Existence Theorems in Probability Theory

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0 Introduction: Existence and Compactness

Mathematicians often prove that certain objects exist, i.e., that certain sets are not empty. The interest in such a task may come from various sources, for example: a physicist may need a model so that he can give a formal treatment to a theory which is intended to capture a portion of reality; or a purely mathematical question may arise regarding the existence of an abstract mathematical entity. Experience tells us that existence problems in mathematics are often difficult.

Let's take an informal look at a common way of solving existence problems in analysis (or in a metric space): We want to show that within a set C there exists an object x with a particular property $\phi(x)$, that is, $(\exists x \in C)\phi(x)$. If we cannot find a solution x directly, we may proceed to find "approximate" solutions; we construct an object which is close to C, but perhaps not in C, and which almost has property ϕ. What is usually done is the following: define a sequence $\langle (x_n) \rangle$ of approximations which get better and better as n increases; if we do things right the sequence has a limit and that limit is the desired x.

This is easier said than done. Existence proofs often involve complicated arguments which verify that a sequence of approximations converges in some sense. The most useful tool in existence proofs is the family of compact sets. Almost everything behaves well when restricted to a compact set. Every nonempty compact set of reals has a maximum and minimum, the continuous image of a compact set is compact, every sequence in a compact metric space has a convergent subsequence, a family of compact sets which has the finite intersection property has nonempty intersection, and so on.

A simple example of an existence proof by approximation is Peano's existence theorem for differential equations: One first constructs a sequence of natural approximations (i.e Euler polygons). Then, using Arzela's theorem, a consequence of compactness that guarantees that under certain conditions a sequence of functions converges, one shows that the limit exists and is precisely the solution wanted. Written in symbolic form, the theorem is a statement of the form

$$(\exists x \in C)(f(x) \in D).$$

The approximation procedure gives us the following property:

$$(\forall \varepsilon > 0)(\exists \in C^{\varepsilon})(f(x) \in D^{\varepsilon}).$$

Here C^{ε} is the set $\{x : \rho(x, C) \leq \varepsilon\}$ with ρ the metric on the space where C lives, and similarly for D^{ε}. Then, if we choose a sequence ε_n approaching 0, we obtain a sequence of approximations. The compactness argument (Arzela's theorem) gives the existence of the limit.

In this paper we present a result (called the Approximation Theorem) which intuitively says "it is enough to approximate", or "if you can find approximate solutions then you can conclude that an exact solution exists without going through the convergence argument." In the above notation, the theorem states that:

$$\text{If } (\forall \varepsilon > 0)(\exists \in C^{\varepsilon})(f(x) \in D^{\varepsilon}) \text{ then } (\exists x \in C)(f(x) \in D). \tag{1}$$

The reader should have no problem showing that condition (1) holds in the following case: C is a compact subset of a complete separable metric space M, D is a closed subset of another complete separable metric space N, and f is a continuous function from M into N.

The main point of this paper is that our Approximation Theorem goes beyond the familiar case of convergence in a compact set. First, we work in metric spaces that are not necessarily separable. Second, we identify new families $\mathcal{C}, \mathcal{D}$ and $\mathcal{F}$ of sets C, D and functions f, such that (1) holds. These are the families of neocompact sets, neoclosed sets, and neocontinuous functions. The family of neocompact sets is much larger than the family of compact sets, and provides a wide variety of new opportunities for proving existence theorems by approximation.

Here we shall concentrate on a particular case of interest in probability theory which serves to illustrate the usefulness of our approach. The setting is the general theory of processes where stochastic processes live on adapted spaces: probability spaces $(\Omega, P, \mathcal{G}, \mathcal{G}_t)_{t\in\mathbf{B}}$ where P is a probability measure on a σ-algebra $\mathcal{G}, \mathbf{B}$ is the set of dyadic rationals in an interval $[0, T)$ where $0 < T \leq \infty$, and $(\mathcal{G}_t)$ is a filtration or flow of σ-subalgebras of $\mathcal{G}$. We work with the metric space $L^0(\Omega, M)$ of all P-measurable functions from Ω into a complete separable metric space M, identifying functions which are equal P-almost surely, with the metric of convergence in probability.

The key concept we introduce is that of a neocompact subset of a space $L^0(\Omega, M)$. The notion of a neocompact set is a generalization of the notion of a compact set which, as explained above, is the key in our approach to the solution of existence problems in analysis. The motivation for this new concept comes from nonstandard analysis and the results that have been obtained using nonstandard techniques in stochastic analysis. But do not be discouraged by the word "nonstandard"; it only appears in this paragraph. All the concepts, results, and proofs in this paper are presented in conventional terms, which require only a familiarity with basic measure theory and topology. Nonstandard analysis' main contribution to probability theory is the introduction of "very rich" spaces where many existence proofs can be simplified. With neocompact sets we are able to define the notion of a rich adapted probability space in conventional terms. The proof that such spaces exist, however, makes use of nonstandard analysis and will be postponed to the paper [9]. In that paper we will give a detailed nonstandard background to what is done here and generalize the current treatment to arbitrary metric spaces.

In this paper we develop a theory of "neometric spaces", which are metric spaces endowed with a collection of neocompact sets. After laying the foundations of the theory we dedicate our efforts to the neometric theory of sets of stochastic processes and stochastic integrals. As a first illustration of our approach, we show how a whole new class of optimality problems can be treated and solved in rich probability spaces. A typical result of this type is that for every continuous stochastic process x on a rich adapted space Ω there is a Brownian motion w on Ω which best approximates x (in the metric of convergence in probability with the sup metric on paths). We then illustrate the use of the Approximation Theorem with some nontrivial applications in the theory of existence of solutions of stochastic differential equations.

In many cases, an existence proof using neocompact sets is an improvement of a conventional weak convergence argument, often producing a stronger result with a much simpler proof. The reason for this is that the set of measures on the metric space M induced by the elements of a neocompact subset of $L^0(\Omega, M)$ is always compact in the topology of weak convergence. The original neocompact set captures more information than the compact set of measures induced by its elements, and the neometric machinery provides a framework for carrying this extra information along in a proof by approximation.

The notion of a neocompact family introduced here is a generalization of the family of neocompact sets introduced in the paper [14]. In that paper a notion of forcing analogous to forcing in set theory was introduced for statements about random variables, and a method of proving existence theorems on rich adapted spaces by forcing was developed. This paper is the result of a long series of refinements and simplifications of the methods in [14]. Our aim has been to extract the essential features needed for applications to existence theorems and to present them in a form which is understandable and can be used without any background from mathematical logic.

The neometric methods developed here have also been successfully tested out in another setting in the paper [7], where they are used to improve the existence theorems of Capiński and Cutland [6] on stochastic Navier-Stokes equations.

In Section 1 we present the basic probability concepts and notation used in this paper. The central notions of a neocompact set and a rich adapted space are introduced in Section 2. Neocompact sets, neoclosed sets, and neocontinuous functions are studied in a general setting in Sections 3 and 4. In Section 5, their study is continued in the context of probability theory. The Approximation Theorem is proved in Section 6. In Sections 7 through 11, a we build a library of neocontinuous functions in stochastic analysis which will later be used in applications. In Section

12, our method is illustrated by proving several optimization theorems. The use of the Approximation Theorem is illustrated in Section 13 with a collection of existence theorems for stochastic integral equations.

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1 Preliminaries

Let $0 < T \leq \infty$ and let $\mathbf{B}$ be the set of dyadic rationals in $[0, T)$. We say that $\Omega = (\Omega, P, \mathcal{G}, \mathcal{G}_t)_{t \in \mathbf{B}}$ is an **adapted probability space** if P is a complete probability measure on $\mathcal{G}, \mathcal{G}_t$ is a σ-subalgebra of $\mathcal{G}$ for each $t \in \mathbf{B}$, and $\mathcal{G}_s \subset \mathcal{G}_t$ whenever $s < t$ in $\mathbf{B}$. Let Ω be an adapted probability space which will remain fixed throughout our discussion. For $s \in [0, T)$ we let $\mathcal{F}_s$ be the P-completion of the σ-algebra $\bigcap\{\mathcal{G}_t : s < t \in \mathbf{B}\}$. Then the filtration $\mathcal{F}_s$ is right continuous, that is, for all $s < \infty$ we have $\mathcal{F}_s = \bigcap\{\mathcal{F}_t : s < t\}$. We say that P is **atomless** if any set of positive measure can be partitioned into two sets of positive measure, and that P is atomless on a σ-algebra $\mathcal{F} \subset \mathcal{G}$ if the restriction of P to $\mathcal{F}$ is atomless.

Throughout this paper we let $M = (M, \rho), N = (N, \sigma)$, and $O = (O, \tau)$ be complete separable metric spaces. $L^0(\Omega, M)$ is the set of all P-measurable functions from Ω into M, identifying functions which are equal P-almost surely. ρ_0 is the metric of convergence in probability on $L^0(\Omega, M)$,

$$\rho_0(x, y) = \inf\{\varepsilon : P[\rho(x(\omega), y(\omega)) \leq \varepsilon] \geq 1 - \varepsilon\}.$$

The space of Borel probability measures on M with the Prohorov metric

$$d(\mu, \nu) = \inf\{\varepsilon : \mu(K) \leq \nu(K^\varepsilon) + \varepsilon \text{ for all closed } K \subset M\}$$

is denoted by $\text{Meas}(M)$. It is again a complete separable metric space, and convergence in $\text{Meas}(M)$ is the same as weak convergence. Each measurable function $x : \Omega \to M$ induces a measure $\text{law}(x) \in \text{Meas}(M)$, and the function

$$\text{law} : L^0(\Omega, M) \to \text{Meas}(M)$$

is continuous. Moreover, if the measure P is atomless on $\mathcal{G}_t$, then for each M the function law maps the set of all $\mathcal{G}_t$-measurable $x \in L^0(\Omega, M)$ onto $\text{Meas}(M)$. A set $C \subset \text{Meas}(M)$ is said to be **tight** if for each $\varepsilon > 0$ there is a compact set $K \subset M$ such that $\mu(K) \geq 1 - \varepsilon$ for all $\mu \in C$. The following result is a useful condition for compactness in $\text{Meas}(M)$.

1.1 *(Prohorov's Theorem) A set $C \subset Meas(M)$ has compact closure if and only if there are sequences $\langle b_m \rangle$ of reals and $\langle K_m \rangle$ of compact subsets of M such that $b_m \to 1$ and*

$$C \subset \bigcap_m \{\mu : \mu(K_m) \geq b_m\}. \tag{2}$$

Moreover, the right side of (2) is compact, because the set $\{\mu : \mu(K) \geq b\}$ is closed in Meas(M) for each closed set K and real b. Good references for the notions and results just introduced are [8] and [4].

We consider products of metric spaces M and N so that graphs of functions from M into N can be treated as subsets of the product space. The product $M \times N$ of two metric spaces (M, ρ) and (N, σ) is defined as the cartesian product with the metric $\rho \times \sigma$ given by

$$(\rho \times \sigma)(x, y) = \max(\rho(x_1, y_1), \sigma(x_2, y_2)).$$

Finite products are defined in a similar way.

We identify the points of the spaces $L^0(\Omega, M) \times L^0(\Omega, N)$ and $L^0(\Omega, M \times N)$ in the natural way. The metrics $\rho_0 \times \sigma_0$ and $(\rho \times \sigma)_0$ for these spaces are different but determine the same topology, because

$$(\rho_0 \times \sigma_0)(x, y) \leq (\rho \times \sigma)_0(x, y) \leq \rho_0(x_1, y_1) + \sigma_0(x_2, y_2).$$

2 Neocompact Sets

We begin this section by introducing the main new concept of this paper, the notion of a neocompact set. A family of neocompact sets is a generalization of the family of compact sets, and retains many of its properties. We shall then look at three special cases of this notion, which are found by considering complete separable metric spaces, probability spaces, and adapted probability spaces. We use script letters $\mathcal{M}, \mathcal{N}, \mathcal{O}$ for complete metric spaces which are not necessarily separable.

Definition 2.1 *Let $\mathbf{M}$ be a collection of complete metric spaces $\mathcal{M}$ which is closed under finite cartesian products, and for each $\mathcal{M} \in \mathbf{M}$ let $\mathcal{B}(\mathcal{M})$ be a collection of subsets of $\mathcal{M}$, which we call* **basic sets***. By a* **neocompact family** *over $(\mathbf{M}, \mathcal{B})$ we mean a triple $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ where for each $\mathcal{M} \in \mathbf{M}, \mathcal{C}(\mathcal{M})$ is a collection of subsets of $\mathcal{M}$ with the following properties, where $\mathcal{M}, \mathcal{N}, \mathcal{O}$ vary over $\mathbf{M}$:*

(a) *$\mathcal{B}(\mathcal{M}) \subset \mathcal{C}(\mathcal{M})$;*

(b) *$\mathcal{C}(\mathcal{M})$ is closed under finite unions; that is, if $A, B \in \mathcal{C}(\mathcal{M})$ then $A \cup B \in \mathcal{C}(\mathcal{M})$.*

(c) *$\mathcal{C}(\mathcal{M})$ is closed under finite and countable intersections;*

(d) *If $C \in \mathcal{C}(\mathcal{M})$ and $D \in \mathcal{C}(\mathcal{N})$ then $C \times D \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$;*

(e) *If $C \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$, then the set*

$$\{x : (\exists y \in \mathcal{N})(x, y) \in C\}$$

belongs to $\mathcal{C}(\mathcal{M})$, and the analogous rule holds for each factor in a finite Cartesian product;

(f) *If $C \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$, and D is a nonempty set in $\mathcal{B}(\mathcal{N})$, then*

$$\{x : (\forall y \in D)(x, y) \in C\}$$

belongs to $\mathcal{C}(\mathcal{M})$, and the analogous rule holds for each factor in a finite Cartesian product.

The sets in $\mathcal{C}(\mathcal{M})$ are called **neocompact sets**.

In particular, we shall call $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ the neocompact family **generated by** $(\mathbf{M}, \mathcal{B})$ *if $\mathcal{C}(\mathcal{M})$ is the smallest collection of sets which satisfies (a)–(f).*

The classical example of a neocompact family is the usual family of compact sets in metric spaces. It is not hard to see that the family of compact sets is closed under all of the rules (a)–(f). Thus if $\mathcal{B}(\mathcal{M})$ is the family of all compact subsets of $\mathcal{M}$, then the family of neocompact sets $\mathcal{C}(\mathcal{M})$ generated by $(\mathbf{M}, \mathcal{B})$ is just $\mathcal{B}(\mathcal{M})$ itself, i.e. every neocompact set is compact.

In fact, for the neocompact family of compact sets, $\mathcal{C}(\mathcal{M})$ is closed under arbitrary intersections, and condition (f) holds for arbitrary nonempty sets D. One reason that compact sets are useful in proving existence theorems is that they have the following property:

If $\mathbf{C}$ is a set of compact sets such that any finite subset of $\mathbf{C}$ has a nonempty intersection, then $\mathbf{C}$ has a nonempty intersection.

In many cases, all that is needed is the following weaker property.

Definition 2.2 *We say that a neocompact family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ has the* **countable compactness property** *if for each $\mathcal{M} \in \mathbf{M}$, every decreasing chain $C_0 \supset C_1 \supset \cdots$ of nonempty sets in $\mathcal{C}(\mathcal{M})$ has a nonempty intersection $\bigcap_n C_n$ (which, of course, also belongs to $\mathcal{C}(\mathcal{M})$).*

Our main point in this paper is that there are important additional cases where the neocompact sets have the countable compactness property, and in such cases the neocompact sets can be used to prove existence theorems in the same way that compact sets are used.

We now turn to a second example, based on probability spaces.

Definition 2.3 *Let $\Omega = (\Omega, P, \mathcal{G})$ be a probability space, and let* **M** *be the family of all the metric spaces $\mathcal{M} = L^0(\Omega, M)$ where M is a complete separable metric space. Given sets $B \subset \mathcal{M}$ and $C \subset Meas(M)$, we let $law(B) = \{law(x) : x \in B\}, law^{-1}(C) = \{x \in \mathcal{M} : law(x) \in C\}$. A subset B of $\mathcal{M}$ will be called* **basic**, *$B \in \mathcal{B}(\mathcal{M})$, if either*

(1) *B is compact, or*

(2) *$B = law^{-1}(C)$ for some compact set $C \subset Meas(M)$.*

We say that Ω is a **rich probability space** *if the measure P is atomless and the neocompact family generated by $(\mathbf{M}, \mathcal{B})$ has the countable compactness property.*

We shall see later that rich probability spaces exist, but that the usual classical examples of probability spaces are not rich in the sense of this paper.

Recall that the image of a compact set B by a continuous function is compact, while the inverse image of a compact set C by a continuous function is closed but need not be compact. The following example shows that if the measure P is atomless and M has more than one point, there will always be compact sets $C \subset \mathrm{Meas}(M)$ such that $\mathrm{law}^{-1}(C)$ is not compact. So in this case the neocompact sets go beyond the compact sets and we have a bigger family to work with.

Example 2.4 *Let $\Lambda = (\Lambda, P, \mathcal{G})$ be an atomless probability space and let $N = \{0, 1\}$. The the space $\mathcal{N} = L^0(\Lambda, N)$ is neocompact in itself but not compact.*

Proof: $\mathcal{N}$ is basic and hence neocompact. Since Λ is atomless there is a countable sequence $\langle S_n \rangle$ of sets of measure 1/2 which are mutually independent. Then the sequence $\langle \mathbf{I}_{S_n} \rangle$ of characteristic functions of $\langle S_n \rangle$ is a sequence in $\mathcal{N}$ with no convergent subsequence, so $\mathcal{N}$ is not compact. □

Finally, we turn to the third example, based on adapted probability spaces.

Definition 2.5 *Let $\Omega = (\Omega, P, \mathcal{G}, \mathcal{G}_t)_{t \in \mathbf{B}}$ be an adapted probability space, and let* **M** *be the family of all the metric spaces $\mathcal{M} = L^0(\Omega, M)$ where M is a complete separable metric space. This time a subset B of $\mathcal{M}$ will be called* **basic**, *$B \in \mathcal{B}(\mathcal{M})$, if either*

(1) *B is compact,*

(2) $B = law^{-1}(C)$ *for some compact set* $C \subset Meas(M)$*, or*

(3) $B = \{x \in law^{-1}(C) : x \text{ is } \mathcal{G}_t - measurable\}$ *for some compact* $C \subset Meas(M)$ *and* $t \in \mathbf{B}$.

We say that Ω *is a* **rich adapted space** *if the measure* P *is atomless on* $\mathcal{G}_0, \Omega$ *admits a Brownian motion, and the neocompact family generated by* $(\mathbf{M}, \mathcal{B})$ *has the countable compactness property.*

The following fact, which is of central importance to our approach, is implicit in the paper [14] and will be proved explicitly in [9].

Theorem 2.6 *Rich probability spaces and rich adapted spaces exist.* □

(In fact, in [9] we prove a stronger result which applies to stochastic processes with values in a nonseparable metric space. Given an uncountable cardinal κ, the notion of a κ-rich probability space or adapted space Ω is defined in the same way as a rich space except that $\mathbf{M}$ is the family of all $L^0(\Omega, M)$ where M is a complete metric space with a dense subset of cardinality less than κ, and countable compactness is replaced by the property that any family of fewer than κ neocompact sets with the finite intersection property has nonempty intersection. Thus rich is ω_1-rich. In [9] we show that for each κ, κ-rich probability and adapted spaces exist.)

We shall see in Example 5.7 that in a rich adapted space, $\mathcal{G}_t$ is always a proper subset of the intersection $\mathcal{F}_t = \bigcap\{\mathcal{G}_s : s > t\}$. The same example will show that rich adapted spaces would not exist if the universal projection condition (f) were strengthened by allowing the set D to be neocompact rather than basic.

The countable compactness property is a powerful tool in proving existence theorems on rich adapted spaces, because it can be applied in a wide variety of situations to show that a set $\bigcap_n C_n$ is nonempty. In many cases, a classical existence proof using compact sets can be generalized to get a new existence theorem using neocompact sets. Neocompact sets are useful because, in spite of the many properties they share with compact sets, there are important neocompact sets which are not compact. We have already seen one such set in Example 2.4. We shall give other examples later, such as the set of all stopping times between 0 and 1, and the set of all Brownian motions, on a rich adapted space.

In this paper we shall see that the class of neocompact sets for a rich adapted space is quite extensive, and because of the countable compactness property, richness is a very strong condition. Rich adapted spaces have plenty of room for a new stochastic process with a desired relationship to a given stochastic process. For example, it will be shown in [15] that every rich adapted space is saturated in the sense of the paper [11].

The requirements that P is atomless on $\mathcal{G}_0$ and that Ω admits a Brownian motion insure that the probability space and the filtration are nontrivial. For instance, it avoids the extreme case where Ω has only one element, in which case $L^0(\Omega, M)$ is isomorphic to M and the neocompact sets are the same as the compact sets.

Note that for a rich probability space or a rich adapted space, each finite set of random variables is compact and hence neocompact.

The family of neocompact sets with respect to an adapted space includes the family of neocompact sets with respect to the corresponding probability space, and consequently any rich adapted space is rich as a probability space.

Before going on, let us show that none of the "ordinary" probability spaces are rich. In the literature, one usually works with a probability space of the form $(M, \mu, \mathcal{G})$ where M is a complete separable metric space and μ is the completion of a Borel probability measure on the family of Borel sets $\mathcal{G}$ in M. Let us call such a probability space **ordinary**. Each separable metric space has a countable open basis $\{O_n : n \in \mathbf{N}\}$. We say that a measurable set A is **independent** of a family of sets $\mathbf{S}$ in a probability space $(\Lambda, P, \mathcal{G})$ if

$$P(A \cap B) = P(A)P(B) \text{ for all } B \in \mathbf{S}.$$

In an atomless ordinary probability space, every measurable set can be approximated in probability by sets in the countable open basis, and therefore no set of measure strictly between 0 and 1 can be independent of this open basis. The following theorem shows that no ordinary probability space is rich, and consequently no ordinary adapted space is rich.

