wedge}$. By Remark 2.4, $\cong$ is symmetric, so $(\mathcal{N}_{\infty})^{\wedge} \cong \mathcal{N}_{\infty}$. Therefore $\mathcal{N}^{o}$ and $\mathcal{N}^{\wedge}$ are elementarily embeddable in $\mathcal{N}_{\infty}$. By Proposition 4.3, if $\mathcal{N}$ is separable then $\mathcal{N}$, $\mathcal{N}^{cl}$, $\mathcal{N}^{o}$, and $\mathcal{N}^{\wedge}$ are strongly separable. $\blacksquare_{4.4}$

The next theorem shows that every pre-complete strongly separable model of T^{R} is isomorphic to one whose event sort is all of $\mathcal{L}$.

Theorem 4.5. *Let $\mathcal{N} = (\mathcal{K}, \mathcal{B})$. Suppose $\mathcal{N}$ is pre-complete and elementarily embeddable in $(\mathcal{M}^{[0,1)}, \mathcal{L})$. Then $\mathcal{N}$ is isomorphic to an elementary pre-submodel of $(\mathcal{M}^{[0,1)}, \mathcal{L})$ whose event sort is all of $\mathcal{L}$.*

Proof. The elementary embedding of $\mathcal{N}$ into $(\mathcal{M}^{[0,1)}, \mathcal{L})$ induces an elementary embedding

$$g : \mathcal{N}^{o} \prec (\mathcal{M}^{[0,1)}, \mathcal{L})^{o}.$$

Since $\mathcal{N}$ is pre-complete, the restriction of the Lebesgue measure algebra $(\mathcal{L}^{o}, \lambda^{o})$ to $g(\mathcal{B}^{o})$ is a measure algebra $(g(\mathcal{B}^{o}), \lambda^{o})$. Since $\mathcal{N}$ is elementarily embeddable in $(\mathcal{M}^{[0,1)}, \mathcal{L})$, $(g(\mathcal{B}^{o}), \lambda^{o})$ is atomless. In both $(g(\mathcal{B}^{o}), \lambda^{o})$ and $(\mathcal{L}^{o}, \lambda^{o})$, every ideal is countably generated. Therefore, by Maharam's theorem ([Mah]), these measure algebras are isomorphic. That is, there is a measure-preserving Boolean isomorphism

$$h : (g(\mathcal{B}^{o}), \lambda^{o})) \cong (\mathcal{L}^{o}, \lambda^{o}).$$

h can be extended to an elementary embedding $\overline{h} : g(\mathcal{N}^{o}) \prec (\mathcal{M}^{[0,1)}, \mathcal{L})^{o}$ as follows. For each $\mathsf{A} \in g(\mathcal{B}^{o})$, let $\overline{h}(\mathsf{A}) = h(\mathsf{A})$. For each $\mathbf{f} \in g(\mathcal{K}^{o})$, let $\overline{h}(\mathbf{f})$ be the unique element of $(\mathcal{M}^{[0,1)}, \mathcal{L})^{o}$ such that

$$(\forall c \in \mathcal{M})[\![\overline{h}(\mathbf{f}) = \mathbf{c}]\!] = h([\![\mathbf{f} = \mathbf{c}]\!]^{o})$$

where $\mathbf{c}$ is the reduction of the constant function with value c. Then for every first order formula $\varphi(\vec{x})$ and tuple $\vec{\mathbf{f}}$ in $g(\mathcal{K})$,

$$[\![\varphi(\overline{h}(\vec{\mathbf{f}}))]\!] = h([\![\varphi(\vec{\mathbf{f}})]\!]^{o}).$$

It follows that the composition $\overline{h} \circ g$ is an elementary embedding

$$\overline{h} \circ g : \mathcal{N}^o \prec (\mathcal{M}^{[0,1)}, \mathcal{L})^o$$

such that $(\overline{h} \circ g)(\mathcal{B}^o) = \mathcal{L}^o$. Since $\mathcal{N}$ is pre-complete, we have

$$\mathcal{N} \cong \mathcal{N}^o \cong (\overline{h} \circ g)(\mathcal{N}^o) = ((\overline{h} \circ g)(\mathcal{K}^o), \mathcal{L}^o) \cong (\mathcal{K}', \mathcal{L})$$

where $\mathcal{K}'$ is the set of $\mathbf{f} \in \mathcal{M}^{[0,1)}$ such that $\mathbf{f}^o \in (\overline{h} \circ g)(\mathcal{K}^o)$. $\blacksquare_{4.5}$

Definition 4.6. Let $\mathcal{H}$ be a countable model of T. We say that $(\mathcal{K}, \mathcal{A})$ is a **countable part of** $(\mathcal{H}^{[0,1)}, \mathcal{L})$ if $(\mathcal{K}, \mathcal{A})$ is countable, $(\mathcal{K}, \mathcal{A}) \prec (\mathcal{H}^{[0,1)}, \mathcal{L})$, $(\mathcal{K}, \mathcal{A})$ has perfect witnesses, and $\mathbf{f}^{-1}(c) \in \mathcal{A}$ for each $\mathbf{f} \in \mathcal{K}$ and $c \in H$ (that is, $\mathbf{f}$ is $\mathcal{A}$-measurable). For each $t \in [0, 1)$, we let $\mathcal{K}(t)$ be the substructure of $\mathcal{H}$ with universe $K(t) = \{\mathbf{f}(t) \colon \mathbf{f} \in \mathcal{K}\}$.

Note that by Remark 4.4, if $\mathcal{P}$ is a countable part of $(\mathcal{H}^{[0,1)}, \mathcal{L})$ then

$$\mathcal{P} \prec \mathcal{P}^{cl} \prec (\mathcal{H}^{[0,1)}, \mathcal{L}).$$

We now prove a collection of lemmas that will allow us to prove things about an arbitrary strongly separable model $\mathcal{N}'$ of T^R by first using Theorem 4.5 to get a pre-structure $\mathcal{N} \prec (\mathcal{H}^{[0,1)}, \mathcal{L})$ such that $\mathcal{N} \cong \mathcal{N}'$ and $\mathcal{N}$ has event sort $\mathcal{L}$, then taking a countable part $(\mathcal{K}, \mathcal{A})$ of $(\mathcal{H}^{[0,1)}, \mathcal{L})$ that is dense in $\mathcal{N}$, and working with the first order structures $\mathcal{K}(t), t \in [0, 1)$. These lemmas will be used repeatedly in Sections 6 through 9.

Lemma 4.7. *Suppose $\mathcal{H}$ is a countable model of T, $\mathcal{N} \prec (\mathcal{H}^{[0,1)}, \mathcal{L})$, and $\mathcal{N}$ has perfect witnesses. Then there exists a countable part $(\mathcal{K}, \mathcal{A})$ of $(\mathcal{H}^{[0,1)}, \mathcal{L})$ that is dense in $\mathcal{N}$.*

Proof. By Proposition 4.3, $\mathcal{N}$ is separable. Therefore there is a countable pre-structure $(\mathcal{K}_0, \mathcal{A}_0) \subseteq \mathcal{N}$ that is dense in $\mathcal{N}$. By listing the first order formulas and using the fact that $\mathcal{N}$ has perfect witnesses, we can construct a chain of countable pre-structures $(\mathcal{K}_n, \mathcal{A}_n), n \in \mathbb{N}$ such that for each n:

- $(\mathcal{K}_n, \mathcal{A}_n) \subseteq (\mathcal{K}_{n+1}, \mathcal{A}_{n+1}) \subseteq \mathcal{N}$;
- for each first order formula $\varphi(x, \vec{y})$ and tuple $\vec{\mathbf{g}}$ in $\mathcal{K}_n$ there exists $\mathbf{f} \in \mathcal{K}_{n+1}$ such that
$$[\![\varphi(\mathbf{f}, \vec{\mathbf{g}})]\!] \doteq [\![(\exists x \varphi)(\vec{\mathbf{g}})]\!];$$
- For each $\mathsf{B} \in \mathcal{A}_n$ there exist $\mathbf{f}, \mathbf{g} \in \mathcal{K}_{n+1}$ such that $\mathsf{B} \doteq [\![\mathbf{f} = \mathbf{g}]\!]$;
- $(\forall c \in H)(\forall \mathbf{f} \in \mathcal{K}_n) \{t : \mathbf{f}(t) = c\} \in \mathcal{A}_{n+1}$.

The union

$$(\mathcal{K}, \mathcal{A}) = \bigcup_n (\mathcal{K}_n, \mathcal{A}_n)$$

is a countable part of $(\mathcal{H}^{[0,1)}, \mathcal{L})$, and $(\mathcal{K}, \mathcal{A})$ is dense in $\mathcal{N}$. $\blacksquare_{4.7}$

Lemma 4.8. *If $\mathcal{H}$ is a countable model of T, and $(\mathcal{K}, \mathcal{A})$ is a countable part of $(\mathcal{H}^{[0,1)}, \mathcal{L})$, then $\mathcal{K}(t) \prec \mathcal{H}$ for almost all t.*

Proof. We note that for each $a \in H$, the set

$$\{t\colon a \in K(t)\}$$

is Borel, because it is equal to the countable union of the sets $\mathbf{h}^{-1}(a)$ where $\mathbf{h} \in \mathcal{K}$. Since $(\mathcal{K}, \mathcal{A}) \prec (\mathcal{H}^{[0,1)}, \mathcal{L})$, for each first order formula $\varphi(\vec{u}, v)$ and each tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, we have

$$\mu([\![(\exists v)\varphi(\vec{\mathbf{f}}, v)]\!]) = \sup_{\mathbf{g} \in \mathcal{K}} \mu([\![\varphi(\vec{\mathbf{f}}, \mathbf{g})]\!]).$$

Therefore for each $\varphi(\vec{u}, v)$ and each $\vec{a} \in H^n$, for almost all t we have

$$\text{if } \vec{a} \in K(t)^n \text{ and } \mathcal{H} \models (\exists v)\varphi(\vec{a}, v) \text{ then } (\exists b \in K(t))\mathcal{H} \models \varphi(\vec{a}, b).$$

Since H is countable, it follows that for almost all t, for every formula $\varphi(\vec{u}, v)$ and every $\vec{a} \in H^n$ we have

$$\text{if } \vec{a} \in K(t)^n \text{ and } \mathcal{H} \models (\exists v)\varphi(\vec{a}, v) \text{ then } (\exists b \in K(t))\mathcal{H} \models \varphi(\vec{a}, b).$$

Then by the Tarski-Vaught condition, we have $\mathcal{K}(t) \prec \mathcal{H}$ for almost all t. $\blacksquare_{4.8}$

Lemma 4.9. *Suppose $\mathcal{H}$ is a countable model of T, and $(\mathcal{K}, \mathcal{A})$ is a countable part of $(\mathcal{H}^{[0,1)}, \mathcal{L})$. Let*

$$\mathcal{K}^+ = \{\mathbf{f} \in \mathcal{H}^{[0,1)} \,:\, \mathbf{f}(t) \in K(t) \textit{ for almost all } t\}.$$

Then $\mathcal{K}^{cl} \subseteq \mathcal{K}^+$. If $\mathcal{A}$ is dense in $\mathcal{L}$, then $\mathcal{K}^{cl} = \mathcal{K}^+$.

The hypothesis that $\mathcal{A}$ is dense in $\mathcal{L}$ is necessary in Lemma 4.9. One can easily get an example to show this using Proposition 3.5 and Lemma 4.7.

Proof of Lemma 4.9. Let $\mathbf{g} \in \mathcal{K}^{cl}$. Then there is a sequence $\mathbf{g}_n$ of elements of $\mathcal{K}$ that converges to $\mathbf{g}$. Therefore for each $r > 0$ there is an $n_r \in \mathbb{N}$ such that $d_{\mathbf{K}}(\mathbf{g}_{n_r}, \mathbf{g}) < r$. Then

$$\lambda(\{t \,:\, \mathbf{g}(t) = \mathbf{g}_{n_r}(t)\}) > 1 - r \text{ and } \{t \,:\, \mathbf{g}(t) \in K(t)\} \supseteq \{t \,:\, \mathbf{g}(t) = \mathbf{g}_{n_r}(t)\}.$$

Since this holds for every $r > 0$, we have $\mathbf{g} \in \mathcal{K}^+$. This shows that $\mathcal{K}^+ \supseteq \mathcal{K}^{cl}$.

We now assume that $\mathcal{A}$ is dense in $\mathcal{L}$, and prove that $\mathcal{K}^+ \subseteq \mathcal{K}^{cl}$. For each element $a \in H$, let $\mathbf{a}$ be the constant function $\mathbf{a}(t) = a$ in $\mathcal{H}^{[0,1)}$. We show next that there is a function in $\mathcal{K}^{cl}$ that agrees with $\mathbf{a}(t)$ whenever possible. Pick an element $\mathbf{h} \in \mathcal{K}$.

Claim 1. For each $a \in H$, the set

$$\mathsf{B}_a = \{t \,:\, a \in K(t)\}$$

is Borel, and the function

$$\mathbf{a_h}(t) = \begin{cases} \mathbf{a}(t), & \text{when } t \in \mathsf{B}_a \\ \mathbf{h}(t) & \text{otherwise} \end{cases}$$

belongs to $\mathcal{K}^{cl}$.

Proof of Claim 1: Let $a \in H$. For each $\mathbf{f} \in \mathcal{K}$, let

$$\mathsf{B}_{a,\mathbf{f}} = \{t \in [0,1) : \mathbf{f}(t) = a\}.$$

List the elements of $\mathcal{K}$, $\mathcal{K} = \{\mathbf{f}_1, \mathbf{f}_2, \ldots\}$. For each n, $\mathbf{f}_n$ is $\mathcal{A}$-measurable, so $\mathsf{B}_{a,\mathbf{f}_n} \in \mathcal{A}$. Hence the set

$$\mathsf{B}_a = \bigcup_n \mathsf{B}_{a,\mathbf{f}_n} = \{t : a \in K(t)\}$$

is Borel.

