Neometric Spaces

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1 Introduction

Recently, in the paper "Existence Theorems in Probability Theory" [9] we presented a new approach for applying methods from nonstandard Analysis to solve existence problems in Probability Theory. We did this in conventional mathematical terms by introducing a new framework: the neometric spaces.

In the introduction to that paper we discussed the main features of traditional existence proofs and indicated that the most difficult step was in going from a sequence of approximations to the existence of a "limit" of such a sequence. Our methods show how to avoid this difficulty in many cases (see for example the recent results in [7]). In rough and provocative terms, nonstandard analysis gives us the limit; it is a mathematical tool which captures in a systematic way the transition from "discrete to continuous" in mathematics.

This is a very basic feature known to nonstandard practitioners. Many theorems have been proved in recent years using this natural idea. Good examples are provided, among others, in the collection [6], the books [1] and [28] and the memoir [15].

In [16] the second author developed a general method of proving existence theorems of the type found in the nonstandard probability literature from his forcing theorem ([16], Theorem 5.3). After a long series of refinements, we managed to present this method in conventional mathematical terms in [9]. In that paper, we merely asked our readers to accept the existence of the so called "rich adapted spaces", and never used the words "Nonstandard Analysis" after the introduction.

Our objective in this paper is quite the opposite: we work within nonstandard analysis to formally prove, as promised in [9], that rich adapted spaces exist. Moreover, we explicitly show how nonstandard analysis provides the inspiration for the main notions and results presented in [9]. The paper [9] gives a large number of applications of rich adapted spaces to probability theory. Nonstandard analysis gives us tools to dig deeper into the structure of subsets of metric spaces. The purpose of this paper is to refine these tools.

The contents of this paper are as follows. In Section 2 we review the basic notions concerning neometric spaces from [9]. In Sections 3 and 4 we study these notions within the nonstandard setting. We give an explicit definition of basic and neocompact sets that captures the way internal sets are used in nonstandard probability practice, and then present a huge neometric family which contains all neometric spaces studied so far. In Section 5 we prove that the huge neometric family contains rich adapted spaces, and hence that rich adapted spaces exist. Section 6 gives a detailed study of the function spaces related to the existence theorems in [9]. In Section 7 we consider neometric spaces whose elements are functions from a probability space into another neometric space. We finish up in Section 8 with a stronger theory of κ-neometric spaces which requires a κ-saturated nonstandard universe where κ is a cardinal greater than ω_1.

In other publications we will present other aspects of our work. In [10] we develop the logical aspects that are behind our results. In [11] we discuss another nonstandard approach to neometric families which uses long sequences. The paper [17] gives a general quantifier elimination result showing that for many neometric families, every neocompact set is a section of a basic set. The paper [18] applies these results to rich probability spaces and rich adapted spaces. The article [20] is an overall survey of the program which is carried out in this series of papers.

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2 Neocompact Sets

A family of neocompact sets is a generalization of the family of compact sets, and retains many of its properties. In this section we review the notions from [9] concerning neocompact families in general. After that we shall concentrate on neocompact families associated with nonstandard universes.

We use script letters $\mathcal{M}, \mathcal{N}, \mathcal{O}$ for complete metric spaces which are not necessarily separable, and let ρ, σ, τ be their metrics. Given two metric spaces $(\mathcal{M}, \rho)$ and $(\mathcal{N}, \sigma)$, the **product metric** is the metric space $(\mathcal{M} \times \mathcal{N}, \rho \times \sigma)$ where

$$(\rho \times \sigma)((x_1 x_2), (y_1, y_2)) = \max(\rho(x_1, y_1), \sigma(x_2, y_2)).$$

Definition 2.1 *Let* $\mathbf{M}$ *be a collection of complete metric spaces* $\mathcal{M}$ *which is closed under finite products, and for each* $\mathcal{M} \in \mathbf{M}$ *let* $\mathcal{B}(\mathcal{M})$ *be a collection of subsets of* $\mathcal{M}$*, which we call* **basic sets**. *By a* **neocompact family** *over* $(\mathbf{M}, \mathcal{B})$ *we mean a triple* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *where for each* $\mathcal{M} \in \mathbf{M}, \mathcal{C}(\mathcal{M})$ *is a collection of subsets of* $\mathcal{M}$ *with the following properties, where* $\mathcal{M}, \mathcal{N}, \mathcal{O}$ *vary over* $\mathbf{M}$*:*

(a) $\mathcal{B}(\mathcal{M}) \subset \mathcal{C}(\mathcal{M})$*;*

(b) $\mathcal{C}(\mathcal{M})$ *is closed under finite unions; that is, if* $A, B \in \mathcal{C}(\mathcal{M})$ *then* $A \cup B \in \mathcal{C}(\mathcal{M})$.

(c) $\mathcal{C}(\mathcal{M})$ *is closed under finite and countable intersections;*

(d) *If* $C \in \mathcal{C}(\mathcal{M})$ *and* $D \in \mathcal{C}(\mathcal{N})$ *then* $C \times D \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$*;*

(e) *If $C \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$, then the set*

$$\{x : (\exists y \in \mathcal{N})(x, y) \in C\}$$

belongs to $\mathcal{C}(\mathcal{M})$, and the analogous rule holds for each factor in a finite Cartesian product;

(f) *If $C \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$, and D is a nonempty set in $\mathcal{B}(\mathcal{N})$, then*

$$\{x : (\forall y \in D)(x, y) \in C\}$$

belongs to $\mathcal{C}(\mathcal{M})$, and the analogous rule holds for each factor in a finite Cartesian product.

The sets in $\mathcal{C}(\mathcal{M})$ are called **neocompact sets**. The neocompact family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ induces a family of metric spaces with extra structure, $\mathcal{M} = (\mathcal{M}, \mathcal{B}(\mathcal{M}), \mathcal{C}(\mathcal{M}))$, which we call **neometric spaces**. A neometric space thus consists of a complete metric space $\mathcal{M} \in \mathbf{M}$ and two families $\mathcal{B}(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ of subsets of $\mathcal{M}$. The properties (a)–(f) not only give conditions on single neometric spaces, but also on finite Cartesian products of neometric spaces.

We call $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ the neocompact family **generated by** $(\mathbf{M}, \mathcal{B})$ if $\mathcal{C}(\mathcal{M})$ is the collection of all sets obtained by finitely many applications of rules (a)–(f).

Example 2.2 *The classical example of a neocompact family is the family $(\mathbf{S}, \mathcal{B}, \mathcal{C})$ generated by $(\mathbf{S}, \mathcal{B})$ where $\mathbf{S}$ is the collection of all complete metric spaces, and for each $\mathcal{M} \in \mathbf{S}, \mathcal{B}(\mathcal{M})$ is equal to the set of all compact subsets of $\mathcal{M}$.*

It is not hard to see that the family of compact sets is closed under all of the rules (a)–(f). Thus the collection of neocompact sets $\mathcal{C}(\mathcal{M})$ generated by $(\mathbf{S}, \mathcal{B})$ is just the family $\mathcal{B}(\mathcal{M})$ of compact sets itself, i.e. every neocompact set is compact.

In fact, the family of compact sets is closed under arbitrary intersections, and condition (f) holds for arbitrary nonempty sets D. One reason that compact sets are useful in proving existence theorems is that they have the following property:

If $\mathbf{C}$ is a set of compact sets such that any finite subset of $\mathbf{C}$ has a nonempty intersection, then $\mathbf{C}$ has a nonempty intersection.

In many cases, all that is needed is the following weaker property.

Definition 2.3 *We say that a neocompact family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ has the* **countable compactness property***, or is* **countably compact***, if for each $\mathcal{M} \in \mathbf{M}$, every decreasing chain $C_0 \supset C_1 \supset \cdots$ of nonempty sets in $\mathcal{C}(\mathcal{M})$ has a nonempty intersection $\bigcap_n C_n$ (which, of course, also belongs to $\mathcal{C}(\mathcal{M})$).*

In particular, the classical neocompact family $(\mathbf{S}, \mathcal{B}, \mathcal{C})$ from Example 2.2 is countably compact. Two other countably compact neometric families studied in [9] are the neocompact families associated with a rich probability space and a rich adapted space. The existence of such spaces will be proved in Section 5. In the next section we shall construct a much larger neocompact family, the huge family $(\mathbf{H}, \mathcal{B}, \mathcal{C})$, which is also countably compact.

Here are the neometric analogues of closed sets, open sets, and continuous functions.

Definition 2.4 *Let $\mathcal{M}, \mathcal{N}$ belong to a neocompact family $\mathbf{M}$.*

A set $C \subset \mathcal{M}$ is **neoclosed** *in $\mathcal{M}$ if $C \cap D$ is neocompact in $\mathcal{M}$ for every neocompact set D in $\mathcal{M}$.*

C is **neoopen** *in $\mathcal{M}$ if $\mathcal{M} - C$ is neoclosed.*

Let $D \subset \mathcal{M}$. A function $f : D \to \mathcal{N}$ is **neocontinuous** *from $\mathcal{M}$ to $\mathcal{N}$ if for every neocompact set $A \subset D$ in $\mathcal{M}$, the restriction $f|A = \{(x, f(x)) : x \in A\}$ of f to A is neocompact in $\mathcal{M} \times \mathcal{N}$.*

The next result involves a pair of neocompact families.

Proposition 2.5 *Let $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ and $(\mathbf{M}', \mathcal{B}', \mathcal{C}')$ be two neocompact families such that $\mathbf{M}' \subset \mathbf{M}$, every neocompact set in $(\mathbf{M}', \mathcal{B}', \mathcal{C}')$ is neocompact in $(\mathbf{M}, \mathcal{B}, \mathcal{C})$, and for each $\mathcal{M} \in \mathbf{M}'$, every set $C \in \mathcal{C}(\mathcal{M})$ is contained in a set $D \in \mathcal{C}'(\mathcal{M})$. Then every neoclosed set in $(\mathbf{M}', \mathcal{B}', \mathcal{C}')$ is neoclosed in $(\mathbf{M}, \mathcal{B}, \mathcal{C})$, and every neocontinuous function with a neoclosed domain in $(\mathbf{M}', \mathcal{B}', \mathcal{C}')$ is neocontinuous in $(\mathbf{M}, \mathcal{B}, \mathcal{C})$.*

Proof: Suppose $\mathcal{M} \in \mathbf{M}'$ and $A \subset \mathcal{M}$ is neoclosed in $(\mathbf{M}', \mathcal{B}', \mathcal{C}')$. Let $C \in \mathcal{C}(\mathcal{M})$. Then C is contained in a set $D \in \mathcal{C}'(\mathcal{M})$. Therefore $A \cap D \in \mathcal{C}'(\mathcal{M})$, and hence $A \cap D \in \mathcal{C}(\mathcal{M})$. It follows that $A \cap C = (A \cap D) \cap C \in \mathcal{C}(\mathcal{M})$. This shows that A is neoclosed in $(\mathbf{M}, \mathcal{B}, \mathcal{C})$.

Now suppose $\mathcal{M}, \mathcal{N} \in \mathbf{M}'$, $A \subset \mathcal{M}$ is neoclosed in $(\mathbf{M}', \mathcal{B}', \mathcal{C}')$, and $f : A \to \mathcal{N}$ is neocontinuous in $(\mathbf{M}', \mathcal{B}', \mathcal{C}')$. Let $C \in \mathcal{C}(\mathcal{M})$ and $C \subset A$. There is a set $D \in \mathcal{C}'(\mathcal{M})$ such that $C \subset D \subset A$. Then the graph F of $f|D$ is neocompact in $(\mathbf{M}', \mathcal{B}', \mathcal{C}')$ and hence neocompact in $(\mathbf{M}, \mathcal{B}, \mathcal{C})$. The graph G of $f|C$ is given by $G = F \cap (C \times \mathcal{N})$, so G is neocompact in $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ and f is neocontinuous in $(\mathbf{M}, \mathcal{B}, \mathcal{C})$. $\square$

We now introduce the notion of a neometric family, which is slightly stronger than the notion of a neocompact family.

Definition 2.6 *We call a neocompact family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ a* **neometric family** *if the projection and distance functions in $\mathbf{M}$ are neocontinuous. That is, the projection functions $\pi_1 : \mathcal{M} \times \mathcal{N} \to \mathcal{M}$ and $\pi_2 : \mathcal{M} \times \mathcal{N} \to \mathcal{N}$ are neocontinuous, the metric* *space* $\mathbf{R}$ *of reals is contained in some member* $\mathcal{R}$ *of* $\mathbf{M}$*, and for each* $\mathcal{M} \in \mathbf{M}$ *the distance function* ρ *of* $\mathcal{M}$ *is neocontinuous from* $\mathcal{M} \times \mathcal{M}$ *into* $\mathcal{R}$*.*

In the classical neocompact family $(\mathbf{S}, \mathcal{B}, \mathcal{C})$ introduced in Example 2.2, a set is neoclosed if and only if it is closed, and a function is neocontinuous if and only if it is continuous. $(\mathbf{S}, \mathcal{B}, \mathcal{C})$ clearly has the countable compactness property. Since the distance and projection functions are continuous for all metric spaces, $(\mathbf{S}, \mathcal{B}, \mathcal{C})$ is a neometric family.

It was shown in [9] that the neocompact families associated with a rich probability space and a rich adapted space are also neometric families.

In [9] we considered existence problems of the form

$$(\exists x \in C) f(x) \in D \tag{1}$$

where C is a neocompact set in $\mathcal{M}$, $f : E \to \mathcal{K}$ is neocontinuous from a neoclosed set $E \supset C$ to $\mathcal{N}$, and D is a neoclosed set in $\mathcal{N}$. We proved a simple but useful approximation theorem which states that every problem of the form (1) which is "approximately true" is true. The proof of the approximation theorem used the following closure property.

Definition 2.7 *A neometric family* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *is said to be* **closed under diagonal intersections** *if the following holds. Let* $\mathcal{M} \in \mathbf{M}$*, let* $A_n \in \mathcal{C}(\mathcal{M})$ *for each* $n \in \mathbf{N}$*, and let* $\lim_{n\to\infty} \varepsilon_n = 0$*. Then*

$$A = \bigcap_n ((A_n)^{\varepsilon_n}) \in \mathcal{C}(\mathcal{M}).$$

In [9] we proved that the neocompact family associated with a rich adapted space is closed under diagonal intersections, and that the following approximation theorem holds for rich adapted spaces. The proof there shows that the theorem is true for every countably compact neometric family closed under diagonal intersections.

Theorem 2.8 *(Approximation Theorem) Let* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *be a countably compact neometric family closed under diagonal intersections. Let* A *be neoclosed in* $\mathcal{M}$ *and let* $f : A \to \mathcal{N}$ *be neocontinuous from* $\mathcal{M}$ *to* $\mathcal{N}$*. Let* B *be neocompact in* $\mathcal{M}$ *and* D *be neoclosed in* $\mathcal{N}$*. Suppose that for each* $\varepsilon > 0$*,*

$$(\exists x \in A \cap B^{\varepsilon}) f(x) \in D^{\varepsilon}. \tag{2}$$

Then (1) holds with $C = A \cap B$*, that is,*

$$(\exists x \in A \cap B) f(x) \in D. \ \square$$

Before leaving this section, we introduce the concept of a neoseparable set, which is analogous to the classical notion of a closed separable set. Neoseparable sets will be of particular interest in the nonstandard setting to be presented in this paper.

Definition 2.9 *A set $A \subset \mathcal{M}$ is said to be* **neoseparable in** $\mathcal{M}$ *if A is the closure of the union of countably many basic subsets of $\mathcal{M}$. In particular, $\mathcal{M}$ itself is neoseparable if some countable union of basic sets is dense in $\mathcal{M}$.*

Note that the closure of a countable union of neoseparable sets in $\mathcal{M}$ is neoseparable in $\mathcal{M}$. Also, since every finite set is basic, every closed separable subset of $\mathcal{M}$ is neoseparable in $\mathcal{M}$.

We shall prove one results about neoseparable sets in an arbitrary countably compact neometric family.

