A Neometric Survey

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1 Introduction

Nonstandard analysis is often used to prove that certain objects exist, i.e., that certain sets are not empty. In the literature one can find many existence theorems whose only known proofs use nonstandard analysis; see, for example, [AFHL].

This article will survey a new method for existence proofs, based on the concept of a neometric space. We shall state definitions and results (usually without proofs) from several other papers, and try to explain how the ideas from these papers fit together as a whole. The purpose of the neometric method is twofold: first, to make the use of nonstandard analysis more accessible to mathematicians, and second, to gain a deeper understanding of why nonstandard analysis leads to new existence theorems. The neometric method is intended to be more than a proof technique—it has the potential to suggest new conjectures and new proofs in a wide variety of settings. However, it bypasses the notion of an internal set and the lifting and pushing down arguments which are the main feature of many nonstandard existence proofs.

The central notion is that of a neocompact family, which is a generalization of the classical family of compact sets. A neocompact family is a family of subsets of metric spaces with certain closure properties. In applications, nonstandard analysis is needed at only one point—to obtain neocompact families which are countably compact. From that point on, the method can be used without any knowledge of nonstandard analysis at all.

This program grew out of earlier work on adapted probability distributions ([K2], [HK]) and a first approach to neocompactness in the paper [K3]. Various aspects of our program will appear in the papers [CK], [FK1], [FK2], [FK3], [FK4], [K4], and [K5]. In this article we shall give an overview of the entire program. We shall explain how the method can be painlessly applied, and discuss the relationship of the method to nonstandard practice and to adapted probability distributions.

Let's take an informal look at a common way of solving existence problems in analysis (or in a metric space): We want to show that within a set C there exists an object x with a particular property $\phi(x)$, that is, $(\exists x \in C)\phi(x)$. If we cannot find a solution x directly, we may proceed to find "approximate" solutions; we construct an object which is close to C, but perhaps not in C, and which almost has property ϕ. What is usually done is the following: define a sequence $\langle(x_n)\rangle$ of approximations which get better and better as n increases; if we do things right the sequence has a limit and that limit is the desired x.

The hard part is to show that the limit exists. In the classical setting, the most common way to get a limit is to show that the sequence x_n is contained in a compact set, and to use the fact that every sequence in a compact set has a convergent subsequence.

A simple example of an existence proof by approximation is Peano's existence theorem for differential equations: One first constructs a sequence of natural approximations (i.e Euler polygons). Then, using Arzela's theorem, a consequence of compactness that guarantees that under certain conditions a sequence of functions converges, one shows that the limit exists and is precisely the solution wanted. Written in symbolic form, the theorem is a statement of the form

$$(\exists x \in C)(f(x) \in D).$$

The approximation procedure gives us the following property:

$$(\forall \varepsilon > 0)(\exists x \in C^{\varepsilon})(f(x) \in D^{\varepsilon}).$$

Here C^{ε} is the set $\{x : \rho(x, C) \leq \varepsilon\}$ with ρ the metric on the space where C lives, and similarly for D^{ε}. Then, if we choose a sequence ε_n approaching 0, we obtain a sequence of approximations. The compactness argument (Arzela's theorem) gives the existence of the limit.

The centerpiece of our method is a result (called the Approximation Theorem) which intuitively says "it is enough to approximate", or "if you can find approximate solutions then you can conclude that an exact solution exists without going through the convergence argument." In the above notation, the theorem states that:

$$\text{If } (\forall \varepsilon > 0)(\exists \in C^{\varepsilon})(f(x) \in D^{\varepsilon}) \text{ then } (\exists x \in C)(f(x) \in D). \tag{1}$$

The reader should have no problem showing that condition (1) holds in the following case: C is a compact subset of a complete separable metric space M, D is a closed subset of another complete separable metric space N, and f is a continuous function from M into N.

The main point of the neometric method is that our Approximation Theorem goes beyond the familiar case of convergence in a compact set. First, we work in metric spaces that are not necessarily separable. Second, we identify new families

$\mathcal{C}, \mathcal{D}$ and $\mathcal{F}$ of sets C, D and functions f, such that (1) holds. These are the families of neocompact sets, neoclosed sets, and neocontinuous functions. The family of neocompact sets is much larger than the family of compact sets, and provides a wide variety of new opportunities for proving existence theorems by approximation.

2 Neometric Families

In this section we summarize the central notion of a neometric family from the paper [FK1], and state the main approximation theorem.

We use script letters

$$\mathcal{M} = (M, \rho), \mathcal{N} = (N, \sigma)$$

for complete metric spaces which are not necessarily separable. Given two metric spaces $\mathcal{M}$ and $\mathcal{N}$, the **product metric** is the metric space $\mathcal{M} \times \mathcal{N} = (M \times N, \rho \times \sigma)$ where

$$(\rho \times \sigma)((x_1, x_2), (y_1, y_2)) = \max(\rho(x_1, y_1), \sigma(x_2, y_2)).$$

The first notion we need is that of a neocompact family.

Definition 2.1 *Let* $\mathbf{M}$ *be a collection of complete metric spaces which is closed under finite products, and for each* $\mathcal{M} \in \mathbf{M}$ *let* $\mathcal{B}(\mathcal{M})$ *be a collection of subsets of* $\mathcal{M}$*, which we call* **basic sets**. *By a* **neocompact family** *over* $(\mathbf{M}, \mathcal{B})$ *we mean a triple* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *where for each* $\mathcal{M} \in \mathbf{M}$, $\mathcal{C}(\mathcal{M})$ *is a collection of subsets of* $\mathcal{M}$ *with the following properties, where* $\mathcal{M}, \mathcal{N}$ *vary over* $\mathbf{M}$*:*

(a) $\mathcal{B}(\mathcal{M}) \subset \mathcal{C}(\mathcal{M})$*;*

(b) $\mathcal{C}(\mathcal{M})$ *is closed under finite unions; that is, if* $A, B \in \mathcal{C}(\mathcal{M})$ *then* $A \cup B \in \mathcal{C}(\mathcal{M})$*;*

(c) $\mathcal{C}(\mathcal{M})$ *is closed under finite and countable intersections;*

(d) *If* $C \in \mathcal{C}(\mathcal{M})$ *and* $D \in \mathcal{C}(\mathcal{N})$ *then* $C \times D \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$*;*

(e) *If* $C \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$*, then the set*

$$\{x : (\exists y \in \mathcal{N})(x, y) \in C\}$$

belongs to $\mathcal{C}(\mathcal{M})$*, and the analogous rule holds for each factor in a finite Cartesian product;*

(f) *If $C \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$, and D is a nonempty set in $\mathcal{B}(\mathcal{N})$, then*

$$\{x : (\forall y \in D)(x, y) \in C\}$$

belongs to $\mathcal{C}(\mathcal{M})$, and the analogous rule holds for each factor in a finite Cartesian product.

The sets in $\mathcal{C}(\mathcal{M})$ are called **neocompact sets**. The neocompact family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ induces a family of metric spaces with extra structure, $(\mathcal{M}, \mathcal{B}(\mathcal{M}), \mathcal{C}(\mathcal{M}))$, which we call **neometric spaces**. A neometric space thus consists of a complete metric space $\mathcal{M} \in \mathbf{M}$ and two families $\mathcal{B}(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ of subsets of $\mathcal{M}$. The properties (a)–(f) not only give conditions on single neometric spaces, but also on finite Cartesian products of neometric spaces.

