

Attractors and neo-attractors for 3D stochastic Navier-Stokes equations

Nigel J. Cutland and H.Jerome Keisler

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Abstract

In [14] nonstandard analysis was used to construct a (standard) global attractor for the 3D stochastic Navier–Stokes equations with general multiplicative noise, living on a Loeb space, using Sell’s approach [26]. The attractor had somewhat *ad hoc* attracting and compactness properties. We strengthen this result by showing that the attractor has stronger properties making it a *neo-attractor* – a notion introduced here that arises naturally from the Keisler-Fajardo theory of neometric spaces [18].

To set this result in context we first survey the use of Loeb space and nonstandard techniques in the study of attractors, with special emphasis on results obtained for the Navier–Stokes equations both deterministic and stochastic, showing that such methods are well-suited to this enterprise.

KEYWORDS attractor, neo-attractor, stochastic Navier–Stokes equations, Loeb space

1 Introduction

The chief topic of this paper is the study of attractors for the time-homogeneous *stochastic Navier-Stokes (sNS) equations* in a bounded domain $\mathbb{D} \subset \mathbb{R}^d$, principally for $d = 3$. These are equations that describe the velocity of an incompressible fluid at each point in \mathbb{D} ; a general version of the sNS equations is

$$\begin{cases} du = [\nu\Delta u - \langle u, \nabla \rangle u + f(u) - \nabla p] dt + g(u) dw_t \\ \operatorname{div} u = 0 \end{cases} \quad (1)$$

Here $u = u(t, \cdot, \cdot)$ where $u(t, x, \omega)$ is the (random) velocity of the fluid at the location $x \in \mathbb{D}$ at time t , so that we have

$$u : [0, \infty) \times \mathbb{D} \times \Omega \rightarrow \mathbb{R}^d$$

where Ω is the domain of an underlying probability space. The initial condition $u(0, \cdot, \cdot) = u_0$ is prescribed (and may be random); the boundary condition is either $u(t, x, \omega) = 0$ for $x \in \partial\mathbb{D}$ or, occasionally, when $d = 2$ we assume periodic boundary conditions. The term p denotes the pressure and ν is the viscosity. The terms f and g represent external forces influencing the fluid, which allow for feedback involving the velocity field u but for the present discussion are homogeneous in time.

The *deterministic* Navier–Stokes equations are the result of taking $g = 0$ and u non-random – so we simply have $u = u(t, x)$.

Although the theory of attractors for 2D *deterministic* Navier–Stokes equations is well understood (see [28] for an exposition), in higher dimensions, or when noise is introduced into the system, or both, even the existence of attractors is still problematic. Nonstandard techniques have proved to be useful in the study of the problems to be addressed. For example, in the paper [14], a new notion of a *process attractor* was introduced. A process attractor $A \subseteq X$ was constructed for a family X of solutions to the 3D stochastic Navier–Stokes equations with a general multiplicative noise. This made essential use of a filtered Loeb probability space using methods from nonstandard analysis, and followed earlier papers [6], [7], [9] that show the usefulness of nonstandard analysis in the study of attractors for Navier–Stokes equations – in all cases giving completely *standard* results. Such techniques are natural because attractors have to do with the behaviour of a system “at infinity” – a notion which can be handled easily in the extended framework of nonstandard analysis. In the case of the Navier–Stokes equations a further advantage stems from the fact that nonstandard analysis has proved to be an effective tool for the study of these equations – both deterministic [3] and stochastic [4]; see the book [8] for a complete exposition.

The purpose of the current paper is two-fold. The first (Part I) is to amplify the remark above by giving a brief survey of the way in which nonstandard analysis provides a useful tool for the study of attractors in general and specifically for the Navier–Stokes equations – both deterministic and stochastic. We include a new and simple proof of Sell’s result establishing the existence of an attractor for 3D Navier–Stokes equations, and its characterization in terms of two-sided solutions. This then provides the context for Part II of the paper.

The second purpose, in Part II, making this a sequel to [14], is to strengthen the results of that paper by introducing the notion of a *neo-attractor* for a system of stochastic Navier–Stokes equations. This is a stronger notion than that discussed in [14], and we show that the attractor A of that paper is actually a neo-attractor. The upshot of this is the existence of an attractor for the 3D stochastic Navier–Stokes equations that has new, stronger and more natural compactness and attraction properties. The general theory of neoattractors that is developed also gives the existence of a neoattractor for any specialized subset Y of the family X having natural closure properties.

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an earlier version of this paper to include the survey to put the new results in context.

1.1 Outline of the paper

In Part I (Section 2) we take a general definition of an attractor for a dynamical system and show how nonstandard analysis can be used to prove one of the basic existence theorems. This is applied in Section 3 to give a simple proof of Sell's existence theorem for an attractor for the 3D deterministic Navier-Stokes equations.

Part II begins with a discussion of the general problem of defining attractors for stochastic systems (Section 4) followed by an outline of the existence results for these that was established in the papers [7], [9] and [14].

We then proceed to new ideas and results. In [14] the compactness and attraction properties of the attractor \mathbb{A} for general 3D stochastic Navier-Stokes equations were somewhat *ad hoc* and unsatisfactory. The class of open neighbourhoods of \mathbb{A} that absorb bounded sets was rather unnatural, while the compactness property required of an attractor was defined in terms of laws of the processes in \mathbb{A} rather than in terms of the set \mathbb{A} itself. In Section 5 we use the Keisler-Fajardo theory of neometric spaces from [18] to define a stronger and more natural notion called a *neo-attractor*. The open neighbourhoods of a neo-attractor that absorb bounded sets are the *neo-open sets*, while the attractor itself is required to be *neo-compact*. We prove a general abstract existence theorem that isolates sufficient conditions for the existence of a neo-attractor. In Section 7 it is shown that the family of solutions X in the paper [14] satisfies these conditions, and that the attractor A of that paper is a neo-attractor in the sense of this paper. The general existence theorem also gives existence of a neo-attractor for any subset Y of solutions having natural closure properties.

Part II depends in an essential way on [14] although we will outline in Section 6 the main ideas to make this paper as self-contained as possible.

1.2 Preliminaries

In Section 2 below we outline the basic ideas of global attractors in a deterministic setting. The book [28] is a good reference for a general background on attractors.

The book [8] for background on stochastic Navier-Stokes equations, and any of the books [1, 2, 13] for more background on nonstandard analysis and Loeb measures. The appendices provide a brief summary of the notions needed from nonstandard analysis, and a short account of the theory of neometric spaces that is needed here. In order to assist the reader we use sans serif symbols (x, y, u, v, T etc) for nonstandard objects and the symbols σ, τ to denote nonstandard time.

PART I: DETERMINISTIC SYSTEMS

2 Attractors

A setting for the study of attractors that includes all particular instances in this paper is that of a dynamical system in a metric space \mathcal{M} described by a semigroup $(S_t)_{t \geq 0}$ of operators, which are assumed to be continuous. To make this precise we have (drawing largely on the exposition in [28])

Definition 2.1 A *semiflow* on a metric space \mathcal{M} is a semigroup of continuous operators $S_t : \mathcal{M} \rightarrow \mathcal{M}$ ($t \geq 0$) such that

- (a) $S_0 = I$ (the identity mapping),
- (b) $S_{t+s} = S_t \circ S_s$ for all $s, t \geq 0$,
- (c) S_t is continuous for all $t \geq 0$.

(Some authors require the resulting mapping $S : [0, \infty) \times \mathcal{M} \rightarrow \mathcal{M}$ to be continuous, but we do not need this.)

The notion of an attractor for such a system is concerned with the asymptotic behaviour of its trajectories. There is a variety of notions of an attractor in the literature, each with its own rationale. For our purposes we adopt the following fairly strong definition.

Definition 2.2 A *global attractor* for a semiflow $(S_t)_{t \geq 0}$ is a set $\mathbb{A} \subseteq \mathcal{M}$ such that

- (a) (**Invariance**) $S_t \mathbb{A} = \mathbb{A}$ for all t .
- (b) \mathbb{A} is compact
- (c) (**Attraction**) For every open neighbourhood U of \mathbb{A} and every bounded set $B \subseteq \mathcal{M}$,

$$S_t B \subseteq U$$

eventually (meaning that there is $t_0 = t_0(B, U)$ such that $S_t B \subseteq U$ for all $t \geq t_0$).

Remarks

1. Some authors describe such an attractor as a (global) set attractor (in contrast to a *point attractor*, where the attraction property (c) applies only to singleton sets $B = \{x\}$).

2. If there is a global attractor it is unique. To see this consider $x \in \mathbb{A}' \setminus \mathbb{A}$ where \mathbb{A}' is another global attractor. Take an open neighbourhood U of \mathbb{A} that excludes x . Since \mathbb{A}' is bounded there is t with $S_t \mathbb{A}' \subseteq U$, which is impossible since $S_t \mathbb{A}' = \mathbb{A}'$ for all t .

A fairly general existence result for global attractors (as given in [28] for example) involves the notion of an *absorbing set*. From the various definitions that appear in the literature the following is appropriate for our needs.

Definition 2.3 *A set E is an **absorbing set** for the semiflow $(S_t)_{t \geq 0}$ if for every bounded set $B \subseteq \mathcal{M}$*

$$S_t B \subseteq E$$

eventually.

To illustrate the applicability of nonstandard techniques for the study of attractors we give a proof of the following variation of a theorem that appears in [28] (Theorem 1.1) .

Theorem 2.4 *Suppose that \mathcal{M} is a metric space with a semiflow $(S_t)_{t \geq 0}$ having a bounded absorbing set E . Assume further that there is some $t > 0$ such that $S_t E$ is relatively compact (i.e. has compact closure). Then there is a global \mathbb{A} attractor for the semiflow given by*

$$\mathbb{A} = \bigcap_{t \in [0, \infty)} S_t(E) \tag{2}$$

Remarks 1. It is clear that for any two bounded absorbing sets E, E' we have

$$\bigcap_{t \in [0, \infty)} S_t(E) = \bigcap_{t \in [0, \infty)} S_t(E')$$

2. It is easy to see that the assumption on the absorbing set E implies the following apparently stronger condition that often appears in the literature:

for each bounded set $B \subseteq \mathcal{M}$ there is $t > 0$ such that $S_t B$ is relatively compact.

3. If $S_t E \subseteq K$ with K compact then clearly K is a *compact* absorbing set – so this apparently stronger hypothesis is also implicit.

Proof. First, as noted above, we may assume that E is compact, so that $*E \subseteq \text{ns}(\mathcal{M})$. Let

$$\mathbb{A} = \bigcap_{t \in [0, \infty)} S_t(E)$$

Then each $S_t(E)$ is compact (since S_t is continuous) so \mathbb{A} is also compact.