Proposition 2.10 *Let $\mathbf{M}$ be a countably compact neometric family. Suppose C is neocompact in $\mathcal{M}$, D is neoseparable and is the closure of an increasing chain $D_1, D_2, \dots$ of basic sets in $\mathcal{M}$, and $C \subseteq D$. Then for each $n \in \mathbf{N}$ there exists $m \in \mathbf{N}$ such that $C \subseteq (D_m)^{1/n}$.*

Proof: Suppose the conclusion fails. That is, there exists $n \in \mathbf{N}$ such that for each $m \in \mathbf{N}$, the set $C - (D_m)^{1/n})$ is nonempty. It is proved in [FK1] that for each basic set B and each real $r > 0$, the set $\{x : \rho(x, B) \geq r\}$ is neoclosed. Therefore for each m the set

$$E_m = \{y \in C : \rho(y, D_m) \geq 1/(2n)\}$$

is neocompact. Since the sets D_m form an increasing chain, the sets E_m form a decreasing chain. Moreover, each set E_m is nonempty because it contains the nonempty set $C - (D_m)^{1/n})$. By countable compactness there is a point $x \in \bigcap_m E_m$. But then $x \in C - D$, contrary to hypothesis. $\square$

3 The Huge Neometric Family

In this section we show that nonstandard universes are a rich source of neometric families. We will construct a huge neometric family $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ associated with each nonstandard universe. By restricting this family, one obtains a variety of smaller neometric families (such as the ones over rich adapted spaces studied in [9]) which arise naturally in a more classical setting. The notion of a neocompact set we shall introduce in this section contains as a special case the notion of a neocompact set in the paper [15].

We assume throughout that $(V(\Xi), V(^*\Xi), ^*)$ is an ω_1-saturated nonstandard universe, where the base set Ξ is some large set which contains the universe of every structure under consideration. We assume familiarity with the basic notions from nonstandard analysis, including transfer principle, the overspill principle, and the notion of an internal set. ω_1-saturation is the principle that for any internal set S, any countable family of internal subsets of S which has the finite intersection property has a nonempty intersection. For each finite hyperreal number $x \in {}^*\mathbf{R}$, the **standard part** $st(x)$ is defined as the unique real number which is infinitely close to x.

In the remainder of this paper we shall frequently use the notions of a ***metric space**, and a ***probability measure**, which are obtained from the corresponding standard notions by transfer. Thus a *metric space is a structure $(\bar{M}, \bar{\rho})$ where $\bar{M}$ is an internal set and $\bar{\rho}$ is an internal function $\bar{\rho} : \bar{M} \times \bar{M} \to {}^*\mathbf{R}$ which satisfies the transfer of the usual rules for a metric. The product of two *metric spaces is defined in the natural way.

If $X, Y \in \bar{M}$, we write $X \approx Y$ if $\bar{\rho}(X, Y) \approx 0$. The **standard part** of an element $X \in \bar{M}$ is the equivalence class

$$^oX = \{Y \in \bar{M} : X \approx Y\}.$$

If $x = {}^oX$, we say that X **lifts** x.

The nonstandard hull construction is a well-known method from [26] which produces a complete metric space from a *metric space and a distinguished point.

Definition 3.1 *Consider a *metric space $(\bar{M}, \bar{\rho})$ and a point $c \in \bar{M}$. The* **galaxy** *of c is the set $G(\bar{M}, c)$ of all points $X \in \bar{M}$ such that $\bar{\rho}(X, c)$ is finite. By the* **nonstandard hull of** $\bar{M}$ *at c we mean the metric space $(\mathcal{H}(\bar{M}, c), \rho)$ where*

$$\mathcal{H}(\bar{M}, c) = \{^oX : X \in G(\bar{M}, c)\}, \rho(^oX, ^oY) = st(\bar{\rho}(X, Y)).$$

Note that any two points $b, c \in \bar{M}$ such that $\bar{\rho}(b, c)$ is finite have the same galaxies and nonstandard hulls, $G(\bar{M}, b) = G(\bar{M}, c)$ and $\mathcal{H}(\bar{M}, b) = \mathcal{H}(\bar{M}, c)$.

The neometric spaces in our huge family **H** will be the closed subspaces of nonstandard hulls. We first need some more definitions.

Given a set $B \subset G(\bar{M}, c)$, the **standard part** of B is the set

$$^oB = \{^oX : X \in B\}$$

of standard parts of elements of B. In the opposite direction, for a set $A \subset \mathcal{H}(\bar{M}, c)$, the **monad** of A is the set

$$\text{monad}(A) = \{X : {}^oX \in A\}.$$

By a Σ_1^0 (Π_1^0) set we mean the union (intersection) of countably many internal subsets of the galaxy $G(\bar{M}, c)$.

Observe that every countable subset of $G(\bar{M}, c)$ is Σ_1^0, and hence every countable subset of $\mathcal{H}(\bar{M}, c)$ is the standard part of a Σ_1^0 set.

For a set $B \subset G(\bar{M}, c)$ and a hyperreal $\varepsilon > 0$, we write

$$\bar{\rho}(X, B) = \inf\{\bar{\rho}(X, Y) : Y \in B\}, B^\varepsilon = \{X : \bar{\rho}(X, B) \leq \varepsilon\}.$$

Proposition 3.2 *For each *metric space $\bar{M}$ and distinguished point $c \in \bar{M}$, the galaxy $G(\bar{M}, c)$ is a Σ_0^1 set, and the monad of the nonstandard hull $\mathcal{H}(\bar{M}, c)$ is the galaxy $G(\bar{M}, c)$.*

Proof: The galaxy $G(\bar{M}, c)$ is the union of the countable chain of internal sets

$$B_m = \{X \in \bar{M} : \bar{\rho}(X, c) \leq m\}.$$

The nonstandard hull $\mathcal{H}(\bar{M}, c)$ is the standard part of the galaxy, and each point of $\bar{M}$ which is infinitely close to a point in $G(\bar{M}, c)$ already belongs to $G(\bar{M}, c)$. □

Definition 3.3 *The* **huge neometric family** $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ *for the nonstandard universe* $(V(\Xi), V(^*\Xi), ^*)$ *is defined as follows.* $\mathbf{H}$ *is the class of all metric spaces* $(\mathcal{M}, \rho)$ *such that $\mathcal{M}$ is a closed subset of some nonstandard hull $\mathcal{H}(\bar{M}, c)$. For each $\mathcal{M} \in \mathbf{H}$, the collections of basic and neocompact subsets of $\mathcal{M}$ are*

$$\mathcal{B}(\mathcal{M}) = \{A \subset \mathcal{M} : A = {}^oB \text{ for some internal set } B \subset G(\bar{M}, c)\},$$

$$\mathcal{C}(\mathcal{M}) = \{A \subset \mathcal{M} : A = {}^oB \text{ for some } \Pi_1^0 \text{ set } B \subset G(\bar{M}, c)\}.$$

Example 3.4 *(The neometric family of nonstandard hulls)*

The nonstandard hull $\mathcal{H}(\bar{M}, c)$ belongs to $\mathbf{H}$ for every *metric space $\bar{M}$ and point $c \in \bar{M}$. For each finite hyperreal $r \geq 0$, the set $\{{}^oX : \bar{\rho}(X, c) \leq r\}$ is basic in $\mathcal{H}(\bar{M}, c)$. It follows that for each real r, the closed ball $\{x \in \mathcal{H}(\bar{M}, c) : \rho(x, c) \leq r\}$ is neocompact in $\mathcal{H}(\bar{M}, c)$. Moreover, the whole space $\mathcal{H}(\bar{M}, c)$ is neoseparable, and has the even stronger property of being the union of a countable chain of basic sets. The family of all nonstandard hulls is closed under finite Cartesian products, and is thus a neometric subfamily of the huge neometric family $\mathbf{H}$.

Example 3.5 *In this example we show that the standard neometric family* $(\mathbf{S}, \mathcal{B}, \mathcal{C})$ *from Example 2.2 may be regarded as a subfamily of* $(\mathbf{H}, \mathcal{B}, \mathcal{C})$.

Recall that $\mathbf{S}$ is the family of all standard complete metric spaces. Given $(M, \rho) \in \mathbf{S}$, we may consider the *metric space $({}^*M, {}^*\rho)$. We abuse notation by identifying M with the set $\{{}^*x : x \in M\}$. Thus M is a closed subset of the nonstandard hull $\mathcal{H}({}^*M, x)$ where x is any element of M, and hence M itself belongs to the huge family $\mathbf{H}$.

We shall return to this example in the next section.

Note that the standard part of the union of two sets is the union of the standard parts, and therefore $\mathcal{B}(\mathcal{M})$ is closed under finite unions. Moreover, finite Cartesian products of basic sets are basic, and every finite subset of $\mathcal{M}$ is basic. On the other hand, the following example shows that the intersection of two basic sets need not be basic.

Example 3.6 *(A set which is neocompact but not basic).*

Let $\bar{N}$ be the set of positive hyperintegers with the discrete metric where any pair of distinct points has distance one, and let $\bar{M}$ be the internal unit interval ${}^*[0, 1]$ with the usual *metric. Then $\bar{M} \times \bar{N}$ has only one galaxy, and the standard part of a point (X, n) is $({}^oX, n)$. Let

$$A = \{(0, n) : n \in \bar{N}\}, B = \{(1/n, n) : n \in \bar{N}\}.$$

Then A and B are internal and have basic standard parts ${}^oA$ and ${}^oB$. However,

$${}^oA \cap {}^oB = \{(0, n) : n \text{ is infinite}\},$$

and by overspill this cannot be the standard part of an internal subset of $\bar{M} \times \bar{N}$. Thus ${}^oA \cap {}^oB$ is neocompact but not basic.

The following lemma is a useful tool in dealing with standard parts of Π_1^0 sets, and will give us a characterization of the monad of a neocompact set.

Lemma 3.7 *Let $A_n, n \in \mathbf{N}$, be a decreasing chain of internal subsets of $\bar{M}$. Then*

$$\bigcap_n ({}^oA_n) = {}^o(\bigcap_n A_n) = {}^o(\bigcap_n (A_n{}^{1/n})).$$

Proof: We first prove that

$$\bigcap_n ({}^oA_n) \subset {}^o(\bigcap_n A_n).$$

Let $x \in \bigcap_n ({}^oA_n)$. Let X lift x. Then for each $n \in \mathbf{N}$ there exists $Y_n \in A_n$ such that ${}^oY_n = x$, and hence $\bar{\rho}(X, Y_n) \approx 0$ and $\bar{\rho}(X, Y_n) \leq 1/n$. By ω_1-saturation, there exists $Y \in \bigcap_n A_n$ such that $\bar{\rho}(X, Y) \leq 1/n$ for all $n \in \mathbf{N}$, so ${}^oY = x$ and $x \in {}^o(\bigcap_n A_n)$.

The inclusion

$$^o(\bigcap_n A_n) \subset {}^o(\bigcap_n ((A_n)^{1/n}))$$

is trivial, and the inclusion

$$^o(\bigcap_n ((A_n)^{1/n})) \subset \bigcap_n ({}^o A_n)$$

follows easily from ω_1-saturation. $\square$

Corollary 3.8 *A set $C \subset \mathcal{M}$ is neocompact if and only if monad(C) is a Π^0_1 set. In fact, if $C = {}^o(\bigcap_n C_n)$ where $\langle C_n \rangle$ is a decreasing chain of internal sets, then*

$$monad(C) = \bigcap_n ((C_n)^{1/n}). \ \square$$

We call the representation of monad(C) in the above proposition a **neocompact normal form** for the monad of C.

The next proposition shows that the definition of the family $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ is unambiguous, i.e., that $\mathcal{B}(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ depend only on the metric space $\mathcal{M}$ and not on the galaxy $G(\bar{M}, c)$ in the *metric space $\bar{M}$. The proof uses Lemma 3.7 and the fact that for each $\mathcal{M} \in \mathbf{H}$, the monad of $\mathcal{M}$ can be recovered from $\mathcal{M}$.

Proposition 3.9 *Let $\mathcal{M} \in \mathbf{H}$ and suppose that $G(\bar{M}, c)$ and $G(\bar{N}, d)$ are galaxies in *metric spaces such that $\mathcal{M}$ is a closed subset of both nonstandard hulls $\mathcal{H}(\bar{M}, c)$ and $\mathcal{H}(\bar{N}, d)$. Then the collections of basic sets $\mathcal{B}(\mathcal{M})$ and neocompact sets $\mathcal{C}(\mathcal{M})$ are the same for $\mathcal{H}(\bar{M}, c)$ as for $\mathcal{H}(\bar{N}, d)$.*

Proof: Each point $x \in \mathcal{M}$ is a standard part of point $X \in \bar{M}$ and a point $Y \in \bar{N}$, that is,

$$x = {}^oX = \{Z \in \bar{M} : Z \approx X\} = {}^oY = \{Z \in \bar{N} : Z \approx Y\}.$$

Therefore the monad of $\mathcal{M}$ is equal to the union

$$\text{monad}(\mathcal{M}) = \bigcup \{x : x \in \mathcal{M}\}.$$

Thus monad$(\mathcal{M}) \subset \bar{M} \cap \bar{N}$, and $\bar{M} \cap \bar{N}$ is nonempty. The intersection of $\bar{M}$ and $\bar{N}$ is a *metric space with respect to both *metrics $\bar{\rho}$ and $\bar{\sigma}$. Moreover, the metrics $\bar{\rho}$ and $\bar{\sigma}$ must be infinitely close to each other on monad$(\mathcal{M})$. If A is basic in $\mathcal{M}$ with respect to $(\bar{M}, c)$, then $A = {}^oB$ with respect to $\bar{\rho}$ for some internal set $B \subset$ monad$(\mathcal{M})$, so $A = {}^oB$ with respect to $\bar{\sigma}$ and A is basic in $\mathcal{M}$ with respect to $(\bar{N}, d)$.

Now suppose that A is neocompact in $\mathcal{M}$ with respect to $(\bar{M}, c)$. Then $A = {}^o(\bigcap_n B_n)$ with respect to $\bar{\rho}$ where each B_n is an internal subset of $\bar{M}$ and $\bigcap_n B_n \subset G(\bar{M}, c)$. By Lemma 3.7,

$$A = {}^o(\bigcap_n (B_n)^{1/n}) = {}^o(\bigcap_n [(B_n)^{1/n} \cap \bar{N}])$$

with respect to $\bar{\rho}$. Then

$$A = {}^o(\bigcap_n [(B_n)^{1/n} \cap \bar{N}])$$

with respect to $\bar{\sigma}$, so A is neocompact in $\mathcal{M}$ with respect to $(\bar{N}, \bar{\sigma})$. $\square$

Corollary 3.10 *(i) If $\mathcal{M}, \mathcal{N} \in \mathbf{H}$ and $C \subset \mathcal{M} \cap \mathcal{N}$, then C is basic, neocompact, or neoseparable in $\mathcal{M}$ if and only if it is basic, neocompact, or neoseparable in $\mathcal{N}$.*

(ii) Let $\mathcal{M}, \mathcal{N} \in \mathbf{H}$ and $\mathcal{M} \subset \mathcal{N}$. If C is neoclosed in $\mathcal{N}$, then $C \cap \mathcal{M}$ is neoclosed in $\mathcal{M}$. If $f : D \to \mathcal{K}$ is neocontinuous from $\mathcal{N}$ to $\mathcal{K}$, then the restriction $f|(D \cap \mathcal{M})$ is neocontinuous from $\mathcal{M}$ to $\mathcal{K}$. $\square$

In view of this corollary, we may call a set C basic, neocompact, or neoseparable if it is basic, neocompact, or neoseparable in any $\mathcal{M} \in \mathbf{H}$ such that $\mathcal{M} \supset C$.

Theorem 3.11 $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ *is a neometric family with the countable compactness property.*

Proof: It is clear that $\mathbf{H}$ is closed under finite Cartesian products, and that every basic set is neocompact, so (a) holds. Properties (b) and (d) are trivial, and property (c) follows easily from ω_1-saturation.