We call $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ the **neocompact family generated by** $(\mathbf{M}, \mathcal{B})$ if $\mathcal{C}(\mathcal{M})$ is the collection of all sets obtained by finitely many applications of the rules (a)–(f).

The classical example of a neocompact family is the family generated by $(\mathbf{S}, \mathcal{B})$ where $\mathbf{S}$ is the collection of all complete metric spaces, and for each $\mathcal{M} \in \mathbf{S}$, $\mathcal{B}(\mathcal{M})$ is equal to the set of all compact subsets of $\mathcal{M}$. It is not hard to see that the family of compact sets is closed under all of the rules (a)–(f). Thus the collection of neocompact sets $\mathcal{C}(\mathcal{M})$ generated by $(\mathbf{S}, \mathcal{B})$ is just $\mathcal{B}(\mathcal{M})$ itself, i.e. every neocompact set is compact.

It is easy to produce neocompact families by first choosing the basic sets and then closing them under the rules (a)–(f). The interesting neocompact families have an extra feature expressed in the following property, which is a familiar property of the family of compact sets in a topological space and plays a key role in the new theory of neometric spaces (see [FK1]).

Definition 2.2 *We say that a neocompact family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ is* **countably compact** *if for each $\mathcal{M} \in \mathbf{M}$, every decreasing chain $C_0 \supset C_1 \supset \cdots$ of nonempty sets in $\mathcal{C}(\mathcal{M})$ has a nonempty intersection $\bigcap_n C_n$ (which, of course, also belongs to $\mathcal{C}(\mathcal{M})$).*

The classical neocompact family $(\mathbf{S}, \mathcal{B}, \mathcal{C})$ of compact sets is clearly countably compact. The interesting question is whether there are other, nontrivial, neocompact families which have it. The only examples we know are built using nonstandard analysis! (See [FK2]).

We now introduce notions for neocompact families analogous to familiar notions for metric spaces, and then introduce the slightly stronger notion of a neometric family.

Definition 2.3 *(a) A set $C \subset \mathcal{M}$ is* **neoclosed** *in $\mathcal{M}$ if $C \cap D$ is neocompact in $\mathcal{M}$ for every neocompact set D in $\mathcal{M}$.*

(b) Let $D \subset \mathcal{M}$. A function $f : D \to \mathcal{N}$ is **neocontinuous** *from $\mathcal{M}$ to $\mathcal{N}$ if for every neocompact set $A \subset D$ in $\mathcal{M}$, the restriction $f|A = \{(x, f(x)) : x \in A\}$ of f to A is neocompact in $\mathcal{M} \times \mathcal{N}$.*

(c) A set A is said to be **neoseparable** *in $\mathcal{M}$ if A is the closure of the union of countably many basic subsets of $\mathcal{M}$.*

Definition 2.4 *We call a neocompact family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ a* **neometric family** *if the distance functions in $\mathbf{M}$ and the projection functions for finite Cartesian products in $\mathbf{M}$ are neocontinuous. That is, the metric space $\mathbf{R}$ of reals is contained in some member $\mathcal{R}$ of $\mathbf{M}$, and for each $\mathcal{M} \in \mathbf{M}$ the distance function ρ of $\mathcal{M}$ is neocontinuous from $\mathcal{M} \times \mathcal{M}$ into $\mathcal{R}$. Moreover, for each $\mathcal{M}, \mathcal{N} \in \mathbf{M}$, the projection functions from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{M}$ and to $\mathcal{N}$ are neocontinuous.*

In the classical family $(\mathbf{S}, \mathcal{B}, \mathcal{C})$ a set is neoclosed if and only if it is closed, and neoseparable if and only if it is closed and separable, and a function is neocontinuous if and only if it is continuous. Since the distance and projection functions on any metric space are continuous, $(\mathbf{S}, \mathcal{B}, \mathcal{C})$ is a neometric family.

The following is a list of facts taken from [FK1]. Taken together, these facts show that the notions of neocompactness, neoclosedness, and neocontinuity behave in a manner analogous to the classical notions of compactness, closedness, and continuity.

Blanket Hypothesis 1 *For the rest of this section, we assume that $\mathbf{M}$ is a collection of complete metric spaces closed under finite Cartesian products, and $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ is a countably compact neometric family such that for each $\mathcal{M} \in \mathbf{M}$, $\mathcal{B}(\mathcal{M})$ contains at least all compact sets in $\mathcal{M}$.*

Basic Facts 1 *1. Every neocompact set in $\mathcal{M}$ is neoclosed and bounded.*

2. Every section of a neocompact set is neocompact. That is, if C is neocompact in $\mathcal{M} \times \mathcal{N}$ and $z \in \mathcal{N}$ then the set $\{x \in \mathcal{M} : (x, z) \in C\}$ is neocompact in $\mathcal{M}$.

3. If $f : D \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$ and $A \subset D$ is neocompact in $\mathcal{M}$, then the set $f(A) = \{f(x) : x \in A\}$ is neocompact in $\mathcal{N}$.

4. If $f : C \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$, C is neoclosed in $\mathcal{M}$, and D is neoclosed in $\mathcal{N}$, then $f^{-1}(D) = \{x \in C : f(x) \in D\}$ is neoclosed in $\mathcal{M}$.

5. Compositions of neocontinuous functions are neocontinuous.

6. Every closed separable subset of $\mathcal{M}$ is neoseparable in $\mathcal{M}$.

7. Every neoclosed set in $\mathcal{M}$ is closed in $\mathcal{M}$.

8. If $f : D \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$, then f is continuous on D.

We now introduce one more property of a well behaved neometric family which is crucial for the deeper applications. This property is called closure under diagonal intersections.

Definition 2.5 *A neometric family* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *is said to be* **closed under diagonal intersections** *if the following holds. Let* $\mathcal{M} \in \mathbf{M}$*, let* $A_n \in \mathcal{C}(\mathcal{M})$ *for each* $n \in \mathbf{N}$*, and let* $\lim_{n\to\infty} \varepsilon_n = 0$*. Then*

$$A = \bigcap_n ((A_n)^{\varepsilon_n}) \in \mathcal{C}(\mathcal{M}).$$

The paper [FK1] has several consequences of closure under diagonal intersections. One example is a neometric analogue of Arzela's theorem. The most important consequence is the approximation theorem which was mentioned in the introduction.

Theorem 2.6 *(Approximation Theorem) Suppose* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *is closed under diagonal intersections. Let* A *be neoclosed in* $\mathcal{M}$ *and* $f : A \to \mathcal{N}$ *be neocontinuous from* $\mathcal{M}$ *to* $\mathcal{N}$*. Let* B *be neocompact in* $\mathcal{M}$ *and* D *be neoclosed in* $\mathcal{N}$*. Suppose that for each* $\varepsilon > 0$*, we have*

$$(\exists x \in A \cap B^{\varepsilon}) f(x) \in D^{\varepsilon}.$$

Then

$$(\exists x \in A \cap B) f(x) \in D.$$

In the paper [A2], Anderson proved a form of the approximation theorem for the classical neometric family $(\mathbf{S}, \mathcal{B}, \mathcal{C})$, and gave several applications. To go further, we need other examples of neometric families which are countably compact and closed under diagonal intersections, and also need a library of useful neocompact sets and neocontinuous functions. In the next section we discuss a neometric family which has been studied in detail and was the was original motivation for our method, the family of neocompact sets in a rich adapted probability space. Other interesting neometric families will be discussed later on in this paper.