Write $\mathbb{T} = *S$, so that we have \mathbb{T}_τ for all $\tau \geq 0, \tau \in * \mathbb{R}$, and observe that:

($\#$) if $x \in *E$ and τ is infinite then $\mathbb{T}_\tau x$ is nearstandard and ${}^\circ \mathbb{T}_\tau x \in \mathbb{A}$.

To see this we have

$$\mathbb{T}_\sigma \mathbf{x} \in {}^*E$$

and so

$${}^\circ(\mathbb{T}_\sigma \mathbf{x}) \in E$$

for any infinite σ by the absorbing property of E and so for any finite t

$$\begin{aligned} {}^\circ(\mathbb{T}_\tau \mathbf{x}) &= {}^\circ(\mathbb{T}_t \mathbb{T}_{\tau-t} \mathbf{x}) \\ &= S_t {}^\circ(\mathbb{T}_{\tau-t} \mathbf{x}) \in S_t E \end{aligned}$$

where we have used the continuity of S_t , and hence ${}^\circ(\mathbb{T}_\tau \mathbf{x}) \in \mathbb{A}$.

To see that \mathbb{A} is invariant take any $a \in \mathbb{A}$. Then $a = \mathbb{T}_\tau \mathbf{x}$ for some infinite τ and $\mathbf{x} \in {}^*E$, so (\sharp) and the continuity of S_t gives

$$S_t a = {}^\circ \mathbb{T}_t a = {}^\circ (\mathbb{T}_t \mathbb{T}_\tau \mathbf{x}) = {}^\circ (\mathbb{T}_{\tau+t} \mathbf{x}) \in \mathbb{A}$$

and

$$a = {}^\circ a = {}^\circ (\mathbb{T}_t \mathbb{T}_{\tau-t} \mathbf{x}) = S_t {}^\circ (\mathbb{T}_{\tau-t} \mathbf{x}) \in S_t \mathbb{A}$$

which shows that $\mathbb{A} = S_t \mathbb{A}$.

Finally we show that \mathbb{A} attracts bounded sets. Take an open neighbourhood U of \mathbb{A} ; it is sufficient to show that $S_t E \subseteq U$ eventually. If this fails then there is a sequence of points $x_n \in E$ with $S_n x_n \notin U$. Then for any infinite N we have ${}^\circ \mathbb{T}_N x_N \in \mathbb{A}$ but $\mathbb{T}_N x_N \notin {}^*U$; since U is open this means that ${}^\circ \mathbb{T}_N x_N \notin U$, a contradiction. ■

Remark It can also be shown that the attractor \mathbb{A} is given by

$$\mathbb{A} = {}^\circ \bigcup_{\tau \text{--infinite}} \mathbb{T}_\tau {}^*E = {}^\circ \bigcap_{n \in \mathbb{N}} \bigcup_{\tau \geq n} \mathbb{T}_\tau {}^*E$$

which was essentially the way an attractor was defined in the papers [7],[9] for example.

Theorem 2.4 has a counterpart in the theory of neo-attractors to be developed later. The same is true for the next result, which gives necessary and sufficient conditions for the existence of a global attractor. We have not been able to find this stated explicitly in the literature although it is easy to derive using standard techniques. For interest we provide a nonstandard proof, using the fact that a sequence (s_n) is relatively compact in M iff and only if *s_N is nearstandard for each infinite N .

Theorem 2.5 *Suppose that \mathcal{M} is a metric space with a semiflow $(S_t)_{t \geq 0}$. The following are equivalent.*

- (a) *There is a global attractor for the semiflow $(S_t)_{t \geq 0}$;*

- (b) *There is a bounded absorbing set E such that for any sequence (x_n) in E and $t_n \rightarrow \infty$ the set $\{S_{t_n}x_n\}$ is relatively compact.*

If (a) (or (b)) holds the global attractor \mathbb{A} is given by

$$\mathbb{A} = \bigcap_{t \geq 0} S_t E \quad (3)$$

for any bounded absorbing set E .

Proof. (a) \Rightarrow (b) For each n let

$$E_n = \mathbb{A}^{\frac{1}{n}} = \{x : d(x, \mathbb{A}) < \frac{1}{n}\}$$

which is open and therefore an absorbing set that is bounded (because \mathbb{A} is compact hence bounded).

Let $E = E_1$, and let $\mathbb{T} = {}^*S$. Take a sequence $x_n \in E$ and $t_n \rightarrow \infty$ and consider an infinite N . We need to show that $\mathbb{T}_{t_N}x_N$ is nearstandard. Since each E_m is absorbing, $\mathbb{T}_{t_N}x_N \in {}^*E_m$ for each finite m and so $\mathbb{T}_{t_N}x_N \in {}^*E_M$ for some infinite M . Then $d(\mathbb{T}_{t_N}x_N, z) < \frac{1}{M} \approx 0$ for some $z \in {}^*\mathbb{A} \subseteq \text{ns}(\mathcal{M})$ and so $\mathbb{T}_{t_N}x_N$ is nearstandard as required.

(b) \Rightarrow (a) Take a bounded absorbing set E as given by (b) and r_0 such that $S_t \bar{E} \subseteq E$ for $t \geq r_0$. Make the following observation.

($\#\#$) If $x_n \in E$ for all n and $t_n \rightarrow \infty$ then $\mathbb{T}_{t_N}x_N$ is nearstandard and ${}^\circ\mathbb{T}_{t_N}x_N \in \mathbb{A}$ for all infinite N .

To see this, the relative compactness of $\{S_{t_n}x_n\}$ means that $\mathbb{T}_{t_N}x_N$ is nearstandard for all infinite N . Then

$${}^\circ\mathbb{T}_{t_N}x_N = {}^\circ(\mathbb{T}_t \mathbb{T}_{r_0} \mathbb{T}_{t_N - t - r_0} x_N) = S_t S_{r_0} {}^\circ(\mathbb{T}_{t_N - t - r_0} x_N) \in S_t (S_{r_0} \bar{E}) \subseteq S_t E$$

since $\mathbb{T}_{t_N - t - r_0} x_N \in {}^*E$ (by absorption) and so ${}^\circ(\mathbb{T}_{t_N - t - r_0} x_N) \in \bar{E}$.

For the compactness of \mathbb{A} take $a_n = S_n x_n \in \mathbb{A}$ and by ($\#\#$) we have ${}^\circ a_N \in \mathbb{A}$ for all infinite N , which shows that \mathbb{A} is sequentially compact, hence compact.

For invariance let $a \in \mathbb{A}$. There is $x_n \in E$ with $a = S_n x_n$ for all n and so $a = \mathbb{T}_N x_N$ and

$$S_t a = {}^\circ\mathbb{T}_t \mathbb{T}_N x_N = {}^\circ(\mathbb{T}_{N+t} x_N) \in \mathbb{A}$$

by ($\#\#$) with $t_n = n + t$, giving $S_t \mathbb{A} \subseteq \mathbb{A}$. Conversely

$$a = {}^\circ a = {}^\circ(\mathbb{T}_t \mathbb{T}_{N-t} x_N) = S_t {}^\circ(\mathbb{T}_{N-t} x_N) \in S_t \mathbb{A}$$

again by ($\#\#$), with $t_n = n - t$, giving $\mathbb{A} \subseteq S_t \mathbb{A}$.

Finally the attraction property is proved exactly as in the previous theorem.

■

Remarks (1) We could also have given a further equivalent condition

- (b+) (i) *There is a bounded absorbing set E ;*
(ii) *For any bounded set B , any sequence (x_n) in B , and $t_n \rightarrow \infty$, the set $\{S_{t_n}x_n\}$ is relatively compact.*

It is clear that (b+) implies (b); and for (b) \Rightarrow (b+) simply note that if B is bounded then taking r with $S_r B \subseteq E$ gives $S_r x_n \in E$ for all n and $\{S_{t_n}x_n\} = \{S_{t_n-r}S_r x_n\}$ is relatively compact (we may assume that $t_n \geq r$ for all n).

(2) It can also be shown that the attractor \mathbb{A} has the description

$$\mathbb{A} = {}^\circ(\text{ns}(\mathcal{M}) \cap \bigcup_{\tau\text{-infinite}} \mathbb{T}_\tau^* E)$$

and an alternative proof can be given starting from this definition of \mathbb{A} .

The following notion of a *subflow* and the corollary below will be useful in later discussion.

Definition 2.6 *Let $(S_t)_{t \geq 0}$ be a semiflow on a metric space \mathcal{M} . By a **subflow** of $(S_t)_{t \geq 0}$ we mean the restriction of $(S_t)_{t \geq 0}$ to a closed subspace \mathcal{N} of \mathcal{M} such that $S_t(\mathcal{N}) \subseteq \mathcal{N}$ for each $t \geq 0$.*

Corollary 2.7 *Assume the hypotheses of Theorem 2.4 or (b) of Theorem 2.5. Then every subflow of $(S_t)_{t \geq 0}$ has a global attractor.*

Proof. $F = E \cap \mathcal{N}$ is a bounded absorbing set for the subflow, and any subset of \mathcal{N} that is relatively compact in \mathcal{M} is relatively compact in \mathcal{N} , so the preceding theorems apply to the restriction of the flow S_t to the subspace \mathcal{N} . ■

Before moving on to questions concerning attractors for stochastic systems, we give a simple proof of Sell's theorem [26] showing the existence of a global attractor for the deterministic 3D Navier–Stokes equations. Here nonstandard techniques are useful in verifying the hypotheses in order to apply Theorem 2.4.

3 Attractors for 3D Navier-Stokes equations

3.1 The 3D Navier–Stokes equations

In this section we consider the time homogeneous deterministic Navier–Stokes equations

$$\begin{cases} du(t) = [\nu \Delta u(t) - \langle u(t), \nabla \rangle u(t) + f(u(t)) - \nabla p(t)] dt \\ \text{div } u = 0 \end{cases} \quad (4)$$

in a bounded domain $\mathbb{D} \subseteq \mathbb{R}^3$ with boundary of class C^2 . Recall from the introduction that the term $f(u(t))$ denotes external forces while $p(t)$ is the pressure. The function $u = u(t, x)$ is the velocity of a fluid at the point $x \in \mathbb{D}$ at time t , and in the above equation $u(t) = u(t, \cdot)$ denotes the whole velocity field at time t . This can be regarded as a functional equation, and we adopt the conventional Hilbert space approach to it as follows. Let \mathbf{H} be the closure of the set

$$\{u \in C_0^\infty(\mathbb{D}, \mathbb{R}^3) : \operatorname{div} u = 0\}$$

in the L^2 norm $|u| = (u, u)^{1/2}$, where $u = \langle u_1, u_2, u_3 \rangle$ and

$$(u, v) = \sum_{j=1}^3 \int_{\mathbb{D}} u_j(x) v_j(x) dx.$$

The letters u, v, w will be used for elements of \mathbf{H} . The subspace \mathbf{V} is the closure of the set $\{u \in C_0^\infty(\mathbb{D}, \mathbb{R}^3) : \operatorname{div} u = 0\}$ in the stronger norm $|u| + \|u\|$ where $\|u\| = ((u, u))^{1/2}$ and

$$((u, v)) = \sum_{j=1}^3 \left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right).$$

\mathbf{H} and \mathbf{V} are Hilbert spaces with scalar products (\cdot, \cdot) and $((\cdot, \cdot))$ respectively, and $|\cdot| \leq c \|\cdot\|$ for some constant c .

