RANDOMIZATIONS OF SCATTERED SENTENCES

H. JEROME KEISLER

Abstract. In 1970, Morley introduced the notion of a sentence φ of the infinitary logic $L_{\omega_1\omega}$ being scattered. He showed that if φ is scattered then the class $I(\varphi)$ of isomorphism types of countable models of φ has cardinality at most $\aleph_1$, and if φ is not scattered then $I(\varphi)$ has cardinality continuum. The absolute form of Vaught's conjecture for φ says that if φ is scattered then $I(\varphi)$ is countable. Generalizing previous work of Ben Yaacov and the author, we introduce here the notion of a separable randomization of φ, which is a separable continuous structure whose elements are random elements of countable models of φ. We improve a result by Andrews and the author, showing that if $I(\varphi)$ is countable then φ has few separable randomizations, that is, every separable randomization of φ is isomorphic to a very simple structure called a basic randomization. We also show that if φ has few separable randomizations, then φ is scattered. Hence if the absolute Vaught conjecture holds for φ, then φ has few separable randomizations if and only if $I(\varphi)$ is countable, and also if and only if φ is scattered. Moreover, assuming Martin's axiom for $\aleph_1$, we show that if φ is scattered then φ has few separable randomizations.

1. Introduction

The notion of a scattered sentence φ of the infinitary logic $L_{\omega_1\omega}$ was introduced by Michael Morley [13] in connection with Vaught's conjecture. The notion of a randomization was introduced by the author in [10] and developed in the setting of continuous model theory by Itaï Ben Yaacov and the author in [6]. The **pure randomization theory** is a continuous theory with a sort $\mathbb{K}$ for random elements and a sort $\mathbb{E}$ for events, and a set of axioms that say that there is an event corresponding to each first order formula with random elements in its argument places, and there is an atomless probability measure on the events. By a **separable randomization** of a first order theory T we mean a separable model of the pure randomization theory in which each axiom of T has probability one.

In [1], Uri Andrews and the author showed that if T is a complete theory with at most countably many countable models up to isomorphism, then T has few separable randomizations, which means that all of its separable randomizations are very simple in a sense explained below. In this paper we generalize that result by replacing the theory T with an infinitary sentence φ, and establish relationships between sentences with countably many countable models, scattered sentences, sentences with few separable randomizations, and Vaught's conjecture.

Date: December 8, 2016.

Let φ be a sentence of $L_{\omega_1\omega}$ whose models have at least two elements, and let $I(\varphi)$ be the class of isomorphism types of countable models of φ. In [13], Morley showed that if φ is scattered then $I(\varphi)$ has cardinality at most $\aleph_1$, and if φ is not scattered then $I(\varphi)$ has cardinality continuum. The absolute form of Vaught's conjecture for φ says that if φ is scattered then $I(\varphi)$ is at most countable.

In the version of continuous model theory developed in [5], the universe of a structure is a complete metric space with distance playing the role of equality, and formulas take values in the unit interval $[0, 1]$ with 0 interpreted as true. A model is separable if its universe has a countable dense subset. The **randomization signature** L^R has two sorts, $\mathbb{K}$ for random elements and $\mathbb{E}$ for events. L^R has a function symbol $[\![\psi(\cdot)]\!]$ of sort $\mathbb{K}^n \to \mathbb{E}$ for each first order formula $\psi(\vec{v})$ with n free variables. The continuous term $[\![\psi(\vec{\mathbf{f}})]\!]$ is interpreted as the event that the formula $\psi(\vec{v})$ is satisfied by the n-tuple $\vec{\mathbf{f}}$ of random elements. In the event sort $\mathbb{E}$, L^R has the Boolean operations and a predicate μ. The continuous formula $\mu(\mathsf{E})$ takes values in $[0, 1]$ and is interpreted as the probability of the event E.

In Theorem 5.1 we show that in any separable model of the pure randomization theory, the function $[\![\psi(\cdot)]\!]$ can be extended in a natural way from the case that $\psi(\vec{v})$ is a first order formula to the case that $\psi(\vec{v})$ is a formula of $L_{\omega_1\omega}$. We can then define a **separable randomization** of an infinitary sentence φ to be a separable model of the pure randomization theory in which $[\![\varphi]\!]$ has probability one.

A **basic randomization of** φ is a very simple kind of separable randomization of φ that is determined up to isomorphism by taking a countable subset $J \subseteq I(\varphi)$ and assigning a probability $\rho(j)$ to each $j \in J$. A basic randomization of φ has a model $\mathcal{M}_j$ of isomorphism type j for each $j \in J$, and a partition of $[0, 1)$ into Borel sets B_j of measure $\rho(j)$. The events are the Borel subsets of $[0, 1)$ with the usual measure, and the random elements are the Borel functions that send B_j into $\mathcal{M}_j$ for each $j \in J$.

We say that φ **has few separable randomizations** if every separable randomization of φ is isomorphic to a basic randomization of φ.

In Theorem 9.6, we show that if $I(\varphi)$ is countable, then φ has few separable randomizations. In Theorem 10.1 we show that if φ has few separable randomizations, then φ is scattered. Therefore, if the absolute form of Vaught's conjecture holds for φ, then φ has few separable randomizations if and only if $I(\varphi)$ is countable, and also if and only if φ is scattered. In Theorem 10.3 we show that if Martin's axiom for $\aleph_1$ holds and φ is scattered, then φ has few separable randomizations.

Section 2 reviews some results we need in the literature about scattered sentences and Vaught's conjecture. Section 3 contains a review of some previous results about randomizations. In Section 4 we introduce the basic randomizations of φ. In Section 5 we introduce the separable randomizations of φ. In Section 6 we develop a key tool for constructing separable randomizations, called a countable generator, and in Section 7 we show that every separable randomization of φ is isomorphic to one that can be constructed in that way. In Section 8 we show that every separable randomization of φ

can be elementarily embedded in some basic randomization if and only if only countably many first order types are realized in countable models of φ. The methods developed in Sections 6 through 8 are used to prove our main results are in Sections 9 and 10. In Section 11 we list some open questions that are related to our results.

2. Scattered Sentences

We fix a countable[1] first order signature L, and all first order structures mentioned are understood to have signature L. We refer to [9] for the infinitary logic $L_{\omega_1\omega}$. Note in particular that every formula of $L_{\omega_1\omega}$ has at most finitely many free variables. By a **countable fragment** L_A of $L_{\omega_1\omega}$ we mean a countable set of formulas of $L_{\omega_1\omega}$ that contains the first order formulas and is closed under subformulas, finite Boolean combinations, quantifiers, and change of free variables.

In general, the class of countable first order structures is a proper class. To avoid this problem, let $\mathbb{M}(L)$ be the class of countable structures with signature L, whose universe is $\mathbb{N}$ or an initial segment of $\mathbb{N}$. Then $\mathbb{M}(L)$ is a set, and every countable structure is isomorphic to some element of $\mathbb{M}(L)$. We define the **isomorphism type** of a countable structure $\mathcal{M}$ to be the set of all $\mathcal{H} \in \mathbb{M}(L)$ such that $\mathcal{H}$ is isomorphic to $\mathcal{M}$.

Consider a sentence φ of $L_{\omega_1\omega}$ that has at least one model. By the Löwenheim-Skolem Theorem, φ has at least one countable model. We let $I(\varphi)$ be the set of all isomorphism types of countable models of φ. By a **Scott sentence** for a countable structure $\mathcal{M}$ we mean an $L_{\omega_1\omega}$ sentence θ such that $\mathcal{M} \models \theta$, and every countable model of θ is isomorphic to $\mathcal{M}$.

Result 2.1. *(Scott's Theorem,* [15]*) Every countable structure has a Scott sentence.*

We let I be the set of all isomorphism types of countable structures of cardinality ≥ 2. Thus $I = I((\exists x)(\exists y)x \neq y)$. For each $i \in I$, we choose once and for all a Scott sentence θ_i for the countable models of isomorphism type i. We say that two countable L-structures $\mathcal{M}, \mathcal{H}$ are α-equivalent if they satisfy the same $L_{\omega_1\omega}$-sentences of quantifier rank at most α. By Scott's theorem, $\mathcal{M}$ and $\mathcal{H}$ are isomorphic if and only if they are α-equivalent for all countable α.

Several equivalent characterizations of scattered sentences were given in [4]. We will take one of these as our definition.

Definition 2.2. An $L_{\omega_1\omega}$ sentence φ is **scattered** if for each countable ordinal α, there are at most countably many α-equivalence classes of countable models of φ. A first order theory T is scattered if the sentence $\bigwedge T$ is scattered.

Result 2.3. *(Morley* [13]*) If φ is scattered then $I(\varphi)$ has cardinality at most $\aleph_1$, and if φ is not scattered than $I(\varphi)$ has cardinality $2^{\aleph_0}$.*

[1] In this paper, "countable" means "of cardinality at most $\aleph_0$".

The **Vaught conjecture for** φ ([18]) says that $I(\varphi)$ is either countable or has cardinality $2^{\aleph_0}$. The **absolute Vaught conjecture for** φ (see Steel [17]) says that if φ is scattered, then $I(\varphi)$ is countable. It is called absolute because its truth does not depend on the underlying model of ZFC. In $ZFC+GCH$ the Vaught conjecture trivially holds for all φ. In $ZFC+\neg CH$, the absolute Vaught conjecture for φ is equivalent to the Vaught conjecture for φ.

Definition 2.4. (Morley [13]) An **enumerated structure** $(\mathcal{M}, a)$ is a countable structure $\mathcal{M}$ with signature L together with a mapping a from $\mathbb{N}$ onto the universe M of $\mathcal{M}$.

Consider a countable fragment L_A and an enumerated structure $(\mathcal{M}, a)$. We take 2^{L_A} to be the Polish space whose elements are the functions from L_A into $\{0,1\}$. We say that a point $t \in 2^{L_A}$ **codes** $(\mathcal{M}, a)$ if for each formula $\psi \in L_A$ with at most the free variables $v_0, \ldots, v_{n-1}$, $t(\psi) = 0$ if and only if $\mathcal{M} \models \psi(a_0, \ldots, a_{n-1})$. Note that each enumerated structure is coded by a unique $t \in 2^{L_A}$.

The lemma below is a variant of Theorem 3.3 in [4], and follows from its proof.

Lemma 2.5. *Let φ be a sentence of $L_{\omega_1\omega}$. The following are equivalent:*

(i) *φ is not scattered.*

(ii) *There is a countable fragment L_A of $L_{\omega_1\omega}$ and a perfect set $P \subseteq 2^{L_A}$ such that each $t \in P$ codes an enumerated model $(\mathcal{M}(t), a(t))$ of φ, and if $s \neq t$ in P then $\mathcal{M}(s)$ and $\mathcal{M}(t)$ do not satisfy the same L_A-sentences.*

3. Randomizations of Theories

3.1. **Continuous Structures.** We assume familiarity with the basic notions about continuous model theory as developed in [5]. We give some brief reminders here.

In continuous model theory, the universe of a structure is a complete metric space, and the universe of a pre-structure is a pseudo-metric space. A structure (or pre-structure) is said to be **separable** if its universe is a separable metric space (or pseudo-metric space). Formulas take truth values in $[0,1]$, and are built from atomic formulas using continuous connectives on $[0,1]$ and the quantifiers sup, inf. The value 0 in interpreted as truth, and a model of a set U of sentences is a continuous structure in which each $\Phi \in U$ has truth value 0.

We extend the notions of embedding and elementary embedding to pre-structures in the natural way. Given pre-structures $\mathcal{P}$, $\mathcal{N}$, we write $h : \mathcal{P} \prec \mathcal{N}$ (h is an **elementary embedding**) if h preserves the truth values of all formulas. If $h\colon \mathcal{P} \prec \mathcal{N}$ where h is the inclusion mapping, we write $\mathcal{P} \prec \mathcal{N}$ and say that $\mathcal{P}$ is an **elementary submodel** of $\mathcal{N}$ (leaving off the 'pre-' for brevity). If $h\colon \mathcal{P} \prec \mathcal{N}$, h preserves distance but is not necessarily one-to-one. Note that compositions of elementary embeddings are elementary embeddings. We write $h\colon \mathcal{P} \cong \mathcal{N}$ if $h : \mathcal{P} \prec \mathcal{N}$ and every element of $\mathcal{N}$ is at distance zero from some element of $h(\mathcal{P})$. We say that $\mathcal{P}$ and $\mathcal{N}$ are **isomorphic**, and write

$\mathcal{P} \cong \mathcal{N}$, if $h\colon \mathcal{P} \cong \mathcal{N}$ for some h. By Remark 2.4 of [1], $\cong$ is an equivalence relation on pre-structures.

We call $\mathcal{N}$ a **reduction of** $\mathcal{P}$ if $\mathcal{N}$ is obtained from $\mathcal{P}$ by identifying elements at distance zero, and call $\mathcal{N}$ a **completion of** $\mathcal{P}$ if $\mathcal{N}$ is a structure obtained from a reduction of $\mathcal{P}$ by completing the metrics. Every pre-structure has a reduction, that is unique up to isomorphism. The mapping that identifies elements at distance zero is called the **reduction mapping**, and is an isomorphism from a pre-structure onto its reduction. Similarly, every pre-structure $\mathcal{P}$ has a completion, that is unique up to isomorphism, and the reduction map is an elementary embedding of $\mathcal{P}$ into its completion.

Following [6], we say that $\mathcal{P}$ is **pre-complete** if the metrics in a reduction of $\mathcal{P}$ are already complete. Thus if $\mathcal{P}$ is pre-complete, the reductions and completions of $\mathcal{P}$ are the same, and $\mathcal{P}$ is isomorphic to its completion.

3.2. **Randomizations.** We assume that:

- L is a countable first order signature.
- T_2 is the theory with the single axiom $(\exists x)(\exists y)x \neq y$.
- T is a theory with signature L that contains T_2.
- φ is a sentence of $L_{\omega_1\omega}$ that implies T_2.