3 Rich Adapted Spaces

Anderson's construction of Brownian motion in [A1] and the lifting method for proving existence theorems for stochastic differential equations on an adapted Loeb space in [K1] were among the earliest applications of the Loeb measure construction in nonstandard analysis. These results were the primary motivation for both the adapted probability distributions in [K2] and [HK] and for the neometric method being discussed in this paper. In this section we shall review the neometric family on a rich adapted space which was introduced in the paper [FK1].

Let $\mathbf{B}$ be the set of dyadic rationals in $\mathbf{R}_+$. We say that $\Omega = (\Omega, P, \mathcal{G}, \mathcal{G}_t)_{t\in\mathbf{B}}$ is a **B-adapted (probability) space** if P is a complete probability measure on $\mathcal{G}$, $\mathcal{G}_t$ is a σ-subalgebra of $\mathcal{G}$ for each $t \in \mathbf{B}$, and $\mathcal{G}_s \subset \mathcal{G}_t$ whenever $s < t$ in $\mathbf{B}$. Let Ω be a $\mathbf{B}$-adapted probability space which will remain fixed throughout our discussion. For $s \in \mathbf{R}_+$ we let $\mathcal{F}_s$ be the P-completion of the σ-algebra $\bigcap\{\mathcal{G}_t : s < t \in \mathbf{B}\}$. Then the filtration $\mathcal{F}_s$ is right continuous, that is, for all $s < \infty$ we have $\mathcal{F}_s = \bigcap\{\mathcal{F}_t : s < t\}$. Each $\mathbf{B}$-adapted space $(\Omega, P, \mathcal{G}, \mathcal{G}_t)_{t\in\mathbf{B}}$ has an associated **right continuous adapted space** $(\Omega, P, \mathcal{F}, \mathcal{F}_t)_{t\in\mathbf{R}_+}$.

We say that P is **atomless** if any set of positive measure can be partitioned into two sets of positive measure, and that P is atomless on a σ-algebra $\mathcal{F} \subset \mathcal{G}$ if the restriction of P to $\mathcal{F}$ is atomless.

We let $M = (M, \rho)$ and $N = (N, \sigma)$ be complete separable metric spaces. We use the corresponding script letter $\mathcal{M} = L^0(\Omega, M)$ to denote the space of all P-measurable functions from Ω into M with the metric ρ_0 of convergence in probability,

$$\rho_0(x, y) = \inf\{\varepsilon : P[\rho(x(\omega), y(\omega)) \le \varepsilon] \ge 1 - \varepsilon\}.$$

(We identify functions which are equal P-almost surely). Note that the product metric $\mathcal{M} \times \mathcal{N}$ is topologically equivalent to the space $L^0(\Omega, M \times N)$.

The space of Borel probability measures on M with the Prohorov metric

$$d(\mu, \nu) = \inf\{\varepsilon : \mu(K) \le \nu(K^\varepsilon) + \varepsilon \text{ for all closed } K \subset M\}$$

is denoted by $\mathrm{Meas}(M)$. It is again a complete separable metric space, and convergence in $\mathrm{Meas}(M)$ is the same as weak convergence. Each measurable function $x : \Omega \to M$ induces a measure $\mathrm{law}(x) \in \mathrm{Meas}(M)$, and the function

$$\mathrm{law} : \mathcal{M} \to \mathrm{Meas}(M)$$

is continuous.

Definition 3.1 *Let $\Omega = (\Omega, P, \mathcal{G}, \mathcal{G}_t)_{t\in\mathbf{B}}$ be a $\mathbf{B}$-adapted space, and let $\mathbf{M}_\Omega$ be the family of all the metric spaces $\mathcal{M} = L^0(\Omega, M)$ where M is a complete separable metric space. A subset B of $\mathcal{M}$ will be called* **basic**, *$B \in \mathcal{B}_\Omega(\mathcal{M})$, if either*

(1) *B is compact, or*

(2) *$B = law^{-1}(C)$ for some compact set $C \subset Meas(M)$, or*

(3) *$B = \{x \in law^{-1}(C) : x \text{ is } \mathcal{G}_t - measurable\}$ for some compact $C \subset Meas(M)$ and $t \in \mathbf{B}$.*

We say that a $\mathbf{B}$*-adapted space* Ω *is* **rich** *if the measure* P *is atomless on* $\mathcal{G}_0, \Omega$ *admits a Brownian motion, and the neocompact family* $(\mathbf{M}_\Omega, \mathcal{B}_\Omega, \mathcal{C}_\Omega)$ *generated by* $(\mathbf{M}_\Omega, \mathcal{B}_\Omega)$ *is countably compact. The sets in* $\mathcal{C}_\Omega(\mathcal{M})$ *are said to be* **neocompact** *for the* $\mathbf{B}$*-adapted space* Ω*.*

It is convenient to identify each complete separable metric space M with the set of all constant functions in $\mathcal{M} = L^0(\Omega, M)$. With this identification we get a notion of a neocontinuous function from M into $\mathcal{N}$, and a neocontinuous function from $\mathcal{N}$ into M.

The simpler notion of a **rich probability space** is defined in the same way as a rich $\mathbf{B}$-adapted space except that condition (3) is left out of the definition of the basic sets.

The paper [FK1] gives examples showing that the usual probability spaces and adapted spaces considered in the classical literature are not rich. Moreover, the universal projection condition (f) cannot be strengthened by allowing the set D to be neocompact rather than basic. We shall discuss the existence of rich probability and adapted spaces later on.

An extensive library of neocompact sets and neocontinuous functions for a rich $\mathbf{B}$-adapted space is developed in [FK1], and is extended further in [CK]. Here is a sampling from this library.

Blanket Hypothesis 2 *For the rest of this section we assume that* Ω *is a rich* $\mathbf{B}$*-adapted space and that* M, N *are complete separable metric spaces.*

Theorem 3.2 *The family of neocompact sets for* Ω *is a neometric family which is closed under diagonal intersections.*

Thus the distance and projection functions are neocontinuous.

Let $T > 0$. A stochastic process $x \in L^0(\Omega, L^0([0,T], M))$ or a continuous stochastic process $x \in L^0(\Omega, C([0,T], M))$ is said to be **adapted** if $x(\cdot, t)$ is $\mathcal{F}_t$-measurable for each $t \in [0,T]$.

Theorem 3.3 *The following sets are neocompact.*

(a) The set $L^0(\Omega, C)$ *where* C *is a compact subset of* M*.*

(b) The set of all $\mathcal{F}_t$*-stopping times in* $L^0(\Omega, [0,T])$*.*

(c) The set of all Brownian motions on $\Omega \times [0,T]$*, that is the set of continuous adapted processes on* Ω *with values in* $\mathbf{R}$ *whose law is the Wiener measure on* $C([0,T], \mathbf{R})$*.*

(d) The set of all $x \in L^0(\Omega, \mathbf{R})$ *such that* $E[|x|] \le r$ *where* $r > 0$ *is fixed.*

The proof of (d) uses closure under diagonal intersections.