To prove the existential quantifier property (e), let C be neocompact in $\mathcal{M} \times \mathcal{N}$. We show that the set

$$D = \{x \in \mathcal{M} : (\exists y \in \mathcal{N})(x, y) \in C\}$$

is neocompact in $\mathcal{M}$. By Corollary 3.8, $\mathrm{monad}(C) = \bigcap_n C_n$ where $\langle C_n \rangle$ is a decreasing chain of internal subsets of $\bar{M} \times \bar{N}$. Let $X \in \bar{M}$ and $x = {}^oX$. Using Lemma 3.7 we see that the following are equivalent:

$$x \in D$$

$$(\exists y \in \mathcal{N})(x, y) \in C$$

$$(\exists Y \in \bar{N})(x, {}^oY) \in C$$

$$(\exists Y \in \bar{N})(X, Y) \in \bigcap_n C_n$$

$$(\forall n)(\exists Y \in \bar{N})(X, Y) \in C_n$$

$$X \in \bigcap_n D_n, \text{ where } D_n = \{Z : (\exists Y \in \bar{N})(Z, Y) \in C_n\}.$$

Since $\bar{N}$ is internal, each set D_n is internal. Therefore $D = {}^o(\bigcap_n D_n)$ is neocompact in $\mathcal{M}$. We now prove property (f), the universal quantifier property. Let C be neocompact in $\mathcal{M} \times \mathcal{N}$ and B be nonempty and basic in $\mathcal{N}$. We must show that the set

$$D = \{x \in \mathcal{M} : (\forall y \in B)(x, y) \in C\}$$

is neocompact in $\mathcal{M}$. Since B is basic, $B = {}^oA$ where A is an internal subset of $\bar{N}$. Again, monad$(C) = \bigcap_n C_n$ where $\langle C_n \rangle$ is a decreasing chain of internal subsets of $\bar{M} \times \bar{N}$. Let $X \in \bar{M}$ and $x = {}^oX$. As in the preceding paragraph, we see that $x \in D$ if and only if $X \in \bigcap_n D_n$, where

$$D_n = \{Z : (\forall Y \in A)(Z, Y) \in C_n\}.$$

Since A is internal, each set D_n is internal. (This is where we need the fact that B is basic). Therefore $D = {}^o(\bigcap_n D_n)$ is neocompact in $\mathcal{M}$. This shows that $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ is a neocompact family.

For each $\mathcal{M} \in \mathbf{H}$, the distance function ρ on $\mathcal{M}$ is neocontinuous because for each neocompact set C in $\mathcal{M} \times \mathcal{M}$ with monad $\bigcap_n C_n$, the graph of $\rho|C$ is equal to the neocompact set

$$\bigcap_n {}^o\{(X, Y, Z) \in \bar{M} \times \bar{M} \times {}^*\mathbf{R} : (X, Y) \in C_n \text{ and } |\bar{\rho}(X, Y) - Z| \leq 1/n\}.$$

The countable compactness property for $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ is an immediate consequence of ω_1-saturation. □

It follows that all the properties of countably compact neometric families stated in the preceding section hold for the huge neometric family. For example, all neoclosed sets are closed.

We conclude this section by briefly considering two other neometric families based on $(\mathbf{H}, \mathcal{B})$. Let $(\mathbf{H}, \mathcal{B}, \mathcal{C}_0)$ be the neocompact family generated by $(\mathbf{H}, \mathcal{B})$, and let $(\mathbf{H}, \mathcal{B}, \mathcal{C}_1)$ be the neocompact family where $\mathcal{C}_1(\mathcal{M})$ is the set of all $C \in \mathcal{C}(\mathcal{M})$ such that C is contained in some basic $B \in \mathcal{B}(\mathcal{M})$. The proof of the preceding theorem shows that $(\mathbf{H}, \mathcal{B}, \mathcal{C}_0)$ and $(\mathbf{H}, \mathcal{B}, \mathcal{C}_1)$ are also countably compact neometric families. Moreover, for each $\mathcal{M} \in \mathbf{H}$ we have

$$\mathcal{C}_0(\mathcal{M}) \subset \mathcal{C}_1(\mathcal{M}) \subset \mathcal{C}(\mathcal{M}).$$

The second inclusion is trivial, and the first inclusion holds because each of the rules (a)–(f) for neocompact sets preserves the property of a set being contained in a basic set. The following question is open: Does $\mathcal{C}_0(\mathcal{M}) = \mathcal{C}_1(\mathcal{M})$ for all $\mathcal{M} \in \mathbf{H}$? That is, can every neocompact subset of a basic set be obtained from basic sets by repeated applications of the rules (a)–(f)? Later on we shall give an example where $\mathcal{C}_1(\mathcal{M}) \neq \mathcal{C}(\mathcal{M})$.

4 Neocompact, Neoseparable, and Neoclosed Sets

In this section we study neocompact sets, neoclosed sets, neoseparable sets, and neocontinuous functions in the huge neometric family $(\mathbf{H}, \mathcal{B}, \mathcal{C})$. We assume throughout this section that $\mathcal{M}$ and $\mathcal{N}$ are closed subsets of the nonstandard hulls $\mathcal{H}(\bar{M}, c)$ and $\mathcal{H}(\bar{N}, d)$, and thus belong to $\mathbf{H}$. In the paper [11] we shall give alternative nonstandard characterizations of the neocompact, neoclosed, and neoseparable sets in $\mathbf{H}$, using sequences indexed by the hyperintegers.

We begin with a simple sufficient condition for being neoclosed.

Proposition 4.1 *For any Π_1^0 set $B \subset \bar{M}$, the set $D = \mathcal{M} \cap {}^oB$ is neoclosed in $\mathcal{M}$.*

Proof: Let $B = \bigcap_n B_n$ and let C be a neocompact set in $\mathcal{M}$ with $\text{monad}(C) = \bigcap_n C_n$, where $\langle B_n \rangle$ and $\langle C_n \rangle$ are decreasing chains of internal subsets of $\bar{M}$. Then $A = {}^o(\bigcap_n (B_n \cap C_n))$ is neocompact in $\mathcal{M}$, and $A \subset D \cap C$. If $x \in D \cap C$, then x has a lifting $X \in B$. Moreover, since $x \in C$, $X \in \text{monad}(C)$, so $X \in \bigcap_n (B_n \cap C_n)$ and $x \in A$. Therefore $A = D \cap C$ and hence D is neoclosed in $\mathcal{M}$. $\square$

In the paper [15], a set D was defined to be neoclosed in $\mathcal{M}$ if it satisfies the condition of the preceding proposition, that $D = \mathcal{M} \cap {}^oB$ for some Π_1^0 set B. Our present notion of neoclosed is weaker, and the above proposition shows that any set which is neoclosed in the sense of [15] is neoclosed in our present sense.

Corollary 3.8 gave us a characterization of the monad of a neocompact set in the huge neometric family. The next result is an analogous characterization of the monad of a neoseparable set. Observe that for $\mathcal{M} \in \mathbf{H}$, a set $A \subset \mathcal{M}$ is neoseparable in $\mathcal{M}$ if and only if A is the closure in $\mathcal{M}$ of a set $B \subset \mathcal{M}$ which is the standard part of a Σ_1^0 set.

Proposition 4.2 *A set $B \subset G(\bar{M}, c)$ is the monad of a neoseparable set if and only if B can be written in the form*

$$B = \bigcap_n \bigcup_m ((B_m)^{1/n})$$

where $\langle B_m \rangle$ is an increasing chain of internal subsets of $G(\bar{M}, c)$.

Proof: Every Σ_1^0 subset of a galaxy $G(\bar{M}, c)$ can be written in the form $\bigcup_m B_m$ where $\langle B_m \rangle$ is an increasing chain of internal sets. Since

$$\bigcap_n \bigcup_m ((B_m)^{1/n}) = \bigcap_n ((\bigcup_m B_m)^{1/n})$$

and ${}^o(\bigcup_m B_m) = \bigcup_m ({}^o B_m)$, $\bigcap_n \bigcup_m ((B_m)^{1/n})$ is the monad of the closure of $\bigcup_m ({}^o B_m)$, and the result follows from the preceding observation. $\square$

If A is neoseparable, we call the representation

$$\mathrm{monad}(A) = \bigcap_n \bigcup_m ((B_m)^{1/n})$$

in the above proposition a **neoseparable normal form** for the monad of A.

The next proposition will lead to a characterization of the basic sets in terms of the neoseparable and neocompact sets.

Proposition 4.3 *(i) If C is neocompact in $\mathcal{M}$, and $D \subset C$ is neoseparable, then $D \subset B \subset C$ for some basic set B.*

(ii) Let D is neoseparable in $\mathcal{M}$, and let $C \subset D$ be neocompact. Then there is a sequence of basic sets $B_n \subset D$ such that

$$C = \bigcap_n ((B_n)^{1/n}).$$

Proof: (i) Let monad(C) have neocompact normal form $\bigcap_n ((\bar{C}_n)^{1/n})$, and let monad($D$) have normal form $\bigcap_n \bigcup_m (\bar{D}_m)^{1/n}$. Then for each m and n, $\bar{D}_m \subset (\bar{C}_n)^{1/n}$. By ω_1-saturation there is an internal set E such that

$$\bigcup_m \bar{D}_m \subset E \subset \bigcap_n ((\bar{C}_n)^{1/n}).$$

Then $B = {}^o E \subset C$. Since $E \subset \mathrm{monad}(\mathcal{M})$, B is basic. Moreover, ${}^o \bar{D}_m \subset B$ for each m, and B is closed, so $D \subset B$.

(ii) We have

$$\mathrm{monad}(C) = \bigcap_n ((C_n)^{1/n}) = \bigcap_n E_n = \bigcap_n ((E_n)^{2/n})$$

for some decreasing chain $\langle C_n \rangle$ of internal subsets of $G(\bar{M}, c)$, where $E_n = (C_n)^{1/n}$. Let

$$\mathrm{monad}(D) = \bigcap_n \bigcup_m ((D_m)^{1/n})$$

be a neoseparable normal form, where D_m is an increasing chain of basic sets. Then for each n,

$$\bigcap_k E_k \subset \bigcup_m ((D_m)^{1/n}).$$

By saturation there exist $k = k(n) \geq n$ and $m = m(n)$ such that $E_k \subset (D_m)^{1/n}$. Then

$$\text{monad}(C) = \bigcap_n E_{k(n)} = \bigcap_n (E_{k(n)} \cap (D_{m(n)})^{1/n})$$

$$\subset \bigcap_n ((E_{k(n)})^{1/n} \cap (D_{m(n)})^{1/n}) \subset \text{monad}(C).$$

Therefore $C = \bigcap_n ((B_n)^{1/n})$ where B_n is the basic set

$$B_n = {}^o(E_{k(n)})^{1/n} \cap D_{m(n)}. \ \square$$

Corollary 4.4 *A set $C \subset \mathcal{M}$ is basic if and only if it is both neocompact and neoseparable.*

Proof: If C is both neocompact and neoseparable, then C is basic by Proposition 4.3. If C is basic, then C is closed and is the standard part of a set which is both Σ_1^0 and Π_1^0, so C is both neocompact and neoseparable. $\square$

Corollary 4.5 *If C is neocompact in $\mathcal{M}$ then every separable subset of C is contained in a basic subset of C.*

Proof: Every closed separable subset of C is neoseparable, and is contained in a basic subset of C by Proposition 4.3. $\square$

Corollary 4.6 *For every neocompact set C and neoseparable set D such that $C \subset D$, there exists a sequence of internal sets $B_n \subset monad(D)$ such that*

$$monad(C) = \bigcap_n ((B_n)^{1/n}). \ \square$$

Note that this representation of the monad of a neocompact set differs from the representation in Corollary 3.8, where $\langle B_n \rangle$ was a decreasing chain of internal sets but B_n was not necessarily included in monad(D).

We now prove an important closure property of the huge neometric family. In the paper [9] we gave a variety of applications of this property to stochastic analysis.

Theorem 4.7 *The huge neometric family $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ is closed under diagonal intersections. That is, whenever $\mathcal{M} \in \mathbf{H}$, A_n is neocompact in $\mathcal{M}$ for each $n \in \mathbf{N}$, and ε_n is a sequence of reals such that $\lim_{n\to\infty} \varepsilon_n = 0$, the set $A = \bigcap_n (A_n)^{\varepsilon_n}$ is neocompact.*

Proof: By [9], Proposition 4.14, each set $(A_n)^{\varepsilon_n}$ is neoclosed, so A is neoclosed. Thus it suffices to prove that A is contained in a neocompact set. Since A_n is neocompact, we have $A_n = {}^o(B_n)$ for some Π^0_1 subset $B_n = \bigcap_m B_{mn}$ of the monad of $\mathcal{M}$. Then

$$A = \bigcap_k ((A_k)^{\varepsilon_k}) = \bigcap_k (({}^o(B_k))^{\varepsilon_k}) \subset \bigcap_k {}^o((B_k)^{2\varepsilon_k}) \subset {}^o\bigcap_k ((B_k)^{3\varepsilon_k}) = {}^oB.$$

B is a Π^0_1 set because for each k,

$$(B_k)^{3\varepsilon_k} = (\bigcap_m (B_{mk}))^{3\varepsilon_k} = \bigcap_m ((B_{mk})^{3\varepsilon_k}).$$

We show that ${}^oB \subset \mathcal{M}$, and hence ${}^oB$ is neocompact in $\mathcal{M}$.

Let $X \in B$ and let $x = {}^oX$. For each $k \in \mathbf{N}$ there exists $Y_k \in B_k$ such that $\bar{\rho}(X, Y_k) \leq 4\varepsilon_k$. Then $y_k = {}^oY_k \in \mathcal{M}$ and $\lim_{k\to\infty} y_k = x$ in $\mathcal{H}(\bar{M}, c)$. Since $\mathcal{M}$ is closed in $\mathcal{H}(\bar{M}, c)$, $x \in \mathcal{M}$. Thus ${}^oB \subset \mathcal{M}$ as required. □

Corollary 4.8 *Every compact set $C \subset \mathcal{M}$ is basic.*

Proof: Let C be compact. Then C is closed and separable, so C is neoseparable. Moreover, C is totally bounded, that is, there is a countable increasing chain of finite subsets C_n of C such that $C = \bigcap_n ((C_n)^{1/n})$. Each finite set is basic, so C is neocompact by the preceding theorem. Therefore C is basic by Corollary 4.4. □

With the preceding corollary as a starting point, we can show how the standard neometric family fits within the huge neometric family. We need a lemma about standard parts of Π^0_1 subsets of ${}^*M$ where M is a standard complete metric space.

Lemma 4.9 *Let M be a complete metric space. For any Π^0_1 set A which is contained in the monad of M in ${}^*M$, ${}^oA$ is compact.*

Proof: Let $A = \bigcap_n A_n$ where A_n is a decreasing chain of internal sets. Let $C = {}^oA$. C is neocompact in the huge neometric family by definition, and therefore C is closed. Since any countably compact metric space is compact, it suffices to show that C is countably compact. Let $\{B_n : n \in \mathbf{N}\}$ be a countable open covering of C. Let D_{mn} be the set of all x such that for some $r > 1/m$, the r-ball centered at x is included in B_n. Then each D_{mn} is open, and B_n contains the closure of D_{mn}. For any $X \in A$ with standard part x, there exist m, n, r such that $r > 1/m$ and the r-ball centered at x is included in B_n. By transfer, the r-ball centered at ${}^*x$ is included in ${}^*B_n$, and hence there exists $s > 1/m$ such that the s-ball centered at X is included in ${}^*B_n$. Therefore $X \in {}^*D_{mn}$, so $\{{}^*D_{mn} : m, n \in \mathbf{N}\}$ covers A. By ω_1-saturation, there exists k such that some finite subset of $\{{}^*D_{mn} : m, n \in \mathbf{N}\}$ covers A_k. Since B_n contains the closure of D_{mn}, some finite subset of $\{B_n : n \in \mathbf{N}\}$ covers C, as required. □

Proposition 4.10 *Let* $\mathbf{S}$ *be the class of all standard complete metric spaces, and for* $M \in \mathbf{S}$ *let*

$$\mathcal{B}(M) = \mathcal{C}(M) = \{C \subset M : C \text{ is compact}\}.$$

Then $\mathbf{S} \subset \mathbf{H}$*, and for each* $M \in \mathbf{H}$*,* $\mathcal{B}(M)$ *and* $\mathcal{C}(M)$ *are the same with respect to* $\mathbf{S}$ *as with respect to* $\mathbf{H}$*.*

Proof: We have already observed in Example 3.6 that $\mathbf{S} \subset \mathbf{H}$. Let $M \in \mathbf{S}$. The preceding lemma shows that every subset of M which is basic in $\mathbf{H}$ is compact. Corollary 4.8 shows that every compact set is basic in $\mathbf{H}$. □

We now turn to the separable members of $\mathbf{S}$.