Let $\mathbf{a}_0 = \mathbf{h}$, and inductively define $\mathbf{a}_n$ by

$$\mathbf{a}_n(t) = \begin{cases} \mathbf{f}_n(t) & \text{when } t \in \mathsf{B}_{a,\mathbf{f}_n} \\ \mathbf{a}_{n-1}(t) & \text{otherwise.} \end{cases}$$

Since $(\mathcal{K}, \mathcal{A})$ is a pre-model of T^R, we see by induction on n that $\mathbf{a}_n$ belongs to $\mathcal{K}$. Moreover, we have

$$\mathbf{a}_n(t) = \begin{cases} \mathbf{a}(t) & \text{when } t \in \bigcup_{m \le n} \mathsf{B}_{a,\mathbf{f}_m} \\ \mathbf{h}(t) & \text{otherwise.} \end{cases}$$

Therefore, for all t we have $\mathbf{a}_n(t) = \mathbf{a}_{\mathbf{h}}(t)$ for all sufficiently large n. It follows that $d_{\mathbf{K}}(\mathbf{a}_n, \mathbf{a}_{\mathbf{h}})$ converges to zero, so Claim 1 holds.

We now let $\mathbf{g}' \in \mathcal{K}^+$ and show that $\mathbf{g}' \in \mathcal{K}^{cl}$. There is a set $\mathsf{B} \in \mathcal{L}$ such that $\lambda(\mathsf{B}) = 1$ and $\mathbf{g}'(t) \in K(t)$ for all $t \in \mathsf{B}$. Define $\mathbf{g}$ by

$$\mathbf{g}(t) = \begin{cases} \mathbf{g}'(t) & \text{when } t \in \mathsf{B}, \\ \mathbf{h}(t) & \text{otherwise.} \end{cases}$$

Then $\mathbf{g} \in \mathcal{H}^{[0,1)}$, $\mathbf{g}(t) \in \mathcal{K}(t)$ for all t, and $d_{\mathbf{K}}(\mathbf{g}, \mathbf{g}') = 0$. So to prove $\mathbf{g}' \in \mathcal{K}^{cl}$ it suffices to find a sequence $\mathbf{g}_n, n \in \mathbb{N}$ of elements of $\mathcal{K}^{cl}$ such that $d_{\mathbf{K}}(\mathbf{g}_n, \mathbf{g})$ converges to 0.

Let $H = \{a^1, a^2, \ldots\}$. Let $\mathbf{g}_0 = \mathbf{h}$, and inductively define $\mathbf{g}_n$ by

$$\mathbf{g}_n(t) = \begin{cases} \mathbf{a}^n_{\mathbf{h}}(t) & \text{when } \mathbf{g}(t) = a^n, \\ \mathbf{g}_{n-1}(t) & \text{otherwise.} \end{cases}$$

Claim 2. $\mathbf{g}_n$ belongs to $\mathcal{K}^{cl}$ for each n.

Proof of Claim 2: We argue by induction on n. We have $\mathbf{g}_0 = \mathbf{h} \in \mathcal{K}$ by hypothesis. Suppose that $\mathbf{g}_{n-1} \in \mathcal{K}^{cl}$. Since $\mathcal{A}$ is dense in $\mathcal{L}$, we have

$$(\mathcal{K}, \mathcal{A}) \prec (\mathcal{K}^{cl}, \mathcal{A}^{cl}) = (\mathcal{K}^{cl}, \mathcal{L}) \prec (\mathcal{H}^{[0,1)}, \mathcal{L}).$$

Therefore $(\mathcal{K}^{cl}, \mathcal{L})$ is a pre-model of T^R. By Claim 1, $\mathbf{a}^n_{\mathbf{h}}$ belongs to $\mathcal{K}^{cl}$. Since $\mathbf{g} \in \mathcal{H}^{[0,1)}$ we have $\{t : \mathbf{g}(t) = a^n\} \in \mathcal{L}$. It follows that $\mathbf{g}_n \in \mathcal{K}^{cl}$, and Claim 2 is proved.

For each $n > 0$, if $t \in \mathsf{B}_{a^n}$ and $\mathbf{g}(t) = a^n$ then for all $m \ge n$ we have

$$\mathbf{g}(t) = a^n = \mathbf{a}^n_{\mathbf{h}}(t) = \mathbf{g}_m(t).$$

Therefore for each $m > 0$,

$$[\![\mathbf{g}_m = \mathbf{g}]\!] \supseteq \{t : \mathbf{g}(t) \in \{a^1, \ldots, a^m\}\} \cap \bigcup_{n=1}^{m} \mathsf{B}_{a^n}.$$

Hence $\lim_{m\to\infty} \mu([\![\mathbf{g}_m = \mathbf{g}]\!]) = 1$, so $\lim_{m\to\infty} d_{\mathbf{K}}(\mathbf{g}_m, \mathbf{g}) = 0$ and $\mathbf{g} \in \mathcal{K}^{cl}$. $\blacksquare_{4.9}$

Lemma 4.10. *Suppose $\mathcal{H}$ is a countable model of T, $(\mathcal{K}, \mathcal{A})$ and $(\mathcal{K}', \mathcal{A}')$ are countable parts of $(\mathcal{H}^{[0,1)}, \mathcal{L})$, and $\mathcal{K}$ and $\mathcal{K}'$ have the same closure in $\mathcal{H}^{[0,1)}$. Then $K(t) = K'(t)$ for almost all t.*

Proof. We will show that for each $a \in H$, the statement [$a \in K(t)$ if and only if $a \in K'(t)$] holds for almost all t. Since H is countable, this will imply that $K(t) = K'(t)$ for almost all t.

Suppose that, on the contrary, there is an $a \in H$ and a Borel set B of positive measure such that for all $t \in \mathsf{B}$, $a \in K'(t) \setminus K(t)$. Then there is an element $\mathbf{f} \in \mathcal{K}'$ and a Borel set $\mathsf{C} \subseteq \mathsf{B}$ such that $\lambda(\mathsf{C}) > 0$ and for all $t \in \mathsf{C}$, $\mathbf{f}(t) = a$. Since $\mathcal{K}$ is dense in the closure of $\mathcal{K}'$, there is an element $\mathbf{g} \in \mathcal{K}$ such that $d_{\mathbf{K}}(\mathbf{f}, \mathbf{g}) < \lambda(\mathsf{C})$. But then there must exist $t \in \mathsf{C}$ such that $\mathbf{g}(t) = \mathbf{f}(t) = a$, so $a \in K(t)$. This contradiction completes the proof. $\blacksquare_{4.10}$

Lemma 4.11. *Suppose $\mathcal{H}$ is a countable model of T and $(\mathcal{K}, \mathcal{A})$ is a countable part of $(\mathcal{H}^{[0,1)}, \mathcal{L})$. Then for each tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$ and each $L_{\omega_1\omega}$ formula $\varphi(\vec{v})$, the set*

$$[\![\varphi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = \{t \colon \mathcal{K}(t) \models \varphi(\vec{\mathbf{f}}(t))\}.$$

belongs to the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$.

Proof. We argue by induction on the complexity of the formula φ. When φ is a first order formula, we have

$$[\![\varphi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} \in \mathcal{A}.$$

Suppose the result holds for all subformulas of φ. If $\varphi = \bigwedge_k \psi_k$, then

$$[\![\varphi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = [\![\bigwedge_k \psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = \bigcap_k [\![\psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{K}},$$

which is a countable intersection of sets in $\sigma(\mathcal{A})$ and hence belongs to $\sigma(\mathcal{A})$. If $\varphi = \neg\psi$, then

$$[\![\varphi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = [\![\neg\psi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = \neg[\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} \in \sigma(\mathcal{A}).$$

Finally, if $\varphi = (\exists v)\psi$, then

$$[\![\varphi(\vec{\mathbf{f}})]\!]^{\mathcal{K}} = [\![(\exists v)\psi(\vec{\mathbf{f}}, v)]\!]^{\mathcal{K}} = \bigcup_{\mathbf{g}\in\mathcal{K}} [\![\psi(\vec{\mathbf{f}}, \mathbf{g})]\!]^{\mathcal{K}},$$

which is a countable union of sets in $\sigma(\mathcal{A})$ and hence belongs to $\sigma(\mathcal{A})$. $\blacksquare_{4.11}$

5. Purely Atomic Types

Definition 5.1. We call a type $p \in S_n(T^R)$ **purely atomic** if there is a finite or countable set $\{q_0, q_1, \ldots\} \subseteq S_n(T)$ such that $\sum_i \nu_p(\{q_i\}) = 1$. A type $p \in S_n(T^R)$ is called **atomless** if the corresponding measure ν_p is atomless.

Note that no type can be both purely atomic and atomless. If in $\mathcal{N}$, $tp(\mathbf{f})$ is purely atomic, $tp(\mathbf{g})$ is atomless, and $\mu([\![\mathbf{h} = \mathbf{f}]\!]) = \mu([\![\mathbf{h} = \mathbf{g}]\!]) = 1/2$, then $tp(\mathbf{h})$ is neither purely atomic nor atomless.

The next proposition relates purely atomic types to strongly separable models of T^R.

Proposition 5.2. *For each type $p \in S_n(T^R)$, the following are equivalent.*

(i) *p is purely atomic.*
(ii) *p is realized in some strongly separable model of T^R.*
(iii) *p is realized in the Borel randomization of some model of T.*

Proof. Assume (i). There is a finite or countable set of types $\{q_0, q_1, \ldots\} \subseteq S_n(T)$ such that $\sum_i \nu_p(q_i) = 1$. By compactness, there is a countable model $\mathcal{H}$ of T such that for each i, q_i is realized by some n-tuple $\vec{a}_i$ in $\mathcal{H}$. Let $\vec{\mathbf{f}}$ be an n-tuple in $\mathcal{H}^{[0,1)}$ such that $\lambda([\![\vec{\mathbf{f}} = \vec{a}_i]\!]) = \nu_p(q_i)$ for each i. Then $\vec{\mathbf{f}}$ realizes p in the strongly separable pre-model $(\mathcal{H}^{[0,1)}, \mathcal{L})$ of T^R, so (ii) holds.

The implication from (ii) to (iii) is trivial. Assume (iii). Then some n-tuple $\vec{\mathbf{f}}$ realizes p in the Borel randomization $(\mathcal{M}^{[0,1)}, \mathcal{L})$ of some model $\mathcal{M}$ of T. By definition, the range of $\vec{\mathbf{f}}$ is a countable set $\{\vec{a}_0, \vec{a}_1, \ldots\} \subseteq M^n$. Therefore $\sum_i \nu_p(\{tp(\vec{a}_i)\}) = 1$, so p is purely atomic and (i) holds. $\blacksquare_{5.2}$

Corollary 5.3. *Suppose p is an atomless type in T^R. Then p can be realized in a separable model of T^R that is not strongly separable.*

Proof. By the compactness theorem, p can be realized in a separable model $\mathcal{N}$ of T^R. Since p is atomless, it is not purely atomic, so by Proposition 5.2, $\mathcal{N}$ is not strongly separable. $\blacksquare_{5.3}$

We will now refine the construction $p = \sum_i \alpha_i p_i^\star$ introduced in Example 2.16, by removing terms of measure zero and combining duplicates. This construction will be used to analyze purely atomic types in T^R.

Definition 5.4. Let $p \in S_n(T^R)$. We say that $p = \sum_i \alpha_i p_i^\star$ is a **nice decomposition** of p if

- $p_0, p_1, \ldots$ are pairwise distinct types in $S_n(T)$;
- $\alpha_i \in (0, 1]$ for each i;
- $\sum_i \alpha_i = 1$.

Note that any two nice decompositions of an n-type $p \in S_n(T^R)$ are the same up to the ordering of the terms. That is, if $p = \sum_i \alpha_i p_i^\star$ and $p = \sum_k \beta_k q_k^\star$ are nice, then

$$\{\alpha_0 p_0^\star, \alpha_1 p_1^\star \ldots\} = \{\beta_0 q_0^\star, \beta_1 q_1^\star \ldots\}.$$

Lemma 5.5. *A type p in T^R has a nice decomposition if and only if it is purely atomic.*

Proof. Suppose p is purely atomic. Let

$$\{p_0, p_1, \ldots\} = \{q \in S_n(T) \colon \nu_p(\{q\}) > 0\}.$$

Since p is purely atomic, this set is finite or countable, and $p = \sum_i \alpha_i p_i^\star$ is a nice decomposition of p where

$$\alpha_i = \nu_p(\{p_i\}).$$

Now suppose p has a nice decomposition $p = \sum_i \alpha_i p_i^\star$. Then for each i the set $\{p_i\}$ is an atom of ν_p, and every set of ν_p-positive measure contains some p_i. Therefore p is purely atomic. $\blacksquare_{5.5}$

Corollary 5.6. *Suppose $\mathcal{M} \models T$ and p is a purely atomic type in T^R with the nice decomposition $p = \sum_i \alpha_i p_i^\star$. Then p is realized in $(\mathcal{M}^{[0,1)}, \mathcal{L})$ if and only if p_i is realized in $\mathcal{M}$ for each i.*

Proof. If p_i is realized in $\mathcal{M}$ for each i, then p is realized in $(\mathcal{M}^{[0,1)}, \mathcal{L})$ by Remark 2.17 (ii). Suppose p is realized in $(\mathcal{M}^{[0,1)}, \mathcal{L})$ by an n-tuple $\vec{\mathbf{g}}$. Then for each i,

$$0 < \alpha_i = \nu_p(\{p_i\}) = \lambda(\{t \,:\, tp(\vec{\mathbf{g}}(t)) = p_i\}).$$

For each i there is an element t_i such that $tp(\vec{\mathbf{g}}(t_i)) = p_i$, so p_i is realized in $\mathcal{M}$. $\blacksquare_{5.6}$

Theorem 3.12 in [BK] gives necessary and sufficient conditions for every separable pre-model of T^R to be strongly separable. In the next proposition, we restate those conditions and give some additional conditions involving types.