Theorem 3.4 *The following sets are neoclosed.*

(a) The set of all $\mathcal{F}_t$-measurable $x \in \mathcal{M}$, where $t \in \mathbf{R}_+$ is fixed.

(b) The set of adapted stochastic processes on $[0, T]$ with values in M.

(c) The set of continuous adapted stochastic processes on $[0, T]$ with values in M.

Theorem 3.5 *The following functions are neocontinuous.*

(a) (Randomization Lemma) The function $g : \mathcal{M} \to \mathcal{N}$ defined by $(g(x))(\omega) = f(x(\omega))$ where $f : M \to N$ is continuous.

(b) The law function from $\mathcal{M}$ to Meas(M).

(c) The stochastic integral function

$$(y, b) \mapsto \int_0^t y(\omega, s) db(\omega, s)$$

where r is finite, y belongs to the neoclosed set of adapted stochastic processes on Ω with values in $[-r, r]$, and b belongs to the neocompact set of Brownian motions.

Moreover, the range of the function (c) is contained in a neocompact set of continuous adapted stochastic processes.

Theorem 3.6 *The following functions are neocontinuous on each uniformly integrable subset of $L^1(\Omega, \mathbf{R})$.*

(a) The expected value function $x \mapsto E[x(\omega)]$.

(b) The conditional expectation function $x \mapsto E[x(\omega)|\mathcal{G}_t]$ where $t \in \mathbf{B}$.

(c) The conditional expectation function $x \mapsto E[x(\omega)|\mathcal{F}_.]$ where the value is a stochastic process.

The paper [FK1] has an example which complements Theorem 3.4 (a) and 3.6 (c). The example shows that for each t, the neocompact set of all $\mathcal{F}_t$-measurable x with law$(x) \in [0, 1]$ cannot be basic, and the continuous function $x \mapsto E[x(\omega)|\mathcal{F}_t]$ cannot be neocontinuous from $L(\Omega, [0, 1])$ to itself.

A variety of optimization and existence theorems for rich $\mathbf{B}$-adapted spaces are proved in [FK1]. In [CK] the method is applied to obtain new optimization and existence theorems for stochastic Navier-Stokes equations. To give an idea of what can be done, we give two examples here.

Theorem 3.7 *Let Ω be a rich $\mathbf{B}$-adapted space and let $T > 0$. For each continuous stochastic process $x \in L^0(\Omega, C([0, T], \mathbf{R})$ there is a Brownian motion on $\Omega \times [0, T]$ whose ρ_0-distance from x is a minimum.*

Proof sketch: The function $f(b) = \rho_0(x, b)$ is neocontinuous from the neocompact set B of Brownian motions on $\Omega \times [0, T]$ into the reals. Therefore its range $f(B)$ is neocompact and hence closed and bounded in the reals, and thus has a minimum.

The next result was proved for adapted Loeb spaces by the lifting method in [K1]. It is a stochastic analogue of the Peano existence theorem, and improves the weak existence theorem of Skorokhod [Sk] for stochastic differential equations,

Theorem 3.8 *Let $T, r > 0$, let g be an adapted stochastic process on $\Omega \times [0, T]$ with values in $C(\mathbf{R}, [-r, r])$, and let b be a Brownian motion on $\Omega \times [0, T]$. Then there exists a continuous adapted stochastic process x such that*

$$x(\omega, t) = \int_0^t g(\omega, s)(x(\omega, s))db(\omega, x).$$

Moreover, the set A of all such solutions x is neocompact, and hence any neocontinuous function f from A into the reals has a minimum (i.e. and optimal solution with respect to f).

Proof sketch: Our library of neocontinuous functions shows that the function

$$h(x, u) = \int_0^t g(\omega, s)(x(\omega, s - u))db(\omega, s),$$

with the convention that $x(\omega, u) = 0$ when $u < 0$, is neocontinuous and its range is contained in a neocompact set D of continuous stochastic processes. Since A is the set of all x such that $h(x, 0) - x = 0$ and $A \subset D$, it follows that A is neocompact.

The set C of all $u \in [0, 1]$ such that $(\exists x \in D)h(x, u) = x$ is also neocompact. Thus C is a closed subset of the unit interval $[0, 1]$. By successively integrating over subintervals $[0, u], [u, 2u], \ldots,$ for each $u \in (0, 1]$ we get an $x \in D$ such that $h(x, u) = x$. It follows that $(0, 1] \subset C$. Since C is closed, we have $0 \in C$, and therefore the set A of solutions is nonempty.

The above proof used the basic facts about neocontinuous functions and neocompact sets in a direct manner. To illustrate the use of the Approximation Theorem, we give a second proof.

Alternative Proof by Approximation: Let h and D be the neocontinuous function and neocompact set from the first proof. By successively integrating over subintervals we see that

$$(\exists (x, u) \in (D \times \{0\})^{\varepsilon})h(x, u) - x \in \{0\}^{\varepsilon}.$$

By the approximation theorem we have

$$(\exists x \in D)h(x, 0) - x = 0$$

as required.

The preceding examples are illustrations of a general approach to the discovery of new conjectures and proofs. In a wide variety of situations, one can ask whether a given set is neocompact, or whether a given function is neocontinuous. Neocompactness can often be proved by checking through the definition of the set to see that it can be constructed from basic sets using the operations (a)—(f). Once neocompactness and neocontinuity are established, the Approximation Theorem immediately suggests a way to prove an existence theorem by proving that approximate solutions exist. In many cases, such as in the preceding theorem, one can then go on to ask if the set of solutions itself is neocompact, and continue the process.

4 Saturated Adapted Spaces

In the paper [HK] the notions of an adapted distribution and of a saturated adapted probability space were introduced. The adapted distribution of a random variable on an adapted space (with values in a complete separable metric space) is the natural analogue of the distribution of a random variable on a probability space. The results of [HK] suggest that two stochastic processes on possibly different spaces may be considered alike if they have the same adapted distribution. For stochastic differential equations and a wide variety of other existence problems, every existence theorem which holds on some adapted space holds on a saturated adapted space. The relationship between saturated and rich adapted spaces was studied in the papers [K4] and [K5]. The key result was a quantifier elimination theorem showing that every neocompact set can be represented in a simple form by means of the adapted distribution.

We begin with the simple notion of a saturated probability space, and then take up the more complicated notions of a saturated **B**-adapted space and a saturated right continuous adapted space.

Definition 4.1 *A probability space Ω is* **saturated** *if for any random variable $x \in L^0(\Omega, M)$ and pair of random variables $\bar{x} \in L^0(\Gamma, M)$ and $\bar{y} \in L^0(\Gamma, N)$ on another probability space Γ such that $law(x) = law(\bar{x})$, there is a random variable $y \in L^0(\Omega, N)$ such that $law(x, y) = law(\bar{x}, \bar{y})$.*

The following theorem was proved in [K5].

Theorem 4.2 *A probability space is rich if and only if it is saturated.*

The main tool in the proof was a quantifier elimination theorem which is of interest in its own right.

Theorem 4.3 *(Quantifier Elimination for Probability Spaces) In a saturated probability space Ω, a set is neocompact in $\mathcal{M}$ if and only if it is of the form*

$$\{x \in \mathcal{M} : law(x, z) \in C\}$$

for some compact set $C \subset Meas(\mathcal{M} \times \mathcal{N})$ and some $z \in \mathcal{N}$.

