Neocompact sets and stochastic Navier-Stokes equations.

Nigel J. Cutland and H. Jerome Keisler

Abstract

We give a detailed exposition of the use of neocompact sets in proving existence of solutions to stochastic Navier-Stokes equations. These methods yield new results concerning optimality of solutions.

1 Introduction

In this paper we give a detailed exposition of the way in which the recent work of S. Fajardo and H. J. Keisler [6] can be used to establish existence of solutions to stochastic Navier-Stokes equations. Fajardo & Keisler [6] develop general methods for proving existence theorems in analysis, with the aim of embracing the many particular existence theorems that can be proved rather easily using nonstandard analysis. The machinery developed centres round the notion of a *neocompact set* - which is a weakening of the notion of a compact set of random variables with values in a metric space M - and the notion of a *rich adapted probability space*, in which any countable chain of nonempty neocompact sets has a nonempty intersection.

In the papers [2, 3] Capiński & Cutland used nonstandard methods to greatly simplify some known existence proofs for the deterministic Navier-Stokes equations and (using similar methods) solved a longstanding problem concerning existence of solutions to general stochastic Navier-Stokes equations. The aim here is to show how the main results of these papers can be obtained using the neocompactness methods developed in [6]. In addition, these methods yield additional information concerning the nature of the set of solutions and existence of optimal solutions.

For the convenience of the reader we begin with a brief summary of the notions and results we shall need from [6].

2 Neocompact sets.

In this section we review the notion of a neocompact set from the paper [6]. Neocompact sets share many of the useful properties of compact sets. We shall introduce the notion in two contexts – probability spaces and adapted spaces.

If $\mathbf{\Omega} = (\Omega, P, \mathcal{G})$ is a probability space and (M, ρ_M) is a complete metric space, then $L^0(\Omega, M)$ will denote the metric space of all measurable functions from Ω into M with the metric of convergence in probability,

$$\rho(x, y) = \inf\{\epsilon > 0 \,:\, P[\rho_M(x(\omega), y(\omega)) < \epsilon] \geq \epsilon\}.$$

For any set A we write $A^\epsilon = \{x : \rho(x, A) \leq \epsilon\}$.

We let Meas(M) be the space of all Borel probability measures on M with the Prohorov metric, and let law : $L^0(\Omega, M) \to$ Meas(M) be the function mapping each random variable x to the measure on M induced by x. This function is continuous.

Definition 2.1 *Let $\mathbf{\Omega}$ be a probability space and let M, M' denote complete separable metric spaces. A set $B \subset L^0(\Omega, M)$ is called* **basic** *if either*

(1) *B is compact, or*

(2) *$B = \{x \,:\, law(x) \in C\}$ for some compact set $C \subset Meas(M)$.*

By the family of **neocompact sets** *over Ω we mean the collection of all subsets of $L^0(\Omega, M)$ obtained by repeated application of the following rules:*

(a) *Every basic set is neocompact.*

(b) *Finite unions of neocompact sets are neocompact.*

(c) *Finite and countable intersections of neocompact sets are neocompact.*

(d) *Finite cartesian products of neocompact sets are neocompact (where we identify $L^0(\Omega, M) \times L^0(\Omega, M')$ with $L^0(\Omega, M \times M')$ in the natural way).*

(e) *If $C \subset L^0(\Omega, M \times M')$ is neocompact, then the set*

$$\{x \,:\, (\exists y)(x, y) \in C\}$$

is neocompact.

(f) *If $C \subset L^0(\Omega, M \times M')$ is neocompact and $D \subset L^0(\Omega, M')$ is basic neocompact and nonempty, then the set*

$$\{x \,:\, (\forall y \in D)(x, y) \in C\}$$

is neocompact.

It is not hard to see that the family of compact sets is closed under all of the rules (a)–(f). In fact, the family of compact sets is closed under arbitrary intersections, and condition (f) holds for arbitrary nonempty sets D. One of the reasons that compact sets are useful in proving existence theorems is that they have the following property:

If $\mathcal{C}$ is a set of compact sets such that any finite subset of $\mathcal{C}$ has a nonempty intersection, then $\mathcal{C}$ has a nonempty intersection.

We define a *rich probability space* as one in which the neocompact sets have a weaker form of this property, called the *countable compactness property.*

Definition 2.2 *A collection $\mathcal{C}$ of sets has the* **countable compactness property** *if the intersection of any countable decreasing chain $C_1 \supset C_2 \supset \cdots$ of nonempty sets in $\mathcal{C}$ is nonempty.*

Definition 2.3 *A probability space $\mathbf{\Omega}$ is said to be* **rich** *if it is atomless and for any complete separable metric space M, the collection of neocompact sets in $L^0(\Omega, M)$ has the countable compactness property.*

We now turn to neocompact sets in adapted spaces. By an *adapted space* we shall mean a structure $\mathbf{\Omega} = (\Omega, P, \mathcal{G}, \mathcal{G}_t)$ where t runs over the dyadic rationals and the $\mathcal{G}_t$ are σ-subalgebras of $\mathcal{G}$ which increase in t. For each real s, we let $\mathcal{F}_s$ be the P-completion of $\bigcap_{t>s} \mathcal{G}_t$.

Definition 2.4 *Let $\mathbf{\Omega}$ be an adapted space. The families of* **basic** *sets and* **neocompact** *sets for the adapted space are defined exactly as for the case of a probability space except that we add to the family of basic sets all sets B of the form*

(3) $B = \{x \in L^0(\Omega, M) : \text{law}(x) \in C \text{ and } x \text{ is } \mathcal{G}_t\text{-measurable}\},$

where C is compact in $Meas(M)$ and t is a dyadic rational.

Definition 2.5 *An adapted space $\mathbf{\Omega} = (\Omega, P, \mathcal{G}, \mathcal{G}_t)$ is said to be* **rich** *if the probability space $(\Omega, P, \mathcal{G}_0)$ is atomless, $\mathbf{\Omega}$ admits a Brownian motion, and for any complete separable metric space M, the collection of neocompact sets in $L^0(\Omega, M)$ has the countable compactness property.*

The following fact is implicit in the paper [12] and will be proved explicitly in [7].

Theorem 2.6 *Rich probability spaces and rich adapted spaces exist.*

This is proved by showing that the adapted Loeb spaces, which were the underlying spaces used in [2] and [3], are rich. Every rich adapted space is also rich as a probability space.

The analogues of closed sets and continuous functions are defined in terms of neocompact sets in the following way.

Definition 2.7 *A set $C \subset L^0(\Omega, M)$ is* **neoclosed** *if $C \cap D$ is neocompact for every neocompact set D.*

A function f mapping a neoclosed set $C \subset L^0(\Omega, M)$ into $L^0(\Omega, N)$ is **neocontinuous** *if for every neocompact set $D \subset C$, the graph of f restricted to D is neocompact.*

It is shown in the paper [6] that for a rich adapted space, every neoclosed set is closed, and every neocontinuous function is continuous. Parallel to the classical case, images of neocompact sets under neocontinuous functions are neocompact, and preimages of neoclosed sets under neocontinuous functions are neoclosed. We shall identify N with the set of constant functions from Ω into N. With this identification, each closed subset of N becomes a neoclosed subset of $L^0(\Omega, N)$, and we obtain the notions of a neocontinuous function from $L^0(\Omega, M)$ into N, and a neocontinuous function from a closed subset of M into $L^0(\Omega, N)$.

In the paper [6], several important sets and functions are shown to be neocompact, neoclosed, or neocontinuous for a rich adapted space.

For example, it is shown that the set of Brownian motions on $\mathbf{\Omega}$ is neocompact, the set of stopping times between 0 and 1 for $\mathbf{\Omega}$ is neocompact, and the set of all $\mathcal{F}_t$-adapted stochastic processes with values in M is neoclosed.