Note that T_2 is just the theory whose models have at least two elements, and $I(\varphi) \subseteq I(T_2) = I$. The randomization theory of T is a continuous theory T^R whose signature L^R has two sorts, a sort $\mathbb{K}$ for random elements of models of T, and a sort $\mathbb{E}$ for events in an underlying probability space. The probability of the event that a first order formula holds for a tuple of random elements will be expressible by a formula of continuous logic. The signature L^R has an n-ary function symbol $[\![\theta(\cdot)]\!]$ of sort $\mathbb{K}^n \to \mathbb{E}$ for each first order formula θ of L with n free variables, a $[0,1]$-valued unary predicate symbol μ of sort $\mathbb{E}$ for probability, and the Boolean operations $\top, \bot, \sqcap, \sqcup, \neg$ of sort $\mathbb{E}$. The signature L^R also has distance predicates $d_{\mathbb{E}}$ of sort $\mathbb{E}$ and $d_{\mathbb{K}}$ of sort $\mathbb{K}$. In L^R, we use $\mathsf{B}, \mathsf{C}, \ldots$ for variables or parameters of sort $\mathbb{E}$, and $\mathsf{B} \doteq \mathsf{C}$ means $d_{\mathbb{E}}(\mathsf{B}, \mathsf{C}) = 0$. For readability we write $\forall, \exists$ for $\sup, \inf$.

The axioms of T^R, which are taken from [6], are as follows:

Validity Axioms

$$\forall \vec{x}([\![\psi(\vec{x})]\!] \doteq \top)$$

where $\forall \vec{x}\, \psi(\vec{x})$ is logically valid in first order logic.

Boolean Axioms The usual Boolean algebra axioms in sort $\mathbb{E}$, and the statements

$$\forall \vec{x}([\![(\neg\theta)(\vec{x})]\!] \doteq \neg[\![\theta(\vec{x})]\!])$$

$$\forall \vec{x}([\![(\varphi \vee \psi)(\vec{x})]\!] \doteq [\![\theta(\vec{x})]\!] \sqcup [\![\psi(\vec{x})]\!])$$

$$\forall \vec{x}([\![(\theta \wedge \psi)(\vec{x})]\!] \doteq [\![\theta(\vec{x})]\!] \sqcap [\![\psi(\vec{x})]\!])$$

Distance Axioms

$$\forall x \forall y\, d_{\mathbb{K}}(x, y) = 1 - \mu[\![x = y]\!], \qquad \forall \mathsf{B} \forall \mathsf{C}\, d_{\mathbb{E}}(\mathsf{B}, \mathsf{C}) = \mu(\mathsf{B} \triangle \mathsf{C})$$

Fullness Axioms (or Maximal Principle)

$$\forall \vec{y} \exists x ([\![\theta(x, \vec{y})]\!] \doteq [\![(\exists x \theta)(\vec{y})]\!])$$

Event Axiom

$$\forall \mathsf{B} \exists x \exists y (\mathsf{B} \doteq [\![x = y]\!])$$

Measure Axioms

$$\mu[\top] = 1 \wedge \mu[\bot] = 0$$
$$\forall \mathsf{B} \forall \mathsf{C} (\mu[\mathsf{B}] + \mu[\mathsf{C}] = \mu[\mathsf{B} \sqcup \mathsf{C}] + \mu[\mathsf{B} \sqcap \mathsf{C}])$$

Atomless Axiom

$$\forall \mathsf{B} \exists \mathsf{C} (\mu[\mathsf{B} \sqcap \mathsf{C}] = \mu[\mathsf{B}]/2)$$

Transfer Axioms

$$[\![\theta]\!] \doteq \top$$

where $\theta \in T$.

By a **separable randomization of** T we mean a separable pre-model of T^R. In this paper we will focus on the **pure randomization theory** T_2^R. T_2^R has the single transfer axiom $[\![(\exists x)(\exists y) x \neq y]\!] \doteq \top$. Note that for any theory $T \supseteq T_2$, any model of T^R is a model of the pure randomization theory. By a **separable randomization** we mean a separable randomization of T_2^R. A separable randomization is called **complete** if it is a model of T_2^R, and **pre-complete** if it is a pre-complete model of T_2^R.

We will use $\mathcal{M}, \mathcal{H}$ to denote models of T_2 with signature L, and use $\mathcal{N}$ and $\mathcal{P}$ to denote models or pre-models of T_2^R with signature L^R. The universe of $\mathcal{M}$ will be denoted by M. A pre-model of T_2^R will be a pair $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ where $\mathcal{K}$ is the part of sort $\mathbb{K}$ and $\mathcal{E}$ is the part of sort $\mathbb{E}$. We write $[\![\theta(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ for the interpretation of $[\![\theta(\vec{v})]\!]$ in a pre-structure $\mathcal{N}$ at a tuple $\vec{\mathbf{f}}$, and write $[\![\theta(\vec{\mathbf{f}})]\!]$ for $[\![\theta(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ when $\mathcal{N}$ is clear from the context.

Result 3.1. *([6], Theorem 2.7) Every model or pre-complete model* $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ *of* T_2^R *has perfect witnesses, i.e.,*

(i) for each first order formula $\theta(x, \vec{y})$ *and each* $\vec{\mathbf{g}}$ *in* $\mathcal{K}^n$ *there exists* $\mathbf{f} \in \mathcal{K}$ *such that*

$$[\![\theta(\mathbf{f}, \vec{\mathbf{g}})]\!] \doteq [\![(\exists x \theta)(\vec{\mathbf{g}})]\!];$$

(ii) for each $\mathsf{B} \in \mathcal{E}$ *there exist* $\mathbf{f}, \mathbf{g} \in \mathcal{K}$ *such that* $\mathsf{B} \doteq [\![\mathbf{f} = \mathbf{g}]\!]$.

We let $\mathcal{L}$ be the family of Borel subsets of $[0, 1)$, and let $([0, 1), \mathcal{L}, \lambda)$ be the usual probability space, where λ is the restriction of Lebesgue measure to $\mathcal{L}$. We let $\mathcal{M}^{[0,1)}$ be the set of functions with countable range from $[0, 1)$ into M such that the inverse image of any element of M belongs to $\mathcal{L}$. The elements of $\mathcal{M}^{[0,1)}$ are called **random elements of** $\mathcal{M}$.

Definition 3.2. The **Borel randomization of** $\mathcal{M}$ is the pre-structure $(\mathcal{M}^{[0,1)}, \mathcal{L})$ for L^R whose universe of sort $\mathbb{K}$ is $\mathcal{M}^{[0,1)}$, whose universe of sort $\mathbb{E}$ is $\mathcal{L}$, whose measure μ is given by $\mu(\mathsf{B}) = \lambda(\mathsf{B})$ for each $\mathsf{B} \in \mathcal{L}$, and whose $[\![\psi(\cdot)]\!]$ functions are

$$[\![\psi(\vec{\mathbf{f}})]\!] = \{t \in [0,1) : \mathcal{M} \models \psi(\vec{\mathbf{f}}(t))\}.$$

(So $[\![\psi(\vec{\mathbf{f}})]\!] \in \mathcal{L}$ for each first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{f}}$ in $\mathcal{M}^{[0,1)}$). Its distance predicates are defined by

$$d_{\mathbb{E}}(\mathsf{B}, \mathsf{C}) = \mu(\mathsf{B} \triangle \mathsf{C}), \quad d_{\mathbb{K}}(\mathbf{f}, \mathbf{g}) = \mu([\![\mathbf{f} \neq \mathbf{g}]\!]),$$

where $\triangle$ is the symmetric difference operation.

Result 3.3. *([6], Corollary 3.6) Every Borel randomization of a countable model of T_2 is a pre-complete separable randomization (in other words, a pre-complete separable model of T_2^R).*

Result 3.4. *([1], Theorem 4.5) Suppose $\mathcal{N}$ is pre-complete and elementarily embeddable in the Borel randomization $(\mathcal{M}^{[0,1)}, \mathcal{L})$ of a countable model of T_2. Then $\mathcal{N}$ is isomorphic to an elementary submodel of $(\mathcal{M}^{[0,1)}, \mathcal{L})$ whose event sort is all of $\mathcal{L}$.*

4. Basic Randomizations

Basic randomizations are generalizations of Borel randomizations. They are very simple continuous pre-structures of sort L^R. Intuitively, a basic randomization is a combination of countably many Borel randomizations of first order structures. [1] considered basic randomizations that are combinations of Borel randomizations of models of a single complete theory T, and called them called product randomizations.

Definition 4.1. Suppose that

- J is a countable subset of I;
- $[0,1) = \bigcup_{j \in J} \mathsf{B}_j$ is a partition of $[0,1)$ into Borel sets of positive measure;
- for each $j \in J$, $\mathcal{M}_j$ has isomorphism type j;
- $\prod_{j \in J} \mathcal{M}_j^{\mathsf{B}_j}$ is the set of all functions $\mathbf{f} \colon [0,1) \to \bigcup_{j \in J} M_j$ such that for all $j \in J$,
$$(\forall t \in \mathsf{B}_j)\mathbf{f}(t) \in M_j \text{ and } (\forall a \in M_j)\{t \in \mathsf{B}_j \colon \mathbf{f}(t) = a\} \in \mathcal{L};$$
- $(\prod_{j \in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ is the pre-structure for L^R whose whose measure and distance functions are as in Definition 3.2. and $[\![\psi(\cdot)]\!]$ functions are
$$[\![\psi(\vec{\mathbf{f}})]\!] = \bigcup_{j \in J} \{t \in \mathsf{B}_j \colon \mathcal{M}_j \models \psi(\vec{\mathbf{f}}(t))\},$$

$(\prod_{i \in J} \mathcal{M}_i^{\mathsf{B}_i}, \mathcal{L})$ is called a **basic randomization**. Given a basic randomization, we let $\mathcal{M}_t = \mathcal{M}_j$ whenever $j \in J$ and $t \in \mathsf{B}_j$. By a **basic randomization of** φ we mean a basic randomization such that $\mathcal{M}_j \models \varphi$ for each $j \in J$.

Remark 4.2.

(1) In a basic randomization, the set $\bigcup_{j\in J} M_j$ is countable, so each $\mathbf{f} \in \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$ has countable range.
(2) If $\mathcal{M}_j \cong \mathcal{H}_j$ for each $j \in J$, then $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L}) \cong (\prod_{j\in J} \mathcal{H}_j^{\mathsf{B}_j}, \mathcal{L})$.
(3) Every basic randomization $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ is isomorphic to a basic randomization $(\prod_{j\in J} \mathcal{H}_j^{\mathsf{B}_j}, \mathcal{L})$ such that for each $j \in J$, $\mathcal{H}_j \in \mathbb{M}(L)$ (so the universe of $\mathcal{H}_j$ is $\mathbb{N}$ or an initial segment of $\mathbb{N}$).
(4) If $\mathcal{M}_j \prec \mathcal{H}_j$ for each $j \in J$, then $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L}) \prec (\prod_{j\in J} \mathcal{H}_j^{\mathsf{B}_j}, \mathcal{L})$. (In this part we do not require that $\mathcal{H}_j$ has isomorphism type j).

Lemma 4.3. *Every basic randomization $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ is a pre-model of the pure randomization theory.*

Proof. All of the axioms for T_2^R except the Fullness Axioms hold trivially. Therefore $\mathcal{P}$ is a pseudo-metric space in both sorts. By Result 3.3, $(\mathcal{M}_j^{[0,1)}, \mathcal{L})$ satisfies the Fullness Axioms for each $j \in J$, and it follows easily that $\mathcal{P}$ also satisfies the Fullness Axioms, and thus is a pre-model of T_2^R. $\blacksquare_{4.3}$

We next introduce useful mappings from a basic randomization $(\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ to the Borel randomizations $(\mathcal{M}_j^{[0,1)}, \mathcal{L})$.

Definition 4.4. Suppose $\mathsf{B} \in \mathcal{L}$ and $\lambda(\mathsf{B}) > 0$. We say that a mapping ℓ **stretches** B to $[0, 1)$ if ℓ is a Borel bijection from B to $[0, 1)$, ℓ^{-1} is also Borel, and for each Borel set $\mathsf{A} \subseteq \mathsf{B}$, $\lambda(\ell(\mathsf{A})) = \lambda(\mathsf{A})/\lambda(\mathsf{B})$.

Let $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ be a basic randomization, and for each $j \in J$, choose an ℓ_j that stretches B_j to $[0, 1)$. Define the mapping $\ell_j \colon \mathcal{P} \to (\mathcal{M}_j^{[0,1)}, \mathcal{L})$ by

$$(\ell_j(\mathbf{f}))(t) = \mathbf{f}(\ell_j^{-1}(t)), \quad \ell_j(\mathsf{E}) = \ell_j(\mathsf{B}_j \cap \mathsf{E}).$$

Remark 4.5. Let $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ be a basic randomization.
(1) For each $j \in J$, there exists a mapping ℓ_j that stretches B_j to $[0, 1)$.
(2) ℓ_j maps $\mathcal{P}$ onto $\mathcal{P}_j := (\mathcal{M}_j^{[0,1)}, \mathcal{L})$.
(3) For each first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{f}}$ of elements of $\mathcal{P}$ of sort $\mathbb{K}$.

$$\lambda([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{P}}) = \sum_{j\in J} \lambda(\mathsf{B}_j)\lambda([\![\psi(\ell_j\vec{\mathbf{f}})]\!]^{\mathcal{P}_j}).$$

(4) $d_{\mathbb{K}}^{\mathcal{P}}(\mathbf{f}, \mathbf{g}) = \sum_{j\in J} \lambda(\mathsf{B}_j) d_{\mathbb{K}}^{\mathcal{P}_j}(\ell_j(\mathbf{f}), \ell_j(\mathbf{g}))$.

Proof. Since $\nu(\mathsf{A}) = \lambda(\mathsf{A})/\lambda(\mathsf{B}_j)$ is a probability measure on B_j, (1) follows from Theorem 17.41 in [8]. (2)–(4) are clear $\blacksquare_{4.5}$

The following result is a generalization of Theorem 7.3 of [1], but the proof we give here is different.

Theorem 4.6. *Every basic randomization is pre-complete and separable.*

Proof. Let $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ be a basic randomization. By Result 3.3, $\mathcal{P}$ is separable and pre-complete in the event sort. For each $j \in J$, pick a mapping ℓ_j that stretches B_j to $[0,1)$. Pick an element $\mathbf{a} \in \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$.