Proposition 2.7 *Let $\Lambda = (\Lambda, P, \mathcal{G})$ be a rich probability space. Then for every countable family $\mathbf{S}$ of measurable sets there exists a set of measure $1/2$ which is independent of $\mathbf{S}$.*

Proof: Let $N = \{0, 1\}$ be the two-element metric space, so that $\mathcal{N} = L^0(\Lambda, N)$ is the space of characteristic functions of measurable sets in Λ. Let $\{x_n : n \in \mathbf{N}\}$ be the set of characteristic functions of sets in $\mathbf{S}$. For each k the set

$$B_k = \{(z_1, \ldots, z_k, y) \in \mathcal{N}^{k+1} :$$
$$P(y = 1) = 1/2 \text{ and } y \text{ is independent of } \{z_1, \ldots, z_k\}\}$$

is of the form $\text{law}^{-1}(C)$ where C is closed and hence compact in the compact space $\text{Meas}(N^{k+1})$. Thus B is neocompact in $\mathcal{N}^{k+1}$. Therefore the set

$$A_k = \{y \in \mathcal{N} : P(y = 1) = 1/2 \text{ and } y \text{ is independent of } \{x_1, \ldots, x_k\}\}$$

is a section of the neocompact set B_k. We will show in Proposition 3.6 below that sections of neocompact sets are neocompact, so A_k is neocompact in $\mathcal{N}$. Since the rich probability space Λ is atomless, each A_k is nonempty. Clearly the sets A_k form a decreasing chain. Then by the countable compactness property, there exists $z \in \bigcap_k A_k$. z is the characteristic function of a set of measure 1/2 which is independent of the family $\mathbf{S}$. $\square$

3 General Neocompact Families

In the next two sections we study neocompact families in general. After that we shall concentrate on the particular cases of rich probability spaces and rich adapted spaces.

Blanket Hypothesis 1 *From now on, we assume that* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *is a neocompact family with the countable compactness property where* $\mathbf{M}$ *is a collection of complete metric spaces closed under finite cartesian products, and for each* $\mathcal{M} \in \mathbf{M}, \mathcal{B}(\mathcal{M})$ *contains at least all compact sets in* $\mathcal{M}$.

It will always be understood that $\mathcal{M}, \mathcal{N}, \mathcal{O}$ are spaces in $\mathbf{M}$.

We now introduce a notion analogous to that of a closed set. It is defined from neocompactness using a property that holds in metric spaces for closed and compact sets.

Definition 3.1 *A set* $C \subset \mathcal{M}$ *is* **neoclosed** *in* $\mathcal{M}$ *if* $C \cap D$ *is neocompact in* $\mathcal{M}$ *for every neocompact set* D *in* $\mathcal{M}$.

Here are some easy facts about neoclosed sets.

3.2 $\mathcal{M}$ *is neoclosed in* $\mathcal{M}$.

3.3 *Every neocompact set in* $\mathcal{M}$ *is neoclosed.*

3.4 *Finite unions and countable intersections of neoclosed sets in* $\mathcal{M}$ *are neoclosed in* $\mathcal{M}$.

Proposition 3.5 *If* C *is neoclosed in* $\mathcal{M} \times \mathcal{N}$ *and* D *is neocompact in* $\mathcal{N}$, *then the set*

$$E = \{x : (\exists y \in D)(x, y) \in C\}$$

is neoclosed in $\mathcal{M}$.

Proof: Let A be neocompact in $\mathcal{M}$. Then

$$E \cap A = \{x \in \mathcal{M} : (\exists y \in \mathcal{N})(x, y) \in C \cap (A \times D)\}$$

is neocompact in $\mathcal{M}$. $\square$

The next proposition shows that sections of neocompact or neoclosed sets are neocompact or neoclosed respectively.

Proposition 3.6 *If B is neocompact (neoclosed) in $\mathcal{M} \times \mathcal{N}$, then for each $y \in \mathcal{N}$ the section*

$$A = \{x \in \mathcal{M} : (x, y) \in B\}$$

is neocompact (neoclosed) in $\mathcal{M}$.

Proof: We prove the neocompact case. The set

$$C = \{x \in \mathcal{M} : (\exists y)(x, y) \in B\}$$

is neocompact by (e). Since $\{y\}$ is neocompact in $\mathcal{N}$, the set $C \times \{y\}$ is neocompact in $\mathcal{M} \times \mathcal{N}$ by (d). Then the set $D = B \cap (C \times \{y\})$ is neocompact in $\mathcal{M} \times \mathcal{N}$ by (c). The desired set A is given by

$$A = \{x \in \mathcal{M} : (\exists y)(x, y) \in D\},$$

so A is neocompact in $\mathcal{M}$ by (e). $\square$

We are now ready to define the notion of a neocontinuous function, which is analogous to the classical notion of a continuous function. For this purpose, we need the product of two metric spaces. Recall that the product $\mathcal{M} \times \mathcal{N}$ is the metric space on the cartesian product with the product metric

$$(\rho \times \sigma)(x, y) = \max(\rho(x_1, y_1), \sigma(x_2, y_2)).$$

Definition 3.7 *Let $D \subset \mathcal{M}$. A function $f : D \to \mathcal{N}$ is* **neocontinuous** *from $\mathcal{M}$ to $\mathcal{N}$ if for every neocompact set $A \subset D$ in $\mathcal{M}$, the restriction $f|A = \{(x, f(x)) : x \in A\}$ of f to A is neocompact in $\mathcal{M} \times \mathcal{N}$.*

Remark 3.8 *$f : D \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$ if and only if $f|A$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$ for every neocompact $A \subset D$ in $\mathcal{M}$.*

Proposition 3.9 *If $f : D \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$ and $A \subset D$ is neocompact in $\mathcal{M}$, then the set*

$$f(A) = \{f(x) : x \in A\}$$

is neocompact in $\mathcal{N}$.

Proof: Let G be the graph $f|A$, which is neocompact. Then $f(A) = \{y \in \mathcal{N} : (\exists x \in \mathcal{M})[(x, y) \in G]\}$, so $f(A)$ is neocompact in $\mathcal{N}$. $\square$

Proposition 3.10 *If $f : C \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}, C$ is neoclosed in $\mathcal{M}$, and D is neoclosed in $\mathcal{N}$, then*

$$f^{-1}(D) = \{x \in C : f(x) \in D\}$$

is neoclosed in $\mathcal{M}$.

Proof: Let B be neocompact in $\mathcal{M}$. Then $A = B \cap C$ is neocompact in $\mathcal{M}$ and the graph G of $f|A$ is neocompact in $\mathcal{M} \times \mathcal{N}$. Thus $f^{-1}(D) \cap B = \{x \in \mathcal{M} : (\exists y \in \mathcal{N})[(x, y) \in G \wedge y \in D]\}$ is neocompact in $\mathcal{M}$ as required. $\square$

Corollary 3.11 *If $f : D \to \mathcal{O}$ is neocontinuous from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{O}$ and $b \in \mathcal{N}$, then g is neocontinuous from $\mathcal{M}$ to $\mathcal{O}$, where g is the function with graph*

$$G = \{(x, z) : (x, b) \in D \text{ and } f(x, b) = z\}.$$

We call g the **section** *of f at b.*

Proof: This follows from the fact that sections of neocompact sets are neocompact. $\square$

Corollary 3.12 *For each $b \in \mathcal{N}$, the constant function $f(x) = b$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$.*

Proof: For each neocompact set $A \subset \mathcal{M}$, the graph of $f|A$ is the neocompact set $A \times \{b\}$. $\square$

Proposition 3.13 *Compositions of neocontinuous functions are neocontinuous.*

Proof: Let $f : C \to D$ be neocontinuous from $\mathcal{M}$ to $\mathcal{N}$, and $g : D \to E$ be neocontinuous from $\mathcal{N}$ to $\mathcal{O}$. Let $A \subset C$ be neocompact in $\mathcal{M}$. Then $B = f(A)$ is neocompact in $\mathcal{N}$. The graphs F of $f|A$ and G of $g|B$ are neocompact in $\mathcal{M} \times \mathcal{N}$ and $\mathcal{N} \times \mathcal{O}$. The graph H of $(g \circ f)|A$ is given by

$$H = \{(x, z) \in \mathcal{M} \times \mathcal{O} : (\exists y \in \mathcal{N})[(x, y) \in F \wedge (y, z) \in G]\}.$$

This set is neocompact in $\mathcal{M} \times \mathcal{O}$, so $g \circ f$ is neocontinuous. $\square$

4 Neometric Families

Now that we have defined the notion of a neocontinuous function, we can restrict our attention to neometric families – neocompact families with neocontinuous projection and distance functions. As we shall see in this section, once we know that the projection and distance functions are neocontinuous, we can obtain many other important neocontinuous functions and neoclosed sets.

Definition 4.1 *We call a neocompact family* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *a* **neometric family**, *and call its members* **neometric spaces**, *if the projection and distance functions in* $\mathbf{M}$ *are neocontinuous.*

That is, the projection functions $\pi_1 : \mathcal{M} \times \mathcal{N} \to \mathcal{M}$ and $\pi_2 : \mathcal{M} \times \mathcal{N} \to \mathcal{N}$ are neocontinuous for all $\mathcal{M}, \mathcal{N} \in \mathbf{M}$, the metric space $\mathbf{R}$ of reals is contained in some member $\mathcal{R}$ of $\mathbf{M}$, and for each $\mathcal{M} \in \mathbf{M}$ the distance function ρ of $\mathcal{M}$ is neocontinuous from $\mathcal{M} \times \mathcal{M}$ into $\mathcal{R}$.

The family of ordinary compact sets is a neometric family because in that case neocontinuity coincides with continuity, and the distance function on any metric space is continuous.

We shall see in the next section that the family of neocompact sets for a rich probability space or a rich adapted space is also a neometric family.

Blanket Hypothesis 2 *In addition to Blanket Hypothesis 1, we assume throughout this section that* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *is a neometric family.*

Proposition 4.2 *The identity function on* $\mathcal{M}$ *is neocontinuous.*

Proof: For each neocompact set $A \subset \mathcal{M}$, $\{(x, x) : x \in A\} = (A \times A) \cap \rho^{-1}\{0\}$ is neocompact because $A \times A$ is neocompact and $\rho^{-1}\{0\}$ is neoclosed. $\square$

Proposition 4.3 *(i) If* $f : D \to \mathcal{N}$ *is neocontinuous from* $\mathcal{M}$ *to* $\mathcal{N}$ *and* $g : D \to \mathcal{O}$ *is neocontinuous from* $\mathcal{M}$ *to* $\mathcal{O}$*, then* $h : D \to \mathcal{N} \times \mathcal{O}$ *is neocontinuous from* $\mathcal{M}$ *to* $\mathcal{N} \times \mathcal{O}$ *where* $h(x) = (f(x), g(x))$.

(ii) The function $(x, y) \mapsto (y, x)$ *is neocontinuous from* $\mathcal{M} \times \mathcal{N}$ *to* $\mathcal{N} \times \mathcal{M}$.

(iii) The function $((x, y), z) \mapsto (x, (y, z))$ *from* $(\mathcal{M} \times \mathcal{N}) \times \mathcal{O}$ *to* $\mathcal{M} \times (\mathcal{N} \times \mathcal{O})$ *and its inverse are neocontinuous.*

Proof: (i) Let $A \subset D$ be neocompact. Then the graph of $h|A$ is the set

$$\{(x, (f(x), g(x)) : x \in A\} = \{u \in A \times (f(A) \times g(A)) :$$

$$\rho(f(\pi_1(u)), \pi_1(\pi_2(u))) = 0 \wedge \rho(g(\pi_1(u)), \pi_2(\pi_2(u))) = 0\}$$

which is neocompact because all compositions of f, g, ρ, π_1, and π_2 are neocontinuous.

(ii) Let f be the function in question. Then $f(u) = (\pi_2(u), \pi_1(u))$, so f is neocontinuous by (i). The proof of

(iii) is similar. □

Lemma 4.4 *Every closed ball in $\mathcal{M}$ is neoclosed, that is, for each $x \in \mathcal{M}$ and $r \in \mathbf{R}$ the set $B = \{y \in \mathcal{M} : \rho(x, y) \leq r\}$ is neoclosed in $\mathcal{M}$.*

Proof: The set $[0, r]$ is compact and hence neocompact. Thus the inverse image $\rho^{-1}([0, r])$ is neoclosed in $\mathcal{M} \times \mathcal{M}$, and B is neoclosed because it is a section of $\rho^{-1}([0, r])$. □

Proposition 4.5 *Every neoclosed set in $\mathcal{M}$ is closed in $\mathcal{M}$.*

Proof: We must show that the intersection of a neoclosed set and a compact set is compact. Since every compact set is neocompact, and the intersection of a neoclosed set and a neocompact set is neocompact, it suffices to show that every neocompact set in $\mathcal{M}$ is closed. Let C be neocompact in $\mathcal{M}$ and let x be in the closure of C. Since closed balls are neoclosed, the sequence $D_n = \{y \in C : \rho(x, y) \leq 1/n\}$ is a decreasing chain of nonempty neocompact sets. By the countable compactness property, $D = \bigcap_n D_n$ is nonempty. But $D \subset \{x\} \cap C$, so $x \in C$ and C is closed. □

Lemma 4.6 *For every neocompact set C in $\mathcal{M}$, the set*

$$D = \{\rho(x, y) : x, y \in C\}$$

is bounded.

Proof: $C \times C$ is neocompact in $\mathcal{M} \times \mathcal{M}$ and ρ is neocontinuous, so by Proposition 3.8, D is neocompact in $\mathcal{R}$. Let π be the distance function for $\mathcal{R}$. Then $\pi \times \pi$ is neocontinuous. By Proposition 3.8, for each $n \in \mathbf{N}$ the set

$$D_n = \{x \in D : x \geq n\} = D \cap \{\max(n, x) : x \in D\}$$

$$= D \cap \{(\pi \times \pi)((n, 0), (0, x)) : x \in D\}$$

is neocompact in $\mathcal{R}$. The sets D_n form a decreasing chain, and $\bigcap_n D_n$ is empty. By the countable compactness property, D_n must be empty for some n, and hence D is bounded. □

Lemma 4.7 *For each $x \in \mathcal{M}$ and $r \in \mathbf{R}$, the set*

$$C = \{y \in \mathcal{M} : \rho(x, y) \geq r\}$$

is neoclosed in $\mathcal{M}$.

Proof: Let A be neocompact in $\mathcal{M}$. By the preceding lemma, the set $\{\rho(x, y) : y \in A\}$ is bounded, and hence is contained in the set $[0, n]$ for some n. Therefore

$$C \cap A = \{y \in \mathcal{M} : \rho(x, y) \in [r, n]\} \cap A.$$

The interval $[r, n]$ is compact and hence neoclosed. By Proposition 3.9, the set $\rho^{-1}([r, n])$ is neoclosed in $\mathcal{M} \times \mathcal{M}$. It follows that $\{y \in \mathcal{M} : \rho(x, y) \in [r, n]\}$ is neoclosed in $\mathcal{M}$, so $C \cap A$ is neocompact and C is neoclosed in $\mathcal{M}$. □

Proposition 4.8 *Let C be a separable subset of $\mathcal{M}$. Then C is neocompact in $\mathcal{M}$ if and only if C is compact.*

Proof: Every compact set is neocompact in $\mathcal{M}$. For the other direction, we suppose that a separable set C is neocompact in $\mathcal{M}$ but not compact. Then C has a countable cover by open balls $B_n, n \in \mathbf{N}$ which has no finite subcover. By the preceding lemma, the complement of each B_n is neoclosed, so the sequence

$$C_n = C - \bigcup\{B_m : m \leq n\}$$

is a countable decreasing chain of nonempty neocompact sets in $\mathcal{M}$. By the countable compactness property, $\bigcap_n C_n$ is nonempty, contradicting the fact that $\{B_n\}$ covers C. Thus C is neocompact in $\mathcal{M}$ if and only if C is compact. □

Corollary 4.9 *If C is a neoclosed separable subset of $\mathcal{M}$, then every closed subset D of C is neoclosed.*

Proof: Let A be neocompact in $\mathcal{M}$. Then $C \cap A$ is neocompact and separable. By the preceding proposition, $C \cap A$ is compact, so $D \cap A$ is compact and hence neocompact, and D is neoclosed. □

Example 4.10 *This example shows that a separable closed subset of $\mathcal{M}$ is not necessarily neoclosed. Suppose C is neocompact but not compact in $\mathcal{M}$. Then there are countable sequences $\langle x_n \rangle$ in C such that no subsequence of $\langle x_n \rangle$ converges, and the countable set $D = \{x_n\}$ is closed but not neoclosed in $\mathcal{M}$.*

Proof: Since C is neocompact but not compact, C is closed but not compact, so there is a sequence $\langle x_n \rangle$ in C with no convergent subsequence. $D = \{x_n\}$ is obviously closed and separable but not compact. Suppose D is neoclosed. Then D is neocompact because $D \subset C$. But then by Proposition 4.8, D is compact. $\Box$

This example still works if the neocompact family is enlarged. Let $\mathcal{M}$, C, and D be as above. For any neometric family $(\mathbf{M}', \mathcal{B}', \mathcal{C}')$ such that $\mathcal{M} \in \mathbf{M}', C \in \mathcal{C}'(\mathcal{M})$ and the countable compactness property holds, the countable closed set D is not neoclosed.

Proposition 4.11 *If $f : D \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$, then f is continuous on D.*

Proof: Suppose $a_n \to$ a in D. Then the set $A = \{a_n \in n \in \mathbf{N}\} \cup \{a\}$ is compact and hence neocompact in $\mathcal{M}$. Therefore the graph of $f|A$ is neocompact in $\mathcal{M} \times \mathcal{N}$ and separable, and hence compact. It follows that $f(a_n) \to f(a)$ in $\mathcal{N}$, so f is continuous. $\Box$

Proposition 4.12 *If D is a separable subset of $\mathcal{M}$, then every continuous function $f : D \to \mathcal{N}$ is neocontinuous.*

Proof: Every neocompact set $C \subset D$ in $\mathcal{M}$ is separable, and thus compact by Proposition 4.8. Therefore $f|C$ is compact and hence neocompact in $\mathcal{M} \times \mathcal{N}$. $\Box$

Proposition 4.13 *If $f : C \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$, then f is uniformly continuous with respect to ρ and σ on every neocompact set $D \subset C$ in $\mathcal{M}$.*

Proof: Let $\varepsilon > 0$ and $n \in \mathbf{N}$. Since ρ and f are neocontinuous, the set

$$E_{n,\varepsilon} = \{(x, y) \in D \times D : \rho(x, y) \leq 1/n \text{ and } \rho(f(x), f(y)) \geq \varepsilon\}$$

is neoclosed in $\mathcal{M} \times \mathcal{M}$. Thus for each $\varepsilon > 0$, $E_{n,\varepsilon}$ is a decreasing chain of neocompact sets in $\mathcal{M} \times \mathcal{M}$. But $\bigcap_n E_{n,\varepsilon} = \emptyset$, so by the countable compactness property there is an $m \in \mathbf{N}$ such that $E_{m,\varepsilon} = \emptyset$. Since this holds for each $\varepsilon > 0$, f is uniformly continuous on D. $\Box$

The **distance** $\rho(x, C)$ between an element x and a nonempty set $C \subset \mathcal{M}$ is defined as

$$\rho(x, C) = \inf\{\rho(x, y) : y \in C\}.$$

It is easily seen that $\rho(x, C)$ is a continuous function of x for every nonempty set C.

Proposition 4.14 *Let C be neocompact in $\mathcal{M}$. Then:*
(i) For each $x \in \mathcal{M}$ there exists $y \in C$ such that $\rho(x,y) = \rho(x,C)$.
(ii) For each $\varepsilon > 0$, the set

$$C^{\varepsilon} = \{x \in \mathcal{M} : \rho(x,C) \leq \varepsilon\}$$

is neoclosed in $\mathcal{M}$.

Proof: (i) Let $r = \rho(x,C)$. By Proposition 4.4, the sequence

$$D_n = \{y \in C : \rho(x,y) \leq r + 1/n\}$$

is a decreasing chain of nonempty neocompact sets in $\mathcal{M}$. By the countable compactness property, there exists $y \in \bigcap_n D_n$. Then $y \in C$ and $\rho(x,y) = r$.

(ii) We prove that C^{ε} is neoclosed. We have

$$C^{\varepsilon} = \{x \in \mathcal{M} : (\exists y \in C)\rho(x,y) \leq \varepsilon\}.$$

By Proposition 3.5 and the neocontinuity of ρ, this set is neoclosed in $\mathcal{M}$. □

The classical fact that C^{ε} is closed whenever C is closed does not carry over to neoclosed sets. Here is an example of a neoclosed set D in the neometric family over a rich probability space and an $\varepsilon > 0$ such that D^{ε} is not neoclosed. This gives us another example of a set which is closed but not neoclosed.

Example 4.15 *Consider the neocompact family associated with a rich probability space Ω, and let $\mathcal{N} = L^0(\Omega, \mathbf{N})$ where $\mathbf{N}$ is the set of natural numbers. Let $\Omega = \Omega_1 \cup \Omega_2$ where $P(\Omega_1) = P(\Omega_2) = 1/2$, let $\langle S_n \rangle$ be a sequence of mutually independent subsets of Ω_2 of measure $1/4$, and let x_n be the random variable*

$$x_n(\omega) = n + 1 \text{ if } \omega \in \Omega_1, 1 \text{ if } \omega \in S_n, \text{ and } 0 \text{ if } \omega \in \Omega_2 - S_n.$$

Let $\varepsilon = 1/2$. Then the set $D = \{x_n : n \in \mathbf{N}\}$ is neoclosed but D^{ε} is not neoclosed in $\mathcal{N}$.

