NEOCLOSED FORCING

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Abstract. A general model-theoretic theory of approximation is presented which encompasses approximation methods found in analysis in both standard and nonstandard settings. We first give a simple version of the main idea, in the classical metric space setting. This was inspired by work of Anderson and Henson. We inductively define the notions of a closed formula, closed forcing, and the set of approximations of a closed formula. It is shown that given a relatively compact sequence, a closed formula is forced if and only if all its approximations are eventually true, and also if and only if the formula is true at every limit point. Then, in the nonstandard setting, we prove harder analogous results using our theory of neometric spaces, where saturation arguments take the place of compactness arguments. These results shed light on well-known nonstandard constructions that produce new theorems about standard objects.

§1. Introduction. One of the main uses of model theory outside of mathematical logic itself has been the introduction in the early sixties of nonstandard analysis by Abraham Robinson (see [R]). He showed how to apply nonstandard models of the appropriate language to a wide variety of problems in analysis. His construction captured the attention of mathematicians because it made the old idea of infinitesimal quantities available to modern mathematics (for a detailed history of the development of these ideas see the last chapter in [R]).

Robinson's original presentation, which relied heavily on the theory of types, has been "cleaned up", so that today one does not have to be a logician in order to understand and use nonstandard analysis. Nevertheless there are close ties between model theory and developments that have originated from nonstandard practice. The purpose of this paper is to develop

2000 *Mathematics Subject Classification.* Primary 03C25, 03H05, Secondary 03c75.

Key words and phrases. neocompact, forcing, nonstandard analysis, approximation .

This research was supported in part by Colciencias, El Banco de la Republica de Colombia.

This research was supported in part by the National Science Foundation and the Vilas Trust Fund.

NS2002 Nonstandard Methods and Applications in Mathematics, June 10-16 2002, Pisa, Italy.
Edited by Unknown

one of these ties: we give a general model theoretic theory of approximation which encompasses approximation methods found in both standard and nonstandard settings. Our primary objective is to explain a range of phenomena that occur throughout standard and nonstandard approximation theory. In the tradition of model theory, we obtain results of a general nature that can replace similar arguments that appear in many different mathematical settings. As an added bonus, these general theorems provide a method for the discovery of new results. We give three examples of this method in Section 4.

The ideas presented here originated in the paper "From Discrete to Continuous Time" ([K2]). In that work the second author of this paper proposed a uniform approach to applications of nonstandard analysis to existence theorems in probability theory, and especially in stochastic analysis. He introduced a language adequate for expressing the main properties of stochastic processes and used a notion of forcing to study the approximation of a continuous time process by a sequence of discrete time processes. The Forcing Theorem in that paper reduced the problem of proving that a statement is true to showing that it is forced.

The work in [K2] led to the development of the theory of neometric spaces in the series of papers [FK1]–[FK4], [K2]–[K6]. In this paper we return to the original forcing idea in [K2] in the light of the subsequent work. The paper "A Neometric Survey" [K5] gives an overview of the series of papers mentioned above, and also has a section called "Forcing and Approximations" which previews the results that are proved in the present paper.

The contents of this paper are as follows. In Section 2 we present a simple analog of the main ideas in the classical metric space setting, which was inspired by the work of Anderson [A] and Henson [H]. We introduce the notions of a closed formula, and closed forcing. Briefly, a condition is an infinite set $p \subseteq \mathbb{N}$, and the property "p forces a formula $\varphi(\langle x_n \rangle)$" is defined by induction on complexity of formulas, starting with the property "every p-limit point of $\langle x_n \rangle$ belongs to the closed set A." Another notion defined by induction on complexity is the set of approximations of a closed formula. The main results are the Closed Approximation Theorem and the Closed Forcing Theorem. Together they show that a closed formula at a convergent sequence is forced if and only if all its approximations are eventually true, and also if and only if the formula is true at every limit point.

In the rest of the paper, beginning in Section 3, we work in a given nonstandard universe. We introduce neoclosed forcing, a more powerful cousin of closed forcing which sheds light on well-known nonstandard constructions that produce new results about standard objects. In Section 3 we review the basic notions and results about neometric spaces which will be needed from the papers [FK1] and [FK2], and define the neoclosed formulas. In Section 4 we define neoclosed forcing, and give three typical examples involving Nash equilibria and stochastic differential equations. In Section 5 we introduce the approximations of a neoclosed formula, and prove the Neoclosed Approximation Theorem, which shows that a neoclosed formula is forced if and only if all its approximations are eventually true. In Section 7 we prove the Neoclosed Forcing Theorem, which says that if a neoclosed formula $\varphi(\langle x_n \rangle)$ is forced, then $\varphi(x_J)$ is true for all sufficiently small infinite J. We apply the Neoclosed Forcing Theorem to prove existence theorems related to the examples of Section 4, and give several general consequences of the theorem.

§2. Closed Forcing. As a warmup, in this section we shall develop a simple version of our forcing and approximation machinery, which is closely related to the papers Anderson [A], Henson [H], and Henson and Iovino [HI].

Anderson [A] introduced a first order language with variables ranging over metric spaces, predicates for closed sets, and symbols for continuous functions, defined a natural notion of an approximation for such formulas, and proved the "Almost-Near" theorem. This theorem says that for a certain class of formulas φ (corresponding to the closed formulas in the sense of this section), for each $\varepsilon > 0$ and compact set C there is an approximation ψ of φ such that each point in C which satisfies ψ is within ε of a point which satisfies φ. Anderson gave a variety of applications of this theorem, for example to almost commuting matrices, to the Peano existence theorem for differential equations, and to cores and approximate competitive equilibria in exchange economies.

Henson [H] introduced a similar notion of approximation for formulas in a language appropriate for Banach spaces, and showed that many results in classical model theory have analogues for Banach spaces where the notion of approximate truth plays a central role.

The forcing language and forcing theorem in this section are simpler "deterministic analogues" of the forcing language and theorem for random variables in the paper [K2]. The existence of this deterministic analogue was already pointed out in [K2] in the remark on page 118.

We let $\mathbf{S}$ be the collection of all complete metric spaces, and let $\mathbb{N}$ denote the set of positive integers.

The product of two metric spaces M and N is the metric space $M \times N$ where the distance between two points in the product is the maximum of the distances in the M and N coordinates. Thus $\mathbf{S}$ is closed under finite products. We use the notation $\vec{v}$ for a finite tuple of variables ranging over a finite product of spaces in $\mathbf{S}$.

We introduce two classes of formulas, the finite formulas and the closed formulas.

Definition 2.1. The language of **closed formulas** has infinitely many variables $u, v, \ldots$ of sort M for each $M \in \mathbf{S}$. It has a constant symbol for each element of each $M \in \mathbf{S}$, a function symbol for each continuous function $f : M \to N$ where $M, N \in \mathbf{S}$, and a relation symbol for every closed subset C of each space $M \in \mathbf{S}$. Terms are built from constants and function symbols in the usual way.

The atomic formulas are expressions of the form $\tau(\vec{v}) \in C$, where τ is a term and C is a closed set of the same sort.

The language has the following connectives and quantifiers:

Countable conjunctions with finitely many free variables,

Finite disjunctions,

Bounded existential quantifiers $(\exists v \in B)\varphi$ where B is compact,

Bounded universal quantifiers $(\forall v \in D)\varphi$ where D is closed and separable.

The language of **finite formulas** has the same vocabulary and the same atomic formulas as the language of closed formulas, but has the following connectives and quantifiers:

Finite conjunctions and disjunctions,

Unbounded existential quantifiers of the form $(\exists v)\varphi$,

Bounded universal quantifiers $(\forall v \in D)\varphi$ where D is finite.

There is no negation symbol. Note that there are finite formulas which are not closed and closed formulas which are not finite. Every term defines a continuous function and thus could be replaced by a term with just one function symbol.

Since the family of complete metric spaces is closed under finite Cartesian products, the language also has symbols for all closed relations and all continuous functions on finitely many variables. Thus every closed relation is already defined by an atomic formula. Since the distance function ρ on each $M \in \mathbf{S}$ maps $M \times M$ continuously into $\mathbb{R}$, the closed forcing language has a symbol for ρ.

The language of closed formulas is adequate for expressing many limit notions which arise in analysis. Here are a few examples which illustrate the kinds of things that can be expressed by closed formulas.

Example 2.2. *(i) An equation $\sigma(\vec{v}) = \tau(\vec{v})$ between two terms can be expressed by the closed formula*

$$\rho(\sigma(\vec{v}), \tau(\vec{v})) \in \{0\},$$

where ρ is the metric.

(ii) An inequality $\sigma(\vec{v}) \leq \tau(\vec{v})$ between two terms of the real sort $\mathbb{R}$ can be expressed by the closed formula

$$min(\sigma(\vec{v}), \tau(\vec{v})) = \sigma(\vec{v}).$$

(iii) If B is compact, the closed formula

$$(\exists v \in B) f(v) = v$$

says that f has a fixed point in C.

(iv) If C is closed and separable, the closed formula

$$u \in C \wedge (\forall v \in C) f(u) \leq f(v)$$

says that f has a minimum at u in B.

We shall give some further examples involving spaces of random variables. Let M be the space of measurable functions from a probability space Ω into the set $\mathbb{R}_+$ of nonnegative real numbers, with the metric of convergence in probability. Let $\{\varphi_n : n \in \mathbb{N}\}$ be the set of continuous piecewise linear functions on $\mathbb{R}_+$ with bounded support and rational vertices—a countable set. Then for each n, the expected value functions $v \mapsto E[\varphi_n(v)]$ and $v \mapsto E[min(n, |v|)]$ are continuous from M into $\mathbb{R}_+$.

Example 2.3. *(i) The inequality $E[|v|] \leq r$ can be expressed by the closed formula*

$$\bigwedge_n E[min(n, |v|)] \leq r.$$

(ii) The property that u and v have the same distribution is expressed by the closed formula

$$\bigwedge_n E[\varphi_n(u)] = E[\varphi_n(v)].$$

(iii) The property that u is independent of v is expressed by the closed formula

$$\bigwedge_m \bigwedge_n E[\varphi_m(u)\varphi_n(v)] = E[\varphi_m(u)]E[\varphi_n(v)].$$

The notion of a formula $\varphi(\vec{v})$ being true for a tuple of constants $\vec{b}$ with the same sort as $\vec{v}$, in symbols $\models \varphi[\vec{b}]$, is defined inductively in the usual way.

The following important result shows that every closed formula defines a closed set.

Theorem 2.4. *For every closed formula $\varphi(\vec{v})$ where $\vec{v}$ has sort M, the set*

$$\{\vec{x} \in M : \models \varphi[\vec{x}]\}$$

is closed.

Proof: First show by induction on the complexity of terms that every term defines a continuous function. It follows that every atomic formula defines a closed set. Then show By induction on the complexity of closed formulas that every closed formula defines a closed set. □

We shall now introduce the notion of closed forcing. As in set theory, we shall define by induction the notion of a "condition" forcing a formula in which the variables are replaced by "names".

A sequence $\langle x_n \rangle$ in a complete metric space M is **relatively compact** iff each subsequence has a convergent subsequence in M. A finite tuple of sequences is denoted by $\langle \vec{x}_n \rangle$. If $\tau(\vec{v})$ is term and $\langle \vec{x}_n \rangle$ is relatively compact, then $\langle \tau(\vec{x}_n) \rangle$ is relatively compact. For each complete metric space M, we introduce a name of sort M for each relatively compact sequence $\langle x_n \rangle$ in M.

By a **condition** we mean an infinite subset of the set $\mathbb{N}$ of positive integers. $p, q, r, \dots$ denote conditions. We shall write $q \sqsubseteq p$, or $p \sqsupseteq q$, if $q \subseteq p \cup F$ for some finite set F. The phrase "for almost all $n \in p$" means "for all but finitely many $n \in p$".

Lemma 2.5. *Given a countable decreasing chain of conditions $p_1 \sqsupseteq p_2 \sqsupseteq \cdots$, there is a condition q such that $q \sqsubseteq p_k$ for all k.*

Proof: For each k, $p_0 \cap \cdots \cap p_k$ contains almost all $n \in p_k$ and thus is infinite. Let $a_0 = 0$. For each $k > 0$ we may choose $a_{k+1} \in p_0 \cap \cdots \cap p_k$ such that $a_{k+1} > a_k$. Then the set $q = \{a_0, a_1, \dots\}$ is such that $q \sqsubseteq p_k$ for all k. □

Given a condition p, we let $p(n)$ be the n^{th} element of p, so that $\langle p(1), p(2), \dots \rangle$ is a strictly increasing sequence of positive integers. For each sequence $\langle x_n \rangle$ and condition p, $\langle x_{p(n)} \rangle$ is the subsequence of $\langle x_n \rangle$ associated with p. Note that if $q \subseteq p$ then $\langle x_{q(n)} \rangle$ is subsequence of $\langle x_{p(n)} \rangle$.

We write $\lim_{n \in p} x_n = y$ if the subsequence $\langle x_{p(n)} \rangle$ converges to y. We say that y is a p-**limit point** of $\langle x_n \rangle$ if some subsequence of $\langle x_{p(n)} \rangle$ converges to y, that is, $\lim_{n \in q} x_n = y$ for some $q \sqsubseteq p$. Thus if $q \sqsubseteq p$, then every q-limit point of $\langle x_n \rangle$ is a p-limit point.

Definition 2.6. We say that a sequence $\langle y_n \rangle$ **approximates** a sequence $\langle x_n \rangle$ in M if $\lim_{n \to \infty} \rho(x_n, y_n) = 0$.

Note that if $\langle x_n \rangle$ is relatively compact and $\langle y_n \rangle$ approximates $\langle x_n \rangle$, then $\langle y_n \rangle$ is relatively compact. Moreover, if x_n converges to x on a set p, then y_n converges to x on p, so $\langle x_n \rangle$ and $\langle y_n \rangle$ have the same p-limit points.

For each closed separable set D, let us pick once and for all an increasing chain of finite sets D_m such that $\bigcup_m D_m$ is dense in D.

DEFINITION 2.7. (Closed Forcing) For each closed formula $\varphi(\vec{v})$, relatively compact sequence $\langle \vec{x}_n \rangle$ of the same sort as $\vec{v}$, and condition p, the relation $p \Vdash \varphi(\langle \vec{x}_n \rangle)$ is defined inductively as follows, where B is compact, C is closed, and D is closed separable.

