# ALMOST EVERYWHERE ELIMINATION OF PROBABILITY QUANTIFIERS

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**Abstract.** We obtain an almost everywhere quantifier elimination for (the noncritical fragment of) the logic with probability quantifiers, introduced by the first author in [10]. This logic has quantifiers like $\exists^{\geq 3/4} y$, which says that "for at least 3/4 of all $y$". These results improve upon the 0-1 law for a fragment of this logic obtained by Knyazev [11]. Our improvements are:

1. We deal with the quantifier $\exists^{\geq r}\mathbf{y}$, where $\mathbf{y}$ is a tuple of variables.
2. We remove the closedness restriction, which requires that the variables in $\mathbf{y}$ occur in all atomic subformulas of the quantifier scope.
3. Instead of the unbiased measure where each model with universe $n$ has the same probability, we work with any measure generated by independent atomic probabilities $p_R$ for each predicate symbol $R$.
4. We extend the results to parametric classes of finite models (for example, the classes of bipartite graphs, undirected graphs, and oriented graphs).
5. We extend the results to a natural (noncritical) fragment of the infinitary logic with probability quantifiers.
6. We allow each $p_R$, as well as each $r$ in the probability quantifier $(\exists^{\geq r}\mathbf{y})$, to depend on the size of the universe.

**§1. Introduction.** The 0–1 law is said to hold for a logic if for each sentence $\varphi$ of the logic, the probability that a model with universe $n = \{0, \ldots, n-1\}$ satisfies $\varphi$ approaches either 0 or 1 as $n$ tends to infinity. One of the nicest theorems in finite model theory is the 0–1 law for first order logic, proved by Glebskii et. al. [6] and independently by Fagin [4].

More recently, 0–1 laws have been obtained for more powerful logics such as the infinitary logic $\mathcal{L}^k_{\infty\omega}$ with only $k$ variables. Kolaitis and Vardi [12], [7] proved the following almost everywhere quantifier elimination theorem.

*For each $k$, there is a class of finite models $\mathbf{C}_k$ of asymptotic measure 1 on which any infinitary formula in $\mathcal{L}^k_{\infty\omega}$ is equivalent to a quantifier-free first order formula with the same free variables.*

If one allows the quantifier-free always true sentence $\mathbf{T}$, the 0–1 law for infinitary sentences in $\mathcal{L}^k_{\infty\omega}$ becomes a special case of the a.e.

quantifier elimination theorem. In fact, the latter theorem explains the 0–1 law in a way that answers Fagin's question "What really causes there to be a 0–1 law?" [5].

The unbiased measure on the set of models with universe $n$, which is the measure in which each model has the same probability, is obtained by giving each atomic sentence the independent probability 1/2. It is well-known that the proof of the above 0–1 laws, as well as the almost everywhere quantifier elimination theorem, still goes through when the unbiased measure is replaced by the measure $\mu_n$ on the models with universe $n$ which is obtained by giving each atomic sentence involving a predicate symbol $R$ an independent probability $p_R \in (0, 1)$.

In this paper we will prove analogous results for an extension of first order logic, introduced by the first author in [10], that allows the use of probability quantifiers like $\exists^{\geq 3/4} y$, which means "for at least 3/4 of all $y$".

We will use the following convention on tuples of variables. When we write a formula in the form $\varphi(\mathbf{x}, \mathbf{y})$, it will be understood that $\mathbf{x}$ and $\mathbf{y}$ are tuples of variables with no repeats and no variables in common, and that the set of free variables of $\varphi$ is equal to the set of variables which occur in either $\mathbf{x}$ or $\mathbf{y}$.

In [11], Knyazev proved a 0–1 law for the fragment of this logic over the unbiased measure, but with three restrictions on the probability quantifier $(\exists^{\geq r}\mathbf{y})\varphi(\mathbf{x}, \mathbf{y})$:

- The probability quantifiers are simple, i.e. $|\mathbf{y}| = 1$.
- The probability quantifiers are closed, i.e. $y$ occurs in all atomic subformulas of the quantifier scope.
- The probability quantifiers are noncritical, i.e. $r \neq \lim_n \mu_n(\varphi)$.

We prove here a 0–1 law and an almost everywhere quantifier elimination theorem for the larger fragment obtained by removing the simplicity and the closedness restrictions. With different probabilities on quantifiers, it is natural to also allow different probabilities on atomic formulas. For this reason we replace the unbiased measure by the more general measure $\mu_n$ described above. This enables us to exhibit the interplay between the atomic probabilities $p_R$ and the probability quantifiers.

Basic concepts and results are introduced in Section 2. In Section 3, we define the noncritical fragment of the finitary logic with probability quantifiers. We prove our main results in Section 4. In Section 5 we extend our main results to parametric classes of finite models, which include the classes of bipartite graphs, undirected graphs, and oriented graphs. In Section 6 our main results are extended to infinitary logics with probability quantifiers. As in the case of ordinary quantifiers, one must restrict attention to formulas with finitely many variables. In the case of probability quantifiers, another restriction is also needed, to formulas in which only finitely many different values occur in the probability quantifiers. Finally, in Section 7 we extend our results to logics in which the atomic probabilities $p_R(n)$ and the quantifier probabilities $r(n)$ depend on the universe size $n$.

**§2. Basic Definitions and Background Results.** Let $\mathbb{N}$ be the set of positive natural numbers. Fix a vocabulary $\nu$ which is a nonempty finite set of relation symbols, and consider only $\nu$-formulas and *finite* $\nu$-models. As usual when considering 0–1 laws, the vocabulary has only predicate symbols, no function symbols. For each predicate symbol $R \in \nu$, fix a probability $p_R \in (0,1)$. For each $n \in \mathbb{N}$, let $\mathbf{M}_n$ be the (finite) set of models with universe $n = \{0, \ldots, n-1\}$, and let $\mu_n$ be the probability measure on $\mathbf{M}_n$ which is generated by independent atomic probabilities $p_R$. That is, if $R \in \nu$ is a predicate symbol of arity $k$, then for each $k$-tuple $\mathbf{a}$ of elements of $n$, the event

$$\{\mathcal{A} \in \mathbf{M}_n \colon \mathcal{A} \models R(\mathbf{a})\}$$

has probability $p_R$, and these events are mutually independent.

Thus, the unbiased measure is generated by $p_R = 1/2$ for all $R \in \nu$.

Let $\mathbf{M} = \bigcup_n \mathbf{M}_n$ be the class of all finite models. For a class $\mathbf{C} \subseteq \mathbf{M}$, $\mu_n(\mathbf{C})$ will denote the probability $\mu_n(\mathbf{C} \cap \mathbf{M}_n)$. We say that the class $\mathbf{C}$ has *asymptotic measure* $r$ if $\lim_n \mu_n(\mathbf{C}) = r$. We define the probability of a sentence $\varphi$ by $\mu_n(\varphi) = \mu_n(\mathbf{C})$ where $\mathbf{C}$ is the class of all finite models of $\varphi$, and we say that $\varphi$ has asymptotic measure $r$ if the class of all finite models of $\varphi$ has asymptotic measure $r$.

We also define probabilities of formulas. Let $\varphi(\mathbf{x})$ be a formula and let $k = |\mathbf{x}|$. Given $n \in \mathbb{N}$ and a tuple of constants $\mathbf{a} \in n^k$, the sentence $\varphi(\mathbf{a})$ has the probability

$$\mu_n(\varphi(\mathbf{a})) = \mu_n\{\mathcal{A} \in \mathbf{M}_n \colon \mathcal{A} \models \varphi(\mathbf{a})\}.$$

The probability of the formula $\varphi(\mathbf{x})$ is obtained by choosing each $\mathbf{a}$ with probability $1/n^k$, so that

$$\mu_n(\varphi(\mathbf{x})) = \sum\{\mu_n(\varphi(\mathbf{a}))/n^k \colon \mathbf{a} \in \{0, \ldots, n-1\}^k\}.$$

In the case that the formula $\varphi(\mathbf{x})$ is a sentence, this definition of $\mu_n(\varphi)$ coincides with the definition in the preceding paragraph.

The classical 0–1 law states that every first order sentence $\varphi$ has asymptotic measure 0 or 1. We will write

$$\varphi(\mathbf{x}) \text{ a.e.}$$

if the sentence $\forall\mathbf{x}\,\varphi(\mathbf{x})$ has asymptotic measure 1. We say that two formulas $\varphi(\mathbf{x})$ and $\psi(\mathbf{x})$ are *equivalent almost everywhere*, written

$$\varphi(\mathbf{x}) \equiv \psi(\mathbf{x}) \text{ a.e.}$$

if the sentence $(\forall\mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x}))$ has asymptotic measure 1. If the sentence $(\forall\mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x}))$ holds in *all* finite models, we write

$$\varphi(\mathbf{x}) \equiv \psi(\mathbf{x})$$

(without the a.e.).

Following [7], we say that a logic $\mathcal{L}$ *reduces to a logic* $\mathcal{L}'$ *weakly almost everywhere* (in symbols $\mathcal{L} \leq_{w.a.e.} \mathcal{L}'$) if every formula $\varphi(\mathbf{x})$ in $\mathcal{L}$ is equivalent almost everywhere to a formula $\varphi'(\mathbf{x})$ in $\mathcal{L}'$. We say that $\mathcal{L}$ reduces to $\mathcal{L}'$ almost everywhere (written $\mathcal{L} \leq_{a.e.} \mathcal{L}'$) if there is a class of models $\mathbf{C} \subseteq \mathbf{M}$ of asymptotic measure 1 such that, for every $\varphi(\mathbf{x}) \in \mathcal{L}$, there is a $\varphi'(\mathbf{x}) \in \mathcal{L}'$, with

$$\mathbf{C} \models \forall\mathbf{x}(\varphi(\mathbf{x}) \leftrightarrow \varphi'(\mathbf{x})).$$

Thus $\mathcal{L} \leq_{a.e.} \mathcal{L}'$ implies $\mathcal{L} \leq_{w.a.e.} \mathcal{L}'$. We say that $\mathcal{L}$ *admits (weak) almost everywhere quantifier elimination* if $\mathcal{L}$ reduces to its quantifier-free fragment (weakly) almost everywhere.

Let $\mathcal{L}_{\omega\omega}$ ($\mathcal{L}_{\omega 0}$) be the set of first order (quantifier-free) formulas, let $\mathcal{L}^k_{\omega\omega}$ ($\mathcal{L}^k_{\omega 0}$) be the fragment that uses only $k$ variables. We allow $\mathcal{L}^0_{\omega 0}$ to include the two quantifier-free sentences $\mathbf{T}$ and $\mathbf{F}$, denoting the always true and always false sentences respectively. Let $\mathcal{L}_{\infty\omega}$ be the infinitary logic, where we allow infinite conjunction and disjunction, let $\mathcal{L}^k_{\infty\omega}$ be the fragment of $\mathcal{L}_{\infty\omega}$ that uses only $k$ variables, and let $\mathcal{L}^\omega_{\infty\omega} = \bigcup_k \mathcal{L}^k_{\infty\omega}$.