Separability in sort $\mathbb{K}$: By 3.3, for each $j \in J$, there is a countable set C_j that is dense in $\mathcal{M}_j^{[0,1)}$. For each finite $F \subseteq J$, let D_F be the set of all $\mathbf{f}$ such that for all $j \in F$, $\mathbf{f}$ agrees with some element of $\ell_j^{-1}C_j$ on B_j, and $\mathbf{f}$ agrees with $\mathbf{a}$ on $[0,1) \setminus \bigcup_{i\in F} \mathsf{B}_j$. Then $D = \bigcup_F D_F$ is a countable subset of $\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$. For each $\varepsilon > 0$, there is a finite $F \subseteq J$ such that $\sum_{j\in F} \mu(\mathsf{B}_j) \geq 1-\varepsilon$. It follows that for each $\mathbf{g} \in \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$, there exists $\mathbf{f} \in D_F$ such that for each $j \in F$, $d_{\mathbb{K}}(\ell_j(\mathbf{f}), \ell_j(\mathbf{g})) < \varepsilon/(|F|+1)$, and therefore by Remark 4.5, $d_{\mathbb{K}}(\mathbf{f}, \mathbf{g}) < 2\varepsilon$. Hence D is dense in $\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$.

Pre-completeness in sort $\mathbb{K}$: Suppose that $\langle \mathbf{f}_n \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence of sort $\mathbb{K}$. By Remark 4.5, for each $j \in J$, $\langle \ell_j(\mathbf{f}_n)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}_j^{[0,1)}$. By Result 3.3, $\mathcal{M}_j^{[0,1)}$ is pre-complete, so there exists $\mathbf{g}_j$ in $\mathcal{M}_j^{[0,1)}$ such that $\lim_{n\to\infty} d_{\mathbb{K}}(\ell_j(\mathbf{f}_n), \mathbf{g}_j) = 0$. Let $\mathbf{g}$ be the function that agrees with $\ell_j^{-1}\mathbf{g}_j$ on B_j for each $j \in J$. Then $\mathbf{g}_j = \ell_j(\mathbf{g}))$ for each $j \in J$, so $\lim_{n\to\infty} d_{\mathbb{K}}(\ell_j(\mathbf{f}_n), \ell_j(\mathbf{g})) = 0$. By Remark 4.5, $\lim_{n\to\infty} d_{\mathbb{K}}(\mathbf{f}_n, \mathbf{g}) = 0$ in $\mathcal{P}$. $\blacksquare_{4.6}$

Definition 4.7. By a **probability density function** on I we mean a function $\rho\colon I \to [0,1]$ such that $\rho(i) = 0$ for all but countably many $i \in I$, and $\sum_i \rho(i) = 1$.

For each basic randomization $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$, the function $\rho(i) = \lambda(\mathsf{B}_i)$ for $i \in J$, and $\rho(i) = 0$ for $i \in I \setminus J$, is called the **density function of** $\mathcal{P}$.

Remark 4.8. It easily seen that ρ is a probability density function on I if and only if ρ is the density function of some basic randomization.

The following result is a generalization of Theorem 7.5 of [1], and is proved in the same way.

Theorem 4.9. *Two basic randomizations are isomorphic if and only if they have the same density function.*

If a continuous structure $\mathcal{N}$ is isomorphic to a basic randomization $\mathcal{P}$, the density function of $\mathcal{P}$ is also called a density function of $\mathcal{N}$. Thus such an $\mathcal{N}$ has a unique density function, which characterizes $\mathcal{N}$ up to isomorphism.

5. Events Defined by Infinitary Formulas

In this section we consider arbitrary complete separable randomizations. By definition, each complete separable randomization has an event function $[\![\psi(\cdot)]\!]^{\mathcal{N}}$ of sort $\mathbb{K}^n \to \mathbb{E}$ for each first order formula $\psi(\vec{v})$ with n free variables. The following theorem extends this to the case where $\psi(\vec{v})$ is a formula of the infinitary logic $L_{\omega_1\omega}$.

Theorem 5.1. *Let $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ be a complete separable randomization, and let Ψ_n be the class of $L_{\omega_1\omega}$ formulas with n free variables. There is a unique family of functions $[\![\psi(\cdot)]\!]^{\mathcal{N}}$, $\psi \in \bigcup_n \Psi_n$, such that:*

(i) *When $\psi \in \Psi_n$, $[\![\psi(\cdot)]\!]^{\mathcal{N}} \colon \mathcal{K}^n \to \mathcal{E}$.*
(ii) *When ψ is a first order formula, $[\![\psi(\cdot)]\!]^{\mathcal{N}}$ is the usual event function for the structure $\mathcal{N}$.*
(iii) $[\![\neg\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}} = \neg[\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$.
(iv) $[\![(\psi_1 \vee \psi_2)(\vec{\mathbf{f}})]\!]^{\mathcal{N}} = [\![\psi_1(\vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqcup [\![\psi_2(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$.
(v) $[\![\bigvee_k \psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{N}} = \sup_k [\![\psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$.
(vi) $[\![(\exists u)\theta(u, \vec{\mathbf{f}})]\!]^{\mathcal{N}} = \sup_{\mathbf{g}\in\mathcal{K}} [\![\theta(\mathbf{g}, \vec{\mathbf{f}})]\!]^{\mathcal{N}}$.

Moreover, for each $\psi \in \Psi_n$, the function $[\![\psi(\cdot)]\!]^{\mathcal{N}}$ is Lipschitz continuous with bound one, that is, for any pair of n-tuples $\vec{\mathbf{f}}, \vec{\mathbf{h}} \in \mathcal{K}^n$ we have

$$d_{\mathbb{E}}([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}, [\![\psi(\vec{\mathbf{h}})]\!]^{\mathcal{N}}) \leq \sum_{m<n} d_{\mathbb{K}}(\mathbf{f}_m, \mathbf{h}_m).$$

Proof. We argue by induction on the complexity of formulas. Assume that the result holds for all subformulas of ψ. If ψ is a first order formula or a negation or finite disjunction, it is clear that the result holds for ψ.

Suppose $\psi = \bigvee_k \psi_k$. We show that the supremum exists. For each $m \in \mathbb{N}$ we have

$$[\![\bigvee_{k=0}^{m} \psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{N}} = \bigsqcup_{k=0}^{m} [\![\psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{N}}.$$

This is increasing in k, so by the completeness of the metric $d_{\mathbb{E}}$ on $\mathcal{E}$, $\lim_{k\to\infty}[\![\bigvee_{j=0}^{k} \psi_j(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ exists and is equal to $\sup_k [\![\psi_k(\vec{\mathbf{f}})]\!]^{\mathcal{N}}$. By hypothesis, the Lipschitz condition holds for each ψ_k. It follows that the Lipschitz condition also holds for ψ.

Now suppose $\psi(\vec{v}) = (\exists u)\theta(u, \vec{v})$. We again show first that the supremum exists. By separability, there is a countable dense subset $D = \{\mathbf{d}_k \colon k \in \mathbb{N}\}$ of $\mathcal{K}$. It follows from the axioms of T_2^R that there is a sequence $\langle \mathbf{g}_k \rangle_{k\in\mathbb{N}}$ of elements of $\mathcal{K}$ such that $\mathbf{g}_0 = \mathbf{d}_0$ and for each k, $\mathbf{g}_{k+1}$ agrees with $\mathbf{g}_k$ on the event $[\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}}$ and agrees with $\mathbf{d}_k$ elsewhere. Then for each $m \in \mathbb{N}$ we have

$$[\![\theta(\mathbf{g}_m, \vec{\mathbf{f}})]\!]^{\mathcal{N}} = \bigsqcup_{k=0}^{m} [\![\theta(\mathbf{d}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}}.$$

So whenever $k \leq m$, we have

$$[\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqsubseteq [\![\theta(\mathbf{g}_m, \vec{\mathbf{f}})]\!]^{\mathcal{N}},$$

and hence

$$\mathsf{E} := \lim_{k\to\infty} [\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}} = \sup_{k\in\mathbb{N}} [\![\theta(\mathbf{g}_k, \vec{\mathbf{f}})]\!]^{\mathcal{N}}$$

exists in $\mathcal{E}$.

Consider any $\mathbf{h} \in \mathcal{K}$. To show that the supremum $\sup_{\mathbf{h}\in\mathcal{K}}[\![\theta(\mathbf{h},\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ exists in $\mathcal{E}$, it suffices to show that $[\![\theta(\mathbf{h},\vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqsubseteq \mathsf{E}$, because it will then follow that E is the desired supremum. Let $\varepsilon > 0$. For some $k \in \mathbb{N}$ we have $d_{\mathbb{K}}(\mathbf{d}_k,\mathbf{h}) < \varepsilon$. Moreover,

$$[\![\mathbf{d}_k = \mathbf{h} \wedge \theta(\mathbf{h},\vec{\mathbf{f}})]\!]^{\mathcal{N}} = [\![\mathbf{d}_k = \mathbf{h} \wedge \theta(\mathbf{d}_k,\vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqsubseteq [\![\theta(\mathbf{g}_k,\vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqsubseteq \mathsf{E}.$$

Then

$$[\![\theta(\mathbf{h},\vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqcap \neg\mathsf{E} \sqsubseteq [\![\mathbf{d}_k \neq \mathbf{h}]\!]^{\mathcal{N}},$$

so

$$\mu([\![\theta(\mathbf{h},\vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqcap \neg\mathsf{E}) \leq \mu([\![\mathbf{d}_k \neq \mathbf{h}]\!]^{\mathcal{N}}) = d_{\mathbb{K}}(\mathbf{d}_k,\mathbf{h}) < \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we have $[\![\theta(\mathbf{h},\vec{\mathbf{f}})]\!]^{\mathcal{N}} \sqsubseteq \mathsf{E}$.

To prove the Lipschitz condition for ψ, we consider a pair of n-tuples $\vec{\mathbf{f}},\vec{\mathbf{h}} \in \mathcal{K}^n$. By the preceding paragraph we have

$$[\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}} = \lim_{k\to\infty}[\![\theta(\mathbf{g}_k,\vec{\mathbf{f}})]\!]^{\mathcal{N}}, \quad [\![\psi(\vec{\mathbf{h}})]\!]^{\mathcal{N}} = \lim_{k\to\infty}[\![\theta(\mathbf{g}_k,\vec{\mathbf{h}})]\!]^{\mathcal{N}}.$$

Therefore for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$d_{\mathbb{E}}([\![\theta(\mathbf{g}_k,\vec{\mathbf{f}})]\!]^{\mathcal{N}}, [\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) < \varepsilon, \quad d_{\mathbb{E}}([\![\theta(\mathbf{g}_k,\vec{\mathbf{h}})]\!]^{\mathcal{N}}, [\![\psi(\vec{\mathbf{h}})]\!]^{\mathcal{N}}) < \varepsilon.$$

By the Lipschitz condition for $\theta(u,\vec{v})$ we have

$$d_{\mathbb{E}}([\![\theta(\mathbf{g}_k,\vec{\mathbf{f}})]\!]^{\mathcal{N}}, [\![\theta(\mathbf{g}_k,\vec{\mathbf{h}})]\!]^{\mathcal{N}}) \leq \sum_{i<n} d_{\mathbb{K}}(\mathbf{f}_j,\mathbf{h}_j).$$

Then by the triangle inequality, for every $\varepsilon > 0$ we have

$$d_{\mathbb{E}}(([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}, d_{\mathbb{E}}([\![\psi(\vec{\mathbf{h}})]\!]^{\mathcal{N}}) < \sum_{i<n} d_{\mathbb{K}}(\mathbf{f}_j,\mathbf{h}_j) + 2\varepsilon,$$

so

$$d_{\mathbb{E}}(([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}, d_{\mathbb{E}}([\![\psi(\vec{\mathbf{h}})]\!]^{\mathcal{N}}) \leq \sum_{i<n} d_{\mathbb{K}}(\mathbf{f}_j,\mathbf{h}_j).$$

$\blacksquare_{5.1}$

Remark 5.2. The proof of Theorem 5.1 only used the metric completeness of the sort $\mathbb{E}$ part of $\mathcal{N}$. Hence the result also holds in the case that $\mathcal{N}$ is a separable randomization that has a metric in sort $\mathbb{K}$ and a complete metric in sort $\mathbb{E}$.

Corollary 5.3. *Suppose that $\mathcal{N},\mathcal{P}$ are complete separable randomizations and $h : \mathcal{N} \cong \mathcal{P}$. Then for every $L_{\omega_1\omega}$ formula $\psi(\vec{v})$ and every tuple $\vec{\mathbf{f}}$ of sort $\mathbb{K}$ in $\mathcal{N}$, we have $h([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) = [\![\psi(h\vec{\mathbf{f}})]\!]^{\mathcal{P}}$.*

Proof. By Theorem 5.1 and an easy induction on the complexity of ψ. $\blacksquare_{5.3}$

When $\mathcal{P}$ is a pre-complete separable randomization, h is the reduction map from $\mathcal{P}$ onto its completion $\mathcal{N}$, and $\psi(\vec{v})$ is a formula of $L_{\omega_1\omega}$, then $[\![\psi(h\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ is uniquely defined by Theorem 5.1. In that case, we will sometimes abuse notation and write $\mu([\![\psi(\vec{\mathbf{f}})]\!]^{\mathcal{P}})$ for $\mu([\![\psi(h\vec{\mathbf{f}})]\!]^{\mathcal{N}})$.

We can now define the notion of a separable randomization of φ.

Definition 5.4. We say that $\mathcal{N}$ is a **complete separable randomization of** φ if $\mathcal{N}$ is a complete separable randomization such that $[\![\varphi]\!]^{\mathcal{N}}$ is the true event $\top$. We call $\mathcal{P}$ a **separable randomization of** φ if the completion of $\mathcal{P}$ is a complete separable randomization of φ. We say that φ **has few separable randomizations** if every complete separable randomization of φ is isomorphic to a basic randomization.

Thus when φ has few separable randomizations, each complete separable randomization $\mathcal{N}$ of φ has a unique density function ρ, and ρ characterizes $\mathcal{N}$ up to isomorphism.