Proof: Each compact subset of Meas($\mathbf{N}$) contains only finitely many of the measures law(x_n), and it follows that each neocompact set in $\mathcal{N}$ contains only finitely many x_n's. Therefore D is neoclosed. Let C be the neocompact set $L^0(\Omega, \{0,1\})$ and let y_n be the product of x_n and the characteristic function of Ω_2. Then for each $n, y_n \in C$ and $\rho_0(x_n, y_n) = 1/2 = \varepsilon$. For any other element $z \in C, \rho_0(z, D) > \varepsilon$, so $C \cap D^{\varepsilon} = \{y_n\}$. The sequence $\langle y_n \rangle$ has no convergent subsequence. Thus by Example 4.10, the set $C \cap D^{\varepsilon}$ is not neoclosed. Therefore $C \cap D^{\varepsilon}$ is not neocompact, and D^{ε} is not neoclosed. □

Up to this point, none of the results in this or the preceding section made use of the universal projection rule (f). Thus if we define a **weak neometric family** $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ to be a family which satisfies all the requirements for a neometric family except possibly the universal projection rule (f), then all of the results up to this point hold for weak neometric families.

The remaining results in this section depend on the universal projection rule (f).

Proposition 4.16 *If C is nonempty and basic in $\mathcal{M}$ then the function $f(x) = \rho(x, C)$ is neocontinuous from $\mathcal{M}$ to $\mathbf{R}$.*

Proof: Let A be neocompact in $\mathcal{M}$. Then $A \times C$ is neocompact in $\mathcal{M} \times \mathcal{M}$, so $B = \rho(A \times C)$ is a neocompact subset of $\mathbf{R}$ and hence is compact. The graph of $f|A$ is given by

$$\{(x, r) \in A \times B : \rho(x, C) \leq r \wedge \rho(x, C) \geq r\}$$

$$= \{(x, r) \in A \times B : (\exists y \in C)\rho(x, y) \leq r \wedge (\forall y \in C)\rho(x, y) \geq r\}.$$

By the neocontinuity of ρ and properties (e) and (f), this set is neocompact in $\mathcal{M} \times \mathbf{R}$. Therefore f is neocontinuous. □

To complement the preceding proposition, we now give an example of a neocompact set A in the neometric family over a rich probability space such that the distance function $\rho_0(x, A)$ is not neocontinuous. This shows that the preceding proposition does not hold under the assumption that C is neocompact rather than basic, and gives us an example of a continuous function which is not neocontinuous.

Example 4.17 *Let Ω be a rich probability space. Let $\langle S_n \rangle$ be a sequence of mutually independent subsets of Ω of measure $1/2$. Let y_n be the characteristic function of S_n, and let*

$$A_n = \{x \in L^0(\Omega, \{0,1\}) : (\forall m < n)\rho_0(x, y_m) = 1/2\}, A = \bigcap_n A_n.$$

Then A is a nonempty neocompact set, the closed set

$$B = \{z \in L^0(\Omega, \{0,1\}) : \rho_0(z, A) = 1/2\}$$

is not neoclosed, and the continuous function $\rho_0(\cdot, A)$ is not neocontinuous.

Proof: Since each singleton $\{y_n\}$ is basic, each A_n is neocompact, and therefore A is neocompact. In fact, A can be represented as a section of a basic relation in $L^0(\Omega, \{0,1\} \times M)$ for some complete separable M. Moreover, $y_n \in A_n$, so A_n is a decreasing chain of nonempty neocompact sets. Then A is nonempty by countable compactness.

Suppose that B is neoclosed. Since $\{0,1\}$ is compact, B is neocompact. Then for each n, the set $B \cap A_n$ is neocompact. We have $y_n \in B \cap A_n$ for each n, so the sets $B \cap A_n$ again form a decreasing chain of nonempty neocompact sets. However, $\bigcap_n (B \cap A_n)$ is empty, because any element of this intersection must belong to A but have distance 1/2 from A. This shows that B cannot be neoclosed. $\square$

Although the function $\rho(\cdot, A)$ fails to be neocontinuous, we shall prove in Section 8 that $\rho(\cdot, A)$ is neo-lower semicontinuous for every nonempty neocompact set A.

We can see from this example that no rich probability space can satisfy the stronger form of the universal projection rule (f) in which the set D is allowed to be neocompact rather than basic. Let A and B be the sets from Example 4.17. Then A is nonempty and neocompact but B is not neocompact. As we saw in the proof of Proposition 4.16,

$$B = \{x : (\exists y \in A)\rho_0(x, y) \leq 1/2 \wedge (\forall y \in A)\rho_0(x, y) \geq 1/2\}.$$

Since B is not neocompact, the set

$$\{x : (\forall y \in A)\rho_0(x, y) \geq 1/2\}$$

is not neocompact. Thus the rule (f) cannot hold for universal projections with respect to the neocompact set A. It follows that the set A is not basic.

The next proposition is a generalization of the universal quantifier rule (f) for neocompact sets.

Proposition 4.18 *Suppose C is neoclosed in $\mathcal{M} \times \mathcal{N}$, B_n is a countable increasing chain of basic sets in $\mathcal{N}$, and B is the $\mathcal{N}$-closure of $\bigcup_n B_n$. Then the set*

$$D = \{x \in \mathcal{M} : (\forall y \in B)(x, y) \in C\}$$

is neoclosed in $\mathcal{M}$. Moreover, if C is neocompact in $\mathcal{M} \times \mathcal{N}$ and $B \neq \emptyset$, then D is neocompact in $\mathcal{M}$.

Proof: We first prove the result in the case that C is neocompact in $\mathcal{M} \times \mathcal{N}$. Since $B \neq \emptyset, D$ is contained in the neocompact set $\{x : (\exists y)(x, y) \in C\}$ in $\mathcal{M}$. By (f), the set

$$D_n = \{x : (\forall y \in B_n)(x, y) \in C\}$$

is neocompact in $\mathcal{M}$. Then $\bigcap_n D_n$ is neocompact in $\mathcal{M}$. Clearly $D \subset \bigcap_n D_n$. We show that $D = \bigcap_n D_n$. Suppose $x \in \bigcap_n D_n$ and let $y \in B$. Then there is a sequence

$\langle y_n \rangle$ converging to y in $\mathcal{N}$ such that $y_n \in B_n$ for each $n \in \mathbf{N}$. We have $(x, y_n) \in C$ for each $n \in \mathbf{N}$, and C is closed, so $(x, y) \in C$ and hence $x \in D$. Therefore $D = \bigcap_n D_n$ and D is neocompact in $\mathcal{M}$.

Now suppose C is neoclosed in $\mathcal{M} \times \mathcal{N}$. Let A be neocompact in $\mathcal{M}$ and let

$$E = D \cap A = \{x \in A : (\forall y \in B)(x, y) \in C\}$$

and

$$E_n = \{x \in A : (\forall y \in B_n)(x, y) \in C\} = \{x \in \mathcal{M} : (\forall y \in B_n)(x, y) \in C \cap (A \times B_n)\}.$$

$C \cap (A \times B_n)$ is neocompact in $\mathcal{M} \times \mathcal{N}$, and by the preceding paragraph, E_n and $\bigcap_n E_n$ are neocompact in $\mathcal{M}$. We see as before that $E = \bigcap_n E_n$, so E is neocompact and D is neoclosed in $\mathcal{M}$. $\square$

Corollary 4.19 *Suppose C is neoclosed in $\mathcal{M} \times \mathcal{N}$, and B is a closed separable subset of $\mathcal{N}$. Then the set*

$$D = \{x \in \mathcal{M} : (\forall y \in B)(x, y) \in C\}$$

is neoclosed in $\mathcal{M}$. Moreover, if C is neocompact in $\mathcal{M} \times \mathcal{N}$ and $B \neq \emptyset$, then D is neocompact in $\mathcal{M}$. $\square$

5 Rich Adapted Spaces

For the remainder of this paper, we confine our attention to neocompact families in rich adapted spaces. It will be understood that any results about rich adapted spaces which do not involve the filtration $\mathcal{G}_t$ also hold for rich probability spaces with the same proof.

If $(M, \rho), (N, \sigma)$, and (O, τ) are complete separable metric spaces and $\Omega = (\Omega, P)$ is a probability space, we use the short notation $(\mathcal{M}, \rho_0) = L^0(\Omega, M), (\mathcal{N}, \sigma_0) = L^0(\Omega, N), (\mathcal{O}, \tau_0) = L^0(\Omega, O)$ for the corresponding spaces of random variables.

Blanket Hypothesis 3 *Hereafter we assume that Ω is a rich adapted space. We shall take $(\mathbf{M}, \mathcal{B})$ to be as in Definition 2.5, and take $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ to be the neocompact family generated by $(\mathbf{M}, \mathcal{B})$.*

In this and the next section, we shall not use the assumption that the neocompact sets satisfy the universal projection rule (f), except in examples.

The following facts are easily proved.

5.1 *A set $C \subset \mathcal{M}$ is contained in a neocompact set if and only if law(C) is contained in some compact set $D \subset$ Meas(M), and also if and only if C is contained in a basic set of the form law$^{-1}(D)$ for some compact D.*

Proof: Each basic set B in $\mathcal{M}$ has the property that $B \subset \text{law}^{-1}(D)$ for some compact $D \subset \text{Meas}(M)$, and each of the rules (a)–(f) for neocompact sets preserves this property. □

5.2 *For every compact subset C of M, the set $L^0(\Omega, C)$ is basic and hence neocompact in $\mathcal{M}$.* □

5.3 *If C is a compact set in Meas(M) and $t \in [0, T)$, then the set*

$$D = \{x \in law^{-1}(C) : x \text{ is } \mathcal{F}_t - measurable\}$$

is neocompact in $\mathcal{M}$.

Proof: D is the countable intersection of basic sets

$$D = \bigcap_{s \in \mathbf{B} \cap (t,T]} \{x \in \mathcal{M} : \text{law}(x) \in C \wedge x \text{ is } \mathcal{G}_s - measurable\}.$$

□

5.4 *For every closed set C in Meas(M), the set*

$$law^{-1}(C) = \{x \in \mathcal{M} : law(x) \in C\}$$

is neoclosed in $\mathcal{M}$. □

5.5 *For every closed subset C of M and $r \in [0, 1]$, the sets $L^0(\Omega, C)$ and*

$$\{x \in \mathcal{M} : P[x(\omega) \in C] \geq r\}$$

are neoclosed in $\mathcal{M}$. □

5.6 *For each $t \in [0, T)$, the sets of $\mathcal{G}_t$-measurable functions in $\mathcal{M}$ and of $\mathcal{F}_t$-measurable functions in $\mathcal{M}$ are neoclosed in $\mathcal{M}$.* □

Here is another example of a function which is continuous but not neocontinuous.

Example 5.7 *Let M be a compact metric space with at least two elements, and let C be the neocompact set*

$$C = \{x \in \mathcal{M} : x \text{ is } \mathcal{F}_t - measurable\}.$$

Then the function $f : \mathcal{M} \to \mathbf{R}$ defined by $f(x) = \rho_0(x, C)$ is continuous but not neocontinuous on a rich adapted space.

Proof: It is clear that f is continuous. We give the proof that f is not neocontinuous in the case that M is the two-element space $M = \{0, 1\}$, so that $\mathcal{M} = L^0(\Omega, M)$ is the neocompact space of characteristic functions of P-measurable subsets of Ω. We assume that f is neocontinuous and get a contradiction. If f is neocontinuous then the set

$$A = \{x \in L^0(\Omega, M) : f(x) = 1/2\}$$

is neoclosed in the neocompact space $\mathcal{M}$ and hence is neocompact. We have $x \in A$ if and only if x is the characteristic function of a set which is independent of $\mathcal{F}_t$. Let t_n be a strictly decreasing sequence of elements of $\mathbf{B}$ with $\lim_{n\to\infty} t_n = t$. The sets

$$B_n = A \cap \{x \in L^0(\Omega, M) : x \text{ is } \mathcal{G}_{t_n}\text{-measurable}\}$$

form a decreasing chain of neocompact sets, and since Ω admits a Brownian motion, each B_n is nonempty. However, if f is neocontinuous then $\bigcap_n B_n$ is empty because any $x \in \bigcap_n B_n$ must be $\mathcal{F}_t$-measurable, so $x \notin A$. This contradicts the countable compactness property for Ω. Thus f cannot be neocontinuous. $\square$

We shall see in Section 8 that the function f is neo-lower semicontinuous.

It follows from the preceding example and Proposition 4.16 that the neocompact set C of $\mathcal{F}_t$-measurable functions in $\mathcal{M}$ is not basic.

This example also shows that $\mathcal{G}_t \neq \mathcal{F}_t$ for every $t < \infty$ in $\mathbf{B}$, because by Proposition 4.16, the function $x \mapsto \rho_0(x, B)$ is neocontinuous where B is the basic set of $\mathcal{G}_t$-measurable functions, but the function $x \mapsto \rho_0(x, C)$ is not. Thus in a rich adapted space, the $\mathcal{G}_t$ filtration is never right continuous, that is, $\mathcal{G}_t$ is a **proper** subset of $\bigcap\{\mathcal{G}_s : s > t\}$ for all $t < \infty$ in $\mathbf{B}$.

Here is a reformulation of Prohorov's theorem in our setting.

Proposition 5.8 *A set $C \subset \mathcal{M}$ is contained in a neocompact set if and only if there are a sequence $\langle b_m \rangle$ of reals converging to 1 and a sequence $\langle K_m \rangle$ of compact sets in M such that $C \subset D$ where*

$$D = \bigcap_m \{y \in \mathcal{M} : P[y(\omega) \in K_m] \geq b_m\}.$$

Moreover, the set D is basic and hence neocompact in $\mathcal{M}$.

Proof: Let C be contained in a neocompact in $\mathcal{M}$. By 5.1 there is a compact set $A \subset \text{Meas}(M)$ such that $C \subset \text{law}^{-1}(A)$. By Prohorov's theorem there are sequences $\langle b_m \rangle$ of reals and $\langle K_m \rangle$ of compact subsets of M such that

$$A \subset B = \bigcap_m \{\mu \in \text{Meas}(M) : \mu(K_m) \geq b_m\}.$$

Then $C \subset \text{law}^{-1}(B)$, and we see from the definition of D that $D = \text{law}^{-1}(B)$, so $C \subset D$ as required. Moreover, B is compact in $\text{Meas}(M)$, so D is basic in $\mathcal{M}$. $\square$

It will be convenient to identify each complete separable metric space M with the set of all constant functions in $\mathcal{M} = L^0(\Omega, M)$. With this identification we get a notion of a neocontinuous function from M into $\mathcal{N}$, and a neocontinuous function from $\mathcal{N}$ into M.

Proposition 5.9 *Let M be a complete separable metric space and identify M with the set of constant functions in $\mathcal{M}$. Then M is neoclosed in $\mathcal{M}$.*

Proof: M has the form $\text{law}^{-1}(D)$ for some closed set $D \subset \text{Meas}(M)$. $\square$

We now prove the important fact that the projection functions for $\mathcal{M} \times \mathcal{N}$ are neocontinuous, and the distance function for $\mathcal{M}$ is neocontinuous from $\mathcal{M} \times \mathcal{M}$ into $\mathbf{R}$. That is, the family of neocompact sets for a rich adapted space is a neometric family. Thus all the results of the preceding section hold for rich adapted spaces.

Proposition 5.10 *(i) The projection functions from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{M}$ and to $\mathcal{N}$ are neocontinuous.*

(ii) The distance function $\rho_0 : \mathcal{M} \times \mathcal{M} \to \mathbf{R}$ is neocontinuous.

Proof: (i) Let π be the projection function from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{N}$. The graph G of π is of the form $\text{law}^{-1} D$ where D is neoclosed, so G is neoclosed by 5.4. For each neocompact set C in $\mathcal{M} \times \mathcal{N}$, the graph of $\pi|C$ is equal to the intersection of G with the neocompact set

$$C \times \{y \in \mathcal{N} : (\exists x \in \mathcal{M})(x, y) \in C\}.$$

Thus the graph of $\pi|C$ is neocompact and π is neocontinuous.

(ii) Note that for all $x, y \in \mathcal{M}$, $\rho_0(x, y) \in [0, 1]$. For each $r \in [0, 1]$ there are closed sets D_r and E_r in $\text{Meas}(M \times M)$ such that for all $(x, y), \rho_0(x, y) \leq r$ iff $\text{law}(x, y) \in D_r, \rho_0(x, y) \geq r$ iff $\text{law}(x, y) \in E_r$. Then the graph of ρ_0 is the neoclosed set

$$\bigcap \{\text{law}^{-1}(D_r) \times [0, r] : r \in \mathbf{B}\} \cap \bigcap \{\text{law}^{-1}(E_r) \times [r, 1] : r \in \mathbf{B}\}$$

where $\mathbf{B}$ is the set of dyadic rationals in $[0, 1]$. Since the range of ρ_0 is contained in the compact set $[0, 1]$, it follows that the restriction of ρ_0 to any neocompact set is neocompact. $\square$

Corollary 5.11 *If C is neoclosed in $\mathcal{M} \times \mathcal{N}$, then the set*

$$D = \{x \in \mathcal{M} : (\forall y \in \mathcal{N})(x, y) \in C\}$$

is neoclosed in $\mathcal{M}$. Moreover, if C is neocompact in $\mathcal{M} \times \mathcal{N}$, then D is neocompact in $\mathcal{M}$.

Proof: Let $\{b_n : n \in \mathbf{N}\}$ be a countable dense set in the separable space N and let $B_n = L^0(\Omega, \{b_1, \ldots, b_n\})$. Then B_n is basic in $\mathcal{N}, \bigcup_n B_n \subset \mathcal{N}$, and $\bigcup_n B_n$ is dense in $\mathcal{N}$. Thus Proposition 4.18 may be applied with $B = \mathcal{N}$. $\square$

Proposition 5.12 *Let M be a complete separable metric space. The function law : $\mathcal{M} \to Meas(M)$ is neocontinuous.*

Proof: Let C be neocompact in $\mathcal{M}$. By 5.1, C is contained in a basic set of the form $\text{law}^{-1}(B)$ for some compact set $B \subset \text{Meas}(M)$. Therefore $\text{law}(C)$ is contained in the compact set B, and the graph G of $\text{law}|C$ is contained in the neocompact set $C \times B$. For each $n \in \mathbf{N}$ there is a finite subset D_n of B such that $B \subset (D_n)^{1/n}$. For each $z \in D_n$ the set

$$E_{n,z} = \{(x, y) \in C \times B : \text{law}(x) \in \{z\}^{1/n} \wedge y \in \{z\}^{1/n}\},$$

is neocompact, and since D_n is finite the set

$$E_n = \bigcup \{E_{n,z} : z \in D_n\}$$

is neocompact. Moreover, $G = \bigcap_n E_n$, so G is neocompact as required. $\square$

Corollary 5.13 *For each neocompact set $C \in \mathcal{C}(\mathcal{M})$, law$(C)$ is compact in Meas(M), and hence C is contained in the basic set law^{-1}(law(C)).*

Proof: By 3.8, 4.8, and the preceding theorem. $\square$

Corollary 5.14 *If C is neoclosed in $\mathcal{M}$ then law(C) is closed in Meas(M).*

Proof: Let C be neoclosed in $\mathcal{M}$. Let D be compact in $\text{Meas}(M)$, and let $E = C \cap \text{law}^{-1}(D)$. Since $\text{law}^{-1}(D)$ is neocompact, E is neocompact. Moreover, $\text{law}(E) = \text{law}(C) \cap D$, and by the preceding corollary, $\text{law}(E)$ is compact. Therefore $\text{law}(C)$ is closed. $\square$

A stopping time is a key notion in stochastic analysis. A **stopping time for** Ω is a random variable $\tau \in L^0(\Omega, [0, T])$ such that for each $t \in [0, T), \min(\tau(\omega), t)$ is $\mathcal{F}_t$-measurable. (In the case $T = \infty, [0, T]$ has the compact metric $\rho(x, y) = |\arctan(x) - \arctan(y)|$.)

Proposition 5.15 *The set of stopping times is neocompact in $L^0(\Omega, [0,T])$.*

Proof: For each $t \in [0,T]$, the set C_t of $\mathcal{F}_t$-measurable $y \in L^0(\Omega,[0,T])$ is neocompact. Therefore the set

$$D_t = \{x \in L^0(\Omega,[0,T]) : \min(x(\omega),t) \in C_t\}$$

$$= \{x \in L^0(\Omega,[0,T]) : (\exists y \in C_t) y(\omega) = \min(x(\omega),t)\}$$

is neocompact. The set of stopping times in $L^0(\Omega,[0,T])$ is equal to the countable intersection $\bigcap_{t\in\mathbf{B}} D_t$, and is therefore neocompact in $L^0(\Omega,[0,T])$. $\square$

Proposition 5.16 *If B is neocompact in $L^0(\Omega,\mathbf{R}^d)$, then the set*

$$C = \{x \in L^0(\Omega,\mathbf{R}^d) : (\exists u \in B)[|x(\omega)| \le |u(\omega)| \textit{ almost surely }]\}$$

is neocompact in $L^0(\Omega,\mathbf{R}^d)$.

Proof: By Corollary 5.13, law(B) is compact in Meas$(\mathbf{R}^d)$. It follows that there is a compact set C' in Meas$(\mathbf{R}^d)$ such that law$(C) \subset C'$. The set

$$A = \{(x,u) \in L^0(\Omega,\mathbf{R}^d \times \mathbf{R}) : |x(\omega)| \le |u(\omega)| \text{ almost surely}\}$$

has the form $\text{law}^{-1}(A')$ for some closed set A' in Meas$(\mathbf{R}^{d+1})$, and is thus neoclosed in $L^0(\Omega,\mathbf{R}^{d+1})$. Therefore

$$E = \{(x,u) \in \text{law}^{-1}(C') \times B : |x(\omega)| \le |u(\omega)| \text{ almost surely}\}$$

is neocompact in $L^0(\Omega,\mathbf{R}^{d+1})$. We have

$$C = \{x : (\exists u)(x,u) \in E\},$$

so C is neocompact in $L^0(\Omega,\mathbf{R}^d)$. $\square$

The following is proved by a similar argument.