By A we denote the self adjoint extension of the projection of $-\Delta$ in \mathbf{H} . Classical theory shows that there is an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of eigenfunctions of A with corresponding eigenvalues $\lambda_k > 0$ such that $\lambda_k \nearrow \infty$. For $u \in \mathbf{H}$ we write $u_k = (u, e_k)$, and write Pr_m for the projection of \mathbf{H} on the subspace \mathbf{H}_m spanned by $\{e_1, \dots, e_m\}$. Since each $e_k \in \mathbf{V}$, then $\mathbf{H}_m \subseteq \mathbf{V}$. If $u = \sum u_k e_k \in \mathbf{V}$ then $\|u\|^2 = \sum \lambda_k u_k^2$, so that the constant c above is $\lambda_1^{-\frac{1}{2}}$.

A trilinear form b is defined by

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\mathbb{D}} u_j(x) \frac{\partial v_i}{\partial x_j}(x) w_i(x) dx = (\langle u, \nabla \rangle v, w)$$

whenever the integrals make sense. Note the following well-known properties of the trilinear form b , where c is a real constant.

$$b(u, v, w) = -b(u, w, v),$$

$$b(u, v, v) = 0,$$

$$|b(u, v, w)| \leq c \|u\| \|v\| \|w\|$$

$$|b(u, v, w)| \leq c |u|^{\frac{1}{4}} \|u\|^{\frac{3}{4}} |v|^{\frac{1}{4}} \|v\|^{\frac{3}{4}} \|w\| \quad (5)$$

The last inequalities are two of the many continuity properties of b that are used in the study of the Navier–Stokes equations, and are proved in [27] for example.

From (5) we have the following crucial lemma (a slight extension of the Crucial Lemma (Lemma 2.7.7) of the book [8]).

Lemma 3.1 (Crucial Lemma) *If $u, v \in \mathbf{V}$ with $\|u\|$ and $\|v\|$ both finite, and $z \in \mathbf{V}$ then*

$$*b(u, v, *z) \approx b(u, v, z)$$

where $u = \circ u$ and $v = \circ v$ (with $u, v \in \mathbf{V}$.)

Proof. The finiteness of $\|u\|$ and $\|v\|$ means that $|u - *u| \approx 0 \approx |v - *v|$. We have $b(u, v, z) = *b(*u, *v, *z)$ and

$$|*b(u, v, *z) - *b(*u, *v, *z)| \leq |*b(u - *u, v, *z)| + |*b(*u, v - *v, *z)|$$

Using (5) gives

$$|*b(u - *u, v, *z)| \leq c|u - *u|^{\frac{1}{4}} \|u - *u\|^{\frac{3}{4}} |v|^{\frac{1}{4}} \|v\|^{\frac{3}{4}} \|z\| \approx 0$$

and similarly $|*b(*u, v - *v, *z)| \approx 0$ which gives the result. ■

3.2 Functional formulation of the Navier–Stokes equations

In the above framework, the deterministic Navier–Stokes equations may be formulated as the following differential equation in \mathbf{V}' (the dual of \mathbf{V}):

$$du = [-\nu Au - B(u) + f(u)]dt, \quad (6)$$

where $B(u) = b(u, u, \cdot)$. Note that the pressure has disappeared, because $\nabla p = 0$ in \mathbf{V}' (using $\operatorname{div} v = 0$ in \mathbf{V} and an integration by parts). Although equation (6) is regarded as an equation in \mathbf{V}' , it turns out that solutions can be found that live in \mathbf{H} (and in fact in \mathbf{V} for almost all times).

The equation (6) is really an *integral* equation, with the integral being the Bochner integral. Thus, when we write

$$u(t_1) = u(t_0) + \int_{t_0}^{t_1} [-\nu Au(t) - B(u(t)) + f(u(t))]dt$$

we mean that for all $v \in \mathbf{V}$ we have

$$(u(t_1), v) = (u(t_0), v) + \int_{t_0}^{t_1} [-\nu(Au(t), v) - (B(u(t)), v) + (f(u(t)), v)]dt$$

as an equation in \mathbb{R} .

The existence of solutions to these equations was established by Leray in the 1930's using the method of finite dimensional ("Galerkin") approximations, a method that is still one of the principal tools in use. An efficient proof of existence was given in [3] using nonstandard methods, using a hyperfinite-dimensional approximation in \mathbf{H}_N for an infinite N , and thereby avoiding the need for specialized compactness results. In all cases the solutions are so called *weak solutions*, and the question of uniqueness of these is still open. (By contrast, so called *strong solutions* that live in \mathbf{V} for all time can only be shown to exist for short time scales, but for these uniqueness is known.)

3.3 Attractors for 3D Navier–Stokes equations

The problem with the very definition of an attractor for the 3D Navier–Stokes equations is a consequence of the problem of uniqueness. If uniqueness was known then a semigroup could be defined on \mathbf{H} by setting $S_t v = u(t)$ where $u(\cdot)$ is the unique solution with initial condition $u(0) = v$. To overcome this possible non-uniqueness, Sell's radical approach in [26] was to take a space of weak solutions \mathcal{W} defined below as the arena of activity rather than the space \mathbf{H} . We will show how nonstandard techniques provide a simple proof of Sell's existence theorem for an attractor in this setting. The solutions constructed by the Galerkin method have certain other properties that can be derived heuristically from the equations themselves (and the same is true for the solutions constructed in [3]). This leads to Sell's definition of a weak solution [26].

Definition 3.2 (Sell [26]) *Let $f : \mathbf{H} \rightarrow \mathbf{H}$ such that $|f(u)| \leq c + d|u|$ with $d < \nu\lambda_1$. A **weak solution to the Navier–Stokes equations (6)** is a function $u : [0, \infty) \rightarrow \mathbf{H}$ such that*

(W1) $u \in L^\infty(0, \infty; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$ for all T

(W2) for all $t \geq t_0 \geq 0$

$$u(t) = u(t_0) + \int_{t_0}^t [-\nu Au(s) - B(u(s)) + f(u(s))] ds \quad (7)$$

as a Bochner integral equation in \mathbf{V}' .

(W3) for almost all $t_0 > 0$ and all $t > t_0$

$$|u(t)|^2 \leq |u(t_0)|^2 \exp(-k_1(t - t_0)) + k_2 \quad (8)$$

where $k_1 = \nu\lambda_1 - d$ and $k_2 = c^2 k_1^{-2}$.

(W4) for almost all $t_0 > 0$ and all $t > t_0$

$$|u(t)|^2 + 2\nu \int_{t_0}^t \|u(s)\|^2 ds \leq |u(t_0)|^2 + 2 \int_{t_0}^t (f(u(s)), u(s)) ds \quad (9)$$

Denote by $\mathcal{W} = \mathcal{W}(f)$ the set of all weak solutions for a given f as above. A norm is defined on \mathcal{W} by

$$|u| = \left(\int_0^\infty |u(t)|^2 \exp(-t) dt \right)^{\frac{1}{2}} = \left(\int_0^\infty |u(t)|^2 \mu(dt) \right)^{\frac{1}{2}}$$

where $\mu(dt) = \exp(-t)dt$

Thus \mathcal{W} is a subspace of the complete metric space M of measurable functions $\xi : [0, \infty) \rightarrow \mathbf{H}$ with norm

$$|\xi| = \left(\int_0^\infty \xi(t)^2 \exp(-t) dt \right)^{\frac{1}{2}} = \left(\int_0^\infty \xi(t)^2 \mu(dt) \right)^{\frac{1}{2}}$$

where $\mu(dt) = \exp(-t)dt$. That is, $M = L^2(0, \infty; \mathbf{H}; \mu)$, which is actually a separable Hilbert space.

Remarks

1. It is well known that a weak solution is weakly continuous in \mathbf{H} ; this follows from (7).
2. Sell [26] has $f \in \mathbf{H}$ so that $c = |f|$ and $d = 0$, giving $k_1 = \nu\lambda_1$ and $k_2 = |f|^2(\nu\lambda_1)^{-2}$ so (W3) is consistent with Sell's corresponding condition.
3. Sell [26] adds the condition that $D_t u \in L^p_{\text{loc}}[0, V')$ for $p = \frac{4}{3}$. This follows however from the fact that

$$D_t u = -\nu Au - B(u) + f$$

together with:

$$\begin{aligned} |Au|_{V'} &= \|u\| \\ |B(u)|_{V'} &\leq c|u|^{\frac{1}{2}} \|u\|^{\frac{3}{2}} \end{aligned}$$

the latter inequality using (5).

4. Sell also defines a wider set of **generalised weak solutions** in order to have a complete metric space. Here however completeness is not needed; we will see below how the attractor we construct for the class \mathcal{W} is also an attractor for the wider class of generalised weak solutions.
5. The above norm is equivalent to the metric used by Sell in [26], and has the same bounded sets.
6. It is well known (see for example [28]) that if $f : H \rightarrow H$ such that $|f(u)| \leq c_0 + d_1|u|$ and $d_1 < \nu\lambda_1$ then for any initial condition $u_0 \in \mathbf{H}$ there is a weak solution $u \in \mathcal{W}(f)$ with $u(0) = u_0$.

3.4 The semiflow and a global attractor

A semiflow S_t is defined on \mathcal{W} as follows.

Definition 3.3 (Sell [26]) *For $u \in \mathcal{W}$ and $t \geq 0$ define $S_t u \in L^\infty(0, \infty; \mathbf{H})$ by*

$$S_t u(s) = u(t + s)$$

for $s \geq 0$. That is, $S_t u$ is obtained by translating u to the left by t and retaining only the portion defined for positive time.