Proposition 5.7. *The following are equivalent.*

(i) *T has a countable saturated model.*
(ii) *T^R has a separable ω-saturated model.*
(iii) *Every separable pre-model of T^R is strongly separable.*
(iv) *$S_n(T)$ is countable for each n.*
(v) *Every type in T^R is purely atomic.*
(vi) *There is no atomless type in T^R.*

Proof. The equivalence of (i)–(iii) is given by Theorem 3.12 in [BK]. The equivalence of (i) and (iv) is due to Vaught [Va].

Assume (iv), and let $p \in S_n(T^R)$. By the Compactness Theorem, p is realized in some separable model $\mathcal{N}$ of T^R. By (iii), $\mathcal{N}$ is strongly separable, so by Proposition 5.2, p is purely atomic. Thus (iv) implies (v).

(v) implies (vi) because no purely atomic type is atomless.

Finally, assume that (iv) fails, so there is an n such that $S_n(T)$ is uncountable. By the Cantor-Bendixson Theorem (Theorem 6.4 in [Kec]), the uncountable Polish space $S_n(T)$ contains a Borel subset P that is perfect, and by Theorem 6.2 in [Kec], there is a continuous injective mapping f from the Cantor space $\{0,1\}^{\mathbb{N}}$ into P. The Cantor space has an atomless Borel probability measure μ. Then $\nu = \mu \circ f^{-1}$ is a Borel probability measure on $S_n(T)$. Let $X \subseteq S_n(T)$ be a Borel set with $r = \nu(X) > 0$. To show that ν is atomless, we must find a Borel set $Y \subseteq X$ with $0 < \nu(Y) < r$. The set $f^{-1}(X)$ is Borel and $\mu(f^{-1}(X)) = r$. Since μ is atomless, there is a Borel set $Z \subseteq f^{-1}(X)$ with $0 < \mu(Z) < r$. Let $Y = f(Z)$. By the Lusin-Souslin theorem (Theorem 15.1 in [Kec]), Y is a Borel subset of X. We have $Z = f^{-1}(Y)$, so $\nu(Y) = \mu(Z)$. Hence $0 < \nu(Y) < r$, and ν is atomless. By Fact 2.15 there is a type $p \in S_n(T^R)$ such that $\nu = \nu_p$, so (vi) fails. Thus (vi) implies (iv). $\blacksquare_{5.7}$

6. Separable Homogeneous Models

In this section we will show that for each complete first order theory T, the number of separable homogeneous models of T^R is equal to the number of countable homogeneous models of T, up to isomorphism. The hard part will be to prove Theorem 6.5, which shows that the strongly separable homogeneous models of T^R are exactly the Borel randomizations of countable homogeneous models of T, up to isomorphism.

Lemma 6.1. *$\mathcal{M}$ is countable homogeneous if and only if $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is separable homogeneous.*

Proof. Suppose first that $\mathcal{M}$ is countable homogeneous. By Lemma 4.2, $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is separable. Let $\vec{\mathbf{f}}, \vec{\mathbf{g}}$ realize the same n-type p in $(\mathcal{M}^{[0,1)}, \mathcal{L})$, let $\mathbf{h} \in \mathcal{M}^{[0,1)}$, and let q be the $(n+1)$-type of $(\vec{\mathbf{f}}, \mathbf{h})$. By Proposition 5.2, p and q are purely atomic. By Lemma 5.5, p has a nice decomposition $p = \sum_i \alpha_i p_i^\star$. Then for each i, $\alpha_i = \lambda(\mathsf{A}_i) = \lambda(\mathsf{B}_i)$ where

$$\mathsf{A}_i = \{t \colon tp(\vec{\mathbf{f}}(t)) = p_i\}, \quad \mathsf{B}_i = \{t \colon tp(\vec{\mathbf{g}}(t)) = p_i\}.$$

Also q has a nice decomposition $q = \sum_j \beta_j q_j^\star$. By grouping the q_j's that contain p_i together for each i, we can write the nice decomposition of q as $q = \sum_i (\sum_j \beta_{ij} q_{ij}^\star)$ where $p_i \subseteq q_{ij}$ for each (i,j). Then for each (i,j) we have $\beta_{ij} = \lambda(\mathsf{C}_{ij})$, where

$$\mathsf{C}_{ij} = \{t \colon tp(\vec{\mathbf{f}}(t), \mathbf{h}(t)) = q_{ij}\} \subseteq \mathsf{A}_i.$$

Note that each of the sets $\mathsf{A}_i, \mathsf{B}_i, \mathsf{C}_{ij}$ belongs to $\mathcal{L}$. For each i, we may partition the set B_i into a union $\mathsf{B}_i = \bigcup_j \mathsf{D}_{ij}$ of Borel sets D_{ij} such that $\beta_{ij} = \lambda(\mathsf{D}_{ij})$. Each of the unions $\bigcup_i \mathsf{A}_i, \bigcup_i \mathsf{B}_i,, \bigcup_i(\bigcup_j \mathsf{C}_{ij})$, and $\bigcup_i(\bigcup_j \mathsf{D}_{ij})$ has measure one. Each of the sets C_{ij} has positive measure, so the type q_{ij} is realized in $\mathcal{M}$. Since $\mathcal{M}$ is countable homogeneous, for each (i,j) and each tuple $\vec{c} \in M^n$ such that $tp(\vec{c}) = p_i$, we may choose an element $d = d(\vec{c}, i, j) \in M$ such that $tp(\vec{c}, d) = q_{ij}$. Let $\mathbf{k}$ be the almost surely unique element of $\mathcal{M}^{[0,1)}$ such that for each (i,j) and $t \in \mathsf{D}_{ij}$, $\mathbf{k}(t) = d(\vec{\mathbf{g}}(t), i, j)$. Then $(\vec{\mathbf{g}}(t), \mathbf{k}(t))$ realizes

q_{ij} for all $t \in \mathsf{D}_{ij}$, and hence $(\vec{\mathbf{g}}, \mathbf{k})$ realizes q. This shows that $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is separable homogeneous.

Now suppose $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is separable homogeneous. $\mathcal{M}$ is countable by Lemma 4.2. Let $\vec{a}, \vec{b}$ be tuples in $\mathcal{M}$ such that $p = tp(\vec{a}) = tp(\vec{b})$, and let $c \in M$. Let $\vec{\mathbf{f}}, \vec{\mathbf{g}}, \mathbf{h}$ be the constant functions in $\mathcal{M}^{[0,1)}$ with values $\vec{a}, \vec{b}, c$ respectively. By Remark 2.17 (iv) , $p^\star = tp(\vec{\mathbf{f}}) = tp(\vec{\mathbf{g}})$. Since $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is separable homogeneous, there exists $\mathbf{k}$ in $\mathcal{M}^{[0,1)}$ such that $tp(\vec{\mathbf{f}}, \mathbf{h}) = tp(\vec{\mathbf{g}}, \mathbf{k})$. Let $q = tp(\vec{a}, c)$. Then $q^\star = tp(\vec{\mathbf{f}}, \mathbf{h}) = tp(\vec{\mathbf{g}}, \mathbf{k})$. Therefore $q = tp(\vec{\mathbf{f}}(t), \mathbf{h}(t)) = tp(\vec{\mathbf{g}}(t), \mathbf{k}(t)) = tp(\vec{b}, \mathbf{k}(t))$ for almost all t, and hence there exists $d \in M$ with $tp(\vec{b}, d) = q$. Thus $\mathcal{M}$ is countable homogeneous. $\blacksquare_{6.1}$

As a brief digression, we use Lemma 6.1 to give a characterization of strongly separable pre-models in terms of types.

Proposition 6.2. *A pre-model $\mathcal{N}$ of T^R is strongly separable if and only if $\mathcal{N}$ is separable and for each n, each n-type $p \in S_n(T^R)$ that is realized in $\mathcal{N}$ is purely atomic.*

Proof. Suppose $\mathcal{N}$ is strongly separable. Then $\mathcal{N}$ is separable by Lemma 3.2 in [BK], and each type that is realized in $\mathcal{N}$ us realized in the Borel randomization of some model of T. By Proposition 5.2, each type that is realized in $\mathcal{N}$ is purely atomic.

For the other direction, suppose $\mathcal{N}$ is separable and each type $p \in S_n(T^R)$ that is realized in $\mathcal{N}$ is purely atomic.

By Lemma 5.5, each type that is realized in $\mathcal{N}$ has a nice decomposition. Let $D = \{\mathbf{f}_0, \mathbf{f}_1, \ldots\}$ be a countable dense subset of $\mathcal{N}$. For each n, let C_n be the set of all n-types $q \in S_n(T)$ such that for some n-tuple $\vec{\mathbf{f}}$ in D, $q^\star$ occurs in a nice decomposition of $tp(\vec{\mathbf{f}})$. Since the nice decompositions of an n-type are unique up to the ordering of the terms, the set C_n is at most countable. By the Compactness Theorem and Fact 2.10, there is a countable homogeneous model $\mathcal{H}$ of T such that each type in $\bigcup_n C_n$ is realized in $\mathcal{H}$. Then by Remark 2.17 (ii), for each n, $p_n = tp(\mathbf{f}_0, \ldots, \mathbf{f}_{n-1})$ is realized in $(\mathcal{H}^{[0,1)}, \mathcal{L})$. Since $\mathcal{H}$ is countable homogeneous, $(\mathcal{H}^{[0,1)}, \mathcal{L})$ is separable homogeneous by Lemma 6.1. Therefore whenever $tp(\mathbf{g}_0, \ldots, \mathbf{g}_{n-1}) = p_n$ in $\mathcal{H}^{[0,1)}$, there is an $\mathbf{h} \in \mathcal{H}^{[0,1)}$ such that $tp(\mathbf{g}_0, \ldots, \mathbf{g}_{n-1}, \mathbf{h}) = p_{n+1}$. It follows by induction that there is a single sequence $(\mathbf{g}_0, \mathbf{g}_1, \ldots)$ such that for each n, $(\mathbf{g}_0, \ldots, \mathbf{g}_{n-1})$ realizes p_n in $(\mathcal{H}^{[0,1)}, \mathcal{L})$. Therefore the mapping $\mathbf{f}_n \mapsto \mathbf{g}_n$ can be extended to an elementary embedding of $\mathcal{N}$ into $(\mathcal{H}^{[0,1)}, \mathcal{L})$. This shows that $\mathcal{N}$ is strongly separable. $\blacksquare_{6.2}$

We now return to our study of separable homogeneous structures.

Lemma 6.3. *Two separable homogeneous continuous structures that realize the same types are isomorphic.*

Proof. Let $\mathcal{N}, \mathcal{P}$ be separable homogeneous continuous structures that realize the same types. By a back and forth argument, there are dense sequences $(\mathbf{f}_0, \mathbf{f}_1, \ldots)$, $(\mathbf{g}_0, \mathbf{g}_1, \ldots)$

in $\mathcal{N},\mathcal{P}$ respectively that realize the same types. By density, there is an isomorphism from $\mathcal{N}$ onto $\mathcal{P}$ which sends each $\mathbf{f}_i$ to $\mathbf{g}_i$. $\blacksquare_{6.3}$

The following result characterizes the set of purely atomic types that are realized in a given separable homogeneous model. It is a converse of Remark 2.17 (ii), and should be compared with Corollary 5.6.

Proposition 6.4. *Suppose $\mathcal{N}$ is a separable homogeneous model of T^R, p is a purely atomic type, and $p=\sum_i \alpha_i p_i^\star$ is a nice decomposition. Then p is realized in $\mathcal{N}$ if and only if $p_i^\star$ is realized in $\mathcal{N}$ for each i.*

Proof. By Remark 2.17 (i), for every model $\mathcal{N}$ of T^R, if $p_i^\star$ is realized in $\mathcal{N}$ for each i then p is realized in $\mathcal{N}$.

For "only if" direction, let $\vec{\mathbf{f}}$ realize $p=\sum_i \alpha_i p_i^\star$ in $\mathcal{N}$. We fix i and show that $p_i^\star$ is realized in $\mathcal{N}$. Since the decomposition is nice, $\alpha_i>0$. Let us say that a tuple $\vec{\mathbf{g}}$ realizes p_i on an event B if $\mathsf{B}\sqsubseteq [\![\varphi(\vec{\mathbf{g}})]\!]$ for each $\varphi\in p_i$. Let S be the set of all $\beta\in[0,1]$ such that some tuple in $\mathcal{N}$ realizes p_i on some event of probability at least β. It is trivial that $0\in S$. We show that $\beta\in S$ implies $\min(1,\beta+\alpha_i)\in S$. It then follows that $1\in S$, and hence that $p_i^\star$ is realized in $\mathcal{N}$.

Suppose $\beta\in S$, so there is a tuple $\vec{\mathbf{g}}$ in $\mathcal{N}$ realizes p_i on an event B such that $\mu(\mathsf{B})\ge\beta$. Since $\vec{\mathbf{f}}$ realizes p, $\vec{\mathbf{f}}$ realizes p_i on some event A of measure α_i. Since μ is atomless, there is an event A' of measure α_i such that $\mu(\mathsf{B}\sqcup\mathsf{A}')=\min(1,\beta+\alpha_i)$. By Fact 2.8 (quantifier elimination), $tp(\mathsf{A})=tp(\mathsf{A}')$ in $\mathcal{N}$. Therefore by separable homogeneity, there is a tuple $\vec{\mathbf{f}}'$ in $\mathcal{N}$ such that $tp(\vec{\mathbf{f}},\mathsf{A})=tp(\vec{\mathbf{f}}',\mathsf{A}')$. Then $\vec{\mathbf{f}}'$ realizes p_i on A'. By Corollary 2.7, there is a tuple $\vec{\mathbf{h}}$ in $\mathcal{N}$ such that $\vec{\mathbf{h}}$ agrees with $\vec{\mathbf{g}}$ on B and agrees with $\vec{\mathbf{f}}'$ on $\neg\mathsf{B}$. Then $\vec{\mathbf{h}}$ realizes p_i on the event $\mathsf{B}\sqcup\mathsf{A}'$ of measure $\min(1,\beta+\alpha_i)$. This shows that $\min(1,\beta+\alpha_i)\in S$, and completes the proof. $\blacksquare_{6.4}$

Our next theorem gives a characterization of strongly separable homogeneous models of T^R.