It is known that the 0–1 law for first order logic (or even the infinitary logic with finitely many variables) come as an easy corollary from the following theorem [6, 12][1].

THEOREM 2.1. *For the unbiased measure, first order logic with k variables admits almost everywhere quantifier elimination, that is,*

$$\mathcal{L}^k_{\omega\omega} \leq_{a.e.} \mathcal{L}^k_{\omega 0}.$$

[1]Theorem 2.1 was essentially proved in [6], though not explicitly mentioned

*In other words, for each $k$, there is a class $\mathbf{C}_k$ of asymptotic measure 1, such that for every formula $\varphi(\mathbf{x})$ in $\mathcal{L}^k_{\omega\omega}$, there is a quantifier-free formula $\theta(\mathbf{x})$ in $\mathcal{L}^k_{\omega 0}$ such that*

$$\mathbf{C}_k \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \longleftrightarrow \theta(\mathbf{x})).$$

⊣

An easy induction can lift this result to the infinitary logic with $k$ variables. Also, taking $\mathbf{x}$ to be empty, the theorem says that each first order sentence with $k$ variables collapses to $\mathbf{T}$ or $\mathbf{F}$ almost everywhere, leading to the 0–1 law. Thus we get:

Corollary 2.2. 1. $\mathcal{L}^k_{\infty\omega} \leq_{a.e.} \mathcal{L}^k_{\omega 0}$.
2. $\mathcal{L}^\omega_{\infty\omega} \leq_{w.a.e.} \mathcal{L}_{\omega 0}$.
3. *The 0–1 law holds for the infinitary logic* $\mathcal{L}^\omega_{\infty\omega}$.

⊣

We point out, however, that it is not true that $\mathcal{L}^\omega_{\infty\omega} \leq_{a.e.} \mathcal{L}_{\omega 0}$, or even $\mathcal{L}_{\omega\omega} \leq_{a.e.} \mathcal{L}_{\omega 0}$. To see this, note that there is no class $\mathbf{C} \subseteq \mathbf{M}$ of asymptotic measure 1 such that, for every first order sentence $\varphi$, either $\mathbf{C} \models \varphi \leftrightarrow \mathbf{T}$, or $\mathbf{C} \models \varphi \leftrightarrow \mathbf{F}$. For example, the first order sentence stating that the model is of size $\geq n$ is equivalent to $\mathbf{T}$ almost everywhere but only on the class of models of size $\geq n$.

**§3. Logic with Probability Quantifiers.** The probability logic $\mathcal{L}_{\omega P}$, introduced in [10], is the first order logic augmented with all probability quantifiers of the forms $(\exists^{\geq r}\mathbf{y})$ where $r \in (0,1)$. To avoid exceptional cases, we do not allow the quantifiers $(\exists^{\geq 0}\mathbf{y})$ and $(\exists^{\geq 1}\mathbf{y})$. We thus have the usual formation rules for first order logic, and the following additional formation rule:

If $\varphi(\mathbf{x},\mathbf{y})$ is a formula with $(\mathbf{x},\mathbf{y})$ being a list of syntactically distinct variables, then $(\exists^{\geq r}\mathbf{y})\varphi(\mathbf{x},\mathbf{y})$ is a formula for each $r \in (0,1)$, with bound variables $\mathbf{y}$. The semantical interpretation of the formula $(\exists^{\geq r}\mathbf{y})\varphi(\mathbf{x},\mathbf{y})$ is:

Let $k = |\mathbf{x}|$, $\ell = |\mathbf{y}|$, $\mathcal{A} \in \mathbf{M}_n$ be a finite model with universe $A$ of size $n$, and $\mathbf{a} \in A^k$ be a $k$-tuple of elements of $A$. Then:

$$\mathcal{A} \models (\exists^{\geq r}\mathbf{y})\varphi(\mathbf{a},\mathbf{y}) \text{ iff } \frac{|\{\mathbf{b} \in A^\ell : \mathcal{A} \models \varphi(\mathbf{a},\mathbf{b})\}|}{n^\ell} \geq r.$$

Thus, $(\exists^{\geq r}\mathbf{y})\varphi(\mathbf{x},\mathbf{y})$ says that the fraction of the tuples $\mathbf{y}$ in the model that satisfy the formula $\varphi(\mathbf{x},\mathbf{y})$ is $\geq r$.

If we further allow infinitary conjunctions we get the infinitary counterpart $\mathcal{L}_{\infty P}$, which is also introduced in [10].

It is clear that the formula

$$(\exists^{\geq r}\mathbf{y})\varphi(\mathbf{x},\mathbf{y}) \rightarrow (\exists^{\geq s}\mathbf{y})\varphi(\mathbf{x},\mathbf{y})$$

is valid whenever $r > s$, and that

$$(\exists^{\geq r}\mathbf{y})\varphi(\mathbf{x},\mathbf{y}) \rightarrow \neg(\exists^{\geq s}\mathbf{y})\neg\varphi(\mathbf{x},\mathbf{y})$$

is valid whenever $r + s > 1$.

As usual, the universal and existential quantifiers are formally applied to single variables. $\forall\mathbf{y}$ means $\forall y_1, \dots \forall y_k$, and similarly for $\exists\mathbf{y}$. Note that for each formula $\varphi(\mathbf{x},\mathbf{y})$ and each $r \in (0,1)$, the formulas

$$\forall\mathbf{y}\,\varphi(\mathbf{x},\mathbf{y}) \rightarrow (\exists^{\geq r}\mathbf{y})\varphi(\mathbf{x},\mathbf{y})$$

and

$$(\exists^{\geq r}\mathbf{y})\varphi(\mathbf{x},\mathbf{y}) \rightarrow \exists\mathbf{y}\,\varphi(\mathbf{x},\mathbf{y})$$

are valid.

The goals of this paper are to exhibit the interplay between the predicate probabilities $p_R$ and the quantifier probabilities $r$, and to obtain almost everywhere quantifier elimination as well as 0–1 laws.

The papers [2], [8], and [9] gave some conditions on a set of generalized quantifiers $\mathbf{Q}$ that lead to a 0–1 law for the logic $\mathcal{L}_{\omega\omega}(\mathbf{Q})$. Those conditions are not fulfilled even for the single quantifier $\exists^{\geq 1/2}$. In fact, the 0–1 law fails here even for the unbiased measure ($p_R = 1/2$ for each $R \in \nu$). An easy counterexample is the sentence $(\exists^{\geq 1/2}x)R(x)$ (in the simple unary vocabulary $\{R\}$), which has an asymptotic probability $\frac{1}{2}$.

If we further allow binary predicates we lose even the convergence property, as seen in the following:

Proposition 3.1. *If the underlying vocabulary contains a binary predicate, then $\mathcal{L}_{\omega P}$ does not have a convergence law for the unbiased measure.*

Proof. Let $R$ be a binary predicate. Consider the sentence

$$\varphi = (\exists x)\left[(\exists^{\geq 1/2}y)R(x,y) \wedge (\exists^{\geq 1/2}y)\neg R(x,y)\right].$$

Since $\varphi$ can be read as $(\exists x)(\exists^{=1/2}y)R(x,y)$, one can see that if $n$ is odd then $\mu_n(\varphi) = 0$. However, to show that $\lim_n \mu_{2n}(\varphi) = 1$, we use independence to get

$$1 - \mu_{2n}(\varphi) = \left[1 - \binom{2n}{n}(1/2)^{2n}\right]^{2n}$$

Then, using Stirling's Formula, we get an asymptotic upper bound of the right hand side of the form $\exp(-cn)$ for some positive constant $c$. ⊣

To avoid these examples, Knyazev [11] considered a fragment of $\mathcal{L}_{\omega P}$ with the following restrictions on the measure $\mu_n$ and the probability quantifier $(\exists^{\geq r}\mathbf{y})\varphi(\mathbf{x}, \mathbf{y})$:

A. $\mu_n$ is taken to be unbiased.
B. The quantifier $(\exists^{\geq r}\mathbf{y})$ is simple, i.e. $|\mathbf{y}| = 1$.
C. The quantifier $(\exists^{\geq r}y)$ is closed, i.e. $y$ occurs in all atomic subformulas of $\varphi(\mathbf{x}, y)$.
D. The quantifier is non-critical, i.e. $r \neq \lim_n \mu_n(\varphi(\mathbf{x}, y))$.

He then proved that the restricted fragment satisfies the 0–1 law for the unbiased measure.

In this section we will remove the first three restrictions A–C, and in the next section we will prove that the resulting fragment admits weak almost everywhere quantifier elimination, as well as a 0–1 law.

For formulas $\varphi(\mathbf{x}, y)$ which satisfy A–C, Knyazev defined the critical value of $\varphi$ as the limit $\lim_n \mu_n(\varphi(\mathbf{x}, y))$. The critical value of a formula plays a crucial role in the study of probability quantifiers on finite structures. In the absence of restrictions A–C, the notion of a critical value will take a more general form, and will be defined formally in Definitions 3.4 and 3.6. Heuristically, a critical value of a formula $\varphi(\mathbf{x}, \mathbf{y})$ is a value $r$ such that the formula $(\exists^{\geq r}\mathbf{y})\varphi(\mathbf{x}, \mathbf{y})$ threatens to violate the 0–1 law. To get an idea of where we are headed, we make some preliminary observations.

First, a critical value of a formula may depend on the variable we want to quantify over, so we will need to introduce the notion of a $\mathbf{y}$-critical value of a formula for each tuple $\mathbf{y}$ of free variables. For example, for the unbiased measure, the formula

$$R_1(x) \wedge R_1(y) \wedge R_2(y)$$

should have the $x$-critical value $1/2$ and the $y$-critical value $1/4$, (i.e. if we quantify over one of these variables using the corresponding critical value for $r$, we may end up with a sentence that does not have an asymptotic measure 0 or 1).

Moreover, a formula may have more than one $y$-critical value for some variable $y$. For example, for the unbiased measure, one can check that the formula

$$(R_1(x) \wedge R_1(y)) \vee (R_1(x) \wedge R_2(x) \wedge \neg R_1(y))$$

should have both the $x$-critical values 1/2 and 1/4.

To define the set of $\mathbf{y}$-critical values of a complicated formula $\varphi$, we will first need to reduce $\varphi$ to an almost everywhere equivalent quantifier-free formula $\varphi^0$, which we will call the *quantifier-free content of* $\varphi$. We can then define the critical values of $\varphi$ to be those of $\varphi^0$.

In our formal definition, the $\mathbf{y}$-critical values of a quantifier-free formula $\alpha(\mathbf{x}, \mathbf{y})$ will be sensitive to the choice of the variable string $\mathbf{y}$, as well as to the equalities that hold between the variables in the string $\mathbf{x}$. We will simultaneously define, by induction on complexity of formulas, the property of a formula being noncritical and the quantifier-free content of a formula.