Some examples of neocontinuous functions which will be used in this paper are the distance function ρ for $L^0(\Omega, M)$, the function $x(\cdot) \mapsto f(x(\cdot))$ where $f : M \to N$ is continuous, the expected value function $x(\cdot) \mapsto E(x(\cdot))$ restricted to a uniformly integrable subset of $L^0(\Omega, \mathbb{R})$, and the stochastic integral function $x \mapsto \int x\, db$ where b is a Brownian motion and x is in a bounded set of adapted processes.

Moreover, compositions of neocontinuous functions are neocontinuous.

In the paper [6] the notion of *neo-lower semicontinuity*, abbreviated *neo-lsc*, is defined as a useful generalisation of the classical notion of lowersemi-continuity. Recall that a function $f : M \to \overline{\mathbb{R}}$ is lower semicontinuous (lsc) if whenever $x_n \to x$ in M then $\underline{\lim}_{n\to\infty} f(x_n) \geq f(x)$; equivalently, f is lsc if for every compact $C \subseteq M$, the upper graph

$$\{(x, r) \in C \times \overline{\mathbb{R}} : f(x) \leq r\}$$

is compact. This is the definition that is generalised in [6]. Regard $\overline{\mathbb{R}} = [-\infty, \infty]$ as a compact metric space with the metric $\rho(r, s) = |\arctan(r) - \arctan(s)|$ and we have:

Definition 2.8 *Let $D \subseteq L^0(\Omega, M)$; a function $f : D \to L^0(\Omega, \overline{\mathbb{R}})$ is* **neo-lsc** *if for every neocompact set $C \subseteq D$ the upper graph*

$$\{(x, y) \in C \times L^0(\Omega, \overline{\mathbb{R}}) : \ f(x) \leq y \text{ a.s.}\}$$

is neocompact.

If $f : D \to \overline{\mathbb{R}}$ it is easy to check that f is neo-lsc if and only if the upper graph $\{(x, r) \in C \times \overline{\mathbb{R}} : f(x) \leq r\}$ is neocompact for each neocompact C.

It is shown in [6] that the expectation operator restricted to positive random variables is neo-lsc, and that a composition $g \circ f$ of two neo-lsc functions is neo-lsc provided that g is monotone. Also, if $f : M \to \overline{\mathbb{R}}$ is lsc then the function $g : L^0(\Omega, M) \to L^0(\Omega, \overline{\mathbb{R}})$ defined by $g(x)(\omega) = f(x(\omega))$ is neo-lsc.

We shall need the following important lemma from [6].

Lemma 2.9 *(Closure Under Diagonal Intersections.) Let $\mathbf{\Omega}$ be either a rich probability space or a rich adapted space. Let A_n be neocompact in $L^0(\Omega, M)$ for each $n \in \mathbb{N}$, and let $\epsilon_n \searrow 0$. Then the set $A = \bigcap_n ((A_n)^{\epsilon_n})$ is neocompact in $L^0(\Omega, M)$.*

The key result from [6] which we shall apply in this paper is the following theorem.

Theorem 2.10 *(Approximation Theorem.) Let* $\mathbf{\Omega}$ *be either a rich probability space or a rich adapted space. Let* A *be neoclosed in* $L^0(\Omega, M)$ *and let* $f : L^0(\Omega, M) \to L^0(\Omega, N)$ *be neocontinuous. Let* $B \subset L^0(\Omega, M)$ *and* $D \subset L^0(\Omega, N)$ *be neocompact. Suppose that for each* $\epsilon > 0$

$$(\exists x \in A \cap B^{\epsilon})\, f(x) \in D^{\epsilon}.$$

Then

$$(\exists x \in A \cap B)\, f(x) \in D.$$

It is easy to check that the analogous result holds with compact, closed, and continuous in place of neocompact, neoclosed, and neocontinuous. We shall use this classical analogue as a warmup in Section 4, and then apply the Approximation Theorem for rich probability spaces and rich adapted spaces later in this paper.

3 The Navier-Stokes equations.

The classical Navier-Stokes equations describe the evolution in time of the velocity field $u : D \to \mathbb{R}^n$ of an incompressible fluid in a domain $D \subseteq \mathbb{R}^n$, so we are thinking of a function $u(x,t) : D \times [0,T] \to \mathbb{R}^n$ given by:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + \langle u, \nabla \rangle u + \nabla p = f \\ \operatorname{div} u = 0 \end{cases} \tag{1}$$

(where $\langle , \rangle$ denotes the inner product in $\mathbb{R}^n$). For convenience we will assume a fixed finite time horizon T but there is no difficulty in extending to the time set $[0, \infty)$. The domain D is bounded with boundary of class C^2 and we will work with the homogeneous Dirichlet boundary condition $u\,|_{\partial D} = 0$. Of course in important applications, $n = 3$ but from the mathematical point of view we can allow $n \leq 4$. In this equation, p denotes the pressure, and f denotes the external forces.

The usual setting for these equations involves the function spaces $\mathbf{H}$, $\mathbf{V}$ which are obtained by closing the set $\{u \in C_0^\infty(D, \mathbb{R}^n) : \operatorname{div} u = 0\}$ in the norms $|\cdot|$ and $|\cdot| + \|\cdot\|$ respectively, where

$$|u| = (u,u)^{\frac{1}{2}}; \quad (u,v) = \sum_{j=1}^n \int_D u^j(\xi) v^j(\xi) \mathrm{d}\xi,$$
$$\|u\| = ((u,u))^{\frac{1}{2}}; \quad ((u,v)) = \sum_{j=1}^n \left(\frac{\partial u}{\partial \xi_j}, \frac{\partial v}{\partial \xi_j}\right).$$

$\mathbf{H}, \mathbf{V}$ are Hilbert spaces. We fix an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ for $\mathbf{H}$ consisting of eigenvectors of the operator $-\Delta$ with eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$. For $u \in \mathbf{H}$ we write $u_k = (u, e_k)$. We let $\mathbf{V}'$ be the dual space of $\mathbf{V}$ with respect to the $|\cdot|$ norm, and let $(\cdot,\cdot)$ denote the duality between $\mathbf{V}'$ and $\mathbf{V}$ extending the scalar product in $\mathbf{H}$. In the equation (1) it is usual to take the force $f \in L^2(0,T;\mathbf{V}')$, and then the equation is understood as a Bochner integral equation in $\mathbf{V}'$. i.e. for each $v \in \mathbf{V}$:

$$(u(t),v) - (u_0,v) = \int_0^t [-\nu(\!(u(s),v)\!) - b(u(s),u(s),v) + (f(s),v)]\mathrm{d}s \tag{2}$$

where u_0 is the given initial condition. The pressure vanishes in this weak formulation because $\langle \nabla p, v\rangle = -\langle p, \operatorname{div} v\rangle = 0$, but of course p can be recovered from a solution to the equation (2). The trilinear form b is the nonlinear term in (1), so that we have:

$$b(u,v,w) = \sum_{i,j=1}^{n} \int_D u^j(\xi)\frac{\partial v^i}{\partial \xi_j}(\xi)w^i(\xi)\mathrm{d}\xi = (\langle u, \nabla\rangle v, w).$$

Note that for $u,v,z \in \mathbf{V}$ we have $b(u,v,z) = -b(u,z,v)$ so that $b(u,v,v) = 0$. There are a number of well known inequalities giving continuity of b in various topologies (see [13] and [14] p.12 for example) and we list here those (for $n \le 4$) that we shall need.

$$|b(u,v,z)| \le c\|u\|\,\|v\|\,\|z\| \tag{3}$$

$$|b(u,v,z)| \le c|u|\,\|v\|\,|Az| \tag{4}$$

$$|b(u,v,z)| \le c|u|\,|Av|\,\|z\| \tag{5}$$

It is customary to write A for the self-adjoint extension of the operator $-\Delta$ on $\mathbf{H}$, and to write $B(u) = b(u,u,\cdot) \in \mathbf{V}'$ for appropriate u. Then it is an easy consequence of (4) and (5) that:

Proposition 3.1 *For all m, $B(\cdot) \in C(K_m, \mathbf{V}'_{weak})$, where K_m is the compact subset of $\mathbf{H}$ given by $K_m = \{u \in \mathbf{H} : \|u\| \le m\}$ with the $\mathbf{H}$-topology.* □

Proof See Proposition 3.4 of [3].

