NONSTANDARD ARITHMETIC AND RECURSIVE COMPREHENSION

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ABSTRACT. First order reasoning about hyperintegers can prove things about sets of integers. In the author's paper *Nonstandard Arithmetic and Reverse Mathematics*, Bulletin of Symbolic Logic 12 (2006), it was shown that each of the "big five" theories in reverse mathematics, including the base theory RCA_0, has a natural nonstandard counterpart. But the counterpart $^*\mathsf{RCA}_0$ of RCA_0 has a defect: it does not imply the Standard Part Principle that a set exists if and only if it is coded by a hyperinteger. In this paper we find another nonstandard counterpart, $^*\mathsf{RCA}_0{}'$, that does imply the Standard Part Principle.

1. INTRODUCTION

In the paper [3], it was shown that each of the "big five" theories of second order arithmetic in reverse mathematics has a natural counterpart in the language of nonstandard arithmetic. In this paper we give another natural counterpart of the weakest these theories, the theory RCA_0 of Recursive Comprehension.

The language L_2 of second order arithmetic has a sort for the natural numbers and a sort for sets of natural numbers, while the language $^*L_1$ of nonstandard arithmetic has a sort for the natural numbers and a sort for the hyperintegers. In nonstandard analysis one often uses first order properties of hyperintegers to prove second order properties of integers. An advantage of this method is that the hyperintegers have more structure than the sets of integers. The method is captured by the Standard Part Principle (STP), a statement in the combined language $L_2 \cup {}^*L_1$ that says that a set of integers exists if and only if it is coded by a hyperinteger. We say that a theory T' in $L_2 \cup {}^*L_1$ is conservative with respect to a theory T in L_2 if every sentence of L_2 provable from T' is provable from T.

For each of the theories $T = \mathsf{WKL}_0, \mathsf{ACA}_0, \mathsf{ATR}_0, \Pi^1_1\text{-}\mathsf{CA}_0$ in the language L_2 of second order arithmetic, [3] gave a theory U of nonstandard arithmetic in the language $^*L_1$ such that:

(1) $\quad U + \mathsf{STP}$ implies T and is conservative with respect to T.

The nonstandard counterpart $^*\mathsf{RCA}_0$ for RCA_0 in [3] does not have property (1). The theory $^*\mathsf{RCA}_0 + \mathsf{STP}$ is not conservative with respect to RCA_0, and $^*\mathsf{RCA}_0$ has only a weakened form of the STP. In this paper we give a new

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nonstandard counterpart ${}^*\mathsf{RCA_0}'$ of $\mathsf{RCA_0}$ that does have property (1). That is, we give a theory U of nonstandard arithmetic in ${}^*L_1$ such that the theory ${}^*\mathsf{RCA_0}' = U + \mathsf{STP}$ implies $\mathsf{RCA_0}$ and is conservative with respect to $\mathsf{RCA_0}$.

Section 2 contains background material. Our main results are stated in Section 3. In Section 4 we give the easy proof that ${}^*\mathsf{RCA_0}'$ implies $\mathsf{RCA_0}$. In Section 5 we give the more difficult proof that ${}^*\mathsf{RCA_0}'$ is conservative with respect to $\mathsf{RCA_0}$. Section 6 contains complementary results showing that various enhancements of ${}^*\mathsf{RCA_0}'$ imply the Weak Koenig lemma, and thus are not conservative with respect to $\mathsf{RCA_0}$. We also discuss some related open questions.

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2. Preliminaries

We refer to [2] for background on models of arithmetic, and to [4] for a general treatment of reverse mathematics in second order arithmetic.

We follow the notation of [3], with one exception. We take the vocabulary of the first order language L_1 of arithmetic to be $\{<, 0, 1, +, \dot{-}, \cdot\}$. The operation $\dot{-}$ is cutoff subtraction, defined by $n + (m \dot{-} n) = \max(m, n)$. Thus $m \dot{-} n = m - n$ if $m \geq n$, and $m \dot{-} n = 0$ if $m < n$. The additional function symbols p_n and $(m)_n$ will be introduced here as defined symbols. (In [3] they were part of the underlying vocabulary of L_1.)

The language L_2 of second order arithmetic is an extension of L_1 with two sorts, $\mathbf{N}$ for natural numbers and $\mathbf{P}$ for sets of natural numbers. In L_2, the symbols of L_1 are taken to be of sort $\mathbf{N}$. L_2 has variables $X, Y, \ldots$ of sort $\mathbf{P}$ and a membership relation $\in$ of sort $\mathbf{N} \times \mathbf{P}$. In either L_1 or L_2, Δ^0_0 is the set of all bounded quantifier formulas, Σ^0_1 is the set of formulas of the form $\exists m\, \varphi$ where $\varphi \in \Delta^0_0$, and so on.

The expressions $m \leq n, m > n, m \geq n$ will be used in the obvious way. We will sometimes use the expression $m = n/r$ as an abbreviation for $m \cdot r = n$. We let $\mathbb{N}$ be the set of (standard) natural numbers. We sometimes also use $\mathbb{N}$ to denote the structure $(\mathbb{N}, <, 0, 1, +, \dot{-}, \cdot)$.

The theory $I\Sigma_1$, Peano Arithmetic with Σ^0_1 induction, has the usual axioms for linear order with first element 0, and the recursive rules for $0, 1, +, \dot{-}$, and $\cdot$, and the Σ^0_1 Induction scheme

$$[\varphi(0, \ldots) \wedge \forall m[\varphi(m, \ldots) \rightarrow \varphi(m+1, \ldots)]] \rightarrow \forall m\, \varphi(m, \ldots)$$

where φ is a Σ^0_1 formula of L_1.

The theory $\mathsf{RCA_0}$ of arithmetic with restricted comprehension is the usual base theory for reverse mathematics. It is the theory in L_2 that has the axioms of $I\Sigma_1$, and the Σ^0_1 Induction scheme and Δ^0_1 Comprehension scheme for formulas of L_2. Each model of $\mathsf{RCA_0}$ will be a pair $(\mathcal{N}, \mathcal{P})$ where $\mathcal{P}$ is a set of subsets of $\mathcal{N}$. The theory $\mathsf{WKL_0}$ is $\mathsf{RCA_0}$ plus the Weak Koenig

Lemma. It is well-known that WKL_0 is not conservative with respect to RCA_0 (see [4]).

The language ${}^*L_1$ is the extension of L_1 that has the sort $\mathbf{N}$ for standard integers and the sort ${}^*\mathbf{N}$ for hyperintegers. It has variables $m, n, \ldots$ of sort $\mathbf{N}$ and $x, y, z, \ldots$ of sort ${}^*\mathbf{N}$. The universe of sort $\mathbf{N}$ is to be interpreted as a subset of the universe of sort ${}^*\mathbf{N}$. ${}^*L_1$ has the same vocabulary $\{<, 0, 1, +, \dot{-}, \cdot\}$ as L_1. All terms are considered to be terms of sort ${}^*\mathbf{N}$, and terms built from variables of sort $\mathbf{N}$ are also considered to be terms of sort $\mathbf{N}$. The atomic formulas are $s = t$, $s < t$ where s, t are terms.

A **bounded quantifier of sort N** is an expression $(\exists m < s)$ or $(\forall m < s)$ where m is a variable of sort $\mathbf{N}$ and s a term of sort $\mathbf{N}$. A **bounded quantifier of sort** ${}^*\mathbf{N}$ is an expression $(\exists x < t)$ or $(\forall x < t)$ where x is a variable of sort ${}^*\mathbf{N}$ which is not of sort $\mathbf{N}$, and t a term. Thus $(\exists x < m)$ is a bounded quantifier of sort ${}^*\mathbf{N}$, but $(\exists m < x)$ is not a bounded quantifier.

A Δ_0^S formula is a formula of ${}^*L_1$ built from atomic formulas using connectives and bounded quantifiers of sorts $\mathbf{N}$ and ${}^*\mathbf{N}$. A Σ_1^S formula is a formula of the form $\exists n\, \varphi$ where φ is a Δ_0^S formula. (The superscript S indicates that the unbounded quantifiers are of the standard sort $\mathbf{N}$.)

Definition 2.1. *The theory* BNA *of Basic Nonstandard Arithmetic has the following axioms in the language* ${}^*L_1$*:*

- *The axioms of* $I\Sigma_1$ *in the language* L_1*,*
- *The sentence saying that* $<$ *is a strict linear order.*
- *The Proper Initial Segment Axioms:*

$$\forall n \exists x (x = n),$$
$$\forall n \forall x [x < n \to \exists m\, x = m],$$
$$\exists y \forall n [n < y].$$

Note that the theory BNA by itself says nothing about the operations $+, \dot{-}, \cdot$ on the nonstandard hyperintegers. We will work with theories that contain BNA and additional axioms.

For each formula φ if L_1, we let ${}^*\varphi$ be a formula of ${}^*L_1$ that is obtained from φ by replacing each bound variable in φ by a variable of sort ${}^*\mathbf{N}$ in a one to one fashion. A **universal sentence** in L_1 is a sentence of the form $\forall \vec{m}\, \varphi(\vec{m})$ where φ has no (bounded or unbounded) quantifiers.

Definition 2.2. *Given a set* Γ *of formulas of* L_1*,* Γ***-Transfer*** *(or* ***Transfer for*** Γ*) is the set of formulas* $\varphi \to {}^*\varphi$ *where* $\varphi \in \Gamma$*.*

$\forall$ ***Transfer****, or* $\forall\mathsf{T}$*, is Transfer for the set of all universal sentences in* L_1*.*

A model of BNA will be an ordered structure of the form $(\mathcal{N}, {}^*\mathcal{N})$ where $\mathcal{N}$ is a model of $I\Sigma_1$, and ${}^*\mathcal{N}$ is a proper end extension of $\mathcal{N}$. In a model $(\mathcal{N}, {}^*\mathcal{N})$ of $\mathsf{BNA} + \forall\mathsf{T}$, ${}^*\mathcal{N}$ will be the non-negative part of an ordered ring. In particular, the commutative, associative, distributive, and order laws will hold for $+, \cdot, <$, and $\dot{-}$ will have the property that $y + (x \dot{-} y) = \max(x, y)$.

The theory ${}^*\Delta\mathsf{PA}$ introduced in [3] has the axioms of BNA plus the following axiom scheme, called **Internal Induction**:

$$\varphi(0,\vec{u}) \wedge \forall x[\varphi(x,\vec{u}) \rightarrow \varphi(x+1,\vec{u})]] \rightarrow \forall x\, \varphi(x,\vec{u})$$

where $\varphi(x,\vec{u})$ is a Δ_0^S formula.

The theory ${}^*\Sigma\mathsf{PA}$ in [3] has the axioms of ${}^*\Delta\mathsf{PA}$ plus the following axiom, called Σ_1^S **Induction**:

$$\varphi(0,\vec{u}) \wedge \forall m[\varphi(m,\vec{u}) \rightarrow \varphi(m+1,\vec{u})]] \rightarrow \forall m\, \varphi(m,\vec{u})$$

where $\varphi(m,\vec{u})$ is a Σ_1^S formula. (See [3], Definition 3.2 and Proposition 3.9).

Note that Internal Induction is an induction over a variable x of sort ${}^*\mathbf{N}$, while Σ_1^S Induction is an induction over a variable m of sort $\mathbf{N}$.

In this paper we will work in the combined language $L_2 \cup {}^*L_1$. We use the notation $(x|y)$ (x divides y) as an abbreviation for the formula $\exists z[x \cdot z = y]$. Using the axioms of $I\Sigma_1$, we can define p_n as the n-th prime in the usual way, and treat p_n as a function symbol of sort $\mathbf{N} \rightarrow \mathbf{N}$. However, we will never write p_x where x is a variable of sort ${}^*\mathbf{N}$. Following [3], we define the **standard set relation** $X = st(x)$ by the formula

$$(\forall n \geq 0)\,[n \in X \leftrightarrow (p_n|x)].$$

The **Upward Standard Part Principle** (Upward STP) is the sentence

$$\forall X \exists x[X = st(x)]$$

which says that every set in $\mathcal{P}$ is coded by a hyperinteger. The **Downward Standard Part Principle** (Downward STP) is the sentence

$$\forall x \exists X[X = st(x)]$$

which says that every hyperinteger codes a set in $\mathcal{P}$. The **Standard Part Principle** STP is the sentence

$$\forall X \exists x[X = st(x)] \wedge \forall x \exists X[X = st(x)].$$

This is the conjunction of the Upward STP and the Downward STP.