We now present analogous results for $\mathbf{B}$-adapted spaces. In order to state these results we need the notion of an adapted function, which was essentially introduced in [HK].

Definition 4.4 *Let $\mathcal{R} = L^0(\Omega, \mathbf{R})$. The class of* **$\mathbf{B}$-adapted functions** *on $\mathcal{M}$ is the least class of functions from $\mathcal{M}$ into $\mathcal{R}$ such that:*

(i) For each bounded continuous function $\phi : M \to \mathbf{R}$, the function $(\hat{\phi}(x))(\omega) = \phi(x(\omega))$ is a $\mathbf{B}$-adapted function on $\mathcal{M}$;

(ii) If $f_1, \ldots, f_m$ are $\mathbf{B}$-adapted functions on $\mathcal{M}$ and $g : \mathbf{R}^m \to \mathbf{R}$ is continuous, then $h(x) = g(f_1(x), \ldots, f_m(x))$ is a $\mathbf{B}$-adapted function on $\mathcal{M}$;

(iii) If f is a $\mathbf{B}$-adapted function on $\mathcal{M}$ and $t \in \mathbf{B}$, then $g(x)(\omega) = E[f(x)|\mathcal{G}_t](\omega)$ is a $\mathbf{B}$-adapted function on $\mathcal{M}$.

Two random variables $x \in L^0(\Omega, M)$ and $\bar{x} \in L^0(\Gamma, M)$ have the same **$\mathbf{B}$-adapted distribution***, in symbols $x \equiv_{\mathbf{B}} \bar{x}$, if $E[f(x)] = E[f(\bar{x})]$ for every $\mathbf{B}$-adapted function f on $\mathcal{M}$.*

A $\mathbf{B}$-adapted space Ω is **saturated** *if for every other $\mathbf{B}$-adapted space Γ, every $x \in L^0(\Omega, M)$, and every pair $\bar{x} \in L^0(\Gamma, M), \bar{y} \in L^0(\Gamma, N)$ such that $x \equiv_{\mathbf{B}} \bar{x}$, there exists $y \in L^0(\Omega, N)$ such that $(x, y) \equiv_{\mathbf{B}} (\bar{x}, \bar{y})$.*

With this definition, the following theorem is proved in [K5] using results from [K4].

Theorem 4.5 *A $\mathbf{B}$-adapted space Ω is rich if and only if it is saturated.*

The main tool for the implication from left to right is the following consequence of our library of neocontinuous functions.

Theorem 4.6 *If Ω is a rich $\mathbf{B}$-adapted space then every $\mathbf{B}$-adapted function for Ω is neocontinuous.*

The main tool for the other direction is the following quantifier elimination theorem which is again of interest in its own right.

Theorem 4.7 *(Quantifier Elimination for* $\mathbf{B}$*-adapted Spaces) Let* Ω *be a rich* $\mathbf{B}$*-adapted space. A set is neocompact in* $\mathcal{M}$ *if and only if it is the intersection of a set of the form*

$$\{x \in \mathcal{M} : law(x, z) \in C\}$$

and countably many sets of the form

$$\{x \in \mathcal{M} : E[f_n(x, z)] \in D_n\}$$

where each f_n *is a* $\mathbf{B}$*-adapted function on* $\mathcal{M} \times \mathcal{N}$*,* $z \in \mathcal{N}$*,* C *is compact in* $Meas(M \times N)$*, and each* D_n *is compact in* $\mathbf{R}$*.*

We now turn to right continuous adapted spaces.

Definition 4.8 *The notion of an* $\mathbf{R}_+$*-adapted function is defined in exactly the same way as a* $\mathbf{B}$*-adapted function except that times are taken from the set* $\mathbf{R}_+$ *and conditional expectations are taken with respect to the right continuous filtration* $\mathcal{F}_t$*.*

We shall say that two random variables x *and* $\bar{x}$ *on right continuous adapted spaces* Ω *and* Γ *have the same* **adapted distribution***, in symbols* $x \equiv \bar{x}$*, if* $E[f(x)] = E[f(\bar{x})]$ *for each* $\mathbf{R}_+$*-adapted function* f*.*

A right continuous adapted space Ω *is* **saturated** *if for every other right continuous adapted space* Γ*, every* $x \in L^0(\Omega, M)$*, and every pair* $\bar{x} \in L^0(\Gamma, M), \bar{y} \in L^0(\Gamma, N)$ *such that* $x \equiv \bar{x}$*, there exists* $y \in L^0(\Omega, N)$ *such that* $(x, y) \equiv (\bar{x}, \bar{y})$*.*

It is shown in [K5] that rich right continuous adapted spaces do not exist, and that nontrivial $\mathbf{R}_+$-adapted functions of the form $E[f(x)|\mathcal{F}_t]$ for a right continuous adapted space can never be neocontinuous.

Here is the main result on right continuous adapted spaces which is proved in [K5].

Theorem 4.9 *For every rich* $\mathbf{B}$*-adapted space, the associated right continuous adapted space is saturated.*

5 The Huge Neometric Family

Our discussion up to this point has not involved nonstandard analysis at all, but we have postponed the proof that rich $\mathbf{B}$-adapted spaces exist until this section. Now it is time to enter the nonstandard world. We present the huge neometric family $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ associated with each $\aleph_1$-saturated nonstandard universe, which was introduced in [FK2]. The huge neometric family is constructed by giving an explicit definition of basic and neocompact sets that captures the way internal sets are used in nonstandard probability practice. The idea is that basic sets should be standard parts of internal sets, and neocompact sets should be standard parts of countable intersections of internal sets. The huge neometric family lives up to its name and contains all neometric spaces studied so far.

We shall see that the huge neometric family contains the neometric family over a **B**-adapted Loeb space. It follows that **B**-adapted Loeb spaces are rich, and therefore that rich **B**-adapted spaces exist.

The huge neometric family is a generalization of the approach to neocompactness originally developed in [K3]. In that paper, the neometric family over a rich **B**-adapted space was introduced in a nonstandard setting. From our current viewpoint, this neometric family is a subfamily of the huge neometric family.

We fix an $\aleph_1$-saturated nonstandard universe. We shall use the notions of a ***metric space** and a ***probability measure**, which are obtained from the corresponding standard notions by transfer: a *metric space is a structure $(\bar{M}, \bar{\rho})$ where $\bar{M}$ is an internal set and $\bar{\rho}$ is an internal function $\bar{\rho} : \bar{M} \times \bar{M} \to {}^*\mathbf{R}$ which satisfies the transfer of the usual rules for a metric. We now quickly review the nonstandard hull construction.

If $X, Y \in \bar{M}$, we write $X \approx Y$ if $\bar{\rho}(X, Y) \approx 0$. The **standard part** of an element $X \in \bar{M}$ is the equivalence class

$$^oX = \{Y \in \bar{M} : X \approx Y\}.$$

If $x = {}^oX$, we say that X **lifts** x.