(a) $p \Vdash \tau(\langle \vec{x}_n \rangle) \in C$ iff $\tau(\vec{a}) \in C$ for each p-limit point $\vec{a}$ of $\langle \vec{x}_n \rangle$.
(b) $p \Vdash (\varphi_1 \vee \varphi_2)(\langle \vec{x}_n \rangle)$ iff $(\forall q \sqsubseteq p)(\exists r \sqsubseteq q)\, r \Vdash \varphi_1(\langle \vec{x}_n \rangle)$ or $r \Vdash \varphi_2(\langle \vec{x}_n \rangle)$.
(c) $p \Vdash \bigwedge_m \varphi_m(\langle \vec{x}_n \rangle)$ iff $p \Vdash \varphi_m(\langle \vec{x}_n \rangle)$ for all m.
(d) $p \Vdash ((\exists v \in B)\varphi)(\langle \vec{x}_n \rangle)$ iff

$(\forall q \sqsubseteq p)(\exists r \sqsubseteq q)(\exists$ relatively compact $\langle y_n \rangle)\, r \Vdash y_n \in B \wedge \varphi(\langle \vec{x}_n, y_n \rangle)$.

(e) $p \Vdash ((\forall v \in D)\varphi)(\langle \vec{x}_n \rangle)$ iff $(\forall m \in \mathbb{N})(\forall \langle y_n \rangle \in (D_m)^{\mathbb{N}}))p \Vdash \varphi((\langle \vec{x}_n, y_n \rangle)$.

The next lemma can be proved in a straightforward manner by induction on complexity of formulas. Part (iii) shows that "weak closed forcing" is equivalent to closed forcing.

LEMMA 2.8. *Suppose $\langle \vec{x}_n \rangle$ is relatively compact and $\langle \vec{y}_n \rangle$ approximates $\langle \vec{x}_n \rangle$.*
(i) If $p \Vdash \varphi(\langle \vec{x}_n \rangle)$ then $p \Vdash \varphi(\langle \vec{y}_n \rangle)$.
(ii) If $p \Vdash \varphi(\langle x_n \rangle)$ and $q \sqsubseteq p$, then $q \Vdash \varphi(\langle x_n \rangle)$.
(iii)

$$p \Vdash \varphi(\langle \vec{x}_n \rangle)$$

if and only if

$$(\forall q \sqsubseteq p)(\exists r \sqsubseteq q)\, r \Vdash \varphi(\langle \vec{x}_n \rangle).$$

□

Given a set C and a point x in a metric space (M, ρ), let

$$\rho(x, C) = \inf\{\rho(x, u) : u \in C\}$$

be the distance from x to C. Here is a characterization of closed forcing for atomic closed formulas.

PROPOSITION 2.9. *Let $\tau(\vec{v}) \in B$ be an atomic closed formula (i.e. an atomic formula where B is closed), and let $\langle \vec{x}_n \rangle$ be relatively compact. Then*

$$p \Vdash \tau(\langle \vec{x}_n \rangle) \in B$$

if and only if

$$\lim_{n \in p} \rho(\tau(\vec{x}_n), B) = 0.$$

Proof: First suppose that $\lim_{n\in p}\rho(\tau(\vec{x}_n),B)=0$. Let $\vec{a}$ be a p-limit point of $\langle\vec{x}_n\rangle$. Then for some $q\sqsubseteq p$, $\lim_{n\in q}\vec{x}_n=\vec{a}$. The function $\vec{x}\mapsto\rho(\tau(\vec{x}),B)$ is continuous, so

$$0=\lim_{n\in q}\rho(\tau(\vec{x}_n),B)=\rho(\tau(\vec{a}),B).$$

The set B is closed, so $\tau(\vec{a})\in B$. This proves that $p\Vdash\varphi(\langle\vec{x}_n\rangle)$.

For the converse, suppose that not $\lim_{n\in p}\rho(\tau(\vec{x}_n),B)=0$. Then there exists $q\sqsubseteq p$ and $b>0$ such that $\rho(\tau(\vec{x}_n),B)\geq b$ for all $n\in q$. Since $\langle\vec{x}_n\rangle$ is relatively compact, there exists a q-limit point $\vec{a}$ of $\langle\vec{x}_n\rangle$. $\vec{a}$ is also a p-limit point. We have $\rho(\tau(\vec{a}),B)\geq b$, so $\tau(\vec{a})\notin B$. Therefore p does not force $\tau(\langle\vec{x}_n\rangle)\in B$. $\square$

We shall now introduce the notion of an approximation of a closed formula, and prove results which give a relationship between forcing, truth, and approximate truth.

Given a real number $r\geq 0$, let

$$C^r=\{x\in\mathcal{M}:\ \rho(x,C)\leq r\}.$$

Recall that if C is closed, then C^r is closed. Thus for any finite formula $\varphi(\vec{x},y)$, real r, and compact set B, the property $(\exists y\in B^r)\varphi(\vec{x},y)$ can be expressed by the finite formula

$$(\exists y)[y\in B^r\wedge\varphi(\vec{x},y)].$$

Definition 2.10. The set $\mathcal{A}(\varphi)$ of **approximations** of a closed formula $\varphi(\vec{v})$ is defined by induction on the complexity of φ as follows.

$\mathcal{A}(\tau(\vec{v})\in B)=\{\tau(\vec{v})\in B^{1/n}:n\in\mathbb{N}\}$.

$\mathcal{A}(\bigwedge_m\varphi_m)=\{\bigwedge_{m\leq n}\psi_m:n\in\mathbb{N}\text{ and }\psi_m\in\mathcal{A}(\varphi_m)\text{ for all }m\leq n\}$.

$\mathcal{A}(\varphi\vee\psi)=\{\varphi_0\vee\psi_0:\ \varphi_0\in\mathcal{A}(\varphi)\text{ and }\psi_0\in\mathcal{A}(\psi)\}$.

$\mathcal{A}((\exists v\in B)\varphi)=\{(\exists v\in B^{1/n})\psi:\ \psi\in\mathcal{A}(\varphi)\text{ and }n\in\mathbb{N}\}$.

$\mathcal{A}((\forall v\in D)\varphi)=\{(\forall v\in D_m)\psi:m\in\mathbb{N}\text{ and }\psi\in\mathcal{A}(\varphi)\}$.

We observe that each approximation of a closed formula is a finite formula (up to logical equivalence). In fact, for each closed formula φ, $\mathcal{A}(\varphi)$ is a countable set of finite formulas. Moreover, if $\psi\in\mathcal{A}(\varphi)$ then $\models\varphi\Rightarrow\psi$. The approximations in $\mathcal{A}(\varphi)$ are in general not closed formulas, because for a compact set C, the set $C^{1/n}$ is not necessary compact.

A typical approximation of formula (iv) in Example 2.2 has the form

$$u\in C^{1/k}\wedge(\forall v\in C_m)f(u)\leq f(v)+1/n.$$

We leave the approximations of the other examples of closed formulas as an exercise for the reader. The following result is an analogue of Henson's Perturbation Principle for Banach spaces in [H].

PROPOSITION 2.11. *(Perturbation Principle) For each closed formula* $\varphi(\vec{v})$, *compact set* D, *and approximation* $\psi \in \mathcal{A}(\varphi)$ *there is a real* $\delta > 0$ *such that whenever* $\vec{x}, \vec{y} \in D, \models \varphi[\vec{x}]$, *and* $\rho(\vec{x}, \vec{y}) \leq \delta$, *we have* $\models \psi[\vec{y}]$.

Proof: The proof is a straightforward induction on the complexity of φ. The atomic case follows from Proposition 2.9 and the fact that every continuous function is uniformly continuous on compact sets. $\Box$

The following lemma shows that the set of approximations of a closed formula is directed in the natural sense. A sequence of formulas $\langle \psi_n \rangle$ is said to be **cofinal** in $\mathcal{A}(\varphi)$ if $\psi_n \in \mathcal{A}(\varphi)$ for each $n \in \mathbb{N}$, $\models \psi_n \Rightarrow \psi_m$ whenever $m \leq n$, and for each approximation $\psi \in \mathcal{A}(\varphi)$ there exists $n \in \mathbb{N}$ such that $\models \psi_n \Rightarrow \psi$.

LEMMA 2.12. *For each closed formula* φ *there is a cofinal sequence in* $\mathcal{A}(\varphi)$.

Proof: An easy induction on the complexity of φ. $\Box$

We shall now prove the Closed Approximation Theorem. In the case that φ is a closed sentence, it says that p forces φ if and only if every approximation of φ is true.

THEOREM 2.13. *(Closed Approximation Theorem) Let* $\varphi(\vec{v})$ *be a closed formula and* $\langle \vec{x}_n \rangle$ *be relatively compact sequence. The following are equivalent.*
(i) $p \Vdash \varphi(\langle \vec{x}_n \rangle)$.
(ii) For all $\psi \in \mathcal{A}(\varphi)$, $\models \psi[\vec{x}_n]$ *for almost all* $n \in p$.

Proof: The result for atomic formulas φ follows from Proposition 2.9.
(i) $\Rightarrow$ (ii). We remark that for any set $S \subseteq \mathbb{N}$, $n \in S$ for almost all $n \in p$ if and only if $(\forall q \sqsubseteq p)(\exists r \sqsubseteq q) n \in S$ for almost all $n \in r$. We argue by induction on the complexity of φ. The conjunction step is trivial.

For the finite disjunction step, assume the implication holds for φ and for ψ. Suppose $p \Vdash (\varphi \vee \psi)(\langle \vec{x}_n \rangle)$. Any approximation of $\varphi \vee \psi$ has the form $\varphi_0 \vee \psi_0$ where $\varphi_0 \in \mathcal{A}(\varphi)$ and $\psi_0 \in \mathcal{A}(\psi)$. We must show that $\models (\varphi_0 \vee \psi_0)[\vec{x}_n]$ for almost all $n \in p$.

Suppose this fails. Then there is an infinite set $q \sqsubseteq p$ such that $\models \neg\varphi_0[\vec{x}_n]$ and $\models \neg\psi_0[\vec{x}_n]$ for all $n \in q$. By the inductive hypotheses, it follows that for each $r \sqsubseteq q$, neither $r \Vdash \varphi[\vec{x}_n]$ nor $r \Vdash \psi[\vec{x}_n]$. This contradicts the assumption that $p \Vdash (\varphi \vee \psi)(\langle \vec{x}_n \rangle)$.

The $(\exists v \in B)$ step is similar.

For the $(\forall v \in D)$ case, suppose the implication holds for φ. Assume that $p \Vdash (\forall v \in D)\varphi)(\langle \vec{x}_n \rangle)$. We must show that for each $m \in \mathbb{N}$ and $\psi \in \mathcal{A}(\varphi)$, $\models ((\forall v \in D_m)\psi)[\vec{x}_n]$ for almost all $n \in p$.

Let us suppose not, so that $\models \neg((\forall v \in D_m)\psi)[\vec{x}_n]$ for infinitely many $n \in p$. We will get a contradiction. Choose $y_n \in D_m$ so that $\models \neg\psi[\vec{x}_n, y_n]$

for infinitely many $n \in p$. Since D_m is finite, it is compact, so $\langle y_n \rangle$ is relatively compact. By the definition of forcing, $p \Vdash \varphi(\langle \vec{x}_n, y_n \rangle)$, and by our inductive hypothesis, $\models \psi[\vec{x}_n, y_n]$ almost all $n \in p$. This is the desired contradiction.

(ii) $\Rightarrow$ (i). The proof is again by induction on the complexity of φ.

The induction step for countable conjunctions is trivial.

For the finite disjunction step, assume the implication holds for φ and for ψ. Suppose it is not the case that $p \Vdash (\varphi \vee \psi)(\langle \vec{x}_n \rangle)$. Then there is a condition $q \sqsubseteq p$ such that for every $r \sqsubseteq q$, neither $r \Vdash \varphi(\langle \vec{x}_n \rangle)$ nor $r \Vdash \psi(\langle \vec{x}_n \rangle)$. By inductive hypothesis, there are approximations $\varphi_0 \in \mathcal{A}(\varphi), \psi_0 \in \mathcal{A}(\psi)$ such that $\models \neg\varphi_0[\vec{x}_n]$ and $\models \neg\psi_0[\vec{x}_n]$ for infinitely many $n \in q$. Then it is not the case that $\models (\varphi_0 \vee \psi_0)[\vec{x}_n]$ for almost all $n \in p$.

Assume the implication holds for $\varphi(\vec{u}, v)$. We prove the implication for $(\exists v \in B)\varphi$ where B is compact, and for $(\forall v \in D)\varphi$ where D is closed and separable.

Suppose that for each $\theta \in \mathcal{A}((\exists v \in B)\varphi))$, $\models \theta[\vec{x}_j]$ for almost all $j \in p$. By Lemma 2.12 there is a cofinal sequence $\langle \varphi_k \rangle$ in $\mathcal{A}(\varphi)$. For each k, $(\exists v \in B^{1/k})\varphi_k$ is an approximation of $(\exists v \in B)\varphi$, and any approximation of $(\exists v \in B)\varphi$ is implied by one of these formulas. Therefore there is an increasing sequence $n(\cdot)$ in $\mathbb{N}$ such that $\models ((\exists v \in B^{1/k})\varphi_k)[\vec{x}_j]$ whenever $n(k) \leq j \in p$. For each $j \in \mathbb{N}$ let $m(j)$ be the greatest k with $n(k) \leq j$. Then $\models ((\exists v \in B^{1/m(j)})\varphi_{m(j)})[\vec{x}_j]$ for all $j \in p$. Choose $y_j \in B^{1/m(j)}$ so that $\models \varphi_{m(j)}[\vec{x}_j, y_j]$ whenever $j \in p$. Since B is compact, $\langle y_n \rangle$ is relatively compact and $p \Vdash \langle y_n \rangle \in B$. By inductive hypothesis, $p \Vdash \varphi(\langle \vec{x}_n, y_n \rangle)$, so $p \Vdash (\langle \vec{y}_n \rangle \in B) \wedge \varphi(\langle \vec{x}_n, y_n \rangle)$. It follows that $p \Vdash (\exists v \in B)\varphi(\langle \vec{x}_n \rangle)$.