As in [3], we say that a theory T' in a language L' is **conservative** with respect to a theory T in a language $L \subseteq L'$ if every sentence of L that is provable from T' is provable from T.

It is shown in [3], Section 5, that the theory ${}^*\mathsf{WKL}_0 = {}^*\Sigma\mathsf{PA} + \mathsf{STP}$ implies and is conservative with respect to WKL_0. In fact, ${}^*\mathsf{WKL}_0 + \forall\mathsf{T}$ is still conservative with respect to WKL_0.

There are two possible options for weakening ${}^*\mathsf{WKL}_0$ to get a theory that implies and is conservative with respect to RCA_0: either weaken STP or weaken ${}^*\Sigma\mathsf{PA}$. The theory ${}^*\mathsf{RCA}_0$ introduced in [3] took the first option. The axioms of ${}^*\mathsf{RCA}_0$ are ${}^*\Sigma\mathsf{PA}$ plus the Upward STP and a very weak form of the Downward STP, called Δ_1^0-STP. As mentioned in the introduction, it was proved in [3] that ${}^*\mathsf{RCA}_0$ implies and is conservative with respect to RCA_0.

Yokoyama [6], [7] introduced a theory that is stronger than ${}^*\mathsf{RCA_0}$ but, like ${}^*\mathsf{RCA_0}$, is conservative with respect to RCA_0 and has a very weak form of the Downward STP.

In [1], Avigad considered several theories that extend RCA_0 without the STP, and can be formulated in the combined language $L_2 \cup^* L_1$ or in higher order analogues. When comparing the results of [1] with the present paper, note that the sort $\mathbf{N}$ in [1] corresponds to the sort ${}^*\mathbf{N}$ in this paper, and the standardness predicate S in [1] corresponds to the sort $\mathbf{N}$ in this paper. Thus the induction axiom for quantifier-free formulas of Primitive Recursive Arithmetic in [1] is the same thing as Internal Induction in this paper.

3. Statements of the Main Results

In this section we introduce a theory ${}^*\mathsf{RCA_0}'$ that does what ${}^*\mathsf{RCA_0}$ does but contains the full Standard Part Principle. ${}^*\mathsf{RCA_0}'$ takes the second option for weakening ${}^*\mathsf{WKL_0}$; it is stronger than $\mathsf{BNA} + \mathsf{STP}$ but weaker than ${}^*\mathsf{WKL_0}$. We will prove that ${}^*\mathsf{RCA_0}'$ implies and is conservative with respect to RCA_0.

We remark that the theory $\mathsf{BNA} + \mathsf{STP}$ by itself says very little about sets of natural numbers. It does not even imply that there are infinite sets of sort $\mathbf{P}$. One can get a model $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$ of $\mathsf{BNA} + \mathsf{STP}$ with no infinite sets by taking $\mathcal{N}$ to be $\mathbb{N}$, taking $\mathcal{P}$ to be the set of finite subsets of $\mathbb{N}$, and taking ${}^*\mathcal{N}$ to be the non-negative part of the ring of polynomials over $\mathbb{Z}$ in a variable x such that $\forall n\, n < x$.

In order to get a theory that is strong enough to imply RCA_0, we will add nonstandard induction and comprehension principles. We will see in Section 6 that too strong a comprehension principle will give a theory that already implies WKL_0, and thus cannot be conservative with respect to RCA_0. For this reason, we need to introduce the class of special Δ_0^S formulas.

Definition 3.1. *By a* ***special*** Δ_0^S ***formula*** *we mean a formula of* ${}^*L_1$ *that is built from atomic formulas* $s = t, s < t$, *and divisibility formulas* $(n|t)$ *where* s, t *are terms, using connectives and bounded quantifiers of sort* $\mathbf{N}$. *A* ***special*** Σ_1^S ***formula*** *is a formula of the form* $\exists m\, \varphi$ *where* φ *is a special* Δ_0^S *formula.*

Every special Δ_0^S formula is a Δ_0^S formula, because the formula $(n|t)$ is an abbreviation for the Δ_0^S formula $(\exists x < t+1)\, n \cdot x = t$. Note that for each term s of sort $\mathbf{N}$, the formula $(s|t)$ is equivalent in BNA to the special Δ_0^S formula $(\exists n < s+1)[n = s \wedge (n|t)]$.

Given a formula $\varphi(m, \vec{u})$ of ${}^*L_1$, the expression $st(x) = \varphi(\cdot, \vec{u})$ stands for the formula

$$\forall m\, [(p_m|x) \leftrightarrow \varphi(m, \vec{u})].$$

Intuitively, $st(x) = \varphi(\cdot, \vec{u})$ means that x codes the class $\{m : \varphi(m, \vec{u})\}$.

Definition 3.2. *The theory* ${}^*\mathsf{RCA_0}'$ *has the following axioms:*

- *The axioms of* BNA*,*

- *Special* Σ_1^S *Induction:*
$$\varphi(0,\vec{u}) \wedge \forall m[\varphi(m,\vec{u}) \rightarrow \varphi(m+1,\vec{u})] \rightarrow \forall m\, \varphi(m,\vec{u})$$
where $\varphi(m,\vec{u})$ *is a special* Σ_1^S *formula.*
- *Special* Δ_1^S *Comprehension:*
$$\forall m[\varphi(m,\vec{u}) \leftrightarrow \neg\psi(m,\vec{u})] \rightarrow \exists x\, st(x) = \varphi(\cdot,\vec{u})$$
where φ, ψ *are special* Σ_1^S *formulas in which* x *does not occur.*
- *The Standard Part Principle* STP.

Proposition 3.3. $^*\mathsf{WKL_0}$ *implies* $^*\mathsf{RCA_0}'$.

Proof. The axioms of $^*\mathsf{WKL_0}$ already include $\mathsf{BNA}+\mathsf{STP}$. Σ_1^S Induction already contains Special Σ_1^S Induction. By Lemma 3.5 in [3], $^*\mathsf{WKL_0}$ implies the Δ_1^S Comprehension scheme, which contains Special Δ_1^S Comprehension. □ □

We now state our main results.

Theorem 3.4. $^*\mathsf{RCA_0}'$ *implies* RCA_0.

Theorem 3.5. $^*\mathsf{RCA_0}' + \forall\mathsf{T}$ *is conservative with respect to* RCA_0.

The proofs of these theorems will be given in the next two sections.

Before embarking on the proofs, we make some comments on the theory $^*\mathsf{RCA_0}'$. Theorems 3.4 and 3.5 show that $^*\mathsf{RCA_0}'$ can serve as a base theory for reverse mathematics in the combined language $L_2 \cup^* L_1$. It implies the axioms of RCA_0, and can be express implications between stronger theories in the nonstandard setting.

The coding of sets by means of prime divisors that is used in $^*\mathsf{RCA_0}'$ is inconvenient for some purposes. One can write down theories with the analogous axioms but a different coding of sets. The problem is that a different coding may result in a theory that is not conservative with respect to RCA_0. The coding by means of prime divisors has the major advantage of giving a theory that is conservative with respect to RCA_0. This coding also has the following nice properties:

$$st(x) \cap st(y) \subseteq st(x+y), \quad st(x) \cup st(y) \subseteq st(xy), \quad st(x) \cap st(x+1) = \emptyset.$$

One might also ask whether $^*\mathsf{RCA_0}'$, or $^*\mathsf{RCA_0}' + \forall\mathsf{T}$, can be used in its own right to carry out certain kinds of nonstandard arguments. We give some heuristic arguments suggesting the answer is yes, to a limited extent. Consider first the weaker theory $\mathsf{BNA} + \forall\mathsf{T}$ in the language $^*L_1$. In $\mathsf{BNA} + \forall\mathsf{T}$, the hyperintegers can be extended in the usual way to the ordered field of hyperrational numbers (quotients of hyperintegers). One can define an infinitesimal as a hyperrational number whose absolute value is less than every positive rational, and a finite hyperrational number as one whose absolute value is less than some positive rational. In this theory one can develop Robinson's infinitesimal treatment of limits and derivatives for rational functions.

As we will see in Section 6, in ${}^*\mathsf{RCA_0}'$ one can use special Δ_1^S-Comprehension and then the Downward STP to build sets, and in this way prove that for every finite hyperrational number x there is a unique real number r, the **shadow** of x, such that for each rational q,

$$q < r \Rightarrow q \leq x \text{ and } q < x \Rightarrow q \leq r.$$

This opens up the possibility of proving theorems for hyperrational numbers and taking shadows to draw conclusions about real numbers.

One can also begin with a proof in a theory that is stronger than ${}^*\mathsf{RCA_0}'$ and convert it to a proof in ${}^*\mathsf{RCA_0}'$, with the particular instance of the axiom in the stronger theory that was needed in the original proof replaced by a hypothesis in the new proof in ${}^*\mathsf{RCA_0}'$. Section 6 has some examples of this.

Many methods that are available in ${}^*\mathsf{WKL_0}$ and are commonly used in nonstandard analysis are not available in ${}^*\mathsf{RCA_0}' + \forall\mathsf{T}$. Internal induction and the overspill principle are not available. The comprehension axioms of ${}^*\mathsf{RCA_0}'$ do not allow bounded quantifiers of sort ${}^*\mathbf{N}$. Transfer for Π_1^0 sentences is not available. All of these methods were left out for a good reason. We will show in Section 6 that under ${}^*\mathsf{RCA_0}'$, each of them implies the Weak Koenig Lemma.

4. Proof that ${}^*\mathsf{RCA_0}'$ Implies RCA_0

In this section we prove Theorem 3.4. Note that the Special Σ_1^S Induction and Special Δ_1^S Comprehension Axioms for ${}^*\mathsf{RCA_0}'$ are sentences in the language ${}^*L_1$ which has variables of sort ${}^*\mathbf{N}$ but no variables of sort $\mathbf{P}$, and we must prove the Σ_1^0-Induction and Δ_1^0-Comprehension Axioms of RCA_0, which have variables of sort $\mathbf{P}$ but no variables of sort ${}^*\mathbf{N}$.

Lemma 4.1. *For every formula $\varphi(x,\dots)$ of ${}^*L_1$,*

$$\mathsf{BNA} \vdash \forall x\, \varphi(x,\dots) \to \forall n\, \varphi(n,\dots).$$

Proof. This follows at once from the axiom $\forall n \exists x (x = n)$. □ □

The notation $S(t)$ (meaning "t is standard") is an abbreviation for the Σ_1^S formula $\exists n\, n = t$. For a tuple of terms $\vec{t} = (t_1, \dots, t_k)$, $S(\vec{t})$ means $S(t_1) \wedge \dots \wedge S(t_n)$.

Lemma 4.2. *For any term $t(\vec{x})$,*

$$\mathsf{BNA} \vdash S(\vec{x}) \to S(t(\vec{x})).$$

Proof. Assume $S(\vec{x})$. This means that there exists a tuple $\vec{m}$ such that $\vec{m} = \vec{x}$. By definition, $t(\vec{m})$ has sort $\mathbf{N}$. By the rules of two-sorted logic, we have $S(t(\vec{m}))$ and $t(\vec{x}) = t(\vec{m})$, and thus $S(t(\vec{x}))$. □ □

We need the following definition from [3] (modified by replacing the expression $(x_i)_t > 0$ by the equivalent expression $(p_t|x_i)$).