Theorem 6.5. *$\mathcal{N}$ is a strongly separable homogeneous model of T^R if and only if $\mathcal{N}$ is isomorphic to $(\mathcal{M}^{[0,1)},\mathcal{L})$ for some countable homogeneous $\mathcal{M}\models T$.*

Proof. The "if" direction follows from Lemma 6.1.

For the other direction, we assume that $\mathcal{N}$ is a strongly separable homogeneous model of T^R. Since $\mathcal{N}$ is strongly separable, there is a countable model $\mathcal{H}$ of T and a pre-structure $\mathcal{P}$ such that $\mathcal{N}\cong\mathcal{P}\prec(\mathcal{H}^{[0,1)},\mathcal{L})$. By Theorem 4.5, we may take $\mathcal{P}$ so that the event sort of $\mathcal{P}$ is all of $\mathcal{L}$. By Fact 2.10, we may take $\mathcal{H}$ to be countable homogeneous. By Lemma 4.7, some countable part $(\mathcal{K},\mathcal{A})$ of $(\mathcal{H}^{[0,1)},\mathcal{L})$ is dense in $\mathcal{P}$. To prove the theorem, it suffices to show that $\mathcal{P}\cong(\mathcal{M}^{[0,1)},\mathcal{L})$ for some countable homogeneous $\mathcal{M}\prec\mathcal{H}$. Our plan is to use the results in Section 4 to show that for almost all t, $\mathcal{K}(t)$ is isomorphic to a fixed homogeneous model $\mathcal{M}\prec\mathcal{H}$, and then show that $\mathcal{P}\cong(\mathcal{M}^{[0,1)},\mathcal{L})$. To do this we establish a series of claims.

Claim 1. (Zero–one Law) For every $L_{\omega_1\omega}$ sentence φ, either $[\![\varphi]\!]^{\mathcal{K}} \doteq \top$ or $[\![\varphi]\!]^{\mathcal{K}} \doteq \bot$.

Proof of Claim 1: We first note that $\sigma(\mathcal{A}) \subseteq \mathcal{L}$, so by Lemma 4.11 we have $[\![\varphi]\!]^{\mathcal{K}} \in \mathcal{L}$. Suppose Claim 1 fails. Then for some $L_{\omega_1\omega}$ sentence φ, $0 < \lambda([\![\varphi]\!]^{\mathcal{K}}) < 1$. Hence there are two events $\mathsf{A}, \mathsf{B} \in \mathcal{L}$ such that $\mathsf{A} \subseteq [\![\varphi]\!]^{\mathcal{K}}, \mathsf{B} \subseteq \neg[\![\varphi]\!]^{\mathcal{K}}$, and $0 < \lambda(\mathsf{A}) = \lambda(\mathsf{B})$. By Fact 2.8, A and B have the same type in $\mathcal{P}$. Since $\mathcal{P}$ is separable homogeneous, $(\mathcal{P}, \mathsf{A})$ and $(\mathcal{P}, \mathsf{B})$ are separable homogeneous and realize the same types. Then by Lemma 6.3, there is an automorphism h of $\mathcal{P}$ such that $h(\mathsf{A}) = \mathsf{B}$. Let $(\mathcal{K}', \mathcal{A}')$ be the image of $(\mathcal{K}, \mathcal{A})$ under h. Then $(\mathcal{K}', \mathcal{A}')$ is also a countable part of $(\mathcal{H}^{[0,1)}, \mathcal{L})$ that is dense in $\mathcal{P}$. By Lemma 4.10, $K(t) = K'(t)$ for almost all t. But then by Lemma 4.11,

$$\mathsf{B} = h(\mathsf{A}) \subseteq h([\![\varphi]\!]^{\mathcal{K}}) = [\![\varphi]\!]^{\mathcal{K}'} = [\![\varphi]\!]^{\mathcal{K}},$$

contradicting the assumption that $\mathsf{B} \subseteq \neg[\![\varphi]\!]^{\mathcal{K}}$. This proves Claim 1.

Claim 2. There is a Borel set E such that $\lambda(\mathsf{E}) = 1$ and for all $s, t \in \mathsf{E}$, $\mathcal{K}(s)$ and $\mathcal{K}(t)$ realize the same types.

Proof of Claim 2: By Lemma 4.8, there is a Borel set E_1 such that $\lambda(\mathsf{E}_1) = 1$ and $\mathcal{K}(t) \prec \mathcal{H}$ for all $t \in \mathsf{E}_1$. For each type $q \in S_n(T)$, the $L_{\omega_1\omega}$ sentence $\varphi_q = (\exists\vec{v}) \bigwedge q$ holds in a structure $\mathcal{K}(t)$ if and only if q is realized in $\mathcal{K}(t)$. By Claim 1, for each type q, either $\mathcal{K}(t) \models \varphi_q$ for almost all t, or $\mathcal{K}(t) \models \neg\varphi_q$ for almost all t. Moreover, if φ_q holds in $\mathcal{K}(t)$ for some $t \in \mathsf{E}_1$, then q is realized in $\mathcal{H}$. Since $\mathcal{H}$ is countable, the set

$$Q = \{q \in \bigcup_n S_n(T) \colon (\exists t \in \mathsf{E}_1)\, \mathcal{K}(t) \models \varphi_q\}$$

is countable. Hence there is a Borel set $\mathsf{E} \subseteq \mathsf{E}_1$ such that $\lambda(\mathsf{E}) = 1$ and for each $q \in Q$, either $\mathcal{K}(t) \models \varphi_q$ for all $t \in \mathsf{E}$, or $\mathcal{K}(t) \models \neg\varphi_q$ for all $t \in \mathsf{E}$. Then E satisfies the requirements for Claim 2.

Claim 3. For almost every $t \in \mathsf{E}$, $\mathcal{K}(t)$ is countable homogeneous.

Proof of Claim 3: It is sufficient to prove the following for each n, each pair $\vec{a}, \vec{b} \in H^n$ such that $tp(\vec{a}) = tp(\vec{b})$ in $\mathcal{H}$, and each $c \in H$:

(1) *For almost all t, if $\vec{a}, \vec{b} \in K(t)^n$ and $c \in K(t)$ then there exists $d \in K(t)$ such that $tp(\vec{a}, c) = tp(\vec{b}, d)$.*

Fix $\vec{a}, \vec{b}, c$ such that $tp(\vec{a}) = tp(\vec{b})$ in $\mathcal{H}$. Let A be the Borel set of all $t \in [0, 1)$ such that $\vec{a}, \vec{b} \in K(t)^n$ and $c \in K(t)$. If $\lambda(\mathsf{A}) = 0$, then (1) is trivial, so we assume $\lambda(\mathsf{A}) > 0$. Then there is a partition $\mathsf{A} = \bigcup_m \mathsf{B}_m$ of A into Borel sets, such that for each m there is a pair $\vec{\mathbf{f}}_m, \vec{\mathbf{g}}_m \in \mathcal{K}^n$ and an element $\mathbf{h}_m \in \mathcal{K}$ with $\vec{\mathbf{f}}_m(t) = \vec{a}$, $\vec{\mathbf{g}}_m(t) = \vec{b}$, and $\mathbf{h}_m(t) = c$ for all $t \in \mathsf{B}_m$. Fix m, and let $\vec{\mathbf{e}}_m$ be the n-tuple that agrees with $\vec{\mathbf{g}}_m$ on B_m and agrees with $\vec{\mathbf{f}}_m$ elsewhere. By Corollary 2.7, $\vec{\mathbf{e}}_m$ belongs to $\mathcal{K}^n$. We have $tp(\vec{\mathbf{f}}_m(t)) = tp(\vec{\mathbf{e}}_m(t))$ for all $t \in [0, 1)$. Hence for each first order formula $\varphi(\vec{u})$,

$$\lambda(\mathsf{B}_m \sqcap [\![\varphi(\vec{\mathbf{f}}_m)]\!]) = \lambda(\mathsf{B}_m \sqcap [\![\varphi(\vec{\mathbf{e}}_m)]\!])$$

and

$$\lambda((\neg \mathsf{B}_m) \sqcap [\![\varphi(\vec{\mathbf{f}}_m)]\!]) = \lambda((\neg \mathsf{B}_m) \sqcap [\![\varphi(\vec{\mathbf{e}}_m)]\!]),$$

so $tp(\vec{\mathbf{f}}_m, \mathsf{B}_m) = tp(\vec{\mathbf{e}}_m, \mathsf{B}_m)$. Since $\mathcal{P}$ is separable homogeneous, there is an element $\mathbf{k}_m$ in $\mathcal{P}$ such that

$$tp(\vec{\mathbf{f}}_m, \mathbf{h}_m, \mathsf{B}_m) = tp(\vec{\mathbf{e}}_m, \mathbf{k}_m, \mathsf{B}_m).$$

Then for each first order formula $\psi(\vec{u}, v)$,

$$\lambda(\mathsf{B}_m \sqcap [\![\psi(\vec{\mathbf{f}}_m, \mathbf{h}_m)]\!])) = \lambda(\mathsf{B}_m \sqcap [\![\psi(\vec{\mathbf{e}}_m, \mathbf{k}_m)]\!]).$$

Hence for each $\psi(\vec{u}, v) \in tp(\vec{a}, c)$,

$$\lambda(\mathsf{B}_m) = \lambda(\mathsf{B}_m \sqcap [\![\psi(\vec{b}, \mathbf{k}_m)]\!]),$$

and thus $tp(\vec{a}, c) = tp(\vec{b}, \mathbf{k}_m(t))$ for almost all $t \in \mathsf{B}_m$. Moreover, since $(\mathcal{K}, \mathcal{A})$ is dense in $\mathcal{P}$, we have $\mathbf{k}_m \in \mathcal{K}^{cl}$, so by Lemma 4.9, $\mathbf{k}_m(t) \in K(t)$ for almost all t. This proves (1) and Claim 3.

Claim 4. There is a countable homogeneous model $\mathcal{M} \prec \mathcal{H}$ such that $\mathcal{K}(t) \cong \mathcal{M}$ for almost all t.

Proof of Claim 4: By Claims 2 and 3, there is a Borel set $\mathsf{E}' \subseteq \mathsf{E}$ such that $\lambda(\mathsf{E}') = 1$ and for all $s, t \in \mathsf{E}'$, $\mathcal{K}(s)$ and $\mathcal{K}(t)$ are countable homogeneous models that realize the same types. By Fact 2.10, $\mathcal{K}(s) \cong \mathcal{K}(t)$ for all $s, t \in \mathsf{E}'$. This proves Claim 4.

We will construct an isomorphism $h\colon (\mathcal{M}^{[0,1)}, \mathcal{L}) \cong \mathcal{P}$. If $\mathcal{H}$ is finite, the theorem holds because T^R is separably categorical, so we may assume H is countably infinite. Arrange the elements of H in a list of length ω. This gives us a listing of M and of $K(t)$ for each $t \in [0, 1)$. Also arrange the elements of $\mathcal{K}$ in a list of length ω. For each $t \in [0, 1)$ we will pick enumerations $M = \{a_0(t), a_1(t), \ldots\}$ and $K(t) = \{b_0(t), b_1(t), \ldots\}$ as follows. When $t \notin \mathsf{E}'$, $a_m(t)$ is the m-th element of M and $b_m(t)$ is the m-th element of $K(t)$. When $t \in \mathsf{E}'$ we proceed inductively on m. We assume that $a_0(t), \ldots, a_{3m-1}(t)$ and $b_0(t), \ldots, b_{3m-1}(t)$ have already been constructed so that

$$tp(a_0(t), \ldots, a_{3m-1}(t)) = tp(b_0(t), \ldots, b_{3m-1}(t))$$

in $\mathcal{H}$. We take $a_{3m}(t)$ to be the first element of $M \setminus \{a_0(t), \ldots, a_{3m-1}(t)\}$, and take $b_{3m}(t)$ to be the first element of $K(t)$ such that

$$tp(a_0(t), \ldots, a_{3m-1}(t), a_{3m}(t)) = tp(b_0(t), \ldots, b_{3m-1}(t), b_{3m}(t)).$$

We then take $b_{3m+1}(t)$ to be the first element of $K(t) \setminus \{b_1(t), \ldots, b_{3m-1}(t)\}$, take $b_{3m+2}(t)$ to be $\mathbf{k}(t)$ where $\mathbf{k}$ is the m-th element of $\mathcal{K}$, and and take $a_{3m+1}(t)$ and $a_{3m+2}(t)$ to be the first elements of M such that

$$tp(a_0(t), \ldots, a_{3m-1}(t), a_{3m}(t), a_{3m+1}(t)a_{3m+2}(t)) = tp(b_0(t), \ldots, b_{3m-1}(t), b_{3m}(t), b_{3m+1}(t), b_{3m+2}(t)).$$

This procedure can always be carried out because $\mathcal{M}$ and $\mathcal{K}(t)$ are countable homogeneous and realize the same types. The construction guarantees that for each t, $M = \{a_0(t), a_1(t), \ldots\}$ and $K(t) = \{b_0(t), b_1(t), \ldots\}$, that $\mathcal{K} = \{b_2(\cdot), b_5(\cdot), \ldots\}$, and that for each $t \in \mathsf{E}'$, the mapping $a_m(t) \mapsto b_m(t)$ is an isomorphism from $\mathcal{M}$ onto $\mathcal{K}(t)$. Because $\mathcal{K}$ is a countable set of $\mathcal{L}$-measurable functions and $K(t) = \{\mathbf{f}(t) \colon \mathbf{f} \in \mathcal{K}\}$, we see by induction that for each m the functions $a_m(t)$ and $b_m(t)$ are $\mathcal{L}$-measurable functions of t.