It is clear that S_t maps \mathcal{W} into \mathcal{W} and has the semigroup property. It is easy to see that each S_t is continuous (in fact Lipschitz continuous) because if $u, u' \in \mathcal{W}$ then

$$\begin{aligned} |S_t(u - u')|^2 &= \int_0^\infty |(u - u')(t + s)|^2 \exp(-s) ds \\ &= \exp(t) \int_t^\infty |(u - u')(s)|^2 \exp(-s) ds \\ &\leq \exp(t) |u - u'|^2 \end{aligned}$$

Theorem 3.4 (Sell [26]) *Suppose that $f : H \rightarrow H$ is continuous with $|f(u)| \leq c_0 + d_1|u|$, where $d_1 < \nu\lambda_1$. Then there is a global attractor \mathbb{A} for the set $\mathcal{W} = \mathcal{W}(f)$ of weak solutions to the Navier–Stokes equations with forcing term f , and any subflow also has a global attractor.*

Proof. We check that the hypotheses of Theorem 2.4 are satisfied for the metric space \mathcal{W} . The result will then follow from Theorem 2.4.

The set $E = \{u \in \mathcal{W} : |u|^2 \leq k_2 + 1\}$ is a bounded absorbing set. This follows easily from the energy inequality (8) in the definition of \mathcal{W} , which shows that for any $u \in \mathcal{W}$

$$|S_t u|^2 \leq |u|^2 \exp(-k_1 t) + k_2.$$

So if $B = \{u \in \mathcal{W} : |u| \leq k\} = B(k)$, say, we have

$$S_t B \subseteq E$$

whenever $t \geq 2(k_1)^{-1} \log k$.

To check the compactness property of the maps S_t , we show that if B is bounded then $S_1 B$ is relatively compact. Here the technology of nonstandard analysis proves useful, particularly Loeb measure and integration theory.

Let $B = B(k)$; it is sufficient to show that each $u \in {}^*S_1 B$ is nearstandard to a point $u \in \mathcal{W}$. Let $u = {}^*S_1 v$ with $v \in {}^*B$. That is

$$u(\tau) = v(\tau + 1)$$

for $\tau \geq 0$. Then (using τ and σ to denote nonstandard times)

$$\int_0^1 |\mathbf{v}(\tau)|^2 d\tau \leq e \int_0^1 |\mathbf{v}(\tau)|^2 \exp(-\tau) d\tau \leq ek^2$$

and so there is some $\tau_0 \in {}^*(0, 1)$ with $|\mathbf{v}(\tau_0)|$ finite *and* with ${}^*(8)$ and ${}^*(9)$ holding for all $\tau \geq \tau_0$. (In fact we may take τ_0 with $|\mathbf{v}(\tau_0)| \leq ek^2 + 1$.)

Writing $k_3 = |\mathbf{v}(\tau_0)|^2 + k_2$ this means that

$$|\mathbf{v}(\tau)|^2 \leq k_3 < \infty$$

for all $\tau \geq \tau_0$. So

$$|\mathbf{u}(\tau)| \leq \sqrt{k_3} \tag{10}$$

for all $\tau \geq 0$. Thus $\mathbf{u}(\tau)$ is weakly nearstandard for each $\tau \geq 0$ and we can define the weak standard part ${}^\circ\mathbf{u}(\tau) \in \mathbf{H}$. If $\tau_1 \approx \tau_2$ then from (7) we have, taking $v = e_n$ for finite n

$$\mathbf{u}_n(\tau_2) - \mathbf{u}_n(\tau_1) = \int_{t_0}^{\tau_1} [-\nu\lambda_n\mathbf{u}_n(\tau) - ({}^*B(\mathbf{u}(\tau)), {}^*e_n) + ({}^*f(\mathbf{u}(\tau)), {}^*e_n)] d\tau \approx 0$$

since the integrand is S-integrable (see below). Thus ${}^\circ\mathbf{u}(\tau_2) = {}^\circ\mathbf{u}(\tau_1)$, so for real $t \geq 0$ we may define $u(t) = {}^\circ\mathbf{u}(\tau)$ for any $\tau \approx t$, and $u(t)$ is weakly continuous. We will see that $u \in \mathcal{W}$ and $u \approx \mathbf{u}$ in the norm of \mathcal{W} .

From ${}^*(9)$ for τ_0 we see that $\int_{\tau_0}^{\tau} \|\mathbf{v}(\sigma)\|^2 d\sigma$ is finite for all finite τ and so $\|\mathbf{v}(\tau)\|$ is finite for a.a. finite $\tau \geq \tau_0$ (with respect to the Loeb measure $d_L\tau$). Thus, for a.a. finite $\tau \geq 0$, $\|\mathbf{u}(\tau)\|$ is finite so that $\mathbf{u}(\tau)$ is strongly nearstandard and

$$|{}^\circ\mathbf{u}(\tau)| = |{}^\circ\mathbf{u}(\tau)| \tag{11}$$

Take any standard finite time T . We will show that u has properties (W3) and (W4) of Definition 3.2 on $[0, T]$. Let \mathbf{Y} be the set of all $\tau \in {}^*[0, T]$ with the following properties:

- (i) $\|\mathbf{u}(\tau)\|$ is finite;
- (ii) ${}^*(8)$ and ${}^*(9)$ hold in ${}^*[0, T]$ with $t_0 = \tau$. That is, for all $\tau_1 \in {}^*[\tau, T]$

$$|\mathbf{u}(\tau_1)|^2 \leq |\mathbf{u}(\tau)|^2 \exp(-k_1(\tau_1 - \tau)) + k_2$$

and

$$|\mathbf{u}(\tau_1)|^2 + 2\nu \int_{\tau}^{\tau_1} \|\mathbf{u}(\sigma)\|^2 d\sigma \leq |\mathbf{u}(\tau)|^2 + 2 \int_{\tau}^{\tau_1} ({}^*f(\mathbf{u}(\sigma)), \mathbf{u}(\sigma)) d\sigma$$

Then \mathbf{Y} has Loeb measure T and so the set $D = \text{st}(\mathbf{Y})$ is a subset of $[0, T]$ with full measure (because $\text{st}^{-1}(D) \cap {}^*[0, T] \supseteq \mathbf{Y}$). Now we see that (8) and (9) hold in $[0, T]$ for all $t_0 \in D$. Take any such $t_0 = {}^\circ\tau$ with $\tau \in \mathbf{Y}$. Then for $t \in (t_0, T]$ we have

$$\begin{aligned} |u(t)|^2 &= |{}^\circ\mathbf{u}(t)|^2 \leq |{}^\circ\mathbf{u}(t)|^2 \leq |{}^\circ\mathbf{u}(\tau)|^2 \exp(-k_1(t - \tau)) + k_2 \\ &= |{}^\circ\mathbf{u}(\tau)|^2 \exp(-k_1(t - t_0)) + k_2 \\ &= |u(t_0)|^2 \exp(-k_1(t - t_0)) + k_2 \end{aligned}$$

using (11) to give ${}^\circ|u(\tau)| = |{}^\circ u(\tau)| = |u(t_0)|$.

Similar reasoning shows that for $t \in (t_0, T]$

$$|u(t)|^2 + 2\nu \int_{t_0}^t \|u(s)\|^2 ds \leq |u(t_0)|^2 + 2 \int_{t_0}^t (*f(u(\sigma)), u(\sigma)) d\sigma$$

Now note that $(*f(u(\sigma)), u(\sigma))$ is S-integrable on $[t_0, t]$ since it is bounded, and so

$$\begin{aligned} {}^\circ \int_{t_0}^t (*f(u(\sigma)), u(\sigma)) d\sigma &= \int_{t_0}^t {}^\circ (*f(u(\sigma)), u(\sigma)) d_L \sigma = \int_{t_0}^t ({}^\circ (*f(u(\sigma))), {}^\circ u(\sigma)) d_L \sigma \\ &= \int_{t_0}^t (f(u(\sigma)), u(\sigma)) d_L \sigma = \int_{t_0}^t (f(u(s)), u(s)) ds \end{aligned}$$

where we have used (11) and the continuity of f . This establishes (9) on $[0, T]$.

Next we show that u satisfies the equation (7). Let $v \in \mathbf{V}$. We know that for all $t \geq t_0 \geq 0$

$$(u(t), *v) = (u(t_0), *v) + \int_{t_0}^t [-\nu((u(\sigma), *v)) - (*B(u(\sigma)), *v) + (*f(u(\sigma)), *v)] d\sigma \quad (12)$$

We have ${}^\circ(u(t), *v) = (u(t), v)$ and ${}^\circ(u(t_0), *v) = (u(t_0), v)$. For the integral, we need to show that each integrand is S-integrable on $[t_0, t]$, using Lindström's lemma for the first two terms. For the first term we have

$$\int_{t_0}^t ((u(\sigma), *v))^2 d\sigma \leq \int_{t_0}^t \|u(\sigma)\|^2 d\sigma \cdot \int_{t_0}^t \|*v\|^2 d\sigma < \infty$$

so $((u(\sigma), *v))$ is S-integrable. For the second term we have, using (5), $|(*B(u(\sigma)), *v)| \leq c|u(\sigma)|^{\frac{1}{2}} \|u(\sigma)\|^{\frac{3}{2}} \|v\|$ and so

$$\int_{t_0}^t |(*B(u(\sigma)), *v)|^{\frac{4}{3}} d\sigma \leq (c\|v\|)^{\frac{4}{3}} k_3^{\frac{1}{3}} \int_{t_0}^t \|u(\sigma)\|^2 d\sigma < \infty$$

where we have used (10), showing that $(*B(u(\sigma)), *v)$ is S-integrable. The final term is S-integrable since it is bounded. So the theory of Loeb integration gives

$$\begin{aligned} (u(t), v) - (u(t_0), v) &= {}^\circ[(u(t), *v) - (u(t_0), *v)] \\ &= {}^\circ \left[\int_{t_0}^t [-\nu((u(\sigma), *v)) - (*B(u(\sigma)), *v) + (*f(u(\sigma)), *v)] d\sigma \right] \\ &= \int_{t_0}^t [-\nu({}^\circ((u(\sigma), *v)) - {}^\circ(*B(u(\sigma)), *v) + {}^\circ(*f(u(\sigma)), *v))] d_L \sigma \\ &= \int_{t_0}^t [-\nu((u(\sigma), v)) - (B(u(\sigma)), v) + (f(u(\sigma)), v)] d_L \sigma \\ &= \int_{t_0}^t [-\nu((u(s), v)) - (B(u(s)), v) + (f(u(s)), v)] ds \end{aligned}$$

where the penultimate equality is because for a.a. σ , $\|\mathbf{u}(\sigma)\|$ is finite and hence

- (a) $\mathbf{u}(\sigma)$ is weakly nearstandard in \mathbf{V} and so $(\mathbf{u}(\sigma), *v) \approx ({}^\circ\mathbf{u}(\sigma), v)$;
- (b) $(*B(\mathbf{u}(\sigma)), *v) \approx (B({}^\circ\mathbf{u}(\sigma)), v)$ using Lemma 3.1; and similarly $(*f(\mathbf{u}(\sigma)), *v) \approx (f(u({}^\circ\sigma)), v)$
- (c) ${}^\circ\mathbf{u}(\sigma) = u({}^\circ\sigma)$
- (d) $(*f(\mathbf{u}(\sigma)), *v) \approx (f(u({}^\circ\sigma)), v)$ using the continuity of f and (11).