Definition 5.1 *Consider a *metric space $(\bar{M}, \bar{\rho})$ and a point $c \in \bar{M}$. The* **galaxy** *of c is the set $G(\bar{M}, c)$ of all points $X \in \bar{M}$ such that $\bar{\rho}(X, c)$ is finite. By the* **nonstandard hull of** $\bar{M}$ *at c we mean the metric space $(\mathcal{H}(\bar{M}, c), \rho)$ where*

$$\mathcal{H}(\bar{M}, c) = \{^oX : X \in G(\bar{M}, c)\}, \rho(^oX, ^oY) = st(\bar{\rho}(X, Y)).$$

Note that any two points $b, c \in \bar{M}$ such that $\bar{\rho}(b, c)$ is finite have the same galaxies and nonstandard hulls,

$$G(\bar{M}, b) = G(\bar{M}, c) \text{ and } \mathcal{H}(\bar{M}, b) = \mathcal{H}(\bar{M}, c).$$

The neometric spaces in our huge family **H** will be the closed subspaces of nonstandard hulls. We need more definitions.

Given a set $B \subset G(\bar{M}, c)$, the **standard part** of B is the set

$$^oB = \{^oX : X \in B\}$$

of standard parts of elements of B. In the opposite direction, for a set $A \subset \mathcal{H}(\bar{M}, c)$ the **monad** of A is the set

$$\text{monad}(A) = \{X : {}^oX \in A\}.$$

By a Σ_1^0 (Π_1^0) set we mean the union (intersection) of countably many internal subsets of the galaxy $G(\bar{M}, c)$.

Observe that every countable subset of $G(\bar{M}, c)$ is Σ_1^0, and hence every countable subset of $\mathcal{H}(\bar{M}, c)$ is the standard part of a Σ_1^0 set.

For a set $B \subset G(\bar{M}, c)$ and a hyperreal $\varepsilon > 0$, we write

$$\bar{\rho}(X, B) = \inf\{\bar{\rho}(X, Y) : Y \in B\}, B^\varepsilon = \{X : \bar{\rho}(X, B) \leq \varepsilon\}.$$

Observe also that for each *metric space $\bar{M}$ and distinguished point $c \in \bar{M}$, the galaxy $G(\bar{M}, c)$ is a Σ_1^0 set, and the monad of the nonstandard hull $\mathcal{H}(\bar{M}, c)$ is the galaxy $G(\bar{M}, c)$.

Definition 5.2 *The* **huge neometric family** $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ *is defined as follows.* $\mathbf{H}$ *is the class of all metric spaces* $(\mathcal{M}, \rho)$ *such that* $\mathcal{M}$ *is a closed subset of some nonstandard hull* $\mathcal{H}(\bar{M}, c)$*. For each* $\mathcal{M} \in \mathbf{H}$*, the collections of basic and neocompact subsets of* $\mathcal{M}$ *are*

$$\mathcal{B}(\mathcal{M}) = \{A \subset \mathcal{M} : A = {}^oB \textit{ for some internal set } B \subset G(\bar{M}, c)\},$$

$$\mathcal{C}(\mathcal{M}) = \{A \subset \mathcal{M} : A = {}^oB \textit{ for some } \Pi_1^0 \textit{ set } B \subset G(\bar{M}, c)\}.$$

Note that the standard part of the union of two sets is the union of the standard parts, and therefore $\mathcal{B}(\mathcal{M})$ is closed under finite unions. Moreover, finite Cartesian products of basic sets are basic, and every finite subset of $\mathcal{M}$ is basic. On the other hand, the intersection of two basic sets need not be basic (see Example 3.6 in [FK2]).

The standard neometric family $(\mathbf{S}, \mathcal{B}, \mathcal{C})$ may be regarded as a subfamily of $(\mathbf{H}, \mathcal{B}, \mathcal{C})$. Given a standard complete metric space $(M, \rho) \in \mathbf{S}$, we may consider the *metric space $(^*M, ^*\rho)$. We abuse notation by identifying M with the set $\{^*x : x \in M\}$. Thus M is a closed subset of the nonstandard hull $\mathcal{H}(^*M, x)$ where x is any element of M, and hence M itself belongs to the huge family $\mathbf{H}$.

Here is a list of facts about the huge neometric family taken from [FK2]

Basic Facts 2 *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *belong to* $\mathbf{H}$.

1. $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ *is a countably compact neometric family which is closed under diagonal intersections.*

2. Every compact set $C \subset \mathcal{M}$ is basic.

3. Let $C = {}^o(\bigcap_n C_n)$ be a neocompact set in $\mathcal{M}$ where $\langle C_n \rangle$ is a decreasing chain of internal sets. Then

$$monad(C) = \bigcap_n ((C_n)^{1/n}).$$

4. A set $B \subset G(\bar{M}, c)$ is the monad of a neoseparable set if and only if B can be written in the form

$$B = \bigcap_n \bigcup_m ((B_m)^{1/n})$$

where $\langle B_m \rangle$ is an increasing chain of internal subsets of $G(\bar{M}, c)$.

5. Let M be a standard complete metric space, that is, $M \in \mathbf{S}$. A subset C of M is neocompact with respect to $\mathbf{H}$ if and only if it is compact, neoclosed with respect to $\mathbf{H}$ if and only if it is closed, and neoseparable with respect to $\mathbf{H}$ if and only if it is closed and separable. If $C \subset M$ is closed and $\mathcal{N} \in \mathbf{H}$, a function $f : C \to \mathcal{N}$ is neocontinuous with respect to $\mathbf{H}$ if and only if it is continuous.

6. Let $\mathcal{M}$ be neoseparable. A set $C \subset \mathcal{M}$ is neocompact in $\mathcal{M}$ if and only if C is neoclosed in $\mathcal{M}$ and any countable covering of C by neoopen sets in $\mathcal{M}$ has a finite subcovering.

7. Let $C \subset \mathcal{M}$ be neocompact and $f : C \to \mathcal{N}$. Then f is neocontinuous if and only if there is an internal function F such that ${}^oF(X) = f({}^oX)$ for all $X \in monad(C)$.

We now look at adapted Loeb spaces within the huge neometric family. We first need some notation for internal probability spaces.

Definition 5.3 *Let $(\Omega, \bar{P}, \bar{\mathcal{G}})$ be a *probability space and let $(\Omega, P, \mathcal{G})$ be the corresponding Loeb probability space. $SL^0(\Omega, M)$ denotes the *metric space of all $\bar{\mathcal{G}}$-measurable functions $X : \Omega \to {}^*M$ with the *metric*

$$\bar{\rho}_0(X, Y) = {}^*\inf\{\varepsilon : \bar{P}[{}^*\rho(X(\omega), Y(\omega)) \geq \varepsilon] \leq \varepsilon\}.$$

We say that $X \in SL^0(\Omega, M)$ is a **lifting** *of a function $x : \Omega \to M$, in symbols ${}^oX = x$, if $X(\omega)$ has standard part $x(\omega) \in M$ for P-almost all $\omega \in \Omega$.*

By the fundamental result of Loeb, that a function $x : \Omega \to M$ is Loeb measurable if and only if it has a lifting, we may take $L^0(\Omega, M)$ to be a subset of the standard part of $SL^0(\Omega, M)$. We now introduce adapted Loeb spaces.