Definition 4.3. *Let $\varphi(\vec{m},\vec{X})$ be a formula in L_2, where $\vec{m},\vec{X}$ contain all the variables of φ, both free and bound. The **lifting** $\overline{\varphi}(\vec{m},\vec{x})$ is the formula of $^*L_1$ defined as follows, where $\vec{x}$ is a tuple of variables of sort $^*\mathbf{N}$ of the same length as $\vec{X}$.*

- *Replace each subformula $t \in X_i$, where t is a term, by $(p_t|x_i)$.*
- *Replace each quantifier $\forall X_i$ by $\forall x_i$, and similarly for $\exists$.*

It is clear that if φ is a Δ^0_0 formula of L_2, then $\overline{\varphi}$ is a Δ^S_0 formula of $^*L_1$, and if φ is a Σ^0_1 formula then $\overline{\varphi}$ is a Σ^S_1 formula. The next lemma is a lifting theorem from formulas of L_2 to formulas of $^*L_1$. It was stated for $^*\Delta\mathsf{PA}$ as Lemma 4.6 in [3], but also holds for BNA.

Lemma 4.4. *(i) For each arithmetical formula $\varphi(\vec{m},\vec{X})$ of L_2,*

$$\mathsf{BNA} \vdash st(\vec{x}) = \vec{X} \to [\varphi(\vec{m},\vec{X}) \leftrightarrow \overline{\varphi}(\vec{m},\vec{x})].$$

(ii) For each formula $\varphi(\vec{m},\vec{X})$ of L_2,

$$\mathsf{BNA} + \mathsf{STP} \vdash st(\vec{x}) = \vec{X} \to [\varphi(\vec{m},\vec{X}) \leftrightarrow \overline{\varphi}(\vec{m},\vec{x})].$$

Proof. We first prove the result in the case that φ is an atomic formula. Work in BNA and assume that $st(\vec{x}) = \vec{X}$. If $\varphi(\vec{m})$ is an atomic formula of the form $t(\vec{m}) = u(\vec{m})$ or $t(\vec{m}) < u(\vec{m})$, then $\overline{\varphi}(\vec{m})$ is the same as $\varphi(\vec{m})$, so $\varphi(\vec{m}) \leftrightarrow \overline{\varphi}(\vec{m})$. If $\varphi(\vec{m},X)$ is an atomic formula of the form $t(\vec{m}) \in X$, then $\overline{\varphi}(\vec{m},x)$ is $(p_{t(\vec{m})}|x)$. By Lemma 4.2, $\exists n\, n = t(\vec{m})$. Therefore $t(\vec{m}) \in X \leftrightarrow (p_{t(\vec{m})}|x)$, as required.

The general case is now follows by induction on the complexity of φ, using the Proper Initial Segment Axioms for bounded quantifiers, and using STP for second order quantifiers. □ □

Proof of Theorem 3.5 from Theorem 5.1. Work in $^*\mathsf{RCA_0}'$, and prove the axioms of $\mathsf{RCA_0}$. The axioms of $I\Sigma_1$ already belong to $^*\mathsf{RCA_0}'$. For each instance θ of Σ^0_1 Induction, $\overline{\theta}$ is an instance of special Σ^S_1 Induction, which is obtained by replacing each subformula of the form $t \in X_i$ by $(p_t|x_i)$. Therefore by Lemma 4.4, Σ^0_1 Induction holds. The argument for Δ^0_1 Comprehension is similar. □ □

5. Proof that $^*\mathsf{RCA_0}'$ is Conservative With Respect to $\mathsf{RCA_0}$

In this section we prove Theorem 3.5. Theorem 3.5 is a consequence of the following theorem and a result in [3].

Theorem 5.1. *Let $(\mathcal{N},\mathcal{P},\mathcal{N}_1)$ be a model of $^*\Sigma\mathsf{PA}+$ Upward $\mathsf{STP}+\forall\mathsf{T}$ such that $\mathcal{N}_1$ has cofinality at least $|\mathcal{P}|$. Then $\mathcal{N}_1$ has a substructure $^*\mathcal{N}$ such that $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$ is a model of $^*\mathsf{RCA_0}'+\forall\mathsf{T}$.*

Note that if $(\mathcal{N},\mathcal{P},\mathcal{N}_1)$ is countable, then $\mathcal{N}_1$ automatically has cofinality $\aleph_0 = |\mathcal{P}|$. So Theorem 5.1 shows in particular that for every countable model $(\mathcal{N},\mathcal{P},\mathcal{N}_1)$ of $^*\Sigma\mathsf{PA}$+Upward $\mathsf{STP}+\forall\mathsf{T}$, $\mathcal{N}_1$ has a substructure $^*\mathcal{N}$ such that $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$ is a model of $^*\mathsf{RCA_0}'+\forall\mathsf{T}$.

We first show that Theorem 3.5 follows from Theorem 5.1 above and Theorem 5.7 in [3].

Proof of Theorem 3.5 from Theorem 5.1. Assume Theorem 5.1. Let θ be a sentence of L_2 that is consistent with RCA_0. Then $\mathsf{RCA}_0+\theta$ has a countable model $(\mathcal{N},\mathcal{P})$ such that $\mathcal{N}$ is not isomorphic to $\mathbb{N}$. By Theorem IX.2.1 in [4], there is a set $\mathcal{P}'\supseteq\mathcal{P}$ such that $(\mathcal{N},\mathcal{P}')$ is a model of WKL_0. By Theorem 5.7 in [3], $(\mathcal{N},\mathcal{P}')$ can be expanded to a countable model $(\mathcal{N},\mathcal{P}',\mathcal{N}_1)$ of ${}^*\mathsf{WKL_0}$ such that $\mathcal{N}_1\cong\mathcal{N}$. We now replace $\mathcal{P}'$ by $\mathcal{P}$ and consider the structure $(\mathcal{N},\mathcal{P},\mathcal{N}_1)$. Since $\mathcal{P}\subseteq\mathcal{P}'$ and $\mathcal{N}_1\cong\mathcal{N}$, $(\mathcal{N},\mathcal{P},\mathcal{N}_1)$ is a model of ${}^*\Sigma\mathsf{PA}$+Upward $\mathsf{STP}+\forall\mathsf{T}+\theta$. By Theorem 5.1, $\mathcal{N}_1$ has a substructure ${}^*\mathcal{N}$ such that $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$ is a model of ${}^*\mathsf{RCA_0}'+\forall\mathsf{T}$. Since θ holds in $(\mathcal{N},\mathcal{P})$, it holds in $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$. This shows that θ is consistent with ${}^*\mathsf{RCA_0}'+\forall\mathsf{T}$, so ${}^*\mathsf{RCA_0}'+\forall\mathsf{T}$ is conservative with respect to RCA_0, and Theorem 3.5 holds. □ □

The remainder of this section is devoted to the proof of Theorem 5.1. Assume the hypotheses of Theorem 5.1.

For $x,y\in\mathcal{N}_1$, write $x\sim y$ if there exists $r\in\mathcal{N}$ such that $x\le ry$ and $y\le rx$. Write $x\ll y$ if $rx<y$ for all $r\in\mathcal{N}$. Note that in $\mathcal{N}_1$,

$$x\ll y\rightarrow x<y,$$

$$u\le x\ll y\le z\rightarrow u\ll z,$$

$$u\sim x\ll y\sim z\rightarrow u\ll z,$$

$$x\sim y\leftrightarrow[\neg x\ll y\wedge\neg y\ll x],$$

$$[x\ll z\wedge y\ll z]\rightarrow x+y\ll z.$$

We write $x\lll y$ if $x^k\ll y$ for each $0<k\in\mathbb{N}$. Note that in $\mathcal{N}_1$,

$$\forall x\exists y\, x\lll y,$$

$$x\lll y\rightarrow x\ll y,$$

$$u\le x\lll y\le z\rightarrow u\lll z,$$

$$u\sim x\lll y\sim z\rightarrow u\lll z,$$

$$[x\lll z\wedge y\lll z]\rightarrow xy\lll z.$$

We say that a set $X\in\mathcal{P}$ is **bounded** if $(\mathcal{N},\mathcal{P})\models\exists m\forall n\,[n\in X\rightarrow n<m]$.

We now show that there is a sequence $\langle U_\alpha,\alpha<\kappa\rangle$ of length $\kappa\le|\mathcal{P}|$ such that each U_α is an unbounded element of $\mathcal{P}$, and for each unbounded $X\in\mathcal{P}$ there is a unique $\alpha<\kappa$ such that $X\Delta U_\alpha$ is bounded. To see that such a sequence exists, let $\mathcal{P}'$ be the set of unbounded elements of $\mathcal{P}$. Let $\mathcal{P}''$ be a subset of $\mathcal{P}'$ that contains exactly one element of each equivalence class under the relation "$X\Delta Y$ is bounded". Since $\mathcal{P}''\subseteq\mathcal{P}$, the elements of $\mathcal{P}''$ can be arranged in a sequence $U_\alpha,\alpha<\kappa$ of length $\kappa\le|\mathcal{P}|$. Then U_α has the required properties.

We claim that there is a sequence $\langle u_\alpha,\alpha<\kappa\rangle$ of elements of $\mathcal{N}_1\setminus\mathcal{N}$ such that whenever $\alpha<\beta<\kappa$:

- $u_\alpha\lll u_\beta$.

- $\mathcal{N}_1 \models (u_\alpha$ is a product of distinct primes).
- For each $n \in \mathcal{N}$, $(p_n | u_\alpha)$ in $\mathcal{N}_1$ if and only if $n \in U_\alpha$.

Given u_α for all $\alpha < \beta$, we obtain u_β as follows. By the Upward STP, there is an element $u \in \mathcal{N}_1$ such that $st(u) = U_\beta$. Take an element $x \in \mathcal{N}_1 \setminus \mathcal{N}$. By Internal Induction in $(\mathcal{N}, \mathcal{N}_1)$, there is an element y such that

$$\mathcal{N}_1 \models y = \Pi\{z : z < x \wedge z \text{ is prime } \wedge (z|u)\}.$$

At this point we use the hypothesis that $|\mathcal{P}|$ is at most the cofinality of $\mathcal{N}_1$. Since $\beta < \kappa \leq |\mathcal{P}| \leq$ the cofinality of $\mathcal{N}_1$, we may take an element v in $\mathcal{N}_1$ that is greater than u_α for all $\alpha < \beta$, and take a prime w in $\mathcal{N}_1$ such that $v \lll w$. Then $u_\beta = wy$ has the required properties.

Let f be the unique function from $\mathcal{P}$ into $\mathcal{N}_1$ such that:

- $f(\emptyset) = 1$.
- For each bounded $Y \in \mathcal{P}$,
$$f(Y) = \Pi\{p_n : n \in Y\} \in \mathcal{N}.$$
- $f(U_\alpha) = u_\alpha$ for each $\alpha < \kappa$.
- Whenever $X, Y \in \mathcal{P}$, $X \cap Y = \emptyset$, and Y is bounded,
$$f(X \cup Y) = f(X)f(Y).$$

Note that whenever $X, Y \in \mathcal{P}$ and $X \Delta Y$ is bounded, $f(X) \sim f(Y)$, and in fact $bf(X) = af(Y)$ where $a = f(X \setminus Y)$ and $b = f(Y \setminus X)$.

Lemma 5.2. *Whenever $X \in \mathcal{P}$ and $n \in \mathcal{N}$, $n \in X$ if and only if $(p_n | f(X))$ in $\mathcal{N}_1$.*

Proof. The result is clear if X is bounded, and also if $X = U_\alpha$ for some α. If X is unbounded, then $X \Delta U_\alpha$ is bounded for some $\alpha < \kappa$. We observe that $bf(X) = af(U_\alpha)$ where a is the product of primes p_n with $n \in X \setminus U_\alpha$ and b is the product of primes p_n with $n \in U_\alpha \setminus X$. The result follows from this observation. □ □

Let

$$Q = \{f(X) : X \in \mathcal{P} \text{ and } X \text{ is unbounded}\}.$$

Then for each $x \in Q$ there exist $a, b \in \mathcal{N}$ and $\alpha < \kappa$ such that $bx = au_\alpha$, and hence $x \sim u_\alpha$.

It will be convenient to have the freedom to subtract elements of $\mathcal{N}_1$ from each other. By $\forall$ Transfer, $\mathcal{N}_1$ satisfies the associative, commutative, and distributive laws for $+$ and $\cdot$. We may therefore we introduce the ordered ring $\mathcal{Z}$ generated by $\mathcal{N}$, and the ordered ring $\mathcal{Z}_1$ generated by $\mathcal{N}_1$, with the vocabulary of L_1 and the additional binary operation $-$. Thus $\mathcal{N}$ is the non-negative part of $\mathcal{Z}$, and $\mathcal{N}_1$ is the non-negative part of $\mathcal{Z}_1$.