Proposition 5.17 *Let I be the set of increasing functions $y \in C([0,1],\mathbf{R})$ with $y(0)=0$, and let B be neocompact in $L^0(\Omega,I)$. For an increasing y let $\Delta y(s,t) = y(t)-y(s)$. Then the set*

$$C = \{x \in L^0(\Omega,I) : (\exists u \in B)[\Delta x(\omega)(s,t) \le \Delta u(\omega)(s,t) \textit{ for all } s \le t] \textit{ a.s.}\}$$

is neocompact in $L^0(\Omega,\mathbf{R})$. $\square$

The following result is sometimes useful in showing that a function is neocontinuous.

Proposition 5.18 *Let $f : C \to D$ be a function such that:*
(i) C is neoclosed in $\mathcal{M}$ and D is neoclosed in $\mathcal{N}$;
(ii) If $(x, z) \in C \times D$ and law$(x, z) =$ law$(x, f(x))$ then $z = f(x)$;
(iii) If $x_n, x \in C$ and law$(x_n) \to$ law(x), then law$(x_n, f(x_n)) \to$ law$(x, f(x))$.
Then f is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$.

Proof: Conditions (i)–(iii) hold for any neocompact set $B \subset C$ in place of C. We may therefore assume without loss of generality that C is neocompact. It follows from (iii) that there is a unique continuous function $g : \mathrm{law}(C) \to \mathrm{law}(C \times D)$ such that for all $x \in C, g(\mathrm{law}(x)) = \mathrm{law}(x, f(x))$. Let F be the graph of f and let H be the closure of law(F) in Meas$(M \times N)$. By 5.4, $\mathrm{law}^{-1}(H)$ is neoclosed in $\mathcal{M} \times \mathcal{N}$. Let $(x, z) \in \mathrm{law}^{-1}(H) \cap (C \times D)$. Then there is a sequence $\langle x_n, f(x_n)\rangle$ in F such that $\mathrm{law}(x_n, f(x_n)) \to \mathrm{law}(x, y)$. Then $\mathrm{law}(x_n) \to \mathrm{law}(x)$, so by (iii),$\mathrm{law}(x, y) = \mathrm{law}(x, f(x))$. By (ii), $y = f(x)$, so $(x, y) \in F$. Therefore by (i), $F = \mathrm{law}^{-1}(H) \cap (C \times D)$, so F is neoclosed in $\mathcal{M} \times \mathcal{N}$. Since we are assuming that C is neocompact, law(C) is compact in Meas(M) by Corollary 5.13. Since g is continuous, $g(\mathrm{law}(C))$ is compact in Meas$(M \times N)$ and thus $f(C)$ is contained in the neocompact set $\{y \in D : (\exists x \in C)\mathrm{law}(x, y) \in g(\mathrm{law}(C))\}$ in $\mathcal{N}$. Thus F is neocompact in $\mathcal{M} \times \mathcal{N}$, and f is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$. $\Box$

Here is an easy consequence of Proposition 5.18.

Lemma 5.19 *(Randomization Lemma). Let M, N, and K be complete separable metric spaces.*
(i) If $f : M \to N$ is continuous then the function $g : \mathcal{M} \to \mathcal{N}$ defined by $(g(x))(\omega) = f(x(\omega))$ is neocontinuous.
(ii) If $f : M \times K \to N$ is continuous then the function $g : \mathcal{M} \times K \to \mathcal{N}$ defined by $(g(x, y))(\omega) = f(x(\omega), y)$ is neocontinuous.
(iii) If $f : M \times K \to N$ is continuous and $y \in L^0(\Omega, K)$, then the function $h : \mathcal{M} \to \mathcal{N}$ defined by $(h(x))(\omega) = f(x(\omega), y(\omega))$ is neocontinuous. $\Box$

For example, the function $\min(x(\omega), y(\omega))$ is neocontinuous. Here is a generalization of Proposition 5.18 which allows a parameter u.

Proposition 5.20 *Let $u \in \mathcal{K}$ and let $f : C \to D$ be a function such that:*
(i) C is neoclosed in $\mathcal{M}$ and D is neoclosed in $\mathcal{N}$;
(ii) If $(x, z) \in C \times D$ and law$(u, x, z) =$ law$(u, x, f(x))$ then $z = f(x)$;
(iii) If $x_n, x \in C$ and law$(u, x_n) \to$ law(u, x), then law$(u, x_n, f(x_n)) \to$ law$(u, x, f(x))$.
Then f is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$. $\Box$

6 An Approximation Theorem

In this section we consider existence problems of the form

$$(\exists x \in C) f(x) \in D \tag{3}$$

where C is a neocompact set in $\mathcal{M}$, f is a neocontinuous function from a neoclosed set $E \supset C$ to $\mathcal{N}$, and D is a neoclosed set in $\mathcal{N}$. We prove a useful approximation theorem which states that every problem of the form (3) which is "approximately true" is true. This result, when combined with a large library of neocontinuous functions, leads to very short proofs of a variety of existence theorems. The method may be helpful in the discovery of new results, as well as in the development of proofs which capture an approximation idea in a natural way. As an illustration we shall give some examples concerning stochastic differential equations at the end of this paper.

Since compositions of neocontinuous functions are neocontinuous, other types of existence problems can be put in the form (3). For instance, the problem

$$(\exists x \in C) f(x) = g(x)$$

is equivalent to the problem

$$(\exists x \in C)\rho_0(f(x), g(x)) \in \{0\}$$

which has the form (3) because the distance function ρ_0 is neocontinuous. Similarly, the problem

$$(\exists x \in C)[f_1(x) = g_1(x) \wedge f_2(x) = g_2(x)]$$

is equivalent to the problem

$$(\exists x \in C) \max[\rho_0(f_1(x), g_1(x)), \rho_0(f_2(x), g_2(x))] = 0.$$

We first prove the simple special case of the approximation theorem where the domain of f and the target set D in (3) are both neocompact. The theorem is useful because it is often much easier to solve the approximate existence problem (4) below than the original existence problem (3). It will already be sufficient for most of the applications to stochastic differential equations given at the end of this paper.

Theorem 6.1 *(Simple Approximation Theorem) Let A and B be neocompact in $\mathcal{M}$ and $f : A \to \mathcal{N}$ be neocontinuous from $\mathcal{M}$ to $\mathcal{N}$. Let D be neocompact in $\mathcal{N}$. Suppose that for each $\varepsilon > 0$,*

$$(\exists x \in A \cap B^\varepsilon) f(x) \in D^\varepsilon. \tag{4}$$

Then (3) holds with $C = A \cap B$, that is,

$$(\exists x \in A \cap B) f(x) \in D.$$

Proof: Since f and ρ are neocontinuous, the set

$$E = \{(\varepsilon, x, y, z) \in [0,1] \times A \times B \times D : \rho(x,y) \leq \varepsilon \wedge \rho(f(x), z) \leq \varepsilon\}$$

is neocompact. By property (e), the set

$$F = \{\varepsilon \in [0,1] : (\exists x \in A)(\exists y \in B)(\exists z \in D)(\varepsilon, x, y, z) \in E\}$$

is neocompact and hence closed. The hypothesis (4) says that $(0,1] \subset F$. Therefore $0 \in F$, and hence (3) holds with $C = A \cap B$ as required. $\square$

The following principle is the key lemma for the general case of the approximation theorem.

Definition 6.2 *Given a sequence of sets A_n and a sequence of positive reals ε_n such that*

$$\lim_{n \to \infty} \varepsilon_n = 0,$$

the set

$$A = \bigcap_n ((A_n)^{\varepsilon_n})$$

is called the **diagonal intersection** *of A_n with respect to ε_n.*

Lemma 6.3 *(Closure under diagonal intersections). Let A_n be neocompact in $\mathcal{M}$ for each $n \in \mathbf{N}$ and let $\lim_{n\to\infty} \varepsilon_n = 0$. Then the set $A = \bigcap_n (A_n)^{\varepsilon_n}$ is neocompact in $\mathcal{M}$.*

Proof: By Proposition 4.14, for each n the set $(A_n)^{\varepsilon_n}$ is neoclosed in $\mathcal{M}$. Therefore A is neoclosed in $\mathcal{M}$. By Corollary 5.13, for each n the set $C_n = \text{law}(A_n)$ is compact. The Prohorov metric d on $\text{Meas}(M)$ has the property that

$$d(\text{law}(x), \text{law}(y)) \leq \rho_0(x,y).$$

Therefore for each n, $\text{law}((A_n)^{\varepsilon_n}) \subset (C_n)^{\varepsilon_n}$, and hence $\text{law}(A) \subset C$ where $C = \bigcap_n ((C_n)^{\varepsilon_n})$. Since $\varepsilon_n \to 0$ and each C_n is totally bounded, the set C is totally bounded. Since C is also closed, it is compact. Therefore A is contained in the basic neocompact set $\text{law}^{-1}(C)$ in $\mathcal{M}$, and hence A is neocompact in $\mathcal{M}$. $\square$

In the above proposition, if $A_1 \supset A_2 \supset \cdots$ then A is just the intersection $\bigcap_n A_n$, but in the general case A will properly contain $\bigcap_n A_n$.

Before stating the general case of the approximation theorem, we give some other applications of closure under diagonal intersections. The first application gives a very natural example of a neocompact set.

Theorem 6.4 *For each $d \in \mathbf{N}$ and positive real r, the set*

$$C = \{x \in L^0(\Omega, \mathbf{R}^d) : E[|x|] \leq r\}$$

is basic and hence neocompact.

Proof: We give the proof for $d = 1$. We first show that C is contained in a neocompact set. Let A_n be the neocompact set $L^0(\Omega, [-n, n]\}$. By Chebychev's inequality, for each n and each $x \in C$,

$$P[|x(\omega)| \leq n] \geq 1 - r/n.$$

Then

$$C \subset \bigcap_n ((A_n)^{r/n}).$$

By closure under diagonal intersections, the set on the right is neocompact.

We now show that C is neoclosed. Whether $x \in C$ depends only on the law of x, so $C = \mathrm{law}^{-1}(D)$ for some set $D \subset \mathrm{Meas}(\mathbf{R})$. Since C is contained in a neocompact set, law(C) is contained in a compact set, so we may take D to be contained in a compact set. We show that D is closed. Let z_n be a sequence in D which converges to some $z \in \mathrm{Meas}(\mathbf{R})$. By the Skorokhod representation theorem (see [8], p. 102), there is a sequence of random variables x_n in C and an $x \in L^0(\Omega, \mathbf{R})$ such that $\mathrm{law}(x_n) = z_n$, $\mathrm{law}(x) = z$, and x_n converges to x almost surely. By Fatou's lemma,

$$E[|x|] = E[\liminf(|x_n|)] \leq \liminf(E[|x_n|]) \leq r,$$

so $x \in C$ and hence $z = \mathrm{law}(x) \in D$. Thus D is closed, as we wished to show.

It follows that D is compact, and thus C is a basic set. □

Another application of closure under diagonal intersections is a neocompact version of the Arzela theorem.

Theorem 6.5 *Suppose that: (i) C is neoclosed in $\mathcal{M}$;*

(ii) $f_n : C \to \mathcal{N}$ is neocontinuous on C for each $n \in \mathbf{N}$;

(iii) The family $\{f_n : n \in \mathbf{N}\}$ is equicontinuous on C, that is, for each $k \in \mathbf{N}$ and $x \in C$ there exists $\ell(k, x) \geq k$ such that whenever $z \in C$ and $\rho_0(x, z) \leq 1/\ell(k, x)$, we have $\sigma_0(f_n(x), f_n(z))) \leq 1/k$ for all $n \in \mathbf{N}$;

(iv) f_n approaches f uniformly on C, that is, for each $k \in \mathbf{N}$ there exists $m_k \in \mathbf{N}$ such that for all $x \in C$ and all $n \geq m_k, \sigma_0(f_n(x), f(x)) \leq 1/k$.

Then f is neocontinuous on C.

Proof: We may assume without loss of generality that C is neocompact. Let F_n be the graph of f_n and F be the graph of f. We must show that F is neocompact. We first show that F is neoclosed. Since f_n is neocontinuous, each F_n is neocompact. Therefore for each n and $k, (F_n)^{1/k}$ is neoclosed. We show that F is equal to the neoclosed set

$$G = \bigcap_k \bigcap_{n \geq m_k} ((F_n)^{1/k}).$$

Assumption (iii) insures that $F \subset G$. Suppose $(x, y) \in G$. Then for each $k \in \mathbf{N}$ and all $n \geq m_{\ell(k,x)}$ there exists $z \in C$ such that $\rho_0(x, z) \leq 1/\ell(k, x) \leq 1/k$ and $\sigma_0(y, f_n(z)) \leq 1/\ell(k, x)$. By assumption (ii), $\sigma_0(f_n(x), f_n(z)) \leq 1/k$. Therefore $\sigma_0(f_n(x), y) \leq 2/k$. Then by (iii) we have $y = f(x)$, whence $(x, y) \in F$.

By closure under diagonal intersection, the set $H = \bigcap_k ((F_{m_k})^{1/k})$ is neocompact. Since G is neoclosed and $F = G \subset H, F$ is neocompact. □

Lemma 6.6 *Let C be neocompact in $\mathcal{M}$ and let $\langle x_n \rangle$ be a sequence in $\mathcal{M}$ such that* $\lim_{n\to\infty} \rho_0(x_n, C) = 0$. *Then the set $D = C \cup \{x_n : n \in \mathbf{N}\}$ is neocompact in $\mathcal{M}$.*

Proof: Choose a decreasing sequence ε_n such that $\varepsilon_n \geq \rho_0(x_n, C)$ and $\varepsilon_n \to 0$ as $n \to \infty$. Let $C_n = C \cup \{x_m : m \leq n\}$. Then each C_n is neocompact, and $D \subset \bigcap_n (C_n)^{\varepsilon_n}$. We also have the opposite inclusion $D \supset \bigcap_n (C_n)^{\varepsilon_n}$, because if $y \notin D$ then $y \notin \{x_m : m \in \mathbf{N}\}$, and $\rho_0(y, C) > 0$ by Proposition 4.14, so $y \notin (C_n)^{\varepsilon_n}$ for some n. By closure under diagonal intersections, D is neocompact in $\mathcal{M}$. □

We now come to the general case of the approximation theorem. A still more general theorem is proved in [10].

Theorem 6.7 *(Approximation Theorem) Let A be neoclosed in $\mathcal{M}$ and $f : A \to \mathcal{N}$ be neocontinuous from $\mathcal{M}$ to $\mathcal{N}$. Let B be neocompact in $\mathcal{M}$ and D be neoclosed in $\mathcal{N}$. Suppose that for each $\varepsilon > 0$, equation (4) holds, that is,*

$$(\exists x \in A \cap B^{\varepsilon}) f(x) \in D^{\varepsilon}.$$

Then (3) holds with $C = A \cap B$, that is,

$$(\exists x \in A \cap B) f(x) \in D.$$

Proof: By hypothesis there is a sequence $\langle x_n \rangle$ in A such that for each $n, x_n \in B^{1/n}$ and $f(x_n) \in D^{1/n}$. By the preceding lemma, for each m the set

$$C_m = C \cup \{x_n : m \leq n \in \mathbf{N}\}$$

is a neocompact subset of A. Since f is neocontinuous on A, the graph

$$G_m = \{(x, f(x)) : x \in C_m\}$$

is neocompact in $\mathcal{M} \times \mathcal{N}$.

Since f is neocontinuous and C_1 is neocompact in $\mathcal{M}$, the set $f(C_1)$ is neocompact in $\mathcal{N}$. We may choose $y_n \in D$ such that $\rho(f(x_n), y_n) \le 1/n$. Then $y_n \in (f(C_1))^{1/n}$. Using the preceding lemma again, the set

$$E = D \cap [f(C_1) \cup \{y_n : n \in \mathbf{N}\}]$$

is neocompact in $\mathcal{N}$. We have $y_m \in E$ and $f(x_m) \in E^{1/m}$ for each m. By Proposition 4.14, $E^{1/m}$ is neoclosed, so

$$H_m = G_m \cap (A \times E^{1/m})$$

is a decreasing chain of neocompact sets. We have $(x_m, f(x_m)) \in H_m$, so H_m is nonempty. By the countable compactness property there exists $(x, z) \in \bigcap_m H_m$. Then $x \in C, z \in D$, and $z = f(x)$ as required. □

It is an easy exercise to prove the analogue of the above approximation theorem with compact, closed, and continuous in place of neocompact, neoclosed, and neocontinuous. Anderson [2] gave some interesting applications of this compact analogue of the approximation theorem. Another application is the existence proof for standard Navier-Stokes equations in the first section of [7]. That paper then goes on to apply the neometric form of the approximation theorem to obtain new existence and optimality results for stochastic Navier-Stokes equations.

It is worth reminding the reader at this point that many of our theorems do not involve the filtration $\mathcal{G}_t$ or $\mathcal{F}_t$, and that such results hold for rich probability spaces as well as rich adapted spaces. In particular, the above approximation theorem also holds for rich probability spaces.

Corollary 6.8 *Let A be neoclosed in $\mathcal{M} \times \mathcal{N}$ and $f : A \to \mathcal{K}$ be neocontinuous from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{K}$. Let B be neocompact in $\mathcal{M}$ and D be neoclosed in $\mathcal{K}$. Suppose that $y \in \mathcal{N}$ and for each $\varepsilon > 0$ there exists y_ε within ε of y such that*

$$(\exists x \in B^\varepsilon)[(x, y_\varepsilon) \in A \wedge f(x, y_\varepsilon) \in D^\varepsilon]. \tag{5}$$

Then

$$(\exists x \in B)[(x, y) \in A \wedge f(x, y) \in D]. \tag{6}$$

Proof: Apply the approximation theorem with $\hat{B} = B \times \{y\}$ in place of B. □

We remark that the set

$$\{x \in B : [(x,y) \in A \wedge f(x,y) \in D]\}$$

of solutions of (6) is neocompact in $\mathcal{M}$ for each $y \in \mathcal{N}$, and the set

$$\{y \in \mathcal{N} : (\exists x \in B)[(x,y) \in A \wedge f(x,y) \in D]\}$$

is neoclosed in $\mathcal{N}$ by Proposition 3.5.

7 Uniform Integrability

In this section we extend the theory of neocontinuity to include integrals. The next example shows that some care will be needed. For this example, recall that $L^1(\Omega, \mathbf{R})$ is the space of all integrable functions from Ω into $\mathbf{R}$ with the metric $\rho_1(x,y) = E[|x-y|]$, and that $L^1(\Omega, \mathbf{R})$ is a subset of $L^0(\Omega, \mathbf{R})$.

Example: Let $x_n, n \in \mathbf{N}$ be a sequence in $L^1(\Omega, \mathbf{R})$ and $y \in L^1(\Omega, \mathbf{R})$ such that x_n converges to y in probability but not in mean, that is, $x_n \to y$ in $L^0(\Omega, \mathbf{R})$ but not in $L^1(\Omega, \mathbf{R})$. Then the set $C = \{x_n : n \in \mathbf{N}\} \cup \{y\}$ is a subset of $L^1(\Omega, \mathbf{R})$ which is compact and hence neocompact in $L^0(\Omega, \mathbf{R})$. The function $x \mapsto E[x]$ maps C into $\mathbf{R}$ but is not continuous in $L^0(\Omega, \mathbf{R})$. Thus the expected value function is defined but cannot be neocontinuous from $L^1(\Omega, \mathbf{R})$ into $\mathbf{R}$ in the neometric family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$.

There are two ways around this difficulty. One way, which we develop in this section, is to restrict our attention to uniformly integrable sets of random variables. We shall see that functions such as the integral with respect to P are neocontinuous on every uniformly integrable set. The other way, developed in the next section, is to introduce a neometric analogue of lower semicontinuity. We shall see that the integral and many related functions satisfy this analogue.

Let $p \in [1, \infty)$ and let $b \in M$. $L^p(\Omega, M)$ is the set of all P-measurable functions $x : \Omega \to M$ such that $\rho(x(\omega), b))^p$ is P-integrable, identifying functions which are equal P-almost surely. ρ_p is the metric of convergence in p-th mean,

$$\rho_p(x,y) = E[\rho(x(\omega), y(\omega))^p]^{1/p}.$$

The set $L^p(\Omega, M)$ and the metric ρ_p do not depend on the choice of the reference point $b \in M$. For $q < p$ in $\{0\} \cup [1, \infty)$, $L^p(\Omega, M)$ is a subset of $L^q(\Omega, M)$ but with a different metric.

For $r \in \mathbf{R}$ and $n \in \mathbf{N}$ let $\phi_n(r) = r$ if $r \geq n, \phi_n(r) = 0$ if $r < n$. We say that a subset C of $\mathcal{M}$ is **uniformly p-integrable** if there is a sequence a_n such that $\lim_{n\to\infty} a_n = 0$ and

$$E[\phi_n(\rho(x(\omega), b)^p)] \leq a_n$$

for each $x \in C$ and $n \in \mathbf{N}$. Thus C is uniformly p-integrable in the present sense if and only if the set of real valued random variables

$$\{\rho(x(\omega), b) : x \in C\}$$

is uniformly p-integrable in the usual sense (cf. [4]). Note that uniform p-integrability does not depend on the choice of the point $b \in M$, and that a set C is uniformly p-integrable if and only if every countable subset of C is uniformly p-integrable. Each uniformly p-integrable subset of $\mathcal{M}$ is contained in $L^p(\Omega, M)$.

Examples. Each finite subset of $L^p(\Omega, M)$ is uniformly p-integrable. In the special case that the set $\{\rho(x, y) : x, y \in M\}$ is bounded, the whole space $\mathcal{M}$ is uniformly p-integrable and $L^p(\Omega, M) = \mathcal{M}$. For each $u \in L^p(\Omega, \mathbf{R}^d)$, the set

$$\{x \in L^0(\Omega, \mathbf{R}^d) : |x(\omega)| \leq |u(\omega)| \text{ almost surely}\}$$

is uniformly p-integrable, and is also neocompact by Proposition 5.16.