Proposition 4.11 *Let* M *be a standard complete metric space, that is,* $M \in \mathbf{S}$*. The following are equivalent:*
(i) M *is separable.*
(ii) M *is neoseparable in* $\mathbf{H}$*.*

Proof: We have already observed that (i) implies (ii). Suppose M is neoseparable, with normal form

$$\text{monad}(M) = \bigcap_n (\bigcup_m (B_m)^{1/n}).$$

Then for each m, B_m is an internal subset of monad(M). Let $C_m = {}^o(B_m)$. By the Lemma 4.9, C_m is compact, and hence C_m has a countable dense subset. Moreover, $\bigcup_m C_m$ is dense in M. Therefore M is separable. □

The next result characterizes the neocompact sets in terms of the neoclosed sets in a neoseparable $\mathcal{M}$.

Proposition 4.12 *Let* $\mathcal{M}$ *be neoseparable. A set* C *is neocompact in* $\mathcal{M}$ *if and only if* C *is neoclosed in* $\mathcal{M}$ *and every countable covering of* C *by neoopen sets in* $\mathcal{M}$ *has a finite subcover.*

Proof: The condition that every countable covering of C by neoopen sets has a finite subcover is equivalent to the condition that for every decreasing chain $\langle D_n \rangle$ of neoclosed sets such that each D_n meets C, the intersection $\bigcap_n D_n$ meets C.

By the countable compactness property, if C is neocompact then C is neoclosed in $\mathcal{M}$ and every countable covering of C by neoopen sets in $\mathcal{M}$ has a finite subcovering. Suppose that C is neoclosed in $\mathcal{M}$ and every countable covering of C by neoopen sets in $\mathcal{M}$ has a finite subcovering. Since $\mathcal{M}$ is neoseparable, there is an increasing chain of basic sets B_m such that $\mathcal{M} = \bigcap_k \bigcup_m (B_m)^{1/k}$. For each m and k let $D_{m,k}$ be the neoclosed set $\{x \in \mathcal{M} : \rho(x, B_m) \geq 1/k\}$ in $\mathcal{M}$. Then for each k, $\bigcap_m D_{m,k} = \emptyset$.

Therefore there exists $m(k)$ such that $C \cap D_{m(k),k} = \emptyset$, and hence $C \subset (B_{m(k)})^{1/k}$. Thus C is contained in the set $E = \bigcap_k((B_{m(k)})^{1/k})$, and $E \subset \mathcal{M}$. By closure under diagonal intersections, E is neocompact. Since C is neoclosed in $\mathcal{M}, C$ is also neocompact. $\square$

Proposition 4.12 holds for any neometric family with the countable compactness and diagonal intersection properties, because these two properties were the only ones which were used in the proof.

Proposition 4.13 *(i) Suppose $\mathcal{M}$ is a countable union of basic sets (for example, $\mathcal{M} = \mathcal{H}(\bar{M}, c)$). Then $\mathcal{C}(\mathcal{M})$ is the smallest collection of sets which contains $\mathcal{B}(\mathcal{M})$ and is closed under countable intersections.*

(ii) Let $\mathcal{M}$ be neoseparable. Then $\mathcal{C}(\mathcal{M})$ is the smallest collection of sets which contains $\mathcal{B}(\mathcal{M})$ and is closed under diagonal intersections.

Proof: (i) $\mathcal{C}(\mathcal{M})$ clearly contains $\mathcal{B}(\mathcal{M})$ and is closed under countable intersections. Let $\mathcal{M} = \bigcup_n B_n$ where each B_n is basic, and let $C \in \mathcal{C}(\mathcal{M})$, so that monad$(C) = \bigcap_n C_n$ where each C_n is internal. For each n we have $B_n = {}^oA_n$ where A_n is internal. We may take the A_n to be increasing and the C_n to be decreasing.

We claim that $C \subset B_k$ for some $k \in \mathbf{N}$. Suppose not. Then for each k there exists $x_k \in C - B_k$. Let X_k be a lifting of x_k. Then for some $\varepsilon_k > 0$, $X_k \in C_k - (A_k)^{\varepsilon_k}$. By saturation there exists $X \in \bigcap_k(C_k - (A_k)^{\varepsilon_k})$. Let $x = {}^oX$. Then $x \in C$ but $x \notin B_k$ for each $k \in \mathbf{N}$, contradicting the assumption that $C \subset \mathcal{M}$. This proves the claim.

We now claim that

$$C = {}^o(\bigcap_n (C_n \cap A_k)).$$

It is clear that C contains the right side. Suppose $x \in C$. Then $x \in B_k$, so x has a lifting $X \in A_k$. Since $X \in$ monad(C),$X \in C_n$ for each $n \in \mathbf{N}$. This proves our second claim. Finally, by saturation,

$${}^o(\bigcap_n (C_n \cap A_k)) = \bigcap_n {}^o(C_n \cap A_k).$$

Therefore C is the intersection of a countable chain of basic sets ${}^o(C_n \cap A_k)$ in $\mathcal{M}$, and (i) is proved.

(ii) By Theorem 4.7, $\mathcal{C}(\mathcal{M})$ contains $\mathcal{B}(\mathcal{M})$ and is closed under diagonal intersections. Let $C \in \mathcal{C}(\mathcal{M})$. By Proposition 4.3 (ii) there is a sequence of basic sets $B_n \in \mathcal{B}(\mathcal{M})$ such that C is equal to the diagonal intersection $\bigcap_n((B_n)^{1/n})$. $\square$

We conclude this section with a useful necessary and sufficient condition for neocontinuity in the huge neometric family.

Definition 4.14 *Let $C \subset \mathcal{M}$. An internal function $F : \bar{M} \to \bar{N}$ is said to be a* **uniform lifting** *of a function $f : C \to \mathcal{N}$ if ${}^oF(X) = f(x)$ whenever X lifts $x \in C$. $f : C \to \mathcal{N}$ is said to be* **uniformly liftable** *from $\mathcal{M}$ into $\mathcal{N}$ if it has a uniform lifting.*

Uniformly liftable total functions on spaces of random variables were studied in [15].

Theorem 4.15 *Let $C \subset \mathcal{M}$. Every uniformly liftable function $f : C \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$ in the huge neometric family.*

Proof: Let F uniformly lift f and let $A \subset C$ be neocompact in $\mathcal{M}$. By Corollary 3.8, the monad of A has a neocompact normal form

$$\mathrm{monad}(A) = \bigcap_n ((B_n)^{1/n}).$$

The sequence $\langle F_n \rangle$ where $F_n = F|((B_n)^{1/n})$ is a decreasing chain of internal sets, and

$$F|(\mathrm{monad}(A)) = \bigcap_n F_n.$$

Since F uniformly lifts f, whenever $X \in \mathrm{monad}(A)$ and $x = {}^oX$ we have ${}^o(X, F(X)) = (x, f(x))$. Therefore ${}^o(\bigcap_n F_n) = f|A$. This shows that $f|A$ is neocompact, so f is neocontinuous. □

Example 4.16 *Let M be a standard Banach space, and consider the nonstandard hull $\mathcal{H}({}^*M, 0)$ of the galaxy of 0 in ${}^*M$. The norm function $x \mapsto \|x\|$, the addition function $(x, y) \mapsto x + y$, and the scalar multiplication function $x \mapsto \alpha x, \alpha \in \mathbf{R}$, have uniform liftings and hence are neocontinuous in the huge neometric family $\mathbf{H}$.*

Theorem 4.17 *Let $C \subset \mathcal{M}$ be neocompact. Then a function $f : C \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$ if and only if f is uniformly liftable.*

Proof: One direction follows from Theorem 4.16. For the other direction, suppose $f : C \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$. Then the graph G of f is neocompact in $\mathcal{M} \times \mathcal{N}$. The monad of G is a Π^0_1 set, so that $\mathrm{monad}(G) = \bigcap_n G_n$ where $\langle G_n \rangle$ is a decreasing chain of internal subsets of $\bar{M} \times \bar{N}$. For each $n \in \mathbf{N}$ let A_n be the set of all internal functions F from $\bar{M}$ into $\bar{N}$ such that for each $m \le n$, and each $X \in \bar{M}$, if $(\exists Y)(X, Y) \in G_m$ then $(X, F(X)) \in G_m$. Since $\langle G_n \rangle$ is a decreasing chain, we see from the transfer of the axiom of choice that $\langle A_n \rangle$ is a decreasing chain of nonempty internal sets. Therefore there exists $F \in \bigcap_n A_n$. Then F is an internal function from $\bar{M}$ into $\bar{N}$ and for each $X \in \mathrm{monad}(C)$, we have $(X, F(X)) \in \mathrm{monad}(G)$. Thus F is a uniform lifting of f. □

Corollary 4.18 *If $f : C \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$ and $A \subset C$ is basic, then $f(A)$ is basic.*

Proof: The restriction of f to A has a uniform lifting F. Let $A = {}^o\bar{A}$ where $\bar{A}$ is internal. Then $F(\bar{A})$ is internal and $f(A) = {}^oF(\bar{A})$. $\square$

5 Rich Adapted Spaces

In [9] we studied an important example of a neocompact family, built upon the metric spaces $L^0(\Omega, M)$ of random variables from an adapted space Ω into a separable metric space M. The adapted space Ω was said to be rich if the corresponding neocompact family has the countable compactness property. In this section we review these notions and, as promised in [9], show that rich adapted spaces exist (and in fact are obtained from the Loeb measure construction).

Given probability space $(\Omega, P, \mathcal{G})$ and a metric space M, a function $x : \Omega \to M$ is called **$\mathcal{G}$-measurable** if $x^{-1}(U) \in \mathcal{G}$ for every open set $U \subset M$, and $L^0(\Omega, M)$ is the set of all $\mathcal{G}$-measurable functions from Ω into M, identifying functions which are equal P-almost surely. ρ_0 is the metric of convergence in probability on $L^0(\Omega, M)$,

$$\rho_0(x, y) = \inf\{\varepsilon : P[\rho(x(\cdot), y(\cdot)) \leq \varepsilon] \geq 1 - \varepsilon\}.$$

An **atom** of a probability space Ω is a set A of positive measure such that every measurable subset of A has measure either 0 or $P[A]$. A measurable set $B \subset \Omega$ is **atomless** in Ω if no subset of B is an atom of Ω. The probability space Ω is said to be atomless if the set $B = \Omega$ is atomless in Ω.

Throughout this section we let $M = (M, \rho)$ and $N = (N, \sigma)$ be complete separable metric spaces. The space of Borel probability measures on M with the Prohorov metric

$$d(\mu, \nu) = \inf\{\varepsilon : \mu(K) \leq \nu(K^\varepsilon) + \varepsilon \text{ for all closed } K \subset M\}$$

is denoted by $\mathrm{Meas}(M)$. It is again a complete separable metric space, and convergence in $\mathrm{Meas}(M)$ is the same as weak convergence. Each measurable function $x : \Omega \to M$ induces a measure $\mathrm{law}(x) \in \mathrm{Meas}(M)$. The function

$$\mathrm{law} : L^0(\Omega, M) \to \mathrm{Meas}(M)$$

is continuous, and in fact,

$$d(\mathrm{law}(x), \mathrm{law}(y)) \leq \rho_0(x, y).$$

Moreover, if Ω is atomless, then for each M the function law maps the set of all $\mathcal{G}$-measurable $x \in L^0(\Omega, M)$ onto $\mathrm{Meas}(M)$. A useful condition for compactness in

Meas(M) is Prohorov's theorem, that a closed set $C \subset \text{Meas}(M)$ is compact if and only if it is tight.

Definition 5.1 *Let $\Omega = (\Omega, P, \mathcal{G})$ be a probability space, and let $\mathbf{M}_\Omega$ be the family of all the metric spaces $\mathcal{M} = L^0(\Omega, M)$ where M is a complete separable metric space. A subset B of $\mathcal{M}$ will be called* **basic** *with respect to the probability space Ω, in symbols $B \in \mathcal{B}_\Omega(\mathcal{M})$, if either*

1. *B is compact, or*
2. *$B = law^{-1}(C)$ for some compact $C \subset Meas(M)$.*

We say that a $C \subset \mathcal{M} \in \mathbf{M}_\Omega$ is **neocompact** with respect to the probability space Ω, in symbols $C \in \mathcal{C}_\Omega(\mathcal{M})$, if C belongs to the neocompact family generated by the basic sets with respect to Ω. Thus $(\mathbf{M}_\Omega, \mathcal{B}_\Omega, \mathcal{C}_\Omega)$ is the neometric family generated by $(\mathbf{M}_\Omega, \mathcal{B}_\Omega)$. We say that Ω is a **rich probability space** if Ω is atomless, and the family of neocompact sets with respect to Ω has the countable compactness property.

We now review the notion of a Loeb probability space (e.g. see [22] and [16]). A *probability space

$$\bar{\Omega} = (\Omega, \bar{P}, \bar{\mathcal{G}})$$

is an element of the set $^*A$ where A is the set of all σ-additive probability spaces in the standard universe. Note that Ω is an internal set, $\bar{\mathcal{G}}$ is a finitely additive algebra of sets, and $\bar{P}$ is a finitely additive function from $\bar{\mathcal{G}}$ into the hyperreal interval $^*[0,1]$. The Loeb space associated with $\bar{\Omega}$ is obtained from the following result of Loeb [23].

Proposition 5.2 *Let $\bar{\Omega} = (\Omega, \bar{P}, \bar{\mathcal{G}})$ be a *probability space. There is a unique σ-additive probability space*

$$\Omega = (\Omega, P, \mathcal{G}),$$

called the **Loeb space** *of $\bar{\Omega}$, such that $\mathcal{G}$ is the σ-algebra generated by $\bar{\mathcal{G}}$ and $P(A) = {}^o\bar{P}(A)$ for each $A \in \bar{\mathcal{G}}$.* □

Note that both $\bar{\Omega}$ and Ω have the same sample set, also denoted by Ω.

Definition 5.3 *Let $\bar{\Omega}$ be a *probability space. A function $X : \Omega \to {}^*M$ will be called* **$\bar{\mathcal{G}}$-measurable** *if X is internal and $X^{-1}(U) \in \bar{\mathcal{G}}$ for each *open set $U \subset {}^*M$. Let $SL^0(\Omega, M)$ be the *metric space of all $\bar{\mathcal{G}}$-measurable functions $X : \Omega \to {}^*M$ with the *metric*

$$\bar{\rho}_0(X, Y) = {}^*\inf\{\varepsilon : \bar{P}[{}^*\rho(X(\omega), Y(\omega)) \geq \varepsilon] \leq \varepsilon\}.$$

In the space $SL^0(\Omega, M)$, the distance between any two points is at most one, so all points belong to the same galaxy. We say that $X \in SL^0(\Omega, M)$ is a **lifting** *of a function $x : \Omega \to M$, in symbols ${}^oX = x$, if $X(\omega)$ has standard part $x(\omega) \in M$ for P-almost all $\omega \in \Omega$. If $X \in SL^0(\Omega, M)$ is a lifting of some x, we say that X is* **near-standard** *and write $X \in ns^0(\Omega, M)$.*

A function $x \in L^0(\Omega, M)$ is **simple** if it has finite range, and a function $X \in SL^0(\Omega, M)$ is ***simple** if it has *finite range. If M is separable, then every $\mathcal{G}$-measurable function is a ρ_0-limit of simple functions.

The following is a well known fundamental result in Loeb measure theory.

Proposition 5.4 *(Loeb [23], Anderson [3]) Let $\bar{\Omega}$ be a *probability space. For any $x \in \Omega \to M$, the following are equivalent:*

(i) x is Loeb measurable, i.e. $x \in L^0(\Omega, M)$.

(ii) x has a lifting $X \in ns^0(\Omega, M)$.

(iii) For each infinite $n \in {}^*\mathbf{N}$, x has a *simple lifting $X \in ns^0(\Omega, M)$ whose range has *cardinality at most n.

(iv) x is a limit of simple functions with respect to ρ_0.