Definition 5.3. *We define ${}^*\mathcal{Z}$ to be the substructure of $\mathcal{Z}_1$ generated by $\mathcal{Z} \cup Q$, and ${}^*\mathcal{N}$ to be the non-negative part of ${}^*\mathcal{Z}$.*

Note that ${}^*\mathcal{Z}$ is again an ordered ring. Since $\forall$ Transfer holds in $(\mathcal{N}, \mathcal{P}, \mathcal{N}_1)$ and ${}^*\mathcal{N} \subseteq \mathcal{N}_1$, $\forall$ Transfer holds in $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$. We show that $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$ is a model of ${}^*\mathsf{RCA_0}'$. It is clear that the axioms of BNA hold in $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$. The next several lemmas will be used to prove that STP holds in $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$.

We call a finite subset $\mathcal{P}_0 \subseteq \mathcal{P}$ **neat** if each $X \in \mathcal{P}_0$ is unbounded and for each $X, Y \in \mathcal{P}_0$, if $X \Delta Y$ is bounded then $X = Y$. A finite subset $Q_0 \subseteq Q$ is called neat if $Q_0 = \{f(X) : X \in \mathcal{P}_0\}$ for some neat set $\mathcal{P}_0 \subseteq \mathcal{P}$. We collect some easy observations about neat sets in a lemma.

Lemma 5.4. *(i) For every finite set $Q_1 \subseteq Q$ there is a neat finite set $Q_0 \subseteq Q$ such that*

$$Q_1 \subseteq \{ny : n \in \mathcal{N} \text{ and } y \in Q_0\}.$$

(ii) Suppose Q_0 is neat, $x, y \in Q_0$, and $x < y$. Then $x \lll y$,

(iii) Suppose Q_0 is neat, x, y are finite products of elements of Q_0, and $x < y$. Then $x \ll y$.

Proof. (i) Let $\mathcal{P}_1$ be the finite subset of $\mathcal{P}$ such that f maps $\mathcal{P}_1$ onto Q_1. The relation "$X \Delta Y$ is bounded" partitions $\mathcal{P}_1$ into finitely many equivalence classes $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$. For each i let $X_i = \bigcap \mathcal{Q}_i$. Then the set $Q_0 = \{f(X_1), \cdots f(X_k)\}$ has the required properties.

(ii) Let $x = f(X), y = f(Y)$. For some $\alpha, \beta < \kappa$, $X \Delta U_\alpha$ and $Y \Delta U_\beta$ are bounded. Therefore $x \sim u_\alpha$ and $y \sim u_\beta$. If $\alpha = \beta$ then $X \Delta Y$ is bounded, and since Q_0 is neat we would have $x = y$, contradicting $x < y$. If $\beta < \alpha$, then $y \sim u_\beta \lll u_\alpha \sim x$, so $y \lll x$, again contradicting $x < y$. We must therefore have $\alpha < \beta$, and by the above argument, $x \lll y$.

(iii) Write x and y as finite products of elements on Q_0 in decreasing order,

$$x = x_0 \cdots x_k, \quad x_0 \geq \cdots \geq x_k, \quad y = y_0 \cdots y_\ell, \quad y_0 \geq \cdots \geq y_\ell.$$

Since $x \neq y$, there must be a least j such that $x_j \neq y_j$ (adding 1's to the end of the shorter product if necessary). Then $x = zx'$ and $y = zy'$ where $z = 1$ if $j = 0$, and $z = x_0 \cdots x_{j-1} = y_0 \cdots y_{j-1}$ if $j > 0$. Hence $x' < y'$. We cannot have $x_j > y_j$, because then by (ii), $y_i \lll x_j$ whenever $j \leq i \leq \ell$, so $y' \lll x_j \leq x'$, contradicting $x' < y'$. Therefore $x_j < y_j$. Using (ii) again, we have $x_i \lll y_j$ for each $i \geq j$, so $x' \lll y_j \leq y'$ and hence $x' \ll y'$. Then for each $r \in \mathcal{N}$, $rx' < y'$, so $rx = zrx' < zy' = y$. Therefore $x \ll y$, as required. □ □

We now give a useful representation for an arbitrary element of ${}^*\mathcal{Z}$.

Definition 5.5. *Let Q_0 be a neat subset of Q and let $x \in \mathcal{Z}_1$. An equation*

$$x = m_0 x_0 + \cdots + m_k x_k$$

*is said to be **neat for** x **over** Q_0 if $k \in \mathbb{N}$ and:*

- *The equation is true.*
- *$m_i \in \mathcal{Z}$ for each $i \leq k$.*

- *Each x_i is a finite product of elements of Q_0 (counting 1 as the empty product).*
- $x_0 < \cdots < x_k$.

Lemma 5.6. *Each element $x \in {}^*\mathcal{Z}$ has a neat equation.*

Proof. We show that the set of $x \in \mathcal{Z}_1$ that have neat equations contains $\mathcal{Z} \cup Q$ and is closed under $+, -$, and $\cdot$. If $m \in \mathcal{Z}$, then $m = m \cdot 1$ itself is a neat equation for m with $k = 0$. If $x \in Q$, then $x = 0 \cdot 1 + 1 \cdot x$ is a neat equation for x with $k = 1$. Suppose

$$x = m_0x_0 + \cdots + m_kx_k, \quad y = n_0y_0 + \cdots + n_\ell y_\ell$$

are neat equations over neat sets Q_0 and Q_1 respectively. By Lemma 5.4 (i), we can assume without loss of generality that the union $Q_0 \cup Q_1$ is neat. Since ${}^*\mathcal{Z}$ is an ordered ring, we may collect terms in the usual way to obtain neat equations for $x + y$, $x - y$, and $x \cdot y$ over $Q_0 \cup Q_1$. □ □

The next lemma shows that the set of non-zero values of m_ix_i is unique in a neat equation for an element $x \in {}^*\mathcal{Z}$.

Lemma 5.7. *Suppose $x \in {}^*\mathcal{Z}$ and*

$$x = m_0x_0 + \cdots + m_kx_k, \quad x = m'_0x'_0 + \cdots + m'_\ell x'_\ell$$

are two neat equations for x. Then

$$\{m_ix_i : i \le k\} \cup \{0\} = \{m'_jx'_j : j \le \ell\} \cup \{0\}.$$

Proof. The result is trivial if $x = 0$. Assume $x \neq 0$. By removing zero terms, we may assume that $m_i \neq 0$ for each $i \le k$, and $m'_j \neq 0$ for each $j \le \ell$. We argue by induction on k, and prove that $\ell = k$ and $m_ix_i = m'_ix'_i$ for each $i \le k$.

We assume the result holds for all $k' < k$ and prove it for k. We first prove that $m_kx_k = m'_\ell x'_\ell$. We have $m_k \neq 0$ and $m'_\ell \neq 0$. By Lemma 5.4 (iii), for each $i < k$ we have $x_i \ll x_k$. Let $y = x - m_kx_k$ and $y' = x - m'_\ell x'_\ell$. If $k = 0$ then $y = 0$. If $k > 0$ then

$$y = m_0x_0 + \ldots + m_{k-1}x_{k-1}$$

is a sum of elements u such that $|u| \ll x_k$. Therefore $|y| \ll x_k$, and

$$|x| = |y + m_kx_k| \le |y| + |m_k|x_k < x_k + |m_k|x_k = (1 + |m_k|)x_k.$$

We also have

$$x_k \le |m_k|x_k = |x - y| \le |x| + |y| \le 2|x|.$$

The analogous results also holds for x'_ℓ,

$$|x| < (1 + |m'_\ell|)x'_\ell, \quad x'_\ell < 2|x|.$$

It follows that

$$x'_\ell \le 2|x| < (1 + |m_k|)x_k, \quad x_k \le 2|x| < (1 + |m'_\ell|)x'_\ell,$$

so $x_k \sim x'_\ell$. Using Lemma 5.4, we can find a neat subset Q_0 of Q, a finite product z of elements of Q_0, and elements $a, b \in \mathcal{N}$ such that $x_k = az$ and $x'_\ell = bz$. It follows that

$$|m_k x_k - m'_\ell x'_\ell| = |(x - y) - (x - y')| = |y - y'| \ll x_k.$$

Then $|(m_k a - m'_\ell b)z| \ll az$, so we must have $m_k a = m'_\ell b$. This proves that

$$m_k x_k = m'_\ell x'_\ell.$$

If $k = 0$ we are done. Suppose that $k > 0$. Then $y = y'$, and we have two neat equations for y. The desired conclusion now follows from the induction hypothesis. This completes the proof. □ □

We will use the above lemma to characterize the divisibility relation $(m|x)$ where $m \in \mathcal{N}$ and $x \in {}^*\mathcal{N}$. It is clear that if m divides x in ${}^*\mathcal{N}$, then m divides x in $\mathcal{N}_1$. That is, if ${}^*\mathcal{N} \models \exists z\, mz = x$ then $\mathcal{N}_1 \models \exists z\, mz = x$. However, the converse is false. For example, if $f(X) \in Q$ and $2 \notin X$, then 2 divides $1 + f(X)$ in the sense of $\mathcal{N}_1$ but Lemma 5.11 below shows that 2 does not divide $1 + f(X)$ in the sense of ${}^*\mathcal{N}$.

Note that for $n, m \in \mathcal{N}$, n divides m in ${}^*\mathcal{N}$ if and only if n divides m in $\mathcal{N}_1$, and also if and only if n divides m in $\mathcal{N}$.

From now on, the expression $(y|x)$ will be used in the sense of ${}^*\mathcal{N}$, so that $(y|x)$ means ${}^*\mathcal{N} \models \exists z\, yz = x$. When x and y belong to ${}^*\mathcal{Z}$, we will use $(y|x)$ to mean that $|y|$ divides $|x|$ in ${}^*\mathcal{N}$.

It is clear that $(0|x)$ if and only if $x = 0$, and that $(m|0)$ for all m. This observation reduces the question of whether $(m|x)$ to the case that $m > 0$ and $x > 0$.

The next four lemmas together will give a criterion for $(m|x)$ when $m \in \mathcal{N}$ and $x \in {}^*\mathcal{N}$.

Lemma 5.8. *Suppose $q \in \mathcal{N}$, $X \in \mathcal{P}$, and $x = f(X)$. Then $(p_q|x)$ if and only if $q \in X$.*

Proof. If $(p_q|x)$, then $q \in X$ by Lemma 5.2. If $q \in X$, then $Y = X \setminus \{q\}$ belongs to $\mathcal{P}$, and $x = f(\{q\})f(Y) = p_q f(Y)$, so $(p_q|x)$. □ □

For $q, n \in \mathcal{N}$ and $0 < n$ let $(n)_q$ be the largest m such that $(p_q)^m$ divides n.