We can now make precise what is taken to be a weak solution to (1)

Definition 3.2 *Given $u_0 \in \mathbf{H}$ and $f \in L^2(0,T;\mathbf{V}')$ the function $u : [0,T) \to \mathbf{H}$ is a* **weak solution** *of the Navier-Stokes equations if*

(i) $u \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$,

(ii) for all $v \in \mathbf{V}$, for all $t \ge 0$, u satisfies equation (2)

The regularity condition (i) emerges naturally by consideration of the time evolution of the energy $\frac{1}{2}|u|^2$.

The spaces $\mathbf{H}$ and $\mathbf{V}$ involved in the formulation of the Navier-Stokes equations are two from the spectrum of Hilbert spaces $\mathbf{H}^\alpha$ given by

$$\mathbf{H}^\alpha = \{u \in \mathbf{H} : \sum_{k=1}^{\infty} \lambda_k^\alpha u_k^2 < \infty\}$$

for $\alpha \geq 0$, with norm $|u|_\alpha = \left(\sum_{k=1}^\infty \lambda_k^\alpha u_k^2\right)^{\frac{1}{2}}$. The dual spaces $\mathbf{H}^{-\alpha}$ are represented by the sets

$$\mathbf{H}^{-\alpha} = \{u \in \mathbb{R}^{\mathbb{N}} : \sum_{k=1}^{\infty} \lambda_k^{-\alpha} u_k^2 < \infty\}$$

with the corresponding norms $|u|_{-\alpha}$. It is easily checked that $\mathbf{H} = \mathbf{H}^0$, $\mathbf{V} = \mathbf{H}^1$ and $\mathbf{V}' = \mathbf{H}^{-1}$.

We write $\mathbf{H}_n = \mathrm{span}\{e_1, \ldots, e_n\}$ and denote the projection from $\mathbf{H}$ onto $\mathbf{H}_n$ by Π_n. For $u \in \mathbf{H}$, $u^{(n)}$ denotes $\Pi_n u$.

4 Existence of weak solutions

In this section we prove

Theorem 4.1 *For every $u_0 \in \mathbf{H}$ there is a weak solution u to the Navier-Stokes equations with $u(0) = u_0$ and with u in the space M_0 defined below.*

This is, of course, a classical result. We shall present a proof using the classical analogue of the Approximation Theorem (Theorem 2.10) and compare it with a classical proof. The point of doing this is to show how the neocompactness machinery allows a direct generalization of our proof to give the existence result for the stochastic Navier-Stokes equations [3], for which there is no classical proof using compactness.

For the development here and in subsequent sections we define some spaces and auxiliary functions that will be important for the construction of solutions. We put

$$M = C([0,T], \mathbf{H}^{-2}) \cap \{y : y(0) \in \mathbf{H}\}$$

which is a complete metric space with the metric $|\cdot|_M$ given by

$$|y_1 - y_2|_M = \sup_{t \leq T} |y_1(t) - y_2(t)|_{-2} + |y_1(0) - y_2(0)|.$$

For $y \in M$ define $\theta(y)$ by

$$\theta(y) = \nu \int_0^T \|y(t)\|^2 \mathrm{d}t + \sup_{t \leq T} |y(t)|^2.$$

Let M_0 be the subset of M given by

$$M_0 = M \cap \{y : \theta(y) \leq K + |y(0)|^2\}$$

where $K = \nu^{-1} \int_0^T |f(t)|^2_{\mathbf{V}'} \mathrm{d}t$.

Proposition 4.2 *M_0 is a closed subspace of M.*

Proof This follows easily from the fact that if u, $u^n \in \mathbf{H}^{-2}$ with $u^n \to u$ in $\mathbf{H}^{-2}$, then $|u|_\alpha \leq \underline{\lim}|u^n|_\alpha$ for any α. Apply this to $|y^n(t)| = |y^n(t)|_0$ and $\|y^n(t)\| = |y^n(t)|_1$ for any sequence $y^n \in M_0$ with $y^n \to y \in M$, and use Fatou's lemma ($\int_0^T \underline{\lim}\|y^n(t)\|^2 dt \leq \underline{\lim} \int_0^T \|y^n(t)\|^2 dt$) and the fact that $\sup_{t\leq T} \underline{\lim}|y^n(t)|^2 \leq \underline{\lim} \sup_{t\leq T} |y^n(t)|^2$. □

Remark It is easy to check that $M_0 \subseteq C([0,T], \mathbf{H}^{-\alpha})$ for all $\alpha > 0$, although we will not need this. It is a consequence of the fact that for $y \in M_0$ each $y_k(t)$ is continuous, and $|y(t)|$ is bounded.

Now for each $k > 0$ define

$$M_k = M \cap \{y : \theta(y) \leq k\}$$

Clearly each M_k is closed in M and $M_0 \subseteq \bigcup_{k>0} M_k = M_\infty$, say.

We now define a function $\gamma : M_\infty \to M$ by

$$\gamma(y)(s) = \int_0^s [-\nu Ay(t) - B(y(t)) + f(t)]\mathrm{d}t.$$

To see that γ takes its values in M, use the following key facts about A and B:

$$\begin{array}{l}\text{(i) } |Au|_{-1} = \|u\| \\ \text{(ii) } |B(u)|_{-1} \leq c\|u\|^2\end{array} \tag{6}$$

with (ii) holding by (3). These properties of A and B ensure that for $y \in M_\infty$ we have

$$\begin{aligned}\gamma(y) &\in C([0,T], \mathbf{H}^{-1}_{weak}) \cap L^\infty(0,T;\mathbf{H}^{-1}) \\ &\subseteq C([0,T], \mathbf{H}^{-\alpha})\end{aligned}$$

for any $\alpha > 1$ and in particular for $\alpha = 2$.

Remark This shows that the choice of $\mathbf{H}^{-2}$ in the definition of M was somewhat arbitrary. We could have taken $\mathbf{H}^{-\alpha}$ for any $\alpha > 1$.

We observe that $u = y$ is a weak solution to the Navier-Stokes equations with initial condition $u_0 = x$ if and only if $y = x + \gamma(y)$.

The next two facts are the key to the construction of a solution.

Proposition 4.3 *The function γ is continuous on each set M_k (in the topology of M).*

Proof This is routine Bochner integration theory using the facts (6) above. □

Proposition 4.4 *For each $k > 0$ the set $\gamma(M_k)$ is relatively compact in M.*

Proof The facts (6) can be used to show that there are constants d and $(d_m)_{m\geq 1}$ such that

$$\begin{aligned}\gamma(M_k) \subseteq \{z \in M : z(0) = 0 \ \& \ \sup_{t\leq T} |z(t)|_{-1} \leq d \\ \& \ |z_m(s) - z_m(t)|^2 \leq d_m|s-t| \text{ for all } s,t \leq T \text{ and } m \in \mathbb{N}\}.\end{aligned}$$

It is then routine to check that the set on the right is compact in M. □

As with classical proofs, we now consider the finite dimensional approximations to (2) - known as the *Galerkin approximations*. For this purpose we define a sequence of functions $\gamma_n : M_\infty \to M$ by

$$\begin{aligned}\gamma_n(y)(s) &= \int_0^s [-\nu A y^{(n)}(t) - B^{(n)}(y(t)) + f^{(n)}(t)]\mathrm{d}t \\ &= \Pi_n \gamma(y)(s).\end{aligned}$$

It is clear that $\gamma_n(y) \in C([0,T], \mathbf{H}_n)$.