Moreover, whenever $x = {}^oX$ and $y = {}^oY$, we have $\rho_0(x, y) = {}^o\bar{\rho}_0(X, Y)$, so that $\bar{\rho}_0$ is a uniform lifting of ρ_0, and $ns^0(\Omega, M)$ is the monad of $L^0(\Omega, M)$ in $SL^0(\Omega, M)$.

We shall need the following property of atomless Loeb probability spaces, which is called saturation and proved in [14], Corollary 4.5.

Proposition 5.5 *Let Ω be an atomless Loeb probability space. Suppose Γ is another probability space, and M and N are complete separable metric spaces. For any random variables $x \in L^0(\Omega, M)$ and $(\bar{x}, \bar{y}) \in L^0(\Gamma, M \times N)$ such that law(x) = law$(\bar{x})$, there exists $y \in L^0(\Omega, N)$ such that law(x, y) = law$(\bar{x}, \bar{y})$.* $\square$

We now look at the space $L^0(\Omega, M)$ within the huge neometric family.

Theorem 5.6 *Let $\bar{\Omega}$ be a *probability space. The set $L^0(\Omega, M)$ is neoseparable with respect to the *metric $\bar{\rho}_0$ on $SL^0(\Omega, M)$, and the metric space $\mathcal{M} = (L^0(\Omega, M), \rho_0)$ belongs to the huge neometric family* $\mathbf{H}$.

Proof: Let $\{m_0, m_1, \ldots\}$ be a countable dense subset of M, and let M_n be the finite set $M_n = \{m_0, m_1, \ldots, m_n\}$. Then ${}^*(M_n) = M_n$, so every $\bar{\mathcal{G}}$-measurable function $X \in SL^0(\Omega, M_n)$ is near-standard, that is, $SL^0(\Omega, M_n) \subset ns^0(\Omega, M)$. On the other hand, every $\mathcal{G}$-measurable function $x \in L^0(\Omega, M_n)$ has a lifting $X \in SL^0(\Omega, M_n)$. Therefore the standard part of $SL^0(\Omega, M_n)$ is the basic set $L^0(\Omega, M_n)$.

Since $\bigcup_n M_n$ is dense in M, and measurable functions with values in M can be approximated in ρ_0 by simple functions, the countable union $\bigcup_n L^0(\Omega, M_n)$ of basic sets is dense in the closed set $L^0(\Omega, M)$. This shows that $L^0(\Omega, M)$ is neoseparable. $\square$

Proposition 5.7 *([15], Propositions 2.4 and 2.7) Let $\bar{\Omega}$ be a *probability space. The function law from $L^0(\Omega, M)$ into Meas(M) has a uniform lifting LAW from $SL^0(\Omega, M)$ into *Meas(M). Moreover, for each $X \in SL^0(\Omega, M)$, we have $X \in ns^0(\Omega, M)$ if and only if LAW(X) is near-standard in *Meas(M).* $\square$

Corollary 5.8 *Let Ω be a Loeb probability space. For each $\mathcal{M} \in \mathbf{M}_\Omega$, every set $C \in \mathcal{C}(\mathcal{M})$ is contained in a set $D \in \mathcal{B}_\Omega(\mathcal{M})$.*

Proof: Let $C \in \mathcal{C}(\mathcal{M})$. Since the law function is uniformly liftable, it is neocontinuous in $(\mathbf{H}, \mathcal{B}, \mathcal{C})$. Then law$(C)$ is neocompact in $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ and separable, so law(C) is compact. Thus C is contained in the set $D = \text{law}^{-1}(\text{law}(C))$, which belongs to $\mathcal{B}_\Omega(\mathcal{M})$. $\square$

We now turn to adapted spaces. In the classical literature one usually considers adapted spaces with time indexed by the set $\mathbf{N}$ of natural numbers or the set $[0, \infty]$ of nonnegative extended reals. The paper [9] worked with adapted spaces with time indexed by the set $\mathbf{B}$ of dyadic rationals in $[0, \infty]$, and used results about $\mathbf{B}$-adapted spaces to draw conclusions about $[0, \infty]$-adapted spaces. Here we shall take a more general approach where time is indexed by an arbitrary linearly ordered set.

Let $\langle \mathbf{L}, \leq \rangle$ be a subset of an internal linearly ordering $\langle \bar{\mathbf{L}}, \leq \rangle$. For convenience we also assume that $\mathbf{L}$ contains a least element 0 and a greatest element ∞. For example, $\mathbf{L}$ can be any standard linear ordering with a first and last element, such as $[0, \infty], \mathbf{B}$, or $\mathbf{N} \cup \{\infty\}$, or any internal linear ordering such as $^*[0, \infty], {}^*\mathbf{B}$, or $^*\mathbf{N} \cup \{\infty\}$. We say that $\Omega = (\Omega, P, \mathcal{G}_t)_{t \in \mathbf{L}}$ is an **L-adapted space** if $\mathcal{G}_t$ is a σ-algebra of subsets of Ω for each $t \in \mathbf{L}$, $\mathcal{G}_s \subset \mathcal{G}_t$ whenever $s < t$ in $\mathbf{L}$, and P is a complete probability measure on $\mathcal{G}_\infty$. We shall write $\mathcal{G} = \mathcal{G}_\infty$, so that $(\Omega, P, \mathcal{G})$ is the probability space associated with the adapted space Ω.

Definition 5.9 *Let $\mathcal{E}$ and $\mathcal{F}$ be σ-subalgebras of $\mathcal{G}_\infty$ with $\mathcal{E} \subset \mathcal{F}$. $\mathcal{F}$ is said to be* **atomless over** $\mathcal{E}$ *if for every $U \in \mathcal{F}$ of positive probability, there is a set $V \subset U$ in $\mathcal{F}$ such that*

$$0 < P[V|\mathcal{E}] < P[U|\mathcal{E}]$$

on a set of positive probability. Following [14], we say that an $\mathbf{L}$-adapted space Ω is **atomless** *if $(\Omega, P, \mathcal{G}_0)$ is an atomless probability space, and $\mathcal{G}_t$ is atomless over $\mathcal{G}_s$ whenever $s < t$ in $\mathbf{L}$.*

Definition 5.10 *Let $\Omega = (\Omega, P, \mathcal{G}_t)_{t\in\mathbf{L}}$ be an $\mathbf{L}$-adapted space, and let $\mathbf{M}_\Omega$ be the family of all the metric spaces $\mathcal{M} = L^0(\Omega, M)$ where M is a complete separable metric space. A subset B of $\mathcal{M}$ will be called* **basic** *with respect to Ω, in symbols $B \in \mathcal{B}_{\Omega,\mathbf{L}}(\mathcal{M})$, if either*

1. *B is compact, or*
2. $$B = \{x \in law^{-1}(C) : x \text{ is } \mathcal{G}_t\text{-measurable}\}$$

   for some compact $C \subset Meas(M)$ and $t \in \mathbf{L}$.

In the case that $t = \infty$, 5.10.2 says that the set

$$B = law^{-1}(C)$$

is basic for each compact $C \subset \mathrm{Meas}(M)$.

We let $(\mathbf{M}_\Omega, \mathcal{B}_{\Omega,\mathbf{L}}, \mathcal{C}_{\Omega,\mathbf{L}})$ be the neometric family generated by $(\mathbf{M}_\Omega, \mathcal{B}_{\Omega,\mathbf{L}})$.

We say that an $\mathbf{L}$-adapted space Ω is **rich** if Ω is atomless and the neocompact family

$$(\mathbf{M}_\Omega, \mathcal{B}_{\Omega,\mathbf{L}}, \mathcal{C}_{\Omega,\mathbf{L}})$$

has the countable compactness property.

By an **L-adapted Loeb space** we mean an $\mathbf{L}$-adapted space $\Omega = (\Omega, P, \mathcal{G}_t)_{t\in\mathbf{L}}$ such that $(\Omega, P, \mathcal{G})$ is a Loeb probability space, $\mathcal{G}_s \subset \mathcal{G}_t$ whenever $s < t \in \mathbf{L}$, and each $\mathcal{G}_t$ is a σ-algebra generated by an internal subalgebra $\bar{\mathcal{G}}_t$ of $\bar{\mathcal{G}}$. We also write $\bar{\mathcal{G}} = \bar{\mathcal{G}}_\infty$.

By applying Proposition 5.4 to the Loeb probability space $(\Omega, P, \mathcal{G}_t)$, we obtain the following lifting theorem for adapted Loeb spaces.

Corollary 5.11 *Let Ω be an $\mathbf{L}$-adapted Loeb space and let $t \in \mathbf{L}$. A random variable $x \in L^0(\Omega, M)$ is $\mathcal{G}_t$-measurable if and only if x has a $\bar{\mathcal{G}}_t$-measurable lifting.* $\square$

We now wish to show that every atomless $\mathbf{L}$-adapted Loeb space is rich. To do this we first prove a pair of lemmas about Loeb probability spaces, and then use the lemmas to show that the basic sets in an $\mathbf{L}$-adapted Loeb space are basic in the huge neometric family.

Lemma 5.12 *Let $\bar{\Omega}$ be a *probability space whose associated Loeb space Ω is atomless. Then there is an infinitesimal ε such that every *atom of $\bar{\Omega}$ has $\bar{P}-$ measure at most ε.*

Proof: If $\bar{\Omega}$ is *atomless then the result holds trivially for every infinitesimal $\varepsilon \geq 0$. Suppose $\bar{\Omega}$ has at least one *atom. Every *atom of $\bar{\Omega}$ has infinitesimal $\bar{P}-$measure, because if A is an atom of $\bar{\Omega}$ and $\alpha = {}^o(\bar{P}(A)) > 0$, then A would be an atom of Ω with Loeb measure α. Since $\bar{\Omega}$ is internal, the set of $\bar{P}$-measures of *atoms of $\bar{\Omega}$ has a *supremum $\varepsilon > 0$. Then ε must be infinitesimal and every *atom of $\bar{\Omega}$ has $\bar{P}$-measure at most ε. □

Lemma 5.13 *Let Ω be an atomless Loeb probability space. Let $x \in \mathcal{M} \in \mathbf{M}_\Omega$. There is a lifting X of x and an infinitesimal ε such that every $y \in \mathcal{M}$ with $law(y) = law(x)$ has a lifting Y where $LAW(Y)$ is within ε of $LAW(X)$ in ${}^*Meas(M)$.*

Proof: By the preceding lemma, we may choose an infinitesimal ε large enough so that every *atom of $\bar{\Omega}$ has $\bar{P}$-measure less than ε^2. By Proposition 5.4, x has a *simple lifting $X \in ns^0(\Omega, M)$ whose range has *cardinality at most $1/\varepsilon$.

Suppose $y \in \mathcal{M}$ and $\mathrm{law}(y) = \mathrm{law}(x)$. Let Z be a lifting of y. Let d be the metric of $\mathrm{Meas}(M)$. We claim that for each positive $n \in \mathbf{N}$, there exists $Y_n \in SL^0(\Omega, M)$ such that

$$\bar{\rho}_0(Y_n, Z) \leq 3/n \tag{3}$$

and

$${}^*d(\mathrm{LAW}(Y_n), \mathrm{LAW}(X)) \leq \varepsilon. \tag{4}$$

We now prove this claim. There is a finite set $A = \{a_1, \ldots, a_k\} \subset M$ such that $x(\omega) \in A^{1/n}$ with probability greater than $1 - 1/n$. For $i = 1, \ldots, k$ let

$$A_i = \{b \in M : \rho(b, a_i) \leq 1/n\},$$

$$B_i = A_i - (A_1 \cup \cdots \cup A_{i-1}),$$

$$B_0 = M - (A_1 \cup \cdots \cup A_k).$$

Then $B_0, \ldots, B_k$ partitions M and $P[x(\omega) \in B_0] \leq 1/n$. Since $\mathrm{law}(y) = \mathrm{law}(x)$, we also have

$$P[y(\omega) \in B_i] = P[x(\omega) \in B_i]$$

for each i.

We now introduce corresponding internal subsets of ${}^*M$. For each $\delta > 0$ and $i = 1, \ldots, k$, let

$$C_{i,\delta} = \{b \in {}^*M : {}^*\rho(b, a_i) \leq (1/n) + \delta\},$$

$$D_{i,\delta} = C_{i,\delta} - (C_{1,\delta} \cup \cdots \cup C_{i-1,\delta}),$$

$$D_{0,\delta} = {}^*M - (C_{1,\delta} \cup \cdots \cup C_{k,\delta}).$$

By induction on i, one can show that for all sufficiently large infinitesimal δ, we have $X(\omega) \in D_{i,\delta}$ if and only if $x(\omega) \in B_i$ and $Z(\omega) \in D_{i,\delta}$ if and only if $y(\omega) \in B_i$ for P-almost all ω. Pick such an infinitesimal δ. Since each *atom of Ω has $\bar{P}$-measure less than ε^2, there is an internal partition $E_0, \ldots, E_k$ of Ω such that for each $i \leq k$, $Z(\omega) \in D_{i,\delta}$ if and only if $\omega \in E_i$ for P-almost all ω, and

$$|\bar{P}[E_i] - \bar{P}[X(\omega) \in D_{i,\delta}]| \leq \varepsilon^2.$$

Again using the fact that all *atoms have *measure $\leq \varepsilon^2$, it follows that there is a *simple function $Y_n \in SL^0(\Omega, M)$ such that:

$$\text{For each } i, \ \text{range}(Y_n|E_i) = D_{i,\delta} \cap (\text{range}(X)) \tag{5}$$

$$\text{For each } b \in \text{range}(X), \bar{P}[Y_n(\omega) = b] \text{ is within } \varepsilon^2 \text{ of } \bar{P}[X(\omega) = b]. \tag{6}$$

Since the range of X has *cardinality at most $1/\varepsilon$, condition (6) holds. Whenever $i \leq k, \omega \in E_i$, and $Z(\omega) \in D_{i,\delta}$, we have

$$(^*\rho(Y_n(\omega), Z(\omega)) \leq 2((1/n) + \delta).$$

Moreover, $\bar{P}[E_0] \leq 1/n$. Therefore

$$\bar{P}[^*\rho(Y_n(\omega), Z(\omega)) \leq 3/n] \geq 1 - 3/n,$$

so condition (5) holds and the claim is proved.

By saturation, there exists $Y \in SL^0(\Omega, M)$ such that $\bar{\rho}_0(Y, Z) \approx 0$ and condition (6) holds. Then Y is a lifting of y with the required property, and the proof is complete. $\square$

Theorem 5.14 *Let Ω be an $\mathbf{L}$-adapted Loeb space, and let $\mathcal{M} = L^0(\Omega, M) \in \mathbf{M}_\Omega$. Then $\mathcal{B}_{\Omega,\mathbf{L}}(\mathcal{M}) \subset \mathcal{B}(\mathcal{M})$, that is, every basic set in $(\mathbf{M}_\Omega, \mathcal{B}_{\Omega,\mathbf{L}})$ is basic in $(\mathbf{H}, \mathcal{B}, \mathcal{C})$.*

Proof: We have already shown that $\mathcal{M}$ belongs to $\mathbf{H}$. Let $B \in \mathcal{B}_{\Omega,\mathbf{L}}(\mathcal{M})$. If B is compact, then $B \in \mathcal{B}(\mathcal{M})$ by Corollary 4.8. Assume that B is not compact, so that

$$B = \{x \in \mathcal{M} : \text{law}(x) \in C \text{ and } x \text{ is } \mathcal{G}_t\text{-measurable}\}$$

for some compact set $C \subset \text{Meas}(M)$ and some $t \in \mathbf{L}$.

Suppose first that C is a one-element set, $C = \{c\}$. Let $x \in B$ and let X be a $\bar{\mathcal{G}}_t$-measurable lifting of x. By Lemma 5.13, there is an infinitesimal ε such that every $y \in B$ has a lifting Y which belongs to the internal set

$$D = \{Y : Y \text{ is } \bar{\mathcal{G}}_t\text{-measurable and } ^*d(\text{LAW}(Y), \text{LAW}(X)) \leq \varepsilon\}.$$

It follows that $B = {}^oD$, so $B \in \mathcal{B}(\mathcal{M})$. The family $\mathcal{B}(\mathcal{M})$ is closed under finite unions, and thus we have $B \in \mathcal{B}(\mathcal{M})$ whenever the set C is finite.