For each $\mathbf{f} \in \mathcal{M}^{[0,1)}$ let $h(\mathbf{f})$ be the unique function $\mathbf{g} \colon [0,1) \to H$ such that for each t and m, $\mathbf{f}(t) = a_m(t)$ if and only if $\mathbf{g}(t) = b_m(t)$. Then $(h(\mathbf{f}))(t) \in \mathcal{K}(t)$ for all t, and $\mathcal{K} \subseteq h(\mathcal{M}^{[0,1)})$. Since $\mathcal{P}$ has event sort $\mathcal{L}$, $\mathcal{A}$ is dense in $\mathcal{L}$. Then by Lemma 4.9, h maps $\mathcal{M}^{[0,1)}$ into $\mathcal{K}^{cl}$, which is the sort $\mathbf{K}$ part of $\mathcal{P}$. For each first order formula φ and tuple $\vec{\mathbf{f}}$ in $\mathcal{M}^{[0,1)}$,

$$[\![\varphi(\vec{\mathbf{f}})]\!] \doteq [\![\varphi(h(\vec{\mathbf{f}}))]\!].$$

In the event sort, let h be the identity function on $\mathcal{L}$. Since $\mathcal{P}$ has event sort $\mathcal{L}$, we have $h \colon (\mathcal{M}^{[0,1)}, \mathcal{L}) \prec \mathcal{P}$. The set $\mathcal{M}^{[0,1)}$ is closed in $\mathcal{M}_\infty^{[0,1)}$, and h preserves distances, so $h(\mathcal{M}^{[0,1)})$ is closed in $\mathcal{M}_\infty^{[0,1)}$. Since $\mathcal{K} \subseteq h(\mathcal{M}^{[0,1)})$ and $(\mathcal{K}, \mathcal{A})$ is dense in $\mathcal{P}$, it follows that $h \colon (\mathcal{M}^{[0,1)}, \mathcal{L}) \cong \mathcal{P}$. $\blacksquare_{6.5}$

Corollary 6.6. *The mapping*

$$\Theta \colon \mathcal{M} \mapsto \textit{completion of } (\mathcal{M}^{[0,1)}, \mathcal{L})$$

is a bijection from the set of isomorphism types of countable homogeneous models of T onto the set of isomorphism types of strongly separable homogeneous models of T^R, and this mapping preserves elementary embeddability.

Proof. By Lemma 6.1, Θ maps countable homogeneous models to strongly separable homogeneous models. If $\Theta(\mathcal{M}) \cong \Theta(\mathcal{H})$, then $\Theta(\mathcal{M})$ and $\Theta(\mathcal{H})$ realize the same types, so by Corollary 5.6, $\mathcal{M}$ and $\mathcal{H}$ realize the same types, and by Fact 2.10, $\mathcal{M} \cong \mathcal{H}$. Thus Θ is one-to-one up to isomorphism. It is clear that Θ preserves elementary embeddability. Theorem 6.5 shows that Θ is onto. $\blacksquare_{6.6}$

Example 6.7. Baldwin and Lachlan [BL] showed that if T is ω_1-categorical but not ω-categorical, then all the countable models of T are countable homogeneous and form an elementary chain of length $\omega + 1$. Corollary 6.6 shows that in that case, the strongly separable homogeneous models of T^R also form an elementary chain of length $\omega + 1$.

Corollary 6.8. *Let κ be the number of countable homogeneous models of T. Then T^R has exactly κ separable homogeneous models, and exactly κ strongly separable homogeneous models.*

Proof. By Theorem 6.5, T^R has exactly κ strongly separable homogeneous models. Suppose first that T has countably many complete types. Then by Fact 2.11, T has a countable saturated model. By Proposition 5.7, every separable model of T^R is strongly separable, so T has exactly κ separable homogeneous models.

Now suppose that T has uncountably many complete types. Then for some n, $S_n(T)$ is uncountable. $S_n(T)$ is a Polish space, and every uncountable Polish space has cardinality 2^ω, so T has 2^ω complete types. By Fact 2.10, every type $p \in S_n(T)$ is realized in some countable homogeneous model of T. Moreover, every countable model realizes only countably many complete types. Since the signature is countable, T has at most 2^ω countable models. T^R also has a countable signature, so T^R has at most 2^ω countable pre-models, and hence at most 2^ω separable models. It follows that T has exactly 2^ω countable homogeneous models, so $\kappa = 2^\omega$. Then T^R has exactly 2^ω strongly separable homogeneous models, and exactly 2^ω separable homogeneous models. $\blacksquare_{6.8}$

The paper [KM] gives an example of a complete first order theory T with exactly m countable homogeneous models for each positive integer m. By the above corollary, such a theory has exactly m separable homogeneous models.

In Theorem 6.5, we used nice decompositions of purely atomic types to characterize the class of all strongly separable homogeneous models of the randomization theory. An open problem is to find a similar characterization of the class of all separable homogeneous models of the randomization theory.

7. Product Randomizations

We now introduce a construction that is like the Borel randomization $(\mathcal{M}^{[0,1)}, \mathcal{L})$ but has a finite or countable family of elementary substructures of $\mathcal{M}$ in place of $\mathcal{M}$.

Definition 7.1. Let $\mathcal{M} \models T$, let I be a finite or countable non-empty set, let $[0,1) = \bigcup_{i\in I} \mathsf{B}_i$ be a partition of $[0,1)$ into Borel sets[3], and for each $i \in I$ let $\mathcal{M}_i \prec \mathcal{M}$. We define

$$\prod_{i\in I} \mathcal{M}_i^{\mathsf{B}_i} = \{\mathbf{f} \in \mathcal{M}^{[0,1)} : (\forall i \in I)(\text{for almost all } t \in \mathsf{B}_i)\mathbf{f}(t) \in M_i\}.$$

It is clear that $(\prod_{i\in I} \mathcal{M}_i^{\mathsf{B}_i}, \mathcal{L})$ is a pre-structure and $(\prod_{i\in I} \mathcal{M}_i^{\mathsf{B}_i}, \mathcal{L}) \subseteq (\mathcal{M}^{[0,1}, \mathcal{L})$. We call the $(\prod_{i\in I} \mathcal{M}_i^{\mathsf{B}_i}, \mathcal{L})$ a **product randomization** in $\mathcal{M}$.

Intuitively, an element of $\prod_{i\in I} \mathcal{M}_i^{\mathsf{B}_i}$ is an experiment in which an element $i \in I$ is chosen with probability $\lambda(\mathsf{B}_i)$ and then an element of $\mathcal{M}_i$ is chosen at random. We say that $\prod_{i\in I} \mathcal{M}_i^{\mathsf{B}_i}$ is the result of sampling from $\mathcal{M}_i$ with probability $\lambda(\mathsf{B}_i)$ for each $i \in I$. We will see in Theorem 8.8 that for a given family of models $\mathcal{M}_i, i \in I$, the product randomization is characterized up to isomorphism by the real numbers $\lambda(\mathsf{B}_i), i \in I$.

Remark 7.2. (i) If two product randomizations have isomorphic parts, then they are isomorphic. Formally, if $\mathcal{M}_i \cong \mathcal{H}_i$ for each $i \in I$, then

$$(\prod_{i\in I} \mathcal{M}_i^{\mathsf{B}_i}, \mathcal{L}) \cong (\prod_{i\in I} \mathcal{H}_i^{\mathsf{B}_i}, \mathcal{L}).$$

(ii) Two product randomizations that agree on the parts of positive measure are isomorphic.

[3]We allow the possibility that some of the sets B_i are empty.

(iii) A product randomization is unchanged if a part is split into several parts with the same elementary substructure. Formally, two product randomizations

$$(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L}),\quad (\prod_{j\in J}\mathcal{H}_j^{\mathsf{C}_j},\mathcal{L})$$

are equal if for each $j\in J$ there exists an $i\in I$ such that $\mathsf{C}_j\subseteq\mathsf{B}_i$ and $\mathcal{H}_j=\mathcal{M}_i$.

Theorem 7.3. *Every product randomization in $\mathcal{M}$ is a pre-complete elementary substructure of the Borel randomization $(\mathcal{M}^{[0,1)},\mathcal{L})$.*

Proof. Let $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})$ be a product randomization in $\mathcal{M}$. Let $\mathbf{f}_m, m\in\mathbb{N}$ be a Cauchy sequence in $\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i}$ with respect to the pseudo-metric $d_{\mathbf{K}}$. By Fact 2.5, $(\mathcal{M}^{[0,1)},\mathcal{L})$ is pre-complete. Therefore $\mathbf{f}_m$ converges to some function $\mathbf{f}$ in $\mathcal{M}^{[0,1)}$. So $\lambda(\{t:\mathbf{f}_m(t)=\mathbf{f}(t)\})$ converges to 1 as $m\to\infty$. Therefore for each i, $\mathbf{f}(t)\in M_i$ for almost all $t\in\mathsf{B}_i$, so $\mathbf{f}\in\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i}$. This shows that $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})$ is pre-complete.

We will now show that $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})$ has perfect witnesses. Since $\mathcal{L}$ is atomless, it will then follow from Fact 2.9 that $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})\prec(\mathcal{M}^{[0,1)},\mathcal{L})$. Since each model of T has at least two elements, it is clear that $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})$ has property 2.6 (ii). To prove 2.6 (i), let $\varphi(y,\vec{x})$ be a first order formula and let $\vec{\mathbf{g}}$ be a tuple in $\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i}$ of length $|\vec{x}|$. Then there is a countable partition $\mathsf{C}_0,\mathsf{C}_1,\ldots$ of $[0,1)$ such that each m, $C_m\in\mathcal{L}$ and $\vec{\mathbf{g}}$ is constant on C_m. So there is a function $\mathbf{f}:[0,1)\to M$ such that for each $i\in I$ and each m, $\mathbf{f}$ is constant on $C_m\cap\mathsf{B}_i$ with a value in M_i, and for each $t\in\mathsf{B}_i$,

$$\mathcal{M}_i\models\varphi(\mathbf{f}(t),\vec{\mathbf{g}}(t))\leftrightarrow(\exists y)\varphi(y,\vec{\mathbf{g}}(t)).$$

Then $\mathbf{f}\in\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i}$. Since $\mathcal{M}_i\prec\mathcal{M}$ for each $i\in I$, for all $t\in[0,1)$ we have

$$\mathcal{M}\models\varphi(\mathbf{f}(t),\vec{\mathbf{g}}(t))\leftrightarrow(\exists y)\varphi(y,\vec{\mathbf{g}}(t)),$$

so

$$(\mathcal{M}^{[0,1)},\mathcal{L})\models[\![\varphi(\mathbf{f},\vec{\mathbf{g}})]\!]\doteq[\![(\exists y)\varphi(y,\mathbf{f},\vec{\mathbf{g}})]\!].$$

Since $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})\subseteq(\mathcal{M}^{[0,1},\mathcal{L})$,

$$(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})\models[\![\varphi(\mathbf{f},\vec{\mathbf{g}})]\!]\doteq[\![(\exists y)\varphi(y,\mathbf{f},\vec{\mathbf{g}})]\!].$$

Thus Condition 2.6 (i) holds. This shows that $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})$ has perfect witnesses. $\blacksquare_{7.3}$

The next theorem shows that up to isomorphism, a product randomization in $\mathcal{M}$ depends only on the structures $\mathcal{M}_i$ and the measures of the Borel sets B_i. For this reason, a product randomization $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})$ can be regarded as a combination of the randomizations $(\mathcal{M}_i^{[0,1)},\mathcal{L}), i\in I$, weighted by the measures $\lambda(\mathsf{B}_i)$. The proof will use the following fact from descriptive set theory.

Fact 7.4. *(By Theorem 17.41 in Kechris [Kec]) Suppose* $\mathsf{B},\mathsf{C}\in\mathcal{L}$ *and* $\lambda(\mathsf{B})=\lambda(\mathsf{C})>0$. *Then there is a Borel bijection* $h:\mathsf{B}\to\mathsf{C}$ *such that* $\lambda(\mathsf{D})=\lambda(h(\mathsf{D}))$ *for every Borel set* $\mathsf{D}\subseteq\mathsf{B}$.

Theorem 7.5. *Let* I *be a finite or countable non-empty set, let* $\mathcal{M}_i\prec\mathcal{M}$ *for each* $i\in I$, *and let* $[0,1)=\bigcup_{i\in I}\mathsf{B}_i$ *and* $[0,1)=\bigcup_{i\in I}\mathsf{C}_i$ *be two partitions of* $[0,1)$ *into Borel sets. Suppose that* $\lambda(\mathsf{B}_i)=\lambda(\mathsf{C}_i)$ *for each* $i\in I$. *Then the product randomizations* $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})$ *and* $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{C}_i},\mathcal{L})$ *are isomorphic.*

Proof. By Remark 7.2 (ii), we can rearrange things so that $\lambda(\mathsf{B}_i)=\lambda(\mathsf{C}_i)>0$ for each $i\in I$, and the new product randomizations will be isomorphic to the original ones. By Fact 7.4, for each $i\in I$ there is a Borel bijection $h_i:\mathsf{B}_i\to\mathsf{C}_i$ such that $\lambda(\mathsf{D})=\lambda(h_i(\mathsf{D}))$ for every Borel set $\mathsf{D}\subseteq\mathsf{B}_i$. Then the union $h=\bigcup_{i\in I}h_i$ is a Borel bijection $h:[0,1)\to[0,1)$ such that $h(\mathsf{B}_i)=\mathsf{C}_i$ for each $i\in I$, and $\lambda(\mathsf{D})=\lambda(h(\mathsf{D}))$ for each $\mathsf{D}\in\mathcal{L}$. Hence the mapping $\mathbf{f}\mapsto\mathbf{f}\circ h^{-1},\mathsf{B}\mapsto h(\mathsf{B})$ gives an isomorphism from $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{B}_i},\mathcal{L})$ to $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{C}_i},\mathcal{L})$. $\blacksquare_{7.5}$

The following theorem gives a key sufficient condition for a pre-model of T^R to be isomorphic to a product randomization.