Classical ODE theory (see [13] for example) shows:

Theorem 4.5 *For each $n > 0$ and $x \in \mathbf{H}$ there is a unique $y \in M_0$ such that $y = x^{(n)} + \gamma_n(y)$ (so $y \in M_{K+|x|^2}$).*

Proof The equation $y = x^{(n)} + \gamma_n(y)$ for $x \in \mathbf{H}$ is simply the Galerkin approximation to the Navier-Stokes equations in dimension n, with initial condition $x^{(n)}$, which has a unique solution $y \in C([0,T], \mathbf{H}_n)$; consideration of the time evolution of the energy $|y(t)|^2$ shows that

$$|y(s)|^2 + \nu \int_0^s \|y(t)\|^2 \mathrm{d}t \leq |y(0)|^2 + \nu^{-1} \int_0^s |f(t)|^2_{\mathbf{V}'} \mathrm{d}t$$

so that $y \in M_0$. Clearly, $y \in M_{K+|x|^2}$. □

The final result in preparation for the fundamental existence result Theorem 4.1 is:

Proposition 4.6 *For $k > 0$, $\gamma_n(y) \to \gamma(y)$ uniformly on M_k.*

Proof By Proposition 4.4, $\overline{\gamma(M_k)}$ is a compact subset of M. Dini's theorem tells us that $\Pi_n z \to z$ uniformly on any any compact set. □

We now complete the proof of 4.1 by using the classical analogue of the Approximation Theorem (2.10).

Proof of Theorem 4.1 Using the classical analogue of the Approximation Theorem, it is sufficient to prove that there is a compact $D \subseteq M$ such that

$$(\forall \varepsilon > 0)(\exists y \in D^\varepsilon \cap M_0)(|x + \gamma(y) - y|_M < \varepsilon).$$

For this we take $D = x + \overline{\gamma(M_k)}$ where $k = K + |x|^2$, which is compact by Proposition 4.4. Now, given $\varepsilon > 0$ take n such that $|\gamma_n(y) - \gamma(y)| < \frac{1}{2}\varepsilon$ for all $y \in M_k$ (using Proposition

4.6), and $|x - x^{(n)}| < \frac{1}{2}\varepsilon$. Take the unique $y \in M_0$ with $y = x^{(n)} + \gamma_n(y) \in M_k$, given by Proposition 4.5. Then

$$|x + \gamma(y) - y| \leq |\gamma(y) - \gamma_n(y)| + |x - x^{(n)}| < \varepsilon$$

and since $x + \gamma(y) \in D$ we have $y \in D^\varepsilon$ and the proof is complete. □

This proof is really not very different from a classical proof using the preliminary results Propositions 4.4, 4.5 and 4.6 above, which runs as follows:

For $x \in \mathbf{H}$, take $y_n = x^{(n)} + \gamma_n(y_n) \in M_0 \cap M_k$ as given by Proposition 4.5, and let $z_n = x + \gamma(y_n) \in D$. By Proposition 4.4, there is a convergent subsequence $z_{n_i} \to z$, say with $z \in D$. Then

$$|y_{n_i} - z| \;\leq\; |x^{(n_i)} + \gamma_{n_i}(y_{n_i}) - (x + \gamma(y_{n_i}))| + |z_{n_i} - z| \;\to\; 0$$

using Proposition 4.6. So z belongs to the closed set M_0 . By Proposition 4.3 we have

$$x + \gamma(z) = x + \lim \gamma(y_{n_i}) = \lim z_{n_i} = z,$$

so that z is the required solution.

5 Existence of statistical solutions

The idea of a *statistical solution* to the Navier-Stokes equations was developed by Foias [8], and the idea is as follows. Suppose that in the Navier-Stokes equations (1) or (2) the initial condition is given by a probability measure μ_0 on $\mathbf{H}$, with the informal idea that there is some underlying probability P such that for $A \subseteq \mathbf{H}$

$$\mu_0(A) = P(u_0 \in A).$$

Then, informally, as time evolves we can think of measures μ_t given by

$$\mu_t(A) = P(u(t) \in A). \tag{7}$$

However, this idea is only heuristic, because we do not have any meaning for the value $u(t)$ for a random initial condition u_0 - since in dimension $n \geq 3$ the uniqueness problem for equation (2) is still open. If we did have uniqueness of solutions then equation (7) could be made precise by writing $S_t(u)$ for the value at time t of the solution to (2) with initial condition u, and then the family of measures μ_t would be given by

$$\mu_t(A) = \mu_0(S_t^{-1}(A)).$$

Even though there is no such function S_t to make this precise, by arguing informally Foias [8] derived the following equation that would be satisfied by the family μ_t if S_t did exist:

$$\int_{\mathbf{H}} \varphi(u) d\mu_t(u) - \int_{\mathbf{H}} \varphi(u) d\mu(u) =$$

$$\int_0^t \int_{\mathbf{H}} [-\nu((u,\varphi'(u))) - b(u,u,\varphi'(u)) + (f(s),\varphi'(u))] d\mu_s(u) ds \tag{8}$$

where φ is any test function of the form $\varphi(u) = \exp^{i(u,v)}$ with $v \in \mathbf{V}$. This is called the *Foias equation*, and it makes sense because there is no reference to the possibly non-existent function S_t that was used informally in Foias' derivation. Solutions to the Foias equation (see below) are called *statistical solutions* to the Navier-Stokes equations. In dimensions $n \geq 3$ Foias' derivation of equation (8) is only heuristic so it does *not* guarantee the existence of statistical solutions, and these must be constructed by some other means, as Foias did in [8].

Here is the precise definition of a statistical solution.

Definition 5.1 *Suppose that a Borel probability measure μ on $\mathbf{H}$ is given, with $\int_{\mathbf{H}} |u|^2 \mathrm{d}\mu < \infty$. Then a family of probability measures $(\mu_t)_{t\geq 0}$ is a* **statistical solution** *of the Navier-Stokes equations with initial condition μ if $\mu_0 = \mu$ and*

(i) the function $t \mapsto \int_{\mathcal{H}} |u|^2 d\mu_t(u)$ is $L^\infty(0,T)$ for all $T < \infty$,

(ii) $\int_0^T \int_{\mathcal{H}} \|u\|^2 d\mu_t(u) dt < \infty$ for all $T < \infty$,

(iii) for all $t \geq 0$ and test functions φ as above equation (8) holds.

Foias' proof [8] of existence of statistical solutions to the Navier-Stokes equations in dimensions $n \leq 4$ is long and complicated. A new and very short proof in [2] was based on the uniqueness of solutions to the Galerkin approximation in an infinite hyperfinite dimension N, which made Foias' heuristic derivation completely rigorous in this setting. Essentially what was proved there was the existence of a random solution to (2) for a given random initial condition. This is easily proved using the neocompactness methods from [6], together with the results of the previous section, as follows.

Theorem 5.2 *Let $x(\omega) \in \mathbf{H}$ be a random variable defined on a rich probability space (Ω, P). Then there is a random variable $y : \Omega \to M_0$ such that for almost all $\omega \in \Omega$, $y(\omega)$ is a (weak) solution to the Navier-Stokes equations with initial condition $x(\omega)$.*

Proof Since $\mathbf{H}$ is separable there is an increasing sequence of compact sets C_n with $x(\omega) \in \cup_{n\geq 1} C_n$ a.s. Let $D_n = \left(C_n + \overline{\gamma(M_{k_n})}\right) \cap M_0$, where $k_n = K + \sup_{z\in C_n} |z|^2$. Then by Proposition 4.4, D_n is compact, and by Theorem 4.1 we have

$$(\forall z \in C_n)(\exists y \in D_n)[y = z + \gamma(y)]. \tag{9}$$

By [6], the function

$$y(\cdot) \mapsto \gamma(y(\cdot))$$

is neocontinuous and the set

$$D = \{y(\cdot) : (\forall n)(P[y(\omega) \in D_n] \geq P[x(\omega) \in C_n])\}$$

is neocompact. It follows that there are sequences $x_n(\cdot)$ and $y_n(\cdot)$ of simple random variables (i.e. random variables with finite ranges) such that x_n takes its values in C_n, $x_n(\omega) \to x(\omega)$ almost everywhere, $y_n(\cdot) \in D$, and

$$y_n(\omega) = x_n(\omega) + \gamma(y_n(\omega))$$

for almost all $\omega \in \Omega$.