Lemma 5.9. *Suppose $r \in \mathcal{N}$ and $y_i = f(Y_i) \in Q$ for each $i \le k$. Let $y = y_0 \cdots y_k$. Then $(r|y)$ if and only if $0 < r$ and*

$$(\forall q < r)(r)_q \le |\{i \le k : q \in Y_i\}|.$$

Proof. Assume $0 < r$. For each $i \le k$ let $U_i = \{q \in Y_i : (p_q|r)\}$ and $Z_i = Y_i \setminus U_i$. U_i is bounded and $U_i, Z_i \in \mathcal{P}$ by Δ^0_1 Comprehension. Let $n_i = f(U_i) \in \mathcal{N}$, and let $z_i = f(Z_i)$. Let $n = n_0 \cdots n_k$ and $z = z_0 \cdots z_k$. Note that $n_i \le r$, so $n < r^{k+1} + 1$. We have $y_i = n_i z_i$ and thus $y = nz$. Since $U_i \cap Z_i$ is empty for each i, z is relatively prime to r in $\mathcal{N}_1$. Therefore $(r|y)$ if and only if $(r|n)$, which in turn holds if and only if $(r)_q \le (n)_q$ for each q. By Lemma 5.8, for each q we have $(n)_q = |\{i \le k : q \in Y_i\}|$ if $(p_q|r)$, and $(n)_q = 0$ otherwise. This proves the lemma. □ □

Lemma 5.10. *Suppose $m, r \in \mathcal{N}$ and $y_i = f(Y_i) \in Q$ for each $i \le k$. Let $y = y_0 \cdots y_k$. Then $(r|my)$ if and only if either $m = 0$, or $0 < r$ and there exists $n < r^{k+1} + 1$ such that $(r|mn)$ and $(n|y)$.*

Proof. Assume $0 < m$ and $0 < r$. It is clear that $(r|mn)$ and $(n|y)$ implies $(r|my)$. Suppose $(r|my)$. Let n and z be as in the proof of Lemma 5.9. Then $n < r^{k+1} + 1$, and $y = nz$, so $(n|y)$. Moreover, z is relatively prime to r in $\mathcal{N}_1$ and $(r|mnz)$, so $(r|mn)$. □ □

Lemma 5.11. *Let $x \in {}^*\mathcal{Z}$ and let*

$$x = m_0x_0 + \cdots + m_kx_k$$

be a neat equation for x. If $r \in \mathcal{N}$, then $(r|x)$ if and only if $(r|m_ix_i)$ for each $i \le k$.

Proof. We prove the nontrivial direction. Suppose $(r|x)$, and take $z \in {}^*\mathcal{Z}$ such that $rz = x$. z has a neat equation

$$z = n_0z_0 + \cdots + n_\ell z_\ell.$$

Then

$$rz = rn_0z_0 + \cdots + rn_\ell z_\ell.$$

is a neat equation for rz. By Lemma 5.7, $k = \ell$, and $rn_iz_i = m_ix_i$ for each $i \le k$, and the result follows. □ □

We now work in $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$ and prove the axioms of ${}^*\mathsf{RCA_0}'$. We have already shown that the axioms of BNA hold.

Lemma 5.12. *The STP holds in $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$.*

Proof. Lemma 5.8 shows that $X = st(f(X))$ for every $X \in \mathcal{P}$. This proves the upward STP.

For the Downward STP, we must show that for each $x \in {}^*\mathcal{N}$ the set $st(x) = \{q \in \mathcal{N} : (p_q|x)\}$ belongs to $\mathcal{P}$. For $x \in {}^*\mathcal{Z}$ we write $st(x) = st(|x|)$. Let

$$x = m_0x_0 + \cdots + m_kx_k$$

be a neat equation for x. By Lemma 5.11,

$$st(x) = st(m_0x_0) \cap \cdots \cap st(x_km_k).$$

For each i, $st(m_i) \in \mathcal{P}$ by Δ^0_1 Comprehension in $(\mathcal{N}, \mathcal{P})$. Fix a positive $i \le k$ and let $x_i = y_0 \cdots y_\ell$ where each $y_j \in Q$. Then $y_j = f(Y_j)$ for some $Y_j \in \mathcal{P}$. By Lemmas 5.9 and 5.10, $(p_q|m_ix_i)$ if and only if either $(p_q|m_i)$ or $q \in Y_j$ for some $j \le \ell$. Therefore

$$st(m_ix_i) = st(m_i) \cup st(y_0) \cup \cdots \cup st(y_\ell).$$

We have $st(m_i) \in \mathcal{P}$ and $st(y_j) \in \mathcal{P}$ for each $j \le \ell$. Since $\mathcal{P}$ is closed under finite unions and finite intersections, it follows that $st(x) \in \mathcal{P}$, and the Downward STP is proved. □ □

For Theorem 5.1, it remains to prove Special Σ_1^S Induction and Special Δ_1^S Comprehension. To prepare for this we prove two more lemmas. The next lemma says that each term of sort $^*\mathbf{N}$ with constants from $^*\mathcal{N}$ and variables $\vec{m}$ can be represented as one of a finite set of "neat polynomials".

Let Q_0 be a neat subset of Q. By a **neat polynomial over** Q_0 we mean an expression $P(\vec{m}, \vec{d})$ of the form

$$P_0(\vec{m}, \vec{d})z_0 + \cdots + P_h(\vec{m}, \vec{d})z_h,$$

where $h \in \mathbb{N}$, $\vec{d}$ is a tuple of constants from $\mathcal{N}$, each $P_i(\vec{m}, \vec{d})$ is a polynomial in $\vec{m}$ with coefficients in $\mathcal{Z}$, each z_i is a finite product of elements of Q_0, and $z_0 < \ldots < z_h$. For readability, we will suppress the parameters $\vec{d}$, writing $P(\vec{m})$ instead of $P(\vec{m}, \vec{d})$.

Recall that by Lemmas 5.4 and 5.6, for each tuple $\vec{x}$ of elements of $^*\mathcal{N}$ there is a neat set Q_0 such that each member of $\vec{x}$ has a neat equation over Q_0.

Lemma 5.13. *Let $\vec{x}$ be a tuple of constants from $^*\mathcal{N}$, $\vec{m}$ be a tuple of variables of sort $\mathbf{N}$, and $t(\vec{m}, \vec{x})$ be a term in $^*L_1$. Let Q_0 be a neat set such that each member of $\vec{x}$ has a neat equation over Q_0. Then there is a finite sequence $P^{(0)}(\vec{m}), \ldots, P^{(k)}(\vec{m})$ of neat polynomials over Q_0, and a finite sequence $\psi_0(\vec{m}), \ldots, \psi_k(\vec{m})$ of quantifier-free formulas of L_1 with constants from $\mathcal{N}$, such that*

$$\mathcal{N} \models \forall \vec{m}[\psi_0(\vec{m}) \vee \cdots \vee \psi_k(\vec{m})]$$

and for each $i \leq k$,

$$(\mathcal{N}, {^*\mathcal{N}}) \models \forall \vec{m}[\psi_i(\vec{m}) \rightarrow t(\vec{m}, \vec{x}) = P^{(i)}(\vec{m})].$$

Proof. We argue by induction on the complexity of $t(\vec{m}, \vec{x})$. If $t(\vec{m}, \vec{x})$ is a single variable m of sort $\mathbf{N}$, the result holds with $P^{(0)} = m$ and ψ_0 being the true formula. If $t(\vec{m}, \vec{x})$ is a single constant $x \in {^*\mathcal{N}}$, the result holds with $P^{(0)}$ being a neat equation for x over Q_0. Assume the result holds for a term $t(\vec{m}, \vec{x})$ with the neat polynomials and formulas

$$P^{(0)}(\vec{m}), \ldots, P^{(k)}(\vec{m}), \quad \psi_0(\vec{m}), \ldots, \psi_k(\vec{m}),$$

and also holds for a term $u(\vec{m}, \vec{x})$ with the neat polynomials and formulas

$$R^{(0)}(\vec{m}), \ldots, R^{(\ell)}(\vec{m}), \quad \theta_0(\vec{m}), \ldots, \theta_\ell(\vec{m}).$$

Then the lemma holds for the sum $t(\vec{m}, \vec{x}) + u(\vec{m}, \vec{x})$ with the neat polynomials and quantifier-free formulas

$$P^{(i)} + R^{(j)}, \quad \psi_i \wedge \theta_j, \quad i \leq k \text{ and } j \leq \ell.$$

Similarly, the lemma holds for the product $t(\vec{m}, \vec{x}) \cdot u(\vec{m}, \vec{x})$ with the neat polynomials and quantifier-free formulas

$$P^{(i)} \cdot R^{(j)}, \quad \psi_i \wedge \theta_j, \quad i \leq k \text{ and } j \leq \ell.$$

To deal with the cutoff difference of two terms, we need a quantifier-free formula that expresses the property that the value of one neat polynomial is greater than the value of another. By adding terms with zero coefficients, each pair of neat polynomials $P^{(i)}(\vec{m}), R^{(j)}(\vec{m})$ over Q_0 can be put in the form

$$P_0(\vec{m},\vec{c})z_0 + \cdots + P_h(\vec{m},\vec{c})z_h,$$
$$R_0(\vec{m},\vec{c})z_0 + \cdots + R_h(\vec{m},\vec{c})z_h,$$

with the same sequence $z_0, \ldots, z_h$ of finite products of elements of Q_0. There is a quantifier-free formula $\varphi_{i,j}(\vec{m})$ with parameters in $\mathcal{N}$ that states that for some $a \le h$, $P_a(\vec{m},\vec{c}) > R_a(\vec{m},\vec{c})$, and $P_b(\vec{m},\vec{c}) = R_b(\vec{m},\vec{c})$ whenever $a < b \le h$. We have $z_0 < \ldots < z_h$, and by Lemma 5.4, $z_0 \ll \ldots \ll z_h$. It follows that for all $\vec{m}$ in $\mathcal{N}$, $\varphi_{i,j}(\vec{m})$ holds if and only if $P^{(i)}(\vec{m}) > R^{(j)}(\vec{m})$. Therefore the lemma holds for the cutoff difference $t(\vec{m},\vec{x}) \dot{-} u(\vec{m},\vec{x})$ with the sequence of neat polynomials

$$P^{(i)} - R^{(j)}, \quad i \le k \text{ and } j \le \ell$$

followed by the zero polynomial, and the sequence of quantifier-free formulas

$$\psi_i \wedge \theta_j \wedge \varphi_{i,j}, \quad i \le k \text{ and } j \le \ell$$

followed by the "otherwise" formula

$$\neg \bigvee_{i=0}^{k} \bigvee_{j=0}^{\ell} \psi_i \wedge \theta_j \wedge \varphi_{i,j}.$$

□ □

The next lemma reduces a special Δ_0^S formula with constants from ${}^*\mathcal{N}$ and variables of sort $\mathbf{N}$ to a Δ_0^0 formula in L_2 with constants from $\mathcal{N}$ and $\mathcal{P}$ and variables of sort $\mathbf{N}$.

Lemma 5.14. *Let $\vec{x}$ be a tuple of constants from ${}^*\mathcal{N}$, and $\vec{m}$ be a tuple of variables of sort $\mathbf{N}$. For each special Δ_0^S formula $\varphi(\vec{m},\vec{x})$ there is a tuple $\vec{d}$ of constants from $\mathcal{N}$, a tuple $\vec{Y}$ of sets in $\mathcal{P}$, and a Δ_0^0 formula $\widehat{\varphi}(\vec{m},\vec{d},\vec{Y})$ in L_2 such that in $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$,*

$$\forall \vec{m}\, [\varphi(\vec{m},\vec{x}) \leftrightarrow \widehat{\varphi}(\vec{m},\vec{d},\vec{Y})].$$

Proof. By Lemma 5.4, there is a neat set Q_0 such that each member of $\vec{x}$ has a neat equation over Q_0. Let $\mathcal{P}_0 = f^{-1}(Q_0)$ and let $\vec{Y}$ be a tuple of sets that enumerates $\mathcal{P}_0$.

Let $t(\vec{m},\vec{x})$ be a term of sort ${}^*\mathbf{N}$ in ${}^*L_1$. Let

$$P^{(0)}(\vec{m}), \ldots, P^{(k)}(\vec{m}), \quad \psi_0(\vec{m}), \ldots, \psi_k(\vec{m})$$

be as in Lemma 5.13, and let $\vec{d}$ be the tuple of constants from ${}^*\mathcal{N}$ that occur in these polynomials and formulas. Let

$$P^{(\ell)}(\vec{m}) = P_0^{(\ell)}(\vec{m},\vec{d})z_0 + \cdots + P_{h_\ell}^{(\ell)}(\vec{m},\vec{d})z_{h_\ell}.$$

We first prove the lemma for atomic formulas of the form $0 < t(\vec{m}, \vec{x})$. Let $\widehat{\varphi}(\vec{m}, \vec{d}, \vec{Y})$ be the quantifier-free formula that says that for each $\ell \le k$, if $\psi_\ell(\vec{m})$ then there is an $i \le h$ such that $P_i^{(\ell)}(\vec{m}, \vec{d}) > 0$ and $P_j^{(\ell)}(\vec{m}, \vec{d}) = 0$ whenever $i < j \le h_\ell$. Then the lemma holds when φ is $0 < t(\vec{m}, \vec{x})$, with the formula $\widehat{\varphi}(\vec{m}, \vec{d}, \vec{Y})$. Note that in this case, $\vec{Y}$ does not occur at all in the formula $\widehat{\varphi}(\vec{m}, \vec{d}, \vec{Y})$.