Now suppose that for each $\theta \in \mathcal{A}((\forall v \in D)\varphi)$, $\models \theta[\vec{x}_j]$ for almost all $j \in p$. We wish to prove that $p \Vdash ((\forall v \in D)\varphi)(\langle \vec{x}_n \rangle)$. Fix an $m \in \mathbb{N}$ and consider any sequence $\langle y_n \rangle \in (D_m)^{\mathbb{N}}$. Since D_m is finite, $\langle y_n \rangle$ is relatively compact. For each $\psi \in \mathcal{A}(\varphi)$, $(\forall v \in D_m)\psi \in \mathcal{A}((\forall v \in D)\varphi)$, so $\models \psi[\vec{x}_j, y_j]$ for almost all $j \in p$. By inductive hypothesis, $p \Vdash \varphi(\vec{x}_n, y_n)$. This shows that $p \Vdash ((\forall v \in D)\varphi)(\langle \vec{x}_n \rangle)$. $\square$

The following result characterizes forcing in terms of truth of formulas. A similar fact was stated as an exercise for the reader in [K2, p. 118]. In the case that φ is a sentence (i.e. a formula with no free variables), it says that p forces φ if and only if φ is true.

THEOREM 2.14. *(Closed Forcing Theorem) Let $\varphi(\vec{v})$ be a closed formula, and let $\langle \vec{x}_n \rangle$ be a relatively compact sequence of sort $\vec{v}$. Then $p \Vdash \varphi(\langle \vec{x}_n \rangle)$ if and only if $\models \varphi[\vec{a}]$ for every p-limit point $\vec{a}$ of $\langle \vec{x}_n \rangle$.*

Proof: By induction on the complexity of φ. The result holds for atomic formulas by definition.

We assume the result for $\varphi(\vec{v})$ and $\psi(\vec{v})$, and prove it for $(\varphi \vee \psi)(\vec{v})$.

Suppose that $\models (\varphi \vee \psi)[\vec{a}]$ for every p-limit point $\vec{a}$ of $\langle \vec{x}_n \rangle$. Let $q \sqsubseteq p$. There exists an $r \sqsubseteq q$ which converges to a p-limit point $\vec{a}$. Then either $\models \varphi[\vec{a}]$ or $\models \psi[\vec{a}]$, and by inductive hypothesis, either $r \Vdash \varphi(\langle \vec{x}_n \rangle)$ or $r \Vdash \psi(\langle \vec{x}_n \rangle)$. It follows that $p \Vdash (\varphi \vee \psi)(\langle \vec{x}_n \rangle)$.

Now suppose that $p \Vdash (\varphi \vee \psi)(\langle \vec{x}_n \rangle)$. Let $\vec{a}$ be a p-limit point of $\langle \vec{x}_n \rangle$ There exists a $q \sqsubseteq p$ which converges to $\vec{a}$. Then for each $r \sqsubseteq q$, either $r \Vdash \varphi(\langle \vec{x}_n \rangle)$ or $r \Vdash \psi(\langle \vec{x}_n \rangle)$. But $\vec{a}$ is the only r-limit point, so by inductive hypothesis, either $\models \varphi[\vec{a}]$ or $\models \psi[\vec{a}]$. Hence $\models (\varphi \vee \psi)[\vec{a}]$.

The $\bigwedge_m$ and $(\forall v \in D)$ cases are trivial.

Assume the result for $\varphi(\vec{u}, v)$, and let B be compact. Suppose that $\models ((\exists v \in B)\varphi)[\vec{a}]$ holds for every p-limit point $\vec{a}$ of $\langle \vec{x}_n \rangle$. Let $q \sqsubseteq p$. There exists $r \sqsubseteq q$ such that $\langle \vec{x}_n \rangle$ converges to a point $\vec{a}$ on r. $\vec{a}$ is a p-limit point of $\langle \vec{x}_n \rangle$, so $\models ((\exists v \in B)\varphi)[\vec{a}]$, and $\models \varphi[\vec{a}, b]$ for some $b \in B$. Let $y_n = b$ for all n. By inductive hypothesis, $r \Vdash y_n \in B \wedge \varphi(\langle \vec{x}_n, y_n \rangle)$, so $p \Vdash (\exists v \in B)\varphi(\langle \vec{x}_n \rangle)$.

Now suppose that $p \Vdash ((\exists v \in B)\varphi)(\langle \vec{x}_n \rangle)$. Then

$$(\forall q \sqsubseteq p)(\exists r \sqsubseteq q)(\exists \text{ relatively compact } \langle y_n \rangle)\, r \Vdash y_n \in B \wedge \varphi(\langle \vec{x}_n, y_n \rangle).$$

Let $\langle \vec{x}_n \rangle$ converge to $\vec{a}$ on some set $q \sqsubseteq p$. Then for some $r \sqsubseteq q$ and some relatively compact $\langle y_n \rangle$, $r \Vdash \varphi(\langle \vec{x}_n, y_n \rangle)$. $\langle y_n \rangle$ has an r-limit point $b \in B$. Then $(\vec{a}, b)$ is an r-limit point of $(\langle \vec{x}_n, y_n \rangle)$. By inductive hypothesis, $b \in B$ and $\models \varphi[\vec{a}, b]$. Therefore $\models ((\exists v \in B)\varphi)[\vec{a}]$ as required. □

COROLLARY 2.15. *Suppose* $\lim_{n \in p} \vec{x}_n = \vec{a}$. *Then* $p \Vdash \varphi(\langle \vec{x}_n \rangle)$ *if and only if* $\models \varphi[\vec{a}]$. □

COROLLARY 2.16. *Let* $\varphi(\vec{v})$ *be a closed formula. For each constant tuple* $\vec{x}$ *and each condition* p*, the following are equivalent:*

$p \Vdash \varphi[\vec{x}]$;

$\models \psi[\vec{x}]$ *for all* $\psi \in \mathcal{A}(\varphi)$;

$\models \varphi[\vec{x}]$.

Proof: This follows at once from the Closed Forcing Theorem and the Closed Approximation Theorem. □

As a special case we have the classical analogue of the Approximation Theorem in [FK1].

COROLLARY 2.17. *Let* A *be closed in* $\mathcal{M}$ *and* $f : A \to \mathcal{N}$ *be continuous from* $\mathcal{M}$ *to* $\mathcal{N}$. *Let* B *be compact in* $\mathcal{M}$ *and* D *be closed in* $\mathcal{N}$. *Suppose that for each* $\varepsilon > 0$,

$$(\exists v \in A \cap B^{\varepsilon}) f(v) \in D^{\varepsilon}.$$

Then

$$(\exists v \in A \cap B) f(v) \in D.$$

Proof: Apply the preceding corollary with $\mathcal{M} = A$ and take φ to be the formula $(\exists v \in A \cap B) f(v) \in D$. □

We now prove a form of Anderson's Almost Near theorem from [A]. Given a closed formula $\varphi(v)$ and a compact set C, it says that each $x \in C$ which "almost" satisfies φ is "near" some $y \in C$ which really does satisfy φ. Here "near" means within ε, and to "almost" satisfy φ means to satisfy a sufficiently good approximation of φ.

Proposition 2.18. *(Almost Near Theorem) Let $\varphi(v)$ be a closed formula where v has sort $\mathcal{M}$, let C be a compact set in $\mathcal{M}$, and let*

$$D = C \cap \{x : \models \varphi[x]\}.$$

Then for every real $\varepsilon > 0$ there exists an approximation $\psi \in \mathcal{A}(\varphi)$ such that

$$C \cap \{x : \models \psi[x]\} \subseteq D^{\varepsilon}.$$

Proof: Suppose the result fails for ε. List all the approximations of φ in a countable sequence $\langle \psi_k \rangle$. Then there is a sequence $\langle x_n \rangle$ in C such that for all n, $\models \psi_n[x_n]$ but $x_n \notin D^{\varepsilon}$. We may assume without loss of generality that $\langle x_n \rangle$ is convergent, and let $x = \lim_{n\to\infty} x_n \in C$. By the Closed Approximation Theorem we have $\mathbb{N} \Vdash \varphi(\langle x_n \rangle)$. By the Closed Forcing Theorem, $\models \varphi[x]$. But then $x \in D$ by hypothesis, which is impossible since $x_n \notin D^{\varepsilon}$. □

Corollary 2.19. *Let $\varphi(v)$ and $\theta(v)$ be closed formulas and C a compact set such that*

$$C \cap \{x : \models \varphi[x]\} \subseteq \{x : \models \theta[x]\}.$$

Then for every approximation $\theta_0 \in \mathcal{A}(\theta)$ there is an approximation $\varphi_0 \in \mathcal{A}(\varphi)$ such that

$$C \cap \{x : \models \varphi_0[x]\} \subseteq \{x : \models \theta_0[x]\}.$$

Proof: Apply the Perturbation Principle and the Almost Near Theorem. □

§3. Neometric Spaces. In this section we give a brief summary of the theory of neometric spaces in the setting of a nonstandard universe, as developed in the papers [FK1] and [FK2]. We omit the proofs in this section, and instead refer to proofs in [FK1] and [FK2]. At the end of this section we introduce the neoclosed formulas, and state the key result that every neoclosed formula defines a neoclosed set.

We assume in this section that the reader is familiar with the basic notions concerning superstructures in nonstandard analysis. See [Li] or [ACH] for the necessary background. We fix once and for all an ω_1-saturated nonstandard universe, and begin with a review of the family of neometric spaces in this nonstandard universe introduced in the paper [FK2].

Given an internal *metric space $(\bar{M}, \bar{\rho})$, the **standard part** ^oX of an element $X \in \bar{M}$ is defined as the set of all $Y \in \bar{M}$ such that $\bar{\rho}(X, Y) \approx 0$. For a subset $C \subseteq \bar{M}$, the **standard part of** C is the set $^oC = \{^oX : X \in C\}$. For each point $z \in \bar{M}$, the **nonstandard hull around** z is the set $\mathcal{H}(\bar{M}, z) = \{^oX : \bar{\rho}(X, z) \text{ is finite}\}$, with the metric $\rho(^oX, {}^oY) = st(\bar{\rho}(X, Y))$. Each nonstandard hull is a complete metric space. A subset B of $\bar{M}$ will be called **limited** if the distance between any two points of B is finite, or equivalently, ^oB is contained in some nonstandard hull $\mathcal{H}(\bar{M}, z)$.

Definition 3.1. By a **neometric space** we will mean a closed subspace of the nonstandard hull of a *metric space around some point z.

Thus each neometric space is a complete metric space. The family of all neometric spaces is called the **huge neometric family** in [FK2].

Throughout this section, $\mathcal{M}, \mathcal{N}, \ldots$ will stand for neometric spaces, and $\bar{M}, \bar{N}, \ldots$ will be the *-metric spaces which they came from.

$\mathcal{M}$ is a **neometric subspace of** $\mathcal{N}$, in symbols $\mathcal{M} \subseteq \mathcal{N}$, if they are both subspaces of the same nonstandard hull $\mathcal{H}(\bar{M}, z)$ and $\mathcal{M}$ is a metric subspace of $\mathcal{N}$.

The **monad** of a subset $A \subseteq \mathcal{M}$ is the set

$$\text{monad}(A) = \{X \in \bar{M} : {}^oX \in A\}.$$

Note that for any set $A \subseteq \mathcal{M}$, $A = {}^o(\text{monad}(A))$.

A point $X \in \bar{M}$ is **near-standard in** $\mathcal{M}$ if it belongs to the monad of $\mathcal{M}$, that is, $^oX \in \mathcal{M}$. Note that the monad of $\mathcal{M}$ is limited, and so is any subset of monad$(\mathcal{M})$.

We adopt the usual convention of identifying a point x of a standard metric space M with the standard part of its internal counterpart, $^o({}^*x)$. With this convention, each standard complete metric space M in the original superstructure is a closed subset of a nonstandard hull of $\bar{M}$, and thus is a neometric space. This is important for applications. For example, we often want the product of a neometric space with the real line to again be a neometric space.

We now introduce the neocompact sets, which are analogues of compact sets.

By a Π^0_1 set we mean the intersection of a countable collection of internal subsets of $\bar{M}$.

Definition 3.2. A set $C \subseteq \mathcal{M}$ is **neocompact** if C is the standard part of some Π^0_1 set $A \subseteq \bar{M}$.

If we need to specify the neometric space we are working in, we will say "neocompact in $\mathcal{M}$". It is easily seen that finite unions of neocompact sets, and finite Cartesian products of neocompact sets, are neocompact. Here are some examples of neocompact sets.

EXAMPLE 3.3. *(i) Every compact set is neocompact. ([FK2], Corollary 4.8)*

(ii) A separable set is neocompact if and only if it is compact. ([FK1] Proposition 4.8.)

(iii) Let $x \in \mathcal{M}$ and $0 < r \in \mathbb{R}$. In general, the closed ball $\{y \in \mathcal{M} : \rho(x, y) \le r\}$ will not be neocompact, but if $\mathcal{M}$ is the nonstandard hull in $\bar{M}$ around some point, then the closed ball is neocompact.

(iv) Let Ω be a Loeb probability space, M be a complete separable metric space, and $\mathcal{M}$ be the space of all M-valued random variables $x : \Omega \to M$, with the metric of convergence in probability. For each compact set $C \subseteq M$, the set of all $x \in \mathcal{M}$ such that $law(x) \in C$ is neocompact in $\mathcal{M}$. ([FK2], Theorem 5.14.)

The following lemma is useful for proving results about neocompact sets.

LEMMA 3.4. *A set $C \subseteq \mathcal{M}$ is neocompact if and only if monad(C) is a Π^0_1 set. ([FK2], Corollary 3.8.)*

Here are two important consequences.

PROPOSITION 3.5. *(i) If $C \subseteq \mathcal{M} \subseteq \mathcal{N}$, then C is neocompact in $\mathcal{M}$ if and only if C is neocompact in $\mathcal{N}$.*

(ii) Finite intersections of neocompact sets are neocompact.

Another consequence is the following fact, which says that neocompact sets behave like compact sets.

THEOREM 3.6. *(Countable Compactness Property) If B_m is a decreasing chain of nonempty neocompact sets, then $\bigcap_m B_m$ is a nonempty neocompact set.*

Proof: Let $B = \bigcap_m B_m$. By Lemma 3.4, for each m, monad(B_m) is equal to a nonempty Π^0_1 set $\bigcap_n C_{mn}$. It is easily seen that

$$\bigcap_m \left(\bigcap_n (C_{mn}) \right) = \bigcap_m (\text{monad}(B_m)) = \text{monad}(B),$$

so B is neocompact. For each k there is a point X_k which belongs to C_{mn} for all $m, n < k$. By ω_1-saturation, there is a point X which belongs to C_{mn} for all $m, n \in \mathbb{N}$. Therefore ${}^oX \in B$, so B is nonempty. □

The next result says that the family of neocompact sets in $\mathcal{M}$ is closed under "diagonal intersections". It is proved in [FK2], Theorem 4.7.