To prepare the way, we first compute the asymptotic probabilities of some formulas built from equalities. Given an equivalence relation $E$ on the set $\{1, .., k\}$ of indices of the (syntactically) distinct variables $\mathbf{x} = (x_1, .., x_k)$, we define the formula $D_E(\mathbf{x})$ which says that $x_i = x_j$ exactly when $(i, j) \in E$, i.e.

$$D_E(\mathbf{x}) := \bigwedge_{(i,j)\in E} (x_i = x_j) \wedge \bigwedge_{(i,j)\notin E} (x_i \neq x_j).$$

We use $D(\mathbf{x})$ to denote $D_E(\mathbf{x})$ when $E$ is the equality, i.e.

$$D(\mathbf{x}) := \bigwedge_{i\neq j} (x_i \neq x_j).$$

Thus $D(\mathbf{x})$ says that the elements of $\mathbf{x}$ are pairwise distinct. We define the tuple $\mathbf{x}_E = (x_{E(1)}, \dots, x_{E(k)})$, where $E(i)$ denotes the equivalence class of $i$.

Thus $\mathbf{x}_E$ is a renaming of the variables $\mathbf{x}$, where variables are identified iff their indices belong to the same equivalence class. Intuitively we assume an interpretation of $\mathbf{x}$ and $\mathbf{y}$ that respects the equivalence relation $E$ but otherwise gives distinct values to distinct variables.

We define $D_E(\mathbf{x}_E)$ to be the formula that says that $x_{E(i)} \neq x_{E(j)}$ whenever $(i, j) \notin E$.

For example, if $E$ is the equivalence relation on $\{1, 2, 3\}$ with equivalence classes $\{1, 3\}$ and $\{2\}$, then

$$(x_1, x_2, x_3)_E = (x_{\{1,3\}}, x_{\{2\}}, x_{\{1,3\}}),$$

$D_E(\mathbf{x})$ is equivalent to $x_1 = x_3 \wedge x_1 \neq x_2$, and $D_E(\mathbf{x}_E)$ is equivalent to $x_{\{1,3\}} \neq x_{\{2\}}$.

For each equivalence relation $E$ on $\{1, \dots, k\}$, $D_E(\mathbf{x}_E)$ is a conjunction of fewer than $k^2$ inequalities, so $\mu_n(\neg D_E(\mathbf{x}_E)) < k^2/n$, and therefore

$$\lim_{n\to\infty} \mu_n(D_E(\mathbf{x}_E)) = 1.$$

It follows as a special case that $\lim_{n\to\infty} \mu_n(D(\mathbf{x})) = 1$. Moreover, if $E$ is an equivalence relation but is not the equality relation, then $\lim_{n\to\infty} \mu_n(D_E(\mathbf{x})) = 0$.

By a *literal* we mean an atomic or negated atomic formula.

We now define the notion of a quantifier-free type. Given $k \geq 1$ and a $k$-tuple of variables $\mathbf{x}$, a *quantifier-free type* $\beta(\mathbf{x})$ is a maximal consistent conjunction of literals in the variables $\mathbf{x}$. Since the vocabulary $\nu$ is finite, each quantifier-free type $\beta(\mathbf{x})$ is a finite conjunction of literals, and hence is a first order quantifier-free formula. Moreover, the set of all quantifier-free types in $\mathbf{x}$ is also finite.

For each quantifier-free type $\beta(\mathbf{x})$, the relation $E$ on $\{1, \ldots, k\}$ defined by

$$E = \{(i, j)\colon \beta(\mathbf{x}) \models x_i = x_j\}$$

is an equivalence relation, which we call the equivalence relation *induced by* $\beta(\mathbf{x})$.

Note that if $\beta(\mathbf{x})$ induces $E$, then $\beta(\mathbf{x})$ logically implies $D_E(\mathbf{x})$, and $\beta(\mathbf{x}_E)$ logically implies $D_E(\mathbf{x}_E)$.

LEMMA 3.2. *Let $\beta(\mathbf{x})$ be a quantifier-free type and let $E$ be the equivalence relation induced by $\beta(\mathbf{x})$. Then $\lim_{n\to\infty} \mu_n(\beta(\mathbf{x}_E))$ exists and is greater than* 0.

PROOF. $\beta(\mathbf{x}_E)$ is equivalent to the conjunction

$$D_E(\mathbf{x}_E) \wedge \bigwedge_{i=1}^{m} \theta_i$$

where each $\theta_i$ is either of the form $R(\mathbf{z})$ or of the form $\neg R(\mathbf{z})$. Let $p = \prod_{i=1}^{m} p_i$ where

$$p_i = \begin{cases} p_R & \text{if } \theta_i \text{ is of the form } R(\mathbf{z}) \\ 1 - p_R & \text{if } \theta_i \text{ is of the form } \neg R(\mathbf{z}). \end{cases}$$

We have $m > 0$ and $p > 0$. By the definition of $\mu_n$, for each $k$-tuple $\mathbf{a}$ of constants such that $D_E(\mathbf{a})$ holds, we have $\mu_n(\beta(\mathbf{a}_E)) = p$. Therefore

$$p \cdot \mu_n(D_E(\mathbf{x}_E)) \leq \mu_n(\beta(\mathbf{x}_E)) \leq p + \mu_n(\neg D_E(\mathbf{x}_E)).$$

Since $\lim_{n\to\infty} \mu_n(D_E(\mathbf{x}_E)) = 1$, it follows that

$$\lim_{n\to\infty} \mu_n(\beta(\mathbf{x}_E)) = p.$$

$\dashv$

Note that for each quantifier-free type $\beta(\mathbf{x})$ we have $\mu_n(\beta(\mathbf{x})) > 0$ for all $n \geq |\mathbf{x}|$, so we can introduce the conditional probability of a formula given $\beta(\mathbf{x})$ in the natural way.

For a formula $\varphi(\mathbf{x}, \mathbf{y})$ and a quantifier-free type $\beta(\mathbf{x})$, and for each $n \geq |\mathbf{x}|$, we define the $\mu_n$ *conditional probability* of $\varphi(\mathbf{x}, \mathbf{y})$ given $\beta(\mathbf{x})$ as

$$\mu_n[\varphi(\mathbf{x}, \mathbf{y}) | \beta(\mathbf{x})] = \frac{\mu_n(\varphi(\mathbf{x}, \mathbf{y}) \wedge \beta(\mathbf{x}))}{\mu_n(\beta(\mathbf{x}))}.$$

In the next lemma we show that the conditional probability of a quantifier-free formula given a quantifier-free type converges as $n \to \infty$. The proof gives a formula for the limit.

LEMMA 3.3. *Let $\alpha(\mathbf{x}, \mathbf{y})$ be a quantifier-free formula and let $\beta(\mathbf{x})$ be a quantifier-free type. Then $\lim_{n\to\infty} \mu_n[\alpha(\mathbf{x}, \mathbf{y}) | \beta(\mathbf{x})]$ exists.*

PROOF. Let $E$ be the equivalence relation induced by $\beta(\mathbf{x})$. Then $\beta(\mathbf{x}_E)$ is a quantifier-free type in $\mathbf{x}_E$, and

$$\mu_n[\alpha(\mathbf{x}_E, \mathbf{y}) | \beta(\mathbf{x}_E)] = \frac{\mu_n(\alpha(\mathbf{x}_E, \mathbf{y}) \wedge \beta(\mathbf{x}_E))}{\mu_n(\beta(\mathbf{x}_E))}.$$

We note that $\beta(\mathbf{x}_E)$ implies $D_E(\mathbf{x}_E)$, and for each $k$-tuple $\mathbf{a}$ of constants such that $D_E(\mathbf{a})$ holds, we have

$$\alpha(\mathbf{a}, \mathbf{y}) \equiv \alpha(\mathbf{a}_E, \mathbf{y}), \quad \beta(\mathbf{a}) \equiv \beta(\mathbf{a}_E).$$

Therefore

$$\mu_n[\alpha(\mathbf{x}, \mathbf{y}) | \beta(\mathbf{x})] = \mu_n[\alpha(\mathbf{x}_E, \mathbf{y}) | \beta(\mathbf{x}_E)].$$

We may assume that each atomic subformula of $\alpha(\mathbf{x}_E, \mathbf{y})$ contains a variable in $\mathbf{y}$, since each atomic subformula which contains only variables in $\mathbf{x}_E$ can be replaced by $\mathbf{T}$ if it follows from $\beta(\mathbf{x}_E)$, and by $\mathbf{F}$ otherwise. Starting from $\alpha(\mathbf{x}_E, \mathbf{y})$, form the equality-free formula $\alpha_E$ by replacing each equality between distinct variables by $\mathbf{F}$, and replacing each equality between the same variable by $\mathbf{T}$.

Now expand $\alpha_E$ into the full disjunctive normal form without equality:

$$\alpha_E \equiv \bigvee_{i=1}^{m} \bigwedge_{j=1}^{l} \theta_{ij},$$

where each conjunction corresponds to a row in the truth table representation of $\alpha_E$.

We claim that

$$\lim_{n\to\infty}(\mu_n[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})]) = \sum_{i=1}^{m}\prod_{j=1}^{l} p_{ij} = p_\beta,$$

where $p_\beta$ is defined by the above equation, and

$$p_{ij} = \begin{cases} p_R & \text{if} \quad \theta_{ij} \text{ is of the form } R(\mathbf{z}) \\ 1-p_R & \text{if} \quad \theta_{ij} \text{ is of the form } \neg R(\mathbf{z}). \end{cases}$$

By convention, $p_\beta = 0$ if $\alpha_E \equiv \mathbf{F}$, and $p_\beta = 1$ if $\alpha_E \equiv \mathbf{T}$.

We now prove the claim. For each tuple $(\mathbf{a}_E, \mathbf{b})$ of distinct elements of $n$, the sentences $\alpha(\mathbf{a}_E, \mathbf{b}) \wedge \beta(\mathbf{a}_E)$ and $\alpha_E(\mathbf{a}_E, \mathbf{b}) \wedge \beta(\mathbf{a}_E)$ are equivalent in all models $\mathcal{A} \in \mathbf{M}_n$, and the sentence $\alpha_E(\mathbf{a}_E, \mathbf{b})$ has probability

$$\mu_n(\alpha_E(\mathbf{a}_E, \mathbf{b})) = p_\beta.$$

Since distinct atomic sentences have independent probabilities,

$$\mu_n(\alpha_E(\mathbf{a}_E, \mathbf{b})) = \frac{\mu_n(\alpha_E(\mathbf{a}_E, \mathbf{b}) \wedge \beta(\mathbf{a}_E))}{\mu_n(\beta(\mathbf{a}_E))}.$$

Therefore the conditional probability $\mu_n[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})]$ differs from $p_\beta$ by at most the probability $q(n)$ that the elements of $(\mathbf{x}_E, \mathbf{y})$ are not all distinct. But $q(n) \le (|\mathbf{x}| + |\mathbf{y}|)^2/n$, so $\lim_{n\to\infty} q(n) = 0$ and the claim follows. ⊣

The above lemma lets us make the following definition.