The final equality holds because the standard part mapping ${}^\circ : *[t, t_0] \rightarrow [t, t_0]$ is measure preserving.

We are now done provided we can show that $|\mathbf{u} - *u| \approx 0$. We have $|\mathbf{u}(\tau)|, |*u(\tau)| \leq c_1$ for all τ and for a.a. $\tau \in *[0, T]$ we have $\|\mathbf{u}(\tau)\|$ and $\|*u(\tau)\|$ finite so that ${}^\circ|\mathbf{u}(\tau) - *u(\tau)| = |{}^\circ\mathbf{u}(\tau) - {}^\circ(*u(\tau))|$ for such τ . The weak continuity of u means that ${}^\circ(*u(\tau)) = u({}^\circ\tau)$ for all τ and since ${}^\circ\mathbf{u}(\tau) = u({}^\circ\tau)$ also we obtain, for finite T ,

$$\begin{aligned} \int_0^T |\mathbf{u}(\tau) - *u(\tau)|^2 \exp(-\tau) d\tau &\approx \int_0^T {}^\circ|\mathbf{u}(\tau) - *u(\tau)|^{2\circ} \exp(-\tau) d_L\tau \\ &= \int_0^T |{}^\circ\mathbf{u}(\tau) - {}^\circ(*u(\tau))|^2 \exp(-{}^\circ\tau) d_L\tau = 0 \end{aligned}$$

Thus

$$\int_0^T |\mathbf{u}(\tau) - *u(\tau)|^2 \exp(-\tau) d\tau \leq T^{-1}$$

for all finite T and by overflow we have the same for some infinite time T . This is enough to show that $|\mathbf{u} - *u| \approx 0$ and so $\mathbf{u} \approx u$ as required. ■

Remarks (1) This proof is very close in many respects to the original non-standard proof of existence of weak solutions to the Navier–Stokes equations given in the paper [3]. In that paper the *existence* of solutions was established by constructing internal solutions to the Galerkin approximation in dimension N where N is infinite. The process of taking standard parts and showing that this gives a standard weak solution is almost identical to the above proof that ${}^\circ\mathbf{u}(t)$ is a solution.

(2) In [26] Sell proves the existence of a global attractor for a wider class of *generalized weak solutions* – these are solutions that may have a singularity at $t = 0$ but are weak solutions away from 0. If u is a generalized weak solution then $S_t u \in \mathcal{W}$ for any $t > 0$, so it is clear that \mathbb{A} is a global attractor for this larger class of solutions.

(3) If the definition of a weak solution (Definition 3.2) is tightened by adding to the list (W1)-(W4) further equalities or inequalities that are preserved under S_t then we will have a subflow provided the resulting set is closed in \mathcal{W} , and this will then have a global attractor. In Part II when discussing solutions to the stochastic Navier–Stokes equations we will need to impose additional conditions on the set of weak stochastic solutions in just this way.

3.5 Two-sided solutions

A neat characterisation of the global attractor above is given by considering two-sided solutions. In [26] Sell remarked that each solution u in the global attractor \mathbb{A} is the restriction to non-negative time of a solution defined for all time – which we call a *two-sided solution*. Here we use the ideas in the previous section to show the converse – so that \mathbb{A} is precisely the set of restrictions of two-sided solutions. First we must make the necessary definitions.

Definition 3.5 *Let $f : \mathbf{H} \rightarrow \mathbf{H}$ such that $|f(u)| \leq c + d|u|$ with $d < \nu\lambda_1$. A **two-sided weak solution to the Navier–Stokes equations** (6) is a function $v \in L^\infty(-\infty, \infty; \mathbf{H}) \cap L^2(-T, T; \mathbf{V})$ for all T with the properties (W2)-(W4) of Definition 3.2 holding for all $t_0 \in (-\infty, \infty)$.*

Denote by $\bar{\mathcal{W}} = \bar{\mathcal{W}}(f)$ the set of all weak solutions for a given f as above. A norm is defined on $\bar{\mathcal{W}}$ by

$$|v| = \left(\int_{-\infty}^{\infty} |v(t)|^2 \exp(-|t|) dt \right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} |v(t)|^2 \mu(dt) \right)^{\frac{1}{2}}$$

where $\mu(dt) = \exp(-|t|)dt$

Thus $\bar{\mathcal{W}}$ is a subset of the Hilbert space $\bar{M} = L^2(-\infty, \infty; \mathbf{H}; \bar{\mu})$ where $\bar{\mu}(dt) = \exp(-|t|)dt$. It will be useful to define the **right-shift operator** R_t on \bar{M} for any $t \in \mathbb{R}$ by

$$(R_t v)(s) = v(s - t)$$

for any function $v \in \bar{M}$. For future reference notice that for $t > 0$

$$S_t((R_t v) \upharpoonright [0, \infty)) = v \upharpoonright [0, \infty) \tag{13}$$

Now we have:

Theorem 3.6 *Let $f : H \rightarrow H$ such that $|f(u)| \leq c + d|u|$ with $d < \nu\lambda_1$. The set $\bar{\mathcal{W}} = \bar{\mathcal{W}}(f)$ of all two-sided weak solutions is bounded and compact, and the global attractor \mathbb{A} for one-sided solutions (Theorem 3.4) is given by*

$$\mathbb{A} = \bar{\mathcal{W}} \upharpoonright [0, \infty)$$

Proof. From (8) it follows that for any $v \in \bar{\mathcal{W}}$

$$|u(t)| \leq \sqrt{k_2}$$

for all t . This is because $v \in L^\infty(-\infty, \infty; \mathbf{H})$, so we can find arbitrarily large negative t_0 with $|v(t_0)| \leq |v|_\infty$ and (8) holding, so that for a given t we can make $|v(t_0)|^2 \exp(-k_1(t - t_0))$ arbitrarily small. So $\bar{\mathcal{W}}$ is bounded – in fact $|v| \leq 2|v|_\infty \leq 2\sqrt{k_2}$.

Notice also that if $v \in \bar{\mathcal{W}}$ then $R_t \in \bar{\mathcal{W}}$ also, and $v \upharpoonright [0, \infty) \in E$. So if $u = v \upharpoonright [0, \infty)$ then from (13)

$$u = S_t((R_t v) \upharpoonright [0, \infty)) \in S_t E$$

and thus $u \in \mathbb{A}$, showing that $\bar{\mathcal{W}} \upharpoonright [0, \infty) \subseteq \mathbb{A}$.

For the converse, take $u \in \mathbb{A}$ so that $*u \in T_\tau *E$ for some infinite τ . Let $*u = T_\tau \mathbf{v}$ with $\mathbf{v} \in *E$ and consider the left translate $\bar{\mathbf{v}} : *[-\tau, \infty) \rightarrow *\mathbf{H}$ of \mathbf{v} given by

$$\bar{\mathbf{v}}(\sigma) = \mathbf{v}(\sigma + \tau)$$

Then $|\bar{\mathbf{v}}(\tau_0)|$ is finite for almost all $\tau_0 \in *[-\tau, -\tau + 1]$ and from this it follows using *(8) that

$${}^\circ|\bar{\mathbf{v}}(\tau)| \leq \sqrt{k_2}$$

for all finite times τ . So we may define $v(t) = {}^\circ\bar{\mathbf{v}}(t)$ for real times t , and then

$$|v(t)| \leq \sqrt{k_2}$$

for all t .

Adapting the proof of Theorem 3.4 shows that $v \in \mathcal{W}$ and for $t \geq 0$ we have

$$v(t) = {}^\circ\bar{\mathbf{v}}(t) = {}^\circ\mathbf{v}(t + \tau) = {}^\circ T_\tau \mathbf{v}(t) = {}^\circ(*u(t)) = u(t)$$

so

$$u = v \upharpoonright [0, \infty)$$

showing that $\mathbb{A} \subseteq \bar{\mathcal{W}} \upharpoonright [0, \infty)$.

The proof that $\bar{\mathcal{W}}$ is compact follows the same lines as the proof that $S_1 B$ is relatively compact in the proof of Theorem 3.4, and is omitted. ■

3.5.1 Another approach

In the paper [6] an alternative way to overcome the problem of lack of uniqueness in order to define a notion of attractor for 3D Navier–Stokes equations. The idea was to work with internal solutions to the Galerkin approximation on \mathbf{H}_N for some fixed infinite N . Here we have uniqueness of solutions, so there is a well defined internal semigroup T_τ defined on \mathbf{H}_N . This was used to define various multi-valued semiflows on \mathbf{H} by means of somewhat *ad hoc* standard part operations, leading to existence of compact attractors in \mathbf{H} , but in a rather weak sense – so as is to be expected, the results are less pleasing than in dimension 2. Rather more satisfactory results were obtained for an approach that used small initial pieces of trajectories of solutions as phase space – this is an idea that is intermediate between the above and that of Sell. For full details, and further variations on this theme, see [6].

PART II: STOCHASTIC SYSTEMS

4 Stochastic attractors for Navier–Stokes equations

For stochastic systems the problem of defining attractors stems from the continual injection of noise into the system as it evolves – so it is unreasonable to expect there to be any set that attracts the random paths as $t \rightarrow \infty$. However, there are several ways to formulate the idea of an attractor for a system of stochastic differential equations in a way that circumvents this problem. One is to consider *measure attractors* (see [7],[25] for example); another is to work with the notion of *stochastic attractor* developed by Crauel & Flandoli [10] — but only in 2D. In the case of 3D stochastic Navier–Stokes equations these approaches are not available because of the possible nonuniqueness. Here it makes sense to consider extending the approach of Sell [26] discussed in Section 3 that was used for 3D deterministic Navier–Stokes equations.

We will briefly review results that have been achieved using the first two of these approaches for the 2D equations, and then present new results concerning the extension of Sell’s approach to 3D stochastic Navier–Stokes equations. In each case, to avoid unnecessary additional complications, the drift and noise coefficients f, g in (1) are taken to be time-independent, so the equations considered are

$$du = [-\nu Au - B(u) + f(u)]dt + g(u)dw_t \quad (14)$$

Here, as noted in the introduction u is a stochastic process, so in the functional formulation we have

$$u : [0, \infty) \times \Omega \rightarrow \mathbf{H}$$

where Ω is the domain of an underlying probability space. The equation (14) is an infinite dimensional stochastic differential equation (SDE), or SPDE, with $w = w(t, \omega)$ a Wiener process that models noise and provides the random forcing term $g(u)dw_t$. The theory of SDEs and SPDEs requires that the underlying probability space (Ω, \mathcal{F}, P) has some extra structure – namely a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, such that w is adapted; that is $w(t, \cdot)$ is \mathcal{F}_t -measurable for each t . A solution $u(t, \omega)$ is also required to be adapted.