THEOREM 3.7. *(Diagonal Intersection Property) Suppose that C_n is neocompact for each $n \in \mathbb{N}$, and let r_n be a sequence of non-negative real numbers which converges to 0. Then the set $\bigcap_n((C_n)^{r_n})$ is neocompact.*

The following lemma from the paper [FK1] is the main consequence of the diagonal intersection property which we will need.

LEMMA 3.8. *If C is neocompact and $\lim_{n\to\infty} \rho(x_n, C) = 0$, then the set $C \cup \{x_n : n \in \mathbb{N}\}$ is neocompact.*

Proof: Choose a decreasing sequence ε_n such that $\varepsilon_n \geq \rho(x_n, C)$ and $\varepsilon_n \to 0$ as $n \to \infty$. Let $C_n = C \cup \{x_m : m \leq n\}$ and $D = C \cup \{x_m : m \in \mathbb{N}\}$. Then each C_n is neocompact, and $D \subseteq \bigcap_n(C_n)^{\varepsilon_n}$. We claim that the opposite inclusion $D \supseteq \bigcap_n(C_n)^{\varepsilon_n}$ also holds. To see this, suppose $y \notin D$. Then $y \notin C$, and since C is closed, $\rho(y, C) = \delta > 0$. Take n large enough so that $\varepsilon_m < \delta$ for all $m \geq n$. Since $y \notin \{x_m : m \in \mathbb{N}\}$, there is an $\eta > 0$ such that $\eta < \rho(y, x_m)$ for all $m < n$ and $\eta < \delta$. Then $\eta \leq \rho(y, D)$. For sufficiently large $m \in \mathbb{N}$, $\varepsilon_m < \eta$ and $C_m \subseteq D$, so $y \notin (C_m)^{\varepsilon_m}$. This proves the claim. By closure under diagonal intersections, D is neocompact. □

We now introduce the neoclosed sets.

DEFINITION 3.9. A set $C \subseteq \mathcal{M}$ is **neoclosed** if $C \cap D$ is neocompact for every neocompact set D. The complement of a neoclosed set in $\mathcal{M}$ is called **neoopen**.

PROPOSITION 3.10. *(i) Every neocompact set is neoclosed and bounded. ([FK1], Lemma 4.6.)*

(ii) Every neoclosed set is closed. ([FK1], Proposition 4.5.)

Note that neocompact is weaker than compact, but neoclosed is stronger than closed.

EXAMPLE 3.11. *(i) For each neocompact set C and positive real number r, the set C^r is neoclosed. ([FK1], Proposition 4.14.)*

(ii) For each $x \in \mathcal{M}$ and $0 < r \in \mathbb{R}$, the set $\{y \in \mathcal{M} : \rho(x, y) \geq r\}$ is neoclosed. ([FK1], Lemma 4.7.)

(iii) Let $\mathcal{M}$ be the space of all square integrable stochastic processes on an adapted Loeb space, with the L^2 norm. The sets of all adapted processes and of all square integrable martingales are neoclosed in $\mathcal{M}$ ([FK2], page 167).

Neoclosed sets can often be found using the following proposition.

PROPOSITION 3.12. *([FK2], Proposition 4.1.) For each internal or Π^0_1 set $A \subseteq \bar{M}$, the standard part*

$$\mathcal{M} \cap {}^oA = \mathcal{M} \cap \{{}^oX : X \in A\}$$

is neoclosed in $\mathcal{M}$. □

We now introduce the analogue of the continuous functions.

Definition 3.13. A function $f : \mathcal{M} \to \mathcal{N}$ is said to be **neocontinuous** if for every neocompact set A in $\mathcal{M}$, the restriction $f|A = \{(x, f(x)) : x \in A\}$ of f to A is neocompact in $\mathcal{M} \times \mathcal{N}$.

Proposition 3.14. *(i) Every neocontinuous function is continuous. ([FK1], Proposition 4.11.)*

(ii) If $f : \mathcal{M} \to \mathcal{N}$ is neocontinuous and B is neocompact in $\mathcal{M}$, then $f(B)$ is neocompact in $\mathcal{N}$. ([FK1], Proposition 3.9.)

(iii) If $f : \mathcal{M} \to \mathcal{N}$ is neocontinuous and C is neoclosed in $\mathcal{N}$, then $f^{-1}(C)$ is neoclosed in $\mathcal{M}$. ([FK1], Proposition 3.10.)

(iv) Compositions of neocontinuous functions are neocontinuous. ([FK1], Proposition 3.13.)

Example 3.15. *(i) The distance function ρ is neocontinuous from $\mathcal{M} \times \mathcal{M}$ to $\mathbb{R}$. ([FK2], p. 145.)*

(ii) Let M be a Polish space, let Ω be a Loeb probability space, and let $\mathcal{M}$ be the space of all random variables $x : \Omega \to M$, with the metric of convergence in probability.

(a) For every continuous function $f : M \to M$, the function $x(\cdot) \to f(x(\cdot))$ is neocontinuous from $\mathcal{M}$ to $\mathcal{M}$. ([FK1], Lemma 5.19.)

(b) The law function is neocontinuous from $\mathcal{M}$ to the space $\mathcal{B}(M)$ of Borel probability measures on M with the Prohorov metric. ([FK1], Proposition 5.12.)

(c) For each closed set $C \subseteq M$ and $r \in [0,1]$, the set $\{x \in \mathcal{M} : P[x(\omega) \in C] \geq r\}$ is neoclosed.

(iii) Let $\overline{\Omega} = (\Omega, \mathcal{F}_t)_{t\in[0,1]}$ be an adapted Loeb space. The stochastic integral function

$$(f, m) \mapsto \int_0^t f(\omega, s)dm(\omega, s)$$

is neocontinuous from $\mathcal{A} \times \mathcal{M}$ to $\mathcal{M}$ where $\mathcal{A}$ is the L^2 space of uniformly bounded adapted stochastic processes on $\overline{\Omega}$ and $\mathcal{M}$ is the L^2 space of continuous square integrable martingales on $\overline{\Omega}$. ([K6], p. 267.)

Proof of (ii) (c): By the Portmanteau theorem ([B], page 11), the set $D = \{\mu : \mu(C) \geq r\}$ is closed in $\mathcal{B}(M)$. We have $P[x(\omega) \in C] \geq r$ if and only if $x \in law^{-1}(D)$. By (b) and Proposition 3.14 (iii), the set $law^{-1}(D)$ is neoclosed.

Neocontinuous functions can often be built using the next proposition. By a **uniform lifting** of a function $f : \mathcal{M} \to \mathcal{N}$ we mean an internal function $F : \bar{M} \to \bar{N}$ such that whenever $X \in \text{monad}(\mathcal{M})$, we have $F(X) \in \text{monad}(\mathcal{N})$ and ${}^o(F(X)) = f({}^oX)$.

PROPOSITION 3.16. *(i) If $f : \mathcal{M} \to \mathcal{N}$ has a uniform lifting, then f is neocontinuous. ([FK2], Theorem 4.16.)*

(ii) If the whole space $\mathcal{M}$ is neocompact, then a function $f : \mathcal{M} \to \mathcal{N}$ is neocontinuous if and only if it has a uniform lifting. ([Fk2], Theorem 4.18.)

EXAMPLE 3.17. *If $\mathcal{M}$ comes from an internal Banach space $\bar{M}$, then the addition and scalar product functions on $\mathcal{M}$ are neocontinuous.*

Finally, we introduce the neoseparable sets, which are analogues of closed separable sets.

DEFINITION 3.18. A set $C \subseteq \mathcal{M}$ is **basic in** $\mathcal{M}$ if C is the closure of the standard part of an internal set $A \subseteq \bar{M}$. A set $C \subseteq \mathcal{M}$ is **neoseparable in** $\mathcal{M}$ if C is the closure of a countable union of basic sets $B_n, n \in \mathbb{N}$.

Note that a finite union of basic sets is basic, and hence any neoseparable set can be represented as the closure of a countable increasing chain $B_0 \subseteq B_1 \subseteq \cdots$ of basic sets.

PROPOSITION 3.19. *A set is both neocompact and neoseparable if and only if it is basic. ([FK2], Corollary 4.4.)*

EXAMPLE 3.20. *(i) Every closed separable set is neoseparable.*

(ii) Every nonstandard hull is neoseparable.

(iii) Let Ω be a Loeb probability space and let $\mathcal{M}$ be the space of all random variables $x : \Omega \to \mathbb{R}$, with the metric of convergence in probability. The set of all integrable $x \in \mathcal{M}$ is neoseparable in $\mathcal{M}$. ([FK2], Theorem 6.4.)

The next result characterizes the neoseparable sets in terms of their monads.

PROPOSITION 3.21. *A set $D \subseteq \mathcal{M}$ is neoseparable if and only if there is an increasing chain of internal sets $E_1, E_2, \ldots \subseteq \bar{M}$ such that monad$(D) = \bigcap_m \bigcup_n ((E_n)^{1/m})$. ([FK2], Proposition 4.2.)*

We now turn to the neoclosed formulas. These formulas were discussed in the survey paper [K5], where they were called positive bounded formulas because of their similarity to the positive bounded formulas in the Banach space setting of Henson [H]. The word positive indicates that the language does not have a negation, and bounded indicates that the quantifiers are bounded However, neoclosed formulas can be infinite, while the positive bounded formulas of [H] are finite.

DEFINITION 3.22. The language of **neoclosed formulas** for a given nonstandard universe has infinitely many variables $u, v, \ldots$ of sort $\mathcal{M}$ for each neometric space $\mathcal{M}$, an n-ary function symbol for each neocontinuous function $f : \mathcal{M}_1 \times \cdots \times \mathcal{M}_n \to \mathcal{N}$, a constant symbol for each element $c \in \mathcal{M}$, and a unary predicate symbol of sort $\mathcal{M}$ for each neoclosed set A in $\mathcal{M}$. Terms are built in the usual way.

The **atomic neoclosed formulas** are $\tau(\vec{v}) \in A$ where τ is a term and A is a neoclosed set of the same sort.

The connectives and quantifiers are:

Countable conjunctions with finitely many free variables,

Finite disjunctions,

Bounded existential quantifiers of the form $(\exists v \in B)\varphi$ where B is neocompact,

Bounded universal quantifiers $(\forall v \in D)\varphi$ where D is neoseparable.

EXAMPLE 3.23. *All the formulas in Example 2.2, with "neoclosed" in (iii) and "neoseparable" in (iv), are neoclosed. If $\mathcal{M}$ is the set of random variables on a Loeb probability space, then all the formulas in Example 2.3 are neoclosed.*

The next theorem is a key result saying that every neoclosed formula defines a neoclosed set. It follows from the closure principles for neoclosed sets that were proved in [FK1] and [FK2].

THEOREM 3.24. *For every neoclosed formula $\varphi(\vec{v})$, the set $\{\vec{x} : \models \varphi[\vec{x}]\}$ is neoclosed.*

Proof: First show that every term defines a neocontinuous function. The basis step, that the identity and projection functions are neocontinuous, follows from Proposition 3.16. The induction step comes from the fact that compositions of neocontinuous functions are neocontinuous (Proposition 3.14 (iv)). Then by Proposition 3.14 (iii), every atomic formula defines a neoclosed set.

Next, it is shown in [FK2], Theorem 3.11, that the family of neocompact sets is closed under finite unions and cartesian products, finite and countable intersections, the existential projection $(\exists y)((x, y) \in C)$, and the universal projection $(\forall y \in D)((x, y) \in C)$ where D is nonempty and basic.

Using the results in the above two paragraphs, one can show by induction on complexity of formulas that every neoclosed formula defines a neoclosed set. The existential quantifier step is given in [FK1], Proposition 3.5, and the universal quantifier step in [FK1], Proposition 4.18. $\square$

The above theorem can be used to show that particular sets are neoclosed. For example, the sets defined by the formulas in Example 3.23 are neoclosed.

§4. Neoclosed forcing. We are now done stating results from earlier papers, and will start proving things again. In this section we introduce neoclosed forcing in a nonstandard universe. We first introduce neotight sequences, which are the neometric analogues of relatively compact sequences. As usual, a **sequence in** $\mathcal{M}$ is a function $\langle x_n \rangle$ from the set $\mathbb{N}$ of standard natural numbers into $\mathcal{M}$.

DEFINITION 4.1. A sequence $\langle x_n \rangle$ in $\mathcal{M}$ is said to be **neotight** if there is a neocompact set B such that $x_n \in B$ for all $n \in \mathbb{N}$.

The next lemma contains some elementary facts about neotight sequences.

LEMMA 4.2. *Suppose that* $f : \mathcal{M} \to \mathcal{N}$ *is neocontinuous.*
(i) If $\langle x_n \rangle$ *is neotight in* $\mathcal{M}$ *then* $\langle f(x_n) \rangle$ *is neotight in* $\mathcal{N}$.
(ii) If $\langle x_n \rangle$ *and* $\langle y_n \rangle$ *are neotight in* $\mathcal{M}$ *and* $\langle y_n \rangle$ *approximates* $\langle x_n \rangle$*, then* $\langle f(y_n) \rangle$ *approximates* $\langle f(x_n) \rangle$.

Proof: (i) $\langle x_n \rangle$ is contained in some neocompact set B. Then $\langle f(x_n) \rangle$ is contained in the neocompact set $f(B)$ and hence is neotight.
(ii) There is a neocompact set C containing both sequences $\langle x_n \rangle$ and $\langle y_n \rangle$. Suppose that $\langle f(y_n) \rangle$ does not approximate $\langle f(x_n) \rangle$. Then for some $\varepsilon > 0$, $\sigma(f(x_n), f(y_n)) \geq \varepsilon$ for infinitely many n. Since the distance functions ρ and σ are neocontinuous,

$$B_k = \{(x, y) \in C \times C : \rho(x, y) \leq 1/k \text{ and } \sigma(f(x), f(y)) \geq \varepsilon\}$$

is a decreasing chain of nonempty neocompact sets. By countable compactness this chain has a nonempty intersection. But if (x, y) belongs to this intersection, then $x = y$ but $f(x) \neq f(y)$, contradiction. □

The following lemma is a consequence of the Diagonal Intersection Property.