DEFINITION 3.4. *Let $\alpha(\mathbf{x},\mathbf{y})$ be a quantifier-free formula. For each quantifier-free type $\beta(\mathbf{x})$, we define*

$$\mu[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})] = \lim_{n\to\infty} \mu_n[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})].$$

*The* **$\mathbf{y}$-critical values** *of $\alpha(\mathbf{x},\mathbf{y})$ are the values $\mu[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})]$ where $\beta(\mathbf{x})$ is a quantifier-free type.*

PROPOSITION 3.5. *Every quantifier-free formula has finitely many $\mathbf{y}$-critical values for each tuple of variables $\mathbf{y}$. Moreover, each $\mathbf{y}$-critical value is equal to a polynomial in $p_R, R \in \nu$, with integer coefficients. In particular, if each probability $p_R$ is a rational number, then each $\mathbf{y}$-critical value of a quantifier-free formula is a rational number.*

PROOF. This follows easily from the proof of Lemma 3.3. ⊣

We now give our main definition, which simultaneously defines the set of noncritical formulas of $\mathcal{L}_{\infty P}$ and the quantifier-free content $\varphi^0$ of a formula $\varphi$. The limits mentioned in this definition exist by Lemma 3.3.

Definition 3.6. *Let $\varphi$ be a formula in the infinitary logic $\mathcal{L}_{\infty P}$ with only finitely many free variables.*

1. *If $\varphi$ is quantifier-free, we stipulate that $\varphi$ is noncritical and that $\varphi^0$ is $\varphi$ itself.*
2. *If $\varphi = \neg\psi$, then $\varphi$ is noncritical if and only if $\psi$ is noncritical, and we define $\varphi^0 = \neg(\psi^0)$.*
3. *If $\varphi = \bigwedge_i \psi_i$, then $\varphi$ is noncritical if and only if $\psi_i$ is noncritical for each $i$, and we define $\varphi^0 = \bigwedge_i((\psi_i)^0)$.*
4. *If $\varphi(\mathbf{x}) = \exists y\, \psi(\mathbf{x}, y)$, then $\varphi$ is noncritical if and only if $\psi$ is noncritical, and we define*
$$\varphi^0(\mathbf{x}) = \bigvee\{\beta(\mathbf{x})\colon \mu[\psi^0(\mathbf{x}, y)|\beta(\mathbf{x})] > 0\} \vee \bigvee_{i<|\mathbf{x}|} \psi^0(\mathbf{x}, x_i).$$
5. *If $\varphi(\mathbf{x}) = (\exists^{\geq r}\mathbf{y})\psi(\mathbf{x}, \mathbf{y})$, then $\varphi$ is noncritical if and only if $\psi$ is noncritical and $r$ is not a $\mathbf{y}$-critical value for $\psi^0$, and we define*
$$\varphi^0(\mathbf{x}) = \bigvee\{\beta(\mathbf{x})\colon \mu[\psi^0(\mathbf{x}, \mathbf{y})|\beta(\mathbf{x})] > r\}.$$

The *noncritical fragment* $\mathcal{L}^-_{\omega P}$ is defined as the set of all noncritical formulas of $\mathcal{L}_{\omega P}$. The *noncritical fragment* $\mathcal{L}^-_{\infty P}$ is defined as the set of all noncritical formulas of $\mathcal{L}_{\infty P}$ with only finitely many free variables.

**Warning:** *The noncritical fragment $\mathcal{L}^-_{\omega P}$ and the quantifier-free content function $\varphi \mapsto \varphi^0$ depend on the given atomic formula probabilities $p_R, R \in \nu$.*

Note that in the special case when $\psi(\mathbf{x}, \mathbf{y})$ is noncritical and
$$\mu[\psi^0(\mathbf{x}, \mathbf{y})|\beta(\mathbf{x})] = 0 \text{ for all } \beta(\mathbf{x}),$$
the formula $(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x}, \mathbf{y})$ is also noncritical and has quantifier-free content $\mathbf{F}$.

On the other hand, if $\psi(\mathbf{x}, \mathbf{y})$ is noncritical and
$$\mu[\psi^0(\mathbf{x}, \mathbf{y})|\beta(\mathbf{x})] = 1 \text{ for all } \beta(\mathbf{x}),$$
the formula $(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x}, \mathbf{y})$ is also noncritical and has quantifier-free content $\mathbf{T}$.

The language $\mathcal{L}_{\omega P}$ has uncountably many formulas because there are uncountably many probability quantifiers. But if we restrict the language to formulas with rational probability quantifiers, then the set of formulas is countable. We now observe that if each atomic probability $p_R, R \in \nu$ is also rational, the set of critical formulas and the quantifier-free content are computable.

PROPOSITION 3.7. *Suppose that $p_R$ is a rational number for each $R \in \nu$. Call a formula $\varphi \in \mathcal{L}_{\omega P}$* ***rational*** *if for every probability quantifier $(\exists^{\geq r}\mathbf{y})$ in $\varphi$, $r$ is a rational number, and let $\mathcal{L}_{\omega\mathbb{Q}}$ be the set of rational formulas of $\mathcal{L}_{\omega P}$. Then each of the following is primitive recursive:*

1. *The relation "$\alpha$ is quantifier-free and $r$ is a $\mathbf{y}$-critical value of $\alpha$".*
2. *The set $\mathcal{L}_{\omega\mathbb{Q}} \cap \mathcal{L}^-_{\omega P}$ of rational noncritical formulas.*
3. *The quantifier-free content function $\varphi \mapsto \varphi^0$ restricted to $\mathcal{L}_{\omega\mathbb{Q}} \cap \mathcal{L}^-_{\omega P}$.*

PROOF. The proof of Lemma 3.3 gives a primitive recursive algorithm for computing the $\mathbf{y}$-critical values of a quantifier-free formula, establishing (*1*). Parts (*2*) and (*3*) follow easily from Part (*1*) and Definition 3.6. ⊣

**§4. Elimination of Probability Quantifiers.** In this section we prove our main result, which shows that each noncritical formula $\varphi$ is almost everywhere equivalent to its quantifier-free content $\varphi^0$. As we mentioned in the introduction, it is well-known that the proof of Theorem 2.1 goes through for the measure $\mu_n$ instead of the unbiased measure, so that every first order formula is almost everywhere equivalent to a quantifier-free formula. In fact, from that proof one can easily see that every first order formula is almost everywhere equivalent to its quantifier-free content as defined in Definition 3.6. We will not repeat that proof here, but we will need the following special case of the result.

LEMMA 4.1. *For each quantifier-free first order formula $\psi(\mathbf{x}, y)$, $\exists y\, \psi(\mathbf{x}, y)$ is almost everywhere equivalent to its quantifier-free content $(\exists y\, \psi(\mathbf{x}, y))^0$.* ⊣

The formula below for the conditional probability of the negation is easily checked.

LEMMA 4.2. *For any quantifier-free formula $\varphi(\mathbf{x}, \mathbf{y})$ and quantifier-free type $\beta(\mathbf{x})$, we have*

$$\mu[\neg\varphi(\mathbf{x}, \mathbf{y})|\beta(\mathbf{x})] = 1 - \mu[\varphi(\mathbf{x}, \mathbf{y})|\beta(\mathbf{x})].$$

⊣

We next observe that every quantifier-free first order formula can be represented in the following normal form. We omit the proof, which is routine.

LEMMA 4.3. *Each quantifier-free first order formula $\varphi(\mathbf{x}, \mathbf{y})$ can be represented in the* **normal form**

$$\varphi(\mathbf{x}, \mathbf{y}) \equiv \bigvee_{i=1}^{h} (\alpha_i(\mathbf{x}, \mathbf{y}) \wedge \beta_i(\mathbf{x})),$$

*where the $\beta_i(\mathbf{x})$ are quantifier-free types in $\mathbf{x}$, and each atomic subformula of each $\alpha_i(\mathbf{x}, \mathbf{y})$ contains a variable from $\mathbf{y}$. This representation is unique up to renumbering and logical equivalence.* ⊣

The next lemma gives the asymptotic probabilities of the formulas $\alpha_i(\mathbf{x}, \mathbf{y})$.

LEMMA 4.4. *Let $\alpha(\mathbf{x}, \mathbf{y})$ be a first order quantifier-free formula in which every atomic subformula contains a variable from $\mathbf{y}$. Let $\beta(\mathbf{x})$ be a quantifier-free type and let $E$ be the equivalence relation induced by $\beta(\mathbf{x})$. Then*

$$\lim_{n\to\infty} \mu_n(\alpha(\mathbf{x}_E, \mathbf{y})) = \mu[\alpha(\mathbf{x}, \mathbf{y})|\beta(\mathbf{x})].$$

PROOF. Since

$$\mu[\alpha(\mathbf{x}, \mathbf{y})|\beta(\mathbf{x})] = \mu[\alpha(\mathbf{x}_E, \mathbf{y})|\beta(\mathbf{x}_E)],$$

and $\beta(\mathbf{x}_E)$ is a quantifier-free type which implies $u \neq v$ for every pair of syntactically distinct variables $u, v$ in the tuple $\mathbf{x}_E$, we may assume that $E$ is the equality relation. By hypothesis, $\alpha(\mathbf{x}, \mathbf{y})$ and $\beta(\mathbf{x})$ have no atomic subformulas in common. Therefore for any constants $\mathbf{a}$ and $\mathbf{b}$ which have no elements in common, we have

$$\mu_n(\alpha(\mathbf{a}, \mathbf{b}) \wedge \beta(\mathbf{a})) = \mu_n(\alpha(\mathbf{a}, \mathbf{b})) \cdot \mu_n(\beta(\mathbf{a})).$$

Let $q(n)$ be the probability that some element of $\mathbf{y}$ equals some element of $\mathbf{x}$. Then

$$|\mu_n(\alpha(\mathbf{x}, \mathbf{y}) \wedge \beta(\mathbf{x})) - \mu_n(\alpha(\mathbf{x}, \mathbf{y})) \cdot \mu_n(\beta(\mathbf{x}))| \leq q(n),$$

$$|\mu_n[\alpha(\mathbf{x}, \mathbf{y})|\beta(\mathbf{x})] - \mu_n(\alpha(\mathbf{x}, \mathbf{y}))| \leq q(n)/\mu_n(\beta(\mathbf{x})).$$

We have $q(n) \leq |x| \cdot |y|/n$, so $\lim_{n\to\infty} q(n) = 0$. The result now follows form Lemmas 3.2 and 3.3 by taking the limit as $n \to \infty$. ⊣

The next lemma is implicit in the paper [11]. For completeness we give a proof here.