Theorem 7.6. *Suppose* $\mathcal{M}$ *is a countable model of* T *and* $(\prod_{i\in I}\mathcal{M}_i^{\mathsf{A}_i},\mathcal{L})$ *is a product randomization in* $\mathcal{M}$. *Let* $(\mathcal{K},\mathcal{A})$ *be a countable part of* $(\mathcal{M}^{[0,1)},\mathcal{L})$ *with closure* $\mathcal{P}=(\mathcal{K}^{cl},\mathcal{L})$. *If* $\mathcal{K}(t)\cong\mathcal{M}_i$ *for each* $i\in I$ *and* $t\in\mathsf{A}_i$, *then* $\mathcal{P}\cong(\prod_{i\in I}\mathcal{M}_i^{\mathsf{A}_i},\mathcal{L})$.

Proof. By Remark 4.4 (iv), $\mathcal{P}$ is a pre-complete model of T^R.

By hypothesis, $\mathcal{A}$ is dense in $\mathcal{L}$. Then by Lemma 4.9, the closure $\mathcal{K}^{cl}$ of $\mathcal{K}$ is the set of all $\mathbf{f}\in\mathcal{M}^{[0,1)}$ such that $\mathbf{f}(t)\in K(t)$ for almost all t. Let $\mathcal{C}$ be the set of all $\mathbf{g}\in\mathcal{K}^{cl}$ such that $\mathbf{g}(t)\in K(t)$ for all t. Then every $\mathbf{f}\in\mathcal{K}^{cl}$ is at distance zero from an element $\mathbf{g}\in\mathcal{C}$ (take a $\mathbf{g}$ that agrees with $\mathbf{f}$ on a Borel set D of λ-measure one, and agrees with an element of $\mathcal{K}$ on the complement of D). Thus $\mathcal{K}\subseteq\mathcal{C}\subseteq\mathcal{K}^{cl}$, and $\mathcal{C}$ is dense in $\mathcal{K}^{cl}$.

By Theorem 7.3, the set $\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{A}_i}$ is closed in $\mathcal{M}^{[0,1)}$. Let $\mathcal{D}$ be the set of all $\mathbf{f}'\in\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{A}_i}$ such that $\mathbf{f}'(t)\in\mathcal{M}_i$ for all $i\in I$ and $t\in\mathsf{A}_i$. Then each function in $\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{A}_i}$ is at distance zero from some $\mathbf{f}'\in\mathcal{D}$, so $\mathcal{D}$ is dense in $\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{A}_i}$.

We now show that the product randomization $(\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{A}_i},\mathcal{L})$ is isomorphic to $\mathcal{P}$. Let us list the elements of the countable set $\mathcal{K}$, $\mathcal{K}=\{\mathbf{f}_1,\mathbf{f}_2,\ldots\}$. Let $\langle\mathbf{g}_1',\mathbf{g}_2',\ldots\rangle$ be a sequence that is dense in $\mathcal{D}$.

Claim 1. There is a sequence $\langle\mathbf{g}_1,\mathbf{g}_2,\ldots\rangle$ in $\mathcal{C}$, and a sequence $\langle\mathbf{f}_1',\mathbf{f}_2',\ldots\rangle$ in $\mathcal{D}$, such that for each $n\in\mathbb{N}$,

$$(\mathcal{M}^{[0,1)},\mathcal{L},\mathbf{f}_1,\ldots,\mathbf{f}_n,\mathbf{g}_1,\ldots,\mathbf{g}_n)\equiv(\mathcal{M}^{[0,1)},\mathcal{L},\mathbf{f}_1',\ldots,\mathbf{f}_n',\mathbf{g}_1',\ldots,\mathbf{g}_n').$$

Claim 1 implies that there is an isomorphism

$$h:\mathcal{P}\cong(\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{A}_i},\mathcal{L})$$

such that $h(\mathbf{f}_n) = \mathbf{f}'_n$ and $h(\mathbf{g}_n) = \mathbf{g}'_n$ for each n, and hence $\mathcal{P}$ is isomorphic to $(\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{A}_i}, \mathcal{L})$.

We will use a back-and forth construction, and argue by induction on n. We will actually prove the following statement that is stronger than Claim 1:

Claim 2. There is a sequence $\langle \mathbf{g}_1, \mathbf{g}_2, \ldots\rangle$ in $\mathcal{C}$, and a sequence $\langle \mathbf{f}'_1, \mathbf{f}'_2, \ldots\rangle$ in $\mathcal{D}$, such that the following statement $S(n)$ holds for each $n \in \mathbb{N}$:

For all $i \in I$ and $t \in \mathsf{A}_i$,

$$(\mathcal{K}(t), (\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{g}_1, \ldots, \mathbf{g}_n)(t)) \cong (\mathcal{M}_i, (\mathbf{f}'_1, \ldots, \mathbf{f}'_n, \mathbf{g}'_1, \ldots, \mathbf{g}'_n)(t)).$$

It is clear that the displayed equation in Claim 1 follows from $S(n)$, so Claim 2 implies Claim 1.

Proof of Claim 2: Note that the statement $S(0)$ just says that $\mathcal{K}(t) \cong \mathcal{M}_i$ for all $i \in I$ and $t \in \mathsf{A}_i$, and is true by hypothesis. Let $n \in \mathbb{N}$ and assume that we already have functions $\mathbf{g}_1, \ldots, \mathbf{g}_{n-1}$ in $\mathcal{C}$ and $\mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}$ in $\mathcal{D}$ such that the statement $S(n-1)$ holds. Thus for each $i \in I$ and each $t \in \mathsf{A}_i$, there is an isomorphism

$$h_{it} : (\mathcal{K}(t), (\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}, \mathbf{g}_1, \ldots, \mathbf{g}_{n-1})(t)) \cong (\mathcal{M}_i, (\mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}, \mathbf{g}'_1, \ldots, \mathbf{g}'_{n-1})(t)).$$

We will find functions $\mathbf{g}_n \in \mathcal{C}, \mathbf{f}'_n \in \mathcal{D}$ such that $S(n)$ holds.

Some care is needed to insure that $\mathbf{g}_n$ and $\mathbf{f}'_n$ are measurable. For instance, we cannot simply take $\mathbf{f}'_n(t) = h_{it}(\mathbf{f}_n(t))$ for each $i \in I$ and $t \in \mathsf{A}_i$, because that function may not be measurable.

Note that every function from $[0, 1)$ into $\mathcal{M}$ that is constant on each set in a Borel partition $\langle \mathsf{C}_k, k \in \mathbb{N}\rangle$ of $[0, 1)$ belongs to $\mathcal{M}^{[0,1)}$. Our plan will be to find a Borel partition $\langle \mathsf{C}_k, k \in \mathbb{N}\rangle$ that is a refinement of $\langle \mathsf{A}_i, i \in I\rangle$, and functions $\mathbf{g}_n, \mathbf{f}'_n\colon [0, 1) \to M$ such that $\mathbf{g}_n(t) \in \mathcal{K}(t)$ for all t, $\mathbf{f}'(t) \in \mathcal{M}_i$ for all $i \in I$ and $t \in \mathsf{A}_i$, and all of the functions involved are constant on each partition set C_k (this insures that $\mathbf{g}_n \in \mathcal{C}$ and $\mathbf{f}'_n \in \mathcal{D}$), and $S(n)$ holds.

Since M is countable and each function in $\mathcal{M}^{[0,1)}$ is Borel, there is a partition $\langle \mathsf{E}_j, j \in \mathbb{N}\rangle$ of $[0, 1)$ into Borel sets and such that:

- for each j, each of the functions

$$\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}, \mathbf{g}_1, \ldots, \mathbf{g}_{n-1}, \mathbf{g}'_1, \ldots, \mathbf{g}'_n$$

  is constant on E_j, and
- there is a function $\alpha : \mathbb{N} \to I$ such for each j, $\mathsf{E}_j \subseteq \mathsf{A}_{\alpha(j)}$ (i.e., the E_j's refine the A_i's.)).

Fix $j \in \mathbb{N}$, consider a point $t \in \mathsf{E}_j$, and let $i = \alpha(j)$. Then $t \in \mathsf{A}_i$. Hence, by taking $b = h_{it}(\mathbf{f}_n(t))$ and $c = h_{it}^{-1}(\mathbf{g}'_n(t))$, we see that there exist $b \in M_i$ and $c \in K(t)$ such that

$$(\mathcal{K}(t), (\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}, \mathbf{g}_1, \ldots, \mathbf{g}_{n-1})(t), \mathbf{f}_n(t), c) \cong (\mathcal{M}_i, (\mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}, \mathbf{g}'_1, \ldots, \mathbf{g}'_{n-1})(t), b, \mathbf{g}'_n(t)).$$

For each $j \in \mathbb{N}$ and $b, c \in M$, let $\mathsf{C}'(j, b, c)$ be the set of all $t \in \mathsf{E}_j$ such that $c \in K(t)$, $b \in M_i$, and the above isomorphism relation holds. Then we always have $\mathsf{C}'(j, b, c) \subseteq \mathsf{E}_j$, and

$$[0,1) = \bigcup\{\mathsf{C}'(j,b,c) \,:\, (j,b,c) \in \mathbb{N} \times M \times M\}.$$

We now show that each of the sets $C'(j,b,c)$ is Borel. Let $(j,b,c) \in \mathbb{N} \times M \times M$ and $i = \alpha(j)$. If $b \notin M_i$ then $C'(j,b,c) = \emptyset$, so we may assume that $b \in M_i$. Since $\mathbf{f}^{-1}(c) \in \mathcal{A}$ for each $\mathbf{f} \in \mathcal{K}$, there is a function $\mathbf{c} \in \mathcal{C}$ such that $\mathbf{c}(t) = c$ whenever $c \in \mathcal{K}(t)$. Since A_i is Borel and $b \in M_i$, there is a function $\mathbf{b} \in \mathcal{D}$ such that $\mathbf{b}(t) = b$ whenever $t \in \mathsf{A}_i$. Therefore $\mathsf{C}'(j,b,c)$ is the set of all $t \in \mathsf{E}_j$ such that $c \in K(t)$, $b \in M_i$, and

$$(\mathcal{K}(t), (\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}, \mathbf{g}_1, \ldots, \mathbf{g}_{n-1}, \mathbf{f}_n, \mathbf{c})(t)) \cong (\mathcal{M}_i, (\mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}, \mathbf{g}'_1, \ldots, \mathbf{g}'_{n-1}), \mathbf{b}, \mathbf{g}'_n)(t)).$$

Since M and $\mathcal{K}$ are countable, the set

$$\{t \,:\, c \in K(t)\} = \bigcup_{\mathbf{f} \in \mathcal{K}} \mathbf{f}^{-1}(c)$$

is Borel. It follows from Fact 2.12 and Lemma 4.11 on $L_{\omega_1\omega}$ formulas that $\mathsf{C}'(j,b,c)$ is Borel.

We now cut the family of Borel sets $\mathsf{C}'(\cdot)$ down to a family of Borel sets $\mathsf{C}(\cdot)$ that form a partition of $[0,1)$. Let β be a bijection from the countable set $\mathbb{N} \times M \times M$ onto $\mathbb{N}$, and when $k = \beta(j,b,c)$ put $\mathsf{C}'_k = C'(j,b,c)$. Let $\mathsf{C}_0 = \mathsf{C}'_0$ and $\mathsf{C}_{k+1} = \mathsf{C}'_{k+1} \setminus \bigcup_{\ell \le k} \mathsf{C}_\ell$. Then C_k is Borel, $\mathsf{C}_k \subseteq \mathsf{C}'_k$ for each k, and $\langle \mathsf{C}_k, k \in \mathbb{N}\rangle$ is a partition of $[0,1)$. We put $\mathsf{C}(j,b,c) = \mathsf{C}_k$ when $k = \beta(j,b,c)$.

Let $\mathbf{f}'_n$ be the function that has the constant value b on each set $\mathsf{C}(j,b,c)$, and let $\mathbf{g}_n$ be the function that has the constant value c on each set $\mathsf{C}(j,b,c)$. Then $\mathbf{f}'_n$ and $\mathbf{g}_n$ are Borel and thus belong to $\mathcal{M}^{[0,1)}$, and we have $\mathbf{f}'_n \in \mathcal{D}$ and $\mathbf{g}_n \in \mathcal{C}$. The construction insures that the functions $\mathbf{f}'_n$ and $\mathbf{g}_n$ satisfy the condition $S(n)$. This completes the induction and proves Claim 2.

Therefore Claim 1 holds. As we have already observed, it follows that $\mathcal{P}$ is isomorphic to the product randomization $(\prod_{i \in I} (\mathcal{M}_i)^{\mathsf{A}_i}, \mathcal{L})$. $\blacksquare_{7.6}$

8. Theories with $\le \omega$ Countable Models

We will say that a theory T **has $\le \omega$ countable models** if there is a finite or countable set S of countable models of T such that every countable model of T is isomorphic to some member of S. Our main result in this section, Theorem 8.6, will characterize all the separable models of T^R when T has $\le \omega$ countable models.