Using the facts that $s < t$ if and only if $0 < t \dot{-} s$, and $s = t$ if and only if $\neg(s < t) \wedge \neg(t < s)$, we see that the lemma holds for all atomic formulas of the forms $s < t$ and $s = t$.

We next deal with the formulas of the form $(r|t(\vec{m}, \vec{x}))$. We may assume that r belongs to the tuple of variables $\vec{m}$. Fix an assignment $\vec{a}$ for $\vec{m}$ in $\mathcal{N}$. Let b be the resulting assignment for r. In the case that $t(\vec{a}, \vec{x}) = 0$, the formula $(b|t(\vec{a}, \vec{x}))$ is true. In the case that $b = 0$ and $t(\vec{a}, \vec{x}) \neq 0$, the formula $(b|t(\vec{a}, \vec{x}))$ is false. Suppose that $b \neq 0$ and $t(\vec{a}, \vec{x}) \neq 0$. By Lemma 5.13, there is an $\ell \le k$ such that $\psi_\ell(\vec{a}, \vec{d})$ holds. Then $t(\vec{a}, \vec{x}) = P^{(\ell)}(\vec{a})$. For each $i \le h_\ell$, let $t_i = P_i^{(\ell)}(\vec{a}, \vec{d})$. We have $t_i \in \mathcal{Z}$. Since Q_0 is neat, we have a neat equation

$$t(\vec{a}, \vec{x}) = t_0 z_0 + \cdots + t_h z_h$$

over Q_0.

For each $i \le h_\ell$, z_i is a finite product $z_i = z_{i,0} \cdots z_{i,k_i}$ of elements of Q_0, and for each $j \le k_i$, $z_{i,j} = f(Z_{i,j})$ for some $Z_{i,j}$ in the sequence $\vec{Y}$. Applying Lemma 5.11, we see that $(b|t(\vec{a}, \vec{x}))$ if and only if $(b|P_i^{(\ell)}(\vec{a}, \vec{d}) z_i)$ for each $i \le h_\ell$. Fix an $i \le h_\ell$. By Lemma 5.10, we have $(b|P_i^{(\ell)}(\vec{a}, \vec{d}) z_i)$ if and only if either $P_i^{(\ell)}(\vec{a}, \vec{d}) = 0$, or there exists $n < (c^{k_i+1}) + 1$ such that $(b|nP_i^{(\ell)}(\vec{a}, \vec{d}))$ and $(n|z_i)$. By Lemma 5.9, we have $(n|z_i)$ if and only if

$$(\forall q < n)(n)_q \le |\{j \le k_i : q \in Z_{i,j}\}|.$$

This shows that $(r|t(\vec{m}, \vec{x}))$ is expressible by a Δ_0^0 formula $\widehat{\varphi}(\vec{m}, \vec{d}, \vec{Y})$ in L_2, so the lemma is proved for the case that $\varphi(\vec{m}, \vec{c}, \vec{x})$ is of the form $(r|t(\vec{m}, \vec{x}))$.

The lemma for an arbitrary special Δ_0^S formula $\varphi(\vec{m}, \vec{c}, \vec{x})$ now follows by a straightforward induction on the complexity of φ. □ □

Lemma 5.15. *Special Σ_1^S Induction holds in $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$.*

Proof. Let $\varphi(\vec{n}, \vec{x})$ be a special Σ_1^S formula where $\vec{x}$ is a tuple of constants from ${}^*\mathcal{N}$. Then $\varphi(\vec{n}, \vec{x})$ is $\exists m\, \psi(m, \vec{n}, \vec{x})$ where ψ is a special Δ_0^S formula. By Lemma 5.14 there is a tuple $\vec{d}$ of constants from $\mathcal{N}$, a tuple $\vec{Y}$ of sets in $\mathcal{P}$, and a Δ_0^0 formula $\widehat{\psi}(m, \vec{n}, \vec{d}, \vec{Y})$ in L_2 such that

$$\forall m \forall \vec{n}\, [\psi(m, \vec{n}, \vec{x}) \leftrightarrow \widehat{\psi}(m, \vec{n}, \vec{d}, \vec{Y})].$$

Let $\widehat{\varphi}(\vec{n}, \vec{d}, \vec{Y})$ be the Σ_1^0 formula $\exists m\, \widehat{\psi}(m, \vec{n}, \vec{d}, \vec{Y})$. Then

$$\forall \vec{n}\, [\varphi(\vec{n}, \vec{x}) \leftrightarrow \widehat{\varphi}(\vec{n}, \vec{d}, \vec{Y})].$$

Thus Special Σ_1^S Induction for $\varphi(\vec{n},\vec{x})$ follows from Σ_1^0 Induction for $\widehat{\varphi}(\vec{n},\vec{d},\vec{Y})$ in $(\mathcal{N},\mathcal{P})$. □ □

Lemma 5.16. *Special Δ_1^S Comprehension holds in $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$.*

Proof. This is proved by an argument similar to the preceding lemma, using Δ_1^0 Comprehension in $(\mathcal{N},\mathcal{P})$ and the upward STP. □ □

It now follows from Lemmas 5.12, 5.15 and 5.16 that $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$ is a model of ${}^*\mathsf{RCA_0}'$, so Theorem 5.1 is proved.

6. Open Questions and Complementary Results

6.1. **Open Questions.** A general question is: How much one can strengthen ${}^*\mathsf{RCA_0}'$ and still be conservative with respect to RCA_0? Here are some natural cases.

Question 6.1. *If one strengthens ${}^*\mathsf{RCA_0}'$ or ${}^*\mathsf{RCA_0}' + \forall\mathsf{T}$ by adding Σ_1^S Induction, is the resulting theory still conservative with respect to RCA_0?*

Question 6.2. *If one strengthens ${}^*\mathsf{RCA_0}'$ by adding Transfer for universal formulas (rather than sentences), is the resulting theory still conservative with respect to RCA_0?*

The above two theories do not imply WKL_0. To see this, let $(\mathcal{N},\mathcal{P})$ be a model of RCA_0 plus the negation of the Weak Koenig Lemma whose first order part is $\mathcal{N}=\mathbb{N}$. An example of such a model is the minimal model where $\mathcal{P}$ is the set of recursive subsets of $\mathbb{N}$ (see [4], Section VIII.1). By the compactness theorem, $\mathcal{N}$ has an elementary extension $\mathcal{N}_1$ of cofinality at least $|\mathcal{P}|$ such that $(\mathcal{N},\mathcal{P},\mathcal{N}_1)$ satisfies the Upward STP. Since $\mathcal{N}=\mathbb{N}$, $(\mathcal{N},\mathcal{P},\mathcal{N}_1)$ is also a model of ${}^*\Sigma\mathsf{PA}+\forall\mathsf{T}$. Theorem 5.1 gives us a substructure ${}^*\mathcal{N}$ of $\mathcal{N}_1$ such that $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$ is a model of ${}^*\mathsf{RCA_0}' + \forall\mathsf{T}$. Using $\mathcal{N}=\mathbb{N}$, it is easily seen that $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$ also satisfies Σ_1^S Induction and Transfer for universal formulas.

Question 6.3. *If one strengthens ${}^*\mathsf{RCA_0}' + \forall\mathsf{T}$ by adding a symbol for exponentiation to the vocabulary, is the resulting theory still conservative with respect to RCA_0?*

Our results in this paper depend on the particular way we code sets of natural numbers by hyperintegers, via prime divisors. Another general question is

Question 6.4. *What are the nonstandard counterparts of RCA_0 when one uses a different method of coding sets of natural numbers by hyperintegers?*

6.2. **Coding Real Numbers by Hyperrational Numbers.** In this subsection we consider a question related to Question 6.4, concerning the representation of real numbers as shadows of hyperrational numbers. Following [4], in RCA_0 the **rational numbers** are introduced in the usual way as quotients of integers, and a **real number** is defined as a sequence $\langle q_n\rangle$ of rational numbers such that $|q_k - q_n| \leq 2^{-k}$ whenever $k < n \in \mathcal{N}$, and two real numbers $\langle q_n \rangle, \langle r_n \rangle$ are defined to be equal if $(\forall n)|q_n - r_n| \leq 2^{1-n}$. In $^*\mathsf{RCA_0}' + \forall\mathsf{T}$, the **hyperrational numbers** are introduced in the usual way as quotients of hyperintegers. Both the real numbers and the hyperrational numbers are ordered fields which contain the rational numbers. A hyperrational number x/y is **finite** if $\exists n\, |x/y| < n$.

Definition 6.5. *In* $^*\mathsf{RCA_0}' + \forall\mathsf{T}$*, a real number* r *is a* ***shadow*** *of a hyperrational number* x/y *if for all rational numbers* q*,*

$$q < r \Rightarrow q \leq x \text{ and } q < x \Rightarrow q \leq r.$$

By the ***Upward Shadow Principle*** *we mean the statement that every real number is the shadow of some hyperrational number.*

By the ***Downward Shadow Principle*** *we mean the statement that every finite hyperrational number has a shadow.*

We will see below that the Downward Shadow Principle is provable in $^*\mathsf{RCA_0}' + \forall\mathsf{T}$. Our question concerns the Upward Shadow Principle.

Question 6.6. *Is the theory*

$$^*\mathsf{RCA_0}' + \forall\mathsf{T} + \textit{Upward Shadow Principle}$$

conservative with respect to $\mathsf{RCA_0}$*? Does it imply* $\mathsf{WKL_0}$*?*

It is obvious that in $^*\mathsf{RCA_0}' + \forall\mathsf{T}$, every hyperrational number has at most one shadow (up to equality).

Proposition 6.7. *The Downward Shadow Principle is provable in* $^*\mathsf{RCA_0}' + \forall\mathsf{T}$*.*

Proof. Work in $^*\mathsf{RCA_0}' + \forall\mathsf{T}$. Let x/y be a finite hyperrational number. By Special Δ^S_1-comprehension, there exists z such that

$$st(z) = \{(n,k) : (k/2^n) \leq (x/y) < ((k+1)/2^n)\}.$$

By the Downward STP, there is a set Z such that $Z = st(z)$. For each n let $q_n = k/2^n$ where k is the unique number such that $(n,k) \in Z$. By Theorem 3.4, Δ^0_1-Comprehension holds. By Δ^0_1-Comprehension, the sequence $\langle q_n \rangle$ exists. It is easily seen that whenever $n < m$, $q_n \leq q_m \leq (x/y) < q_n + 2^{-n}$, so $\langle q_n \rangle$ is a real number. It is clear that $\langle q_n \rangle$ is a shadow of (x/y). □ □

Proposition 6.8. *The Upward Shadow Principle is provable in* $^*\mathsf{WKL_0}$.

Proof. Work in $^*\mathsf{WKL_0}$. It follows from Internal Induction that the hyperrational numbers form an ordered field. Let $\langle q_n \rangle$ be a real number. We may assume that $\langle q_n \rangle$ is positive. By STP, there exists u such that $st(u) = \langle q_n \rangle$. Let z be a positive infinite hyperinteger. Then

$$(\forall n)(\exists x < z)(\exists y < z)(\forall m < n)(q_m \leq (x/y) < q_m + 2^{-m}),$$

and the inner part

$$(\exists x < z)(\exists y < z)(\forall m < n)(q_m \leq (x/y) < q_m + 2^{-m})$$

is expressible as a Δ_0^S formula $\theta(n,u,z)$. By Overspill, there is an infinite v such that $\theta(v,u,z)$. Therefore

$$(\exists x < z)(\exists y < z)(\forall m)(q_m \le (x/y) < q_m + 2^{-m}).$$

It follows that $\langle q_n \rangle$ is a shadow of (x/y). □ □

Proposition 6.9. *The theory*

$$^*\mathsf{RCA_0}' + \forall\mathsf{T} + \text{(Every shadow is rational)}$$

is conservative with respect to RCA_0. *Hence the Upward Shadow Principle is not provable in* $^*\mathsf{RCA_0}' + \forall\mathsf{T}$.