4.1 Measure attractors

This approach is currently applicable only to $d = 2$ since it is necessary that the equation (14) has a unique solution. The functional formulation of the 2D stochastic Navier–Stokes equations outlined in Sections 3.1 and 3.2 applies equally in the 2D setting. To ensure that for each initial condition $u \in \mathbf{H}$ there

is a unique solution $u(t, \omega) = v(t, \omega; u)$ with $u(0) = u$ (so $v(0, \omega; u) = u$) it is assumed that f, g satisfy an appropriate Lipschitz condition. A semigroup S_t is now defined on $\mathcal{M}_1(\mathbf{H})$, the set of Borel probability measures on \mathbf{H} , where $S_t\mu = \mu_t$ is defined for $\mu \in \mathcal{M}_1(\mathbf{H})$ by

$$\int_{\mathbf{H}} \vartheta(u) d\mu_t(u) = \int_{\mathbf{H}} [\mathbb{E}_P \vartheta(v(t, \cdot; u))] d\mu(u)$$

for all bounded weakly continuous functions $\theta : \mathbf{H} \rightarrow \mathbf{H}$.

An attractor for the dynamical system $(\mathcal{M}_1(\mathbf{H}), S_t)$ is called a **measure attractor**. The existence of measure attractors for the sNS equations was first investigated by Schmalfuß, in [25] for example. The paper [7] with Capiński establishes existence of a measure attractor for (14) under quite general conditions:

Theorem 4.1 *Suppose that f, g are Lipschitz and satisfy an appropriate growth condition¹. Then there is a measure attractor $\mathbb{A} \subset \mathcal{M}_1(\mathbf{H})$ for the stochastic Navier–Stokes equations (14). That is*

- (a) \mathbb{A} is weakly compact;
- (b) $S_t\mathbb{A} = \mathbb{A}$ for all t ;
- (c) for each open set $\mathcal{O} \supseteq \mathbb{A}$, and for each $r > 0$

$$S_t B^r \subseteq \mathcal{O}$$

for all sufficiently large t , where $B^r = \{\mu \in X : \int |u|^2 d\mu(u) \leq r\}$

The methods in [7] do not make essential use of Loeb spaces although at some points they can be employed to assist the construction.

4.2 Stochastic attractors

For a stochastic system such as (14) the idea of a *stochastic attractor* developed by Crauel & Flandoli [10] takes into account the fact that at all times new noise is introduced into the evolution of each path of any solution to (14). A **stochastic attractor** is defined to be a random set $\mathbb{A}(\omega)$ that, at time 0, attracts trajectories “starting at $-\infty$ ” (compared to the usual idea of an attractor being a set “at time ∞ ” that attracts trajectories starting at time 0).

This idea is spelled out below, and involves the introduction of a one parameter group $\theta_t : \Omega \rightarrow \Omega$ of measure preserving maps, which should be thought of as a shift of the noise to the left by t . In proving the existence of a stochastic attractor for the system (14) the nonstandard framework makes it particularly easy to consider $-\infty$.

¹For example, a sufficient condition is that $|f(u)|_{-1} \leq c + \delta_1 \|u\|$ and $|g(u)|_{\mathbf{H}, \mathbf{H}} \leq c + \delta_2 \|u\|$ for some $\delta_1, \delta_2 > 0$ with $2\delta_1 + \delta_2^2 \cdot \text{tr}Q < 2\nu$, where Q is the covariance of the \mathbf{H} -valued Wiener process w .

Making this precise, suppose that φ is a stochastic flow of solutions to (14). That is, φ is a measurable function

$$\varphi : [0, \infty) \times \mathbf{H} \times \Omega \rightarrow \mathbf{H}$$

such that $\varphi(\cdot, \cdot, \omega)$ is continuous for a.a. ω , and for each fixed initial condition u_0 the process $u(t, \omega) = \varphi(t, u_0, \omega)$ is a solution to (14) with $u(0, \omega) = u_0$.

The notion of a semigroup in the usual definition of a deterministic attractor, along with the notion of an attractor itself, is now replaced by the following.

Definition 4.2 (i) *The flow φ is a **crude cocycle** if for each $s \in \mathbb{R}^+$ there is a full set Ω_s such that for all $\omega \in \Omega_s$*

$$\varphi(s + t, x, \omega) = \varphi(t, \varphi(s, x, \omega), \theta_s \omega)$$

holds for each $x \in \mathbf{H}$ and $t \in \mathbb{R}^+$.

(ii) *A cocycle is **perfect** if Ω_s does not depend on s .*

(iii) *Given a perfect cocycle φ , a **global stochastic attractor** is a random compact subset $A(\omega)$ of \mathbf{H} such that for almost all ω*

$$\varphi(t, A(\omega), \omega) = A(\theta_t \omega), \quad t \geq 0,$$

$$\lim_{t \rightarrow \infty} \text{dist}(\varphi(t, B, \theta_{-t} \omega), A(\omega)) = 0$$

for each bounded set $B \subset \mathbf{H}$.

Note that the existence of a perfect cocycle is necessary for the possibility of having a stochastic attractor. Constructing a perfect cocycle is difficult for infinite dimensional systems, particularly for those that are truly stochastic (as compared to random dynamical systems in which paths may be treated individually).

4.2.1 Existence of a stochastic attractor for the Navier–Stokes equations

A stochastic attractor was constructed for the stochastic Navier–Stokes equation with $d = 2$ by Crauel & Flandoli [10], but their version of (14) reduced to a random equation that could be solved pathwise, giving essentially a pathwise construction of the random attractor $A(\omega)$. The first example of a stochastic attractor for a truly *stochastic* version of the Navier–Stokes equations was constructed in [9] using Loeb space methods, seemingly in an essential way. In the following, for simplicity the Wiener process was taken to be one dimensional.

Theorem 4.3 (Capiński & Cutland[9]) (a) *Suppose that $(g(u) - g(v), u - v) = 0$ and $(g(u), u) = 0$.² With appropriate Lipschitz and growth conditions on*

²For example $g(u) = \langle h, \nabla \rangle u$ for some $h \in \mathbf{H}$.

f, g , there is an adapted Loeb space carrying a stochastic flow of solutions to the system (14) that is a perfect cocycle, and there is a stochastic attractor $\mathbb{A}(\omega)$ (compact in the strong topology of \mathbf{H}) for this system.

(b) If g has the additional property that $((g(v), v)) = 0$ for $v \in \mathbf{V}$ the stochastic attractor is bounded and weakly compact in \mathbf{V} .

The proof of this result is quite long and complicated, and uses heavily the fact that solutions to (14) may be obtained as standard parts of Galerkin approximations of dimension N , infinite. A delicate extension of the Kolmogorov continuity theorem as adapted to a nonstandard setting by Lindström [1] is at the heart of the construction of the perfect cocycle. An outline of the main steps and ideas of the proof is given in Chapter 2 of [13].

4.3 Sell's approach for stochastic systems

As explained in detail in Section 3, Sell's radical approach [26] to the problem of attractors for the deterministic Navier–Stokes equations for $d = 3$, bearing in mind the possible nonuniqueness of solutions, was to replace the phase space \mathbf{H} by a space \mathcal{W} of entire solutions to the Navier–Stokes equations equipped with the semigroup S_t on \mathcal{W} defined by

$$(S_t u)(s) = u(t + s).$$

For the 3-dimensional stochastic case, Sell's idea was used by Flandoli & Schmalfuß in the paper [15] for the Navier–Stokes equations with a special form of multiplicative noise, using a mild solution concept. The equation considered allowed essentially a pathwise solution, and then a random attractor was obtained by combining Sell's approach with the idea of pulling back in time to $-\infty$, as developed by Crauel & Flandoli [10]. In a later paper [16] Flandoli & Schmalfuß consider in the same framework the Navier–Stokes equations with an irregular forcing term, but no feedback.

An alternative way to extend Sell's approach to the stochastic Navier–Stokes equations was developed in [14], and for systems of the form (14) with a very general multiplicative noise. The only assumption on the coefficients f, g is that they are continuous and fulfil a mild growth condition. For simplicity the Wiener process w is taken to be a 1-dimensional Brownian motion but this is not an essential restriction.

The idea is to consider a set X of solutions to (14) on a filtered probability space $\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and consider a semigroup $(S_t)_{t \geq 0}$ acting on X in the same way as Sell's semigroup on deterministic solutions. For this to make sense however some additional structure must be assumed for the underlying probability space Ω , because a simple time shift of an adapted process will not be adapted. Thus we now assume that the space $\mathbf{\Omega}$ is equipped with a family of measure preserving maps $\theta_t : \Omega \rightarrow \Omega$ for $t \geq 0$ with the following properties:

- (θ1) $\theta_0 = \text{identity}$ and $\theta_t \circ \theta_s = \theta_{t+s}$;
- (θ2) $\theta_t \mathcal{F}_s = \mathcal{F}_{t+s}$ for all $s, t \geq 0$;
- (θ3) $w(t+s, \theta_t \omega) - w(t, \theta_t \omega) = w(s, \omega)$ for all $s \geq 0$.

Note that the property (θ3) tells us that for a fixed t the increments of the process $w(t+s, \theta_t \omega)$ are the same as those of the process $w(s, \omega)$. Thus θ_t can be thought of as a shift of the noise to the right by t .

The family (θ_t) allows the following definition of a semiflow S_r of stochastic processes.

Definition 4.4 (Semiflow of Processes) (a) Suppose that $u = u(t, \omega)$ is an adapted stochastic process defined for $t > 0$. Then for any $r \geq 0$ the adapted process $v = S_r u$ is defined by

$$v(t, \omega) = u(r+t, \theta_r \omega).$$

(b) By the **semiflow** $(S_t)_{t \geq 0}$ on a filtered space Ω we mean that there is a measure preserving family $(\theta_t)_{t \geq 0}$ obeying (θ1) – (θ3) from which S_t is defined as above.

Note that for any process u the process $S_t u$ lives on the same space as u . For any theory of attractors involving the semiflow S_t and a set of solutions X of solutions to (14) on a filtered probability space Ω there are a number of difficulties to be addressed in the 3D situation, stemming largely from the possible non-uniqueness of solutions. First, in most standard existence proofs for (14) the space Ω that carries the solution has to be constructed carefully, and both it and the Wiener process depend on the solution itself. Here however we need a single space that carries solutions for all initial conditions (including those that are random); moreover, we need not just one solution for each initial condition but a sufficiently rich supply of solutions. This in itself requires a rather large probability space. Moreover, the space Ω must be rich enough to carry all processes that are in some sense infinite limits under the action of S_t . In the paper [14] the underlying space was an adapted Loeb space that was shown to meet all these requirements. (At a deeper level this is a consequence of the fact that Loeb spaces are *universal* – which can be described informally as saying that anything that happens anywhere happens on a Loeb space).