LEMMA 4.5. *Let $\psi(\mathbf{x}, y)$ be a first order quantifier-free formula such that $y$ appears in every atomic subformula. Let*

$$L = \lim_{n\to\infty} \mu_n(\psi(\mathbf{x}, y))$$

*(which exists by Lemma 4.4). Then*

1. *If $r < L$, then $D(\mathbf{x}) \to (\exists^{\geq r} y)\psi(\mathbf{x}, y)$ a.e.*
2. *If $r > L$, then $D(\mathbf{x}) \to \neg(\exists^{\geq r} y)\psi(\mathbf{x}, y)$ a.e.*

PROOF. Let $\beta(\mathbf{x})$ be a quantifier-free type which induces the equality relation. By Lemma 4.4, $L = \mu[\psi(\mathbf{x}, y)|\beta(\mathbf{x})]$. As in the proof of Lemma 3.3, we can compute $L$ by starting from $\psi(\mathbf{x}, y)$, forming the equality-free formula $\alpha(\mathbf{x}, y)$ by replacing each equality between distinct variables by $\mathbf{F}$, and replacing each equality between the same variable by $\mathbf{T}$, and putting $\alpha$ into the full disjunctive normal form without equality:

$$\alpha \equiv \bigvee_{i=1}^{m} \bigwedge_{j=1}^{l} \theta_{ij}.$$

We have

$$L = \sum_{i=1}^{m} \prod_{j=1}^{l} p_{ij},$$

where

$$p_{ij} = \begin{cases} p_R & \text{if } \theta_{ij} \text{ is of the form } R(\mathbf{z}) \\ 1 - p_R & \text{if } \theta_{ij} \text{ is of the form } \neg R(\mathbf{z}). \end{cases}$$

It follows from the theory of independent Bernoulli trials that for each real $\varepsilon > 0$ there exist $c > 0$ and $N \in \mathbb{N}$ such that for each $n \geq N$ and each tuple of distinct constants $\mathbf{a} \in n^{|\mathbf{x}|}$,

$$\mu_n\left(\left|\frac{|\{y \in n \colon \psi(\mathbf{a}, y)\}|}{n} - L\right| > \varepsilon\right) \leq e^{-cn}.$$

But $\lim_{n\to\infty} n^{|\mathbf{x}|} e^{-cn} = 0$. Thus for each $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu_n\left(\forall \mathbf{x}\left(D(\mathbf{x}) \to (\exists^{\geq L-\varepsilon} y)\psi(\mathbf{x}, y)\right)\right) = 1,$$

so (*1*) holds. Similarly,

$$\lim_{n\to\infty} \mu_n\left(\forall \mathbf{x}\left(D(\mathbf{x}) \to \neg(\exists^{\geq L+\varepsilon} y)\psi(\mathbf{x}, y)\right)\right) = 1,$$

and (*2*) follows. ⊣

Lemma 4.5 deals with the simple case, where the probability quantifier is on a single variable $y$. To handle the general case, we prove a stronger lemma, which almost everywhere eliminates the quantifiers $(\exists^{\geq r}\mathbf{y})$.

LEMMA 4.6. *Let* $\psi(\mathbf{x},\mathbf{y})$ *be a first order quantifier-free formula, in which each atomic subformula contains at least one of the variables in* $\mathbf{y}$. *Let* $L = \lim_{n\to\infty}\mu_n(\psi(\mathbf{x},\mathbf{y}))$. *Then*

1. *If* $r < L$, *then* $D(\mathbf{x}) \to (\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y})$ *a.e.*
2. *If* $r > L$, *then* $D(\mathbf{x}) \to \neg(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y})$ *a.e.*

PROOF. By induction on $|\mathbf{y}|$.
*Basis:* $(|\mathbf{y}| = 1)$ This is Lemma 4.5.

*Induction Step:* (Assume the result for $|\mathbf{y}|$ and quantify over $\mathbf{y}, z$) For (*1*), we assume that $r < L$, and put $\psi(\mathbf{x},\mathbf{y},z)$ in the full disjunctive normal form, so that the disjuncts are exclusive. We'll show by an example that if (*1*) is proved for the exclusive disjuncts, it will then follow that it holds for the full disjunction.

Say $\psi(\mathbf{x},\mathbf{y},z) = \psi_1(\mathbf{x},\mathbf{y},z) \vee \psi_2(\mathbf{x},\mathbf{y},z)$, where $\psi_1(\mathbf{x},\mathbf{y},z)$ and $\psi_2(\mathbf{x},\mathbf{y},z)$ are exclusive and may or may not contain $z$. Thus we have that $L = L_1 + L_2$ where $L_1 = \lim_{n\to\infty}\mu_n(\psi_1(\mathbf{x},\mathbf{y},z))$, and $L_2 = \lim_{n\to\infty}\mu_n(\psi_2(\mathbf{x},\mathbf{y},z))$.

Since $L - r = L_1 + L_2 - r > 0$, we can find $r_1, r_2$ such that $r = r_1 + r_2$, $0 < r_1 < L_1$, and $0 < r_2 < L_2$.

Now it is readily seen that the implication

$$(\exists^{\geq r_1}\mathbf{y}z)\psi_1(\mathbf{x},\mathbf{y},z) \wedge (\exists^{\geq r_2}\mathbf{y}z)\psi_2(\mathbf{x},\mathbf{y},z) \to (\exists^{\geq r}\mathbf{y}z)\psi(\mathbf{x},\mathbf{y},z)$$

is logically valid.

Thus, if both

$$D(\mathbf{x}) \to (\exists^{\geq r_1}\mathbf{y}z)\psi_1(\mathbf{x},\mathbf{y},z) \text{ a.e.}$$

and

$$D(\mathbf{x}) \to (\exists^{\geq r_2}\mathbf{y}z)\psi_2(\mathbf{x},\mathbf{y},z) \text{ a.e. },$$

then

$$D(\mathbf{x}) \to (\exists^{\geq r}\mathbf{y}z)\psi(\mathbf{x},\mathbf{y},z) \text{ a.e.}$$

So it is enough to prove (*1*) for the disjuncts, i.e. without loss of generality we assume that $\psi(\mathbf{x},\mathbf{y},z)$ is a conjunction of literals.

Let's write it as:

$$\psi(\mathbf{x},\mathbf{y},z) = \bigwedge_{i=1}^{m} \alpha_i(\mathbf{x},\mathbf{y},z) \wedge \bigwedge_{i=1}^{k} \beta_i(\mathbf{x},\mathbf{y}),$$

where $z$ appears in each $\alpha_i(\mathbf{x},\mathbf{y},z)$.

From the form of $\psi(\mathbf{x},\mathbf{y},z)$ we can see that $L = \prod_{j=1}^{m+k} p_j$ for suitable values of $p_j,\ j = 1,\dots,m+k$. Since $r < L$, we can again find $r_1$ and $r_2$ such that $r = r_1 r_2$, $\prod_{j=1}^{m} p_j > r_1 > 0$ and $\prod_{j=m+1}^{m+k} p_j > r_2 > 0$.

Now one can check that the implication

$$(\exists^{\geq r_2}\mathbf{y})\left(D(\mathbf{y}) \to (\exists^{\geq r_1} z)\psi(\mathbf{x},\mathbf{y},z)\right) \longrightarrow (\exists^{\geq r}\mathbf{y}z)\psi(\mathbf{x},\mathbf{y},z)$$

is valid almost everywhere. Thus, we just need to prove that

$$D(\mathbf{x}) \to (\exists^{\geq r_2}\mathbf{y})\left(D(\mathbf{y}) \to (\exists^{\geq r_1} z)\psi(\mathbf{x},\mathbf{y},z)\right) \text{ a.e.}$$

But we have that:

$$\begin{aligned}
& D(\mathbf{y}) \to (\exists^{\geq r_1} z)\psi(\mathbf{x},\mathbf{y},z) \\
= \quad & D(\mathbf{y}) \to (\exists^{\geq r_1} z)\left(\bigwedge_{i=1}^{m} \alpha_i(\mathbf{x},\mathbf{y},z) \wedge \bigwedge_{i=1}^{k} \beta_i(\mathbf{x},\mathbf{y})\right) \\
\equiv\ & D(\mathbf{y}) \to \left(\left((\exists^{\geq r_1} z)\bigwedge_{i=1}^{m} \alpha_i(\mathbf{x},\mathbf{y},z)\right) \wedge \bigwedge_{i=1}^{k} \beta_i(\mathbf{x},\mathbf{y})\right) \\
\equiv \quad & D(\mathbf{y}) \to \bigwedge_{i=1}^{k} \beta_i(\mathbf{x},\mathbf{y}) \text{ a.e. },
\end{aligned}$$

where in the last a.e. equivalence we used Lemma 4.5, which states that

$$D(\mathbf{y}) \to (\exists^{\geq r_1} z)\bigwedge_{i=1}^{m} \alpha_i(\mathbf{x},\mathbf{y},z) \text{ a.e. },$$

since

$$\lim_{n\to\infty} \mu_n \left(\bigwedge_{i=1}^{m} \alpha_i(\mathbf{x},\mathbf{y},z)\right) = \prod_{j=1}^{m} p_j > r_1.$$

Now, since

$$\lim_{n\to\infty} \mu_n \left(\bigwedge_{i=1}^{m} \beta_i(\mathbf{x},\mathbf{y})\right) = \prod_{j=m+1}^{m+k} p_j > r_2,$$

we can use the induction hypothesis to get:

$$D(\mathbf{x}) \to (\exists^{\geq r_2}\mathbf{y}) \bigwedge_{i=1}^{k} \beta_i(\mathbf{x},\mathbf{y}) \text{ a.e.}$$

Thus, we get

$$D(\mathbf{x}) \to (\exists^{\geq r_2}\mathbf{y}) \big(D(\mathbf{y}) \to (\exists^{\geq r_1} z)\psi(\mathbf{x},\mathbf{y},z)\big) \text{ a.e. },$$

which completes the proof of (*1*).

For (*2*), we assume that $r > L$, or equivalently, $1-r < 1-L$. By Lemma 4.2, $1-L$ is the $\mathbf{y}$-critical value of $\neg\psi(\mathbf{x},\mathbf{y},z)$. Choose $s$ such that $1-r < s < 1-L$. Applying (*1*) to the formula $\neg\psi(\mathbf{x},\mathbf{y},z)$, we see that

$$D(\mathbf{x}) \to (\exists^{\geq s}\mathbf{y}z)\neg\psi(\mathbf{x},\mathbf{y},z) \text{ a.e.}$$

We have $r+s > 1$, and it follows that

$$D(\mathbf{x}) \to \neg(\exists^{\geq r}\mathbf{y}z)\psi(\mathbf{x},\mathbf{y},z) \text{ a.e. },$$

and we conclude that (*2*) holds. ⊣

This removes Restriction B (quantifier simplicity). As an easy corollary we have:

LEMMA 4.7. *Let $\psi(\mathbf{x},\mathbf{y})$ be a first order quantifier-free formula, in which each atomic subformula contains at least one of the variables in $\mathbf{y}$. Let $E$ be an equivalence relation on the indices of $\mathbf{x}$, and let $L = \lim_{n\to\infty} \mu_n(\psi(\mathbf{x}_E,\mathbf{y}))$. Then*

1. *If $r < L$, then $D_E(\mathbf{x}) \to (\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y})$ a.e.*
2. *If $r > L$, then $D_E(\mathbf{x}) \to \neg(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y})$ a.e.*

⊣

We now deal with general quantifier-free formulas. The following lemma allows us to remove Restriction C (quantifier closedness).