Lemma 7.1 *Let $q < p$ in $\{0\} \cup [1, \infty)$. A sequence in $L^p(\Omega, M)$ converges in $L^p(\Omega, M)$ if and only if it converges in $L^q(\Omega, M)$ and is uniformly p-integrable.*

Proof: This follows easily from the special case where $M = \mathbf{R}$, where the result is well known. □

Lemma 7.2 *If $p \in [1, \infty)$ and C is a uniformly p-integrable subset of $L^p(\Omega, \mathbf{R}^d)$, then C is contained in a uniformly p-integrable basic set in $L^0(\Omega, \mathbf{R}^d)$.*

Proof: Let $\langle a_n \rangle$ be a sequence such that $\lim_{n\to\infty} a_n = 0$ and C is contained in the set

$$B = \{x : E[\phi_n(|x|^p)] \leq a_n \text{ for all } n \in \mathbf{N}\}.$$

Then B is uniformly p-integrable. Using elementary facts about uniformly integrable sets (cf. [8]), we see that if $x_m \in B$ and $\lim_{m\to\infty} \text{law}(x_m) = \text{law}(x)$, then

$$\lim_{m\to\infty} E[\phi_n(|x_m|^p)] = E[\phi_n(|x|^p)],$$

so $x \in B$. Thus law(B) is closed. Also, every sequence $\langle x_m \rangle$ in B has a subsequence $\langle x_{k_m} \rangle$ such that law(x_{k_m}) converges, so law(B) is compact. It follows that $B = \text{law}^{-1}(\text{law}(B))$, so B is basic. □

Corollary 7.3 *Let $p \in [1,\infty)$. Every uniformly p-integrable set C in $\mathcal{M}$ is contained in a uniformly p-integrable neoclosed set D in $\mathcal{M}$.*

Proof: Let $b \in M$, let $f(x)(\omega) = \rho(x(\omega), b)$, and let $A = f(C)$. Then A is uniformly p-integrable in $L^0(\Omega, \mathbf{R})$, so A is contained in a uniformly p-integrable basic set B in $L^0(\Omega, \mathbf{R})$. f is neocontinuous, so the set $D = f^{-1}(B)$ contains C and is uniformly p-integrable and neoclosed in $\mathcal{M}$. □

The following two results are consequences of Proposition 5.18.

Proposition 7.4 *The integral function $x \mapsto E[x(\omega)]$ is neocontinuous on every uniformly integrable subset of $L^0(\Omega, \mathbf{R}^d)$.*

Proof: Let C be neocompact and uniformly integrable in $L^0(\Omega, \mathbf{R}^d)$, and for $x \in C$ let $f(x) = E[x(\omega)]$. Conditions 5.18 (i) and (ii) are easily seen to hold. Let $x_n, x \in C$ and $\text{law}(x_n) \to \text{law}(x)$. It follows from uniform integrability that $f(x_n) \to f(x)$ in $\mathbf{R}^d$. Therefore $\text{law}(x_n, f(x_n)) \to \text{law}(x, f(x))$, so condition 5.18 (iii) holds. By Proposition 5.18, f is neocontinuous. □

Proposition 7.5 *Let $p \in [1,\infty)$. The distance function ρ_p is neocontinuous on every uniformly p-integrable set $D \subset L^p(\Omega, M) \times L^p(\Omega, M)$.*

Proof: Similar to the preceding proof. □

Theorem 7.6 *Let $t \in \mathbf{B}$ and define $f : L^1(\Omega, \mathbf{R}) \to L^1(\Omega, \mathbf{R})$ by $f(x) = E[x|\mathcal{G}_t]$. Then*

(i) f is neocontinuous on every uniformly integrable subset of $L^0(\Omega, \mathbf{R}^d)$.

(ii) For each $r > 0$, the set $B = \{f(x) : E[|x|] \leq r\}$ is neocompact in $L^0(\Omega, \mathbf{R})$.

Proof: (i) Let A be neocompact in $L^0(\Omega, \mathbf{R}^d)$ and uniformly integrable. Then the set $\{f(x) : x \in A\}$ is uniformly integrable in $L^0(\Omega, \mathbf{R}^d)$, and by Lemma 7.2 is contained in a uniformly integrable neocompact set F in $L^0(\Omega, \mathbf{R}^d)$. The set G of $\mathcal{G}_t$-measurable characteristic functions $x \in L^0(\Omega, \{0,1\})$ is basic in $L^0(\Omega, \mathbf{R}^d)$. Since the integral function $x \mapsto E[x]$ is neocontinuous on every uniformly integrable set,

$$H = \{(x, y, z) \in A \times F \times G : y \text{ is } \mathcal{G}_t\text{-measurable and } E[yz] - E[xz] = 0\}$$

is neocompact. Therefore by property (f), the set

$$I = \{(x, y) \in A \times F : (\forall z \in G)(x, y, z) \in H\}$$

is neocompact. By definition of conditional expectation, I is the graph of the restriction $f|A$. Therefore f is neocontinuous on every uniformly integrable set.

(ii) By Theorem 6.4, the set $C = \{x : E[|x|] \le r\}$ is neocompact. (Note, however, that C is not uniformly integrable, and f is not continuous on C.) We have $B = f(C)$. By 5.6, the set D of $\mathcal{G}_t$-measurable functions is neoclosed, so $C \cap D$ is neocompact. We show that $B = C \cap D$. For all x we have $f(x) \in D$ and $E[|f(x)|] \le E[|x|]$. Therefore $B \subset C \cap D$. Moreover, for all $x \in D$ we have $f(x) = x$, so $C \cap D \subset B$. $\square$

We conclude this section with another example of a function which is continuous but not neocontinuous.

Example 7.7 *Let N be a compact subset of $\mathbf{R}$ with at least two elements and let $t \in [0,T)$. The function $g : L^0(\Omega, N) \to L^0(\Omega, \mathbf{R})$ defined by $g(x) = E[x|\mathcal{F}_t]$ is continuous but is not neocontinuous in a rich adapted space.*

Proof: We give the proof when $N = \{0,1\}$, so that $\mathcal{N} = L^0(\Omega, N)$ is the neocompact space of characteristic functions of P-measurable subsets of Ω. g is clearly continuous. We assume that g is neocontinuous and get a contradiction. Let the measure μ on $\mathbf{R}$ be the point mass at $1/2$. If g is neocontinuous then the set $A = \{x \in L^0(\Omega, N) : \text{law}(g(x)) = \mu\}$ is neoclosed in the neocompact space $\mathcal{N}$ and hence is neocompact. We have $x \in A$ if and only if $E[x|\mathcal{F}_t](\omega) = 1/2$ for almost all ω. Let t_n be a strictly decreasing sequence of elements of $\mathbf{B}$ with $\lim_{n\to\infty} t_n = t$. As in Example 5.7, the sets

$$B_n = A \cap \{x \in L^0(\Omega, N) : x \text{ is } \mathcal{G}_{t_n}\text{-measurable}\}$$

form a decreasing chain of nonempty neocompact sets. However, if g is neocontinuous then $\bigcap_n B_n$ is empty because any $x \in \bigcap_n B_n$ must be $\mathcal{F}_t$-measurable, so $g(x)(\omega) = x(\omega) \in N$ and $x \notin A$. This contradicts the countable compactness property for Ω. Thus g cannot be neocontinuous. $\square$

8 Semicontinuity

In this section we shall continue the study of expected values in the neometric setting. We start with the fact that, by Fatou's lemma, the expected value function is lower semicontinuous from $L^0(\Omega, [0,\infty])$ to $[0,\infty]$. To improve that result we shall introduce a neometric analogue of lower semicontinuity.

It will be convenient to use the compact topological space $\bar{\mathbf{R}} = [-\infty, \infty]$ of extended reals. We make $\bar{\mathbf{R}}$ into a metric space by defining the distance $\sigma(r,s) = |\arctan(s) - \arctan(r)|$, with $\arctan(-\infty) = -\pi/2$, $\arctan(\infty) = \pi/2$. For any metric space $\mathcal{M}$ and set $D \subset \mathcal{M}$, a function $f : D \to \bar{\mathbf{R}}$ is said to be **lower**

semicontinuous, or LSC, if whenever $x_n \to x$ in D, we have $\liminf_{n\to\infty}(f(x_n)) \geq f(x)$. It is easy to see that f is LSC if and only if for every compact set $C \subset \mathcal{M}$, the upper graph $\{(x,r) \in C \times \bar{\mathbf{R}} : f(x) \leq r\}$ is compact.

We shall use the notation $\bar{\mathbf{R}}_+ = [0,\infty], \bar{\mathcal{R}} = L^0(\Omega, \bar{\mathbf{R}})$, and $\bar{\mathcal{R}}_+ = L^0(\Omega, \bar{\mathbf{R}}_+)$.

Definition 8.1 *Let $D \subset \mathcal{M}$. A function $f : D \to \bar{\mathcal{R}}$ is* **neo-LSC** *on $\mathcal{M}$ if for every neocompact set $C \subset D$, the upper graph*

$$\{(x,y) \in C \times \bar{\mathcal{R}} : f(x)(\omega) \leq y(\omega) \ \textit{almost surely}\}$$

is neocompact.

In the special case that f maps D into $\mathbf{R}$, the definition simplies to the following.

Proposition 8.2 *Let $D \subset \mathcal{M}$ and let $f : D \to \mathbf{R}$. Then f is neo-LSC if and only if the set*

$$\{(x,z) \in C \times \bar{\mathbf{R}} : f(x) \leq z\}$$

is neocompact for every neocompact set $C \subset D$. Moreover, if f is neo-LSC then f is LSC.

Proof: Let $C \subset D$ be neocompact,

$$A = \{(x,y) \in C \times \bar{\mathcal{R}} : f(x) \leq y(\omega) \text{ almost surely}\}$$

and

$$B = \{(x,z) \in C \times \bar{\mathbf{R}} : f(x) \leq z\}.$$

If A is neocompact, then B is neocompact because

$$B = A \cap (C \times \bar{\mathbf{R}})$$

and $C \times \bar{\mathbf{R}}_+$ is neoclosed. If B is neocompact, then A is neocompact because

$$A = \{(x,y) : (\exists z)[(x,z) \in B \text{ and } z \leq y(\omega) \text{ almost surely}]\}.$$

This shows that f is neo-LSC if and only if B is neocompact for every C.

Now suppose f is neo-LSC. If C is compact, then $C \times \bar{\mathbf{R}}$ is compact, and therefore the set B is compact. This shows that f is LSC. □

Recall from Proposition 4.16 that if $B \subset \mathcal{M}$ is basic then the distance from a random variable x to B is neocontinuous in x. Examples 4.17 and 5.7 show that the distance from x to a neocompact set is not necessarily neocontinuous, but the next result shows that it is at least neo-LSC.

Proposition 8.3 *If $C \subset \mathcal{M}$ is neocompact, then the function*

$$f(x) = \rho_0(x, C)$$

is neo-LSC from $\mathcal{M}$ into $\mathbf{R}$.

Proof: Let $A \subset \mathcal{M}$ be neocompact. The upper graph

$$D = \{(x, z) \in A \times \bar{\mathbf{R}}_+ : \rho_0(x, C) \leq z\}$$

is neocompact because

$$D = \{(x, z) : (\exists y)[\rho_0(x, y) \leq z \wedge y \in C]\}.$$

Thus f is neo-LSC by Proposition 8.2. □

We now turn to the expected value function.

Theorem 8.4 *(i) The expected value function*

$$E : \bar{\mathcal{R}}_+ \to \bar{\mathbf{R}}$$

is neo-LSC.

(ii) For each $t \in \mathbf{B}$, the conditional expectation function $f : \bar{\mathcal{R}}_+ \to \bar{\mathcal{R}}$ defined by $f(x) = E[x|\mathcal{G}_t]$ is neo-LSC.

Proof: We prove (ii). The proof of (i) will be similar but with $\mathcal{G}$ instead of $\mathcal{G}_t$. Let A be neocompact in $\bar{\mathcal{R}}_+$, and let

$$G = \{(x, y) \in A \times \bar{\mathcal{R}} : f(x)(\omega) \leq y(\omega) \text{ almost surely}\}.$$

We must show that G is neocompact. By the randomization lemma, for each $n \in \mathbf{N}$ the function $x \mapsto \min(n, x)$ is neocontinuous from $\bar{\mathcal{R}}_+$ to $L^0(\Omega, [0, n])$. The set $L^0(\Omega, [0, n])$ is uniformly integrable, so by Theorem 7.6, the function $y \mapsto E[y|\mathcal{G}_t]$ is neocontinuous on $L^0(\Omega, [0, n])$. Therefore for each n the function

$$f_n(x) = E[\min(n, x)|\mathcal{G}_t]$$

is neocontinuous and hence neo-LSC from $\bar{\mathcal{R}}_+$ to $\bar{\mathcal{R}}$. Since $\min(n, x) \leq x$ everywhere, we have

$$f_n(x)(\omega) \leq f(x)(\omega)$$

almost surely. Moreover, $\lim_{n\to\infty} \min(n, x) = x$, so by Fatou's lemma,

$$\liminf_{n\to\infty}(f_n(x)(\omega)) \geq f(x)(\omega)$$

almost surely. Therefore

$$\sup_{n\to\infty}(f_n(x)(\omega)) = f(x)(\omega)$$

almost surely. Then

$$G = \bigcap_n \{(x,y) \in A \times \bar{\mathcal{R}} : f_n(x)(\omega) \le y(\omega) \text{ almost surely}\},$$

and hence G is neocompact as required. $\square$

We cannot replace $\mathcal{G}_t$ by $\mathcal{F}_t$ in the preceding theorem. The proof of Example 7.7 shows that for each $t \in [0,T)$, the conditional expectation function $x \mapsto E[x|\mathcal{F}_t]$ is not even neo-LSC on $L^0(\Omega,\{0,1\})$.

Proposition 8.5 *If D is a separable subset of $\mathcal{M}$, then every LSC function $f : D \to \bar{\mathbf{R}}$ is neo-LSC.*

Proof: Every neocompact set $C \subset D$ in $\mathcal{M}$ is separable, and thus compact by Proposition 4.8. Therefore the upper graph

$$G = \{(x,r) \in C \times \bar{\mathbf{R}} : f(x) \le r\}$$

is compact and hence neocompact, so f is neo-LSC. $\square$

All continuous functions into $\bar{\mathbf{R}}$ are LSC. Here is the analogous result for neo-LSC functions.

Proposition 8.6 *Let $D \subset \mathcal{M}$. Every neocontinuous function $f : D \to \bar{\mathcal{R}}$ is neo-LSC. A function $f : D \to \bar{\mathcal{R}}$ is neocontinuous if and only if both f and $-f$ are neo-LSC.*

Proof: We may assume that D is neocompact. Suppose first that f is neocontinuous. Then the set

$$F = \{(x, f(x)) : x \in C\}$$

is neocompact. The set

$$L = \{(z,y) \in \bar{\mathcal{R}} \times \bar{\mathcal{R}} : z(\omega) \le y(\omega) \text{ almost surely}\}$$

is also neocompact by 5.2. Therefore the upper graph

$$G = \{(x,y) \in C \times \bar{\mathcal{R}} : f(x)(\omega) \le y(\omega) \text{ almost surely}\}$$

$$= \{(x,y) \in C \times \bar{\mathcal{R}} : (\exists z \in \bar{\mathcal{R}})((x,z) \in F \wedge (z,y) \in L)\}$$

is neocompact, and f is neo-LSC. Similarly, $-f$ is neocontinuous and hence neo-LSC.

Now suppose that both f and $-f$ are neo-LSC. Then both the upper and lower graphs

$$G = \{(x,y) \in C \times \bar{\mathcal{R}} : f(x)(\omega) \le y(\omega) \text{ almost surely}\}$$

and

$$H = \{(x,y) \in C \times \bar{\mathcal{R}} : f(x)(\omega) \ge y(\omega) \text{ almost surely}\}$$

are neocompact. Therefore the graph $F = G \cap H$ of $f|C$ is neocompact, so f is neocontinuous. □

The next result is a randomization lemma for lower semicontinuous functions.

Proposition 8.7 *If $f : M \to \bar{\mathbf{R}}$ is LSC then the function $g : \mathcal{M} \to \bar{\mathcal{R}}$ defined by $g(x)(\omega) = f(x(\omega))$ is neo-LSC.*

Proof: Let C be neocompact in $\mathcal{M}$. By Proposition 5.8, there are a sequence $\langle b_m \rangle$ of reals converging to 1 and a sequence $\langle K_m \rangle$ of compact sets in M such that $C \subset D$ where

$$D = \bigcap_m \{x \in \mathcal{M} : P[x(\omega) \in K_m] \ge b_m\}.$$

Moreover, D is neocompact in $\mathcal{M}$. It suffices to show that the set

$$G = \{(x,y) \in D \times \bar{\mathcal{R}} : f(x(\omega)) \le y(\omega) \text{ almost surely}\}$$

is neocompact. Since f is lower semicontinuous, for each m the set

$$A_m = \{(x,y) \in K_m \times \bar{\mathbf{R}} : f(x) \le y\}$$

is compact. By 5.5, the set

$$B_m = \{(x,y) \in D \times \bar{\mathcal{R}} : P[(x,y)(\omega) \in A_m] \ge b_m\}$$

is neoclosed. Also, $G = \bigcap_m B_m$, so G is neoclosed. For each m, the set $\hat{A}_m = L^0(\Omega, A_m)$ is neocompact and $B_m \subset (\hat{A}_m)^{b_m}$. Hence $G \subset \bigcap_m (\hat{A}_m)^{b_m}$. By the diagonal intersection property, the set $\bigcap_m (\hat{A}_m)^{b_m}$ is neocompact. Therefore G is neocompact as required. □

We next prove a result on the composition of two neo-LSC functions. If $B \subset \bar{\mathcal{R}}$, we say that a function $g : B \to \bar{\mathcal{R}}$ is **monotone** if whenever $x \in B$ and $x(\omega) \le y(\omega)$ almost surely, we have $y \in B$ and $g(x)(\omega) \le g(y)(\omega)$ almost surely. For example, the conditional expectation function $x \mapsto E[x|\mathcal{G}_t]$ is monotone with domain $B = \bar{\mathcal{R}}_+$.

Proposition 8.8 *Suppose that $f : A \to B$ is neo-LSC on $\mathcal{M}$, and $g : B \to \bar{\mathcal{R}}$ is neo-LSC on $\bar{\mathcal{R}}$ and monotone. Then the composition $g \circ f : A \to \bar{\mathcal{R}}$ is neo-LSC on $\mathcal{M}$.*

Proof: Let $C \subset A$ be neocompact. Since f and g are neo-LSC and g is monotone, we see in turn that each of the sets

$$F = \{(x, y) \in C \times \bar{\mathcal{R}} : f(x)(\omega) \le y(\omega) \text{ almost surely}\},$$

$$D = \{y \in \bar{\mathcal{R}} : (\exists x)(x, y) \in F\},$$
$$G = \{(y, z) \in D \times \bar{\mathcal{R}} : g(y) \le z \text{ almost surely}\},$$
$$H = \{(x, y, z) : (x, y) \in F \text{ and } (y, z) \in G\},$$
$$J = \{(x, z) : (\exists y)(x, y, z) \in H\}$$

are neocompact. Using the monotonicity of g again, one can check that

$$J = \{(x, z) \in C \times \bar{\mathcal{R}} : ((g \circ f)(x))(\omega) \le z(\omega) \text{ almost surely}\}.$$

Since this set is neocompact, $g \circ f$ is neo-LSC. □

Corollary 8.9 *If $f : D \to \bar{\mathcal{R}}_+$ is neo-LSC on $\mathcal{M}$, then the functions $x \mapsto E[f(x)]$ and $x \mapsto E[f(x)|\mathcal{G}_t]$ are neo-LSC on $\mathcal{M}$.*

Corollary 8.10 *Let $p \in [1, \infty)$. The distance function ρ_p is neo-LSC on $L^p(\Omega, M) \times L^p(\Omega, M)$.*

Proposition 8.11 *The composition of a neocontinuous function and a neo-LSC function is neo-LSC. That is, if $f : A \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$ and $g : f(A) \to \bar{\mathcal{R}}$ is neo-LSC on $\mathcal{N}$, then $g \circ f : A \to \bar{\mathcal{R}}$ is neo-LSC on $\mathcal{M}$.*

Proof: The proof is similar to the proof of Proposition 8.8 but is simpler. □

Corollary 8.12 *For each open set $O \subset M$, the function $x \mapsto P[x(\omega) \in O]$ is neo-LSC from $\mathcal{M}$ to $[0, 1]$.*

Proof: By the Portmanteau theorem ([4], p. 11), the function $f(\mu) = \mu(O)$ is LSC from Meas(M) to $[0, 1]$, and $P[x(\omega) \in O] = f(\text{law}(x))$. □

Corollary 8.13 *For each set $D \subset \mathcal{K}$, neocontinuous function $f : D \to \mathcal{M}$, and neocompact set $C \subset \mathcal{M}$, the function $g : D \to \mathbf{R}$ defined by $g(x) = \rho_0(f(x), C)$ is neo-LSC from $\mathcal{K}$ into $\bar{\mathbf{R}}$.* □

9 Spaces of Stochastic Processes

Sections 9, 10, and 11 contain technical material in stochastic analysis. Our aim in these sections is to build a library of neocompact sets and neocontinuous functions in a rich adapted space. On a first reading, it may be helpful to skip ahead to Section 12 where we begin to apply neocompact sets.

We assume throughout this section that M is a complete separable metric space. To illustrate our method in the simplest case, we work with stochastic processes on the unit time interval $[0,1]$, and take $\mathbf{B}$ to be the set of binary rationals in $[0,1)$. One can also consider stochastic processes with the time line $[0,\infty)$ instead of $[0,1]$. Another possibility, which was used in [14], is to take the set of binary rationals in $[0,\infty)$ as the time line.

Definition 9.1 *By a* **stochastic process** *on* Ω *with values in* M *we mean a measurable function* x *from* $\Omega \times [0,1]$ *into* M*, where* $\Omega \times [0,1]$ *has the product of the rich measure* P *on* Ω *and Lebesgue measure on* $[0,1]$*.*

There are several ways to build metric spaces of stochastic processes. The simplest approach is to form a new metric space N whose elements are functions from $[0,1]$ into M, and then consider metric spaces of random variables with values in N. Here are some possibilities for N.

$C([0,1],M)$ is the space of continuous functions from $[0,1]$ into M with the metric $\sup\{\rho(x(t),y(t)) : t \in [0,1]\}$ of uniform convergence.