The need for a very large underlying probability space Ω raises a second set of problems. The resulting set of solutions X on Ω is also large and consequently it is unrealistic to expect an attractor \mathbb{A} that is compact; in fact various *noncompactness* results were proved in [14]. However, the attractor \mathbb{A} in [14] has a compact set of laws, but this is somewhat unsatisfactory. Similarly for the attracting property: in [14] this was established only for an *ad hoc* family of open neighbourhoods of \mathbb{A} .

In the remaining sections of this paper we to show that the attractor \mathbb{A} in [14] has stronger compactness and attraction properties that are described naturally using the framework of neometric spaces, in which we develop the notion of a *neoattractor* – which is *neocompact* and has the attraction property for all *neo-open* neighbourhoods of \mathbb{A} . This is the subject of the next section.

5 Neometric spaces and Neoattractors

Before developing the theory of neoattractors, later to be applied to the stochastic Navier–Stokes equations, we recall the basic definitions involved in the theory of neometric spaces, as developed in the papers [17] and [18] in the setting of a nonstandard universe. For the sake of completeness we provide in Appendix 2 a brief summary of the theory of neometric spaces.

5.1 Neometric spaces

Roughly speaking, a neometric space \mathcal{M} is a metric space with extra structure given by its parent, which is a \ast -metric space \mathbf{M} . We start by defining the standard part of an element of the parent space.

Given a \ast -metric space (\mathbf{M}, δ) (which is internal), the **standard part** ${}^\circ x$ of an element $x \in \mathbf{M}$ is defined by

$${}^\circ x = \{y \in \mathbf{M} : \delta(x, y) \approx 0\}.$$

Note that this is a generalisation of the notion of the standard part of a near-standard element of a standard metric space. For a subset $C \subseteq \mathbf{M}$, the **standard part of C** is the set ${}^\circ C = \{{}^\circ x : x \in C\}$. For each point $z \in \mathbf{M}$, the **nonstandard hull around z** is the set

$$\mathcal{H}(\mathbf{M}, z) = \{{}^\circ x : \delta(x, z) \text{ is finite}\}$$

with the metric $\rho({}^\circ x, {}^\circ y) = \text{st}(\delta(x, y))$. Each nonstandard hull is a complete metric space.

Definition 5.1 *By a **neometric space** we will mean a closed subspace \mathcal{M} of the nonstandard hull of a \ast -metric space \mathbf{M} around some point z . \mathbf{M} is called the **parent** of \mathcal{M} .*

Note that this notion is relative to a particular nonstandard universe. Each neometric space is a complete metric space. Hereafter, $\mathcal{M}, \mathcal{N}, \dots$ will stand for neometric spaces, and $\mathbf{M}, \mathbf{N}, \dots$ will be their parents.

The **monad** of a subset $A \subseteq \mathcal{M}$ is the set

$$\text{monad}(A) = \{x \in \mathbf{M} : {}^\circ x \in A\}.$$

Note that this is an extension of the idea of the monad of a point in a standard metric space. For any set $A \subseteq \mathcal{M}$, $A = {}^\circ(\text{monad}(A))$. The monad operation commutes with arbitrary unions, intersections, and complements.

A point $x \in \mathbf{M}$ is **near-standard** in \mathcal{M} , in symbols $x \in \text{ns}(\mathcal{M})$, if x belongs to the monad of \mathcal{M} , that is, ${}^\circ x \in \mathcal{M}$. (Thus $\text{ns}(\mathcal{M})$ is an alternative notation for $\text{monad}(\mathcal{M})$.) Note that the monad of \mathcal{M} is contained in the nonstandard hull around a point.

We adopt the usual convention of identifying a point x of a standard metric space M with the standard part of its internal counterpart, ${}^\circ(*x)$. With this convention, each standard complete metric space M in the original superstructure is a closed subset of a nonstandard hull of $*M$, and thus is a neometric space. This is important for applications; for example, the product of a neometric space with the real line is a neometric space.

Definition 5.2 *By a Π_1^0 set we mean the intersection of a countable collection of internal subsets of \mathbf{M} .*

*A set $A \subseteq \mathbf{M}$ is said to be **countably determined** if there is a countable family of internal sets B_1, B_2, \dots such that A is a finite or infinite Boolean combination of B_1, B_2, \dots*

Thus any Π_1^0 set is countably determined. It is clear that the family of countably determined sets in \mathbf{M} is closed under complements, and under finite and countable unions and intersections.

The *neocompact* subsets of a neometric space are analogues of compact sets.

Definition 5.3 *Let \mathcal{M} be a neometric space.*

*A set $C \subseteq \mathcal{M}$ is **neocompact** (in \mathcal{M}) if C is the standard part of some Π_1^0 set $A \subseteq \text{ns}(\mathcal{M})$.*

Note that if $\mathcal{N} \subseteq \mathcal{M}$, then a set $C \subseteq \mathcal{N}$ is neocompact in \mathcal{N} if and only if it is neocompact in \mathcal{M} . It is easily seen that finite unions of neocompact sets, and finite Cartesian products of neocompact sets, are neocompact; \aleph_1 -saturation shows that countable intersections of neocompact sets are also neocompact. In a standard separable metric space, the neocompact sets are the same as the compact sets.

We now introduce the neoclosed sets.

Definition 5.4 *Let $X \subseteq \mathcal{M}$. A set $C \subseteq X$ is **neoclosed in X** if $C \cap D$ is neocompact for every neocompact set $D \subseteq X$ in \mathcal{M} . The complement of a neoclosed set in X is called **neopen** in X .*

Note that if $X \subseteq \mathcal{M}$ and C is neoclosed in \mathcal{M} , then $C \cap X$ is neoclosed in X . However, $C \cap X$ is not necessarily neoclosed in \mathcal{M} . If X is itself neoclosed in \mathcal{M} then a set $D \subseteq X$ is neoclosed in X if and only if it is neoclosed in \mathcal{M} .

Whereas neocompact is weaker than compact (Proposition 9.2(a)), neoclosed is stronger than closed.

Examples 5.5 (a) ([17], **Proposition 4.14.**) For each neocompact set C and positive real number r , the set $C^{\leq r} = \{x : \rho(x, C) \leq r\}$ is neoclosed in \mathcal{M} .

(b) ([17], **Lemma 4.7**) For each $x \in \mathcal{M}$ and $r \in (0, \infty)$, the open ball $\{y : \rho(x, y) < r\}$ is neoopen in \mathcal{M} .

The analogue of the continuous functions is as follows.

Definition 5.6 A function $f : \mathcal{M} \rightarrow \mathcal{N}$ is said to be **neocontinuous** if for every neocompact set C in \mathcal{M} , the restriction $f \upharpoonright C = \{(x, f(x)) : x \in C\}$ of f to C is neocompact in $\mathcal{M} \times \mathcal{N}$.

Neocontinuous is a stronger notion than continuous, but many naturally occurring continuous functions are actually neocontinuous.

Examples 5.7 (a) ([18], **p. 145.**) The distance function ρ is neocontinuous from $\mathcal{M} \times \mathcal{M}$ to \mathbb{R} .

(b) The projection function is neocontinuous from $\mathcal{M} \times \mathcal{N}$ to \mathcal{M} .

5.2 Neo-attractors

Let \mathcal{M} be a neometric space and let $X \subseteq \mathcal{M}$. By a **neo-semiflow** on X we mean a function $S : [0, \infty) \times \mathcal{M} \rightarrow \mathcal{M}$ such that $S_t(\cdot)$ is neocontinuous for each particular $t \in [0, \infty)$, and for all $x \in X$ and $s, t \in [0, \infty)$, we have $S_t(x) \in X$, $S_0(x) = x$, and $S_{s+t}(x) = S_s(S_t(x))$.

We assume throughout this section that S is a neo-semiflow on X . Since S_t is a neocontinuous function for each $t \in [0, \infty)$, it follows from Proposition 9.7 that $S_t C$ is neocompact whenever C is neocompact.

In the example in Section 7 and thereafter, S will actually be a **neo-continuous semiflow** on X , that is, a neo-semiflow which is a neocontinuous function from $[0, \infty) \times \mathcal{M}$ into \mathcal{M} . However, in this section it is enough for S to be a neo-semiflow.

Definition 5.8 A **neo-attractor** for S on X in \mathcal{M} is a set $\mathbb{A} \subseteq X$ such that:

(a) \mathbb{A} is neocompact in \mathcal{M} .

(b) $S_t \mathbb{A} = \mathbb{A}$ for all $t \in [0, \infty)$.

(c) For each bounded set $B \subseteq X$ and neoopen set $\mathcal{O} \supseteq \mathbb{A}$ in X , $S_t B \subseteq \mathcal{O}$ eventually (that is, there exists $r \in [0, \infty)$ such that $S_t B \subseteq \mathcal{O}$ for all $t \in [r, \infty)$.)

It is clear that if $X \neq \emptyset$ then every neo-attractor is nonempty, because condition (c) fails when $A = \mathcal{O} = \emptyset$.

Proposition 5.9 There is at most one neo-attractor for S on X .

Proof. Suppose \mathbb{A}, \mathbb{A}' are neo-attractors for S on X in \mathcal{M} , and let $x \in \mathbb{A} \setminus \mathbb{A}'$. Since \mathbb{A} is neocompact, \mathbb{A} is bounded. The set $\mathcal{O} = X \setminus \{x\}$ is a superset of \mathbb{A}' which is neopen in X , so $S_t \mathbb{A} \subseteq \mathcal{O}$ for some $t \in [0, \infty)$. But $x \in \mathbb{A} = S_t \mathbb{A}$, which contradicts $x \notin \mathcal{O}$. ■

Note that the above uniqueness result holds even if condition (a) of Definition 5.8 is weakened to the condition that \mathbb{A} is bounded.

Definition 5.10 An **absorbing set** for S on X is a set $E \subseteq X$ such that $S_t B \subseteq E$ eventually for each bounded set $B \subseteq X$.

Lemma 5.11 Let B and E be bounded absorbing sets for S on X . If $\lim_{k \rightarrow \infty} t_k = \infty$, then

$$\bigcap_{k \in \mathbb{N}} S_{t_k} E = \bigcap_{t \geq 0} S_t E = \bigcap_{t \geq 0} S_t B.$$

Proof. It suffices to prove that the first set is contained in the third. Since B is absorbing and E is bounded, there exists $b \geq 0$ such that $S_u E \subseteq B$ for all $u \geq b$. Let $t \in [0, \infty)$. Take $k \in \mathbb{N}$ such that $t_k - t \geq b$. Then $S_{t_k} E = S_t S_{t_k - t} E \subseteq S_t B$, so $\bigcap_{k \in \mathbb{N}} S_{t_k} E \subseteq \bigcap_{t \geq 0} S_t B$. ■

Lemma 5.12 Let E be a bounded absorbing set for S on X , and let $\mathbb{A} = \bigcap_{t \geq 0} S_t E$. Then $S_t \mathbb{A} \subseteq \mathbb{A}$ for each $t \in [0, \infty)$.