LEMMA 4.8. *Let $\psi(\mathbf{x},\mathbf{y})$ be a first order quantifier-free formula. and assume that $r \in (0,1)$ is not a $\mathbf{y}$-critical value for $\psi$. Then*

$$(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y}) \equiv \big((\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y})\big)^0 \quad \text{a.e.}$$

*That is, $(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y})$ is almost everywhere equivalent to its quantifier-free content.*

PROOF. We first write $\psi$ in the normal form of Lemma 4.3:

$$\psi(\mathbf{x},\mathbf{y}) \equiv \bigvee_{i=1}^{m} (\alpha_i(\mathbf{x},\mathbf{y}) \wedge \beta_i(\mathbf{x})).$$

For each $i$, we have $\beta_i(\mathbf{x}) \to D_E(\mathbf{x})$ where $E$ is the equivalence relation induced by $\beta_i(\mathbf{x})$. Therefore by Lemma 4.7, whenever $r < \mu[\alpha_i(\mathbf{x},\mathbf{y})|\beta_i(\mathbf{x})]$, we have

$$\beta_i(\mathbf{x}) \to (\exists^{\geq r}\mathbf{y})\alpha_i(\mathbf{x},\mathbf{y}) \text{ a.e. },$$

and hence

$$\beta_i(\mathbf{x}) \equiv (\exists^{\geq r}\mathbf{y})\alpha_i(\mathbf{x},\mathbf{y}) \wedge \beta_i(\mathbf{x}) \text{ a.e.}$$

On the other hand, when $r > \mu[\alpha_i(\mathbf{x},\mathbf{y})|\beta_i(\mathbf{x})]$, we have

$$\beta_i(\mathbf{x}) \to \neg(\exists^{\geq r}\mathbf{y})\alpha_i(\mathbf{x},\mathbf{y}) \text{ a.e. },$$

and hence

$$\mathbf{F} \equiv (\exists^{\geq r}\mathbf{y})\alpha_i(\mathbf{x},\mathbf{y}) \wedge \beta_i(\mathbf{x}) \text{ a.e.}$$

Since each $\beta_i(\mathbf{x})$ is a quantifier-free type, for each $i$ we have

$$\psi(\mathbf{x},\mathbf{y}) \wedge \beta_i(\mathbf{x}) \equiv \alpha_i(\mathbf{x},\mathbf{y}) \wedge \beta_i(\mathbf{x}),$$

so

$$\mu[\psi(\mathbf{x},\mathbf{y})|\beta_i(\mathbf{x})] = \mu[\alpha_i(\mathbf{x},\mathbf{y})|\beta_i(\mathbf{x})].$$

It follows that

$$\begin{aligned}(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y}) &\equiv (\exists^{\geq r}\mathbf{y})\bigvee_{i=1}^{m}(\alpha_i(\mathbf{x},\mathbf{y}) \wedge \beta_i(\mathbf{x}))\\ &\equiv \bigvee_{i=1}^{m}\big((\exists^{\geq r}\mathbf{y})\alpha_i(\mathbf{x},\mathbf{y}) \wedge \beta_i(\mathbf{x})\big)\\ &\equiv \bigvee\{\beta_i(\mathbf{x})\colon r < \mu[\alpha_i(\mathbf{x},\mathbf{y}))|\beta_i(\mathbf{x})]\} \text{ a.e.}\\ &= \bigvee\{\beta_i(\mathbf{x})\colon r < \mu[\psi(\mathbf{x},\mathbf{y}))|\beta_i(\mathbf{x})]\}\\ &= \big((\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y})\big)^0.\end{aligned}$$

⊣

Using this lemma, we obtain our main result, which extends Theorem 2.1 to the probability logic $\mathcal{L}^-_{\omega P}$.

THEOREM 4.9. *Every formula $\varphi$ in $\mathcal{L}^-_{\omega P}$ is a.e. equivalent to its quantifier-free content.*

PROOF. By induction on the complexity of $\varphi$. The basis step is trivial, and the induction steps for connectives are easy.

For the induction step for existential quantifiers, we let $\varphi(\mathbf{x}) = \exists y\, \psi(\mathbf{x}, y)$ where $\psi(\mathbf{x}, y)$ is noncritical, and assume the inductive hypothesis that $\psi(\mathbf{x}, y)$ is a.e. equivalent to $\psi^0(\mathbf{x}, y)$. Then $\varphi(\mathbf{x})$ is a.e. equivalent to $\exists y\, \psi^0(\mathbf{x}, y)$. By definition, $\varphi(\mathbf{x})$ has the same quantifier-free content as $\exists y\, \psi^0(\mathbf{x}, y)$, and by Lemma 4.1, $\exists y\, \psi^0(\mathbf{x}, y)$ is a.e. equivalent to its quantifier-free content, so $\varphi(\mathbf{x})$ is a.e. equivalent to its quantifier-free content.

It remains to give the induction step for probability quantifiers. For this step, we let $\varphi(\mathbf{x})$ be a noncritical formula of the form $(\exists^{\geq r}\mathbf{y})\psi(\mathbf{x}, \mathbf{y})$, assume the induction hypothesis that $\psi(\mathbf{x}, \mathbf{y})$ is a.e. equivalent to $\psi^0(\mathbf{x}, \mathbf{y})$, and use Lemma 4.8 to get:

$$\varphi(\mathbf{x}) = (\exists^{\geq r}\mathbf{y})\psi(\mathbf{x}, \mathbf{y}) \equiv (\exists^{\geq r}\mathbf{y})\psi^0(\mathbf{x}, \mathbf{y}) \equiv ((\exists^{\geq r}\mathbf{y})\psi^0(\mathbf{x}, \mathbf{y}))^0 \text{ a.e.}$$

⊣

Corollary 4.10. 1. $\mathcal{L}^-_{\omega P} \leq_{w.a.e.} \mathcal{L}_{\omega 0}$.
2. *The 0–1 law holds for the logic* $\mathcal{L}^-_{\omega P}$.

Proof. Part *1* is a corollary of Theorem 4.9, and part *2* follows because the quantifier-free content of a sentence must be **T** or **F**. ⊣

We conclude this section with a discussion of almost everywhere reducibility for fragments of $\mathcal{L}^-_{\omega P}$. For $k \in \mathbb{N}$, let $\mathcal{L}^{k-}_{\omega P}$ be the set of formulas of $\mathcal{L}^-_{\omega P}$ with at most $k$ variables. For each finite subset $P_0 \subseteq (0, 1)$, let $\mathcal{L}^{k-}_{\omega P_0}$ be the set of formulas of $\mathcal{L}^{k-}_{\omega P}$ with probability quantifiers only from $P_0$. Thus $\mathcal{L}_{\omega P}$ is the union of the fragments $\mathcal{L}^{k-}_{\omega P_0}$ over all $k \in \mathbb{N}$ and all finite $P_0 \subseteq (0, 1)$.

We observe that $\mathcal{L}^{1-}_{\omega P} \leq_{a.e.} \mathcal{L}_{\omega 0}$ does not hold. To see this, suppose that $\mathcal{L}^{1-}_{\omega P} \leq_{a.e.} \mathcal{L}_{\omega 0}$ holds for the unbiased measure, and $R \in \nu$ is a unary relation. Then there is a set $C$ of finite models of asymptotic measure 1 such that for each $\mathcal{A} \in C$ and $n \in \mathbb{N}$ we have $\mathcal{A} \models (\exists^{\geq(1/2-1/n)}y)R(y)$. But then each $\mathcal{A} \in C$ satisfies $\mathcal{A} \models (\exists^{\geq 1/2}y)R(y)$, which contradicts the fact that $(\exists^{\geq 1/2}y)R(y)$ has asymptotic probability 1/2.

Theorem 4.11. *For each $k \in \mathbb{N}$ and finite set $P_0 \subseteq (0, 1)$ we have $\mathcal{L}^{k-}_{\omega P_0} \leq_{a.e.} \mathcal{L}_{\omega 0}$.*

Proof. Let $F$ be the set of all formulas $\varphi(\mathbf{x})$ in $\mathcal{L}^{k-}_{\omega P_0}$ such that for some quantifier-free formula $\alpha$, either

$$\varphi(\mathbf{x}) = \exists y \alpha(\mathbf{x}, y)$$

or

$$\varphi(\mathbf{x}) = (\exists^{\geq r}\mathbf{y})\alpha(\mathbf{x}, \mathbf{y})$$

for some $r \in P_0$. Since $k$ and $P_0$ are finite, the set $F$ is finite. By Theorem 4.9 and the fact that finite intersections of sets of asymptotic measure 1 have asymptotic measure 1, there is a set of finite structures $C$ of asymptotic measure 1 such that for each formula $\varphi(\mathbf{x}) \in F$ and each $\mathcal{A} \in C$,

$$\mathcal{A} \models \forall \mathbf{x} \left( \varphi(\mathbf{x}) \leftrightarrow \varphi^0(\mathbf{x}) \right).$$

Now let $S$ be the set of all formulas $\psi(\mathbf{x}) \in \mathcal{L}^{k-}_{\omega P_0}$ such that

$$\mathcal{A} \models \forall \mathbf{x} \left( \psi(\mathbf{x}) \leftrightarrow \psi^0(\mathbf{x}) \right)$$

for all $\mathcal{A} \in C$. Then every quantifier-free formula belongs to $S$, and also $F \subseteq S$. An easy induction on the complexity of formulas will show that every formula of $\psi(\mathbf{x}) \in \mathcal{L}^{k-}_{\omega P_0}$ belongs to $S$.

The probability quantifier step of this induction is as follows. Suppose that $\psi(\mathbf{x}, \mathbf{y}) \in S$ and $\varphi(\mathbf{x}) = (\exists^{\geq r} \mathbf{y}) \psi(\mathbf{x}, \mathbf{y})$ is noncritical. Then $\varphi^0(\mathbf{x}) = ((\exists^{\geq r} \mathbf{y}) \psi^0(\mathbf{x}, \mathbf{y}))^0$. Let $\mathcal{A} \in C$. Since $\psi(\mathbf{x}, \mathbf{y}) \in S$ we have

$$\mathcal{A} \models \forall \mathbf{x} \forall \mathbf{y} \left( \psi(\mathbf{x}, \mathbf{y}) \leftrightarrow \psi^0(\mathbf{x}, \mathbf{y}) \right),$$

and hence

$$\mathcal{A} \models \forall \mathbf{x} \left( (\exists^{\geq r} \mathbf{y}) \psi(\mathbf{x}, \mathbf{y}) \leftrightarrow (\exists^{\geq r} \mathbf{y}) \psi^0(\mathbf{x}, \mathbf{y}) \right).$$

Since $(\exists^{\geq r} \mathbf{y}) \psi^0(\mathbf{x}, \mathbf{y}) \in F \subseteq S$, we have

$$\mathcal{A} \models \forall \mathbf{x} \left( (\exists^{\geq r} \mathbf{y}) \psi^0(\mathbf{x}, \mathbf{y}) \leftrightarrow \varphi^0(\mathbf{x}) \right).$$

Therefore

$$\mathcal{A} \models \forall \mathbf{x} \left( \varphi(\mathbf{x}) \leftrightarrow \varphi^0(\mathbf{x}) \right),$$

that is, $\varphi(\mathbf{x}) \in S$. The other steps of the induction are similar. $\dashv$

**§5. Parametric Classes.** In [14], W. Oberschelp showed that the first order 0–1 law holds for the class of all finite models of certain universal first order sentences, called parametric sentences (see also [3], Section 4.2). In this section we get similar generalizations of the almost everywhere quantifier elimination theorem and the 0–1 law for the probability logic $\mathcal{L}^{-}_{\omega P}$.