Proof. It is enough to show that in the model of $^*\mathsf{RCA_0}' + \forall\mathsf{T}$ constructed in the proof of Theorem 3.5, the shadow of each finite hyperrational number (x/y) is rational. By Lemmas 5.4 and 5.6, x and y have neat equations

$$x = m_0x_0 + \cdots + m_kx_k, \quad y = n_0 + \cdots + n_\ell y_\ell$$

over the same neat set Q_0. If $x \ll y$, then the shadow of (x,y) is zero. Suppose not $x \ll y$. We cannot have $y \ll x$, because (x/y) is finite. Therefore $x \sim y$, and $x_k \sim y_\ell$. Since x_k and y_ℓ are finite products of elements of Q_0, we must have $x_k = y_\ell$. We say that a hyperrational number x/y is **infinitesimal** if $|x| \ll |y|$. One can now show that there are infinitesimal hyperrational numbers ε, δ such that

$$x = (m_k + \varepsilon)x_k, \quad y = (n_\ell + \delta)x_k,$$

and hence that $|(x/y) - (m_k/n_\ell)|$ is infinitesimal, so (m_k/n_ℓ) is the shadow of (x/y). □ □

6.3. **Theories that Imply** WKL_0**.** In this subsection we will show that several theories that appear to be only slightly stronger than $^*\mathsf{RCA_0}'$ actually imply WKL_0. Let T_0 be the theory

$$T_0 = \mathsf{RCA}_0 + \mathsf{BNA} + \mathsf{STP}.$$

We shall give some rather weak statements U in the language $^*L_1$ such that

$$T_0 + U \text{ implies } \mathsf{WKL}_0.$$

For any such statement U, it follows from Theorem 3.4 that $^*\mathsf{RCA_0}' + U$ implies WKL_0, and thus $^*\mathsf{RCA_0}' + U$ cannot be conservative with respect to RCA_0.

A key idea in these results will be to keep track of the Overspill scheme. Recall from [3] that **Overspill** is the set of formulas

$$\forall n\, \varphi(n,\vec{y}) \rightarrow \exists x\, [\neg S(x) \wedge \varphi(x,\vec{y})],$$

where $\varphi(x,\vec{y})$ is a Δ_0^S formula of $^*L_1$.

It is sometimes helpful to interpret Overspill as a statement about the undefinability of $S(x)$. In a model $(\mathcal{N}, {}^*\mathcal{N})$ of BNA, we say that $S(x)$ is **definable** by a Δ_0^S formula $\varphi(x,\vec{y})$ if $\exists \vec{y} \forall x [S(x) \leftrightarrow \varphi(x,\vec{y})]$.

Remark 6.10. In a model of BNA, Overspill holds if and only if $S(x)$ is not definable by a Δ_0^S formula.

Proof. Let $\varphi(x, \vec{y})$ be a Δ_0^S formula. Then the following are equivalent in BNA:
Overspill holds for $\varphi(x, \vec{y})$.
$\forall \vec{y}[\forall n\, \varphi(n, \vec{y}) \rightarrow \exists x\, [\neg S(x) \wedge \varphi(x, \vec{y})]]$.
$\neg \exists \vec{y}[\forall n \varphi(n, \vec{y}) \wedge \forall x[\varphi(x, \vec{y}) \rightarrow S(x)]]$.
$\neg \exists \vec{y} \forall x[\varphi(x, \vec{y}) \leftrightarrow S(x)]$.
$S(x)$ is not definable by $\varphi(x, \vec{y})$. □ □

The following result shows that $^*\mathsf{RCA_0}'$+Transfer for Π_1^0 sentences implies $\mathsf{WKL_0}$. This can be compared with Theorem 3.5 and the discussion after Question 6.2, which give other forms of Transfer that do not imply $\mathsf{WKL_0}$ in $^*\mathsf{RCA_0}'$.

Proposition 6.11. *In the theory T_0, each scheme in the following list implies the next.*

(1) *Transfer for Π_1^0 sentences*
(2) *Internal Induction*
(3) *Overspill*
(4) $\mathsf{WKL_0}$

Proof. We work in T_0. First assume Transfer for Π_1^0 sentences. Let $\varphi(y, \vec{u})$ be a Δ_0^S formula, and assume that

$$\varphi(0, \vec{u}) \wedge \forall y[\varphi(y, \vec{u}) \rightarrow \varphi(y+1, \vec{u})].$$

Then

$$\forall x[\varphi(0, \vec{u}) \wedge (\forall y < x)[\varphi(y, \vec{u}) \rightarrow \varphi(y+1, \vec{u})]].$$

By Σ_1^0 Induction,

$$\forall \vec{n} \forall m[\varphi(0, \vec{n}) \wedge (\forall q < m)[\varphi(q, \vec{n}) \rightarrow \varphi(q+1, \vec{n})] \rightarrow (\forall q < m)\, \varphi(q, \vec{n})].$$

By Transfer for Π_1^0 sentences,

$$\forall \vec{u} \forall x[\varphi(0, \vec{u}) \wedge (\forall y < x)[\varphi(y, \vec{u}) \rightarrow \varphi(y+1, \vec{u})] \rightarrow (\forall y < x)\, \varphi(y, \vec{u})].$$

Therefore $\forall x\, (\forall y < x)\, \varphi(y, \vec{u})$, and hence $\forall y\, \varphi(y, \vec{u})$, so Internal Induction holds.

The proof of Lemma 3.7 in [3], with minor changes, shows that in BNA, Internal Induction implies Overspill.

The proof of Theorem 5.4 in [3] shows that Overspill implies the Weak Koenig Lemma. □ □

Remark 6.12. *It follows from Theorem 5.7 in* [3] *that* $^*\mathsf{WKL_0}$+*Transfer for first order sentences is conservative with respect to* $\mathsf{WKL_0}$, *so* $^*\mathsf{RCA_0}'$ *plus each of the theories in Proposition 6.11 is conservative with respect to* $\mathsf{WKL_0}$.

By Proposition 6.11, any extension of ${}^*\mathsf{RCA_0}'$ which is conservative with respect to RCA_0 must have models $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$ such that Δ_0^0-Induction fails in ${}^*\mathcal{N}$. The next proposition shows that in the model of ${}^*\mathsf{RCA_0}'$ constructed in the proof of Theorem 3.5, Δ_0^0-Induction fails dramatically in ${}^*\mathcal{N}$.

Proposition 6.13. *In the model $(\mathcal{N},\mathcal{P},{}^*\mathcal{N})$ constructed in Theorem 5.1, $S(x)$ is definable by a Δ_0^S formula $\varphi(x)$ whose only free variable is x, and thus Overspill fails. In particular, $S(x)$ is definable by the Δ_0^S formula*

$$(\forall y<x)[(2|y)\vee(2|y+1)].$$

Proof. The sentence $\forall m[(2|m)\vee(2|m+1)]$ is provable from $I\Sigma_1$ and thus holds in $\mathcal{N}$, Therefore $\forall x[S(x)\rightarrow\varphi(x)]$.

For the other direction, suppose $\neg S(x)$, that is, $x\in{}^*\mathcal{N}$ but $x\notin\mathcal{N}$. By definition, the set U_0 belongs to $\mathcal{P}$ and is unbounded. Then $st(u_0)=U_0$, so $u_0\notin\mathcal{N}$. Let $z=min(x-1,u_0)$. Then $z<x$ and $z\notin\mathcal{N}$. By Lemma 5.6, z must have a neat equation $z=m_0+m_1z_1$ where $m_0\in\mathcal{Z}$, $0<m_1\in\mathcal{N}$, $z_1=u_0/a$ where $a\in\mathcal{N}$, a is a product of distinct primes in $\mathcal{N}$, and a divides u_0. We may assume that z_1 is not divisible by 2, because if it is we can replace a by $2a$ and m_1 by $2m_1$. In $\mathcal{N}$ we may write $m_1=bn_1$ where b is a power of 2 and n_1 is not divisible by 2. Let $y=m_0+n_1z_1$. Then $y<x$, n_1z_1 is not divisible by 2, and $y=m_0+n_1z_1$ is a neat equation. By Lemma 5.11, neither y nor $y+1$ is divisible by 2. Therefore $\neg\varphi(x)$, and the result is proved. □ □

We now look at what happens when a weak comprehension axiom is added to ${}^*\mathsf{RCA_0}'$. We recall some notation from [3]. An S-arithmetical formula is a finite string of quantifiers of sort $\mathbf{N}$ followed by a Δ_0^S formula. Δ_0^S Comprehension (Δ_0^S-CA) is the scheme

$$\exists z\forall m[(p_m|z)\leftrightarrow\varphi(m,\vec{u})] \tag{2}$$

where $\varphi(m,\vec{u})$ is a Δ_0^S formula in which z does not occur. S-ACA is the stronger scheme (2) where $\varphi(m,\vec{u})$ is an S-arithmetical formula. It is shown in [3], Lemma 3.4, that Δ_0^S-CA is provable in ${}^*\Delta\mathsf{PA}$, and hence in ${}^*\mathsf{WKL_0}$. It is shown in [3], Section 7, that the theory ${}^*\mathsf{ACA_0}={}^*\mathsf{WKL_0}+S$-$\mathsf{ACA}$ implies and is conservative with respect to ACA_0.

The next result shows that Theorem 3.5 would fail if we added the Δ_0^S-CA scheme to ${}^*\mathsf{RCA_0}'$.

Proposition 6.14. *Let T_1 be the theory*

$$T_1=T_0+\Delta_0^S\text{-}\mathsf{CA}.$$

(i) Any model of T_1 in which Overspill fails satisfies S-ACA and ACA_0.
(ii) T_1 implies WKL_0.

Proof. It is clear that (i) and Proposition 6.11 implies (ii). To prove (i), we work in T_1 and prove S-ACA. Suppose that some instance of the Overspill scheme fails. By Remark 6.10, $S(x)$ is definable by a Δ_0^S formula $\varphi(x,\vec{y})$. Then for some $\vec{y}$ we have $\forall x[S(x)\leftrightarrow\varphi(x,\vec{y})]$. By the Proper Initial Segment

Axioms, there is an H such than $\neg S(H)$. It follows that each Σ_1^S formula $\exists m\, \psi(m, n, \vec{u})$ with parameters $\vec{u}$ is equivalent to the Δ_0^S formula

$$(\exists x < H)[\psi(x, n, \vec{u}) \wedge \varphi(x, \vec{y})].$$

Call this formula $\theta(n, \vec{u}, H)$. Then by Δ_0^S-CA,

$$\exists z \forall n[(p_n|z) \leftrightarrow \theta(n, \vec{u}, H)],$$

and hence

$$\exists z \forall n[(p_n|z) \leftrightarrow \exists m\, \psi(m, n, \vec{u})].$$

This proves Σ_1^S-CA. By the proof of Proposition 7.4 in [3], $\mathsf{BNA} + \Sigma_1^S$-$\mathsf{CA}$ implies S-ACA. ACA_0 now follows from the proof of Theorem 7.6 in [3]. □ □

Remark 6.15. *By Lemma 3.4 in* [3], ${}^*\mathsf{WKL_0}$ *implies* T_1, *so* ${}^*\mathsf{RCA_0}' + T_1$ *is conservative with respect to* WKL_0.

Proposition 6.14 shows that any model of T_1 either satisfies Overspill or satisfies ACA_0. We note that T_1 does not imply ACA_0, because ${}^*\mathsf{WKL_0}$ implies T_1 but does not imply ACA_0. We will see that T_1 also does not imply Overspill. In fact, Proposition 6.16 will show that a much stronger theory T_2 does not imply Overspill.

We consider some stronger comprehension and induction schemes. Π^*_∞-CA is the scheme

$$\exists x \forall m[(p_m|x) \leftrightarrow \varphi(m, \vec{u})]$$

where $\varphi(m, \vec{u})$ is any formula of ${}^*L_1$ in which x does not occur.