We now return to the general case where C is a compact set. We first prove that $B \in \mathcal{C}(\mathcal{M})$. For each $n \in \mathbf{N}$ let $\bar{B}_n$ be the internal set

$$\bar{B}_n = \{X \in \mathrm{LAW}^{-1}(({}^*C)^{1/n}) : X \text{ is } \bar{\mathcal{G}}_t\text{-measurable}\}.$$

By a classical result of Robinson, a subset C of a standard metric space is compact if and only if $C = {}^o({}^*C)$ (e.g. see [29]). Then by Lemma 3.7, $C = {}^o(\bigcap_n(({}^*C)^{1/n}))$. Therefore by Proposition 5.7 and Corollary 5.11,

$$B = {}^o(\bigcap_n(\bar{B}_n)),$$

so $B \in \mathcal{C}(\mathcal{M})$.

We now show that B is neoseparable in $(\mathbf{H}, \mathcal{B}, \mathcal{C})$. Since C is compact, there is an increasing chain $\langle C_n \rangle$ of finite subsets of C such that $C = \bigcap_n((C_n)^{1/n})$. We have already shown that for each of the finite sets C_n, the set

$$B_n = \{x \in \mathrm{law}^{-1}(C_n) : x \text{ is } \mathcal{G}_t\text{-measurable}\}$$

belongs to $\mathcal{B}(\mathcal{M})$. Since B is closed and $B_n \subset B$, B contains the closure of $\bigcup_n B_n$. To prove the opposite inclusion, let $x \in B$. Then x is a random variable on the atomless Loeb space

$$\Omega_t = (\Omega, P, \mathcal{G}_t).$$

We have $\mathrm{law}(x) \in C$, so we may choose $\alpha_n \in C_n$ such that $d(\mathrm{law}(x), \alpha_n) \leq 1/n$. Put $\alpha_0 = \mathrm{law}(x)$. By the Skorokhod representation theorem (see [8], p. 102), there is a random variable y on some probability space with values in $M^{\mathbf{N}}$ such that $\mathrm{law}(y_n) = \alpha_n$ for each n, and $y_n \to y_0$ almost everywhere. Then $\mathrm{law}(y_0) = \mathrm{law}(x)$. By Proposition 5.5, there is a random variable z on Ω_t with values in $M^{\mathbf{N}}$ such that $z_0 = x$ and $\mathrm{law}(z) = \mathrm{law}(y)$. Then $\mathrm{law}(z_n) = \alpha_n$ for each n, and $\rho_0(z_n, x) \to 0$. Moreover, z_n is $\mathcal{G}_t$-measurable, so $z_n \in B_n$ and x is in the closure of $\bigcup_n B_n$ as required. Therefore B is both neocompact and neoseparable, and hence B belongs to $\mathcal{B}(\mathcal{M})$ by Corollary 4.4. $\square$

Theorem 5.15 *Every atomless* $\mathbf{L}$*-adapted Loeb space* Ω *is rich. Thus for each standard linearly ordered set* $\mathbf{L}$*, rich* $\mathbf{L}$*-adapted spaces exist.*

Proof: Let $\mathcal{M} = L^0(\Omega, M) \in \mathbf{M}_\Omega$. By Theorem 5.14, $\mathcal{B}_{\Omega,\mathbf{L}}(\mathcal{M}) \subset \mathcal{B}(\mathcal{M})$. It follows that $\mathcal{C}_{\Omega,\mathbf{B}}(\mathcal{M}) \subset \mathcal{C}(\mathcal{M})$. By Theorem 3.11, the huge neometric family

$(\mathbf{H}, \mathcal{B}, \mathcal{C})$ has the countable compactness property. It follows that the neometric family $(\mathbf{M}_\Omega, \mathcal{B}_{\Omega,\mathbf{B}}, \mathcal{C}_{\Omega,\mathbf{B}})$ has the countable compactness property. $\square$

Taking $\mathbf{L}$ to be the one-point linear order, we get the corresponding result for probability spaces.

Corollary 5.16 *Every atomless Loeb probability space is rich.* $\square$

The paper [9] contains many examples of neocompact and neoclosed sets and neocontinuous functions in the neometric family $(\mathbf{M}_\Gamma, \mathcal{B}_\Gamma, \mathcal{C}_\Gamma)$ where Γ is a rich $\mathbf{B}$-adapted space. Let $\mathcal{F}_t$ be the right continuous filtration $\mathcal{F}_t = \bigcap_{s>t} \mathcal{G}_s$ obtained from Ω. Then, for example, the set of $\mathcal{F}_t$-adapted processes in $L^0(\Gamma, C([0,1], M))$ is neoclosed, and the conditional expectation function $x \mapsto E[x|\mathcal{F}_t]$ is neocontinuous from each uniformly integrable subset of $L^0(\Gamma, \mathbf{R})$ into the space $L^0(\Gamma, L^1([0,1], \mathbf{R}))$ of stochastic processes. It follows from Theorem 5.14 that for a rich $\mathbf{B}$-adapted Loeb space, all these examples also have the corresponding property in the huge neometric family.

Corollary 5.17 *Let Ω be an atomless $\mathbf{L}$-adapted Loeb space. Every neocompact set, neoclosed set, neoseparable set, and neocontinuous function with neoclosed domain in $(\mathbf{M}_\Omega, \mathcal{B}_{\Omega,\mathbf{L}}, \mathcal{C}_{\Omega,\mathbf{L}})$ is also neocompact, neoclosed, neoseparable, or neocontinuous, respectively, in the huge neometric family $(\mathbf{H}, \mathcal{B}, \mathcal{C})$.*

Proof: By Theorem 5.14, Corollary 5.8, and Proposition 2.5 $\square$

It was shown in [9], Example 5.7, that a rich $\mathbf{B}$-adapted space is nowhere right continuous, that is, $\mathcal{F}_t \neq \mathcal{G}_t$ for each $t \in \mathbf{B}$. Similarly, a rich $\mathbf{R}_+$-adapted space is nowhere right continuous. Each $\mathbf{B}$-adapted space $\Omega_\mathbf{B} = (\Omega, P, \mathcal{G}, \mathcal{G}_t)_{t\in\mathbf{B}}$ has a corresponding right continuous $\mathbf{R}_+$-adapted space $\Omega_\mathbf{R} = (\Omega, P, \mathcal{G}, \mathcal{F}_t)_{t\in\mathbf{R}_+}$. The example from [9] shows that $\Omega_\mathbf{R}$ can never be rich.

The paper [14] introduced the notion of adapted distribution and the related notion of a saturated adapted space, and it was shown that for each $\mathbf{B}$-adapted Loeb space $\Omega_\mathbf{B}$, the associated right continuous adapted space $\Omega_\mathbf{R}$ is saturated. The paper [18] explains the relationship between rich and saturated adapted spaces. It is shown that for a countable subset $\mathbf{L}$ of $\mathbf{R}_+$, an $\mathbf{L}$-adapted space is rich if and only if it is saturated. Moreover, if $\Omega_\mathbf{B}$ is a rich $\mathbf{B}$-adapted space, then the associated right continuous adapted space $\Omega_\mathbf{R}$ is saturated.

We conclude this section by looking at a particularly well behaved example of a $\mathbf{B}$-adapted Loeb space.

Example 5.18 *(The hyperfinite adapted space).*

Let Ω_0 be a hyperfinite set with more than one element, let N be an infinite hyperinteger, and let $\mathbf{B}_N$ be the hyperfinite set of all multiples of 2^{-N} between 0 and 2^N. Then $\mathbf{B}$ is a subset of $\mathbf{B}_N$. Let $\Omega = (\Omega_0)^{\mathbf{B}_N}$ be the hyperfinite set of all internal functions from $\mathbf{B}_N$ into Ω_0, let $\bar{\mathcal{G}}$ be the algebra of all internal subsets of Ω, and let $\bar{P}$ be the counting probability measure on $\bar{\mathcal{G}}$, which gives each element of Ω the same weight. If $\omega, \upsilon \in \Omega$ and $t \in \mathbf{B}_N$, we write $\omega \sim_t \upsilon$ if $\omega(s) = \upsilon(s)$ for all $s \leq t$ in $\mathbf{B}_N$. For $0 < t < \infty$ in $\mathbf{B}$ let $\bar{\mathcal{G}}_t$ be the algebra of all internal subsets U of Ω such that U is closed under the equivalence relation $\sim_t$. Finally, pick an infinitesimal $\iota \in \mathbf{B}_N$ and let $\bar{\mathcal{G}}_0$ be the algebra of all internal subsets U of Ω such that U is closed under $\sim_\iota$. Since $\bar{\mathcal{G}}$ is the algebra of all internal subsets of Ω, $SL^0(\Omega, M)$ is the set of all internal functions $X : \Omega \to {}^*M$. By the **hyperfinite adapted space determined by** Ω_0 and ι we mean the $\mathbf{B}$-adapted Loeb space $(\Omega, P, \mathcal{G}, \mathcal{G}_t)_{t \in \mathbf{B}}$.

Hyperfinite adapted spaces have been studied extensively in the literature, for example in [2], [15], [19], and [1].

One may wonder why we did not simply take $\iota = 0$, so that $\bar{\mathcal{G}}_0$ is just the algebra of all internal U closed under $\sim_0$. The problem with this choice is that it would make the algebra $\mathcal{G}_0$ be finite when Ω_0 is finite. We wish to allow the possibility that $(\Omega, P, \mathcal{G}_0)$ is atomless even when Ω_0 is finite, as in the next proposition.

Proposition 5.19 *([2]) Let Ω be the hyperfinite adapted space determined by Ω_0 and ι.*

(i) $\mathcal{G}_t$ is atomless over $\mathcal{G}_s$ for each $t > s$ in $\mathbf{B}$.

(ii) If either Ω_0 is infinite or $N\iota$ is infinite then $(\Omega, P, \mathcal{G}_0)$ is atomless, and Ω is an atomless $\mathbf{B}$-adapted space.

If either Ω_0 is infinite or $N\iota$ is infinite then the hyperfinite adapted space Ω determined by Ω_0 and ι is rich.

Proof: By Proposition 5.19 and Theorem 5.15. □

6 Function Spaces with Separable Targets

In this section we shall continue to let $\bar{\Omega}$ be a *probability space with associated Loeb probability space Ω, and let $M, N, \ldots$ be complete separable metric spaces.

We shall introduce another neometric family, whose underlying metric spaces are the spaces of Loeb measurable functions f from Ω into M such that the distance of f from any element of M is Loeb integrable. In the case that $M = \mathbf{R}$, the underlying metric space is just the space of Loeb integrable functions from Ω into $\mathbf{R}$. We shall identify each element $b \in M$ with the constant function $\Omega \times \{b\}$, which is an element of both $L^0(\Omega, M)$ and $ns^0(\Omega, M)$. If $X : \Omega \to {}^*\mathbf{R}$ is *measurable with respect to

$\bar{P}$, $\bar{E}[X]$ denotes the *expected value of X with respect to $\bar{P}$. If $x : \Omega \to \mathbf{R}$ is Loeb integrable, $E[x]$ is the expected value of x with respect to P.

We let $L^1(\Omega, M)$ be the metric space of all $x \in L^0(\Omega, M)$ such that $\rho(x(\cdot), a)$ is Loeb integrable for each $a \in M$, with the metric

$$\rho_1(x, y) = E[\rho(x(\cdot), y(\cdot))],$$

integrating with respect to the Loeb measure P on Ω.

We have $L^1(\Omega, M) \subset L^0(\Omega, M)$ as sets of equivalence classes of functions, but some care is needed because the two spaces have different metrics.

Recall that a function $F : \Omega \to {}^*\mathbf{R}$ is said to be S-**integrable** if F is internal, $\bar{E}[|F(\omega)|]$ is finite, and

$${}^o\bar{E}[|F(\omega)|] = \lim_{n\to\infty} {}^o\bar{E}[\min(|F(\omega)|, n)].$$

Loeb [23] showed that a function $f \in L^0(\Omega, \mathbf{R})$ is integrable with respect to P if and only if f has an S-integrable lifting F, in which case $E[f(\omega)] = {}^o\bar{E}[F(\omega)]$.

Definition 6.1 *Let $\bar{\rho}_1$ be the *metric on the set $SL^0(\Omega, M)$ defined by*

$$\bar{\rho}_1(X, Y) = \bar{E}[{}^*\rho(X(\cdot), Y(\cdot))].$$

(We allow the possibility that $\bar{\rho}_1(X, Y) = \infty$, but this possibility could be avoided by truncating $\bar{\rho}_1$ at a sufficiently small infinite hyperreal number J.) Let $SL^1(\Omega, M)$ be the set of all $X \in SL^0(\Omega, M)$ such that ${}^*\rho(X(\cdot), a)$ is S-integrable for each $a \in M$. We let $ns^1(\Omega, M)$ be the set

$$ns^1(\Omega, M) = ns^0(\Omega, M) \cap SL^1(\Omega, M).$$

In the following lemma we collect some easy consequences of the triangle inequality.

Lemma 6.2 *Let Ω be a Loeb probability space and M a complete separable metric space.*

(i) If $X \in L^0(\Omega, M)$ and ${}^*\rho(X(\cdot), a)$ is S-integrable for some $a \in M$, then ${}^*\rho(X(\cdot), b)$ is S-integrable for every $b \in M$.

(ii) If $X, Y \in SL^1(\Omega, M)$, then ${}^*\rho(X(\cdot), Y(\cdot))$ is S-integrable and $\bar{\rho}_1(X, Y)$ is finite.

(iii) If $X \in SL^1(\Omega, M), Y \in SL^0(\Omega, M)$, and $\bar{\rho}_1(X, Y) \approx 0$, then $Y \in SL^1(\Omega, M)$.

(iv) $SL^1(\Omega, \mathbf{R})$ is the set of all S-integrable $X \in SL^0(\Omega, \mathbf{R})$. □

We need the following known result from Loeb integration theory, which is analogous to Proposition 5.4 (see [1] or [28]).

Proposition 6.3 *For any $x \in \Omega \to M$, the following are equivalent:*

(i) $x \in L^1(\Omega, M)$.

(ii) x has a lifting in $ns^1(\Omega, M)$.

(iii) For each infinite $n \in {}^*\mathbf{N}$, x has a *simple lifting $X \in ns^1(\Omega, M)$ whose range has *cardinality at most n.

(iv) x is a limit of simple functions with respect to ρ_1.

Moreover, whenever X lifts x and Y lifts y in $ns^1(\Omega, M)$, we have

$$\rho_1(x, y) = {}^o\bar{\rho}_1(X, Y),$$

so that $\bar{\rho}_1$ is a uniform lifting of ρ_1, and $ns^1(\Omega, M)$ is the monad of $L^1(\Omega, M)$ in $(SL^0(\Omega, M), \bar{\rho}_1)$. □

Theorem 6.4 *The set $L^1(\Omega, M)$ is neoseparable with respect to the *metric $\bar{\rho}_1$ on $SL^0(\Omega, M)$, and the metric space $(L^1(\Omega, M), \rho_1)$ belongs to the huge neometric family* $\mathbf{H}$.

Proof: The argument is similar to the proof of Theorem 5.6. Let $\{m_0, m_1, \ldots\}$ be a countable dense subset of M, and let M_n be the finite set $M_n = \{m_0, m_1, \ldots, m_n\}$. Then $L^1(\Omega, M_n)$ is a basic set for each $n \in \mathbf{N}$. Every integrable function $x \in L^1(\Omega, M)$ can be approximated in the metric ρ_1 by functions in $L^1(\Omega, M_n)$, so $L^1(\Omega, M)$ is the closure of the countable union $\bigcup_n L^0(\Omega, M_n)$ of basic sets, and hence is neoseparable. □

The remaining results in this section give relationships between the neometric spaces $L^1(\Omega, M)$ and $L^0(\Omega, M)$ in the huge family $\mathbf{H}$. These results will show that the study of the neometric spaces $L^1(\Omega, M)$ can be reduced to the study of uniformly integrable sets in the neometric spaces $L^0(\Omega, M)$. This reduction allowed us to get by without introducing the neometric spaces on $L^1(\Omega, M)$ at all in the paper [9].