DEFINITION 5.1. *A first order sentence $\pi$ is said to be* ***parametric*** *if it is a finite conjunction of sentences of the form* $\forall \mathbf{x}[D(\mathbf{x}) \rightarrow \tau(\mathbf{x})]$ *where $\tau(\mathbf{x})$ is a Boolean combination of literals $\tau_1, \ldots, \tau_k$ such that each element of $\mathbf{x}$ occurs at least once in each $\tau_i$. The class of finite models of a parametric sentence is called a* ***parametric class****.*

For example, the class of all bipartite graphs, and the class of all oriented graphs, are parametric. The class of all models in which certain relations are symmetric and irreflexive is also parametric. In particular, the class of all undirected graphs is parametric.

If $k$ is the maximum arity of the relations in $\nu$, then any parametric sentence which has a model of cardinality $\geq k$ has models of all cardinalities.

We assume throughout this section that $\pi$ is a parametric formula

$$\bigwedge_{i=1}^{k} \forall \mathbf{z}_i [D(\mathbf{z}_i) \to \tau_i(\mathbf{z}_i)]$$

which has models in all finite cardinalities. For any $n$ and formula $\varphi(\mathbf{x})$, we define

$$(\mu_n|\pi)(\varphi(\mathbf{x})) = \frac{\mu_n(\varphi(\mathbf{x}) \wedge \pi)}{\mu_n(\pi)}.$$

Since $\pi$ has models of all finite cardinalities, $(\mu_n|\pi)$ is a probability measure on the class of all models with universe $n$.

All of the notions and results in Sections 3 and 4 will carry over to the class of finite models of $\pi$ with the measure $(\mu_n|\pi)$ in place of $\mu_n$. Instead of considering all quantifier-free types $\beta(\mathbf{x})$, we consider only the quantifier-free types which are consistent with $\pi$. We will define the $\pi$-noncritical fragments of $\mathcal{L}_{\omega P}$ and $\mathcal{L}_{\infty P}$, denoted by $\mathcal{L}^{-}_{\omega P}(\pi)$ and $\mathcal{L}^{-}_{\infty P}(\pi)$. The following lemma lets us compute $(\mu_n|\pi)(\varphi(\mathbf{x}))$ from $\mu_n(\varphi(\mathbf{x}))$, and to pass to the limit as $n \to \infty$

LEMMA 5.2. *For each $k$ and each $n \geq k$, there is a constant $c_{n,k} \geq 1$ such that for each $k$-tuple of distinct variables $\mathbf{x}$:*

*(i) For every quantifier-free type $\beta(\mathbf{x})$ which is consistent with $\pi$ and induces the identity relation on $\mathbf{x}$,*

$$(\mu_n|\pi)(\beta(\mathbf{x})) = \mu_n(\beta(\mathbf{x})) \cdot c_{n,k}.$$

*(ii) $\lim_{n\to\infty} c_{n,k}$ exists.*

PROOF. Let $\mathbf{y}$ be a tuple of distinct variables such that $\mathbf{x} \cap \mathbf{y} = \emptyset$ and $|\mathbf{x} \cup \mathbf{y}| = n$. Let $\theta(\mathbf{x}, \mathbf{y})$ be the conjunction of $D(\mathbf{x}, \mathbf{y})$ and all instances $\tau_i(\mathbf{z}_i)$ of parts of $\pi$ where $\mathbf{z}_i$ is a subsequence of $\mathbf{x}, \mathbf{y}$ that contains at least one variable from $\mathbf{y}$. Let $\mathbf{a}, \mathbf{b}$ be sequences of distinct elements of $n$ of length $|\mathbf{a}| = |\mathbf{x}|$ and $|\mathbf{b}| = |\mathbf{y}|$ such that $\mathbf{a} \cup \mathbf{b} = n$. Using the hypothesis that $\pi$ is a parametric sentence, we note that in any model $\mathbf{A} \in \mathbf{M}_n$, for each quantifier-free type $\beta(\mathbf{x})$, $\beta(\mathbf{a}) \wedge \pi$ is equivalent to $\beta(\mathbf{a}) \wedge \theta(\mathbf{a}, \mathbf{b})$. Since each atomic subformula of $\theta$ is either an inequality in $D(\mathbf{x},\mathbf{y})$ or contains a variable in $\mathbf{y}$, $\beta(\mathbf{x})$ is independent of $\theta(\mathbf{x},\mathbf{y})$. Let $c_{n,k} = \mu_n(\theta(\mathbf{x},\mathbf{y}))/\mu_n(\pi)$. It is clear that $c_{n,k}$ depends only on $n$ and $k$ (and $\pi$), and not on the $k$-tuple of variables $\mathbf{x}$.

(i) We have $c_{n,k} \geq 1$, and for any quantifier-free type $\beta(\mathbf{x})$ consistent with $\pi$ we have

$$\mu_n(\beta(\mathbf{a}) \wedge \pi) = \mu_n(\beta(\mathbf{a})) \cdot \mu_n(\theta(\mathbf{a},\mathbf{b})) = \mu_n(\beta(\mathbf{a})) \cdot \mu_n(\pi) \cdot c_{n,k}.$$

Therefore

$$\mu_n(\beta(\mathbf{x}) \wedge \pi) = \mu_n(\beta(\mathbf{x})) \cdot \mu_n(\pi) \cdot c_{n,k},$$

and (i) follows.

(ii) Let $B$ be the set of all quantifier-free types $\beta(\mathbf{x})$ such that $\beta(\mathbf{x})$ induces the identity relation on $\mathbf{x}$ and is consistent with $\pi$. Let $\psi(\mathbf{x})$ be the conjunction of $D(\mathbf{x})$ and all instances $\tau_i(\mathbf{x}_i)$ of parts of $\pi$ where $\mathbf{x}_i$ is a subsequence of $\mathbf{x}$. Then

$$\frac{1}{c_{n,k}} = \frac{\mu_n(\pi)}{\mu_n(\theta(\mathbf{x},\mathbf{y}))} = \frac{\mu_n(\theta(\mathbf{x},\mathbf{y}) \wedge \psi(\mathbf{x}))}{\mu_n(\theta(\mathbf{x},\mathbf{y}))} = \mu_n(\psi(\mathbf{x})|\theta(\mathbf{x},\mathbf{y}))$$
$$= \mu_n(\psi(\mathbf{x})) = \sum\{\mu_n(\beta(\mathbf{x})) : \beta \in B\}.$$

By Lemma 3.2, the limit of the right side exists and is $> 0$, so (ii) holds. ⊣

The analogue of Lemma 3.2 is:

LEMMA 5.3. *Let $\beta(\mathbf{x})$ be a quantifier-free type which is consistent with $\pi$ and let $E$ be the equivalence relation induced by $\beta(\mathbf{x})$. Then*

$$\lim_{n\to\infty} (\mu_n|\pi)(\beta(\mathbf{x}_E))$$

*exists and is greater than* 0.

PROOF. By Lemmas 3.2 and 5.2. ⊣

The conditional probability $(\mu_n|\pi)[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})]$ is then defined in the obvious way when $\beta(\mathbf{x})$ is consistent with $\pi$. We have the following analogue of Lemma 3.3:

LEMMA 5.4. *Let $\alpha(\mathbf{x},\mathbf{y})$ be a quantifier-free formula and let $\beta(\mathbf{x})$ be a quantifier-free type consistent with $\pi$. Then*

$$\lim_{n\to\infty} (\mu_n|\pi)[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})]$$

*exists.*

PROOF. By Lemmas 3.2, 3.3, and 5.2. ⊣

This lemma lets us make the following definition.

Definition 5.5. *Let $\alpha(\mathbf{x},\mathbf{y})$ be a quantifier-free formula. For each quantifier-free type $\beta(\mathbf{x})$ which is consistent with $\pi$, we define*

$$(\mu|\pi)[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})] = \lim_{n\to\infty}(\mu_n|\pi)[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})].$$

*The $\pi,\mathbf{y}$-**critical values** of $\alpha(\mathbf{x},\mathbf{y})$ are the values $(\mu|\pi)[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})]$ where $\beta(\mathbf{x})$ is a quantifier-free type consistent with $\pi$.*

We can now define the $\pi$-quantifier-free content $\varphi^\pi$ of $\varphi$ and the $\pi$-noncritical fragments of $\mathcal{L}_{\omega P}$ and $\mathcal{L}_{\infty P}$ by a small modification of Definition 3.6.

Definition 5.6. *Let $\varphi$ be a formula in the infinitary logic $\mathcal{L}_{\infty P}$ with only finitely many free variables.*

1. *If $\varphi$ is quantifier-free, we stipulate that $\varphi$ is $\pi$-noncritical and that $\varphi^\pi$ is $\varphi$ itself.*
2. *If $\varphi = \neg\psi$, then $\varphi$ is $\pi$-noncritical if and only if $\psi$ is $\pi$-noncritical, and we define $\varphi^\pi = \neg(\psi^\pi)$.*
3. *If $\varphi = \bigwedge_i \psi_i$, then $\varphi$ is $\pi$-noncritical if and only if $\psi_i$ is $\pi$-noncritical for each $i$, and we define $\varphi^\pi = \bigwedge_i((\psi_i)^\pi)$.*
4. *If $\varphi(\mathbf{x}) = \exists y\,\psi(\mathbf{x},y)$, then $\varphi$ is $\pi$-noncritical if and only if $\psi$ is $\pi$-noncritical, and we define*
$$\varphi^\pi(\mathbf{x}) = \bigvee\{\beta(\mathbf{x})\colon (\mu|\pi)[\psi^\pi(\mathbf{x},y)|\beta(\mathbf{x})] > 0\} \vee \bigvee_{i<|\mathbf{x}|}\psi^\pi(\mathbf{x},x_i).$$
5. *If $\varphi(\mathbf{x}) = (\exists^{\geq r}\mathbf{y})\psi(\mathbf{x},\mathbf{y})$, then $\varphi$ is $\pi$-noncritical if and only if $\psi$ is $\pi$-noncritical and $r$ is not a $\pi,\mathbf{y}$-critical value for $\psi^\pi$, and we define*
$$\varphi^\pi(\mathbf{x}) = \bigvee\{\beta(\mathbf{x})\colon (\mu|\pi)[\psi^\pi(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})] > r\}.$$

The $\pi$-*noncritical fragment* $\mathcal{L}^-_{\omega P}(\pi)$ is defined as the set of all $\pi$-noncritical formulas of $\mathcal{L}_{\omega P}$. The $\pi$-*noncritical fragment* $\mathcal{L}^-_{\infty P}(\pi)$ is defined as the set of all $\pi$-noncritical formulas of $\mathcal{L}_{\infty P}$ with only finitely many free variables. Two formulas $\varphi(\mathbf{x}),\psi(\mathbf{x})$ of $\mathcal{L}^-_{\omega P}(\pi)$ are said to be a.e.($\pi$)-equivalent if

$$\lim_{n\to\infty}(\mu_n|\pi)(\forall\mathbf{x}[\varphi(\mathbf{x})\leftrightarrow\psi(\mathbf{x})]) = 1.$$

The following analogue of Theorem 4.9 is obtained by a routine modification of its proof, using Lemma 5.2.