Definition 5.4 *By a* **B-adapted Loeb space** *we mean an* $\mathbf{B}$*-adapted space* $\Omega = (\Omega, P, \mathcal{G}_t)_{t \in \mathbf{B}}$ *such that* $(\Omega, P, \mathcal{G})$ *is a Loeb probability space,* $\mathcal{G}_s \subset \mathcal{G}_t$ *whenever* $s < t \in \mathbf{B}$*, and each* $\mathcal{G}_t$ *is a* σ*-algebra generated by an internal subalgebra* $\bar{\mathcal{G}}_t$ *of* $\bar{\mathcal{G}}$*.*

The following theorem from [FK2] shows that $\mathbf{B}$-adapted Loeb spaces are rich and hence that rich $\mathbf{B}$-adapted spaces exist. As we have emphasized in the introduction, this is the one place where nonstandard analysis is needed in order to prove of existence theorems via neocompact sets.

Theorem 5.5 *Let* Ω *be an atomless* $\mathbf{B}$*-adapted Loeb space.*

(i) The set $L^0(\Omega, M)$ *is neoseparable with respect to the* **metric* $\bar{\rho}_0$ *on* $SL^0(\Omega, M)$*, and the metric space* $\mathcal{M} = (L^0(\Omega, M), \rho_0)$ *belongs to the huge neometric family* $\mathbf{H}$*.*

(ii) Every basic set, neocompact set, neoclosed set, neoseparable set, and neocontinuous function with neoclosed domain in $(\mathbf{M}_\Omega, \mathcal{B}_\Omega, \mathcal{C}_\Omega)$ *is also basic, neocompact, neoclosed, neoseparable, or neocontinuous, respectively, in the huge neometric family* $(\mathbf{H}, \mathcal{B}, \mathcal{C})$*.*

(iii) Ω *is rich.*

Corollary 5.6 *Every atomless Loeb probability space is rich.*

The paper [FK2] gave several other examples of natural neometric families within the huge neometric family. We mention three of them here.

Theorem 5.7 *Let* M *be a standard Banach space. In the huge neometric family, the nonstandard hull* $\mathcal{H}({}^*M, 0)$ *of the galaxy of* 0 *in* ${}^*M$ *is neoseparable, each closed ball in* $\mathcal{H}({}^*M, 0)$ *is neocompact, and the norm function* $x \mapsto \|x\|$*, the addition function* $(x, y) \mapsto x + y$*, and the scalar multiplication function* $x \mapsto \alpha x, \alpha \in \mathbf{R}$*, are neocontinuous.*

Theorem 5.8 *Let* Ω *be an atomless Loeb probability space. The set* $L^1(\Omega, \mathbf{R})$ *of Loeb integrable functions on* Ω *is neoseparable with respect to the* **metric* $\bar{\rho}_1(X, Y) = \bar{E}[\bar{\rho}(X(\cdot), Y(\cdot))]$*, and the metric space* $(L^1(\Omega, \mathbf{R}), \rho_1)$ *belongs to the huge neometric family* $\mathbf{H}$*.*

The next example from [FK2] concerns Loeb integrable functions with values in a neoseparable space rather than in the separable space of reals. We first need some definitions.

Definition 5.9 *Let* Ω *be an atomless Loeb probability space and let* $(\bar{M}, \bar{\rho})$ *be a* **metric space. Let* $\bar{\rho}_1$ *be the* **metric on the set* $SL^0(\bar{\Omega}, \bar{M})$ *defined by* $\bar{\rho}_1(X, Y) =$

$\bar{E}[\bar{\rho}(X(\cdot), Y(\cdot))]$. *For each* $c \in \bar{M}$, *let* $SL^1(\bar{\Omega}, \bar{M}, c)$ *be the set of all* $X \in SL^0(\bar{\Omega}, \bar{M})$ *such that* $\bar{\rho}(X(\cdot), c)$ *is* S*-integrable.*

Choose a point c *in the monad of* $\mathcal{M}$. *We let* $\mathcal{L}^1(\Omega, \mathcal{M})$ *denote the metric space of all functions* $x : \Omega \to \mathcal{M}$ *such that* x *has a lifting in* $SL^1(\bar{\Omega}, \bar{M}, c)$, *with the metric* ρ_1 *such that* $\rho_1(x, y) = {}^o\bar{\rho}_1(X, Y)$ *whenever* $X, Y \in SL^1(\bar{\Omega}, \bar{M}, c)$, X *lifts* x, *and* Y *lifts* y.

Theorem 5.10 *Let* $(\mathcal{M}, \rho)$ *be neoseparable in the huge neometric family* $\mathbf{H}$. *The set* $\mathcal{L}^1(\Omega, \mathcal{M})$ *is neoseparable with respect to the* **metric* $\bar{\rho}_1$, *and the metric space* $(\mathcal{L}^1(\Omega, \mathcal{M}), \rho_1)$ *belongs to* $\mathbf{H}$.

6 Forcing and Approximations

The paper [FK3] in this volume develops another approach to our program. It centers on the notion of a long sequence in the huge neometric family, that is, a sequence indexed by the hyperintegers. The notions of neocompactness, neoclosedness, and neocontinuity have natural characterizations in terms of long sequences, and long sequence arguments have a flavor much like the more traditional lifting and pushing down arguments in nonstandard analysis.

One of the central ideas in the paper [K3] was a notion of forcing for formulas in an infinitary language built from neocompact sets and neocontinuous functions. This notion of forcing is defined by an induction on formulas which is reminiscent of forcing in set theory. However, the "names" in the statements to be forced are sequences of elements, the "conditions" are infinite sets of natural numbers, and proofs by forcing resemble classical proofs by convergence. Forcing was applied in that paper to prove several existence theorems in stochastic analysis. Long sequences played an important role in the treatment of forcing.

The forthcoming paper [FK4] generalizes the treatment of forcing introduced in [K3], and introduces a second kind of forcing which applies only to positive bounded formulas but appears to be easier to use. The paper [FK4] also introduces a notion of an approximation for positive bounded formulas, and uses the results about forcing to generalize the approximation theorem 2.6 stated earlier in this paper. This approximation theorem is closely related to a theorem of Anderson [A2] in a classical compact setting, and the work of Henson [He] and Henson and Iovino [HI] in the setting of Banach space model theory. In this section we shall present the main notions and results from [FK4] on forcing and approximations of positive bounded formulas.

We begin by introducing the language **PB** of **positive bounded formulas**. We shall always work in the huge neometric family **H**.

Definition 6.1 *The language* **PB** *of positive bounded formulas has the following symbols:*

Infinitely many variables $u, v, \ldots$ of sort $\mathcal{M}$ for each neometric space $\mathcal{M} \in \mathbf{H}$,

An n-ary function symbol for each total neocontinuous function

$$f : \mathcal{M}_1 \times \cdots \times \mathcal{M}_n \to \mathcal{N},$$

A constant symbol for each element $c \in \mathcal{M}$,

A unary predicate symbol of sort $\mathcal{M}$ for each neoclosed set C in $\mathcal{M}$.

Terms are built in the usual way by applying function symbols to variables and constants of the appropriate sorts. The atomic formulas of **PB** *are $\tau(\vec{v}) \in C$ where τ is a term and C is a neoclosed set of the same sort. The formulas of* **PB** *are built from atomic formulas using finite and countable conjunctions, finite disjunctions, existential quantifiers of the form $(\exists v \in C)\phi$ where C is neocompact, and universal quantifiers $(\forall v \in D)\phi$ where D is neoseparable.*

Since the distance functions are neocontinuous, an equation $\tau(\vec{u}) = \pi(\vec{v})$ between two terms can be expressed by the **PB** formula $\rho(\tau(\vec{u}), \pi(\vec{v})) \in \{0\}$.