$L^0([0,1],M)$ is the space of all Lebesgue measurable functions $x : [0,1] \to M$ with the metric ρ_0 of convergence in probability, identifying functions which are equal a.e.

If $p \in [1,\infty)$, the metric space $L^p([0,1],M)$ is the set of Lebesgue measurable functions $x : [0,1] \to M$ such that $(\rho(x(\cdot),a))^p$ is Lebesgue integrable for every $a \in M$, with the metric ρ_p defined by

$$\rho_p(x(\cdot),y(\cdot)) = \int_0^1 (\rho(x(t),y(t))^p dt^{1/p}.$$

$D([0,1],M)$ is the space of right continuous left limit (rcll) functions from $[0,1]$ into M, with a complicated metric called the Skorokhod J_1 metric. We shall leave the treatment of this space in our framework for a future publication.

Example 9.2 *Some neocontinuous functions on spaces of stochastic processes:*

Let $p \in \{0\} \cup [1,\infty)$. Since the identity function from $C([0,1],M)$ into $L^p([0,1],M)$ is continuous, the identity function from $L^0(\Omega, C([0,1],M))$ into $L^0(\Omega, L^p([0,1],M))$

is neocontinuous by the randomization lemma 5.19. It follows that any neocompact set in $L^0(\Omega, C([0,1], M))$ is neocompact in $L^0(\Omega, L^p([0,1], M))$. Moreover, if $D \subset L^0(\Omega, C([0,1], M))$ and $f : D \to \mathcal{N}$ is neocontinuous from $L^0(\Omega, L^p([0,1], M))$ to $\mathcal{N}$, then f is neocontinuous from $L^0(\Omega, C([0,1], M))$ to $\mathcal{N}$.

The elements of $L^0(\Omega, C([0,1], M))$ are called **continuous** stochastic processes. If x is a continuous stochastic process and $t \in [0,1]$, we let x_t be the random variable $x_t(\omega) = x(\omega)(t)$. Since the application function $(a,t) \mapsto a(t)$ is continuous from $C([0,1], M) \times [0,1]$ into M, we see from the randomization lemma 5.19 that $(x,t) \mapsto x_t$ is neocontinuous from $L^0(\Omega, C([0,1], M)) \times [0,1]$ into $L^0(\Omega, M)$. Since the supremum function $a \mapsto \sup\{a(t) : t \in [0,1]\}$ is continuous from $C([0,1], \mathbf{R})$ to $\mathbf{R}$, the function $x \mapsto \sup\{x(\omega)(t) : t \in [0,1]\}$ is neocontinuous from $L^0(\Omega, C([0,1], \mathbf{R}))$ into $\mathcal{R}$.

Let N be either $L^p([0,1], M)$ or $C([0,1], M)$. If $t \in$ (0,1] and $x \in \mathcal{N}$, we let $x|t$ be the restriction $(x|t)(\omega) = (x(\omega))|[0,t]$. For each t, the function $x \mapsto x|t$ is neocontinuous from $L^0(\Omega, N)$ to $L^0(\Omega, N_t)$ where N_t is either $L^p([0,t], M)$ or $C([0,t], M)$.

If $x \in L^0(\Omega, N)$, a **representative** of x is a measurable function $\tilde{x} : \Omega \times [0,1] \to M$ such that $\tilde{x}(\omega, t) = x(\omega)(t)$ almost surely in $P\times$ Lebesgue measure.

Definition 9.3 *Let $p \in \{0\} \cup [1, \infty)$, and let N be either $C([0,1], M)$ or $L^p([0,1], M)$. A stochastic process $x \in \mathcal{N}$ is said to be* **adapted** *if x has a representative $\tilde{x}$ such that $\tilde{x}(\cdot, t)$ is $\mathcal{F}_t$-measurable for all $t \in [0,1)$. $\mathcal{A}^p(\Omega, M)$ will denote the set of all adapted processes in $L^0(\Omega, L^p([0,1], M))$, and $\mathcal{A}(\Omega, M)$ denotes $\mathcal{A}^0(\Omega, M)$.*

Proposition 9.4 *Let $p \in \{0\} \cup [1, \infty)$, and let N be either $C([0,1], M)$ or $L^p([0,1], M)$. Then the set of adapted processes is neoclosed in $\mathcal{N}$.*

Proof: For each $t \in (0,1)$, the set of $\mathcal{F}_t$-measurable elements of $L^0(\Omega, N_t)$ is neoclosed in $\mathcal{N}_t$, and therefore the set

$$C_t = \{x \in \mathcal{N} : x|t \text{ is } \mathcal{F}_t - \text{measurable}\}$$

is neoclosed in $\mathcal{N}$. The set of adapted processes in $\mathcal{N}$ is equal to the countable intersection $\bigcap_{0<t\in\mathbf{B}} C_t$, and thus is neoclosed in $\mathcal{N}$. □

Example 9.5 *Let $\mathcal{A}^C(\Omega, M)$ be the neoclosed set of adapted processes in $L^0(\Omega, C([0,1], M))$ and let $ST(\Omega)$ be the set of stopping times in $L^0(\Omega, [0,1])$. If $x \in \mathcal{A}^C(\Omega, M)$ and $\tau \in ST(\Omega), x^\tau$ denotes the* **stopped process**

$$x^\tau(\omega, t) = x(\omega, \min(\tau(\omega), t)).$$

The deterministic function $(a,t)\mapsto a^t$ is continuous from $C([0,1],M)\times[0,1]$ into M, so the function $(x,\tau)\mapsto x^\tau$ is neocontinuous from $\mathcal{A}^C(\Omega,M)\times ST(\Omega)$ into $\mathcal{A}^C(\Omega,M)$ by the randomization lemma. Since $ST(\Omega)$ is neocompact by Proposition 5.15, it follows that for every neocompact set $B\subset\mathcal{A}^C(\Omega,M)$, the set of stopped processes

$$\{x^\tau : x\in B \text{ and } \tau\in ST(\Omega)\}$$

is neocompact in $\mathcal{A}^C(\Omega,M)$.

We conclude this section by showing that on each uniformly integrable set in $L^0(\Omega,\mathbf{R})$, the function $x(\omega)\mapsto E[x(\omega)|\mathcal{F}_t]$ is a neocontinuous function into the space of adapted stochastic processes $\mathcal{A}^1(\Omega,\mathbf{R})$. This is in contrast to Example 7.7, where we saw that the corresponding function with t held fixed is not neocontinuous. In proving neocontinuity of this function, we shall use our neocompact analogue of Arzela's theorem.

Definition 9.6 *Let*

$$step_n : L^0(\Omega, M^{\mathbf{B}_n}) \to L^0(\Omega, L^p([0,1],M))$$

be defined by

$$step_n(x)(\omega,t) = x_s(\omega) \text{ whenever } s\in\mathbf{B}_n \text{ and } t\in[s,s+2^{-n}).$$

That is, $\text{step}_n(x)(\omega)$ is the right continuous step function with steps in $\mathbf{B}_n$ whose value at each $s\in\mathbf{B}_n$ is $x_s(\omega)$. It follows easily from Proposition 5.18 that step_n is neocontinuous.

The next theorem is like Proposition 7.6 but with $\mathcal{F}_t$ and a variable t instead of $\mathcal{G}_t$ and a fixed t.

Theorem 9.7 *Define the function $f : L^1(\Omega,\mathbf{R})\to\mathcal{A}^1(\Omega,\mathbf{R})$ by*

$$f(x)(\omega,t) = E[x(\cdot)|\mathcal{F}_t](\omega).$$

(i) The function f is neocontinuous on each uniformly integrable set $C\subset L^0(\Omega,\mathbf{R})$.
(ii) For each $r>0$, the set $\{f(x) : E[|x|]\le r\}$ is neocompact in $\mathcal{A}^1(\Omega,\mathbf{R})$.

Proof: Let ρ be the metric for $L^1([0,1],\mathbf{R})$. For $t\in\mathbf{B}$ let $g(x)(\omega,t)=E[x(\cdot)|\mathcal{G}_t](\omega)$. Then $f(x(\omega))(s)=\lim_{t\downarrow s} g(x)(\omega,t)$ for all $s\in[0,1)$. For each $n\in\mathbf{N}$ let f_n be the function

$$f_n(x(\omega)) = \text{step}_n((E[x(\cdot)|\mathcal{G}_t](\omega) : t\in\mathbf{B}_n)).$$

Then $f_n(x(\omega))(t) = g(x)(\omega, t)$ for all $t \in \mathbf{B}_n$.

(i) Let $C \subset L^0(\Omega, \mathbf{R})$ be a uniformly integrable set. Each f_n is neocontinuous on C. The family $\{f_n : n \in \mathbf{N}\}$ is equicontinuous on C because for each $x, y \in C$ and each n,

$$\rho_0(f_n(x), f_n(y)) \le E[\sum_{t \in \mathbf{B}_n} |E[x|\mathcal{G}_t] - E[y|\mathcal{G}_t]||]/2^n$$

$$\le \sum_{t \in \mathbf{B}_n} E[|E[x-y|\mathcal{G}_t]|]/2^n \le \sum_{t \in \mathbf{B}_n} E[|x-y|]/2^n = E[|x-y|]$$

and C is uniformly integrable.

We show that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly on C. For $a, b \in \mathbf{R}$ let $U(a,b)(\omega)$ be the number of times the path $g(x)(\omega, t)$ crosses from below a to above b when $t \in \mathbf{B}$. Doob's upcrossing inequality (cf. [8]) shows that

$$E[U(a,b)] \le \frac{E[|x|]}{b-a}.$$

Since C is uniformly integrable, the set of values $E[|x|]$ for $x \in C$ is bounded. Let $\varepsilon > 0$ and choose $c > 0$ so that $E[|x|]/c \le \varepsilon$ for all $x \in C$. By Doob's martingale inequality,

$$P[\sup_{t \in \mathbf{B}} |g(x)(\omega, t)| \ge c] \le \varepsilon \text{ for all } x \in C.$$

For $x \in C$ let $S(x)$ be the set of all ω such that $\sup_{t\in\mathbf{B}} |g(x)(\omega,t)| \le c$, so that $P[S(x)] \ge 1 - \varepsilon$. Now let $x \in C$ and $\omega \in S(x)$, and let $V_n(x, \omega)$ be the set of times $t \in [0, 1)$ at which the paths $f_n(x)(\omega, t)$ and $f(x)(\omega, t)$ differ by at least ε. If the set $V_n(x, \omega)$ has Lebesgue measure at least ε/c, then it must meet at least $2^n(\varepsilon/c)$ of the subintervals of $[0, 1]$ bounded by points of $\mathbf{B}_n$. Within each of these subintervals, the path $g(x)(\omega, t)$ must have an upcrossing of some subinterval of $[-c, c]$ of length $\varepsilon/2$. In view of the upcrossing inequality above, there is a $k \in \mathbf{N}$ such that for all $n \ge k$, the set $V_n(x, \omega)$ has Lebesgue measure $\le \varepsilon/2c$, and thus for $x \in C$ and $\omega \in S(x)$,

$$\rho(f_n(x)(\omega, \cdot), f(x)(\omega, \cdot)) = \int_0^1 |f_n(x)(\omega, t)) - f(x)(\omega, t)| dt \le \varepsilon(1 - \varepsilon/c) + \varepsilon \le 2\varepsilon.$$

For all $x \in C$ and $n \ge k$, $P[S(x)] \ge 1 - \varepsilon$, and hence $\rho_0(f_n(x), f(x)) \le 2\varepsilon$. Therefore $f_n(x) \to f(x)$ uniformly on C. It now follows by the analogue of Arzela'a theorem, Theorem 6.5, that the function f is neocontinuous on C.

(ii) Let $D = \{x : E[|x|] \le r\}$. By Theorem 6.4, D is neocompact, and by Proposition 7.6, $f_n(D)$ is neocompact in $L^0(\Omega, L^1([0, 1], \mathbf{R}))$ for each n. The first part of the proof shows that $f_n(x) \to f(x)$ uniformly on D. Therefore

$$f(D) \subset \bigcap_n (f_n(D))^{\varepsilon_n}$$

for some sequence $\varepsilon_n \to 0$. The opposite inclusion is easily seen to hold as well. By closure under diagonal intersections, $f(D)$ is neocompact in $L^0(\Omega, L^1([0,1], \mathbf{R}))$. $\square$

For each $d \in \mathbf{N}$ let

$$\mathcal{L}^d = L^0(\Omega, L^1([0,1]^d, \mathbf{R})).$$

The preceding theorem can be generalized to show that for each d, the function

$$f : \mathcal{L}^d \to \mathcal{L}^{d+1}$$

defined by

$$f(y)(\omega, \vec{s}, t) = E[y(\cdot, \vec{s})|\mathcal{F}_t](\omega)$$

is neocontinuous on each uniformly integrable subset of $\mathcal{L}^d$. The paper [11] introduced a whole class of bounded functions from $\mathcal{L}^1$ into $\mathcal{L}^d$, called **conditional processes**. This class is defined as the smallest class of functions which contains the functions

$$\hat{\Phi}(x)(\omega, t_1, \ldots, t_d) = \Phi(x(\omega, t_1), \ldots, x(\omega, t_d))$$

where Φ is bounded and continuous on $\mathbf{R}^d$, and is closed under (repeated) composition by continuous real functions and by the conditional expectation function

$$f(y)(\omega, \vec{s}, t) = E[y(\cdot, \vec{s})|\mathcal{F}_t](\omega).$$

Since compositions of neocontinuous functions are neocontinuous, and all conditional processes are bounded, it follows that each conditional process in the sense of [11] is neocontinuous for each rich adapted probability space Ω.

10 Martingale Integrals

For simplicity, we confine our discussion here to martingales with continuous paths on the unit interval $[0,1]$. With some additional complications, the more general case of local martingales with paths in $D([0,\infty), \mathbf{R}^d)$ can be treated in a similar manner.

Martingales are integrable stochastic processes with values in $\mathbf{R}^d$ such that for each pair of times $s \leq t$, the expected value at time t conditioned on time s equals the value at time s. Brownian motions are examples of martingales.

Definition 10.1 *A process $z \in L^2(\Omega, C([0,1], \mathbf{R}^d))$ is said to be a (continuous square integrable)* **martingale** *if z is adapted and*

$$E[z_1(\cdot)|\mathcal{F}_t](\omega) = z_t(\omega) \text{ almost surely for all } t \in [0,1).$$

We let $\mathcal{M}(\Omega, \mathbf{R}^d)$ denote the set of martingales $z \in L^2(\Omega, C([0,1], \mathbf{R}^d))$ such that $z(\omega, 0) = 0$.

Proposition 10.2 *$\mathcal{M}(\Omega, \mathbf{R}^d) \cap A$ is neocompact in $L^0(\Omega, C([0,1], \mathbf{R}^d))$ for every 2-uniformly integrable neocompact set A in $L^0(\Omega, C([0,1], \mathbf{R}^d))$.*

Proof: (Cf. [13]). Recall that for each $t \in \mathbf{B}$, $(\bigcup_{s<t} \mathcal{F}_s) \subset \mathcal{G}_t \subset \mathcal{F}_t$. We claim that a continuous process $z \in L^2(\Omega, C([0,1], \mathbf{R}^d))$ is a martingale if and only if $z_s = E[z_1|\mathcal{G}_s]$ for all $s \in \mathbf{B}$. To see this, suppose first that z is a martingale and $s \in \mathbf{B}$. Then $z_s = E[z_1|\mathcal{F}_s]$, and by continuity z_s is $\mathcal{G}_s$-measurable and

$$z_s = E[z_s|\mathcal{G}_s] = E[E[z_1|\mathcal{F}_s]|\mathcal{G}_s] = E[z_1|\mathcal{G}_s].$$

For the converse suppose that $z_s = E[z_1|\mathcal{G}_s]$ for all $s \in \mathbf{B}$. Then for all $t \in [0,1)$,

$$z_t = \lim_{s \downarrow t} z_s = \lim_{s \downarrow t} E[z_1|\mathcal{G}_s] = E[z_1|\mathcal{F}_t],$$

so z is an $\mathcal{F}_t$-martingale. This proves the claim.

For each $s \in \mathbf{B}$, the function $z \mapsto E[z_1|\mathcal{G}_s]$ is neocontinuous on each uniformly 2-integrable set by Proposition 7.6. Let A be a uniformly 2-integrable neocompact set in $L^0(\Omega, C([0,1], \mathbf{R}^d))$. Then

$$B_s = \{z \in A : z_s - E[z_1|\mathcal{G}_s] = 0\}$$

is neocompact. By the preceding paragraph,

$$\mathcal{M}(\Omega, \mathbf{R}^d) \cap A = \bigcap_{s \in \mathbf{B}} B_s,$$

which is a countable intersection of neocompact sets and hence is neocompact. □

We now turn to stochastic integrals.

To illustrate our methods in a simple case we shall consider integrals with respect to continuous square integrable martingales, and then specialize to integrals with respect to d-dimensional Brownian motions. We also restrict our attention to the case of bounded adapted integrands. There are analogous results for the more general case of integrals of predictable processes with respect to local semimartingales.

Definition 10.3 *Let c, d, and $k \in \mathbf{N}$ remain fixed throughout our discussion and let $\mathbf{K} = [-k, k]$. Let $\mathbf{K}^{cd}$ be the space of all $c \times d$ matrices with entries in $\mathbf{K}$, with the Euclidean norm $|u| = \sqrt{\sum_{i,j}(u_{i,j})^2}$. Recall that $\mathcal{A}(\Omega, M)$ is the set of all adapted processes in $L^0(\Omega, L^0([0,1], M))$, i.e., the set of measurable adapted real valued processes with values in M. The* **stochastic integral**

$$\int_0^t g(\omega, s) dz(\omega, s), g \in \mathcal{A}(\Omega, \mathbf{K}^{cd}), z \in \mathcal{M}(\Omega, \mathbf{R}^d)$$

is the unique continuous function from $\mathcal{A}(\Omega, \mathbf{K}^{cd}) \times \mathcal{M}(\Omega, \mathbf{R}^d)$ *into* $\mathcal{M}(\Omega, \mathbf{R}^c)$ *such that if* $g(\cdot, u)$ *is constant and* $\mathcal{F}_s$*-measurable for* $u \in [s, t)$*, then*

$$\int_0^t g(\omega, u) dz(\omega, u) = \int_0^s g(\omega, u) dz(\omega, u) + g(\omega, s)[z(\omega, t) - z(\omega, s)].$$

Definition 10.4 *We say that a set* C *is* **neocompact in** $\mathcal{A}(\Omega, M)$ *if* $C \subset \mathcal{A}(\Omega, M)$ *and* C *is neocompact in* $L^0(\Omega, L^0([0,1], M))$*. We say that* C *is* **neocompact in** $\mathcal{M}(\Omega, \mathbf{R}^d)$ *if* $C \subset \mathcal{M}(\Omega, \mathbf{R}^d)$ *and* C *is uniformly 2-integrable and neocompact in* $L^0(\Omega, C([0,1], \mathbf{R}^d))$*.*

Definition 10.5 *Let* $\mathbf{B}_n$ *be the set of multiples of* 2^{-n} *in* $[0, 1)$*. The* **quadratic variation** $[z, z]$ *of a one dimensional continuous martingale* $z \in \mathcal{M}(\Omega, \mathbf{R})$ *is the* L^1 *limit of the sequence of sums*

$$[z, z]_n(\omega)(t) = \sum \{[z(\omega)(s) - z(\omega)(s - 2^{-n})]^2 : s \in \mathbf{B}_n \text{ and } s \leq t\}.$$

This limit exists and $[z, z]$ *is a continuous increasing process (cf. [8]).*

We use the classical equation

$$E[(\int_0^t g(\omega, s) dz(\omega, s))^2] = E[\int_0^t (g(\omega, s))^2 d[z, z]]. \tag{7}$$

It can be shown that for every uniformly 2-integrable set $C \subset \mathcal{M}(\Omega, \mathbf{R}^d)$, the stochastic integral function is neocontinuous from $\mathcal{A}(\Omega, \mathbf{K}^{cd}) \times C$ to $\mathcal{M}(\Omega, \mathbf{R}^c)$, showing first that the quadratic variation is neocontinuous from C to $L^1(\Omega, C([0,1], \mathbf{R}))$. (For analogous results in the nonstandard setting, see [12] and [1]). Here we shall prove the easier result that for each $z \in \mathcal{M}(\Omega, \mathbf{R}^d)$, the stochastic integral with respect to z is neocontinuous from $\mathcal{A}(\Omega, \mathbf{K}^{cd})$ to $\mathcal{M}(\Omega, \mathbf{R}^c)$. This will be sufficient for us to obtain existence theorems for stochastic differential equations.

Lemma 10.6 *Let* $z \in \mathcal{M}(\Omega, \mathbf{R})$*. The function*

$$\theta(x)(\omega, t) = \int_0^t (x(\omega, s))^2 d[z, z](\omega, s)$$

is neocontinuous from $\mathcal{A}(\Omega, \mathbf{K}\ )$ *to* $L^0(\Omega, I)$ *and its range is contained in a neocompact set, where* I *is the set of increasing functions in* $C([0,1], \mathbf{R})$ *with value* 0 *at* 0*.*

Proof: The deterministic function $x \mapsto \int_0^t (x(s))^2 dy(s)$ is continuous from $L^0([0,1], \mathbf{K}) \times I$ to I. By the randomization lemma 5.19, θ is neocontinuous from $\mathcal{A}(\Omega, \mathbf{K}\ )$ to $L^0(\Omega, I)$. For each $x \in \mathcal{A}(\Omega, \mathbf{K}\ )$ and $s < t$ in $[0,1]$ we have

$$(\theta(x)(\omega, t) - \theta(x)(\omega, s)) \leq k^2([z,z](\omega, t) - [z,z](\omega, s)).$$

By Proposition 5.17, the range of θ is contained in a neocompact set in $L^0(\Omega, I)$. □

Lemma 10.7 *Let $k > 0$ and let C be a neocompact set in $\mathcal{M}(\Omega, \mathbf{R})$. The set A of all $y \in \mathcal{M}(\Omega, \mathbf{R})$ such that for some $z \in C$,*

$$(y(\omega, t) - y(\omega, s))^2 \leq k^2 (z(\omega, t) - z(\omega, s))^2 \textit{ for all } s \leq t \textit{ in } [0,1] \textit{ a.s.}$$

is neocompact in $\mathcal{M}(\Omega, \mathbf{R})$.