Now let $C = \{x(\cdot)\}$, which is neocompact. Then we have proved

$$(\forall \varepsilon > 0)(\exists y \in D^\varepsilon)(\exists z \in C^\varepsilon)(z + \gamma(y) = y)$$

(simply take $z = x_n$ within ε of x and $y = y_n$).

The Approximation Theorem gives a random variable $y(\cdot) \in D$ with

$$y(\omega) = x(\omega) + \gamma(y(\omega))$$

for almost all ω, which is as required. □

To see that this is sufficient to give statistical solutions, we have

Theorem 5.3 *Suppose that y is an M_0-valued random variable on a probability space (*not *necessarily a rich space) with*

$$y(\omega) = x(\omega) + \gamma(y(\omega))$$

for a.a. ω, and such that

$$E(|x(\omega)|^2) < \infty.$$

Then the family of measures μ_t on $\mathbf{H}$ given by $\mu_t(A) = P(y(\omega, t) \in A)$ is a statistical solution to the Navier-Stokes equations.

Proof Simply follow through Foias' heuristic, as in [2]. □

6 Stochastic Navier-Stokes equations

We show here how the existence proof for stochastic Navier-Stokes equations in dimensions $n \leq 4$ can be presented using the machinery of [6]. The pattern of the proof is similar to the corresponding existence result for the deterministic case (Sec. 4), that used the classical analogue of this machinery. In that case we showed how the essential facts that made this work also led to a simple proof that avoids that machinery. In the stochastic case however, it seems unlikely that the proof below can be recast in a way that avoids the use of some form of the neocompactness machinery, because of the need for some kind of enriched probability space to carry the solution in general.

We begin by reviewing from [3] the formulation of the stochastic Navier-Stokes equations. Suppose that $Q : \mathbf{H} \to \mathbf{H}$ is a linear nonnegative trace class operator and that $w(t)$, $t \geq 0$ is an $\mathbf{H}$-valued Wiener process with covariance Q, defined on an adapted probability space

$\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. If $f: [0,T] \times \mathbf{V} \to \mathbf{V}'$ and $g: [0,T] \times \mathbf{V} \to \mathcal{L}(\mathbf{H}, \mathbf{V}')$ then the stochastic Navier-Stokes equations with full feed-back take the form

$$du(t) = [-\nu Au(t) - B(u(t)) + f(t,u(t))]dt + g(t,u(t))dw(t) \tag{10}$$

which is to be understood as an integral equation, using the Bochner integral for the drift term and the stochastic integral of Ichikawa [10] for the noise term. The initial condition u_0 can be random in $\mathbf{H}$, although to begin with we consider only fixed initial conditions $u_0 \in \mathbf{H}$. Thus we have the following definition:

Definition 6.1 *A stochastic process $u(t,\omega)$ on $\mathbf{\Omega}$ is a* **solution to the stochastic Navier-Stokes equations** *(10) if it is adapted and has almost all paths in the space*

$$Z = C(0,T;\mathbf{H}_{weak}) \cap L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$$

and the integral equation

$$u(t) = u_0 + \int_0^t [-\nu Au(s) - B(u(s)) + f(s,u(s))]ds + \int_0^t g(s,u(s))dw(s) \tag{11}$$

holds as an identity in $\mathbf{V}'$.

Now take a rich adapted probability space $\mathbf{\Omega} = (\Omega, P, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0})$ carrying an $\mathbf{H}$-valued Wiener process w with covariance Q. A process is said to be *adapted for* $\mathbf{\Omega}$ if it is adapted with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ where $\mathcal{F}_t$ is the P-completion of $\bigcap_{s>t} \mathcal{G}_s$.

Recall that K_m is the compact subset of $\mathbf{H}$ given by $K_m = \{u \in \mathbf{H} : \ \|u\| \leq m\}$ with the $\mathbf{H}$-topology.

The existence theorem proved in [3], in the special case that $\mathbf{\Omega}$ is an adapted Loeb space, is

Theorem 6.2 *Let $\mathbf{\Omega}$ be a rich adapted space. Suppose that $u_0 \in \mathbf{H}$ and $f : [0,T] \times \mathbf{V} \to \mathbf{V}'$ and $g : [0,T] \times \mathbf{V} \to \mathcal{L}(\mathbf{H}, \mathbf{H})$ are jointly measurable functions with the following properties:*

(i) $f(t,\cdot) \in C(K_m, \mathbf{V}'_{weak})$ for all m,

(ii) $g(t,\cdot) \in C(K_m, \mathcal{L}(\mathbf{H}, \mathbf{H})_{weak})$ for all m,

(iii) $|f(t,u)|_{\mathbf{V}'} + |g(t,u)|_{\mathbf{H},\mathbf{H}} \leq a(t)(1+|u|)$, for some function $a \in L^2(0,T)$.

Then for each fixed $u_0 \in \mathbf{H}$ the stochastic Navier-Stokes equation (11) has a solution u on $\mathbf{\Omega}$, with u independent of $\mathcal{G}_0$, and such that

$$E\left(\sup_{t\leq T} |u(t)|^2 + \int_0^T \|u(t)\|^2 dt\right) < \infty.$$

(In fact u is in the space N_0 defined below).

We present a proof of this using the neocompactness machinery; first we set up some notation.

Let N be the set
$$N = \{y \in L^0(\Omega, M) \,:\, y \text{ is adapted}\},$$
and let $G^\perp$ be the set
$$G^\perp = \{y \in L^0(\Omega, M) \,:\, y \text{ is independent of } \mathcal{G}_0\}.$$
Let
$$\mathcal{E}(y) = \int_0^T \|y(t)\|^2 \mathrm{d}t + \sup_{t \le T} |y(t)|^2,$$
and define
$$N_0 = N \cap \{y : \ y(0) \in \mathbf{H} \ \& \ E(\mathcal{E}(y)) \le K_1 + K_2|y(0)|^2 \ \}$$
where K_1 and K_2 are constants to be specified below. (In this definition by "$y(0) \in \mathbf{H}$" we mean that $y(0)$ is non-random and in $\mathbf{H}$.)

Proposition 6.3 *N, $G^\perp$, and N_0 are neoclosed subsets of $L^0(\Omega, M)$.*

Proof It is shown in [6] that the set of adapted processes is neoclosed; i.e. N is neoclosed.

To show that $G^\perp$ is neoclosed, first show that the set
$$G_0^\perp = \{x \in L^0(\Omega, \mathbb{R}) : |x| \le \pi/2 \text{ and } x \text{ is independent of } \mathcal{G}_0\}$$
is neoclosed. We have
$$G_0^\perp = \{x : (\forall z \in C)(E(xz) = E(x)E(z) \text{ and } |x| \le \pi/2)\}$$
where C is the set of $\mathcal{G}_0$-measurable indicator functions, which is a basic neocompact set. Then use the universal quantifier rule (f) and the fact that the expectation E is neocontinous on uniformly bounded sets of random variables. For $G^\perp$ itself, note that
$$y \in G^\perp \quad \Leftrightarrow \quad (\forall k)(\forall q \in [0,T])(\arctan(y_k(q,\cdot)) \in G_0^\perp)$$
(q ranging over rationals). The function $y(\cdot) \mapsto \arctan(y_k(q,\cdot))$ is neocontinuous. Now use the facts that neoclosed sets are closed under countable intersections and preimages by neocontinuous functions.