LEMMA 4.3. *(i) If* C *is neocompact and* $\lim_{n\to\infty} rho(x_n, C) = 0$*, then* $\langle x_n \rangle$ *is neotight.*
(ii) If $\langle y_n \rangle$ *is neotight and* $\langle x_n \rangle$ *approximates* $\langle y_n \rangle$*, then* $\langle x_n \rangle$ *is neotight.*

Proof: (i) The set $C \cup \{x_n : n \in \mathbb{N}\}$ is neocompact by Lemma 3.8.
(ii) If B is a neocompact set containing each y_n, then $\lim_{n\to\infty} \rho(x_n, B) = 0$, and $\langle x_n \rangle$ is neotight by part (i). □

We now define forcing for neoclosed formulas. As in the case of closed forcing, we add to our language a name for each neotight sequence $\langle x_n \rangle$ in each neometric space $\mathcal{M}$, and we call $\mathcal{M}$ the sort space of $\langle x_n \rangle$. As before, a condition is an infinite subset of $\mathbb{N}$. Our rule for forcing an atomic formula will be analogous to the rule for closed forcing given by Proposition 2.9.

To prepare the way, for each neoseparable set D let us choose once and for all a countable increasing chain $D_0 \subseteq D_1 \subseteq \cdots$ of basic sets such that D is the closure of the union $\bigcup_m D_m$.

Definition 4.4. (Neoclosed Forcing) For each neoclosed formula $\varphi(\vec{v})$, neotight sequence of tuples $\langle\vec{x}_n\rangle$ of the same sort as $\vec{v}$, and condition p, the forcing relation $p \Vdash \varphi(\langle\vec{x}_n\rangle)$ is defined inductively as follows, where A is neoclosed, B is neocompact, and D is neoseparable.

$p \Vdash \tau(\langle\vec{x}_n\rangle) \in A$ iff $\lim_{n\in p} \rho(\tau(\vec{x}_n), A) = 0$.

$p \Vdash \bigwedge_m \varphi_m(\langle\vec{x}_n\rangle)$ iff $(\forall m) p \Vdash \varphi_m(\langle\vec{x}_n\rangle)$.

$p \Vdash (\varphi \vee \psi)(\langle\vec{x}_n\rangle)$ iff

$$(\forall q \sqsubseteq p)(\exists r \sqsubseteq q)[r \Vdash \varphi(\langle\vec{x}_n\rangle) \text{ or } r \Vdash \psi(\langle\vec{x}_n\rangle)].$$

$p \Vdash ((\exists v \in B)\varphi)(\langle\vec{x}_n\rangle)$ iff

$$(\forall q \sqsubseteq p)(\exists r \sqsubseteq q)(\exists \text{ neotight } \langle y_n\rangle)\, r \Vdash (y_n \in B \wedge \varphi(\langle\vec{x}_n, y_n\rangle)).$$

$p \Vdash ((\forall v \in D)\varphi)(\langle\vec{x}_n\rangle)$ iff

$$(\forall m)(\forall \langle y_n\rangle \in (D_m)^{\mathbb{N}})\, p \Vdash \varphi(\langle\vec{x}_n, y_n\rangle).$$

The definition of neoclosed forcing was given in [K5] with an apparently stronger rule for the universal quantifier step. We shall return to this point in Proposition 5.6 where we shall see that the two formulations are equivalent.

The next lemma is proved by a straightforward induction on the complexity of neoclosed formulas.

Lemma 4.5. *Suppose $\langle\vec{x}_n\rangle$ is neotight and $\langle\vec{y}_n\rangle$ approximates $\langle\vec{x}_n\rangle$.*
(i) If $p \Vdash \varphi(\langle\vec{x}_n\rangle)$, then $p \Vdash \varphi(\langle\vec{y}_n\rangle)$.
(ii) If $p \Vdash \varphi$ and $q \sqsubseteq p$, then $q \Vdash \varphi$.
(iii)

$$p \Vdash \varphi(\langle\vec{x}_n\rangle)$$

if and only if

$$(\forall q \sqsubseteq p)(\exists r \sqsubseteq q) r \Vdash \varphi(\langle\vec{x}_n\rangle).$$

□

Neoclosed forcing as defined here is similar to the kind of forcing introduced in the paper [K2]. The rules for forcing atomic formulas, conjunctions, disjunctions, and existential quantifiers are the same. However, the forcing in [K2] was defined only for spaces of random variables on Loeb probability spaces. The forcing language in [K2] had a more restricted family of atomic formulas, but had a negation symbol. The forcing rule for universal quantifiers in [K2] was defined using the negation rule.

In many applications of nonstandard analysis in the literature, one constructs a standard object by pushing down a hyperfinite approximation of some kind. These applications can often be translated into arguments showing that a neoclosed formula is forced by a sequence of standard objects.

The following example is motivated by the paper [MS].

EXAMPLE 4.6. *Let Ω be an atomless Loeb probability space, M be a compact metric space, $\mathcal{M}$ be the space of M-valued random variables on Ω, and N be the space of all Borel probability measures on $M \times M$ with the Prohorov metric. Let g be a measurable function from Ω to the space $C(M \times N, R)$ of continuous real valued functions on $M \times N$ with the sup norm, and let $u \in \mathcal{M}$. A random variable $x \in \mathcal{M}$ is a* **Nash equilibrium** *for g, u if there is a $y \in \mathcal{M}$ such that $y(\omega) = x(\omega)$ almost surely and*

$$g(\omega)(y(\omega), law(y, u)) \geq \sup_{a \in M} g(\omega)(a, law(y, u))$$

for all $\omega \in \Omega$. It is shown in [MS], Theorem 1, that a Nash equilibrium for (g, u) exists when u is a simple function. Using results from Section 3, one can see that the following formula, which defines the set of Nash equilibria, is neoclosed.

$$(\forall a \in M)\, P[g(\omega)(x(\omega), law(x, u)) \geq g(\omega)(a, law(x, u))] = 1.$$

Let $\{a_j : j \in \mathbb{N}\}$ be a countable dense subset of M. The following "weaker" formula is also neoclosed and defines the set of Nash equilibria.

$$(1) \quad \bigwedge_j \bigwedge_k P[g(\omega)(x(\omega), law(x, u)) + 1/k \geq g(\omega)(a_j, law(x, u))] \geq 1 - 1/k.$$

If $\langle x_n, u_n \rangle$ is a sequence in $\mathcal{M} \times \mathcal{M}$ such that for each j, k,

$$P[g(\omega)(x_n(\omega), law(x_n, u_n)) + 1/k \geq g(\omega)(a_j, law(x_n, u_n))] \geq 1 - 1/k$$

for all sufficiently large n, then $\mathbb{N}$ forces the formula (1) at $\langle x_n, u_n \rangle$.

In the next two examples, let $\overline{\Omega}$ be an adapted Loeb space with time set $[0, 1]$, let w be a Brownian motion on $\overline{\Omega}$, let x range over the space $\mathcal{M}$ of continuous adapted stochastic processes on $\overline{\Omega}$, and let $f(\cdot, \cdot)$ be a bounded continuous real function.

EXAMPLE 4.7. *The stochastic integral equation*

$$(2) \qquad \forall t \in [0, 1] \left(x(\omega, t) = \int_0^t f(s, x(\omega, s)) dw(\omega, s) \right)$$

is a neoclosed formula in x, which defines a neocompact set. If $\langle x_n \rangle$ is a neotight sequence in $\mathcal{M}$ such that

$$\lim_{n \to \infty} \rho \left(x_n(\omega, t), \int_0^t f(s, x_n(\omega, s)) dw(\omega, s) \right) = 0$$

for each rational $t \in [0, 1]$, then $\mathbb{N}$ forces the formula (2) at $\langle x_n \rangle$.

EXAMPLE 4.8. *Let Φ be a countable set of bounded continuous real functions such that whenever $E[g(x(\omega))] = E[g(y(\omega))]$ for all $g \in \Phi$, x and y have the same distribution. Let $(g_k, t_k), k \in \mathbb{N}$ be an enumeration of $\Phi \times (\mathbb{Q} \cap [0,1])$. Let C_0 be the set defined by formula (2). For each k, let*

$$b_k = \sup\{E[g_k(x(\cdot, t_k))] : x \in C_k\},$$
$$C_{k+1} = \{x \in C_k : E[g_k(x(\cdot, t_k))] = b_k\}.$$

The intersection $C = \bigcap_k C_k$ is of interest because it is shown in [K1], [K6] that C is a nonempty neocompact set, and each $x \in C$ is a strong Markov process which solves (2). Moreover, the formula

(3)

$$\forall u \in [0,1] \left(x(\omega, u) = \int_0^u f(s, x(\omega, s)) dw(\omega, s) \right) \wedge \bigwedge_k E[g_k(x(\cdot, t_k))] = b_k$$

is neoclosed and defines the set C. If $\langle x_n \rangle$ is as in Example 4.7 and for each k we have

$$\lim_{n \to \infty} E[(x_n(\cdot . t_k))] = b_k,$$

then $\mathbb{N}$ forces the formula (3) at $\langle x_n \rangle$.

We will revisit the above three examples in Section 7, after the statement of the Neoclosed Forcing Theorem. In each case, the example together with the Neoclosed Forcing Theorem will give an existence result.

§5. Neoclosed Approximations. The main advantage of neoclosed forcing compared to the forcing in [K2] is that there is a useful characterization of neoclosed forcing in terms of approximations of formulas.

We now define the set of approximations of a neoclosed formula. These approximations will not be neoclosed formulas, but will instead be **finite formulas**, which have only finite conjunctions and disjunctions, but allow atomic formulas of the form $\tau(\vec{v}) \in A^{1/n}$ and existential quantifiers of the form $(\exists v \in B^{1/n})\varphi$.

DEFINITION 5.1. The set $\mathcal{A}(\varphi)$ of **approximations** of a neoclosed formula $\varphi(\vec{v})$ is defined by induction on the complexity of φ as follows:

$\mathcal{A}(\tau(\vec{v}) \in A) = \{\tau(\vec{v}) \in A^{1/n} : n \in \mathbb{N}\}$.

$\mathcal{A}(\bigwedge_m \varphi_m) = \{\bigwedge_{m \le n} \psi_m : n \in \mathbb{N}$ and $\psi_m \in \mathcal{A}(\varphi_m)$ for all $m \le n\}$.

$\mathcal{A}(\varphi \vee \psi) = \{\varphi_0 \vee \psi_0 :\ \varphi_0 \in \mathcal{A}(\varphi)$ and $\psi_0 \in \mathcal{A}(\psi)\}$.

$\mathcal{A}((\exists v \in B)\varphi) = \{(\exists v \in B^{1/n})\psi :\ \psi \in \mathcal{A}(\varphi)$ and $n \in \mathbb{N}\}$.

$\mathcal{A}((\forall v \in D)\varphi) = \{(\forall v \in D_m)\psi : m \in \mathbb{N}$ and $\psi \in \mathcal{A}(\varphi)\}$.

The set $\widehat{\mathcal{A}}(\varphi)$ of **strong approximations** of a neoclosed formula φ is defined as above except that the $\exists$ clause is replaced by

$\widehat{\mathcal{A}}((\exists v \in B)\varphi) = \{(\exists v \in B)\psi :\ \psi \in \widehat{\mathcal{A}}(\varphi)\}$.

Note that for each neoclosed formula φ, $\mathcal{A}(\varphi)$ and $\widehat{\mathcal{A}}(\varphi)$ are countable sets of finite formulas. The approximations of φ are not necessarily neoclosed formulas, because $A^{1/n}$ might not be neoclosed when A is neoclosed, and $B^{1/n}$ might not be neocompact when B is neocompact. However, if φ is built from atomic formulas of the form $\tau(\vec{v}) \in C$ where C is neocompact, then every *strong approximation* of φ is a neoclosed formula.

If $\psi \in \mathcal{A}(\varphi)$ and $\widehat{\psi} \in \widehat{\mathcal{A}}(\varphi)$ then $\models \varphi \Rightarrow \psi$ and $\models \varphi \Rightarrow \widehat{\psi}$. Moreover, for each approximation $\psi \in \mathcal{A}(\varphi)$ there is a strong approximation $\widehat{\psi} \in \widehat{\mathcal{A}}(\varphi)$ such that $\models \widehat{\psi} \Rightarrow \psi$.

PROPOSITION 5.2. *(Perturbation Principle) For each neoclosed formula $\varphi(\vec{v})$, neocompact set B, and strong approximation $\psi \in \widehat{\mathcal{A}}(\varphi)$ there is a real $\delta > 0$ such that whenever $\vec{x}, \vec{y} \in B, \models \varphi[\vec{x}]$, and $\rho(\vec{x}, \vec{y}) \leq \delta$, we have $\models \psi[\vec{y}]$.*

Proof: The proof is a straightforward induction on the complexity of φ. The atomic case follows from the fact that every neocontinuous function is uniformly continuous on neocompact sets. □

A sequence of formulas $\langle \psi_n \rangle$ is said to be **cofinal** in $\mathcal{A}(\varphi)$ if $\psi_n \in \mathcal{A}(\varphi)$ for each $n \in \mathbb{N}$, $\models \psi_n \Rightarrow \psi_m$ whenever $m \leq n$, and for each approximation $\psi \in \mathcal{A}(\varphi)$ there exists $n \in \mathbb{N}$ such that $\models \psi_n \Rightarrow \psi$. Cofinal sequences in $\widehat{\mathcal{A}}(\varphi)$ are defined similarly.

LEMMA 5.3. *For each neoclosed formula φ there exist cofinal sequences in $\mathcal{A}(\varphi)$ and in $\widehat{\mathcal{A}}(\varphi)$.*

Proof: An easy induction on the complexity of φ. □

THEOREM 5.4. *(Neoclosed Approximation Theorem) Let $\varphi(\vec{v})$ be a neoclosed formula, p be a condition, and $\langle \vec{x}_n \rangle$ be a neotight sequence. The following are equivalent.*

(i) $p \Vdash \varphi(\langle \vec{x}_n \rangle)$.

(ii) For all $\psi \in \widehat{\mathcal{A}}(\varphi)$, $\models \psi[\vec{x}_n]$ for almost all $n \in p$.

(iii) For all $\psi \in \mathcal{A}(\varphi)$, $\models \psi[\vec{x}_n]$ for almost all $n \in p$.

Proof: We first prove (i) $\Rightarrow$ (ii) by induction on the complexity of φ. The atomic step and conjunction step are trivial.

For the finite disjunction step, assume the implication holds for φ and for ψ. Suppose $p \Vdash (\varphi \vee \psi)(\langle \vec{x}_n \rangle)$, and let $\varphi_0 \in \widehat{\mathcal{A}}(\varphi)$ and $\psi_0 \in \widehat{\mathcal{A}}(\psi)$. Then for every $q \sqsubseteq p$ there exists $r \sqsubseteq q$ such that either $r \Vdash \varphi$ or $r \Vdash \psi$. By inductive hypothesis, in either case we have $\models (\varphi_0 \vee \psi_0)[\vec{x}_j]$ for almost all $j \in r$. Therefore $\models (\varphi_0 \vee \psi_0)[\vec{x}_j]$ for almost all $j \in p$.