Corollary 5.5. *Let* $\mathcal{P} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ *be a basic randomization with completion* $\mathcal{N}$*, and let* $h\colon \mathcal{P} \cong \mathcal{N}$ *be the reduction map. For each* $L_{\omega_1\omega}$ *formula* $\psi(\vec{v})$ *and tuple* $\vec{\mathbf{f}}$ *in* $\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$*,* $[\![\psi(h\vec{\mathbf{f}})]\!]^{\mathcal{N}}$ *is the reduction of the event*

$$\bigcup_{j\in J}\{t \in \mathsf{B}_j : \mathcal{M}_j \models \psi(\vec{\mathbf{f}}(t))\}.$$

Hence $\mathcal{P}$ *is a basic randomization of* φ *if and only if* $\mathcal{P}$ *is a basic randomization and* $\mathcal{P}$ *is a separable randomization of* φ*.*

Proof. In the case that $\psi(\vec{v})$ is an atomic formula, the result holds by definition. A routine induction on the complexity of formulas gives the result for arbitrary $L_{\omega_1\omega}$ formulas. $\blacksquare_{5.5}$

Note that the complete separable randomizations of the sentence $\bigwedge T$ are exactly the separable models of the continuous theory T^R. With more overhead, we could have taken an alternative approach in which the complete separable randomizations of an $L_{\omega_1\omega}$ sentence φ are exactly the separable models of a theory φ^R in an infinitary continuous logic such as the logic in [7]. The idea would be to consider a countable fragment L_A of $L_{\omega_1\omega}$, and have the randomization signature $(L_A)^R$ contain a function symbol $[\![\psi(\cdot)]\!]$ for each formula $\psi(\vec{v})$ of L_A. Then Theorem 5.1 shows that every separable randomization can be expanded in a unique way to a model with the signature $(L_A)^R$ that satisfies the infinitary sentences corresponding to the conditions (i)–(v). In this approach, φ^R would be the theory in infinitary continuous logic with the axioms of the pure randomization theory plus the above infinitary sentences and an axiom stating that $[\![\varphi]\!] \doteq \top$.

6. Countable Generators of Randomizations

In this section we give a general method of constructing pre-complete separable randomizations. In the next section we will show that every pre-complete separable randomization is isomorphic to one that can be constructed in that way.

Definition 6.1. Assume that $(\Omega, \mathcal{E}, \nu)$ is an atomless probability space such that the metric space $(\mathcal{E}, d_{\mathbb{E}})$ is separable, and for each $t \in \Omega$, $\mathcal{M}_t$ is a countable model of T_2.

A **countable generator** (in $\langle \mathcal{M}_t \rangle_{t\in\Omega}$ over $(\Omega, \mathcal{E}, \nu)$) is a countable set C of elements $\mathbf{c} \in \prod_{t\in\Omega} M_t$ such that:

(a) $M_t = \{\mathbf{c}(t) \colon \mathbf{c} \in C\}$ for each $t \in \Omega$, and

(b) For every first order atomic formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{b}}$ in C,

$$\{t \in \Omega : \mathcal{M}_t \models \psi(\vec{\mathbf{b}}(t))\} \in \mathcal{E}.$$

Theorem 6.2. *Let C be a countable generator in $\langle \mathcal{M}_t \rangle_{t\in\Omega}$ over $(\Omega, \mathcal{E}, \nu)$. There is a unique pre-structure $\mathcal{P}(C) = (\mathcal{K}, \mathcal{E})$ such that:*

(c) *$\mathcal{K}$ is the set of all $\mathbf{f} \in \prod_{t\in\Omega} M_t$ such that $\{t \in \Omega \colon \mathbf{f}(t) = \mathbf{c}(t)\} \in \mathcal{E}$ for each $\mathbf{c} \in C$;*

(d) *$\top, \bot, \sqcup, \sqcap, \neg$ are the usual Boolean operations on $\mathcal{E}$, and μ is the measure ν;*

(e) *for each first order formula $\psi(\vec{x})$ and tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, we have*

$$[\![\psi(\vec{\mathbf{f}})]\!] = \{t \in \Omega : \mathcal{M}_t \models \psi(\mathbf{f}(t))\};$$

(f) $d_{\mathbb{E}}(\mathsf{B}, \mathsf{C}) = \nu(\mathsf{B} \triangle \mathsf{C}), \quad d_{\mathbb{K}}(\mathbf{f}, \mathbf{g}) = \mu([\![\mathbf{f} \neq \mathbf{g}]\!])$.

Moreover, $\mathcal{P}(C)$ is a pre-complete separable randomization.

Proof of Theorem 6.2. It is clear that $\mathcal{P}(C)$ is unique. We first show by induction on the complexity of formulas that condition (b) holds for all first order formulas ψ. The steps for logical connectives are trivial. For the quantifier step, suppose (b) holds for $\psi(u, \vec{v})$. Then by (a) and (c)–(f),

$$[\![(\exists u)\psi(u, \vec{\mathbf{b}})]\!] = \{t \colon \mathcal{M}_t \models (\exists u)\psi(u, \vec{\mathbf{b}}(t))\} = \{t \colon (\exists c \in M_t)\mathcal{M}_t \models \psi(c, \vec{\mathbf{b}}(t))\} =$$

$$= \{t \colon (\exists \mathbf{c} \in C)\mathcal{M}_t \models \psi(\mathbf{c}(t), \vec{\mathbf{b}}(t))\} = \bigcup_{\mathbf{c}\in C} [\![\psi(\mathbf{c}, \vec{\mathbf{b}})]\!] \in \mathcal{E},$$

so (b) holds for $(\exists u)\psi(u, \vec{v})$. By the definition of $\mathcal{K}$, for each tuple $\vec{\mathbf{g}}$ in $\mathcal{K}$ and $\vec{\mathbf{b}}$ in C, we have $[\![\vec{\mathbf{g}} = \vec{\mathbf{b}}]\!] \in \mathcal{E}$. Then for every first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{g}}$ in $\mathcal{K}$,

$$[\![\psi(\vec{\mathbf{g}})]\!] = \bigcup \{[\![\psi(\vec{\mathbf{b}}) \wedge \vec{\mathbf{g}} = \vec{\mathbf{b}}]\!] \colon \vec{\mathbf{b}} \text{ is a tuple in } C\}.$$

We therefore have

(b') For each first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{g}}$ in $\mathcal{K}$, $[\![\psi(\vec{\mathbf{g}})]\!] \in \mathcal{E}$.

This shows that $\mathcal{P}(C)$ is a pre-structure with signature L^R.

It is easily seen that $\mathcal{P}(C)$ satisfies all the axioms of T_2^R except possibly the Fullness and Event Axioms. We next show that $\mathcal{P}(C)$ has perfect witnesses. Once this is done, it follows at once that $\mathcal{P}(C)$ also satisfies the Fullness and Event Axioms, and hence is a pre-model of T_2^R.

Consider a first order formula $\psi(u, \vec{v})$ and a tuple $\vec{\mathbf{g}}$ in $\mathcal{K}$. For each $t \in \Omega$, there is a least $n(t) \in \mathbb{N}$ such that $\mathcal{M}_t \models (\exists u)\psi(u, \vec{\mathbf{g}}(t)) \to \psi(\mathbf{c}_{n(t)}(t), \vec{\mathbf{g}}(t))$. Since (b') holds and

$C \subseteq \mathcal{K}$, the function $\mathbf{f}$ such that $\mathbf{f}(t) := \mathbf{c}_{n(t)}(t)$ belongs to $\mathcal{K}$. Therefore

$$[\![\psi(\mathbf{f}, \vec{\mathbf{g}})]\!] \doteq [\![(\exists u)\psi(u, \vec{\mathbf{g}})]\!].$$

Now consider an event $\mathsf{E} \in \mathcal{E}$. Since each $\mathcal{M}_t \models T_2$, we have $[\![(\exists u)u \neq \mathbf{c}_0]\!] \doteq \top$. Therefore there exists $\mathbf{f} \in \mathcal{K}$ such that $[\![\mathbf{f} \neq \mathbf{c}_0]\!] \doteq \top$. Then the function $\mathbf{g}$ such that $\mathbf{g}(t) = \mathbf{f}(t)$ for $t \in \mathsf{E}$ and $\mathbf{g}(t) = \mathbf{c}_0(t)$ for $t \notin \mathsf{E}$ belongs to $\mathcal{K}$, and $[\![\mathbf{f} = \mathbf{g}]\!] \doteq \mathsf{E}$. This shows that $\mathcal{P}(C)$ has perfect witnesses, so $\mathcal{P}(C)$ is a pre-model of T_2^R.

We now show that $\mathcal{P}(C)$ is pre-complete. This means that when d is either $d_{\mathbb{K}}$ or $d_{\mathbb{E}}$, for every Cauchy sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ with respect to d, there exists x such that $d(x_n, x) \to 0$ as $n \to \infty$. This is clear for $d_{\mathbb{E}}$ because $(\Omega, \mathcal{E}, \nu)$ is countably additive. Suppose $\langle \mathbf{f}_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence for $d_{\mathbb{K}}$. Let $C = \{\mathbf{c}_k \colon k \in \mathbb{N}\}$, and $C_m = \{\mathbf{c}_0, \ldots, \mathbf{c}_m\}$. For each $k \in \mathbb{N}$, $\langle [\![\mathbf{f}_n = \mathbf{c}_k]\!] \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to $d_{\mathbb{E}}$. Therefore there exists $\mathsf{B}_k \in \mathcal{E}$ such that $\lim_{n \to \infty} d_{\mathbb{E}}([\![\mathbf{f}_n = \mathbf{c}_k]\!], \mathsf{B}_k) = 0$. Then $\mu(\mathsf{B}_k) = \lim_{n \to \infty} \mu([\![\mathbf{f}_n = \mathbf{c}_k]\!])$. We now cut the sets B_k down to disjoint sets with the same unions. Let $\mathsf{A}_0 = \mathsf{B}_0$, and for each m, let $\mathsf{A}_{m+1} = \mathsf{B}_{m+1} \setminus \bigcup_{k=0}^m \mathsf{B}_k$. Note that for all m,

$$\bigcup_{k=0}^m \mathsf{A}_k = \bigcup_{k=0}^m \mathsf{B}_k, \quad \mathsf{A}_k \subseteq \mathsf{B}_k, \quad (\forall k < m)\mathsf{A}_k \cap \mathsf{A}_m = \emptyset.$$

Claim. $\mu(\bigcup_{k=0}^\infty \mathsf{A}_k) = 1$.

Proof of Claim: Fix an $\varepsilon > 0$. We show that there exists m such that $\mu(\bigcup_{k=0}^m \mathsf{B}_k) > 1 - \varepsilon$. Note that for each m,

$$\mu(\bigcup_{k=0}^m \mathsf{B}_k) = \lim_{n \to \infty} \mu([\![\mathbf{f}_n \in C_m]\!]).$$

Therefore it suffices to show that

$$(\exists m)(\forall n)\mu([\![\mathbf{f}_n \in C_m]\!]) > 1 - \varepsilon.$$

Suppose this is not true. Then

$$(\forall m)(\exists n)\, \mu([\![\mathbf{f}_n \notin C_m]\!]) \geq \varepsilon.$$

Since $C = \bigcup_m C_m$,

$$(\forall n)(\exists h)\, \mu([\![\mathbf{f}_n \in C_h]\!]) \geq 1 - \varepsilon/2,$$

so

$$(\forall m)(\exists n)(\exists h)\mu([\![\mathbf{f}_n \in (C_h \setminus C_m)]\!]) \geq \varepsilon/2.$$

It follows that there are sequences $n_0 < n_1 < \ldots$ and $m_0 < m_1 < \ldots$ such that

$$(\forall k)\mu([\![\mathbf{f}_{n_k} \in (C_{m_{k+1}} \setminus C_{m_k})]\!]) \geq \varepsilon/2.$$

Therefore

$$(\forall k)(\forall h > k)d_{\mathbb{K}}(\mathbf{f}_{n_k}, \mathbf{f}_{n_h}) \geq \varepsilon/2.$$

This contradicts the fact that $\langle \mathbf{f}_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence, and the Claim is proved.

By Condition (c), there is an $\mathbf{f}$ in $\mathcal{P}(C)$ such that $\mathbf{f}$ agrees with $\mathbf{c}_k$ on A_k for each $k \in \mathbb{N}$. For each n and h we have

$$d_{\mathbb{K}}(\mathbf{f}_n, \mathbf{f}) = \mu([\![\mathbf{f}_n \neq \mathbf{f}]\!]) = \sum_{k=0}^{\infty} \mu([\![\mathbf{f}_n \neq \mathbf{f}]\!] \cap \mathsf{A}_k) = \sum_{k=0}^{\infty} \mu([\![\mathbf{f}_n \neq \mathbf{c}_k]\!] \cap \mathsf{A}_k) \leq$$

$$\leq \sum_{k=0}^{h} \mu([\![\mathbf{f}_n \neq \mathbf{c}_k]\!] \cap \mathsf{A}_k) + \mu(\bigcup_{k>h} \mathsf{A}_k) \leq \sum_{k=0}^{h} \mu([\![\mathbf{f}_n \neq \mathbf{c}_k]\!] \cap \mathsf{B}_k) + \mu(\bigcup_{k>h} \mathsf{A}_k)$$

$$\leq \sum_{k=0}^{h} d_{\mathbb{E}}([\![\mathbf{f}_n = \mathbf{c}_k]\!], \mathsf{B}_k) + \mu(\bigcup_{k>h} \mathsf{A}_k).$$

By the Claim, for each $\varepsilon > 0$ we may take h such that $\mu(\bigcup_{k>h} \mathsf{A}_k) < \varepsilon/2$. For all sufficiently large n we have

$$\sum_{k=0}^{h} d_{\mathbb{E}}([\![\mathbf{f}_n = \mathbf{c}_k]\!], \mathsf{B}_k) < \varepsilon/2,$$

and hence $d_{\mathbb{K}}(\mathbf{f}_n, \mathbf{f}) < \varepsilon$. It follows that $\lim_{n\to\infty} d_{\mathbb{K}}(\mathbf{f}_n, \mathbf{f}) = 0$, so $\mathcal{P}(C)$ is pre-complete.