The set of y with $y(0) \in \mathbf{H}$, constant, is neoclosed because the function $y \mapsto y(0)$ is neocontinuous and the space $\mathbf{H}$ is closed and separable (so the set of constant random variables in $L^0(\Omega, \mathbf{H})$ is neoclosed).

The set of y with $E(\mathcal{E}(y)) \le K_1 + K_2|y(0)|^2$ is neoclosed since the function $\mathcal{E} : M \to \mathbb{R}$ is lsc and so $E(\mathcal{E}(y(\cdot))$ is neo-lsc, and the function $y \mapsto y(0)$ is neocontinuous.

N_0 is the intersection of these two sets with N and thus is also neoclosed. $\square$

Now for each $k > 0$ define

$$N_k = N \cap \{y : E(\mathcal{E}(y)) \leq k\}.$$

Clearly each N_k is neoclosed in N, and $N_0 \subseteq \cup_{k>0} N_k = N_\infty$ say.

We now define a function $\tilde{\gamma} : N_\infty \to N$ by

$$\tilde{\gamma}(y)(s) = \int_0^s [-\nu A y(t) - B(y(t)) + f(t, y(t))]\mathrm{d}t + \int_0^s g(t, y(t)) dw(t) \tag{12}$$

Note that $u = y$ is a solution to the stochastic Navier-Stokes equations with initial condition $u_0 = x$ if and only if $y = x + \tilde{\gamma}(y)$. Moreover, we have

$$\tilde{\gamma} : G^\perp \cap N_\infty \to G^\perp \cap N,$$

because the Wiener process w has increments that are independent of $\mathcal{F}_0$ (by definition).

We wish to prove neocompact analogues of Propositions 4.4 and 4.3 for the sets N_k. For this we must deal with the two integrals in $\tilde{\gamma}$ separately. We may write

$$\tilde{\gamma}(y) = h(y) + I(y)$$

where $I(y)$ denotes the infinite dimensional stochastic integral

$$I(y) = \int_0^\cdot g(s, y(s))\mathrm{d}w(s).$$

and $h(y)$ is the Bochner integral term

$$h(y)(s, \omega) = \int_0^s [-\nu A y(t, \omega) - B(y(t, \omega)) + f(t, y(t, \omega))]\mathrm{d}t = \gamma(y(\omega))(s).$$

Here $\gamma : M_\infty \to M$ is slightly more general than the function γ in Section 4 because of the feedback in f, but Propositions 4.3 and 4.4 are still valid; i.e. γ here is also continuous on M_k and $\gamma(M_k)$ is relatively compact for each $k > 0$ (with the obvious modifications to the definition of M_k).

First we deal with h. Define the set

$$\hat{N}_k = \{y \in L^0(\Omega, M) : E(\mathcal{E}(y)) \leq k\}$$

which is neoclosed, and let

$$\hat{M}_m = \{y \in L^0(\Omega, M) : \mathcal{E}(y(\cdot)) \leq m \text{ a.s. }\} = L^0(\Omega, M_m).$$

First we note that:

Lemma 6.4 *Assume the hypotheses of Theorem 6.2. Then for each $m > 0$ the function h is neocontinuous from $\hat{M}_m$ into $L^0(\Omega, M)$ and $h(\hat{M}_m)$ is contained in a neocompact subset of $L^0(\Omega, M)$.*

Proof Since $h(y)(\omega) = \gamma(y(\omega))$ and γ is continuous, it follows that h is neocontinuous on each $\hat{M}_m$. Since $\gamma(M_m)$ is relatively compact, it follows that $h(\hat{M}_m)$ is contained in a neocompact set. □

Lemma 6.5 *Assume the hypotheses of Theorem 6.2. Then for each $k > 0$ the function h is neocontinuous from $\hat{N}_k$ into $L^0(\Omega, M)$ and $h(\hat{N}_k)$ is contained in a neocompact subset of $L^0(\Omega, M)$.*

Proof For each m let D_m be a neocompact set with $h(\hat{M}_m) \subseteq D_m$. Fix k, and take $y \in \hat{N}_k$. By Chebyshev's inequality, for each m we have $P(\mathcal{E}(y) \leq m) \geq 1 - \frac{k}{m}$. Define y^m by

$$y^m = \begin{cases} y & \text{on the set } \{\mathcal{E}(y) \leq m\} \\ 0 & \text{otherwise} \end{cases}$$

Then we have $y^m \in \hat{M}_m$ and $h(y) = h(y^m)$ on $\{\mathcal{E}(y) \leq m\}$. Hence $\rho(h(y), h(y^m)) \leq \frac{k}{m}$ and so $h(y) \in (D_m)^{k/m}$. Thus $h(\hat{N}_k) \subseteq D$ where

$$D = \bigcap_m (D_m)^{\frac{k}{m}}$$

which is neocompact by closure under diagonal intersections.

To see that h is neocontinuous on $\hat{N}_k$, take a neocompact set $C \subseteq \hat{N}_k$ and for each m let $C^m = \{y^m : y \in C\} \subseteq \hat{M}_m$. The function $y \mapsto y^m$ is neocontinuous, and so C^m is neocompact. The graph of h restricted to C is the neocompact set

$$\bigcap_m \{(y, v) \in C \times D \,:\, (\exists z \in C^m)\Big(\rho(y, z) + \rho(v, h(z)) \leq \frac{2k}{m}\Big)\}.$$

□

Corollary 6.6 *Assume the hypotheses of Theorem 6.2. Then for each k the function h is neocontinuous from N_k into N, and $h(N_k)$ is contained in a neocompact subset of N with respect to $L^0(\Omega, M)$.*

Proof This follows immediately from Lemmas 6.4 and 6.5, since $N_k \subseteq \hat{N}_k$. □

To obtain the same result for I we must first prepare the way by dealing with a class of bounded integrands. For each $m > 0$ define

$$J_m = N \cap \{y : \sup_{t \leq T} |y(t)|^2 \leq m \text{ a.s.}\};$$

clearly J_m is neoclosed.

Lemma 6.7 *Assume the hypotheses of Theorem 6.2. Then for each m the function I is neocontinuous from J_m into N, and $I(J_m)$ is contained in a neocompact subset of N with respect to $L^0(\Omega, M)$.*

Proof Let $I_n(y)$ denote the finite dimensional stochastic integral

$$I_n(y) = (\int_0^\cdot g(s, y(s))\mathrm{d}w(s))^{(n)}.$$

The results of [6] show that for each n and m, the function I_n is neocontinuous from J_m into $L^0(\Omega, M)$. The bound (iii) on g insures that for fixed m the set $\cup_n \mathrm{law}(I_n(J_m))$ is tight, so the function I maps J_m into a neocompact set D in $L^0(\Omega, M)$.

For each neocompact set $C \subseteq J_m$ in $L^0(\Omega, M)$, the graph of $I|C$ is the neocompact set

$$\bigcap_n \{(y, z) \in C \times D \, : \, \Pi_n(z) = I_n(y)\}.$$

Therefore I is neocontinuous on J_m. □

Now we must extend Lemma 6.7 to N_k.

Let ST be the set of all stopping times $\tau \in L^0(\Omega, [0, T])$ and for any process y let y^τ denote the stopped process $y(\omega)(t \wedge \tau)$. It is shown in [FK] that ST is neocompact, and that the function $(y, \tau) \mapsto y^\tau$ is neocontinuous from $N \times ST$ into N.