The next theorem says that every positive bounded formula defines a neoclosed set.

Theorem 6.2 *For every* **PB** *formula $\phi(\vec{v})$, the set $\{\vec{x} : \phi(\vec{x}) \text{ is true }\}$ is neoclosed in the sort space of $\vec{v}$.*

A sequence of k-tuples $\langle \vec{x}_n \rangle$ in $\mathcal{M}$ is said to be **neotight** if it is contained in a neocompact set in $\mathcal{M}$. By a **condition** we mean an infinite set $p \subset \mathbf{N}$. In the following, p, q, and r will denote conditions.

We now introduce positive bounded forcing.

Definition 6.3 *For each* **PB** *formula $\phi(\vec{v})$, neotight sequence $\langle \vec{x}_n \rangle$ of the same sort as $\vec{v}$, and condition p, the* **forcing relation** *$p \Vdash \phi(\langle \vec{x}_n \rangle)$ is defined inductively as follows:*

$p \Vdash f(\langle \vec{x}_n \rangle) \in C$ *iff* $\lim_{n \in p} \rho(f(\vec{x}_n), C) = 0$.

$p \Vdash \bigwedge_m \phi_m(\langle \vec{x}_n \rangle)$ *iff* $(\forall m) p \Vdash \phi_m(\langle \vec{x}_n \rangle$.

$p \Vdash (\forall v \in D)\phi(\langle \vec{x}_n \rangle, v)$ *iff* $(\forall$ *neotight* $\langle y_n \rangle$ *in* $D) p \Vdash \phi(\langle \vec{x}_n, y_n \rangle)$.

$p \|\!- (\phi \vee \psi)(\langle \vec{x}_n \rangle)$ *iff*

$$(\forall q \subset p)(\exists r \subset q) r \|\!- \phi(\langle \vec{x}_n \rangle) \vee r \|\!- \psi(\langle \vec{x}_n \rangle).$$

$p \|\!- (\exists v \in C)\phi(\langle \vec{x}_n \rangle, v)$ *iff*

$$(\forall q \subset p)(\exists r \subset q)(\exists \text{ neotight } \langle y_n \rangle \text{ in } C) r \|\!- \phi(\langle \vec{x}_n, y_n \rangle).$$

The main technical result about positive bounded forcing in [FK4] uses long sequences, which are defined in [FK3] in this volume. This theorem and its corollary capture the analogy between forcing and classical proofs by convergence.

Theorem 6.4 *Let $\phi(\vec{v})$ be a positive bounded formula, let $\langle \vec{x}_n \rangle$ be a neotight sequence in $\mathcal{M}$, and let $\langle \vec{x}_J \rangle$ be a long sequence which is an $\mathcal{M}$-extension of $\langle \vec{x}_n \rangle$. If $p \| \!- \phi(\langle \vec{x}_n \rangle)$ then $\phi(\vec{x}_J)$ is true for all sufficiently small infinite $J \in {}^*p$.*

Corollary 6.5 *Let $\phi(\vec{v})$ be a positive bounded formula and let $\lim_{n \in p} \vec{x}_n = \vec{x}$ in $\mathcal{M}$. If $p \| \!- \phi(\langle \vec{x}_n \rangle)$ then $\phi(\vec{x})$ is true.*

We now define the set of approximations of a **PB** formula.

Definition 6.6 *The set $\mathcal{A}(\phi)$ of* **approximations** *of a* **PB** *formula $\phi(\vec{v})$ is defined by induction on the complexity of ϕ as follows. For each neoseparable set D, let $\langle D_m \rangle$ be a chain of basic sets such that $\bigcup_m D_m$ is dense in D.*

$$\mathcal{A}(\tau(\vec{v}) \in C) = \{\tau(\vec{v}) \in C^{1/n} : n \in \mathbf{N}\}.$$

$$\mathcal{A}(\bigwedge_m \phi_m) = \{ \bigwedge_{m \leq n} \psi_m : n \in \mathbf{N} \text{ and } \psi_m \in \mathcal{A}(\phi_m) \text{ for all } m \leq n\}.$$

$$\mathcal{A}(\phi \vee \psi) = \{\phi_0 \vee \psi_0 : \phi_0 \in \mathcal{A}(\phi) \text{ and } \psi_0 \in \mathcal{A}(\psi)\}.$$

$$\mathcal{A}((\exists v \in C)\phi) = \{(\exists v \in C^{1/n})\psi : \psi \in \mathcal{A}(\phi) \text{ and } n \in \mathbf{N}\}.$$

$$\mathcal{A}((\forall v \in D)\phi) = \{(\forall v \in D_m)\psi : \psi \in \mathcal{A}(\phi) \text{ and } m \in \mathbf{N}\}.$$

Note that each approximation of a **PB** formula ϕ is a consequence of ϕ. The approximations of ϕ are finite formulas but are not necessarily positive bounded, because if C is neocompact, the set $C^{1/n}$ is neoclosed but not necessarily neocompact.

The next theorem from [FK4] characterizes positive bounded forcing in terms of approximate truth.

Theorem 6.7 *Let $\phi(\vec{v})$ be a positive bounded formula and $\langle \vec{x}_n \rangle$ be neotight. The following are equivalent.*

(i) $p \Vdash \phi(\langle \vec{x}_n \rangle)$.

(ii) For all $\psi \in \mathcal{A}(\phi)$, $\psi(\vec{x}_n)$ is true for all but finitely many $n \in p$,

Corollary 6.8 *(Positive Bounded Approximation Theorem) Let $\phi(\vec{v})$ be a* **PB** *formula and $\vec{c}$ be a tuple of constants. Then $\phi(\vec{c})$ is true if and only if $\psi(\vec{c})$ is true for every approximation $\psi \in \mathcal{A}(\phi)$.*

In the case that Ω is an atomless **B**-adapted Loeb space, the Approximation Theorem 2.6 is a special case of the Positive Bounded Approximation Theorem. To see this, write the formula

$$(\exists x \in A \cap B) f(x) \in D$$

in the equivalent form

$$(\exists x \in B)[x \in A \wedge f(x) \in D].$$

The latter formula is positive bounded by Theorem 5.5.

Here are two further results about approximations from [FK4] which are analogous to theorems from [HI] and [A2].

Theorem 6.9 *(Perturbation Principle) For each* **PB** *formula $\phi(\vec{v})$, neocompact set D, and approximation $\psi \in \mathcal{A}(\phi)$, there is a real $\delta > 0$ such that whenever $\vec{x}, \vec{y} \in D$, $\phi(\vec{x})$ holds, and $\rho(\vec{x}, \vec{y}) \leq \delta$, we have $\psi(\vec{y})$.*

Theorem 6.10 *(Almost-Near Theorem) Let $\phi(v)$ be a* **PB** *formula where v has sort $\mathcal{M}$, let C be a neocompact set in $\mathcal{M}$, and let D be a neoseparable set in $\mathcal{M}$ such that every $x \in C$ such that $\phi(x)$ is true belongs to D. Then for every real $\varepsilon > 0$ there is an approximation $\psi \in \mathcal{A}(\phi)$ such that every $x \in C$ such that $\psi(x)$ is true belongs to D^{ε}.*

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