For the $(\exists v \in B)$ step, assume the implication holds for $\varphi(\vec{u}, v)$, and $p \Vdash ((\exists v \in B)\varphi)(\langle \vec{x}_n \rangle)$. Let $\varphi_0 \in \widehat{\mathcal{A}}(\varphi)$. Then for each $q \sqsubseteq p$ there exists $r \sqsubseteq q$ and a sequence $\langle y_n \rangle$ in B such that $r \Vdash \varphi(\langle \vec{x}_n, y_n \rangle)$. By inductive hypothesis, $\models \varphi_0[\vec{x}_n, y_n]$ for almost all $n \in r$. Therefore $\models ((\exists v \in B)\varphi_0)[\vec{x}_n]$ for almost all $n \in r$, and it follows that $\models ((\exists v \in B)\varphi_0)[\vec{x}_n]$ for almost all $n \in p$.

For the $(\forall v \in D)$ step, suppose D is neoseparable, and recall that D is the closure of $\bigcup_m D_m$ where D_m is an increasing sequence of basic sets. Suppose further $\varphi_0 \in \widehat{\mathcal{A}}(\varphi)$ and for each $m \in \mathbb{N}$, $\models \neg(\forall v \in D_m)\varphi_0[\vec{x}_n]$ for infinitely many $n \in p$. Choose $\langle y_n \rangle$ in D_m such that $\models \neg\varphi_0[\vec{x}_n, y_n]$ for infinitely many $n \in p$. By inductive hypothesis, not $p \Vdash \varphi(\langle \vec{x}_n, y_n \rangle)$. Therefore not $p \Vdash (\forall v \in D)\varphi(\langle \vec{x}_n \rangle)$.

(iii) $\Rightarrow$ (i). The proof is again by induction on the complexity of φ. For each φ, choose a cofinal sequence $\langle \varphi_k : k \in \mathbb{N} \rangle$ in $\mathcal{A}(\varphi)$. The induction steps for atomic formulas and countable conjunctions are trivial.

The induction step for $\varphi \vee \psi$ is exactly the same as the corresponding step in the proof of the Closed Approximation Theorem.

Assume the result holds for $\varphi(\vec{u}, v)$. We prove the result for $(\exists v \in B)\varphi$ where B is neocompact, and for $(\forall v \in D)\varphi$ where D is neoseparable.

Suppose that for each $\theta \in \mathcal{A}((\exists v \in B)\varphi)$, $\models \theta[\vec{x}_j]$ for almost all $j \in p$. Then there is an increasing sequence $n(\cdot)$ in $\mathbb{N}$ such that $\models ((\exists v \in B^{1/k})\varphi_k)[\vec{x}_j]$ whenever $n(k) \leq j \in p$. For each $j \in \mathbb{N}$ let $m(j)$ be the greatest k with $n(k) \leq j$. Then $\models ((\exists v \in B^{1/m(j)})\varphi_{m(j)})[\vec{x}_j]$ for all $j \in p$. Choose $y_j \in B^{1/m(j)}$ so that $\models \varphi_{m(j)}[\vec{x}_j, y_j]$ whenever $j \in p$. Since $n(\cdot)$ is increasing, $\lim_{j \to \infty} 1/m(j) = 0$, so there is a sequence $\langle z_n \rangle$ in B which approximates $\langle y_n \rangle$. Since B is neocompact, $\langle z_n \rangle$ is neotight, and hence $\langle y_n \rangle$ is neotight by Lemma 4.3. (This is the one place where the diagonal intersection property is needed in the proof). Then $p \Vdash \varphi(\langle \vec{x}_n, y_n \rangle)$ by inductive hypothesis. Therefore $p \Vdash \varphi(\langle \vec{x}_n, z_n \rangle)$ by Lemma 4.5. This shows that $p \Vdash (\exists v \in B)\varphi(\langle \vec{x}_n \rangle)$.

Suppose that for each $\theta \in \mathcal{A}((\forall v \in D)\varphi))$, $\models \theta[\vec{x}_j]$ for almost all $j \in p$. Then for each $k \in \mathbb{N}$ there exists $n(k)$ such that $\models (\forall v \in D_k)\varphi_k[\vec{x}_j]$ for all $n(k) \leq j \in p$. We may take $n(k)$ to be strictly increasing. Let $\langle y_n \rangle$ be a sequence in some D_k. Then $\langle y_n \rangle$ is neotight, and $\models \varphi_k[\vec{x}_j, y_j]$ for all $n(k) \leq j \in p$. By inductive hypothesis, $p \Vdash \varphi(\langle \vec{x}_n, y_n \rangle)$. Since this holds for each k and each sequence $\langle y_n \rangle$ in D_k, $p \Vdash (\forall v \in D)\varphi(\langle \vec{x}_n \rangle)$. $\square$

As a consequence of the Neoclosed Approximation Theorem we show that apparently stronger quantifier rules for neoclosed forcing are equivalent to our official rules. We used weaker rules in our definition in order to get a stronger Neoclosed Forcing Theorem in the next section.

Lemma 5.5. *Suppose C is neocompact in $\mathcal{M}$, D is neoseparable in $\mathcal{M}$, and $C \subseteq D$. Then for each $n \in \mathbb{N}$ there exists $m(n) \in \mathbb{N}$ such that $C \subseteq ((D_{m(n)})^{1/n})$.*

Proof: Suppose the conclusion fails. That is, there exists $n \in \mathbb{N}$ such that for each $m \in \mathbb{N}$, the set $C \setminus ((D_m)^{1/n})$ is nonempty. By Example 3.11, for each m the set

$$E_m = \{y \in C : \rho(y, D_m) \geq 1/(2n)\}$$

is neocompact. Since the sets D_m form an increasing chain, the sets E_m form a decreasing chain. Moreover, each set E_m is nonempty because it contains the nonempty set $C \setminus ((D_m)^{1/n})$. By countable compactness there is a point $x \in \bigcap_m E_m$. But then $x \in (C \setminus D)$, contrary to hypothesis. $\square$

Proposition 5.6. *Let B be neocompact, D be neoseparable, $\langle \vec{x}_n \rangle$ be neotight, and p be a condition.*

(i)

$$p \Vdash ((\exists v \in B)\varphi)(\langle \vec{x}_n \rangle)$$

if and only if

$$(\exists \langle y_n \rangle \text{ in } B^{\mathbb{N}})\, p \Vdash \varphi(\langle \vec{x}_n \rangle, \langle y_n \rangle).$$

(ii)

$$p \Vdash ((\forall v \in D)\varphi)(\langle \vec{x}_n \rangle)$$

if and only if

$$(\forall \text{ neocompact } C \subseteq D)(\forall \langle y_n \rangle \text{ in } C^{\mathbb{N}})\, p \Vdash \varphi(\langle \vec{x}_n \rangle, \langle y_n \rangle).$$

Proof: (i) We prove the nontrivial direction. Suppose that

$$p \Vdash ((\exists v \in B)\varphi)(\langle \vec{x}_n \rangle).$$

Let $\langle \varphi_m \rangle$ be a cofinal sequence of strong approximations of φ. By the Neoclosed Approximation Theorem, for each m we have

$$\models ((\exists v \in B)\varphi_m)[\vec{x}_n]$$

for almost all $n \in p$. It follows that there is a sequence $\langle y_n \rangle$ in B such that

$$\models \varphi_m[\vec{x}_n, y_n]$$

for almost all $n \in p$. By the Neoclosed Approximation Theorem again,

$$p \Vdash \varphi(\langle \vec{x}_n \rangle, \langle y_n \rangle).$$

We prove the nontrivial direction of (ii). Suppose that $p \Vdash ((\forall v \in D)\varphi)(\langle \vec{x}_n \rangle)$. Let $\langle \psi_k \rangle$ be a cofinal sequence of approximations of φ. By the Neoclosed Approximation Theorem, for each $k \in \mathbb{N}$ there exists $n(k)$ such that $\models (\forall v \in D_k)\varphi_k[\vec{x}_j]$ for all $n(k) \leq j \in p$. We may take $n(k)$ to be strictly increasing. Let C be a neocompact subset of D and let $\langle y_n \rangle$ be a sequence in C. By Lemma 5.5, for each m there exists $k(m)$ such that $C \subseteq D_{k(m)}^{1/m}$. We may take the sequence $k(m)$ to be increasing. We may therefore choose a sequence $\langle z_j \rangle$ such that for each m, $z_j \in D_{k(m)}$

and $\rho(y_j, z_j) \leq 2/m$ whenever $n(k(m)) \leq j < n(k(m+1))$. Then $\langle z_j \rangle$ approximates $\langle y_j \rangle$, so $\langle z_j \rangle$ is neotight by Lemma 4.3. Moreover, for each k we have $\models \varphi_k[\vec{x}_j, z_j]$ for all $n(k) \leq j \in p$. By the Neoclosed Approximation Theorem, $p \Vdash \varphi(\langle \vec{x}_n, z_n \rangle)$. By Lemma 4.5, we have $p \Vdash \varphi(\langle \vec{x}_n, y_n \rangle)$ as required. $\square$

§6. Neometric Convergence. We now define the neometric analogue of a set which contains every p-limit point of a sequence, and prove several lemmas about the notion. This will play the same role for neoclosed forcing that the set of p-limits played for closed forcing in Section 2.

DEFINITION 6.1. Let $\langle x_n \rangle$ be a neotight sequence in $\mathcal{M}$, let A be neoclosed in $\mathcal{M}$, and let p be a condition. We write $x_n \hookrightarrow^p A$, and say that $\langle x_n \rangle$ neometrically p-**converges to** A, if every neocompact set that contains x_n for infinitely many $n \in p$ contains an element of A.

Note that if $x_n \hookrightarrow^p A$ and $q \sqsubseteq p, B \supseteq A$, then $x_n \hookrightarrow^q B$. Also, if $x_n \in A$ for each $n \in p$, then $x_n \hookrightarrow^p A$. We first look at the case where $\langle x_n \rangle$ is relatively compact.

PROPOSITION 6.2. *Suppose $\langle x_n \rangle$ is relatively compact and A is neoclosed in $\mathcal{M}$. Then $x_n \hookrightarrow^p A$ if and only if A contains every p-limit point of $\langle x_n \rangle$.*

Proof: Suppose $x_n \hookrightarrow^p A$, and let a be a p-limit point of $\langle x_n \rangle$. Then $\lim_{n \in q} x_n = a$ for some $q \sqsubseteq p$. For each k let $C_k = \{a\} \cup \{x_n : k \leq n \in q\}$. Then each C_k is a neocompact (even compact) set which contains x_n for infinitely many $n \in p$, and therefore C_k meets A. Thus $A \cap C_k$ is a decreasing chain of nonempty neocompact sets. By countable compactness, $\bigcap_k (A \cap C_k)$ is nonempty. But $\bigcap_k C_k = \{a\}$, so $a \in A$.

Let B be the set of all p-limit points of $\langle x_n \rangle$. Then B is closed. Since $\langle x_n \rangle$ is relatively compact, B is compact, and hence B is neocompact. By hypothesis, $B \subseteq A$. Let C be a neocompact set which contains x_n for all $n \in q$ where $q \sqsubseteq p$. By relative compactness there exists $r \subseteq q$ and $b \in \mathcal{M}$ such that $\lim_{n \in r} x_n = b$. Then $b \in C \cap B$, so C meets B. Hence $x_n \hookrightarrow^p B$ and therefore $x_n \hookrightarrow^p A$. $\square$

We now turn to the general case where $\langle x_n \rangle$ is neotight.

LEMMA 6.3. *Suppose A is neoclosed in $\mathcal{M}$, $\langle x_n \rangle$ is neotight, and $x_n \hookrightarrow^p A$. If $\langle y_n \rangle$ approximates $\langle x_n \rangle$, then $y_n \hookrightarrow^p A$.*

Proof: Let B be a neocompact set which contains x_n for all n and let C be a neocompact set which contains y_n for infinitely many $n \in p$. For each k, the set $C^{1/k}$ is neoclosed, so $B \cap C^{1/k}$ is a neocompact set which contains x_n for infinitely many $n \in p$. By hypothesis, $B \cap C^{1/k}$ meets A for each k. Thus $B \cap C^{1/k} \cap A$ is a decreasing chain of nonempty neocompact sets. By countable compactness the intersection of this chain is nonempty. Therefore C meets A as required. □

Corollary 6.4. *Suppose A is neoclosed in $\mathcal{M}$ and $\langle x_n \rangle$ is neotight. If $\lim_{n\in p} \rho(x_n, A) = 0$, then $x_n \hookrightarrow^p A$.*

Proof: Choose $y_n \in A$ such that $\lim_{n\in p} \rho(x_n, y_n) = 0$. Then $\langle y_n \rangle$ approximates $\langle x_n \rangle$. $\langle y_n \rangle$ is neotight by Lemma 4.3. Since each $y_n \in A$, $y_n \hookrightarrow^p A$. By Lemma 6.3, $x_n \hookrightarrow^p A$. □

We will see later that the converse of the above corollary is false. Here are some necessary conditions for $x_n \hookrightarrow^p A$.

Lemma 6.5. *Suppose A is neoclosed in $\mathcal{M}$, $\langle x_n \rangle$ is neotight, and $x_n \hookrightarrow^p A$.*

(i) If B is neocompact and contains x_n for almost all $n \in p$, then $x_n \hookrightarrow^p (A \cap B)$.

(ii) Every neoopen set which contains A contains x_n for almost all $n \in p$.

(iii) For every neoseparable set $D \supseteq A$, $\lim_{n\in p} \rho(x_n, D) = 0$.

Proof: Let C be a neocompact set containing each x_n.

(i) Let C be a neocompact set which contains x_n for infinitely many $n \in p$. Then $C \cap B$ is also neocompact and contains x_n for infinitely many $n \in p$. Therefore $C \cap B$ meets A, so C meets $A \cap B$ and $x_n \hookrightarrow^p (A \cap B)$.

(ii) Let O be a neoopen set containing A. Suppose that $x_n \in C \setminus O$ for infinitely many $n \in p$. The set $C \setminus O$ is neocompact, and hence meets A. But this is impossible because $A \subseteq O$. Therefore O contains x_n for almost all $n \in p$, as required.