Π^*_∞-IND is the scheme

$$[\varphi(0, \vec{u}) \wedge \forall m[\varphi(m, \vec{u}) \rightarrow \varphi(m+1, \vec{u})]] \rightarrow \forall m\, \varphi(m, \vec{u})$$

where $\varphi(m, \vec{u})$ is any formula of ${}^*L_1$.

Proposition 6.16. *The theory*

$$T_2 = {}^*\mathsf{RCA_0}' + \Pi^*_\infty\text{-}\mathsf{CA} + \Pi^*_\infty\text{-}\mathsf{IND} + \forall\mathsf{T}$$

does not imply Overspill.

Proof. We build a model of T_2 in which Overspill fails. Let $(\mathcal{N}, \mathcal{P})$ be the standard model of second order arithmetic where $\mathcal{N} = \mathbb{N}$ and $\mathcal{P}$ is the power set of $\mathbb{N}$. By the compactness theorem, $\mathcal{N}$ has an elementary extension $\mathcal{N}_1$ of cofinality at least $|\mathcal{P}| = 2^{\aleph_0}$. By Theorem 5.1 and Proposition 6.13, there is a substructure ${}^*\mathcal{N}$ of $\mathcal{N}_1$ such that $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$ is a model of ${}^*\mathsf{RCA_0}' + \forall\mathsf{T}$ and Overspill fails. Since $\mathcal{N} = \mathbb{N}$, $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$ also satisfies the other axioms of T_2. □ □

Our final result shows that Theorem 3.5 would fail if we added a symbol for every primitive recursive function to the vocabulary. In fact, when we do this we get a theory that implies WKL_0.

Let $L_1(\mathsf{PR})$ be the language L_1 with a new function symbol for every primitive recursive function, and similarly for L_2 and ${}^*L_1$. Let $\mathsf{RCA}_0(\mathsf{PR})$

be the theory obtained by adding to RCA_0 the defining equation for each primitive recursive function. It is well-known that $\mathsf{RCA}_0(\mathsf{PR})$ is conservative with respect to RCA_0. Let $\forall\mathsf{T}(\mathsf{PR})$ be Transfer for the set of all universal sentences of $L_1(\mathsf{PR})$.

Proposition 6.17. *The theory* $\mathsf{RCA}_0(\mathsf{PR}) + \forall\mathsf{T}(\mathsf{PR}) + \mathsf{BNA} + \mathsf{STP}$ *implies Overspill and* WKL_0.

Proof. We will give a proof of Overspill that uses $\forall\mathsf{T}(\mathsf{PR})$. This can be contrasted with the proof of Overspill in Lemma 3.7 of [3] using Internal Induction.

By the Proper Initial Segment axioms, it suffices to prove Overspill for Δ_0^S formulas in ${}^*L_1(\mathsf{PR})$ all of whose variables have sort ${}^*\mathbf{N}$. Every such formula is the star of a Δ_0^0 formula of $L_1(\mathsf{PR})$. Let $\varphi(n,\vec{m})$ be a Δ_0^0 formula of $L_1(\mathsf{PR})$. We work in ${}^*\mathsf{RCA_0}'(\mathsf{PR})$ and prove Overspill for the starred formula ${}^*\varphi(y,\vec{x})$.

Every Δ_0^0 formula $\psi(\vec{r})$ of $L_1(\mathsf{PR})$ defines a primitive recursive predicate. So $L_1(\mathsf{PR})$ has a function symbol $\alpha_\psi(\vec{r})$ such that

$$\forall\vec{r}[\psi(\vec{r}) \leftrightarrow \alpha_\psi(\vec{r}) = 0].$$

We show by induction on the complexity of ψ that

$$\forall\vec{z}[{}^*\psi(\vec{z}) \leftrightarrow \alpha_\psi(\vec{z}) = 0]. \tag{3}$$

If ψ is an atomic formula, then (3) follows from $\forall\mathsf{T}(\mathsf{PR})$. If (3) holds for φ and ψ, then it follows from $\forall\mathsf{T}(\mathsf{PR})$ that (3) holds for $\varphi\wedge\psi$, $\varphi\vee\psi$, and $\neg\varphi$. Suppose $\psi(\vec{r})$ is $(\forall n < r_i)\,\varphi(n,\vec{r})$. Then

$$\forall n\forall\vec{r}[\varphi(n,\vec{r}) \leftrightarrow \alpha_\varphi(n,\vec{r}) = 0]$$
$$\forall\vec{r}[\psi(\vec{r}) \leftrightarrow \alpha_\psi(\vec{r}) = 0]$$
$$\forall\vec{r}[\alpha_\psi(\vec{r}) = 0 \leftrightarrow (\forall n < r_i)\alpha_\varphi(n,\vec{r}) = 0].$$

By $\forall\mathsf{T}(\mathsf{PR})$,

$$\forall\vec{z}[\alpha_\psi(\vec{z}) = 0 \rightarrow (\forall y < z_i)\alpha_\varphi(y,\vec{z}) = 0].$$

Let $\beta(\vec{r})$ be the function

$$\beta(\vec{r}) = (\mu n < r_i)\alpha_\varphi(n,\vec{r}) > 0.$$

Then β is primitive recursive, and using $\forall\mathsf{T}(\mathsf{PR})$ again we have

$$\forall\vec{r}[\alpha_\psi(\vec{r}) > 0 \rightarrow [\beta(\vec{r}) < r_i \wedge \alpha_\varphi(\beta(\vec{r}),\vec{r}) > 0]],$$
$$\forall\vec{z}[\alpha_\psi(\vec{z}) > 0 \rightarrow [\beta(\vec{z}) < z_i \wedge \alpha_\varphi(\beta(\vec{z}),\vec{z}) > 0]],$$
$$\forall\vec{z}[\alpha_\psi(\vec{z}) = 0 \leftrightarrow (\forall y < z_i)\alpha_\varphi(y,\vec{z}) = 0].$$

Now suppose (3) holds for $\varphi(n,\vec{r})$, that is,

$$\forall y\forall\vec{z}[{}^*\varphi(y,\vec{z}) \leftrightarrow \alpha_\varphi(y,\vec{z}) = 0].$$

Then the following are equivalent:

$${}^*\psi(\vec{z}), \quad (\forall y < z_i)\,{}^*\varphi(y,\vec{z}), \quad (\forall y < z_i)\alpha_\varphi(y,\vec{z}) = 0, \quad \alpha_\psi(\vec{z}) = 0.$$

This proves that (3) holds for ψ, and completes the induction.

Now let $\varphi(n,\vec{m})$ be a Δ^0_0 formula, and assume that

$$\forall n\, {}^*\varphi(n,\vec{x}).$$

We must prove

$$\exists y[\neg S(y) \wedge {}^*\varphi(y,\vec{x})]. \tag{4}$$

We have $\forall n\, \alpha_\varphi(n,\vec{x}) = 0$. Let $\gamma(m,\vec{r})$ be the primitive recursive function

$$\gamma(m,\vec{r}) = (\mu n < m)\alpha_\varphi(n,\vec{r}) > 0.$$

Then the following universal sentences hold:

$$\forall \vec{r}\, \gamma(0,\vec{r}) = 0,$$

$$\forall m \forall \vec{r} \forall n[[\gamma(m,\vec{r}) = m \wedge \alpha_\varphi(m,\vec{r}) = 0] \rightarrow \gamma(m+1,\vec{r}) = m+1].$$

By $\forall\mathsf{T}(\mathsf{PR})$, the stars of these sentences hold. Therefore by Special Σ^S_1 Induction,

$$\forall m\, \gamma(m,\vec{x}) = m.$$

We note that the following universal sentences hold, and by $\forall\mathsf{T}(\mathsf{PR})$ their stars hold:

$$\forall m \forall \vec{r} \forall n\, [[n < \gamma(m,\vec{r}) \wedge n < m] \rightarrow \alpha_\varphi(n,\vec{r}) = 0],$$

$$\forall m \forall \vec{r} \forall n[\alpha_\varphi(\gamma(m,\vec{r}),\vec{r}) = 0 \rightarrow \gamma(m,\vec{r}) = m].$$

By the proper Initial Segment axioms there exists z such that $\neg S(z)$. Let $u = \gamma(z,\vec{x})$. We cannot have $S(u)$, because then $\alpha_\varphi(u,\vec{x}) = 0$ and $u = z$, contradicting $\neg S(z)$. So $\neg S(u)$. Hence $0 < u$, and there exists $y = u - 1$. We have $\neg S(y)$. Since $y < u = \gamma(z,\vec{x})$, we have $\alpha_\varphi(y,\vec{x}) = 0$. Then by (3), ${}^*\varphi(y,\vec{x})$. This shows that (4) holds, and proves Overspill.

WKL_0 now follows by Proposition 6.11. □ □

Remark 6.18. The proof of Theorem 5.7 in [3] goes through when symbols for the primitive recursive functions are added to the vocabulary. It follows that the analogue of ${}^*\mathsf{WKL_0} + \forall\mathsf{T}$ in this vocabulary is conservative with respect to WKL_0, and hence the theory $\mathsf{RCA}_0(\mathsf{PR}) + \forall\mathsf{T}(\mathsf{PR}) + \mathsf{BNA} + \mathsf{STP}$ is conservative with respect to WKL_0.

Since the proof of a single sentence is finite, there is a finite set of primitive recursive functions such that the corresponding fragment of $\mathsf{RCA}_0(\mathsf{PR}) + \forall\mathsf{T}(\mathsf{PR}) + \mathsf{BNA} + \mathsf{STP}$ already implies the Weak Koenig Lemma, and hence implies WKL_0. Question 6.3 asks whether this happens for the fragment obtained by adding just the exponential function.

7. Conclusion

This paper and [3] together show that for each of the "big five" theories T of reverse mathematics there is a theory T' such that:

(a) T' implies and is conservative with respect to T,

(b) T' is of the form $\mathsf{BNA} + \mathsf{STP} + U$ where U is a theory in the language ${}^*L_1$ of nonstandard arithmetic.

Let us call such a theory T' a nonstandard counterpart of T. The paper [3] gave nonstandard counterparts of each of the theories $\mathsf{WKL}_0, \mathsf{ACA}_0, \mathsf{ATR}_0$, and Π^1_1-CA_0. For RCA_0, [3] gave a nonstandard theory ${}^*\mathsf{RCA}_0$ which had property (a) but did not have property (b). In this paper give a nonstandard counterpart of RCA_0, namely the theory

$${}^*\mathsf{RCA}_0{}' = \mathsf{BNA} + \mathsf{STP} + \text{ Special } \Sigma^S_1\text{-}\mathsf{IND} + \text{ Special } \Delta^S_1\text{-}\mathsf{CA}.$$

Moreover, the stronger theory ${}^*\mathsf{RCA}_0{}' + \forall\mathsf{T}$, where $\forall\mathsf{T}$ is the Transfer scheme for universal sentences, is also a nonstandard counterpart of RCA_0. The main arguments were in Section 5, where we showed that ${}^*\mathsf{RCA}_0{}' + \forall\mathsf{T}$ is conservative with respect to RCA_0. To do this we used a result of Tanaka [5] and a special algebraic construction to show that every countable model $(\mathcal{N}, \mathcal{P})$ of RCA_0 can be expanded to a model $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$ of ${}^*\mathsf{RCA}_0{}' + \forall\mathsf{T}$.

As mentioned in the Introduction, in nonstandard analysis one often uses first order properties of hyperintegers to prove second order properties of integers, and the hyperintegers have more structure than the sets of integers. The objective of the theory ${}^*\mathsf{RCA}_0{}' + \forall\mathsf{T}$ is to capture the structure that the hyperintegers can have in a nonstandard counterpart of RCA_0.

In Section 6 we asked how much one can strengthen ${}^*\mathsf{RCA}_0{}' + \forall\mathsf{T}$ and still be conservative with respect to RCA_0. We showed that several theories that appear to be only slightly stronger than ${}^*\mathsf{RCA}_0{}'$ already imply WKL_0 and thus cannot be conservative with respect to RCA_0. We also posed some open questions asking whether certain other theories stronger than $\mathsf{RCA}_0' + \forall\mathsf{T}$ are conservative with respect to RCA_0.

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