Proposition 6.8 *For each $k > 0$ the set $I(N_k)$ is contained in a neocompact subset of N with respect to $L^0(\Omega, M)$.*

Proof By Lemma 6.7, for each $m \in \mathbb{N}$ there is a neocompact set $D_m \subseteq N$ such that

$$I(J_m) \subseteq D_m.$$

If $y \in N_k$ then $E(\sup_{t \le T} |y(t)|^2) \le k$, and by Chebyshev's inequality we have

$$P(\sup_{t \le T} |y(t)|^2 \le m) \ge 1 - \tfrac{k}{m}.$$

For $y \in N$ let $\tau_{m,y}(\omega)$ be the first time t such that either $t = T$ or $\sup_{s \le t} |y(s)|^2 \ge m$. Then $\tau = \tau_{m,y}$ is a stopping time; if $y \in N_k$, then y^τ belongs to J_m and $P(y = y^\tau) \ge 1 - \frac{k}{m}$; i.e. $\rho(y, y^\tau) \le \frac{k}{m}$. Moreover $I(y) = I(y^\tau)$ on the set $\{y = y^\tau\}$ and so $\rho(I(y), I(y^\tau)) \le \frac{k}{m}$. But $I(y^\tau) \in D_m$ and hence $I(y) \in (D_m)^{k/m}$ in the space $L^0(\Omega, M)$. It follows that $I(N_k)$ is contained in the set

$$D = N \cap \bigcap_m (D_m)^{\frac{k}{m}}.$$

By closure under diagonal intersections, D is neocompact in $L^0(\Omega, M)$. □

Proposition 6.9 *Assume the hypotheses of Theorem 6.2. Then for each k the function I is neocontinuous from N_k into N.*

Proof Let C be a neocompact subset of N_k. By the preceding proposition, there is neocompact $D \supseteq I(N_k)$. Since C and ST are neocompact, the set $\tilde{C} = \{y^\tau : y \in C \wedge \tau \in ST\}$ is neocompact. Moreover, the argument in the proof of Proposition 6.8 shows that for each $y \in C$, for each m there is a stopping time τ such that $y^\tau \in \tilde{C} \cap J_m$ and $\rho(y, y^\tau) \leq \frac{k}{m}$ and $\rho(I(y), I(y^\tau)) \leq \frac{k}{m}$. It follows that the graph of $I|C$ is the neocompact set

$$\bigcap_m \{(y,v) \in C \times D : (\exists z \in \tilde{C} \cap J_m)\Big(\rho(y,z) + \rho(v, I(z)) \leq \frac{2k}{m}\Big)\},$$

so I is neocontinuous on N_k. □

The following proposition is the key to the construction of a solution. It is the neocompact analogue of propositions 4.4 and 4.3, and follow immediately from Corollary 6.6 and Propositions 6.8, 6.9.

Proposition 6.10 *Assume the hypotheses of Theorem 6.2. Then for each $k > 0$ the function $\tilde{\gamma}$ is neocontinuous from N_k into N and the set $\tilde{\gamma}(N_k)$ is contained in a neocompact subset of N with respect to $L^0(\Omega, M)$.*

Next we define the Galerkin approximation to equation (11) and show that the stochastic version of Theorem 4.5 holds. We set

$$\tilde{\gamma}_n(y) = \Pi_n \tilde{\gamma}(y).$$

Then we have

Theorem 6.11 *There are constants K_1 and K_2 such that for each $n > 0$ and $x \in \mathbf{H}$ there is $y \in N_0 \cap G^\perp$ such that $y = x^{(n)} + \tilde{\gamma}_n(y)$ (so $y \in G^\perp \cap N_{K_1 + K_2|x|^2}$) .*

Proof The equation $y = x^{(n)} + \tilde{\gamma}_n(y)$ is a finite dimensional SDE which can be solved by routine standard methods and has a solution. It was shown in [3] that there are K_1 and K_2, independent of n (and which can be given explicitly in terms of the parameters ν, T, trQ, $\int_0^T a^2$), such that $E(\mathcal{E}(y)) \leq K_1 + K_2|y(0)|^2$ whenever $y = x^{(n)} + \tilde{\gamma}_n(y)$. The proof of this estimate uses the same ideas as are used for estimating the energy for the Galerkin approximation to the deterministic Navier-Stokes equations, coupled with the Burkholder-Davis-Gundy inequality and an application of Gronwall's lemma. (In [3] it was established for an infinite integer n, by transfer of finite dimensional methods which are used in the present situation.) □

Theorem 6.12 (Neo-Dini's Theorem) *For a rich adapted space $\mathbf{\Omega}$, let $D \subseteq L^0(\Omega, M)$ be a neocompact set and suppose that $f_n : D \to \mathbb{R}$ is a sequence of neocontinuous functions with $f_n(x) \searrow 0$ monotonically as $n \to \infty$ for each fixed $x \in D$. Then $f_n \to 0$ uniformly on D.*

Proof Suppose f_n does not converge uniformly to 0 on D. Then there is an $\epsilon > 0$ such that for arbitrarily large $n \in \mathbb{N}$, the set

$$D_n = \{x \in D : |f_n(x)| \geq \epsilon\}$$

is nonempty. Since $f_n(x)$ converges monotonically for each x, the sets D_n form a decreasing chain. Each D_n is neocompact because D is neocompact and f_n is neocontinuous. By the countable compactness property, $\bigcap_n D_n$ is nonempty, contrary to hypothesis. □

We now use the neo-Dini's theorem to prove a stochastic analogue of Proposition 4.6.

Proposition 6.13 *For $k > 0$, $\tilde{\gamma}_n(y) \to \tilde{\gamma}(y)$ uniformly on N_k.*

Proof By Proposition 6.10, $\tilde{\gamma}(N_k) \subseteq D$ for some neocompact set $D \subseteq N$. The projection function Π_n is continuous on M and therefore neocontinuous on $L^0(\Omega, M)$. By the above neo-Dini's Theorem, $\Pi_n(z) \to z$ uniformly on D, and the result follows. □

Proof of Theorem 6.2 We argue as in the proof of Theorem 4.1. Let $x = u_0$. It suffices to show that there exists $y \in G^\perp \cap N_k$ where $k = K_1 + K_2|x|^2$ such that $y = x + \tilde{\gamma}(y)$. By Proposition 12, there is a neocompact set $D \subseteq N$ such that $\tilde{\gamma}(N_k) \subseteq D$. By 6.11 and 6.13, for each ϵ >0 there exists $n \in \mathbb{N}$ and $y_n \in G^\perp \cap N_k$ such that

$$y_n = x^{(n)} + \tilde{\gamma}_n(y_n),$$

and in the space N, y_n is within ϵ of $x + \tilde{\gamma}(y_n)$ which belongs to D. Thus

$$(\forall \epsilon > 0)(\exists z \in D^\epsilon \cap G^\perp \cap N_k)(\rho(z, x + \tilde{\gamma}(z)) \leq \epsilon).$$

By Proposition 6.10, $\tilde{\gamma}$ is neocontinuous on N_k. The Approximation Theorem now gives the required solution $y \in G^\perp \cap N_k$ such that $y = x + \tilde{\gamma}(y)$. □

We conclude with an existence theorem for stochastic Navier-Stokes equations with a random initial condition.

Theorem 6.14 *Let $\mathbf{\Omega}$ be a rich adapted space. Suppose that $u_0(\omega)$ is a $\mathcal{G}_0$-measurable random variable on $\mathbf{\Omega}$ with values in $\mathbf{H}$, and f, g satisfy the hypotheses of Theorem 6.2. Then the stochastic Navier-Stokes equation (11) has a solution $u(t, \omega)$ on $\mathbf{\Omega}$ with initial value $u(0, \omega) = u_0(\omega)$. Moreover, if $E\left[|u_0(\omega)|^2\right] < \infty$, then*

$$E\left(\sup_{t \leq T} |u(t)|^2 + \int_0^T \|u(t)\|^2 dt\right) < \infty. \tag{13}$$

Proof We shall use the fact that Theorem 6.2 gives us a solution with a deterministic initial value which is independent of $\mathcal{G}_0$.