(iii) Let D be the closure of $\bigcup_m D_m$ where D_m is an increasing chain of basic sets. Assume that (ii) fails, so that for some $\varepsilon > 0$ and some $q \sqsubseteq p$, $\rho(x_n, D) > \varepsilon$ for all $n \in q$. For each m, the set $C_m = \{y \in C : \rho(y, D_m) \geq \varepsilon\}$ is neocompact and contains x_n for all $n \in q$. Then the intersection $\bigcap_m C_m$ is neocompact, contains x_n for all $n \in q$, and is disjoint from A. This contradicts the hypothesis that $x_n \hookrightarrow^p A$. □

Lemma 6.6. *Suppose $f : \mathcal{M} \to \mathcal{N}$ is neocontinuous, A is neoclosed in $\mathcal{N}$, and $\langle x_n \rangle$ is neotight in $\mathcal{M}$. Then*

$$f(x_n) \hookrightarrow^p A$$

if and only if

$$x_n \hookrightarrow^p f^{-1}(A).$$

Proof: Let B be a neocompact set in $\mathcal{M}$ which contains each x_n.

First suppose $x_n \hookrightarrow^p f^{-1}(A)$. Let C be a neocompact set in $\mathcal{N}$ such that $f(x_n) \in C$ for infinitely many $n \in p$. Then $f^{-1}(C)$ is neoclosed in $\mathcal{M}$, so $D = B \cap f^{-1}(C)$ is neocompact, and $x_n \in D$ for infinitely many $n \in p$.

Therefore D meets $f^{-1}(A)$, and hence $f(D)$, which is a subset of C, meets A. Thus $f(x_n) \hookrightarrow^p A$.

Now suppose $f(x_n) \hookrightarrow^p A$. Let C be a neocompact set in $\mathcal{M}$ such that $x_n \in C$ for infinitely many $n \in p$. Then $f(C)$ is neocompact in $\mathcal{N}$ and $f(x_n) \in f(C)$ for infinitely many $n \in p$, so $f(C)$ meets A. Then C meets $f^{-1}(A)$, and hence $x_n \hookrightarrow^p f^{-1}(A)$. □

We conclude this section by showing that the neometric convergence relation $x_n \hookrightarrow^p A$ is closely related to the notion of a long sequence which was studied in the paper [FK3].

By a **long sequence in** $\mathcal{M}$ we mean a function $\langle x_J \rangle$ from the set ${}^*\mathbb{N}$ of hyperintegers into $\mathcal{M}$ such that for some internal function $\langle X_J \rangle$ from ${}^*\mathbb{N}$ into $\bar{M}$, we have $x_J = {}^o(X_J)$ for all $J \in {}^*\mathbb{N}$. The internal function $\langle X_J \rangle$ is called a **lifting** of $\langle x_J \rangle$. Note that if $x_J = {}^o(X_J)$ is a long sequence in $\mathcal{M}$, then for each $K \in {}^*\mathbb{N}$, the truncated sequence $y_J = x_{\min(J,K)}$ is also a long sequence in $\mathcal{M}$.

Long sequences give us useful examples of neocompact sets.

Proposition 6.7. *([FK3]). Let $\langle x_J \rangle$ be a long sequence in $\mathcal{M}$ and p be a condition. Then the set*

$$\{x_J : J \leq K, J \in {}^*p\}$$

is basic and the set

$$\{x_J : J \leq K, J \in {}^*p \setminus p\}$$

is neocompact.

Proof: $\{x_J : J \leq K \wedge J \in {}^*p\}$ is the standard part of the internal subset $\{X_J : J \leq K \wedge J \in {}^*p\}$ of monad$(\mathcal{M})$, and $\{x_J : J \leq K \wedge J \in {}^*p \setminus p\}$ is the standard part of the Π^0_1 subset $\{X_J : J \leq K \wedge J \in {}^*p \setminus p\}$ of monad$(\mathcal{M})$. □

Example 6.8. *The converse of Corollary 6.4 is false; there is a neocompact set A and a neotight sequence $\langle x_n \rangle$ such that $x_n \hookrightarrow^p A$ but $\rho(X_n, A) = 1$ for all n.*

Proof: Let $\mathcal{M}$ be a neometric space which neotight sequence $\langle x_n \rangle$ such that $\rho(x_m, x_n) = 1$ for all $m < n \in \mathbb{N}$. (For example, one can take $\mathcal{M}$ to be the nonstandard hull of a hyperfinite dimensional internal Euclidean space, or the space of random variables on an atomless Loeb probability space). Extend $\langle x_n \rangle$ to a long sequence $\langle x_J \rangle$. For some infinite K, $\rho(x_H, x_J) = 1$ for all $H < J \leq K$. By Proposition 6.7, the set $A = \{x_J : J \leq K, J \notin \mathbb{N}\}$ is neocompact, and $\rho(x_n, A) = 1$ for all n. However, an overspill argument shows that A meets any neocompact set which contains infinitely many x_n, so $x_n \hookrightarrow^p A$ for every condition p. □

Here is a characterization of neotight sequences in terms of long sequences.

PROPOSITION 6.9. *A sequence $\langle x_n \rangle$ in neotight in $\mathcal{M}$ if and only if it can be extended to a long sequence in $\mathcal{M}$.*

Proof: If $\langle x_n \rangle$ can be extended to a long sequence $\langle x_J \rangle$ in $\mathcal{M}$, then by Proposition 6.7, each x_n belongs to the basic set $\{x_J : J \in {}^*\mathbb{N}\}$. For the converse, suppose each x_n belongs to a neocompact set C. Let monad$(C) = \bigcap_m B_m$ where B_m is a decreasing chain of internal sets. Then for each finite n there exists an $X_n \in \bigcap_m B_m$ with standard part x_n. By ω_1-saturation the sequence $\langle X_n \rangle$ can be extended to an internal function $\langle X_J \rangle$ from ${}^*\mathbb{N}$ into $\bigcap_m B_m$. This is a lifting of a long sequence $\langle x_J \rangle$ in $\mathcal{M}$ which extends $\langle x_n \rangle$. $\square$

The following notion of "almost everywhere in p" is convenient.

DEFINITION 6.10. Let p be an infinite subset of $\mathbb{N}$, and let $\Phi(J)$ be a statement where J varies over the set ${}^*\mathbb{N}$ of hyperintegers. We say that $\Phi(J)$ **holds** a.e.(p), or that $\Phi(J)$ **holds for all sufficiently small infinite** $J \in {}^*p$, if there is an infinite $K \in {}^*\mathbb{N}$ such that $\Phi(J)$ holds whenever $J \in {}^*p \setminus p$ and $J \le K$.

LEMMA 6.11. *If $\langle x_J \rangle$ and $\langle y_J \rangle$ are long sequences in $\mathcal{M}$ such that $x_n = y_n$ for all $n \in p$, then $x_J = y_J$ a.e.(p).*

Proof: Let $\langle X_J \rangle$ and $\langle Y_J \rangle$ be liftings of $\langle x_J \rangle$ and $\langle y_J \rangle$. Then $\bar{\rho}(X_n, Y_n) \le 1/n$ for all $n \in p$. By overspill, there is an infinite K such that $\bar{\rho}(X_J, Y_J) \le 1/J$, and hence $x_J = y_J$, whenever $J \in {}^*p$ and $J \le K$. $\square$

LEMMA 6.12. *Suppose that for each $m \in \mathbb{N}$, property $\Phi_m(J)$ holds a.e.(p). Then property $\bigwedge_m \Phi_m(J)$ holds a.e.(p).*

Proof: By hypothesis, for each $m \in \mathbb{N}$ there is an infinite hyperinteger $K(m)$ such that $\Phi_m(J)$ holds for all $J \in {}^*p \setminus p$ such that $J \le K(m)$. By ω_1-saturation there is an infinite hyperinteger K such that $K \le K(m)$ for all $m \in \mathbb{N}$. Then $\bigwedge_m \phi_m(J)$ holds for all $J \in {}^*p \setminus p$ such that $J \le K$. $\square$

We now characterize neometric convergence in terms of long sequences.

THEOREM 6.13. *Let $\langle x_n \rangle$ be neotight in $\mathcal{M}$, $\langle x_J \rangle$ be a long sequence in $\mathcal{M}$ extending $\langle x_n \rangle$ and A be neoclosed. Then*

$$x_n \hookrightarrow^p A$$

if and only if

$$x_J \in A \text{ a.e.}(p).$$

Proof: Assume $x_J \in A$ a.e.(p). Let $C \subseteq \mathcal{M}$ be a neocompact set which contains x_n for infinitely many $n \in p$, and let $q = \{n \in p : x_n \in C\}$. Let $\langle y_n \rangle$ be a sequence such that $y_n \in C$ for all $n \in \mathbb{N}$ and $y_n = x_n$ whenever $n \in q$. Then $\langle y_n \rangle$ is neotight in the subspace C of $\mathcal{M}$, and by Proposition 6.9, it can be extended to a long sequence $\langle y_J \rangle$ in C. By Lemma 6.11, there is an infinite $K \in {}^*\mathbb{N}$ such that $x_J = y_J$ whenever $J \in {}^*q$ and $J \leq K$. We may also choose K so that $x_J \in A$ whenever $J \in {}^*p \setminus p$ and $J \leq K$. Thus for any $J \in {}^*q \setminus q$ such that $J \leq K$, we have $x_J = y_J \in A \cap C$, so A meets C and $x_n \hookrightarrow^p A$.

Now suppose $x_n \hookrightarrow^p A$. Since $\langle x_n \rangle$ is neotight, there is a neocompact set B such that $x_n \in B$ for all $n \in \mathbb{N}$. By Lemma 6.5 (i), $x_n \hookrightarrow^p (A \cap B)$. The set $A \cap B$ is neocompact. By Proposition 3.21 there is a decreasing chain of internal sets $C_1, C_2, \ldots \subseteq \bar{M}$ such that monad$(A \cap B) = \bigcap_m C_m$.

Let $\langle X_J \rangle$ lift $\langle x_J \rangle$. We claim that for each $m \in \mathbb{N}$, $X_J \in C_m$ a.e.(p). To see this, suppose that it is not the case that $X_J \in C_m$ a.e.(p). Then there are arbitrarily small $J \in {}^*p \setminus p$ such that $X_J \notin C_m$. Let $q = \{n \in p : X_n \notin C_m\}$. By underspill, q is infinite, and thus is a condition $q \sqsubseteq p$. Let D be the internal set

$$D = \{X_J : J \in {}^*q\} \setminus C_m.$$

Each element of D is near-standard in $\mathcal{M}$, and $X_n \in D$ for all $n \in q$. Moreover, D is disjoint from C_m, and monad$(A \cap B) \subseteq C_m$. Therefore ${}^o(D)$ is a neocompact set in $\mathcal{M}$ which contains x_n for all $n \in q$ but is disjoint from $A \cap B$. This contradicts $x_n \hookrightarrow^p (A \cap B)$ and proves the claim.

By Lemma 6.12 we have $X_J \in \bigcap_m C_m$ a.e.(p), so $x_J \in (A \cap B)$ a.e.(p). □

Corollary 6.14. *Suppose $\langle x_n \rangle$ is neotight, and for each $m \in \mathbb{N}$, A_m is a neoclosed set and $\langle x_n \rangle \hookrightarrow^p A_m$. Then $\langle x_n \rangle \hookrightarrow^p (\bigcap_m (A_m))$.* □

§7. Neoclosed Forcing Theorem. Consider the simple neoclosed formula

$$(\exists \vec{v} \in B) f(\vec{v}) \in C, \tag{4}$$

where B is neocompact and C is neoclosed. The approximations of this formula are

$$(\exists \vec{v} \in B^{1/n}) f(\vec{v}) \in C^{1/n}. \tag{5}$$

The paper [FK1] has an Approximation Theorem showing that condition (5) for all $n \in \mathbb{N}$ implies condition (4). We now come to the Neoclosed Forcing Theorem, which says that $x_n \hookrightarrow^p A$ is a necessary condition for forcing any neoclosed formula which defines A. A consequence of the Neoclosed Forcing Theorem, Corollary 7.3, is a generalization of the Approximation

Theorem in [FK1] which applies to arbitrary neoclosed formulas rather than only to neoclosed formulas of the simple form (4).

THEOREM 7.1. *(Neoclosed Forcing Theorem) For any neoclosed formula* $\varphi(\vec{v})$*, neotight sequence* $\langle \vec{x}_n \rangle$ *of the same sort as* $\vec{v}$*, and condition* p*, if*

$$p \Vdash \varphi(\langle \vec{x}_n \rangle) \tag{6}$$

then

$$\vec{x}_n \hookrightarrow^p A_\varphi \tag{7}$$

where A_φ *is the neoclosed set defined by* φ*. Moreover, if* $\langle \vec{x}_J \rangle$ *is a long sequence extending* $\langle x_n \rangle$*, then*

$$\models \varphi(\vec{x}_J) \tag{8}$$

for all sufficiently small infinite $J \in {}^*p$*.*

Before giving the proof of this theorem, we illustrate how the theorem can be used by revisiting Examples 4.6—4.8. The following result appears to be new.

EXAMPLE 4.6 CONTINUED. *Let* Ω*,* $\mathcal{M}$*,* M*, and* g *be as in Example 4.6. Then for every* $u \in \mathcal{M}$ *there exists a Nash equilibrium for* g, u*.*

Proof: Let $\langle x_n, u_n \rangle$ be as in Example 4.6, and extend it to a long sequence $\langle x_J, u_J \rangle$. Be Example 4.6, $\mathbb{N}$ at $\langle x_k \rangle$ forces the formula (1), which defines the neoclosed set of Nash equilibria for (g, u_k). By the Neoclosed Forcing Theorem, for all sufficiently small infinite J, x_J satisfies (1) and thus is a Nash equilibrium for (g, u_J). If $\langle u_n \rangle$ is a sequence of simple functions converging in probability to u, then $u_J = u$ for infinite J, so x_J is a Nash equilibrium for (g, u). Since it is proved in [MS] that a Nash equilibrium exists for (g, u_n) when u_n is a simple function, it follows that a Nash equilibrium exists for arbitrary u. $\Box$

The next two results were first proved in [K6] using the Approximation Theorem of [FK1] (which, as remarked above, is a special case of Corollary 7.3). Here we will instead use the Neoclosed Forcing Theorem directly. As in Section 4, we let $\overline{\Omega}$ be an adapted Loeb space with time set $[0, 1]$, let w be a Brownian motion on $\overline{\Omega}$, let x range over the space $\mathcal{M}$ of continuous adapted stochastic processes on $\overline{\Omega}$, and let $f(\cdot, \cdot)$ be a bounded continuous real function.