Note that if T has $\le \omega$ countable models, then $\bigcup_n S_n(T)$ is obviously countable, so T has a countable saturated model by Fact 2.11.

We now give some examples of theories with $\le \omega$ countable models. In Section 9 we will be interested in theories with the additional property that all countable models are homogeneous, so we will keep track of that property here.

Examples 8.1. In each of the following cases, T has $\le \omega$ countable models, and every countable model of T is homogeneous.

- T is ω-categorical.
- T is ω_1-categorical (Baldwin and Lachlan [BL]).
- T is the complete theory of an equivalence relation.
- T is the complete theory of a unary function in which all elements have the same number of pre-images.
- T is the complete theory of a module, is ω-stable, and has $\leq \omega$ countable models (Garavaglia [Ga]).
- T is the disjoint union of countably many ω-categorical relational theories $T_0, T_1, \ldots$[4]. In this case the countable models of T are characterized by the number of elements outside the union.

Examples 8.2. In each of the following cases, T has $\leq \omega$ countable models, but T may have countable models that are not homogeneous.

- T has Morley rank at most 2 (Cutland [Cu]).
- T is the complete theory of a structure $\mathcal{M}_{\vec{a}}$, where $\mathcal{M}$ is a model of a theory with $\leq \omega$ countable models, and $\vec{a}$ is a finite tuple in $\mathcal{M}$.
- (Ehrenfeucht, first published in [Va]) T is the theory of the rationals with order and a constant symbol for each natural number. This theory has three countable models up to isomorphism.
- T is the disjoint union of finitely many relational theories each with $\leq \omega$ countable models.
- For some complete relational theory U with finitely many countable models, T is the theory of an equivalence relation such that the restriction to each equivalence class is a model of U.

Remark 8.3. Vaught's Conjecture, that any theory with $< 2^\omega$ countable models has $\leq \omega$ countable models, has been proved for many special classes of theories (e.g. see Buechler [Bue], Mayer [May], Shelah et.al. [SHM], and Steel [St]).

We now turn to our main result in this section, Theorem 8.6.

The next lemma shows that when T has $\leq \omega$ countable models, every separable model of T^R can be represented as a product randomization of countable models of T.

Lemma 8.4. *Suppose T has $\leq \omega$ countable models, and let $\mathcal{M}$ be a countable saturated model of T. Then every separable model $\mathcal{N}$ of T^R is isomorphic to a product randomization $(\prod_{i\in I} \mathcal{M}_i^{\mathsf{A}_i}, \mathcal{L})$ in $\mathcal{M}$. Moreover, the models $\mathcal{M}_i$ can be taken to be pairwise non-isomorphic.*

Proof. By Proposition 5.7, $\mathcal{N}$ is strongly separable. Then $\mathcal{N}$ is elementarily embeddable in the Borel randomization of some countable model of T. Every countable model of T is elementarily embeddable in $\mathcal{M}$, so by Remark 2.3, $\mathcal{N}$ is elementarily embeddable in

[4]The disjoint union of $T_1, T_2, \ldots$ is the complete theory of a disjoint union of models of $T_1, T_2, \ldots$ with disjoint signatures and an extra unary predicate symbol for each universe

$(\mathcal{M}^{[0,1)}, \mathcal{L})$, and hence $\mathcal{N}$ is isomorphic to a pre-complete model $\mathcal{P} \prec (\mathcal{M}^{[0,1)}, \mathcal{L})$. By Theorem 4.5, we may take $\mathcal{P}$ so that its event sort is $\mathcal{L}$. By Lemma 4.7, there is a countable part $(\mathcal{K}, \mathcal{A})$ of $(\mathcal{M}^{[0,1)}, \mathcal{L})$ that is dense in $\mathcal{P}$. By Remark 4.4 (iv), $\mathcal{P} \cong \mathcal{P}^{cl}$. We may therefore take $\mathcal{P}$ so that $\mathcal{P} = \mathcal{P}^{cl} = (\mathcal{K}^{cl}, \mathcal{A}^{cl})$.

Since T has $\leq \omega$ countable models and $\mathcal{M}$ is countable saturated, there is a finite or countable set $\{\mathcal{M}_i : i \in I\}$ of elementary submodels $\mathcal{M}_i \prec \mathcal{M}$ such that for every countable model $\mathcal{H} \models T$ there is a unique $i \in I$ with $\mathcal{H} \cong \mathcal{M}_i$. Then the $\mathcal{M}_i$ are pairwise non-isomorphic. By Fact 2.12, for each $i \in I$ there is an $L_{\omega_1\omega}$ sentence φ_i that defines $\mathcal{M}_i$. Then by Lemma 4.11, for each $i \in I$ the set

$$\mathsf{A}_i = \{t \in [0,1) : \mathcal{K}(t) \cong \mathcal{M}_i\} = \{t \in [0,1) : \mathcal{K}(t) \models \varphi_i\}$$

is Borel, and by Lemmas 4.8 and 4.9, the countable part $(\mathcal{K}, \mathcal{A})$ can be taken so that the sets $\{\mathsf{A}_i : i \in I\}$ form a partition of $[0,1)$.

Therefore, by Theorem 7.6,

$$\mathcal{N} \cong \mathcal{P} \cong (\prod_{i \in I} (\mathcal{M}_i)^{\mathsf{A}_i}, \mathcal{L}).$$

$\blacksquare_{8.4}$

Our next theorem will show that when T has $\leq \omega$ countable models, the separable models of T^R are characterized up to isomorphism by a countable family of real numbers that assign a probability to each isomorphism type of countable models of T.

Given a theory T with $\leq \omega$ countable models, we let $\mathcal{M}(T)$ be the countable saturated model of T, which is unique up to isomorphism. The **isomorphism type** of a model $\mathcal{H} \prec \mathcal{M}(T)$ is the set of all elementary submodels of $\mathcal{M}(T)$ that are isomorphic to $\mathcal{H}$, and we let $I(T)$ be the set of all isomorphism types of elementary submodels of $\mathcal{M}(T)$. The set $I(T)$ is finite or countable, and a (probability) density function on $I(T)$ is a function $\rho : I(T) \to [0,1]$ such that $\sum_{i \in I(T)} \rho(i) = 1$. We will show that each separable model of T is characterized up to isomorphism by a density function on $I(T)$.

Definition 8.5. Let $\mathcal{N}$ be a separable model of T^R. A **density function for** $\mathcal{N}$ is a density function ρ on $I(T)$ such that $\mathcal{N}$ is isomorphic to some product randomization

$$(\prod_{i \in I(T)} (\mathcal{M}_i)^{\mathsf{A}_i}, \mathcal{L})$$

in $\mathcal{M}(T)$ where $\mathcal{M}_i$ has isomorphism type i and $\lambda(\mathsf{A}_i) = \rho(i)$ for each $i \in I(T)$.

Theorem 8.6. *(Representation Theorem) Suppose that T has $\leq \omega$ countable models.*

(i) *Every separable model of T^R has a unique density function;*
(ii) *any two separable models of T^R with the same density function are isomorphic;*
(iii) *for every density function ρ on $I(T)$, there is a separable model $\mathcal{N}$ of T^R with density function ρ.*

Proof. (ii) follows from Theorem 7.5, and (iii) follows from Theorem 7.3.

(i) Let $\mathcal{N}$ be a separable model of T^R. We show that $\mathcal{N}$ has a density function. By Lemma 8.4, there is a product randomization $(\prod_{j\in J}(\mathcal{M}_j)^{\mathsf{B}_j}, \mathcal{L})$ in $\mathcal{M}(T)$ that is isomorphic to $\mathcal{N}$, and the models $\mathcal{M}_j$ are pairwise non-isomorphic. We may then take J to be a subset of $I(T)$, so $\mathcal{M}_j$ belongs to the isomorphism type j for each $j \in J$. Put $\mathsf{A}_i = \mathsf{B}_i$ for $i \in J$, and $\mathsf{A}_i = \emptyset$ for $i \in I(T) \setminus J$. Then $[0,1) = \bigcup_{i\in I(T)} \mathsf{A}_i$ is a partition of $[0,1)$ into Borel sets, and $(\prod_{j\in J}(\mathcal{M}_j)^{\mathsf{B}_j}, \mathcal{L})$ is equal to $(\prod_{i\in I(T)}(\mathcal{M}_i)^{\mathsf{A}_i}, \mathcal{L})$, so $\mathcal{N}$ is isomorphic to the product randomization $(\prod_{i\in I(T)}(\mathcal{M}_i)^{\mathsf{A}_i}, \mathcal{L})$. Hence the function $\rho : i \mapsto \lambda(\mathsf{A}_i)$ is a density function for $\mathcal{N}$.

The uniqueness of the density function for $\mathcal{N}$ is a consequence of the general result below, Theorem 8.8. $\blacksquare_{8.6}$

Remark 8.7. For a given density function ρ on $I(T)$, the Borel sets A_i can always be taken to be intervals. If $\lhd$ is a well ordering of $I(T)$, we may take A_i to be the interval $[r_i, s_i)$ where $r_i = \sum_{j\lhd i}\rho(j)$ and $s_i = r_i + \rho(i)$, so the model with density function ρ will be isomorphic to the product randomization

$$(\prod_{i\in I(T)}(\mathcal{M}_i)^{[r_i,s_i)}, \mathcal{L}).$$

In the motivating case where T is an ω_1-categorical theory, it is natural to take the set of isomorphism types to have order type $\omega + 1$, where for each $k \le \omega$, i_k is the class of models of dimension k.

We now prove a uniqueness result for product randomizations that applies to all complete first order theories T.

Theorem 8.8. *(Uniqueness) Let $\mathcal{M}$ be a countable model of T and let I be a finite or countable set. Suppose that*

- $\mathcal{M}_i \prec \mathcal{M}$ *for each* $i \in I$*;*
- *the models* $\mathcal{M}_i$ *are pairwise non-isomorphic;*
- $[0,1) = \bigcup_{i\in I}\mathsf{B}_i$ *and* $[0,1) = \bigcup_{i\in I}\mathsf{C}_i$ *are partitions of* $[0,1)$ *into Borel sets;*
- $(\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{B}_i}, \mathcal{L})$ *and* $(\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{C}_i}, \mathcal{L})$ *are isomorphic.*

Then $\lambda(\mathsf{B}_i) = \lambda(\mathsf{C}_i)$ *for each* $i \in I$*.*

Proof. By Theorem 7.3, $\mathcal{P} = (\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{B}_i}, \mathcal{L})$ and $\mathcal{P}' = (\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{C}_i}, \mathcal{L})$ are pre-complete elementary substructures of $(\mathcal{M}^{[0,1)}, \mathcal{L})$. By hypothesis, there is an isomorphism $h : \mathcal{P} \cong \mathcal{P}'$. By Lemma 4.7, there is a countable part $(\mathcal{K}, \mathcal{A})$ of $(\mathcal{M}^{[0,1)}, \mathcal{L})$ that is dense in $\mathcal{P}$. We may also take $(\mathcal{K}, \mathcal{A})$ so that

$$\{\mathsf{B}_i : i \in I\} \cup \{h^{-1}(\mathsf{C}_i) : i \in I\} \subseteq \mathcal{A}.$$

It follows that the image $(\mathcal{J}, h(\mathcal{A})) = h(\mathcal{K}, \mathcal{A})$ is a countable part of $(\mathcal{M}^{[0,1)}, \mathcal{L})$ that is dense in $\mathcal{P}'$.

Claim. For each $i \in I$ we have $K(t) = M_i$ for almost all $t \in \mathsf{B}_i$, and $J(t) = M_i$ for almost all $t \in \mathsf{C}_i$.

Proof of Claim: For each $\mathbf{f} \in \mathcal{K}$ we have $\mathbf{f}(t) \in M_i$ for almost all $t \in \mathsf{B}_i$, so $K(t) \subseteq M_i$ for almost all $t \in \mathsf{B}_i$. Let $c \in M_i$. There exists a $\mathbf{g} \in \prod_{i \in I}(\mathcal{M}_i)^{\mathsf{B}_i}$ that has the constant value c on B_i. Since $(\mathcal{K}, \mathcal{A})$ is dense in $\mathcal{P}$, we have $c \in K(t)$ for almost all $t \in \mathsf{B}_i$. Since M_i is countable, it follows that $M_i \subseteq K(t)$ for almost all $t \in \mathsf{B}_i$. This proves the Claim.

By Fact 2.12, for each $i \in I$ there is an $L_{\omega_1\omega}$ sentence φ_i that defines $\mathcal{M}_i$. Since the $\mathcal{M}_i$ are pairwise non-isomorphic, we have $\mathcal{M}_j \models \varphi_i$ if and only if $j = i$. By Lemma 4.11, for each $i \in I$ the set $\{t : \mathcal{K}(t) \models \varphi_i\}$ is Borel, and $\lambda(\mathsf{B}_i \triangle \{t : \mathcal{K}(t) \models \varphi_i\}) = 0$. Similarly, $\{t : \mathcal{J}(t) \models \varphi_i\}$ is Borel, and $\lambda(\mathsf{C}_i \triangle \{t : \mathcal{J}(t) \models \varphi_i\}) = 0$. We see by induction on complexity that for each $L_{\omega_1\omega}$ formula $\psi(v_1, \ldots, v_n)$, and n-tuple $(\mathbf{f}_1, \ldots, \mathbf{f}_n)$ in $\mathcal{K}$,

$$\lambda(\{t : \mathcal{K}(t) \models \psi(\mathbf{f}_1(t), \ldots \mathbf{f}_n(t))\}) = \lambda(\{t : \mathcal{J}(t) \models \psi(h\mathbf{f}_1(t), \ldots h\mathbf{f}_n(t))\}).$$

Therefore for each $i \in I$ we have

$$\lambda(\mathsf{B}_i) = \lambda(\{t : \mathcal{K}(t) \models \varphi_i\}) = \lambda(\{t : \mathcal{J}(t) \models \varphi_i\}) = \lambda(\mathsf{C}_i).$$

$\blacksquare_{8.8}$

9. Homogeneity and Minimality over Events

In this section we will introduce the notion of a T^R-structure being homogeneous over events, which is weaker than being homogeneous, and the notion a prime T^R-structure being minimal over events, which is weaker than being minimal. We will see that a product randomization of countable homogeneous models of T is not always homogeneous, but is always homogeneous over events. Also, no prime model of T^R is minimal, but the Borel randomization of a prime minimal model of T is minimal over events.