Let $x(\omega) = u_0(\omega)$ be a $\mathcal{G}_0$-measurable initial value. Since $\mathbf{H}$ is separable there is an increasing sequence of compact sets $C_n \subseteq \mathbf{H}$ with $P[x(\omega) \in C_n] \geq 1 - 1/n$ for each $n \geq 1$. Let $k_n = K_1 + K_2 \sup_{z \in C_n} |z|^2$. Let D_n be a neocompact subset of N such that

$$D_n \supseteq \tilde{\gamma}(N_{k_n}).$$

We can take the sets D_n to be increasing. Let G_n be the neocompact set of all $\mathcal{G}_0$-measurable random variables in $L^0(\Omega, C_n)$. Then the set $B_n = G_n + D_n$ is a neocompact subset of N such that

$$B_n \supseteq G_n + \tilde{\gamma}(N_{k_n}).$$

By closure under diagonal intersections (Lemma 2.9), the set

$$B = \bigcap_{n \geq 1} ((B_n)^{1/n})$$

is neocompact.

For each $n \geq 1$ there is a simple function $x_n \in G_n$ such that for each $m \leq n$, whenever $x(\omega) \in C_m$ then $x_n(\omega) \in C_m$ and $x_n(\omega)$ is within $1/n$ of $x(\omega)$. Hence $x_n \in (G_m)^{1/m}$ for all $m \leq n$.

By piecing together finitely many solutions from Theorem 6.2 which are independent of $\mathcal{G}_0$, we see that for each n there is a $y_n \in N_{k_n} \cap B_n$ such that

$$y_n = x_n + \tilde{\gamma}(y_n) \tag{14}$$

So $y_n \in (B_m)^{1/m}$ for each $1 \leq m \leq n$, and it follows (since the sets B_n are increasing) that $y_n \in B$. Thus for each n,

$$(\exists x_n \in \{x\}^{1/n})(\exists y_n \in B)[y_n = x_n + \tilde{\gamma}(y_n)].$$

Since the addition function is neocontinuous on random variables, it follows that the function

$$(z(\cdot), y(\cdot)) \mapsto z(\cdot) + \tilde{\gamma}(y(\cdot))$$

is neocontinuous on $L^0(\Omega, \mathbf{H}) \times N_{k_n}$. So by the Approximation Theorem there is a stochastic process $y(\cdot) \in B$ with

$$y = x + \tilde{\gamma}(y)$$

as required.

Suppose finally that $E\left[|x(\omega)|^2\right] = m < \infty$. By Theorem 6.2, the y_n may be chosen so that $y_n \in N_k$, where $k = K_1 + K_2 m$. Since N_k is neoclosed, we may take the solution y to be in N_k, and therefore (13) holds. □

In Theorem 6.14, it would be more natural to allow the initial value $u_0(\omega)$ to be $\mathcal{F}_0$-measurable rather than $\mathcal{G}_0$-measurable, where $\mathcal{F}_0$ is the completion of $\bigcap_{t>0} \mathcal{G}_t$. The following corollary shows that this can be done if the rich adapted space is good enough. The construction of a rich adapted space in [7] actually produces a rich adapted space

$$\mathbf{\Omega} = (\Omega, P, \mathcal{G}, \mathcal{G}_t)$$

with the following additional property:

For every complete separable M and every $\mathcal{F}_0$-measurable random variable $x \in L^0(\Omega, M)$ there is a σ-algebra $\mathcal{G}'_0 \subseteq \mathcal{F}_0$ such that x is $\mathcal{G}'_0$-measurable and the adapted space $\mathbf{\Omega}$ with $\mathcal{G}'_0$ in place of $\mathcal{G}_0$ is still rich.

Corollary 6.15 *If $\mathbf{\Omega}$ is a rich adapted space with the above additional property, then Theorem 6.14 holds with $\mathcal{F}_0$ in place of $\mathcal{G}_0$.*

7 Optimal solutions

We show in this section how the neocompact machinery provides existence of optimal solutions to the stochastic Navier-Stokes equations - where the term 'optimal' is capable of a wide variety of interpretations. We shall restrict attention to the case of a fixed initial condition $u_0 = x \in \mathbf{H}$, as in Theorem 6.2. Recall that we proved above that there is a solution y to the equation (11) in the set $N_{k(x)}$ where $k(x) = K_1 + K_2|x|^2$ (in fact the solution constructed is in $G^\perp$ also). It is natural to define the following **solution set** S_x for the initial condition $x \in \mathbf{H}$:

$$S_x = \{y \in N_{k(x)} : y \text{ is a solution to the stochastic Navier-Stokes equation and } y(0) = x\}$$

$$= \{y \in N_{k(x)} : y = x + \tilde{\gamma}(y)\}$$

The key point now is the observation that

Theorem 7.1 *For each $x \in \mathbf{H}$ the set S_x is neocompact.*

Proof (i) Let $f(y) = y - x - \tilde{\gamma}(y)$, which is neocontinuous on the neoclosed set $N_{k(x)}$, by Proposition 6.10. Clearly we have

$$S_x = N_{k(x)} \cap f^{-1}(\{0\})$$

which is neoclosed. Moreover, $S_x \subseteq x + \tilde{\gamma}(N_{k(x)})$ which is contained in a neocompact set (by Proposition 6.10 again). Hence S_x is neocompact. □

The following is a general optimality result for solutions to the stochastic Navier-Stokes equations; it is a special case of a general optimization result found in [6].

Theorem 7.2 *Suppose that $g : N_0 \to \overline{\mathbb{R}}$ is a function that is neo-lsc on $N_{k(x)}$; then there is a solution $\hat{y} \in S_x$ with*

$$g(\hat{y}) = \inf\{g(y) : y \in S_x\}$$

i.e. $\hat{y}$ is an optimal solution for the quantity $g(y)$.

Proof Since g is neo-lsc and S_x is neocompact, the upper graph

$$A = \{(y, r) \in S_x \times \overline{\mathrm{IR}} : \; g(y) \leq r\}$$

is neocompact; so the set

$$B = \{r \in \overline{\mathrm{IR}} : (\exists y \in S_x)(y, r) \in A\}.$$

is a neocompact subset of $\overline{\mathrm{IR}}$, which means that it is in fact compact. Thus B has a minimum element s, say, and there is $\hat{y} \in S_x$ with $g(\hat{y}) \leq s$. On the other hand, for *any* $y \in S_x$, putting $r = g(y)$ we have $(y, r) \in A$ and so $r \in B$. Thus $s \leq r = g(y)$ and the optimality of $\hat{y}$ for g is established. □

There are several natural functions g for which optimal solutions might be sought - for example the function $E(\mathcal{E}(y))$ that occurs naturally in the proofs of the previous section, and the expected energy integral

$$\mathbf{E}(y) = \tfrac{1}{2}E(\int_0^T |y(t)|^2 dt$$

and the expected enstrophy integral

$$\mathbf{S}(y) = \tfrac{1}{2}E(\int_0^T \|y(t)\|^2 dt.$$

Each of these functions is neo-lsc.

On the face of it, the optimality theorem appears to be specific to the particular rich adapted space on which we are working. However, it has been shown that rich adapted spaces are universal, so that if y' is a solution on some other adapted space $\mathbf{\Omega}'$ then there is $y \in L^0(\Omega, M)$ with the same adapted distribution as y' and so for any neo-lsc function g of the form $g(y) = Eg_0(y(\cdot))$ where g_0 is lsc, we have the existence of globally optimal solutions. This applies to the natural examples $\mathcal{E}$, $\mathbf{E}$, and $\mathbf{S}$ above.

The optimality result above can be generalised to the case where the initial condition u_0 is random, or specialised to the deterministic setting (where $g = 0$).

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