For $r \in \mathbf{R}$ and $n \in \mathbf{N}$ we let $\mathrm{tail}_n(r) = r$ if $r \geq n, \mathrm{tail}_n(r) = 0$ if $r < n$. Choose an element $b \in M$ which will remain fixed throughout our discussion. We say that a subset C of $L^1(\Omega, M)$ is **uniformly integrable** if there is a sequence a_n such that for each $x \in C$,

$$\lim_{n\to\infty} a_n = 0 \text{ and } E[\mathrm{tail}_n(\rho(x(\cdot), b))] \leq a_n \text{ for all } n \in \mathbf{N}. \tag{7}$$

Uniform integrability does not depend on the choice of the element $b \in M$.

We recall a well known characterization of S-integrability.

Lemma 6.5 *Let $f : \Omega \to {}^*\mathbf{R}$ be internal. The following are equivalent.*
(i) f is S-integrable.
(ii) $\bar{E}[|f|]$ is finite and $\bar{E}[tail_J(|f|)] \approx 0$ for all infinite $J \in {}^*\mathbf{N}$.
(iii) $\bar{E}[|f|]$ is finite and $\lim_{n\to\infty} {}^o(\bar{E}[tail_n(|f|)]) = 0$. $\square$

Theorem 6.6 *In the huge neometric family* $\mathbf{H}$, *a set $C \subset L^1(\Omega, M)$ is basic in $L^1(\Omega, M)$ if and only if it is basic in $L^0(\Omega, M)$ and uniformly integrable.*

Proof: Suppose first that C is basic in $L^1(\Omega, M)$. Then there is an internal set $A \subset ns^1(\Omega, M)$ such that $C = {}^oA$ with respect to the *metric $\bar{\rho}_1$. Then $A \subset ns^0(\Omega, M)$ and $C = {}^oA$ with respect to $\bar{\rho}_0$, so C is basic in $L^0(\Omega, M)$. For each n let

$$\bar{a}_n = \max\{\bar{E}[\mathrm{tail}_n(\bar{\rho}(X(\cdot), b))] : X \in A\},$$

and let $a_n = st(\bar{a}_n)$. By S-integrability, $\bar{a}_n$ is infinitesimal for each infinite n. Therefore by overspill, a_n satisfies condition (7) for all $x \in C$, whence C is uniformly integrable.

Now suppose C is basic in $L^0(\Omega, M)$ and uniformly integrable. There is an internal set $A \subset ns^0(\Omega, M)$ such that $C = {}^oA$ with respect to the *metric $\bar{\rho}_0$, and a sequence $\langle a_n \rangle$ such that (7) holds for all $x \in$ C. For each $Z \in {}^*M$ and $J \in {}^*\mathbf{N}$ let $\mathrm{trunc}_{J,b}(Z) = Z$ if $\bar{\rho}(Z, b) \le J$, $\mathrm{trunc}_{J,b}(Z) = 0$ otherwise. By (7), for each $n, J \in \mathbf{N}$, we have

$$(\forall X \in A)\bar{E}[\mathrm{tail}_n(\bar{\rho}(\mathrm{trunc}_{J,b}(X), b))] \le 2a_n. \tag{8}$$

By overspill, for each $n \in \mathbf{N}$, (8) holds for all sufficiently small infinite $J \in {}^*\mathbf{N}$. By ω_1-saturation, for all sufficiently small infinite J, (8) holds for all $n \in \mathbf{N}$, and therefore $\mathrm{trunc}_{J,b}(X)$ is S-integrable for all $X \in A$. Moreover, for each $X \in A$ and infinite J, $\mathrm{trunc}_{J,b}(X)$ lifts ${}^oX$ in the $\bar{\rho}_0$ metric. Thus for sufficiently small infinite J, the internal set $B = \{\mathrm{trunc}_{J,b}(X) : X \in A\}$ has the properties that $B \subset ns^1(\Omega, M)$ and $C = {}^oB$ with respect to $\bar{\rho}_1$. This shows that C is basic in $L^1(\Omega, M)$. $\square$

Theorem 6.7 *In the huge neometric family* $\mathbf{H}$, *a set $C \subset L^1(\Omega, M)$ is neocompact in $L^1(\Omega, M)$ if and only if it is neocompact in $L^0(\Omega, M)$ and uniformly integrable.*

Proof: Suppose first that C is neocompact in $L^1(\Omega, M)$. Then there is a Π^0_1 set $A = \bigcap_n A_n \subset ns^1(\Omega, M)$ such that $C = {}^oA$ with respect to the *metric $\bar{\rho}_1$. The proof that C is neocompact in $L^0(\Omega, M)$ and uniformly integrable is similar to the first paragraph of the preceding proof, but with

$$\bar{a}_n = max\{\bar{E}[\mathrm{tail}_n(\bar{\rho}(X(\cdot), b))] : X \in A_n\}.$$

Now suppose C is neocompact in $L^0(\Omega, M)$ and uniformly integrable. There is a Π_1^0 set $A = \bigcap_n A_n \subset ns^0(\Omega, M)$ such that $C = {}^oA$ with respect to the *metric $\bar{\rho}_0$, and a sequence $\langle a_n \rangle$ such that (7) holds for all $x \in C$. Let B_n be the internal set

$$B_n = \{X \in (A_n)^{1/n} : \bar{E}[\text{tail}_n(\bar{\rho}(X, b))] \leq 2a_n\}.$$

Then $\bigcap_n B_n \subset ns^1(\Omega, M)$ and $C = {}^o(\bigcap_n B_n)$, so C is neocompact in $L^1(\Omega, M)$. $\square$

Corollary 6.8 *In the huge neometric family* $\mathbf{H}$*, a set* $A \subset L^1(\Omega, M)$ *is neoclosed in* $L^1(\Omega, M)$ *if and only if* $A \cap B \in \mathcal{C}(\mathcal{M})$ *for each uniformly integrable set* $B \in \mathcal{C}(\mathcal{M})$. $\square$

Now consider the neometric family $(\mathbf{M}_\Omega, \mathcal{B}_\Omega, \mathcal{C}_\Omega)$ of neocompact sets for the probability space Ω, which was introduced in Definition 5.1.

Corollary 6.9 *Suppose* $C \subset L^1(\Omega, M)$ *and* $C \cap D \in \mathcal{C}_\Omega(\mathcal{M})$ *for each uniformly integrable set* $D \in \mathcal{C}_\Omega(\mathcal{M})$*. Then* C *is neoclosed in* $L^1(\Omega, M)$ *with respect to the huge neometric family* $\mathbf{H}$.

Proof: Let $B \in \mathcal{C}(\mathcal{M})$ be uniformly integrable. It is shown in [9] that B is contained in a uniformly integrable set which is neoclosed in the family $(\mathbf{M}_\Omega, \mathcal{B}_\Omega, \mathcal{C}_\Omega)$. By Corollary 5.8, B is also contained in a set in $\mathcal{C}_\Omega(\mathcal{M})$. Taking the intersection, we see that B is contained in a uniformly integrable set $D \in \mathcal{C}_\Omega(\mathcal{M})$. Then $C \cap D \in \mathcal{C}_\Omega(\mathcal{M})$, so $C \cap D \in \mathcal{C}(\mathcal{M})$ and hence $C \cap B = (C \cap D) \cap B \in \mathcal{C}(\mathcal{M})$. Therefore C is neoclosed in $L^1(\Omega, M)$. $\square$

Proposition 6.10 *In the huge neometric family* $\mathbf{H}$*, let* C *be neocompact in* $L^1(\Omega, M)$ *and let* $f : C \to \mathcal{N}$*. Then* f *is neocontinuous from* $L^1(\Omega, M)$ *to* $\mathcal{N}$ *if and only if* f *is neocontinuous from* $L^0(\Omega, M)$ *to* $\mathcal{N}$.

Proof: Let G be the graph of $f|C$. We must show that G is neocompact in $L^1(\Omega, M) \times \mathcal{N}$ if and only if G is neocompact in $L^0(\Omega, M) \times \mathcal{N}$. The implication from left to right is trivial. For the other direction, suppose that G is neocompact in $L^0(\Omega, M) \times \mathcal{N}$. Then monad$(G)$ is Π_1^0 with respect to $L^0(\Omega, M) \times \mathcal{N}$. Also, $C = {}^oD$ where D is Π_1^0 with respect to $L^1(\Omega, M)$ and $D \subset ns^1(\Omega, M)$. It follows that

$$A = \text{monad}(G) \cap (D \times \bar{N})$$

is a Π_1^0 set with respect to $L^1(\Omega, M) \times \mathcal{N}$ and $G = {}^oA$, so G is neocompact in $L^1(\Omega, M) \times \mathcal{N}$. $\square$

Corollary 6.11 *Let $\mathcal{M}, \mathcal{N} \in \mathbf{M}_\Omega$, and in the neometric family $(\mathbf{M}_\Omega, \mathcal{B}_\Omega, \mathcal{C}_\Omega)$ let $C \subset \mathcal{M}$ be neoclosed and let $f : C \to \mathcal{N}$ be neocontinuous on each uniformly integrable subset of C. Then in the huge neometric family $\mathbf{H}, f$ is neocontinuous from $L^1(\Omega, M)$ to $\mathcal{N}$.*

Proof: Let $B \subset C$ be neocompact in $L^1(\Omega, C)$. As in the proof of the last corollary, there is a uniformly integrable set $D \in \mathcal{C}_\Omega(\mathcal{M})$ such that $B \subset D \subset$ C. Then $f|D$ is neocontinuous in the neometric family $(\mathbf{M}_\Omega, \mathcal{B}_\Omega, \mathcal{C}_\Omega)$. Therefore in the huge neometric family $\mathbf{H}, f|D$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$. By the preceding proposition, $f|D$ is neocontinuous from $L^1(\Omega, M)$ to $\mathcal{N}$. Then $f|B$ is neocontinuous from $L^1(\Omega, M)$ to $\mathcal{N}$. $\square$

All the preceding notions and results in this section can be readily extended to the case of the space $L^p(\Omega, M)$ where $p \in [1, \infty)$, with uniform p-integrability in place of uniform integrability.

Many results involving uniformly integrable sets in the paper [9] can now be reinterpreted in terms of the neometric spaces $L^p(\Omega, M)$ in the huge neometric family. For example, the result that the conditional expectation function $f(x) = E[x|\mathcal{F}_t]$ is neocontinuous on each uniformly integrable set shows that f is neocontinuous from $L^1(\Omega, \mathbf{R})$ to $L^1(\Omega, L^1([0,1], \mathbf{R}))$. The result that the set $\mathcal{M}(\Omega, \mathbf{R}^d)$ of martingales $z \in L^2(\Omega, C([0,1], \mathbf{R}^d))$ with $z(\omega, 0) = 0$ meets each uniformly 2-integrable neocompact subset of $L^0(\Omega, C([0,1], \mathbf{R}^d))$ in a neocompact set shows that the set $\mathcal{M}(\Omega, \mathbf{R}^d)$ is neoclosed in the neometric space $L^2(\Omega, C([0,1], \mathbf{R}^d))$.

7 Function Spaces with Neoseparable Targets

In the literature on nonstandard analysis, one finds spaces of functions which take values in a neoseparable subspace of a nonstandard hull, rather than in a separable metric space. See, for example, [25] and [30]. In this section we shall see that such spaces are also neoseparable and belong to the huge neometric family. We shall consider spaces of functions from a Loeb probability space Ω to a neoseparable metric space $\mathcal{M}$ which belongs to the huge neometric family $\mathbf{H}$.

We continue to let $\bar{\Omega}$ be a *probability space with associated Loeb probability space Ω. In this section we suppose that $\mathcal{M}$ is a neoseparable subset of some nonstandard hull $\mathcal{H}(\bar{M}, c)$ where $(\bar{M}, \bar{\rho})$ is a *metric space, and ρ is the metric for $\mathcal{M}$.

Several examples of spaces $\mathcal{M} \in \mathbf{H}$ were given in the preceding sections. The case that $\mathcal{M}$ is a nonstandard hull of a *metric space $\bar{M}$ is of particular interest.

Definition 7.1 *A function $X : \bar{\Omega} \to \bar{M}$ is called $\bar{\mathcal{G}}$-***measurable** *if X is internal and $X^{-1}(U) \in \bar{\mathcal{G}}$ for all *open sets $U \subset \bar{M}$. $SL^0(\bar{\Omega}, \bar{M})$ is the *metric space of all $\bar{\mathcal{G}}$-measurable functions $X : \Omega \to \bar{M}$ with the *metric*

$$\bar{\rho}_0(X, Y) = {}^*\inf\{\varepsilon : \bar{P}[\bar{\rho}(X(\omega), Y(\omega)) \geq \varepsilon] \leq \varepsilon\}.$$

We say that $X \in SL^0(\bar{\Omega}, \bar{M})$ is a **lifting** *of a function $x : \Omega \to \mathcal{M}$, in symbols ${}^oX = x$, if $X(\omega)$ has standard part $x(\omega) \in \mathcal{M}$ for P-almost all $\omega \in \Omega$.*

Proposition 5.4 stated three equivalent conditions for a function to belong to the space $L^0(\Omega, M)$ of Loeb measurable functions from Ω into a separable metric space M. In the more general case that $\mathcal{M}$ is neoseparable, these conditions are no longer equivalent. The set of Loeb measurable functions from Ω into $\mathcal{M}$ need not be neoseparable or even contained in a nonstandard hull $\mathcal{H}(SL^0(\bar{\Omega}, \bar{M}), c)$, but we shall see that the three equivalent conditions lead to three neoseparable spaces.

We shall use the notation $\mathcal{L}^0(\Omega, \mathcal{M})$ for the largest of these spaces, the space of all functions $x : \Omega \to \mathcal{M}$ such that x has a lifting in $SL^0(\bar{\Omega}, \bar{M})$. We give $\mathcal{L}^0(\Omega, \mathcal{M})$ the metric ρ_0 such that $\rho_0(x, y) = {}^o\bar{\rho}_0(X, Y)$ whenever $X, Y \in SL^0(\bar{\Omega}, \bar{M})$, X lifts x, and Y lifts y. As usual, we identify two functions $x, y : \Omega \to \mathcal{M}$ if they are equal P-almost everywhere.

A function $x \in \mathcal{L}^0(\Omega, \mathcal{M})$ is **simple** if it has finite range, and a function $X \in SL^0(\bar{\Omega}, \bar{M})$ is ***simple** if it has *finite range.

Theorem 7.2 *Let $(\mathcal{M}, \rho)$ be neoseparable in the huge neometric family* **H**. *Then each of the following three metric spaces are neoseparable and belong to* **H**.

(i) The space $\mathcal{N} = (\mathcal{L}^0(\Omega, \mathcal{M}), \rho_0)$.

(ii) The subspace

$$\mathcal{N}_1 = \{x \in \mathcal{N} : x \textit{ has a } {}^*\textit{simple lifting with respect to } \bar{\rho}_0\}.$$

(iii) The subspace

$$\mathcal{N}_2 = \{x \in \mathcal{N} : x \textit{ is a } \rho_0\textit{-limit of simple functions}\}.$$

Moreover, $\mathcal{N}_2 \subset \mathcal{N}_1 \subset \mathcal{N}$.

Proof: (i) The monad of $\mathcal{M}$ has the neoseparable normal form

$$\operatorname{monad}(\mathcal{M}) = \bigcap_n \bigcup_k (\bar{M}_k)^{1/n},$$

where $\bar{M}_k$ is an increasing chain of internal subsets of $\bar{M}$. Then

$$\bar{N}_k = SL^0(\bar{\Omega}, \bar{M}_k)$$

is internal. For each $X \in SL^0(\bar{\Omega}, \bar{M})$, the following are equivalent with respect to $\bar{\rho}_0$:

$$ {}^oX \in \mathcal{N}, $$

$$ {}^oX(\omega) \in \mathcal{M}\ P\text{-almost surely}, $$

$$ X(\omega) \in \bigcap_n \bigcup_k ((\bar{M}_k)^{1/n})\ P\text{-almost surely}, $$

$$ \bar{P}[X(\omega) \in \bigcup_k ((\bar{M}_k)^{1/n})] \geq 1 - 1/n \text{ for each } n, $$

$$ (\forall n \in \mathbf{N})(\exists k \in \mathbf{N})\bar{P}[X(\omega) \in ((\bar{M}_k)^{1/n})] \geq 1 - 1/n, $$

$$ (\forall n \in \mathbf{N})(\exists k \in \mathbf{N})X \in ((\bar{N}_k)^{1/n}), $$

$$ X \in \bigcap_n \bigcup_k ((\bar{N}_k)^{1/n}). $$

Therefore the monad of $\mathcal{N}$ is equal to $\bigcap_n \bigcup_k (\bar{N}_k)^{1/n}$, so $\mathcal{N}$ is neoseparable.