We have not yet used the hypothesis that $(\mathcal{E}, d_{\mathbb{E}})$ is separable. We use it now to show that $\mathcal{P}(C)$ is separable. The Boolean algebra $\mathcal{E}$ has a countable subalgebra $\mathcal{E}_0$ such that $\mathcal{E}_0$ is dense with respect to $d_{\mathbb{E}}$, and $[\![\psi(\vec{\mathbf{b}})]\!] \in \mathcal{E}_0$ for each first order formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{b}}$ in C. Let D be the set of all $\mathbf{f} \in \mathcal{K}$ such that for some $k \in \mathbb{N}$, $[\![\mathbf{f} \in C_k]\!] = \top$ and $[\![\mathbf{f} = \mathbf{c}_n]\!] \in \mathcal{E}_0$ for all $n \leq k$. Then D is countable and D is dense in $\mathcal{K}$ with respect to $d_{\mathbb{K}}$, so $\mathcal{P}(C)$ is separable. $\blacksquare_{6.2}$

Remark 6.3. Suppose C is a countable generator in $\langle \mathcal{M}_t \rangle_{t\in\Omega}$ over $(\Omega, \mathcal{E}, \nu)$, and let $\mathcal{P}(C) = (\mathcal{K}, \mathcal{E})$. Then:

(1) $C \subseteq \mathcal{K}$.
(2) If $C \subseteq D \subseteq \mathcal{K}$ and D is countable, then D is a countable generator in $\langle \mathcal{M}_t \rangle_{t\in\Omega}$.
(3) For each $t \in \Omega$, $M_t = \{\mathbf{f}(t) : \mathbf{f} \in \mathcal{K}\}$.
(4) If $\mathcal{M}_t \cong \mathcal{H}_t$ for all t, then there is a countable generator D in $\langle \mathcal{H}_t \rangle_{t\in\Omega}$ such that $\mathcal{P}(D) \cong \mathcal{P}(C)$.

Proof. We prove (4). For each t, choose an isomorphism $h_t : \mathcal{M}_t \cong \mathcal{H}_t$. For each $\mathbf{c} \in C$, define $h\mathbf{c}$ by $(h\mathbf{c})(t) = h_t(\mathbf{c}(t))$ and let $D = \{h\mathbf{c} : \mathbf{c} \in C\}$. Then D is a countable generator in $\langle \mathcal{H}_t \rangle_{t\in\Omega}$ and $\mathcal{P}(D) \cong \mathcal{P}(C)$. $\blacksquare_{6.3}$

The next corollary connects countable generators to basic randomizations.

Corollary 6.4. *Let $\mathcal{N} = (\prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ be a basic randomization.*

(i) *There is a countable generator C in $\langle \mathcal{M}_t \rangle_{t\in[0,1)}$ over $([0,1), \mathcal{L}, \lambda)$ such that $C \subseteq \prod_{j\in J} \mathcal{M}_j^{\mathsf{B}_j}$.*
(ii) *If C is as in (i), then $\mathcal{P}(C) = \mathcal{N}$.*

(iii) *If C is a countable generator in $\langle\mathcal{H}_t\rangle_{t\in[0,1)}$ over $([0,1),\mathcal{L},\lambda)$, $C\subseteq\prod_{j\in J}\mathcal{M}_j^{\mathsf{B}_j}$, and $\mathcal{H}_t\prec\mathcal{M}_t$ for all t, then $\mathcal{P}(C)\prec\mathcal{N}$.*

Proof. (i) For each $j\in J$, choose an enumerated structure $(\mathcal{M}_j,a_{j,0},a_{j,1},\ldots)$. Let $C=\{\mathbf{c}_n\colon n\in\mathbb{N}\}$ where $\mathbf{c}_n(t)=a_{j,n}$ whenever $j\in J$ and $t\in\mathsf{B}_j$. C has the required properties.

(ii) Let $\mathcal{P}(C)=(\mathcal{K},\mathcal{L})$. Since $C\subseteq\prod_{j\in J}\mathcal{M}_j^{\mathsf{B}_j}$, for all $j\in J, a\in M_j$, and $\mathbf{c}\in C$ we have $\{t\in\mathsf{B}_j\colon\mathbf{c}(t)=a\}\in\mathcal{L}$. It follows that for each $j\in J$ and $\mathbf{f}$,

$$(\forall a\in M_j)\{t\in\mathsf{B}_j\colon\mathbf{f}(t)=a\}\in\mathcal{L}\Leftrightarrow(\forall\mathbf{c}\in C)\{t\in\mathsf{B}_j\colon\mathbf{f}(t)=\mathbf{c}(t)\}\in\mathcal{L}.$$

Therefore $\mathcal{K}=\prod_{j\in J}\mathcal{M}_j^{\mathsf{B}_j}$, and (ii) holds.

(iii) Let $\mathcal{P}(C)=(\mathcal{K},\mathcal{L})$. For each $\mathbf{f}\in\mathcal{K}$ we have $[0,1)=\bigcup_{\mathbf{c}\in C}\{t\colon\mathbf{f}(t)=\mathbf{c}(t)\}$, and $\{t\colon\mathbf{f}(t)=\mathbf{c}(t)\}\in\mathcal{L}$ for all $\mathbf{c}\in C$. Therefore $\mathcal{K}\subseteq\prod_{j\in J}\mathcal{M}_j^{\mathsf{B}_j}$. Since $\mathcal{H}_t\prec\mathcal{M}_t$, $[\![\psi(\cdot)]\!]$ has the same interpretation in $\mathcal{P}(C)$ as in $\mathcal{N}$ for every first order formula $\psi(\vec{v})$. Therefore $(\mathcal{K},\mathcal{L})$ is a pre-substructure of $\mathcal{N}$. By quantifier elimination (Theorem 2.9 of [6]) we have $\mathcal{P}(C)\prec\mathcal{N}$. $\blacksquare_{6.4}$

The next result gives a very useful "pointwise" characterization of the event corresponding to an infinitary formula in a complete separable randomization that is isomorphic to $\mathcal{P}(C)$.

Proposition 6.5. *Suppose $\mathcal{N}$ is a complete separable randomization, C is a countable generator in $\langle\mathcal{M}_t\rangle_{t\in\Omega}$ over $(\Omega,\mathcal{E},\nu)$, and $h:\mathcal{P}(C)\cong\mathcal{N}$. Then for every $L_{\omega_1\omega}$ formula $\psi(\vec{v})$ and tuple $\vec{\mathbf{f}}$ of sort $\mathbb{K}$ in $\mathcal{P}(C)$, we have*

$$\{t\colon\mathcal{M}_t\models\psi(\vec{\mathbf{f}}(t))\}\in\mathcal{E},\qquad[\![\psi(h\vec{\mathbf{f}})]\!]^{\mathcal{N}}=h(\{t\colon\mathcal{M}_t\models\psi(\vec{\mathbf{f}}(t))\}).$$

Moreover, $\mathcal{N}$ is a separable randomization of φ if and only if $\mu(\{t\colon\mathcal{M}_t\models\varphi\})=1$.

Proof. This is proved by a straightforward induction on the complexity of $\psi(\vec{v})$ using Theorems 5.1 and 6.2. $\blacksquare_{6.5}$

7. A Representation Theorem

In this section we show that every complete separable randomization of φ is isomorphic to $\mathcal{P}(C)$ for some countable generator C in countable models of φ.

We will use the following result, which is a consequence of Theorem 3.11 of [3], and generalizes Proposition 2.1.10 of [2].

Proposition 7.1. *For every pre-complete model $\mathcal{N}'$ of T^R, there is an atomless probability space $(\Omega,\mathcal{E},\nu)$ and a family of models $\langle\mathcal{M}_t\rangle_{t\in\Omega}$ of T such that $\mathcal{N}'$ is isomorphic to a pre-complete model $\mathcal{N}=(\mathcal{K},\mathcal{E})$ of T^R such that $\mathcal{K}\subseteq\prod_{t\in\Omega}M_t$ and $\mathcal{N}$ satisfies Conditions (d), (e), and (f) of Theorem 6.2.*

Proof. Proposition 2.1.10 of [2] gives this result in the case that T is a complete theory, with the additional conclusion that there is a single model $\mathcal{M}$ of T such that $\mathcal{M}_t\prec\mathcal{M}$

for all $t \in \Omega$ [2]. The same argument works in the general case, but without the model $\mathcal{M}$. $\blacksquare_{7.1}$

Proposition 7.2. *Suppose $\mathcal{N}'$ is pre-complete and elementarily embeddable in a basic randomization. Then $\mathcal{N}'$ is isomorphic to a pre-complete elementary submodel $\mathcal{N}$ of a basic randomization $(\prod_{j\in J}\mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ such that the event sort of $\mathcal{N}$ is all of $\mathcal{L}$. Moreover, Conditions (d), (e), and (f) of Theorem 6.2 hold for $\mathcal{N}=(\mathcal{K},\mathcal{L})$ and $(\prod_{j\in J}\mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$.*

Proof. Suppose $\mathcal{N}' \cong \mathcal{N}'' \prec (\prod_{j\in J}\mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$. For each $j \in J$, let ℓ_j be a mapping that stretches B_j to $[0,1)$. Then ℓ_j maps $\mathcal{N}''$ onto a pre-complete elementary submodel $\mathcal{N}_j$ of $(\mathcal{M}_j^{[0,1)}, \mathcal{L})$. By Result 3.4, $\mathcal{N}_j$ is isomorphic to a pre-complete elementary submodel of $(\mathcal{M}_j^{[0,1)}, \mathcal{L})$ with event sort $\mathcal{L}$. Using the inverse mappings ℓ_j^{-1}, it follows that $\mathcal{N}''$ is isomorphic to a pre-complete elementary submodel $\mathcal{N}=(\mathcal{K},\mathcal{L}) \prec (\prod_{j\in J}\mathcal{M}_j^{\mathsf{B}_j}, \mathcal{L})$ with event sort $\mathcal{L}$. It is easily checked that $\mathcal{N}$ satisfies Conditions (d), (e), and (f) of Theorem 6.2. $\blacksquare_{7.2}$

Theorem 7.3. *(Representation Theorem) Every pre-complete separable randomization $\mathcal{N}$ of φ is isomorphic to $\mathcal{P}(C)$ for some countable generator C in a family of countable models of φ. Moreover, if $\mathcal{N}$ is elementarily embeddable in some basic randomization, then C can be taken to be over the probability space $([0,1),\mathcal{L},\lambda)$.*

Proof. Let $\mathcal{N}'$ be a pre-complete separable randomization of φ. By Proposition 7.1, there is an atomless probability space $(\Omega,\mathcal{E},\nu)$ and a family of models $\langle \mathcal{M}_t\rangle_{t\in\Omega}$ such that $\mathcal{N}'$ is isomorphic to a pre-complete model $\mathcal{N}=(\mathcal{K},\mathcal{E})$ of T_2^R where $\mathcal{K}\subseteq \prod_{t\in\Omega} M_t$ and $\mathcal{N}$ satisfies Conditions (d), (e), and (f) of Theorem 6.2. If $\mathcal{N}'$ is elementarily embeddable in a basic randomization, then by Proposition 7.2, we may take $(\Omega,\mathcal{E},\nu)=([0,1),\mathcal{L},\lambda)$.

Since $\mathcal{N}$ is separable, there is a countable pre-structure $(\mathcal{J}_0,\mathcal{A}_0)\prec\mathcal{N}$ that is dense in $\mathcal{N}$. We will use an argument similar to the proofs of Lemmas 4.7 and 4.8 of [1]. By Result 3.1, $\mathcal{N}$ has perfect witnesses. Hence by listing the first order formulas, we can construct a chain of countable pre-structures $(\mathcal{J}_n,\mathcal{A}_n), n\in\mathbb{N}$ such that for each n:

- $(\mathcal{J}_n,\mathcal{A}_n)\subseteq(\mathcal{J}_{n+1},\mathcal{A}_{n+1})\subseteq\mathcal{N}$;
- for each first order formula $\theta(u,\vec{v})$ and tuple $\vec{\mathbf{g}}$ in $\mathcal{J}_n$ there exists $\mathbf{f}\in\mathcal{J}_{n+1}$ such that
$$[\![\theta(\mathbf{f},\vec{\mathbf{g}})]\!] \doteq [\![(\exists u\theta)(\vec{\mathbf{g}})]\!];$$
- For each $\mathsf{B}\in\mathcal{A}_n$ there exist $\mathbf{f},\mathbf{g}\in\mathcal{J}_{n+1}$ such that $\mathsf{B}\doteq[\![\mathbf{f}=\mathbf{g}]\!]$.

The union
$$\mathcal{P}=(\mathcal{J},\mathcal{A})=\bigcup_n(\mathcal{J}_n,\mathcal{A}_n)$$

[2]In [2], $\mathcal{P}$ is called a neat randomization of $\mathcal{M}$

is a countable dense elementary submodel of $\mathcal{N}$ that has perfect witnesses. Therefore for each first order formula $\theta(u,\vec{v})$ and each tuple $\vec{\mathbf{g}}$ in $\mathcal{J}$, there exists $\mathbf{f}\in\mathcal{J}$ such that

$$[\![(\exists u)\theta(u,\vec{\mathbf{g}})]\!]^{\mathcal{N}} = [\![(\exists u)\theta(u,\vec{\mathbf{g}})]\!]^{\mathcal{P}} \doteq [\![\theta(\mathbf{f},\vec{\mathbf{g}})]\!]^{\mathcal{P}} = [\![\theta(\mathbf{f},\vec{\mathbf{g}})]\!]^{\mathcal{N}}.$$

Since $\mathcal{J}$ is countable, there is an event $\mathsf{E}\in\mathcal{E}$ such that $\nu(\mathsf{E})=1$ and for every tuple $\vec{\mathbf{g}}$ in $\mathcal{J}$ there exists $\mathbf{f}\in\mathcal{J}$ so that

$$(\forall t\in\mathsf{E})\mathcal{M}_t\models[(\exists u)\theta(u,\vec{\mathbf{g}}(t))\leftrightarrow\theta(\mathbf{f}(t),\vec{\mathbf{g}}(t))].$$

For each $t\in\Omega$ let $\mathcal{H}_t=\{\mathbf{f}(t)\colon\mathbf{f}\in\mathcal{J}\}$. By the Tarski-Vaught test, we have $\mathcal{H}_t\prec\mathcal{M}_t$, and hence $\mathcal{H}_t\models T_2$, for each $t\in\mathsf{E}$.

Pick a countable model $\mathcal{H}$ of φ. For any set $\mathsf{D}\subseteq\mathsf{E}$ such that $\mathsf{D}\in\mathcal{E}$ and $\nu(\mathsf{D})=1$, let C^{D} be the set of all functions that agree with an element of $\mathcal{J}$ on D and take a constant value in $\mathcal{H}$ on $\Omega\setminus\mathsf{D}$. Let $\mathcal{H}^{\mathsf{D}}_t=\mathcal{H}_t$ for $t\in\mathsf{D}$, and $\mathcal{H}^{\mathsf{D}}_t=\mathcal{H}$ for $t\in\Omega\setminus\mathsf{D}$. Then $\mathcal{H}^{\mathsf{D}}_t$ is a model of T_2 for each $t\in\Omega$, and C^{D} is a countable generator in $\langle\mathcal{H}^{\mathsf{D}}_t\rangle_{t\in\Omega}$. By Theorem 6.2, $\mathcal{P}(C^{\mathsf{D}})$ is a pre-complete separable randomization. The reduction of $(\mathcal{J},\mathcal{A})$ is dense in the reductions of $\mathcal{N}$ and of $\mathcal{P}(C^{\mathsf{D}})$, and both $\mathcal{N}$ and $\mathcal{P}(C^{\mathsf{D}})$ are pre-complete. Therefore $\mathcal{N}\cong\mathcal{P}(C^{\mathsf{D}})$.