EXAMPLE 4.7 CONTINUED. *Suppose there is a neotight sequence* $\langle x_n \rangle$ *in* $\mathcal{M}$ *such that*

$$\lim_{n\to\infty} \rho\left(x_n(\omega, t), \int_0^t f(s, x_n(\omega, s))dw(\omega, s)\right) = 0$$

for each rational $t \in [0,1]$. Then the stochastic integral equation (2),

$$\forall t \in [0,1]\left(x(\omega,t) = \int_0^t f(s,x(\omega,s))dw(\omega,s)\right),$$

has a solution x in $\mathcal{M}$.

Proof: By Example 4.7, $\mathbb{N}$ at $\langle x_k\rangle$ forces the neoclosed formula (2). Let $\langle x_J\rangle$ be a long sequence extending $\langle x_n\rangle$. By the Neoclosed Forcing Theorem, for all sufficiently small infinite J, x_J satisfies (2). □

EXAMPLE 4.8 CONTINUED. *Let b_k, g_k, and t_k be as in Example (4.8). Suppose there is a sequence $\langle x_n\rangle$ which satisfies the hypotheses of Example 4.7 above, and furthermore that for each k we have*

$$\lim_{n\to\infty} E[(x_n(\cdot.t_k))] = b_k.$$

Then the equation (2) has a solution x in $\mathcal{M}$ with the strong Markov property.

Proof: By Example 4.8, $\mathbb{N}$ at $\langle x_k\rangle$ forces the neoclosed formula (3). By the Neoclosed Forcing Theorem, for all sufficiently small infinite J, x_J satisfies (3), and hence is a solution of (2) with the strong Markov property.

Proof of the Neoclosed Forcing Theorem: Part (8) follows from (7) and Theorem 6.13. We argue by induction on the complexity of the formula $\varphi(\vec{v})$. It will always be understood that C is a neocompact set in the sort space of $\vec{v}$, and $\langle\vec{x}_n\rangle$ is a sequence in C. We must prove that for every neoclosed formula φ, property (6) implies property (7).

Suppose first that φ is an atomic neoclosed formula $\tau(\langle\vec{x}_n\rangle) \in B$. Let $f : \mathcal{M} \to \mathcal{N}$ be the neocontinuous function defined by τ. $f(C)$ is neocompact by the Proposition 3.14 (ii). The set defined by φ is $A_\varphi = f^{-1}(B)$, which is neoclosed by the Proposition 3.14 (iii). In this case, (6) says that

$$\lim_{n\in p} \rho(f(\vec{x}_n), B) = 0.$$

Then (7) follows from Lemma 6.6 and Corollary 6.4.

The infinite conjunction step follows from the inductive hypothesis and Corollary 6.14.

For the finite disjunction step, suppose $\varphi = (\psi \vee \theta)$ and the result holds for ψ and for θ. Assume that (6) holds for φ. To prove (7), let D be a neocompact set which contains $\vec{x}_n$ for all n in an infinite set $q \sqsubseteq p$. By (6), there is an $r \sqsubseteq q$ such that either $r \Vdash \psi(\langle\vec{x}_n\rangle)$, or $r \Vdash \theta(\langle\vec{x}_n\rangle)$, say the former. By inductive hypothesis, $\vec{x}_n \hookrightarrow^r A_\psi$. Since $r \sqsubseteq q$, $\vec{x}_n \in D$ for all $n \in r$. Therefore D contains an element of A_ψ. But $A_\varphi = A_\psi \cup A_\theta$, so D meets A_φ, and (7) follows.

For the existential quantifier step, suppose $\varphi = (\exists v \in B)\psi(\vec{u}, v)$ where B is neocompact, and the result holds for ψ. We assume that (6) holds and prove (7). Again let D be a neocompact set which contains $\vec{x}_n$ for all n in an infinite set $q \sqsubseteq p$. By the definition of forcing there exists $r \sqsubseteq q$ and $\langle y_n \rangle$ in B such that

$$r \Vdash \psi(\langle \vec{x}_n, y_n \rangle).$$

By inductive hypothesis,

$$(\vec{x}_n, y_n) \hookrightarrow^r A_\psi.$$

$D \times B$ is neocompact and contains $(\vec{x}_n, y_n)$ for all $n \in q$, so there exists $(\vec{x}, y) \in A_\psi \cap (D \times B)$. Then $\vec{x} \in A_\varphi \cap D$, and thus $\vec{x}_n \hookrightarrow^p A_\varphi$. This proves (7).

For the universal quantifier step, suppose $\varphi = (\forall v \in D)\psi(\vec{u}, v)$ where D is a neoseparable set, and the result holds for ψ.

We first consider the case where the set D is basic. Thus $D = {}^o(E)$ for some internal set $E \subseteq \text{monad}(\mathcal{N})$. By the definition of forcing, for all $\langle y_n \rangle \in D^{\mathbb{N}}$ we have $p \Vdash \psi(\vec{x}_n, y_n)$. By inductive hypothesis, $(\vec{x}_n, y_n) \hookrightarrow^p A_\psi$ for all $\langle y_n \rangle \in D^{\mathbb{N}}$.

The product $C \times D$ is neocompact, so the intersection $A = A_\psi \cap (C \times D)$ is neocompact. Moreover, for all $\langle y_n \rangle \in D^{\mathbb{N}}$, we have $(\vec{x}_n, y_n) \in C \times D$ for all n, and therefore $(\vec{x}_n, y_n) \hookrightarrow^p A$ by Lemma 6.5 (i). By Lemma 3.4, $\text{monad}(A) = \bigcap_m A_m$ for some decreasing chain of internal sets A_m. Extend $\langle x_n \rangle$ to a long sequence $\langle x_J \rangle$ in C, and take a lifting $\langle X_J \rangle$.

Now fix an integer $m \in \mathbb{N}$. Since A_m is internal, there is an internal function $\langle Z_J^m \rangle$ from ${}^*\mathbb{N}$ into E such that for each J,

(9) $$(\vec{X}_J, Z_J^m) \in A_m \text{ if and only if } (\forall U \in E)(\vec{X}_J, U) \in A_m.$$

Let $z_J^m = {}^o(Z_J^m) \in D$. Then $\langle \vec{X}_J, Z_J^m \rangle$ is a lifting of the long sequence $\langle \vec{x}_J, z_J^m \rangle$ in $C \times D$. Since $(\vec{x}_n, z_n^m) \hookrightarrow^p A$, it follows from Theorem 6.13 that $(\vec{x}_J, z_J^m) \in A$ a.e.(p), and therefore $(\vec{X}_J, Z_J^m) \in A_m$ a.e.(p). By Lemma 6.12,

(10) $$\left(\bigwedge_m (\vec{X}_J, Z_J^m) \in A_m \right) \text{ a.e.}(p).$$

Thus there is an infinite K such that for every $m \in \mathbb{N}$, $(\vec{X}_J, Z_J^m) \in A_m$ for every infinite $J \leq K$ in ${}^*p$.

By (9) and (10),

$$(\forall U \in E)(\vec{X}_J, U) \in \left(\bigcap_m (A_m) \right) \text{ a.e.}(p).$$

Therefore

$$(\forall u \in D)(\vec{x}_J, u) \in A \text{ a.e.}(p),$$

and hence

$$\vec{x}_J \in A_\varphi \text{ a.e.}(p).$$

Finally, by Theorem 6.13 again, the required condition (7) holds. This completes the universal quantifier step in the case that D is basic.

In the case that D is neoseparable, we note that condition (6) says that

$$p \Vdash ((\forall v \in D)\psi)(\langle \vec{x}_n \rangle),$$

and by definition of forcing this is equivalent to

$$(\forall m) p \Vdash ((\forall v \in D_m)\psi)(\langle \vec{x}_n \rangle).$$

By our preceding argument, this implies that

$$(\forall m)\vec{x}_n \hookrightarrow^p B_m, \tag{11}$$

where B_m is the neoclosed set defined by $((\forall v \in D_m)\psi)(\langle \vec{x}_n \rangle)$. Since D is the closure of $\bigcup_m D_m$ and the set defined by ψ is closed, the formula $(\forall v \in D)\psi$ is equivalent to $\bigwedge_m (\forall v \in D_m)\psi$, that is, $A_\varphi = \bigcap_m B_m$. By Lemma 6.12, (11) is equivalent to the required condition (7). This completes our induction. □

We shall now give several consequences of the Neoclosed Forcing and Approximation Theorems.

COROLLARY 7.2. *Let $\varphi(\vec{v})$ be a neoclosed formula. If $p \Vdash \varphi(\langle \vec{x}_n \rangle)$ for some condition p and some neotight sequence $\langle \vec{x}_n \rangle$, then $\models \varphi[\vec{a}]$ for some point $\vec{a}$.* □

COROLLARY 7.3. *Let $\varphi(\vec{v})$ be a neoclosed formula and suppose $\lim_{n\in p} \vec{x}_n = \vec{a}$. Then for each condition p, the following are equivalent.*
(i) $p \Vdash \varphi(\langle \vec{x}_n \rangle)$.
(ii) $\models \varphi[\vec{a}]$.
(iii) $\models \psi[\vec{a}]$ for every $\psi \in \mathcal{A}(\varphi)$.
(iv) $\models \psi[\vec{a}]$ for every $\psi \in \widehat{\mathcal{A}}(\varphi)$.

Proof: Since $\lim_{n\in p} \vec{x}_n = \vec{a}$, we have $\vec{x}_n \hookrightarrow^p \{a\}$. By Lemma 4.5, (i) holds if and only if $p \Vdash \varphi(\langle \vec{a} \rangle)$. The result now follows from the Neoclosed Approximation and Forcing Theorems. □

COROLLARY 7.4. *Suppose $\varphi(v)$ is neoclosed, D is a neoclosed set such that every sequence in D is neotight, and for each approximation $\psi \in \mathcal{A}(\varphi)$ and $n \in \mathbb{N}$ there exists $x \in D^{1/n}$ such that $\models \psi[x]$. Then there exists $x \in D$ such that $\models \varphi[x]$.*

Proof: Let $\langle\psi_n\rangle$ be a cofinal set of approximations of φ. For each $n \in \mathbb{N}$, choose $x_n \in D^{1/n}$ such that $\models \psi_n[x_n]$, and choose $y_n \in D$ with $\rho(x_n, y_n) \leq 1/n$. Then $\langle y_n\rangle$ is neotight in D, and by Lemma 4.3, $\langle x_n\rangle$ is neotight. By the Neoclosed Approximation Theorem, $\mathbb{N} \Vdash \varphi(\langle x_n\rangle)$. Moreover, $\mathbb{N} \Vdash \langle x_n\rangle \in D$. Therefore there exists $x \in D$ such that $\models \varphi[x]$. □

THEOREM 7.5. *(Almost Near Theorem) Let $\varphi(v)$ be a neoclosed formula where v has sort $\mathcal{M}$, let B be a neocompact set in $\mathcal{M}$, and let D be a neoseparable set in $\mathcal{M}$ such that $B \cap \{x : \models \varphi[x]\} \subseteq D$. Then for every real $\varepsilon > 0$ there exists an approximation $\psi \in \mathcal{A}(\varphi)$ such that $B \cap \{x : \models \psi[x]\} \subseteq D^\varepsilon$.*

Proof: Suppose the result fails for ε. Let A be the set defined by φ. Thus $B \cap A \subseteq D$. Take a countable cofinal sequence of approximations $\langle\varphi_k\rangle$ of φ. Then there is a sequence $\langle x_n\rangle$ in B such that for all $n, \models \varphi_n[x_n]$ but $x_n \notin D^\varepsilon$. Since B is neocompact, $\langle x_n\rangle$ is neotight. By the Neoclosed Approximation Theorem, we have $\mathbb{N} \Vdash \varphi(\langle x_n\rangle)$. By the Neoclosed Forcing Theorem, $x_n \hookrightarrow^{\mathbb{N}} A$. The set $B \cap A$ is neocompact, so by Lemma 5.5 there is an m such that $B \cap A \subseteq (D_m)^{\varepsilon/2}$. By Example 3.11, the set $\{x \in B : \rho(x, D_m) \geq \varepsilon\}$ is neocompact. But this set contains x_n for all n and is disjoint from A, contradicting $x_n \hookrightarrow^{\mathbb{N}} A$. This contradiction proves the result. □

In the case that $\mathcal{M}$ is a separable metric space, the preceding result reduces to the Almost Near theorem of Anderson [A].

COROLLARY 7.6. *Let $\varphi(v)$ and $\theta(v)$ be neoclosed formulas and B a neocompact set such that*

$$\{x : x \in B \wedge \models \varphi[x]\} \subseteq \{x : \models \theta[x]\},$$

and suppose that $\{x : \models \theta[x]\}$ is neoseparable. Then for every approximation $\theta_0 \in \mathcal{A}(\theta)$ there is an approximation $\varphi_0 \in \mathcal{A}(\varphi)$ such that

$$B \cap \{x : \models \varphi_0[x]\} \subseteq \{x : \models \theta_0[x]\}.$$

□

COROLLARY 7.7. *(Invariance Theorem) Suppose that $\varphi(\vec{v})$ is a neoclosed formula, $\langle\vec{x}_n\rangle$ is neotight, and for each approximation $\psi \in \mathcal{A}(\varphi)$, $\models \psi[\vec{x}_n]$ for almost all $n \in p$. Then for every neocontinuous function f from the sort space $\mathcal{M}$ of $\vec{v}$ to a complete separable metric space $\mathcal{N}$, there exists $\vec{b} \in \mathcal{M}$ and $q \sqsubseteq p$ such that $\models \varphi[\vec{b}]$ and $\lim_{n\in q} f(\vec{x}_n) = f(\vec{b})$.*

Proof: Since f is neocontinuous, $\langle f(\vec{x}_n)\rangle$ is neotight. Since $\mathcal{N}$ is separable, $\langle f(\vec{x}_n)\rangle$ is relatively compact. Therefore for some $q \sqsubseteq p, \langle f(\vec{x}_n)\rangle$

converges on q to an element $c \in \mathcal{N}$. By the Neoclosed Approximation Theorem,

$$q \Vdash [\varphi(\langle \vec{x}_n \rangle) \wedge f(\langle \vec{x}_n \rangle) = c].$$

By the Neoclosed Forcing Theorem, there is a point $\vec{b}$ such that $\models \varphi[\vec{b}]$ and $f(\vec{b}) = c$. $\square$

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