Proof: We may assume that C is nonempty. Since C is uniformly 2-integrable there is a sequence $\langle a_n \rangle$ such that $\lim_{n\to\infty} a_n = 0$ and

$$E[\phi_n(\sup_t (z(\omega, t))^2)] \leq a_n$$

for all $z \in C$ and $n \in \mathbf{N}$. Then A is contained in the uniformly 2-integrable neoclosed set

$$B = \{y \in L^0(\Omega, C([0,1], \mathbf{R}^d)) : E[\phi_n(\sup_t (y(\omega, t)/k)^2)] \leq a_n \text{ for all } n \in \mathbf{N}\}.$$

Let D_n be the set

$$\{(y,z) \in B \times C : (\forall s, t \in \mathbf{B}_n)(y(\omega, t) - y(\omega, s))^2 \leq k^2 (z(\omega, t) - z(\omega, s))^2 \text{ a.s.}\}.$$

D_n is a decreasing chain of uniformly 2-integrable neoclosed sets. Since the paths of each $y \in B$ are continuous,

$$A = \mathcal{M}(\Omega, \mathbf{R}) \cap \{y \in B : (\exists z \in C)(y,z) \in \bigcap_n D_n\}.$$

For each y, the sets

$$E_n(y) = \{z \in C : (y,z) \in D_n\}$$

form a decreasing chain of neocompact sets. Then by the countable compactness property,

$$A = \mathcal{M}(\Omega, \mathbf{R}) \cap \{y \in B : (\bigcap_n E_n(y)) \neq \emptyset\}$$

$$= \mathcal{M}(\Omega, \mathbf{R}) \cap \bigcap_n \{y \in B : E_n(y) \neq 0\}$$

$$= \mathcal{M}(\Omega, \mathbf{R}) \cap \bigcap_n \{y \in B : (\exists z \in C)(y, z) \in D_n\}.$$

By Proposition 3.5, each set in the countable intersection on the right is neoclosed. Therefore

$$A = \mathcal{M}(\Omega, \mathbf{R}) \cap F$$

where F is a uniformly 2-integrable neoclosed set.

For each $\delta > 0$ and $x \in C([0,1], \mathbf{R})$ let $w(\delta, x)$ be the modulus of continuity

$$w(\delta, x) = \sup\{|x(t) - x(s)| : s, t \in [0,1] \text{ and } |t - s| \leq \delta\}.$$

For each $(y, z) \in \bigcap_n D_n$ and $\delta > 0$ we have $y(\omega, 0) = z(\omega, 0) = 0$ and

$$w(\delta, y(\omega)) \leq k w(\delta, z(\omega))$$

almost surely. Let $N = \{x \in C([0,1], \mathbf{R}) : x(0) = 0\}$. A set $S \subset \text{Meas}(N)$ has compact closure if and only if for each $\eta > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu[w(\delta, x) > \varepsilon] < \eta$ for all $\mu \in S$ (cf. [4]). Since C is neocompact, law(C) is compact. Therefore for each η and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $z \in C$,

$$P\{\omega : w(\delta, z(\omega)) > \varepsilon\} < \eta.$$

Then for all $y \in F$,

$$P\{\omega : w(\delta, y(\omega)) > k\varepsilon\} < \eta.$$

It follows that law(F) has compact closure in Meas$(C([0,1], \mathbf{R})$. Therefore F is contained in a neocompact set, and since F is neoclosed, F is neocompact in $L^0(\Omega, C([0,1], \mathbf{R}))$. By Proposition 10.2, $A = \mathcal{M}(\Omega, \mathbf{R}) \cap F$ is uniformly 2-integrable and neocompact in $L^0(\Omega, C([0,1], \mathbf{R}))$, and hence is neocompact in $\mathcal{M}(\Omega, \mathbf{R})$. □

Proposition 10.8 *Let $z \in \mathcal{M}(\Omega, \mathbf{R}^c)$. The function*

$$\phi(x)(\omega, t) = \int_0^t x(\omega, s) dz(\omega, s)$$

is neocontinuous from $\mathcal{A}(\Omega, \mathbf{K}^{cd})$ to $\mathcal{M}(\Omega, \mathbf{R}^c)$ and its range is contained in a neocompact set in $\mathcal{M}(\Omega, \mathbf{R}^c)$.

Proof: Since each coordinate of the stochastic integral is a sum of d one-dimensional stochastic integrals, it suffices to prove the result in the one dimensional case $c = d = 1$. The range of ϕ is contained in the neocompact set A in $\mathcal{M}(\Omega, \mathbf{R}^c)$ from the preceding lemma with $C = \{z\}$. For $t \in \mathbf{B}$, let G_t be the set of all $\mathcal{G}_t$-measurable $x \in L^0(\Omega, \mathbf{K})$, and for $n \in \mathbf{N}$ let H_n be the Cartesian product $\prod\{G_t : t \in \mathbf{B}_n\}$. Each G_t is basic in $L^0(\Omega, \mathbf{K})$. For each n, the function $\phi|\text{step}_n(H_n)$ is neocontinuous because it is given by the finite sum

$$\phi(x)(\omega, t) = \sum\{x(\omega, s)(z(\omega, s + 2^{-n}) - z(\omega, s)) : s \in \mathbf{B}_n \text{ and } s + 2^n \leq t\}$$
$$+x(\omega, s + 2^n)(z(\omega, t) - z(\omega, s + 2^n)).$$

Let $B \subset \mathcal{A}(\Omega, \mathbf{K}\ )$ be neocompact and let

$$D_n = \{(x, y) \in B \times A : (\forall v \in H_n)(\rho_2(\phi(\text{step}_n(v)), y))^2 \leq E[\theta(x - \text{step}_n(v))]\}.$$

By applying property (f) $2^n + 1$ times, we see that D_n is neocompact. By (7), the graph of $\phi|B$ is equal to the neocompact set $\bigcap_n D_n$. Thus ϕ is neocontinuous. □

11 Stochastic Integrals Over Brownian Motions

A d-dimensional **Brownian motion** on Ω is a continuous martingale x in $\mathcal{M}(\Omega, \mathbf{R}^d))$ such that law(x) is the Wiener distribution on $C([0, 1], \mathbf{R}^d)$. Let W_d be the set of d-dimensional Brownian motions on Ω.

Proposition 11.1 *Let N be either $C([0, 1], \mathbf{R}^d)$ or $L^0([0, 1], \mathbf{R}^d)$. For each $p \in [1, \infty)$, the set W_d of d-dimensional Brownian motions on Ω is neocompact and uniformly p-integrable in $\mathcal{N}$.*

Proof: W_d is the intersection of the set $\mathcal{M}(\Omega, \mathbf{R}^d)$ and the neocompact uniformly p-integrable set of processes x such that law(x) is the Wiener distribution. By Proposition 10.2, W_d is neocompact in $\mathcal{N}$. □

We remark that the set W_d is not compact, because it contains a sequence of independent Brownian motions, and such a sequence has no convergent subsequence.

We now consider stochastic integrals with respect to d-dimensional Brownian motions. For stochastic integrals with respect to Brownian motion, Definition 10.3 in the preceding section can be extended to the case that y is adapted with paths in $L^2([0, 1], \mathbf{R}^{cd})$, and gives a continuous function from $\mathcal{A}^2(\Omega, \mathbf{R}^{cd}) \times W_d$ into $\mathcal{A}^2(\Omega, \mathbf{R}^c)$.

Proposition 11.2 *Let ϕ be the stochastic integral function*

$$\phi(y, w)(\omega, t) = \int_0^t y(\omega, s)dw(\omega, s).$$

(i) ϕ is neocontinuous from $\mathcal{A}^2(\Omega, \mathbf{R}^{cd}) \times W_d$ to $\mathcal{A}^2(\Omega, \mathbf{R}^c)$.
(ii) For each k, ϕ is neocontinuous from $\mathcal{A}(\Omega, \mathbf{K}^{cd}) \times W_d$ to $\mathcal{M}(\Omega, \mathbf{R}^c)$.

Proof: It suffices to prove the theorem in the one dimensional case $c = d = 1$.

(i) This is a direct consequence of Proposition 5.18.

(ii) The quadratic variation of a one dimensional Brownian motion $w \in W_1$ is just $[w, w](\omega, t) = t$. Therefore equation (7) becomes

$$E[\int_0^t g(\omega, s) dw(\omega, s)^2] = E[\int_0^t (g(\omega, s))^2 ds].$$

Since the set W_1 is neocompact in $\mathcal{M}(\Omega, \mathbf{R})$, we see from Lemma 10.7 that the set

$$\{\int_0^t y(\omega, s) dw(\omega, s) : y \in \mathcal{A}(\Omega, \mathbf{K}) \text{ and } w \in W_1\}$$

is contained in a neocompact set in $\mathcal{M}(\Omega, \mathbf{R})$. A straightforward modification of the proof of Proposition 10.8 now shows that the function ϕ is neocontinuous in both variables y and w. $\Box$

From now on we let w be a d-dimensional Brownian motion for Ω which remains fixed throughout our discussion. We next wish to show that the function

$$(g, x) \mapsto \int_0^t g(s, x(\omega, s)) dw(\omega, s)$$

is neocontinuous, where g is a measurable but possibly discontinuous function from $[0, 1] \times \mathbf{R}^c$ into $\mathbf{K}^{cd}$, and x is in $\mathcal{M}(\Omega, \mathbf{R}^c)$. In order to prove such a result, we will have to restrict x to a certain neoclosed set. The difficulty is that the function $x \mapsto g(\cdot, x)$ is not continuous, and is not even well defined for all x because g may have two representatives g_1 and g_2 such that $g_1(s, x(\omega, s)) \neq g_2(s, x(\omega, s))$ on a subset of Ω of positive measure. For neocontinuity to make sense, we must first choose an appropriate metric for the space of measurable functions from $[0, 1] \times \mathbf{R}^c$ into $\mathbf{K}^{cd}$. For this purpose we use the normal probability measure on $\mathbf{R}^c$.

Definition 11.3 *For each closed set $\mathbf{H} \subset \mathbf{R}^{c \times d}$, we let $\mathbf{L}(\mathbf{H}\ )$ be the space of measurable functions from $[0, 1] \times \mathbf{R}^c$ into $\mathbf{H}$ with the metric of convergence in probability relative to the product of Lebesgue measure on $[0, 1]$ and the c-dimensional normal measure on $\mathbf{R}^c$.*

Let $\mathbf{J}$ be the compact set of all matrices $u \in \mathbf{K}^{cd}$ such that $1/k \leq det(uu^T)$, and let $\mathcal{I}$ be the set of all stochastic integrals

$$\mathcal{I} = \{\int_0^t y(\omega, s) dw(\omega, s) : y \in \mathcal{A}(\Omega, \mathbf{J})\}.$$

The preceding corollary shows that $\mathcal{I}$ is contained in a neocompact subset of $\mathcal{M}(\Omega, \mathbf{R}^c)$.

We apply the following inequality of Krylov [16].

Lemma 11.4 *There is a constant b depending only on k, c, and d such that for each $x \in \mathcal{I}$ and Borel function $h : [0,1] \times \mathbf{R}^c \to \mathbf{R}$,*

$$E[\int_0^1 |h(t, x(\omega, t))|dt] \leq b\|h\|_{c+1}.\square \tag{8}$$

Definition 11.5 *Let $\mathcal{H}$ be the set of all adapted processes $x \in L^0(\Omega, C([0,1], \mathbf{R}^c))$ such that for each Borel function $h : [0,1] \times \mathbf{R}^c \to \mathbf{R}$, the inequality (8) holds. Thus $\mathcal{I} \subset \mathcal{H}$.*

Lemma 11.6 *The set $\mathcal{H}$ is neoclosed in $L^0(\Omega, C([0,1], \mathbf{R}^c))$.*

Proof: Let M be the complete separable metric space of all Borel functions $h : [0,1] \times \mathbf{R}^c \to \mathbf{R}$ with the metric $\rho(g, h) = \min(1, \|g - h\|_{c+1})$. For each $n \in \mathbf{N}$, let U_n be the compact set of all $h \in M$ such that h has Lipschitz bound n and support $[0,1] \times [-n, n]^c$. For each n, the function

$$(h, x) \mapsto E[\int_0^1 |h(t, x(\omega, t))|dt]$$

is neocontinuous from $U_n \times L^0(\Omega, C([0,1], \mathbf{R}^c))$ to $\mathbf{R}$, and the function $h \to \|h\|_{c+1}$ is continuous from U_n to $\mathbf{R}$. Therefore by Proposition 4.18, the set

$$A_n = \{x : (\forall h \in U_n) E[\int_0^1 |h(t, x(\omega, t))|dt] \leq b\|h\|_{c+1}\}$$

and the intersection $\bigcap_n A_n$ are neoclosed in $L^0(\Omega, C([0,1], \mathbf{R}^c))$. We show that $\mathcal{H} = \bigcap_n A_n$. The inclusion $\mathcal{H} \subset \bigcap_n A_n$ is trivial. Let $x \in \bigcap_n A_n$. Then the function

$$I(h) = E[\int_0^1 h(t, x(\omega, t))dt]$$

on the vector lattice $\bigcup_n U_n$ of Lipschitz functions with bounded support generates a Daniell integral. Every Borel function h is measurable with respect to I, and the inequality (8) for Lipschitz h insures that Lebesgue null sets are null sets with respect to I. By taking limits we see that the inequality (8) holds for all $h \in M$, so $x \in \mathcal{H}$. $\square$

Lemma 11.7 *For any $g \in \mathbf{L}(\mathbf{K}^{cd})$ and $x \in \mathcal{H}$,*

$$E[\sup_t |\int_0^t g(s, x(\omega, s))dw(\omega, s)|^2] \leq 4b\||g(\cdot, \cdot)|^2\|_{c+1}.$$

Proof: By Doob's martingale inequality,

$$E[\sup_t |\int_0^t g(s,x(\omega,s))dw(\omega,s)|^2]$$

$$\leq 4E[|\int_0^1 g(s,x(\omega,s))dw(\omega,s)|^2] = 4E[\int_0^1 |g(s,x(\omega,s))|^2 ds].$$

Now apply the inequality (8) with $h(\cdot,\cdot) = |g(\cdot,\cdot)|^2$. □

Theorem 11.8 *The stochastic integral function*

$$\psi : \mathbf{L}(\mathbf{K}^{cd}) \times \mathcal{H} \to \mathcal{M}(\Omega, \mathbf{R}^c)$$

defined by

$$\psi(g,x)(\omega,t) = \int_0^t g(s,x(\omega,s))dw(\omega,s) \tag{9}$$

is neocontinuous and its range is contained in a neocompact set in $\mathcal{M}(\Omega, \mathbf{R}^c)$.

Proof: By Proposition 10.8, the function

$$\phi : \mathcal{A}(\Omega, \mathbf{K}^{cd}) \to \mathcal{M}(\Omega, \mathbf{R}^c),$$

defined by

$$\phi(y)(\omega,t) = \int_0^t y(\omega,s)dw(\omega,s),$$

is neocontinuous and its range is contained in a neocompact set B in $\mathcal{M}(\Omega, \mathbf{R}^c)$.

For $n \in \mathbf{N}$, let $\mathbf{L}_n$ be the closed set of all $g \in \mathbf{L}(\mathbf{K}^{cd})$ with support $[0,1]\times[-n,n]^c$, and let U_n be the compact set of functions $g \in \mathbf{L}_n$ with uniform Lipschitz bound n. Then $\bigcup_n U_n$ is dense in $\mathbf{L}(\mathbf{K}^{cd})$. By the randomization lemma 5.19 and the neocontinuity of ϕ, the stochastic integral function

$$\psi(h,x) = \phi(h(t,x(\omega,t)))$$

is well defined and neocontinuous on $U_n \times \mathcal{H}$.

We show that ψ is neocontinuous on $\mathbf{L}_m \times \mathcal{H}$ and then do the same for $\mathbf{L}(\mathbf{K}^{cd}) \times \mathcal{H}$. Let G_m be the set

$$G_m = \bigcap_n \{(g,x,y) \in \mathbf{L}_m \times \mathcal{H} \times B : (\forall h \in U_n)\rho_2(y,\psi(h,x))^2 \leq 8b\||g-h|^2\|_{c+1}\}.$$

Then $(g,x,\psi(g,x)) \in G_m$ whenever $(g,x) \in \mathbf{L}_m \times \mathcal{H}$. If $g \in \mathbf{L}_m, x \in \mathcal{H}$, and $\varepsilon > 0$, then for all $f,h \in \bigcup_n U_n$ such that

$$\||g-f|^2\|_{c+1} \leq \varepsilon, \||g-h|^2\|_{c+1} \leq \varepsilon,$$

we have

$$\rho_2(\psi(f,x),\psi(h,x))^2 = \rho_2(0,\psi((f-h),x))^2 \le 4b\||f-h|^2\|_{c+1} \le 8b\varepsilon.$$

Since $\bigcup_n(U_n \cap \mathbf{L}_m)$ is dense in $\mathbf{L}_m$, it follows that G_m is the graph of $\psi|(\mathbf{L}_m \times \mathcal{H})$. By Proposition 4.18, G_m is a countable intersection of neoclosed sets and is thus neoclosed. Since B is neocompact, it follows from Proposition 5.18 that $\psi|(\mathbf{L}_m \times \mathcal{H})$ is neocontinuous.

The truncation function $g \mapsto g \wedge m$, defined by

$$(g \wedge m)(t,y) = g(t,y) \text{ if } y \in [-m,m]^c,$$

$$(g \wedge m)(t,y) = 0 \text{ otherwise}$$

is continuous from $\mathbf{L}(\mathbf{K}^{cd})$ onto $\mathbf{L}_m$. Therefore for each m the function

$$\psi_m(g,x) = \psi(g \wedge m, x)$$

is neocontinuous from $\mathbf{L}(\mathbf{K}^{cd}) \times \mathcal{H}$ to B. Let $C \subset \mathcal{H}$ be neocompact. Since $\mathbf{K}^{cd}$ is bounded, there is a sequence $\langle a_m \rangle$ in $\mathbf{R}$ such that $\lim_{m\to\infty} a_m = 0$ and whenever $m \le n$ and $(g,x) \in \mathbf{L}(\mathbf{K}^{cd}) \times C$, we have $\rho_2(\psi_m(g,x),\psi_n(g,x)) \le a_m$. Let G be the neoclosed set

$$G = \bigcap_m \{(g,x,y) \in \mathbf{L}(\mathbf{K}^{cd}) \times C \times B : \rho_2(\psi_m(g,x),y) \le a_m\}.$$

Then G is the graph of $\psi|(\mathbf{L}(\mathbf{K}^{cd}) \times C)$ and G is neoclosed. Since B is neocompact, it follows from Proposition 5.18 that ψ is neocontinuous on $\mathbf{L}(\mathbf{K}^{cd}) \times \mathcal{H}$. □

12 Optimization Theorems

In this section we give some applications of the following corollary.

Corollary 12.1 *Let C be a nonempty neocompact set in $\mathcal{M}$.*

(i) For every neocontinuous function $f : C \to \mathbf{R}$, the range $f(C)$ has a maximum and minimum.

(ii) For every neo-LSC function $g : C \to \bar{\mathbf{R}}$, the range $g(C)$ has a minimum.

Proof: (i) By Proposition 3.8, $f(C)$ is neocompact in $\mathbf{R}$. By Proposition 4.8, $f(C)$ compact. Since C and hence $f(C)$ is nonempty, it has a maximum and minimum.

(ii) By Proposition 8.5, the upper graph

$$G = \{(x,r) \in C \times \bar{\mathbf{R}} : g(x) \leq r\}$$

is neocompact. Therefore the set

$$A = \{r \in \bar{\mathbf{R}} : (\exists x)(x,r) \in G\}$$

is a neocompact subset of $\bar{\mathbf{R}}$. Since $\bar{\mathbf{R}}$ is separable, A is compact. C is nonempty, so A is nonempty and has a minimum element s, which is also a minimum element of $g(C)$. $\square$

By applying this corollary to our library of neocompact sets and neocontinuous functions, we can quickly obtain many optimization results. Here are a few examples.

Theorem 12.2 *For every continuous stochastic process $x \in L^0(\Omega, C([0,1], \mathbf{R}))$ there exists a stopping time $\tau \in L^0(\Omega, [0,1])$ such that $E[|x_{\tau(\omega)}(\omega)|]$ is a minimum. Moreover, if $E[\sup_t(|x_t(\omega)|)] < \infty$, there exists a stopping time $\sigma \in L^0(\Omega, [0,1])$ such that $E[x_{\sigma(\omega)}(\omega)]$ is a minimum.*

Proof: The set C of stopping times in $L^0(\Omega, [0,1])$ is neocompact, the function $\tau \mapsto |x_\tau|$ is neocontinuous from C to $L^0(\Omega, \mathbf{R}_+)$, and the function $y \mapsto E[y(\omega)]$ is neo-LSC and monotone. Therefore the composition $\tau \mapsto E[|x_\tau|]$ is neo-LSC from C to $\bar{\mathbf{R}}_+$, and thus its range has a minimum.