In particular, C^{E} is a countable generator in $\langle\mathcal{H}^{\mathsf{E}}_t\rangle_{t\in\Omega}$, and $\mathcal{N}\cong\mathcal{P}(C^{\mathsf{E}})$. Now let $\mathsf{D}=\{t\in\mathsf{E}\colon\mathcal{H}^{\mathsf{E}}_t\models\varphi\}$. Since $\mathcal{N}$ is a pre-complete randomization of φ, we see from Proposition 6.5 that $\mu(\mathsf{D})=1$. Then $\mathcal{H}^{\mathsf{D}}_t\models\varphi$ for all $t\in\Omega$, $\mathcal{P}(C^{\mathsf{D}})\cong\mathcal{N}$, and C^{D} is a countable generator in a family of countable models of φ. $\blacksquare_{7.3}$

8. Elementary Embeddability in a Basic Randomization

Let $S_n(T)$ be the set of first order n-types realized in countable models of T, and $S_n(\varphi)$ be the set of first order types realized in countable models of φ. Note that $S_0(\varphi)=\{Th(\mathcal{M})\colon\mathcal{M}\models\varphi\}$.

Theorem 3.12 in [6] and Proposition 5.7 in [1] show that:

Result 8.1. *Let T be complete. The following are equivalent:*

(i) $\bigcup_n S_n(T)$ *is countable.*
(ii) *Every complete separable randomization of T is elementarily embeddable in the Borel randomization of a countable model of T.*
(iii) *For every complete separable randomization $\mathcal{N}$ of T, $n\in\mathbb{N}$, and n-tuple $\vec{\mathbf{f}}$ of sort $\mathbb{K}$ in $\mathcal{N}$, there is a type $p\in S_n(T)$ such that $\mu([\![\bigwedge p(\vec{\mathbf{f}})]\!]^{\mathcal{N}})>0$.*

In Theorem 8.3 below, we generalize this result by replacing a complete theory T and a Borel randomization by an arbitrary $L_{\omega_1\omega}$ sentence φ and a basic randomization.

We will use Proposition 6.2 of [1], which can be formulated as follows.

Result 8.2. *Let T be complete. The following are equivalent:*

(i) *$\mathcal{N}$ is a complete separable randomization of T and for each n and each n-tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, $\sum_{q\in S_n(T)}\mu([\![\bigwedge q(\vec{\mathbf{f}})]\!]^{\mathcal{N}})=1$.*

(ii) $\mathcal{N}$ *is elementarily embeddable in the Borel randomization of a countable model of* T*.*

Theorem 8.3. *The following are equivalent:*

(i) $\bigcup_n S_n(\varphi)$ *is countable.*
(ii) *Every complete separable randomization of* φ *is elementarily embeddable in a basic randomization.*
(iii) *For every complete separable randomization* $\mathcal{N}$ *of* φ*,* $n \in \mathbb{N}$*, and* n*-tuple* $\vec{\mathbf{f}}$ *in* $\mathcal{K}$*, there is a type* $p \in S_n(\varphi)$ *such that* $\mu(\llbracket \bigwedge p(\vec{\mathbf{f}}) \rrbracket^{\mathcal{N}}) > 0$*.*

In (ii), we do not know whether the basic randomization can be taken to be a basic randomization of φ.

Proof of Theorem 8.3. We first assume (i) and prove (ii). Let $\mathcal{N}$ be a complete separable randomization of φ. By Theorem 7.3, there is a countable generator C in a family of countable models $\langle \mathcal{M}_t \rangle_{t \in \Omega}$ of φ over an atomless probability space $(\Omega, \mathcal{E}, \nu)$, such that $\mathcal{N} \cong \mathcal{P}(C) = (\mathcal{K}, \mathcal{E})$. For each $t \in \Omega$, $\mathcal{M}_t$ is a countable model of φ, so $Th(\mathcal{M}_t) \in S_0(\varphi)$. By (i), $S_0(\varphi)$ is countable. Let $\mathsf{B}_T = \{t \in \Omega \colon \mathcal{M}_t \models T\}$. By Proposition 6.5, $\mathsf{B}_T \in \mathcal{E}$. Let $G = \{T \in S_0(\varphi) \colon \nu(\mathsf{B}_T) > 0\}$, and consider any $T \in G$. Let ν_T be the atomless probability measure on $(\Omega, \mathcal{E})$ such that $\nu_T(\mathsf{E}) = \nu(\mathsf{E} \cap \mathsf{B}_T)/\nu(\mathsf{B}_T)$. (Note that ν_T is the conditional probability of ν with respect to B_T.) Let $\mathcal{N}_T$ be the structure $(\mathcal{K}, \mathcal{E})$ with the probability measure ν_T instead of ν. Then $\mathcal{N}_T$ is a pre-complete separable randomization of both φ and T. Let $S_n = S_n(T) \cap S_n(\varphi)$. Since $S_n(\varphi)$ is countable, $(\forall \vec{v}) \bigvee_{q \in S_n} \bigwedge q(\vec{v})$ is a sentence of $L_{\omega_1 \omega}$ and is a consequence of φ. Therefore

$$\nu_T(\llbracket (\forall \vec{v}) \bigvee_{q \in S_n} \bigwedge q(\vec{v}) \rrbracket) = 1.$$

Then for every n-tuple $\vec{\mathbf{f}}$ in $\mathcal{K}$, $\sum_{q \in S_n(T)} \nu_T(\llbracket \bigwedge q(\vec{\mathbf{f}}) \rrbracket) = 1$. Hence by Result 8.2, there is a countable model $\mathcal{H}_T$ of T and an elementary embedding

$$h_T \colon \mathcal{N}_T \prec (\mathcal{H}_T^{[0,1)}, \mathcal{L}).$$

Let $\{\mathsf{A}_T \colon T \in G\}$ be a Borel partition of $[0,1)$ such that $\lambda(\mathsf{A}_T) = \nu(\mathsf{B}_T)$ for each T. Let J be the set of isomorphism types of the models $\{\mathcal{H}_T \colon T \in G\}$. For each $T \in G$ let $\mathcal{H}_j = \mathcal{H}_T$, $h_j = h_T$, and $\mathsf{A}_j = \mathsf{A}_T$ where j is the isomorphism type of $\mathcal{H}_T$. Then $\mathcal{P} = (\prod_{j \in J} \mathcal{H}_j^{\mathsf{A}_j}, \mathcal{L})$ is a basic randomization. For each $j \in J$, let ℓ_j be a mapping that stretches A_j to $[0,1)$, and let $\ell_j \colon \mathcal{P} \to (\mathcal{H}_j^{[0,1)}, \mathcal{L})$ be the mapping defined in Definition 4.4. We then get an elementary embedding of $\mathcal{N}$ into $\mathcal{P}$ by sending each $\mathsf{E} \in \mathcal{E}$ to the set $\bigcup_{j \in J} \ell_j^{-1}(h_j(\mathsf{E}))$, and sending each $\mathbf{f} \in \mathcal{K}$ to the function that agrees with $\ell_j^{-1}(h_j(\mathbf{f}))$ on A_j for each $j \in J$.

We next assume (ii) and prove (iii). Let $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ be a complete separable randomization of φ, and let $\vec{\mathbf{f}}$ be an n-tuple in $\mathcal{K}$. By (ii), there is an elementary embedding

h from $\mathcal{N}$ into a basic randomization $\mathcal{P} = (\prod_{j\in J} \mathcal{H}_j^{\mathsf{A}_j}, \mathcal{L})$. Then there is a $j \in J$ and a set $\mathsf{B} \subseteq \mathsf{A}_j$ such that $\lambda(\mathsf{B}) > 0$ and $(h\vec{\mathbf{f}})$ is constant on B. Let $r = \lambda(\mathsf{B})$. Let p be the type of $h(\vec{\mathbf{f}})$ in $\mathcal{H}_j$. Then for each $\theta(\vec{v}) \in p$ we have $\mathcal{P} \models \mu([\![\theta(h\vec{\mathbf{f}})]\!]) \geq r$. Since h is an elementary embedding, for each $\theta \in p$ we have $\mathcal{N} \models \mu([\![\theta(\vec{\mathbf{f}})]\!]) \geq r$. Therefore

$$\mu([\![\bigwedge p(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) = \inf_{\theta\in p} \mu([\![\theta(\vec{\mathbf{f}})]\!]^{\mathcal{N}}) \geq r > 0,$$

and (iii) is proved.

Finally, we assume that (i) fails and prove that (iii) fails. Since (i) fails, there exists n such that $S_n(\varphi)$ is uncountable. We introduce some notation. Let L_0 be the set of all atomic first order formulas. Let 2^{L_0} be the Polish space whose elements are the functions $s\colon L_0 \to \{0,1\}$. As in Section 2, we say that a point $t \in 2^{L_0}$ **codes** an enumerated structure $(\mathcal{M}, a)$ if for each formula $\theta(v_0, \ldots, v_{n-1}) \in L_0$, $t(\theta) = 0$ if and only if $\mathcal{M} \models \theta[a_0, \ldots, a_{n-1}]$. We note for each $t \in 2^{L_0}$, any two enumerated structures that are coded by t are isomorphic. When t codes an enumerated structure, we choose one and denote it by $(\mathcal{M}(t), a(t))$. For each $L_{\omega_1\omega}$ formula $\psi(v_0, \ldots, v_{n-1})$, let $[\psi]$ be the set of all $t \in 2^{L_0}$ such that $(\mathcal{M}(t), a(t))$ exists and $\mathcal{M}(t) \models \psi[a_0(t), \ldots, a_{n-1}(t)]$.

Claim. There is a perfect set $P \subseteq [\varphi]$ such that for all s, t in P, we have

$$(\mathcal{M}(s), a_0(s), \ldots, a_{n-1}(s)) \equiv (\mathcal{M}(t), a_0(t), \ldots, a_{n-1}(t))$$

if and only if $s = t$.

Proof of Claim: By Proposition 16.7 in [8], for each $L_{\omega_1\omega}$ formula ψ, $[\psi(\vec{v})]$ is a Borel subset of 2^{L_0}. In particular, $[\varphi]$ is Borel. Let E be the set of pairs $(s,t) \in [\varphi] \times [\varphi]$ such that

$$(\mathcal{M}(s), a_0(s), \ldots, a_{n-1}(s)) \equiv (\mathcal{M}(t), a_0(t), \ldots, a_{n-1}(t)).$$

E is obviously an equivalence relation on $[\varphi]$. Since $S_n(\varphi)$ is uncountable, E has uncountably many equivalence classes. We show that E is Borel. Let F be the set of all first order formulas $\theta(v_0, \ldots, v_{n-1})$. For each $\theta \in F$, let

$$E_\theta = \{(s,t) \in [\varphi] \times [\varphi] \colon s \in [\theta] \leftrightarrow t \in [\theta]\}.$$

Since $[\varphi]$ and $[\theta]$ are Borel, E_θ is Borel. Moreover, F is countable, and $E = \bigcap_{\theta\in F} E_\theta$. Therefore E is a Borel equivalence relation. By Silver's theorem in [14], there is a perfect set $P \subseteq [\varphi]$ such that whenever $s, t \in [\varphi]$, we have $(s,t) \in E$ if and only if $s = t$, as required in the Claim.

By Theorem 6.2 in [8], P has cardinality $2^{\aleph_0}$. By the Borel Isomorphism Theorem (15.6 in [8]), there is a Borel bijection β from $[0,1)$ onto P whose inverse is also Borel. Each $s \in P$ codes an enumerated model $(\mathcal{M}(s), a(s))$ of φ. For each $t \in [0,1)$ and $n \in \mathbb{N}$, $a_n(\beta(t)) \in M(\beta(t))$, so for each n the composition $\mathbf{c}_n = a_n \circ \beta$ is a function such that $\mathbf{c}_n(t) \in M(\beta(t))$. Let $C = \{\mathbf{c}_n \colon n \in \mathbb{N}\}$. Then for each t, we have

$$\{\mathbf{c}(t) \colon \mathbf{c} \in C\} = \{a_n(\beta(t)) \colon n \in \mathbb{N}\} = M(\beta(t)),$$

so C satisfies Condition (a) of Definition 6.1.

We next show that C is a countable generator. We will then show that the completion of $\mathcal{P}(C)$ is a separable randomization of φ that is not elementarily embeddable in a basic randomization.

For each $\theta \in L_0$, the set

$$P \cap [\theta] = \{s \in P \colon \mathcal{M}(s) \models \theta[a_0(s), \ldots, a_{n-1}(s)]\}$$

is Borel. Since β and its inverse are Borel functions, it follows that

$$\{t \in [0,1) \colon \mathcal{M}(\beta(t)) \models \theta(\mathbf{c}_0(t), \ldots, \mathbf{c}_{n-1}(t))\} \in \mathcal{L}.$$

Thus C satisfies condition (b) of Definition 6.1, and hence is a countable generator in the family $\langle \mathcal{M}(\beta(t))\rangle_{t\in[0,1)}$ of countable models of φ over the probability space $([0,1), \mathcal{L}, \lambda)$.

By Theorem 6.2 and Proposition 6.5, $\mathcal{P}(C)$ is a pre-complete separable randomization of φ. Then the completion $\mathcal{N}$ of $\mathcal{P}(C)$ is a complete separable randomization of φ. By the properties of P, for each first-order n-type p, there is at most one $t \in [0,1)$ such that $(\mathbf{c}_0(t), \ldots, \mathbf{c}_{n-1}(t))$ realizes p in $\mathcal{M}(\beta(t))$. Then

$$\mu(\llbracket \bigwedge p(\mathbf{c}_0, \ldots, \mathbf{c}_{n-1})\rrbracket^{\mathcal{N}}) = 0.$$

Therefore $\mathcal{N}$ cannot be elementarily embeddable in a basic randomization. This shows that (iii) fails, and completes the proof. $\blacksquare_{8.3}$

9. Sentences with Few Separable Randomizations

In this section we show that any infinitary sentence that has only countably many countable models has few separable randomizations (Theorem 9.6 below). We begin by stating a result from [1].