THEOREM 5.7. *Every formula $\varphi$ in $\mathcal{L}^-_{\omega P}(\pi)$ is a.e.($\pi$)-equivalent to its $\pi$-quantifier-free content.*

COROLLARY 5.8. *The 0–1 law holds for the logic $\mathcal{L}^-_{\omega P}(\pi)$ with respect to the measures $(\mu_n|\pi)$.*

**§6. Infinitary Logic.** We continue to work with a finite vocabulary $\nu$, and an underlying atomic probability $p_R$ for each predicate symbol $R \in \nu$.

We first note that the 0–1 law fails for the infinitary probability logic $\mathcal{L}_{\infty P}$, even if we limit the number of variables used and allow only noncritical formulas. For example, let $R$ be a unary predicate symbol in $\nu$, and for each natural $n$ define the sentence

$$\varphi_n = (\exists^{\geq(1/2-1/n)}x)R(x).$$

Then $\varphi_n$ is a noncritical sentence in $\mathcal{L}^-_{\omega P}$, which has asymptotic measure 0, but one can easily check that

$$\bigvee_n \varphi_n \equiv (\exists^{\geq 1/2}x)R(x)$$

which has asymptotic measure 1/2.

If we allow binary predicates then we can imitate the example given in Proposition 3.1 to get a nonconvergent sentence. However, if we allow only finitely many values of $r$ to occur in quantifiers $\exists^{\geq r}$ within a single infinitary formula, then we get both an almost everywhere quantifier elimination as well as a 0–1 law.

Thus, for each $k \in \mathbb{N}$ and each finite $P_0 \subseteq (0,1)$ let's define $\mathcal{L}^{k-}_{\infty P_0}$ to be the set of formulas of $\mathcal{L}^-_{\infty P}$ (where we allow infinitary conjunctions and disjunctions) with at most $k$ variables and with probability quantifiers only from $P_0$.

Then we have

THEOREM 6.1. *Let $k \in \mathbb{N}$ and let $P_0 \subseteq (0,1)$ be finite.*

1. $\mathcal{L}^{k-}_{\infty P_0} \leq_{a.e.} \mathcal{L}^{k}_{\omega 0}$,
2. *Every formula of $\mathcal{L}^{k-}_{\infty P_0}$ is a.e. equivalent to its quantifier-free content.*

PROOF. The proof is the same as the proof of Theorem 4.11, but with infinitely many formulas at the conjunction step of the induction. ⊣

Define $\mathcal{L}^{\omega-}_{\infty P}$ to be the union of $\mathcal{L}^{k-}_{\infty P_0}$ for all $k \in \mathbb{N}$ and all finite $P_0 \subseteq (0,1)$. Thus each particular formula of $\mathcal{L}^{\omega-}_{\infty P}$ has only finitely many variables and finitely many distinct probabilities occurring in quantifiers.

COROLLARY 6.2. 1. $\mathcal{L}^{\omega-}_{\infty P} \leq_{w.a.e.} \mathcal{L}_{\omega 0}$.
2. *The 0–1 law holds for the logic* $\mathcal{L}^{\omega-}_{\infty P}$.

⊣

We note that the probability logics $\mathcal{L}^{k-}_{\infty P_0}$ and $\mathcal{L}^{\omega-}_{\infty P}$ correspond to the logics $\mathcal{L}^{k}_{\infty\omega}$ and $\mathcal{L}^{\omega}_{\infty\omega}$ respectively. For each noncritical formula $\varphi$ of $\mathcal{L}^{\omega-}_{\infty P}$, the quantifier-free content $\varphi^0$ is a first order quantifier-free formula, so by Proposition 3.5 it has finitely many $\mathbf{y}$-critical values, each of which is equal to a polynomial in $p_R, R \in \nu$ with integer coefficients.

**§7. Logic with Probability Functions.** In this section we treat the case where the atomic probabilities $p_R(n)$ and quantifier probabilities $(\exists^{\geq r(n)}\mathbf{y})$ are allowed to depend on the universe size $n$. $\mathcal{L}_{\omega P(n)}$ will be first order logic augmented by the probability quantifiers $(\exists^{\geq r(n)}\mathbf{y})$ where $r(n) \in (0,1)$ for each $n \in \mathbb{N}$, and $\mathcal{L}_{\infty P(n)}$ will be the corresponding extension of the infinitary logic $\mathcal{L}_{\infty\omega}$.

The dependency of the ratio $r(n)$ on $n$ puts the quantifiers $(\exists^{\geq r(n)}\mathbf{y})$ in correspondence with the general monotone numerical quantifier $(Q^f\mathbf{y})$, which says that the number of tuples $\mathbf{y}$ in a model of size $n$ is $\geq f(n)$. Here, $r(n) = f(n)/n^{\ell}$, when $\ell = |\mathbf{y}|$.

The ordinary existential quantifier $\exists y$ has the same semantic interpretation as the probability quantifier $(\exists^{\geq 1/n}y)$, but our definitions of a noncritical formula and quantifier-free content will differ for these two quantifiers. (It will be easier for $\exists y\varphi(\mathbf{x}, y)$ to be noncritical than for $(\exists^{\geq 1/n}y)\varphi(\mathbf{x}, y)$ to be noncritical).

For the usual infinitary logic $\mathcal{L}^{k}_{\infty\omega}$ with $k$ variables and ordinary existential quantifiers, the proofs of the 0–1 law as well as the almost everywhere quantifier elimination theorem go through when the independent atomic probabilities $p_R(n)$ vary with $n$, as long as they are bounded away from 0 and 1.

If $p_R(n) \to 0$ or 1 as $n \to \infty$, the 0–1 law may not hold, as shown in [15] and [13], where one can find a nearly complete characterization of those $p_R(n)$ for which the 0–1 law holds.

As the atomic probabilities change with $n$, $\mathbf{y}$-critical values also change with $n$, and will now be called $\mathbf{y}$-*critical sequences*. One difficulty we face is that the analogue of Lemma 3.3 will fail, that is, the conditional probability $\mu_n[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})]$ of a quantifier-free formula given a quantifier-free type need not converge.

In the present setting, we define a $\mathbf{y}$-critical sequence as follows.

DEFINITION 7.1. *Let $\alpha(\mathbf{x},\mathbf{y})$ be a quantifier-free first order formula. A sequence $r(n)$ is* $\mathbf{y}$***-critical for*** *$\alpha(\mathbf{x},\mathbf{y})$ if for some quantifier-free type $\beta(\mathbf{x})$ we have*

1. $\liminf(\mu_n[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})] - r(n)) \leq 0$ *and*
2. $\liminf(r(n) - \mu_n[\alpha(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})]) \leq 0$.

The simultaneous definition of the noncritical fragment and the quantifier-free content now takes the following form.

DEFINITION 7.2. *Let $\varphi$ be a formula in the infinitary logic $\mathcal{L}_{\infty P(n)}$ with only finitely many free variables.*

1. *If $\varphi$ is quantifier-free, we stipulate that $\varphi$ is noncritical and that $\varphi^0$ is $\varphi$ itself.*
2. *If $\varphi = \neg\psi$, then $\varphi$ is noncritical if and only if $\psi$ is noncritical, and we define $\varphi^0 = \neg(\psi^0)$.*
3. *If $\varphi = \bigwedge_i \psi_i$, then $\varphi$ is noncritical if and only if $\psi_i$ is noncritical for each $i$, and we define $\varphi^0 = \bigwedge_i((\psi_i)^0)$.*
4. *If $\varphi(\mathbf{x}) = \exists y\, \psi(\mathbf{x}, y)$, then $\varphi$ is noncritical if and only if $\psi$ is noncritical and for each quantifier-free type $\beta(\mathbf{x})$, either*
$$\liminf \mu_n[\psi^0(\mathbf{x},y)|\beta(\mathbf{x})] > 0$$
*or*
$$\models \forall\mathbf{x}\left(\beta(\mathbf{x}) \to \bigvee_{i<|\mathbf{x}|} \psi^0(\mathbf{x},x_i)\right),$$
*and we define*
$$\varphi^0(\mathbf{x}) = \bigvee\{\beta(\mathbf{x})\colon \liminf \mu_n[\psi^0(\mathbf{x},y)|\beta(\mathbf{x})] > 0\} \vee \bigvee_{i<|\mathbf{x}|} \psi^0(\mathbf{x},x_i).$$
5. *If $\varphi(\mathbf{x}) = (\exists^{\geq r(n)}\mathbf{y})\psi(\mathbf{x},\mathbf{y})$, then $\varphi$ is noncritical if and only if $\psi$ is noncritical and $r(n)$ is not a $\mathbf{y}$-critical sequence for $\psi^0$, and we define*
$$\varphi^0(\mathbf{x}) = \bigvee\{\beta(\mathbf{x})\colon \liminf(\mu_n[\psi^0(\mathbf{x},\mathbf{y})|\beta(\mathbf{x})] - r(n)) > 0\}.$$

The *noncritical fragment* $\mathcal{L}^-_{\omega P(n)}$ is defined as the set of all noncritical formulas of $\mathcal{L}_{\omega P(n)}$. The *noncritical fragment* $\mathcal{L}^-_{\infty P(n)}$ is defined as the set of all noncritical formulas of $\mathcal{L}_{\infty P(n)}$ with only finitely many free variables.

For each $k \in \mathbb{N}$ and finite set $P_0$ of sequences $r(n) \in (0,1)$ we define $\mathcal{L}^{k-}_{\infty P_0(n)}$ to be the set of formulas in $\mathcal{L}^{-}_{\infty P(n)}$ with at most $k$ free variables and with probability quantifiers only from $P_0$. We let $\mathcal{L}^{\omega-}_{\infty P(n)}$ be the union of $\mathcal{L}^{k-}_{\infty P_0(n)}$ over all $k \in \mathbb{N}$ and all finite $P_0$.

With this definition, our earlier proofs go through, and we have the following theorem.

THEOREM 7.3. *Let $k \in \mathbb{N}$ and let $P_0$ be a finite set of sequences of elements of $(0,1)$.*

1. $\mathcal{L}^{k-}_{\infty P_0(n)} \leq_{a.e.} \mathcal{L}^{k}_{\omega 0}$.
2. *Every formula $\varphi \in \mathcal{L}^{k-}_{\infty P_0(n)}$ is a.e. equivalent to its quantifier-free content.*

$\dashv$

COROLLARY 7.4. 1. $\mathcal{L}^{\omega-}_{\infty P(n)} \leq_{w.a.e.} \mathcal{L}_{\omega 0}$.

2. *The 0–1 law holds for the logic $\mathcal{L}^{\omega-}_{\infty P(n)}$.*

$\dashv$

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