9.1. **Homogeneity over Events.** In the next paragraph we will define the notion of a T^R-structure being homogeneous over events. In Examples 8.1 we listed some first order theories with $\leq \omega$ countable models in which every countable model is homogeneous. On the other hand, we will see in Theorem 9.5 that unless T is ω-categorical, T^R will have continuum many non-isomorphic separable models that are homogeneous over events but not homogeneous.

Definition 9.1. Two n-tuples $\vec{\mathbf{f}}, \vec{\mathbf{g}}$ in $\mathcal{K}$ **realize the same type over events** if $tp(\vec{\mathbf{f}}, \vec{\mathsf{B}})^{\mathcal{N}} = tp(\vec{\mathbf{g}}, \vec{\mathsf{B}})^{\mathcal{N}}$ for every tuple $\vec{\mathsf{B}}$ in $\mathcal{B}$. $\mathcal{N}$ is **homogeneous over events** if for every pair of n-tuples $\vec{\mathbf{f}}, \vec{\mathbf{g}}$ in $\mathcal{K}$ that realize the same type over events in $\mathcal{N}$, and every $\mathbf{f}'$ in $\mathcal{K}$, there exists $\mathbf{g}'$ in $\mathcal{K}$ such that $(\vec{\mathbf{f}}, \mathbf{f}')$ and $(\vec{\mathbf{g}}, \mathbf{g}')$ realize the same type over events in $\mathcal{N}$.

Remark 9.2. If $\mathcal{P}$ is a separable pre-complete model of T^R, then $\mathcal{P}$ is homogeneous, or homogeneous over events, if and only if its completion is.

Theorem 9.3. *(i) Every product randomization of countable homogeneous models of T is homogeneous over events.*

(ii) Suppose $\mathcal{M}$ is a countable model of T, $\mathcal{P} = (\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{B}_i}, \mathcal{L})$ is a product randomization in $\mathcal{M}$, and $\mathcal{P}$ is homogeneous over events. Then for each $i \in I$ with $\lambda(\mathsf{B}_i) > 0$, $\mathcal{M}_i$ is homogeneous.

Proof. (i) Let $\mathcal{M}$ be a countable model of T and let $\mathcal{P} = (\prod_{i\in\mathbb{N}} \mathcal{M}_i^{\mathsf{B}_i}, \mathcal{L})$, where each $\mathcal{M}_i \prec \mathcal{M}$ is countable homogeneous. $\mathcal{P}$ is pre-complete by Theorem 7.3. We must show that $\mathcal{P}$ is homogeneous over events. Let $\vec{\mathbf{f}}, \vec{\mathbf{g}}$ be n-tuples in $\prod_i \mathcal{M}_i^{B_i}$ which have the same type over events in $\mathcal{P}$, and let $\mathbf{f}' \in \prod_i \mathcal{M}_i^{B_i}$. There is a partition $\mathsf{C}_m, m \in \mathbb{N}$ of $[0, 1)$ such that each C_m belongs to $\mathcal{L}$ and $\vec{\mathbf{f}}, \vec{\mathbf{g}}, \mathbf{f}'$ are constant on C_m. Then for each $m \in \mathbb{N}$ and $i \in I$,

$$tp(\vec{\mathbf{f}}, \mathsf{B}_i \cap \mathsf{C}_m)^{\mathcal{P}} = tp(\vec{\mathbf{g}}, \mathsf{B}_i \cap \mathsf{C}_m)^{\mathcal{P}}.$$

Hence for each first order formula $\varphi(\vec{x})$,

$$\mu[\llbracket\varphi(\vec{\mathbf{f}})\rrbracket \sqcap (\mathsf{B}_i \cap \mathsf{C}_m)] = \mu[\llbracket\varphi(\vec{\mathbf{g}})\rrbracket \sqcap (\mathsf{B}_i \cap \mathsf{C}_m)].$$

It follows that whenever $\lambda(\mathsf{B}_i\cap\mathsf{C}_m) > 0$, the constant values of $\vec{\mathbf{f}}(t)$ and $\vec{\mathbf{g}}(t)$ for $t \in \mathsf{B}_i\cap\mathsf{C}_m$ must realize the same type in $\mathcal{M}_i$. Since each $\mathcal{M}_i$ is countable homogeneous, for each (i, m) there is an element $c(i, m) \in M_i$ such that whenever $t \in \mathsf{B}_i\cap\mathsf{C}_m$ and $\lambda(\mathsf{B}_i\cap\mathsf{C}_m) > 0$, $(\mathcal{M}, \vec{\mathbf{f}}(t), \mathbf{f}'(t)) \equiv (\mathcal{M}, \vec{\mathbf{g}}(t), c(i, m))$. Let $\mathbf{g}'$ be the function that has the constant value $c(i, m)$ on each set $\mathsf{B}_i\cap\mathsf{C}_m$. Then $\mathbf{g}' \in \prod_{i\in I}(M_i)^{\mathsf{B}_i}$, and $(\vec{\mathbf{f}}, \mathbf{f}')$ and $(\vec{\mathbf{g}}, \mathbf{g}')$ realize the same type over events in $\mathcal{P}$. It follows that $\mathcal{P}$ is homogeneous over events.

(ii) Suppose that for some $i \in I$, $\lambda(\mathsf{B}_i) > 0$ but $\mathcal{M}_i$ is not homogeneous. Then there are tuples $\vec{a}, \vec{b}$ and an element c in M_i such that $(\mathcal{M}_i, \vec{a}) \equiv (\mathcal{M}_i, \vec{b})$ but there is no d in M_i such that $(\mathcal{M}_i, \vec{b}, d) \equiv (\mathcal{M}_i, \vec{a}, c)$. Let $\vec{\mathbf{f}}, \vec{\mathbf{g}}, \mathbf{h}$ be functions in $\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{B}_i}$ such that:

- $\vec{\mathbf{f}}(t) = \vec{a}, \vec{\mathbf{g}}(t) = \vec{b}$, and $\mathbf{h}(t) = c$ for all $t \in \mathsf{B}_i$;
- $\vec{\mathbf{f}}(t) = \vec{\mathbf{g}}(t)$ for all $t \in [0, 1) \setminus \mathsf{B}_i$.

Then $\mathbf{f}, \mathbf{g}$ realize the same type over events in $\mathcal{P}$, but there is no $\mathbf{h}'$ in $\prod_{i\in I}(\mathcal{M}_i)^{\mathsf{B}_i}$ such that

$$tp(\vec{\mathbf{g}}, \mathbf{h}', \mathsf{B}_i)^{\mathcal{P}} = tp(\vec{\mathbf{f}}, \mathbf{h}, \mathsf{B}_i)^{\mathcal{P}}.$$

This contradicts the hypothesis that $\mathcal{P}$ is homogeneous over events. $\blacksquare_{9.3}$

Corollary 9.4. *Suppose T has $\leq \omega$ countable models. Then every countable model of T is homogeneous if and only if every separable model of T^R is homogeneous over events.*

Theorem 9.5. *If T is not ω-categorical, then T^R has continuum many non-isomorphic separable models that are homogeneous over events but not homogeneous.*

Proof. If $S_n(T)$ is countable for each n, then by Facts 2.11 and 3.1, T has a prime model $\mathcal{M}_1$ and a countable saturated model $\mathcal{M}_2$. These models are countable homogeneous, and $\mathcal{M}_1 \prec \mathcal{M}_2$ but there is a type $p \in S_n(T)$ that is realized in $\mathcal{M}_2$ but not in $\mathcal{M}_1$. If

$S_n(T)$ is uncountable for some n, then by Fact 2.10, T again has two countable homogeneous models $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \prec \mathcal{M}_2$ and some type $p \in S_n(T)$ is realized in $\mathcal{M}_2$ but not in $\mathcal{M}_1$. In either case, for each $r \in (0,1)$ let $\mathcal{P}_r$ be the product randomization $((\mathcal{M}_1)^{[0,r)} \times (\mathcal{M}_2)^{[r,1)}, \mathcal{L})$. By Theorems 7.3 and 9.3, each $\mathcal{P}_r$ is pre-complete and homogeneous over events. By Theorem 8.8, the pre-models $\mathcal{P}_r, r \in (0,1)$ are pairwise non-isomorphic.

We show that for $0 < r < 1$, $\mathcal{P}_r$ is not homogeneous. Let $a, b \in M_1$ with $a \neq b$. Let $\mathbf{a}$ be the constant function with value a. Let $\mathbf{f}, \mathbf{g}$ be the functions such that $\mathbf{f}$ maps $[0,r)$ to a and $[r,1)$ to b, and $\mathbf{g}$ maps $[1-r,1)$ to a and $[0,1-r)$ to b. Then $(\mathbf{a},\mathbf{f})$ and $(\mathbf{a},\mathbf{g})$ have the same type in $\mathcal{P}_r$. There is a tuple $\vec{\mathbf{h}}$ in $\mathcal{P}_r$ such that $\vec{\mathbf{h}}(t)$ realizes p whenever $\mathbf{f}(t) \neq \mathbf{a}(t)$. But there cannot be a tuple $\vec{\mathbf{h}}'$ in $\mathcal{P}_r$ such that $\vec{\mathbf{h}}'(t)$ realizes p whenever $\mathbf{g}(t) \neq \mathbf{a}(t)$, so there cannot be a tuple $\vec{\mathbf{h}}'$ in $\mathcal{P}_r$ such that $(\mathbf{a}, \mathbf{g}, \vec{\mathbf{h}}')$ has the same type as $(\mathbf{a}, \mathbf{f}, \vec{\mathbf{h}})$ in $\mathcal{P}_r$. Therefore $\mathcal{P}_r$ is not homogeneous. $\blacksquare_{9.5}$

9.2. **Minimality over Events.** In this subsection we answer a question posed by the anonymous referee. Recall that by Proposition 3.5, T^R cannot have a prime model that is minimal. The referee asked whether T^R can have a prime model that is minimal over events.

Definition 9.6. A prime model $\mathcal{N}$ of T^R is **minimal over events** if every elementary submodel of $\mathcal{N}$ that contains all the events of $\mathcal{N}$ is equal to $\mathcal{N}$.

Theorem 9.7. *Let $\mathcal{M}$ be a prime model of T. Then $\mathcal{M}$ is minimal if and only if the completion of $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is minimal over events.*

Proof. $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is prime by Lemma 3.3.

Suppose first that $\mathcal{M}$ is not minimal. Then there is a model $\mathcal{H} \prec \mathcal{M}$ such that $\mathcal{H} \neq \mathcal{M}$, so there is an element $a \in M \setminus H$. By Remark 2.3 we have $(\mathcal{H}^{[0,1)}, \mathcal{L}) \prec (\mathcal{M}^{[0,1)}, \mathcal{L})$, and the constant function with value a is a distance one from every element of $\mathcal{H}^{[0,1)}$, so the completion of $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is not minimal over events.

Now suppose that the completion of $(\mathcal{M}^{[0,1)}, \mathcal{L})$ is not minimal over events. Then there is a pre-complete $(\mathcal{K}, \mathcal{L}) \prec (\mathcal{M}^{[0,1)}, \mathcal{L})$ and an element $\mathbf{f} \in \mathcal{M}^{[0,1)} \setminus \mathcal{K}^{cl}$. By Fact 2.6, $(\mathcal{K}, \mathcal{L})$ has perfect witnesses, so by Lemma 4.7, $(\mathcal{M}^{[0,1)}, \mathcal{L})$ has a countable part $(\mathcal{K}_0, \mathcal{A})$ that is dense in $(\mathcal{K}, \mathcal{L})$. Then $\mathcal{K}_0^{cl} = \mathcal{K}^{cl}$ and $\mathcal{A}_0$ is dense in $\mathcal{L}$. By Lemma 4.9, it is not true that $\mathbf{f}(t) \in \mathcal{K}_0(t)$ for almost all $t \in [0,1)$, but by Lemma 4.8, we have $\mathcal{K}_0(t) \prec \mathcal{M}$ for almost all $t \in [0,1)$. Therefore there exists $t \in [0,1)$ such that $f(t) \in M \setminus \mathcal{K}_0(t)$ and $\mathcal{K}_0(t) \prec \mathcal{M}$. Hence $\mathcal{M}$ is not minimal. $\blacksquare_{9.7}$

Corollary 9.8. *T has a prime model that is minimal if and only if T^R has a prime model that is minimal over events.*

Proof. The forward direction follows at once from Theorem 9.7. For the converse, suppose $\mathcal{N}$ is a prime model of T^R that is minimal over events. By Theorem 3.4 (ii), $\mathcal{N}$ is isomorphic to $(\mathcal{M}^{[0,1)}, \mathcal{L})$ for some prime model $\mathcal{M}$ of T. Then by Theorem 9.7, $\mathcal{M}$ is minimal. $\blacksquare_{9.8}$

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Department of Mathematics,, University of Wisconsin-Madison, Madison, WI 53706, www.math.wisc.edu/∼andrews

E-mail address: andrews@math.wisc.edu

Department of Mathematics,, University of Wisconsin-Madison, Madison, WI 53706, www.math.wisc.edu/∼keisler

E-mail address: keisler@math.wisc.edu