Result 9.1. *([1], Theorem 6.3). If T is complete and $I(T)$ is countable, then T has few separable randomizations.*

Theorem 9.6 below will generalize this result by replacing the complete theory T by an arbitrary $L_{\omega_1\omega}$ sentence φ.

The following lemma is a consequence of Theorem 7.6 in [1]. The underlying definitions are somewhat different in [1], so for completeness we give a direct proof here.

Lemma 9.2. *Let $\mathcal{N} = (\mathcal{H}^{[0,1)}, \mathcal{L})$ be the Borel randomization of a countable model $\mathcal{H}$ of T_2. Suppose $\mathcal{M}_t \cong \mathcal{H}$ for each $t \in [0,1)$, and C is a countable generator in $\langle \mathcal{M}_t\rangle_{t\in[0,1)}$ over $([0,1), \mathcal{L}, \lambda)$. Then $\mathcal{P}(C) \cong \mathcal{N}$.*

Remark 9.3. In the special case that $\mathcal{M}_t = \mathcal{M}$ for all $t \in [0,1)$ and $C \subseteq \mathcal{M}^{[0,1)}$, Corollary 6.4 and Remark 4.2 (ii) immediately give

$$\mathcal{P}(C) = (\mathcal{M}^{[0,1)}, \mathcal{L}) \cong \mathcal{N}.$$

This argument does not work in the general case, where the structures $\mathcal{M}_t$ may vary with t and there is no measurability requirement on the elements of C.

Proof of Lemma 9.2. Let $\mathcal{P}(C) = (\mathcal{J}, \mathcal{L})$. Let H denote the universe of $\mathcal{H}$. Let $\{\mathbf{f}_1, \mathbf{f}_2, \ldots\}$ and $\{\mathbf{g}'_1, \mathbf{g}'_2, \ldots\}$ be countable dense subsets of $\mathcal{J}$ and $\mathcal{H}^{[0,1)}$ respectively.

Claim. There is a sequence $\langle \mathbf{g}_1, \mathbf{g}_2, \ldots \rangle$ in $\mathcal{J}$, and a sequence $\langle \mathbf{f}'_1, \mathbf{f}'_2, \ldots \rangle$ in $\mathcal{H}^{[0,1)}$, such that the following statement $S(n)$ holds for each $n \in \mathbb{N}$:

For all $t \in [0, 1)$,

$$(\mathcal{M}_t, (\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{g}_1, \ldots, \mathbf{g}_n)(t)) \cong (\mathcal{H}, (\mathbf{f}'_1, \ldots, \mathbf{f}'_n, \mathbf{g}'_1, \ldots, \mathbf{g}'_n)(t)).$$

Once the Claim is proved, it follows that for each first order formula $\psi(\vec{u}, \vec{v})$,

$$[\![\psi(\vec{\mathbf{f}}, \vec{\mathbf{g}})]\!]^{\mathcal{P}(C)} = [\![\psi(\vec{\mathbf{f}}', \vec{\mathbf{g}}')]\!]^{\mathcal{N}},$$

and hence there is an isomorphism $h\colon \mathcal{P}(C) \cong \mathcal{N}$ such that $h(\mathsf{E}) = \mathsf{E}$ for all $\mathsf{E} \in \mathcal{L}$, and $h(\mathbf{f}_n) = \mathbf{f}'_n$ and $h(\mathbf{g}_n) = \mathbf{g}'_n$ for all n.

Proof of Claim: Note that the statement $S(0)$ just says that $\mathcal{M}_t \cong \mathcal{H}$ for all $t \in [0, 1)$, and is true by hypothesis. Let $n \in \mathbb{N}$ and assume that we already have functions $\mathbf{g}_1, \ldots, \mathbf{g}_{n-1}$ in $\mathcal{J}$ and $\mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}$ in $\mathcal{H}^{[0,1)}$ such that the statement $S(n-1)$ holds. Thus for each $t \in [0, 1)$, there is an isomorphism

$$h_t : (\mathcal{M}_t, (\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}, \mathbf{g}_1, \ldots, \mathbf{g}_{n-1})(t)) \cong (\mathcal{H}, (\mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}, \mathbf{g}'_1, \ldots, \mathbf{g}'_{n-1})(t)).$$

We will find functions $\mathbf{g}_n \in \mathcal{J}, \mathbf{f}'_n \in \mathcal{H}^{[0,1)}$ such that $S(n)$ holds.

Let Z be the set of all isomorphism types of structures

$$(\mathcal{H}, a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}, a, b),$$

and for each $z \in Z$ let θ_z be a Scott sentence for structures of isomorphism type z. Since H is countable, Z is countable. For each $a \in H$ and $t \in [0, 1)$ let $z(a, t)$ be the isomorphism type of

$$(\mathcal{H}, (\mathbf{f}'_1, \ldots, \mathbf{f}'_{n-1}, \mathbf{g}'_1, \ldots, \mathbf{g}'_{n-1})(t), a, \mathbf{g}'_n(t)).$$

Then $z(a, t) \in Z$.

For each $a \in H$ and $c \in C$, let $\mathsf{B}(a, c)$ be the set of all $t \in [0, 1)$ such that

$$(\mathcal{M}_t, (\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}, \mathbf{g}_1, \ldots, \mathbf{g}_{n-1})(t), \mathbf{f}_n(t), c(t)) \models \theta_{z(a,t)}.$$

By Proposition 6.5, each of the sets $\mathsf{B}(a, c)$ is Borel. By taking $a \in H$ such that $a = h_t(\mathbf{f}_n(t))$, and $c \in C$ such that $c(t) = h_t^{-1}(\mathbf{g}'_n(t))$, we see that for every $t \in [0, 1)$ there exist $a \in H$ and $c \in C$ with $t \in \mathsf{B}(a, c)$. Thus

$$[0, 1) = \bigcup \{\mathsf{B}(a, c)\colon a \in H, c \in C\}.$$

Every countable family of Borel sets with union $[0, 1)$ can be cut down to a countable partition of $[0, 1)$ into Borel sets. Thus there is a partition

$$\langle \mathsf{D}(a, c)\colon a \in H, c \in C \rangle$$

of $[0, 1)$ into Borel sets $\mathsf{D}(a, c) \subseteq \mathsf{B}(a, c)$.

Let $\mathbf{f}'_n$ be the function that has the constant value a on each set $\mathsf{D}(a,c)$, and let $\mathbf{g}_n$ be the function that agrees with c on each set $\mathsf{D}(a,c)$. Then $\mathbf{f}'_n$ is Borel and thus belong to $\mathcal{H}^{[0,1)}$, and $\mathbf{g}_n$ belongs to $\mathcal{J}$. Moreover, whenever $t \in \mathsf{D}(a,c)$ we have $t \in \mathsf{B}(a,c)$ and hence

$$(\mathcal{M}_t, (\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{g}_1, \ldots, \mathbf{g}_n)(t)) \cong (\mathcal{H}, (\mathbf{f}'_1, \ldots, \mathbf{f}'_n, \mathbf{g}'_1, \ldots, \mathbf{g}'_n)(t)).$$

So the functions $\mathbf{f}'_n$ and $\mathbf{g}_n$ satisfy the condition $S(n)$. This completes the proof of the Claim and of Lemma 9.2. $\blacksquare_{9.3}$

Recall that for each $i \in I$, θ_i is a Scott sentence for structures of isomorphism type i.

Lemma 9.4. *Let $\mathcal{P} = (\prod_{j \in J} (\mathcal{H}_j)^{\mathsf{A}_j}, \mathcal{L})$ be a basic randomization. Then for each complete separable randomization $\mathcal{N}$, the following are equivalent:*

(i) *$\mathcal{N}$ is isomorphic to $\mathcal{P}$.*
(ii) *$\mu([\![\theta_j]\!]^{\mathcal{N}}) = \lambda(\mathsf{A}_j)$ for each $j \in J$.*

Proof. Assume (i) and let $h : \mathcal{P} \cong \mathcal{N}$. By Corollary 6.4, $\mathcal{P} = \mathcal{P}(C)$ for some countable generator C in $\langle \mathcal{H}_t \rangle_{t \in [0,1)}$ over $([0,1), \mathcal{L}, \lambda)$. By Proposition 6.5, for each $j \in J$ we have

$$[\![\theta_j]\!]^{\mathcal{N}} = h(\{t \in [0,1) \colon \mathcal{H}_t \models \theta_j\}) = h(\mathsf{A}_j),$$

so (ii) holds.

We now assume (ii) and prove (i). Since the events $\mathsf{A}_j, j \in J$ form a partition of $[0,1)$, $\sum_{j \in J} \lambda(\mathsf{A}_j) = 1$, so by (ii) we have $\sum_{j \in J} \mu([\![\theta_j]\!]^{\mathcal{N}}) = 1$. Therefore $[\![\bigvee_{j \in J} \theta_j]\!]^{\mathcal{N}} = \top$, so $\mathcal{N}$ is a randomization of the sentence $\varphi = \bigvee_{j \in J} \theta_j$. Since $I(\varphi)$ is countable, $\bigcup_n S_n(\varphi)$ is countable. Then by Theorem 8.3, $\mathcal{N}$ is elementarily embeddable in a basic randomization. By Theorem 7.3, $\mathcal{N}$ is isomorphic to $\mathcal{P}(C)$ for some countable generator C in a family $\langle \mathcal{M}_t \rangle_{t \in [0,1)}$ of countable models of φ over the probability space $([0,1), \mathcal{L}, \lambda)$. By Proposition 6.5, for each $j \in J$ the set $\mathsf{B}_j = \{t \in [0,1) \colon \mathcal{M}_t \models \theta_j\} \in \mathcal{L}$ and $\lambda(\mathsf{B}_j) = \mu([\![\theta_j]\!]^{\mathcal{N}}) = \lambda(\mathsf{A}_j)$. By Theorem 4.9, $\mathcal{P} \cong \mathcal{P}' = (\prod_{j \in J} (\mathcal{H}_j)^{\mathsf{B}_j}, \mathcal{L})$. For each $j \in J$, let ℓ_j be a mapping that stretches B_j to $[0,1)$.

Our plan is to use Lemma 9.2 to show that the images of $\mathcal{P}(C)$ and $\mathcal{P}'$ under ℓ_j are isomorphic for each j. Intuitively, this shows that for each j, the part of $\mathcal{P}(C)$ on B_j is isomorphic to the part of $\mathcal{P}'$ on A_j. The isomorphisms on these parts can then be combined to get an isomorphism from $\mathcal{P}(C)$ to $\mathcal{P}'$.

Here are the details. For each j, $\mathcal{P}_j = (\mathcal{H}_j^{[0,1)}, \mathcal{L})$ is the Borel randomization of $\mathcal{H}_j$, and ℓ_j maps $\mathcal{P}'$ to $\mathcal{P}_j$ and maps C to a countable generator $\ell_j(C)$ in $\langle \mathcal{M}'_t \rangle_{t \in [0,1)}$ over $([0,1), \mathcal{L}, \lambda)$, where $\mathcal{M}'_t = \mathcal{M}_{\ell_j^{-1}(t)}$. Note that for each $j \in J$ and $t \in \ell_j(\mathsf{B}_j)$, we have $\mathcal{M}'_t \cong \mathcal{H}_j$. Therefore by Lemma 9.2, we have an isomorphism $h_j \colon \mathcal{P}(\ell_j(C)) \cong \mathcal{P}_j$ for each $j \in J$. By pulling these isomorphisms back we get an isomorphism $h \colon \mathcal{P}(C) \cong \mathcal{P}'$ as follows. For an element $\mathbf{f}$ of $\mathcal{P}(C)$ of sort $\mathbb{K}$, $h(\mathbf{f})$ is the element of $\mathcal{P}'$ that agrees with $\ell_j^{-1}(h_j(\ell_j(\mathbf{f})))$ on the set B_j for each j. Since $\mathcal{N} \cong \mathcal{P}(C)$ and $\mathcal{P}' \cong \mathcal{P}$, (i) holds. $\blacksquare_{9.4}$

Lemma 9.5. *The following are equivalent.*

(i) *φ has few separable randomizations.*
(ii) *For every complete separable randomization $\mathcal{N}$ of φ, there is a countable set $J \subseteq I$ such that $[\![\bigvee_{j\in J}\theta_j]\!]^{\mathcal{N}} = \top$.*
(iii) *For every complete separable randomization $\mathcal{N}$ of φ, $\mu([\![\theta_i]\!]^{\mathcal{N}}) > 0$ for some $i \in I$.*

Proof. It follows from Lemma 9.4 that (i) implies (ii). It is trivial that (ii) implies (iii).

We now assume (ii) and prove (i). Let $\mathcal{N}$ be a complete separable randomization of φ and let J be as in (ii). By removing j from J when $[\![\theta_j]\!]^{\mathcal{N}} = \bot$, we may assume that $\mu([\![\theta_j]\!]^{\mathcal{N}}) > 0$ for each $j \in J$. We also have

$$\sum_{j\in J}\mu([\![\theta_j]\!]^{\mathcal{N}}) = \mu([\![\bigvee_{j\in J}\theta_j]\!]^{\mathcal{N}}) = 1.$$

For each $j \in J$, choose $\mathcal{H}_j \in j$. Choose a partition $\{\mathsf{A}_j : j \in J\}$ of $[0,1)$ such that $\mathsf{A}_j \in \mathcal{L}$ and $\lambda(\mathsf{A}_j) = \mu([\![\theta_j]\!]^{\mathcal{N}})$ for each $j \in J$. Then by Lemma 9.4, $\mathcal{N}$ is isomorphic to the basic randomization $(\prod_{j\in J}\mathcal{H}_j^{\mathsf{A}_j}, \mathcal{L})$. Therefore (i) holds.