If $E[\sup_t(|x_t(\omega)|)] < \infty$, the set

$$D = \{y \in L^0(\Omega, \mathbf{R}) : |y(\omega)| \leq \sup_t(|x_t(\omega)|) \text{ almost surely}\}$$

is uniformly integrable and neocompact. The function $\sigma \mapsto x_\sigma$ is neocontinuous from C to D, and the function $y \mapsto E[y(\omega)]$ is neocontinuous from D to $\mathbf{R}$. Therefore the composition $\sigma \mapsto E[x_\sigma]$ is neocontinuous from C to $\mathbf{R}$, and hence its range has a minimum. $\square$

Theorem 12.3 *Let $p \in \{0\} \cup [1, \infty)$. For each process $x \in L^p(\Omega, N)$ and each nonempty neocompact set $C \subset L^p(\Omega, N)$ there exists $y \in C$ whose distance from x is a minimum in $L^p(\Omega, N)$.*

Proof: The distance function $\rho_p(x,y)$ is neo-LSC on the neocompact set $\{x\} \times C$, so its range has a minimum. $\square$

Corollary 12.4 *Let $p, q \in \{0\} \cup [1, \infty)$ and let N be either $C([0,1], \mathbf{R}^d)$ or $L^q([0,1], \mathbf{R}^d)$. For each process $x \in L^p(\Omega, N)$ there is a d-dimensional Brownian motion b on Ω whose distance from x is a minimum in $L^p(\Omega, N)$.*

Proof: The set of Brownian motions is neocompact in $\mathcal{N}$ and is contained in $L^p(\Omega, N)$. □

The following result is a sort of selection theorem.

Theorem 12.5 *Let M be a compact metric space, and let $f \in L^0(\Omega, C(M, \mathbf{R}))$. There exists $x \in \mathcal{M}$ such that*

$$f(\omega)(x(\omega)) = \sup\{f(\omega)(y) : y \in M\}$$

for almost all $\omega \in \Omega$.

Proof: Let $g : \mathbf{R} \to \mathbf{R}$ be a bounded continuous increasing function (e.g. the arctan function). The function

$$h(x) = E[g(f(\omega)(x(\omega)))]$$

is neocontinuous from $\mathcal{M}$ to $\mathbf{R}$. Since $\mathcal{M}$ is neocompact, the range of h is nonempty and compact, and thus has a maximum at some $x \in \mathcal{M}$. Let

$$s(\omega) = \sup\{f(\omega)(y) : y \in M\}$$

and

$$A = \{\omega \in \Omega : f(\omega)(x(\omega)) < s(\omega)\}$$

. Since M is separable, the set A is measurable, and if $P[A] > 0$ then there exists $y \in M$ such that

$$P[f(\omega)(x(\omega)) < f(\omega)(y)] > 0.$$

Define $z \in \mathcal{M}$ by putting $z(\omega) = x(\omega)$ if $f(\omega)(x(\omega)) \geq f(\omega)(y)$, and $z(\omega) = y$ otherwise. Then $E[g(f(\omega)(x(\omega))] < E[g(f(\omega)(z(\omega))]$, contrary to the choice of x. Therefore $P[A] = 0$ and the result follows. □

In the following theorem, we give $C(\mathbf{R}, \mathbf{R})$ the metric

$$\rho(x, y) = \min(1, \sum_{n=1}^{\infty}\{2^{-n}\sup\{|x(t) - y(t)| : |t| \leq n\})$$

of uniform convergence on compact sets, so that $C(\mathbf{R}, \mathbf{R})$ is a complete separable metric space.

Theorem 12.6 *Let $k > 0$, let $\mathbf{K} = [-k, k]$, let*

$$f \in \mathcal{A}(\Omega, C(\mathbf{R}, \mathbf{K}))$$

be an adapted process, let $g : C([0,1],\mathbf{R}) \to \bar{\mathbf{R}}_+$ *be a lower semicontinuous function, and let* $z \in \mathcal{M}(\Omega,\mathbf{R})$ *be a continuous martingale. Then any neocompact set* C *in* $\mathcal{A}(\Omega,\mathbf{R})$ *contains an element* x *such that*

$$E[g(\int_0^t f(\omega,s,x(\omega,s))dz(\omega,s))]$$

is minimal.

Proof: The functions $x \mapsto f(\omega,s,x(\omega,s))$ and $y \mapsto \int_0^t y(\omega,s)dz(\omega,s)$ are neocontinuous. By the results of Section 8, the function

$$x \mapsto E[g(\int_0^t f(\omega,s,x(\omega,s))dz(\omega,s))]$$

is neo-LSC from C to $\bar{\mathbf{R}}_+$, and by Corollary 12.1 its range has a minimum. □

Theorem 12.7 *Let* $k > 0$*, let* $\mathbf{K} = [-k,k]$*, let* $f \in \mathcal{A}(\Omega,\mathbf{K}^{cd})$ *be an adapted process, and let* $g : C([0,1],\mathbf{R}^c) \to \bar{\mathbf{R}}_+$ *be a lower semicontinuous function. Then the set* W_d *of d-dimensional Brownian motions on* Ω *contains an element* w *such that*

$$E[g\int_0^t f(\omega,s)dw(\omega,s)]$$

is minimal.

Proof: By Proposition 11.2 and the argument for the preceding theorem, the function $w \mapsto E[g\int_0^t f(\omega,s)dw(\omega,s)]$ is neo-LSC from W_d to $\bar{\mathbf{R}}_+$. The set W_d is neocompact, so the range of the function has a minimum. □

13 Existence Theorems

In this section we illustrate the use of the approximation theorem by proving some existence theorems for stochastic differential equations. We continue to assume that $\Omega = (\Omega, P, \mathcal{G}, \mathcal{G}_t)_{t\in\mathbf{B}}$ is a rich adapted probability space. It should be emphasized that the results of this section do not hold for arbitrary adapted probability spaces, and depend heavily on the richness property. For instance, it was shown by Barlow [3] that, even in the case that z is a Brownian motion, Theorem 13.2 below is false without our blanket hypothesis that Ω is a rich adapted space.

As a warmup, we prove an existence theorem for ordinary differential equations with random parameters (cf K1]). We let d be a positive integer, let $k > 0$, and let $\mathbf{K} = [-k, k]$. For each complete separable metric space M, let $C(\mathbf{R}^d, M)$ be the metric space of continuous functions from $\mathbf{R}^d$ into M with the metric

$$\hat{\rho}(x, y) = \min(1, \sum_{n=1}^{\infty} \sup\{\rho(x(u), y(u)) : |u| \leq n\} 2^{-n}).$$

This determines the topology of uniform convergence on compact sets (the compact-open topology). Then $C(\mathbf{R}^d, M)$ is a complete separable metric space. For example, if M is either $\mathbf{R}^c$ or a closed ball about the origin in $\mathbf{R}^c$, then the set of Lipschitz functions with compact support is dense in $C(\mathbf{R}^d, M)$.

Theorem 13.1 *Let $f \in L^0(\Omega, C(\mathbf{R}^d, \mathbf{K}^d))$. There exists $x \in L^0(\Omega, C([0, 1], \mathbf{R}^d))$ such that*

$$x(\omega, t) = \int_0^t f(\omega, s)(x(\omega, s)) ds.$$

If f is adapted, then x may be taken to be adapted.

Proof: Let

$$\mathcal{M} = L^0(\Omega, C([0, 1], \mathbf{R}^d)).$$

Let D be the compact set of $y \in C([0, 1], \mathbf{R}^d)$ such that y has Lipschitz bound k and $y(0) = 0$. By 5.2, the set $\mathcal{D} = L^0(\Omega, D)$ is neocompact in $\mathcal{M}$. The function

$$x \mapsto \int_0^t f(\omega, s)(x(\omega, s)) ds \tag{10}$$

is a composition of neocontinuous functions, and is therefore neocontinuous from $\mathcal{M}$ to $\mathcal{D}$. We wish to prove the formula

$$(\exists x \in \mathcal{D}) x(\omega, t) = \int_0^t f(\omega, s)(x(\omega, s)) ds. \tag{11}$$

For $y \in D$ we use the convention $y(t) = 0$ for $t < 0$. Instead of proving (11), we instead prove the equivalent statement

$$(\exists x \in \mathcal{D})(\exists u \in \{0\})[x(\omega, t) = \int_0^t f(\omega, s)(x(\omega, s - u)) ds]. \tag{12}$$

The function $(x, u) \mapsto x(\omega, t-u)$ is neocontinuous on $\mathcal{M} \times [0, 1]$ because the function $(y, u) \mapsto y(t - u)$ is continuous on $C([0, 1], \mathbf{R}^d) \times [0, 1]$. A typical approximation of (12) says that there exists x within ε of $\mathcal{D}$ and $u \in [0, \varepsilon]$ such that $x(\omega, t)$ is within ε of $\int_0^t f(\omega, s)(x(\omega, s - u)) ds$ in $\mathcal{M}$. Take $u = \varepsilon$. By successively integrating over the subintervals

$$[0, u], [u, 2u], \cdots,$$

we get $x \in \mathcal{D}$ such that

$$x(\omega, t) = \int_0^t f(\omega, s)(x(\omega, s-u))ds.$$

Then (12) holds by the Approximation Theorem.

In the adapted case, the same proof works taking $\mathcal{D}$ to be the set of adapted processes in $L^0(\Omega, D)$, which is neocompact because the set of adapted processes in $\mathcal{M}$ is neoclosed. □

We now apply our method to get a short proof of an existence theorem for stochastic integral equations from [12] and [17]. Recall that $\mathbf{K}^{cd}$ is the set of all $c \times d$ matrices with coefficients in the set $\mathbf{K} = [-k, k]$, and $\mathbf{L}(\mathbf{K}^{cd})$ is the space of measurable functions from $[0,1] \times \mathbf{R}^c$ into $\mathbf{K}^{cd}$.

Theorem 13.2 *For each $g \in \mathcal{A}(\Omega, C(\mathbf{R}^c, \mathbf{K}^{cd}))$ and martingale $z \in \mathcal{M}(\Omega, \mathbf{R}^d)$ there exists a martingale $x \in \mathcal{M}(\Omega, \mathbf{R}^c))$ such that*

$$x(\omega, t) = \int_0^t g(\omega, s)(x(\omega, s))dz(\omega, s). \tag{13}$$

Proof: Let $z \in \mathcal{M}(\Omega, \mathbf{R}^d)$. By Proposition 10.8, the stochastic integral function is neocontinuous from $\mathcal{A}(\Omega, \mathbf{K}^{cd})$ to $\mathcal{M}(\Omega, \mathbf{R}^c)$ and its range

$$\{\int_0^t h(\omega, s)dz(\omega, s) : h \in \mathcal{A}(\Omega, \mathbf{K}^{cd})\}$$

is contained in a neocompact set D in $\mathcal{M}(\Omega, \mathbf{R}^{cd})$. We prove the formula

$$(\exists u \in \{0\})(\exists x \in D)[x(\omega, t) = \int_0^t g(\omega, s)(x(\omega, s-u))dz(\omega, s)]. \tag{14}$$

As in the proof of Theorem 13.1, we easily see that for each $u > 0$ there exists $x \in D$ such that

$$x(\omega, t) = \int_0^t g(\omega, s)(x(\omega, s-u))dz(\omega, s).$$

Therefore each approximation of (14) holds, and (14) follows by the Approximation Theorem. □

The proofs of Theorems 13.1 and 13.2 can easily be combined to obtain an existence theorem for stochastic integral equations with both a drift and a diffusion term,

$$x(\omega, t) = \int_0^t f(\omega, s)(x(\omega, s))ds + \int_0^t g(\omega, s)(x(\omega, s))dz(\omega, s).$$

Since g has ω as an argument in the preceding theorem, we obtain an existence theorem with an adapted control $y(\omega, t)$ as a corollary.

Corollary 13.3 *Let h and y be measurable adapted process with $h \in \mathcal{A}(\Omega, C(\mathbf{R}^{c+e}, \mathbf{K}^{cd}))$ and $y \in \mathcal{A}(\Omega, \mathbf{R}^e)$, and let z be a martingale in $\mathcal{M}(\Omega, \mathbf{R}^d)$. Then there exists a martingale $x \in \mathcal{M}(\Omega, \mathbf{R}^c))$ such that*

$$x(\omega, t) = \int_0^t h(\omega, s)(x(\omega, s), y(\omega, s))dz(\omega, s). \tag{15}$$

Proof: Apply Theorem 13.2 with $g(\omega, t)(\cdot) = h(\omega, t)(\cdot, y(\omega, t))$. □

The set S of pairs

$$(g, x) \in \mathcal{A}(\Omega, C(\mathbf{R}^c, \mathbf{K}^{cd})) \times \mathcal{M}(\Omega, \mathbf{R}^c)$$

such that (13) holds is neoclosed. We use this fact to show that optimal solutions exist.

Corollary 13.4 *Let $f : C([0, 1], \mathbf{R}^c) \to \bar{\mathbf{R}}_+$ be lower semicontinuous and let $z \in \mathcal{M}(\Omega, \mathbf{R}^d)$.*

(i) For each nonempty neocompact subset B of $\mathcal{A}(\Omega, C(\mathbf{R}^c, \mathbf{K}^{cd}))$, the set $\mathbf{T}$ of pairs $(g, x) \in B \times \mathcal{M}(\Omega, \mathbf{R}^c))$ such that (13) holds has an element (g, x) such that $E[f(x(\omega))]$ is minimal.

(ii) For each $h \in \mathcal{A}(\Omega, C(\mathbf{R}^{c+e}, \mathbf{K}^{cd}))$ and each nonempty neocompact set $C \subset \mathcal{A}(\Omega, \mathbf{R}^e)$, the set $\mathbf{U}$ of pairs $(x, y) \in \mathcal{M}(\Omega, \mathbf{R}^c) \times C$ such that (15) holds has an element (x, y) such that $E[f(x(\omega))]$ is minimal (so that y is an optimal control in C).)

Proof: (i) The set $\mathbf{T}$ is nonempty by Theorem 13.2, and neocompact because it is neoclosed and contained in the set

$$B \times \{\int_0^t j(\omega, s)dz(\omega, s) : j \in \mathcal{A}(\Omega, \mathbf{K}^{cd})\},$$

which in turn is contained in a neocompact subset of $B \times \mathcal{M}(\Omega, \mathbf{R}^c)$. Therefore the set $D = \{x : (\exists g)(g, x) \in \mathbf{T}\}$ is neocompact. By Section 8, the function $x \mapsto E[f(x(\omega))]$ is neo-LSC from D into $\bar{\mathbf{R}}_+$, and by Corollary 12.1 its range has a minimal element.

The proof of (ii) is similar. □

All of the applications up to this point used only the simple approximation theorem 6.1 rather than the more general approximation theorem 6.7. The following invariance theorem uses the general approximation theorem, in the form of Corollary 6.9. Recall that Meas(M) is the space of Borel probability measures on M with the Prohorov metric, and

$$\text{law} : \mathcal{M} \to \text{Meas}(M)$$

is the neocontinuous function where $\mathrm{law}(x)$ is the measure on M induced by x. We say that $x_n \to x$ in **distribution** if $\mathrm{law}(x_n) \to \mathrm{law}(x)$ in $\mathrm{Meas}(M)$.

Theorem 13.5 *Suppose that $g_n \to g$ in $\mathcal{A}(\Omega, C(\mathbf{R}^c, \mathbf{K}^{cd}))$, $u_n \to 0$ in $[0,1]$, $D \subset \mathcal{M}(\Omega, \mathbf{R}^c)$ is neocompact, and for each $n \in \mathbf{N}$, $x_n \in D$ and x_n is within $1/n$ of*

$$\int_0^t g_n(\omega, s)(x_n(\omega, s - u_n))dz(\omega, s).$$

Then there is a solution $x \in D$ of (13) such that some subsequence of (g_n, x_n) converges in distribution to (g, x).

Proof: Let B be the compact set $\{g\} \cup \{g_n : n \in \mathbf{N}\}$. Then the set $B \times [0,1] \times D$ is neocompact, so its image under the law function is compact. Therefore the sequence (g_n, x_n) has a subsequence such that the sequence $(\mathrm{law}(g_n), \mathrm{law}(x_n))$ converges to a point $(\mathrm{law}(g), y)$. Now consider the formula

$$(\exists x \in D)[x(\omega, t) = \int_0^t \hat{g}(\omega, s)(x(\omega, s - \hat{u}))dz(\omega, s) \wedge \mathrm{law}(x) = y]. \tag{16}$$

A typical approximation of (16) says that there exists x within ε of D such that x is within ε of

$$\int_0^t \hat{g}(\omega, s)(x(\omega, s - \hat{u}))dz(\omega, s)$$

and $\mathrm{law}(x)$ is within ε of y. By taking x_n for x, we see that each approximation of (16) is satisfied by (g_n, u_n) for sufficiently large n. By Corollary 6.9, (16) holds for $(g, 0)$, so there is an $x \in \mathcal{M}(\Omega, \mathbf{R}^c)$ which solves (13) such that $\mathrm{law}(x) = y$. $\square$

By applying the countable compactness property to Theorem 13.2, we obtain an infinite dimensional analogue.

Theorem 13.6 *Let $j : \mathbf{N} \to \mathbf{N}$ and $k \to \mathbf{N} \to \mathbf{N}$ be increasing sequences, and let $\mathbf{K}_n = [-k(n), k(n)]$. For each $n \in \mathbf{N}$, let $g_n \in \mathcal{A}(\Omega, C(\mathbf{R}^{j(n)}, \mathbf{K}_n))$ and let $z_n \in \mathcal{M}(\Omega, \mathbf{R})$ be a continuous martingale. Then there exists a sequence of martingales $x_n \in \mathcal{M}(\Omega, \mathbf{R})$ such that for each n*

$$x_n(\omega, t) = \int_0^t g_n(\omega, s)(x_1(\omega, s), \ldots, x_{j(n)}(\omega, s))dz_n(\omega, s). \tag{17}$$

Proof: By Proposition 10.8, for each n there is a neocompact set B_n in $\mathcal{M}(\Omega, \mathbf{R})$ containing all stochastic integrals $\int h(\omega, s)dz_n$ where $h \in \mathcal{A}(\Omega, \mathbf{K}_n)$. For each $m \leq n \in \mathbf{N}$, let

$$C_{m,n} = \{(x_1, \ldots, x_m) \in B_1 \times \cdots \times B_m : (\exists x_{m+1} \in B_{m+1}) \cdots (\exists x_{j(n)} \in B_{j(n)})$$

$$\bigwedge_{i\le n}[x_i = \int_0^t g_i(\omega,s)(x_1(\omega,s),\ldots x_{j(i)}(\omega,s))dz_i(\omega,s)]\}.$$

For each m, $C_{m,n}, n \in \mathbf{N}$ is a decreasing chain of neocompact sets in $\mathcal{M}(\Omega, \mathbf{R}^m)$. For each m and n, Theorem 13.2 shows that $C_{m,n} \neq \emptyset$. Let $D_m = \bigcap_n C_{m,n}$. Using the countable compactness property, there exists $x_1 \in D_1$. Continuing inductively, given $(x_1, \ldots, x_m) \in D_m$, for each n there exists x_{m+1} such that $(x_1, \ldots, x_{m+1}) \in C_{m+1,n}$, and we may use the countable compactness property to obtain x_{m+1} such that $(x_1, \ldots, x_{m+1}) \in D_{m+1}$. Then the sequence $(x_1, x_2, \ldots)$ satisfies (17). $\square$

As another illustration of our method, we give a short proof of an existence theorem from [13] (see also [1] and [5]) for differential equations where the coefficient matrix $g(s,x)$ does not depend on ω but is only measurable rather than continuous in x. The analogous weak existence theorem was proved earlier by Krylov [16] using the same inequality which we used in Theorem 11.8. We assume that $g(s,x)$ is nondegenerate, that is, g maps $[0,1] \times \mathbf{R}^c$ into the set

$$\mathbf{J} = \{y \in \mathbf{K}^{cd} : det(yy^T) \geq 1/k\}.$$

Let w be a d-dimensional Brownian motion on Ω, and recall that

$$\mathcal{I} = \{\int_0^t y(\omega,s)dw(\omega,s) : y \in \mathcal{A}(\Omega, \mathbf{J})\} \subset \mathcal{M}(\Omega, \mathbf{R}^c).$$

Theorem 13.7 *For each measurable function $g \in \mathbf{L}(\mathbf{J})$ and d-dimensional Brownian motion w, there exists $x \in \mathcal{M}(\Omega, \mathbf{R}^c)$ such that*

$$x(\omega,t) = \int_0^t g(s, x(\omega,s))dw(\omega,s). \tag{18}$$

Proof: There is a sequence g_n of continuous functions converging to g in $\mathbf{L}(\mathbf{J}\ )$. By Theorem 13.2, for each $n \in \mathbf{N}$ we may choose a solution $x_n \in \mathcal{I}$ of the equation

$$x_n(\omega,t) = \int_0^t g_n(s, x_n(\omega,s))dw(\omega,s).$$

By Theorem 11.8, the stochastic integral function

$$\psi(g,x)(\omega,t) = \int_0^t g(s, x(\omega,s))dw(\omega,s)$$

is neocontinuous from $\mathbf{L}(\mathbf{K}^{cd}) \times \mathcal{H}$ into a neocompact set $B \subset \mathcal{M}(\Omega, \mathbf{R}^c)$, where $\mathcal{H}$ is a neoclosed set of adapted functions containing $\mathcal{I}$. Then the set $C = \{g\} \times B$ is neocompact in $\mathbf{L}(\mathbf{K}^{cd}) \times \mathcal{M}(\Omega, \mathbf{R}^c)$. For each n, $(g, x_n) \in C$, so the distance between (g_n, x_n) and C converges to 0. Thus each approximation to the existence problem

$$(\exists (g, x) \in C) x = \psi(g, x)$$

is true. By the Approximation Theorem, there exists an $x \in B$ such that $x = \psi(g, x)$. which solves (18). $\square$

Like Theorem 13.2, the preceding results can be extended to equations with drift terms. We can draw additional conclusions from the fact that the stochastic integral is neocontinuous. Here is one example, which is proved in the same way as the preceding theorem.

Theorem 13.8 *Suppose* $g \in \mathbf{L}(\mathbf{K}^{cd}), h \in \mathcal{A}(\Omega, \mathbf{K}^{cd})$, *and*

$$g(t, y) + h(\omega, t) \in \mathbf{J} \text{ for all } \omega, t, y.$$

For each d*-dimensional Brownian motion* w *there exists* $x \in \mathcal{M}(\Omega, \mathbf{R}^c)$ *such that*

$$x(\omega, t) = \int_0^t [g(s, x(\omega, s)) + h(\omega, s)] dw(\omega, s). \ \square$$

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