The above proof gives an explicit representation of $\mathcal{N}$ as a neoseparable set; it shows that $\mathcal{N}$ is the ρ_0-closure of the standard part of the Σ_1^0 set $\bigcup_k \bar{N}_k$.

(ii) Argue as in (i) but restrict everything to the internal set of all *simple elements of $SL^0(\bar{\Omega}, \bar{M})$.

(iii) To show that the set is neoseparable, argue as in (i) but replace N_k by the set J_k of all $X \in N_k$ such that the range of X has cardinality at most k. The fact that every limit of simple functions has a *simple lifting is a generalization of a result in [25] and [30]. Let x be a limit of simple functions x_n. We may choose x_n so that $\rho_0(x_n, x) \leq 1/n$. For each n choose a simple lifting X_n of x_n. By saturation there is a *simple function X such that $\bar{\rho}_0(X_n, X) \leq 2/n$ for each n. It follows that X is a lifting of x. $\square$

In the special case that $\mathcal{M}$ is a standard complete separable metric space M, we have $\mathcal{N}_2 = \mathcal{N}_1 = \mathcal{N} = L^0(\Omega, M)$ in the above theorem, and the result reduces to Theorem 5.6. In the case that $\bar{\Omega}$ is *finite, every $\bar{\mathcal{G}}$-measurable function $X : \bar{\Omega} \to \bar{M}$ is *simple, so $\mathcal{N}_1 = \mathcal{N}$.

If M is a standard metric space whose cardinality is less than the first real-valued measurable cardinal, then every Borel probability measure on M has separable support (cf. [5]), and it follows that for every standard probability space $(\Omega, P, \mathcal{G})$, every $\mathcal{G}$-measurable function $X : \Omega \to M$ is a ρ_0-limit of simple functions. By transfer, if the cardinality of $\bar{M}$ is less than the first real-valued measurable cardinal in the sense of the nonstandard universe, then every $\bar{\mathcal{G}}$-measurable function $X : \bar{\Omega} \to \bar{M}$ is a $\bar{\rho}_0$-limit of *simple functions, and hence $\mathcal{N}_1 = \mathcal{N}$ in the above theorem.

We now obtain an analogous result for L^1 spaces.

Definition 7.3 *Let $\bar{\rho}_1$ be the *metric on the set $SL^0(\bar{\Omega}, \bar{M})$ defined by $\bar{\rho}_1(X,Y) = \bar{E}[\bar{\rho}(X(\cdot), Y(\cdot))]$. For each $c \in \bar{M}$, let $SL^1(\bar{\Omega}, \bar{M}, c)$ be the set of all $X \in SL^0(\bar{\Omega}, \bar{M})$ such that $\bar{\rho}(X(\cdot), c)$ is S-integrable.*

By a triangle inequality argument, we see that if b and c are in the same galaxy of $\bar{M}$, then

$$SL^1(\bar{\Omega}, \bar{M}, b) = SL^1(\bar{\Omega}, \bar{M}, c).$$

Moreover, $SL^1(\bar{\Omega}, \bar{M}, c)$ is closed under the relation $\bar{\rho}_1(X,Y) \approx 0$.

Let c be any point in the monad of $\mathcal{M}$. We let $\mathcal{L}^1(\Omega, \mathcal{M})$ denote the metric space of all functions $x : \Omega \to \mathcal{M}$ such that x has a lifting in $SL^1(\bar{\Omega}, \bar{M}, c)$, with the metric ρ_1 such that $\rho_1(x,y) = {}^o\bar{\rho}_1(X,Y)$ whenever $X, Y \in SL^1(\bar{\Omega}, \bar{M}, c)$, X lifts x, and Y lifts y.

Theorem 7.4 *Let $(\mathcal{M}, \rho)$ be neoseparable in the huge neometric family* **H**. *Then each of the following three metric spaces is neoseparable and belongs to* **H**.

(i) The space $\mathcal{K} = \mathcal{L}^1(\Omega, \mathcal{M})$.

(ii) The subspace

$$\mathcal{K}_1 = \{x \in \mathcal{K} : x \textit{ has a } {}^*\textit{simple lifting with respect to } \bar{\rho}_1\}.$$

(iii) The subspace

$$\mathcal{K}_2 = \{x \in \mathcal{K} : x \textit{ is a limit of simple functions in the } \rho_1 \textit{metric}\}.$$

Moreover, $\mathcal{K}_2 \subset \mathcal{K}_1 \subset \mathcal{K}$.

Proof: (i) Let $c \in \mathcal{M}$. Since $\mathcal{M}$ is neoseparable we may take an increasing chain $\bar{M}_k$ of internal subsets of $\bar{M}$ such that

$$\text{monad}(\mathcal{M}) = \bigcap_n \bigcup_k (\bar{M}_k)^{1/n},$$

and $\bar{M}_k$ contains c and is contained in the closed ball

$$\{z \in \mathcal{M} : \bar{\rho}(z,c) \le k\}.$$

Let $\bar{N}_k$ be the internal set $SL^0(\bar{\Omega}, \bar{M}_k)$. We claim that for each $X \in SL^0(\bar{\Omega}, \bar{M})$, the following are equivalent:

1. ${}^oX \in \mathcal{K}$,
2. $X \in SL^1(\bar{\Omega}, \bar{M}, c)$ and ${}^oX(\omega) \in \mathcal{M} P$-almost surely,

3. $X \in SL^1(\bar{\Omega}, \bar{M}, c)$ and $(\forall n \in \mathbf{N})(\exists k \in \mathbf{N})(\bar{\rho}_0(X, \bar{N}_k) \leq 1/n)$.

4. $(\forall n \in \mathbf{N})(\exists k \in \mathbf{N})(\bar{\rho}_1(X, \bar{N}_k) \leq 1/n)$.

The equivalence of 1, 2, and 3 follow from the preceding proof. Assume 3, and let $n \in \mathbf{N}$. Whenever $j \leq k$ we have

$$\bar{\rho}_1(X, \bar{N}_k) \leq \bar{E}[\mathrm{tail}_j(\bar{\rho}(X, c))] + (j+1)\bar{\rho}_0(X, \bar{N}_k).$$

Take j so that $\bar{E}[\mathrm{tail}_j(\bar{\rho}(X, c))] \leq 1/2n$ and k so that $(j+1)\bar{\rho}_0(X, \bar{N}_k) \leq 1/2n$. Then $\bar{\rho}_1(X, \bar{N}_k) \leq 1/n$, so 4 holds.

Assume 4. We have

$$(\bar{\rho}_0(X, \bar{N}_k))^2 \leq \bar{\rho}_1(X, \bar{N}_k),$$

and thus

$$(\forall n \in \mathbf{N})(\exists k \in \mathbf{N})(\bar{\rho}_0(X, \bar{N}_k) \leq 1/n).$$

Since $\bar{M}_k$ is contained in the closed ball of radius k centered at c, for all $Z \in \bar{M}$ such that $\bar{\rho}(Z, c) \geq 2k$ we have

$$\bar{\rho}(Z, c) \leq \bar{\rho}(Z, \bar{M}_k) + k \leq \bar{\rho}(Z, \bar{M}_k) + \bar{\rho}(Z, c)/2,$$

and hence $\bar{\rho}(Z, c) \leq 2\bar{\rho}(Z, \bar{M}_k)$. Then

$$\bar{E}[\mathrm{tail}_2 k(\bar{\rho}(X, c))] \leq 2\bar{\rho}_1(X, \bar{N}_k). \tag{9}$$

It follows from 4 that $\lim_{k\to\infty} {}^o\bar{\rho}_1(X, \bar{N}_k) = 0$, so the left side of (9) converges to 0 and $X \in SL^1(\bar{\Omega}, \bar{M}, c)$. This proves 3.

Therefore

$$\mathrm{monad}(\mathcal{K}) = \bigcap_n \bigcup_k ((\bar{N}_k)^{1/n})$$

with respect to the *metric $\bar{\rho}_1$, so $\mathcal{K}$ is neoseparable.

The proof also shows that $\mathcal{K}$ is the ρ_1-closure of the standard part of the Σ_1^0 set $\bigcup_k \bar{N}_k$.

The proofs of (ii) and (iii) are similar to the proofs in the preceding theorem. □

Again, we have $\mathcal{K}_2 = \mathcal{K}_1 = \mathcal{K}$ if $\mathcal{M}$ is a standard complete separable metric space. Moreover, if $\bar{\Omega}$ is *finite or $\bar{M}$ is smaller than the first real-valued measurable cardinal in the sense of the nonstandard universe, then $\mathcal{K}_1 = \mathcal{K}$.

Analogous results hold for L^p spaces where $p \in [1, \infty)$.

8 κ-Saturated Nonstandard Universes

Throughout this section we let κ be an uncountable regular cardinal. We shall see that in a κ-saturated nonstandard universe, the huge neometric family is not only countably compact but has a stronger property called κ-compact. We shall also introduce the notion of a κ-neoseparable set and obtain analogues of many of our results on neoseparable sets. The proofs are straightforward generalizations of the proofs we have given for the case $\kappa = \omega_1$, and will be omitted. Our main reason for carrying out this generalization is that it extends the theory to spaces $L^0(\Omega, M)$ where M is a standard space which is not separable.

Definition 8.1 *A κ-neocompact family* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *is defined in the same way as a neocompact family except that condition (c) is replaced by*

(c_κ) $\mathcal{C}(\mathcal{M})$ *is closed under intersections of fewer than κsets.*

Thus a neocompact family is the same as an ω_1-neocompact family.

*A κ-***neometric family** *is a κ-neocompact family in which the projection and distance functions are neocontinuous for every $\mathcal{M}$ and $\mathcal{N}$.*

A collection of sets $\{C_i : i \in I\}$ is said to be **downward directed** if for all $i, j \in I$ there exists $k \in I$ such that $C_k \subset C_i \cap C_j$. We say that a κ-neocompact family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ has the κ-**compactness property** if for each $\mathcal{M} \in \mathbf{M}$, every downward directed collection $\{C_i : i \in I\}$ of fewer than κ nonempty sets in $\mathcal{C}(\mathcal{M})$ has a nonempty intersection $\bigcap_n C_n$.

A subset C of a metric space $\mathcal{M} \in \mathbf{M}$ is said to be κ-**separable** if C has a dense subset of size $< \kappa$, and κ-**neoseparable** if it is the closure of the union of a set of fewer than κ basic sets in $\mathcal{M}$.

Proposition 8.2 *Let* $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ *be a κ-neometric family, let $\mathcal{M} \in \mathbf{M}$, and let C be a κ-separable subset of $\mathcal{M}$. Then C is neocompact in $\mathcal{M}$ if and only if C is compact.*
□

We assume hereafter that $(V(\Xi), V({}^*\Xi), {}^*)$ is a κ-saturated nonstandard universe, that is, a nonstandard universe such that for any internal set S, any family of fewer than κ internal subsets of S which has the finite intersection property has a nonempty intersection. By a $\Sigma_1^0(\kappa)$ set in a *metric space $(\bar{M}, \bar{\rho})$ we mean the union of fewer than κ internal subsets of the galaxy $G(\bar{M}, c)$. $\Pi_1^0(\kappa)$ sets are defined analogously.

Definition 8.3 *The* κ-**huge neometric family** $(\mathbf{H}_\kappa, \mathcal{B}_\kappa, \mathcal{C}_\kappa)$ *for the κ-saturated nonstandard universe* $(V(\Xi), V({}^*\Xi), {}^*)$ *is defined as follows.* $\mathbf{H}_\kappa$ *is the class of* *all metric spaces $(\mathcal{M},\rho)$ such that $\mathcal{M}$ is a closed subset of some nonstandard hull $\mathcal{H}(\bar{M},c)$.*

For each $\mathcal{M}\in\mathbf{H}_\kappa$, the collections of basic and neocompact subsets of $\mathcal{M}$ are

$$\mathcal{B}_\kappa(\mathcal{M})=\{A\subset\mathcal{M}:A={}^oB \text{ for some internal set } B\subset G(\bar{M},c)\},$$

$$\mathcal{C}_\kappa(\mathcal{M})=\{A\subset\mathcal{M}:A={}^oB \text{ for some } \Pi^0_1(\kappa) \text{ set } B\subset G(\bar{M},c)\}.$$

Theorem 8.4 *$(\mathbf{H}_\kappa,\mathcal{B}_\kappa,\mathcal{C}_\kappa)$ is a κ-neometric family with the κ-compactness property.* $\square$

Note that a set $A\subset\mathcal{H}(\bar{M},c)$ is κ-neoseparable in $\mathbf{H}_\kappa$ if and only if A is the closure of the standard part of a $\Sigma^0_1(\kappa)$ set (in the topology of the nonstandard hull $\mathcal{H}(\bar{M},c)$).

Proposition 8.5 *Every closed κ-separable subset of $\mathcal{H}(\bar{M},c)$ is κ-neoseparable in $\mathbf{H}_\kappa$.* $\square$

Proposition 8.6 *Let M be a standard complete metric space. The following are equivalent:*

(i) M is κ-separable.

(ii) M is κ-neoseparable in $\mathbf{H}_\kappa$. $\square$

A function $f:\Omega\to M$ is said to be Loeb measurable if $f^{-1}(A)$ is Loeb measurable for every open set $A\subset M$. $L^0(\Omega,M)$ is the space of all Loeb measurable functions from Ω into M with the ρ_0 metric. $L^1(\Omega,M)$ is the set of Loeb integrable functions with the ρ_1 metric. The sets $SL^0(\Omega,M)$ and $SL^1(\Omega,M)$ are defined as before where M is a standard complete metric space. We need the following generalization of Proposition 5.4 for the case that M is a κ-separable complete metric space.

Proposition 8.7 *Let Ω be a Loeb probability space with a measure P and let M be a standard κ-separable complete metric space.*

(i) (Anderson [3]). If $X\in ns^0(\Omega,M)$ then ${}^oX\in L_0(\Omega,M)$ (that is, ${}^oX$ is Loeb measurable).

(ii) (Ross [27]). Suppose $x\in L_0(\Omega,M)$. Then $x={}^oX$ for some $X\in ns^0(\Omega,M)$, and there is a set $A\subset\Omega$ of Loeb measure one such that $x(A)$ is separable.

(iii) (Implicit in [3]) If $X,Y\in ns^0(\Omega,M)$ then $\rho_0({}^oX,{}^oY)={}^o\bar{\rho}_0(X,Y)$. $\square$

Theorem 8.8 *For every Loeb probability space Ω and standard κ-separable complete metric space M, the metric space $\mathcal{M}=L^0(\Omega,M)$ is κ-neoseparable in $\mathbf{H}_\kappa$.* $\square$

Definition 8.9 *Let $\Omega = (\Omega, P, \mathcal{G}_t)_{t \in \mathbf{B}}$ be an adapted Loeb space in a κ-saturated nonstandard universe Ξ. Let $\mathbf{M}_{\kappa,\Omega}$ be the set of all metric spaces $L^0(\Omega, M)$ where M is a standard complete κ-separable metric space. Ω is a κ-**rich adapted space** if the measure P is atomless on $\mathcal{G}_0, \Omega$ admits a Brownian motion with respect to the filtration $\mathcal{F}_t, t \in \mathbf{R}_+$, and the κ-neocompact family generated by $(\mathbf{M}_{\kappa,\Omega}, \mathcal{B}_\Omega)$ has the κ-compactness property.*

Theorem 8.10 *If Ω_0 is infinite then the hyperfinite adapted space Ω associated with Ω_0 is a κ-rich adapted space.* $\square$

Theorem 8.11 *For every Loeb probability space Ω and standard complete κ-separable metric space M, the metric space $L^1(\Omega, M)$ is κ-neoseparable in $\mathbf{H}_\kappa$.* $\square$

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