We assume that (ii) fails and prove that (iii) fails. Since (ii) fails, there is a complete separable randomization $\mathcal{N}$ of φ such that for every countable set $J \subseteq I$, $\mu([\![\bigvee_{i\in I}\theta_j]\!]^{\mathcal{N}}) < 1$. The set $J = \{i \in I : \mu([\![\theta_j]\!]^{\mathcal{N}}) > 0\}$ is countable. By Theorem 7.3, $\mathcal{N}$ is isomorphic to $\mathcal{P}(C)$ for some countable generator C in a family $\langle\mathcal{M}_t\rangle_{t\in\Omega}$ of countable models of φ over a probability space $(\Omega, \mathcal{E}, \nu)$. By Proposition 6.5, the set $\mathsf{E} = \{t : \mathcal{M}_t \models \bigvee_{j\in J}\theta_j\}$ belongs to $\mathcal{E}$, and $\nu(\mathsf{E}) = \mu([\![\bigvee_{j\in J}\theta_j]\!]^{\mathcal{N}}) < 1$. Let $\mathcal{P}'$ be the pre-structure $\mathcal{P}(C)$ but with the measure ν replaced by the measure υ defined by $\upsilon(\mathsf{D}) = \nu(\mathsf{D} \setminus \mathsf{E})/\nu(\Omega \setminus \mathsf{E})$. This is the conditional probability of D given $\Omega \setminus \mathsf{E}$. Then the completion $\mathcal{N}'$ of $\mathcal{P}'$ is a separable randomization of φ such that $\mu([\![\theta_i]\!]^{\mathcal{N}'}) = 0$ for every $i \in I$, so (iii) fails. $\blacksquare_{9.5}$

Here is our generalization of Result 9.1.

Theorem 9.6. *If $I(\varphi)$ is countable, then φ has few separable randomizations.*

Proof. Suppose $J = I(\varphi)$ is countable. Then φ has the same countable models as the sentence $\bigvee_{j\in J}\theta_j$. Let $\mathcal{N}$ be a complete separable randomization of φ. By Theorem 7.3, $\mathcal{N} \cong \mathcal{P}(C)$ for some countable generator C in a family of $\langle\mathcal{M}_t\rangle_{t\in\Omega}$ countable models of φ. By Proposition 6.5,

$$\mu([\![\bigvee_{j\in J}\theta_j]\!]^{\mathcal{N}}) = \mu([\![\bigvee_{j\in J}\theta_j]\!]^{\mathcal{P}(C)}) = \mu(\{t : \mathcal{M}_t \models \bigvee_{j\in J}\theta_j\}) = \mu(\{t : \mathcal{M}_t \models \varphi\}) = 1.$$

Therefore $[\![\bigvee_{j\in J}\theta_j]\!]^{\mathcal{N}} = \top$, so φ satisfies Condition (ii) of Lemma 9.5. By Lemma 9.5, φ has few separable randomizations. $\blacksquare_{9.6}$

10. Few Separable Randomizations Versus Scattered

In this section we prove two main results. First, any infinitary sentence with few separable randomizations is scattered. Second, Martin's axiom for $\aleph_1$ implies that every scattered infinitary sentence has few separable randomizations. We also discuss the connection between these results and the absolute Vaught conjecture.

Theorem 10.1. *If φ has few separable randomizations, then φ is scattered.*

Proof. Suppose φ is not scattered. By Lemma 2.5, there is a countable fragment L_A of $L_{\omega_1\omega}$ and a perfect set $P \subseteq 2^{L_A}$ such that:

- Each $s \in P$ codes an enumerated model $(\mathcal{M}(s), a(s))$ of φ, and
- If $s \neq t$ in P then $\mathcal{M}(s)$ and $\mathcal{M}(t)$ do not satisfy the same L_A-sentences.

By Theorem 6.2 in [8], P has cardinality $2^{\aleph_0}$. By the Borel Isomorphism Theorem (15.6 in [8]), there is a Borel bijection β from $[0, 1)$ onto P whose inverse is also Borel. For each $s \in P$, $(\mathcal{M}(s), a(s))$ can be written as $(\mathcal{M}(s), a_0(s), a_1(s), \ldots)$. For each $t \in [0, 1)$, let $\mathcal{M}_t = \mathcal{M}(\beta(t))$. It follows that:

(i) $\mathcal{M}_t \models \varphi$ for each $t \in [0, 1)$, and
(ii) If $s \neq t$ in P then $\mathcal{M}_s$ and $\mathcal{M}_t$ do not satisfy the same L_A-sentences.

For each $n \in \mathbb{N}$, the composition $\mathbf{c}_n = a_n \circ \beta$ belongs to the Cartesian product $\prod_{t\in[0,1)} M_t$. For each $t \in [0, 1)$, we have

$$\{\mathbf{c}_n(t) \colon n \in \mathbb{N}\} = \{a_n(\beta(t)) \colon n \in \mathbb{N}\} = M(\beta(t)) = M_t.$$

Consider an atomic formula $\psi(\vec{v})$ and a tuple $(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_n}) \in C$. ψ belongs to the fragment L_A. The set

$$\{s \in P \colon \mathcal{M}(s) \models \psi(a_{i_1}(s), \ldots, a_{i_n}(s))\} = \{s \in P \colon s(\psi(v_{i_1}, \ldots, v_{i_n})) = 0\}$$

is Borel in P. Since β and its inverse are Borel functions, it follows that

$$\{t \in [0, 1) \colon \mathcal{M}_t \models \psi(\mathbf{c}_{i_1}(t), \ldots, \mathbf{c}_{i_n}(t)) \in \mathcal{L}.$$

Thus C satisfies conditions (a) and (b) of Definition 6.1, and hence is a countable generator in $\langle \mathcal{M}_t \rangle_{t\in[0,1)}$ over $([0, 1), \mathcal{L}, \lambda)$.

By (ii), for each $i \in I$, there is at most one $t \in [0, 1)$ such that $\mathcal{M}_t \models \theta_i$. By Theorem 6.2 and Proposition 6.5, the randomization $\mathcal{N} = \mathcal{P}(C)$ generated by C is a separable pre-complete randomization of φ. The event sort of $\mathcal{N}$ is $([0, 1), \mathcal{L}, \lambda)$. Therefore, for each $i \in I$, the event $[\![\theta_i]\!]^{\mathcal{N}}$ is either a singleton or empty, and thus has measure zero. So by Lemma 9.5, φ does not have few separable randomizations. $\blacksquare_{10.1}$

Corollary 10.2. *Assume that the absolute Vaught conjecture holds for the $L_{\omega_1\omega}$ sentence φ. Then the following are equivalent:*

(i) *$I(\varphi)$ is countable;*
(ii) *φ has few separable randomizations;*
(iii) *φ is scattered.*

Proof. (i) implies (ii) by Result 9.1. (ii) implies (iii) by Theorem 10.1. The absolute Vaught conjecture for φ says that (iii) implies (i). $\blacksquare_{10.2}$

Our next theorem will show that if ZFC is consistent, then the converse of Theorem 10.1 is consistent with ZFC.

The Lebesgue measure is said to be $\aleph_1$-additive if the union of $\aleph_1$ sets of Lebesgue measure zero has Lebesgue measure zero. Note that the continuum hypothesis implies that Lebesgue measure is not $\aleph_1$-additive. Solovay and Tennenbaum [16] proved the relative consistency of Martin's axiom $MA(\aleph_1)$, and Martin and Solovay [12] proved that $MA(\aleph_1)$ implies that the Lebesgue measure is $\aleph_1$-additive. Hence if ZFC is consistent, then so is ZFC plus the Lebesgue measure is $\aleph_1$-additive. See [11] for an exposition.

Theorem 10.3. *Assume that the Lebesgue measure is $\aleph_1$-additive. If φ is scattered, then φ has few separable randomizations.*

Proof. Suppose φ is scattered. Then there are at most countably many ω-equivalence classes of countable models of φ, so there are at most countably many first order types that are realized in countable models of φ. Thus $\bigcup_n S_n(\varphi)$ is countable.

Let $\mathcal{N}$ be a complete separable randomization of φ. By Theorem 8.3, $\mathcal{N}$ is elementarily embeddable in some basic randomization. By Theorem 7.3, there is a countable generator C in a family $\langle \mathcal{M}_t \rangle_{t\in[0,1)}$ of countable models of φ over $([0,1), \mathcal{L}, \lambda)$ such that $\mathcal{N} \cong \mathcal{P}(C)$. By Proposition 6.5, for each $i \in I(\varphi)$ we have $\mathsf{B}_i := \{t \colon \mathcal{M}_t \models \theta_i\} \in \mathcal{L}$. Moreover, the events B_i are pairwise disjoint and their union is $[0,1)$. By Result 2.3, $I(\varphi)$ has cardinality at most $\aleph_1$.

Let $J := \{i \in I(\varphi) \colon \lambda(\mathsf{B}_i) > 0\}$. Then J is countable. The set $I(\varphi) \setminus J$ has cardinality at most $\aleph_1$, so by hypothesis we have $\lambda(\bigcup_{j\in J} \mathsf{B}_j) = 1$. Pick an element $j_0 \in J$. For $j \neq j_0$ let $\mathsf{A}_j = \mathsf{B}_j$. Let A_{j_0} contain the other elements of $[0,1)$, so $\mathsf{A}_{j_0} = \mathsf{B}_{j_0} \cup ([0,1) \setminus \bigcup_{j\in J} \mathsf{B}_j)$. Then $\langle \mathsf{A}_j \rangle_{j\in J}$ is a partition of $[0,1)$. For each $j \in J$, choose a model $\mathcal{H}_j$ of isomorphism type j. Then $\mathcal{P} = (\prod_{j\in J} \mathcal{H}_j^{\mathsf{A}_j}, \mathcal{L})$ is a basic randomization of φ. For each $j \in J$ we have $\lambda([\![\theta_j]\!]^{\mathcal{N}}) = \lambda(\mathsf{A}_j)$, so by Lemma 9.4, $\mathcal{N}$ is isomorphic to $\mathcal{P}$. This shows that φ has few separable randomizations. $\blacksquare_{10.3}$

Corollary 10.4. *Assume that the Lebesgue measure is $\aleph_1$-additive. Then the following are equivalent.*

(i) *For every φ, the absolute Vaught conjecture holds.*
(ii) *For every φ, if φ has few separabable randomiztions then $I(\varphi)$ is countable.*

Proof. Corollary 10.2 shows that (i) implies (ii).

Assume that (i) fails. Then there is a scattered sentence φ such that $|I(\varphi)| = \aleph_1$. By Theorem 10.3, φ has few separable randomizations. Therefore (ii) fails. $\blacksquare_{10.4}$

11. Some Open Questions

Question 11.1. Suppose $\mathcal{N}$ and $\mathcal{P}$ are complete separable randomizations. If

$$\mu([\![\varphi]\!]^{\mathcal{N}}) = \mu([\![\varphi]\!]^{\mathcal{P}})$$

for every $L_{\omega_1\omega}$ sentence φ, must $\mathcal{N}$ be isomorphic to $\mathcal{P}$?

Question 11.2. Suppose C and D are countable generators in $\langle \mathcal{M}_t \rangle_{t\in\Omega}$, $\langle \mathcal{H}_t \rangle_{t\in\Omega}$ over the same probability space $(\Omega, \mathcal{E}, \nu)$. If $\mathcal{M}_t \cong \mathcal{H}_t$ for ν-almost all $t \in \Omega$, must $\mathcal{P}(C)$ be isomorphic to $\mathcal{P}(D)$?

Question 11.3. (Possible improvement of Theorem 8.3.) If $\bigcup_n S_n(\varphi)$ is countable, must every complete separable randomization of φ be elementarily embeddable in a basic randomization of φ?

Question 11.4. Can Theorem 10.3 be proved in ZFC (without the hypothesis that the Lebesgue measure is $\aleph_1$-additive)?

Added in December, 2016: The above question was answered affirmatively in a forthcoming paper, "Scattered Sentences Have Few Separable Randomizations", by Uri Andrews, Isaac Goldbring, Sherwood Hachtman, H.Jerome Keisler, and David Marker.

References

[1] Uri Andrews and H. Jerome Keisler. Separable Models of Randomizations. Journal of Symbolic Logic 80 (2015), 1149-1181.
[2] Uri Andrews, Isaac Goldbring, and H. Jerome Keisler. Definable Closure in Randomizations. Annals of Pure and Applied Logic 166 (2015), pp. 325-341.
[3] Itaï Ben Yaacov. On Theories of Random Variables. Israel J. Math 194 (2013), 957-1012.
[4] John Baldwin, Sy Friedman, Martin Koerwein, and Michael Laskowski. Three Red Herrings around Vaught's Conjecture. To appear, Transactions of the American Mathematical Society.
[5] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson and Alexander Usvyatsov. Model Theory for Metric Structures. In Model Theory with Applications to Algebra and Analysis, vol. 2, London Math. Society Lecture Note Series, vol. 350 (2008), 315-427.
[6] Itaï Ben Yaacov and H. Jerome Keisler. Randomizations of Models as Metric Structures. Confluentes Mathematici 1 (2009), pp. 197-223.
[7] C.J. Eagle. Omitting Types in Infinitary [0,1]-valued Logic. Annals of Pure and Applied Logic 165 (2014), 913-932.
[8] Alexander Kechris. Classical Descriptive Set Theory. Springer-Verlag (1995).
[9] H. Jerome Keisler. Model Theory for Infinitary Logic. North-Holland 1971.
[10] H. Jerome Keisler. Randomizing a Model. Advances in Mathematics 143 (1999), 124-158.
[11] Kenneth Kunen. Set Theory. Studies in logic 34, London:College Publications (2011).
[12] Donald Martin and Robert Solovay. Internal Cohen Extensions. Annals of Mathematical Logic 2 (1970), 143-178.
[13] Michael Morley. The Number of Countable Models. Jounal of Symbolic logic 35 (1970), 14-18.
[14] Jack H. Silver. Counting the Number of Equivalence Classes of Borel and Coanalytic Equivalence Relations. Ann. Math. Logic 18 (1980), 1-28.
[15] Dana Scott. Logic with Denumerably Long Formulas and Finite Strings of Quantifiers. In *The Theory of Models*, ed. by J. Addison et. al., North-Holland 1965, 329-341.
[16] Robert Solovay and Stanley Tennenbaum. Iterated Cohen Extensions and Souslin's Problem. Annals of Mathematics 94 (1971), 201-245.
[17] John R. Steel. On Vaught's conjecture. In Cabal Seminar 76-77, 193-208, Lecture Notes in Math., 689, Springer (1978).
[18] Robert L. Vaught. Denumerable Models of Complete Theories. Pages 303-321 in Infinitistic Methods, Warsaw 1961.

Department of Mathematics,, University of Wisconsin-Madison, Madison, WI 53706, www.math.wisc.edu/∼keisler
E-mail address: keisler@math.wisc.edu