Proof. Take $b \geq 0$ so that $S_u E \subseteq E$ for all $u \geq b$. For any $t, u \in [0, \infty)$,

$$\mathbb{A} \subseteq S_{u+b} E, \quad S_{t+b} E \subseteq E,$$

so

$$S_t \mathbb{A} \subseteq S_t S_{u+b} E = S_u S_{t+b} E \subseteq S_u E.$$

Since this holds for all $u \in [0, \infty)$, it follows that $S_t \mathbb{A} \subseteq \mathbb{A}$. ■

The next theorem is a neometric generalization of Theorem 2.4. It shows that \mathbb{A} is a neo-attractor in the case that the absorbing set E is neocompact.

Theorem 5.13 Let S be a neo-semiflow on X with a neocompact absorbing set E . Then the set $\mathbb{A} = \bigcap_{u \in [0, \infty)} S_u E$ is a neo-attractor for S on X .

Proof. We have $\mathbb{A} \subseteq E \subseteq X$. Since E is neocompact, $S_t E$ is neocompact for each $t \geq 0$. Then by Lemma 5.11, \mathbb{A} is a countable intersection of neocompact sets, and thus \mathbb{A} is neocompact.

Take $b \geq 0$ such that $S_u E \subseteq E$ for all $u \geq b$. Now let \mathcal{O} be a neopen set in X such that $\mathbb{A} \subseteq \mathcal{O}$. To prove that $S_t B \subseteq \mathcal{O}$ eventually for each bounded set $B \subseteq X$ it suffices to show that $S_t E \subseteq \mathcal{O}$ eventually. Suppose not. Then there is a sequence $t_k, k \in \mathbb{N}$ such that for each k , $S_{t_k} E \setminus \mathcal{O} \neq \emptyset$ and $t_{k+1} \geq t_k + b$. Each set $S_{t_k} E$ is neocompact, and since $X \setminus \mathcal{O}$ is neoclosed in X , $S_{t_k} E \setminus \mathcal{O}$ is neocompact. Moreover, $t = t_{k+1} - t_k \geq b$, so $S_t E \subseteq E$ and hence

$S_{t_{k+1}}E = S_{t_k}S_tE \subseteq S_{t_k}E$. Therefore $\{S_{t_k}E \setminus \mathcal{O}\}$ is a decreasing chain. Then by Lemma 5.11 and countable compactness, $\mathbb{A} \setminus \mathcal{O} = \bigcap_{k \in \mathbb{N}} (S_{t_k}E \setminus \mathcal{O}) \neq \emptyset$, contradicting $\mathbb{A} \subseteq \mathcal{O}$.

Let $t \in [0, \infty)$. We have $S_t\mathbb{A} \subseteq \mathbb{A}$ by Lemma 5.12. It remains to prove that $\mathbb{A} \subseteq S_t\mathbb{A}$. Choose $r \in (b, \infty)$. For each $n \in \mathbb{N}$ let $C_n = S_{n \cdot r}E$. Then C_n is neocompact. Since $S_rE \subseteq E$, $C_{n+1} = S_{n \cdot r}S_rE \subseteq S_{n \cdot r}E = C_n$, so $\{C_n\}$ is a decreasing chain. Since $r > 0$, $n \cdot r \rightarrow \infty$ as $n \rightarrow \infty$. By Lemma 5.11, $\mathbb{A} = \bigcap_{n \in \mathbb{N}} C_n$. By Proposition 9.9,

$$S_t\mathbb{A} = S_t \left(\bigcap_{n \in \mathbb{N}} C_n \right) = \bigcap_{n \in \mathbb{N}} (S_t C_n).$$

Moreover, for each $n \in \mathbb{N}$, $\mathbb{A} \subseteq S_{n \cdot r+t}E = S_t S_{n \cdot r}E = S_t C_n$. Thus $\mathbb{A} \subseteq S_t\mathbb{A}$. ■

Definition 5.14 *Let S be a neo-semiflow on X . By a **neo-subflow** of S we mean the restriction $S \upharpoonright Z$ of S to a neoclosed set Z in X such that $S_t Z \subseteq Z$ for each $t \geq 0$.*

Corollary 5.15 *Assume the hypotheses of Theorem 5.13. Then every neo-subflow of S has a neo-attractor.*

The next theorem gives a weaker sufficient condition for S to have a neo-attractor on X , which is analogous to the necessary and sufficient condition for the deterministic case in Theorem 2.5. We first prove a lemma.

Lemma 5.16 *Let E be a bounded absorbing set for S on X , and let $\mathbb{A} = \bigcap_{t \geq 0} S_t E$. Suppose that E is neoclosed in X and, whenever $\lim_{n \rightarrow \infty} t_n = \infty$, $B = \{x_1, x_2, \dots\} \subseteq E$, and $x_n \in S_{t_n}E$ for each $n \in \mathbb{N}$, there is a neocompact set C such that $B \subseteq C \subseteq X$.*

Then whenever $\lim_{n \rightarrow \infty} t_n = \infty$, $B = \{x_1, x_2, \dots\}$, and $x_n \in S_{t_n}E$ for each $n \in \mathbb{N}$, there is a neocompact set D such that $B \subseteq D \subseteq \mathbb{A} \cup B$.

Proof. Suppose $\lim_{n \rightarrow \infty} t_n = \infty$, $B = \{x_1, x_2, \dots\}$, and $x_n \in S_{t_n}E$ for each $n \in \mathbb{N}$. Take b such that $S_u E \subseteq E$ for all $u \geq b$. For each $k \in \mathbb{N}$, let $B_k = \{x_n : t_n \geq k + b\}$. Then $B \setminus B_k$ is finite. Now fix k for the moment. For each $x_n \in B_k$, $x_n \in S_{t_n}E = S_k S_{t_n - k}E$ so we may choose $y_n \in S_{t_n - k}E \subseteq E$ such that $x_n = S_k(y_n)$. Write $C_k = \{y_n : x_n \in B_k\} \subseteq E$, so that $B_k = S_k C_k \subseteq S_k E$. By hypothesis there is a neocompact set D_k such that $C_k \subseteq D_k \subseteq X$. Since E is neoclosed in X , we may take $D_k \subseteq E$ (or else replace D_k by $D_k \cap E$ which is neocompact). Then $B_k \subseteq S_k D_k \subseteq S_k E$. $S_k D_k$ is neocompact by Proposition 9.7, and since $B \setminus B_k$ is finite, $(S_k D_k) \cup B$ is neocompact.

Carrying out this for all k gives the set $D = \bigcap_{k \in \mathbb{N}} ((S_k D_k) \cup B)$ which is neocompact. By Lemma 5.11, $\mathbb{A} = \bigcap_{k \in \mathbb{N}} S_k E$. Therefore $B \subseteq D \subseteq \mathbb{A} \cup B$. ■

Theorem 5.17 *Let S be a neo-semiflow on X . Assume that:*

- (a) $\text{monad}(X)$ is countably determined,
- (b) For each $u \geq 0$ there is an internal function $\mathbb{T}_u : \mathbb{M} \rightarrow \mathbb{M}$ such that ${}^\circ(\mathbb{T}_u(x)) = S_u({}^\circ x)$ for all $x \in \text{monad}(X)$, and
- (c) There is a bounded absorbing set E for S such that whenever $\lim_{n \rightarrow \infty} t_n = \infty$, $B = \{x_1, x_2, \dots\} \subseteq E$, and $x_n \in S_{t_n}E$ for each $n \in \mathbb{N}$, there is a neocompact set $C \subseteq X$ such that $B \subseteq C$.

Then the set $\mathbb{A} = \bigcap_{u \in [0, \infty)} S_u E$ is a neo-attractor for S on X .

Moreover, any neo-subflow $S \upharpoonright Z$ of S such that $\text{monad}(Z)$ is countably determined has a neo-attractor.

Proof. It is easily seen that if condition (c) holds for E , then it also holds for $D \cap X$ where D is any closed ball containing E . We may therefore assume that $E = D \cap X$ for some closed ball D in \mathcal{M} . Then $\text{monad}(E) = \text{monad}(X) \cap D$ for some Π_1^0 set D , and therefore $\text{monad}(E)$ is countably determined. Moreover, D is neoclosed in \mathcal{M} , so E is neoclosed in X .

We will use Theorem 9.3 to show that \mathbb{A} is neocompact. Let $B = \{x_n : n \in \mathbb{N}\} \subseteq \mathbb{A}$. Then $x_n \in S_n E$ for each $n \in \mathbb{N}$. By hypothesis (c) and Lemma 5.16, there is a neocompact set C such that $B \subseteq C \subseteq \mathbb{A} \cup B = \mathbb{A}$.

We now show that $\text{monad}(\mathbb{A})$ is countably determined. By Lemma 5.11, we have $\mathbb{A} = \bigcap_{n \in \mathbb{N}} S_n E$. It follows from (b) that

$$\text{monad}(S_n E) = \bigcap_{m \in \mathbb{N}} ((\mathbb{T}_n \text{monad}(E))^{1/m}).$$

Then using Lemma 9.1, we see that $\text{monad}(S_n E)$ is countably determined for each $n \in \mathbb{N}$, and therefore

$$\text{monad}(\mathbb{A}) = \text{monad}\left(\bigcap_{n \in \mathbb{N}} S_n E\right) = \bigcap_{n \in \mathbb{N}} \text{monad}(S_n E)$$

is countably determined. Then by Theorem 9.3, \mathbb{A} is neocompact.

We next prove that $\mathbb{A} = S_t \mathbb{A}$ for each $t \in [0, \infty)$. We have $S_t \mathbb{A} \subseteq \mathbb{A}$ by Lemma 5.12. Let $x \in \mathbb{A}$. Then for each $n \in \mathbb{N}$, $x \in S_{t+n} E = S_t S_n E$, so there exists $y_n \in S_n E$ such that $x = S_t(y_n)$. For each $k \in \mathbb{N}$, let $B_k = \{y_n : k \leq n \in \mathbb{N}\}$. By (c) and Lemma 5.16, there is a neocompact set C_k such that $B_k \subseteq C_k \subseteq \mathbb{A} \cup B_k$. Then $x \in S_t C_k$. We may choose the sets C_k to be a decreasing chain. Then the set $C = \bigcap_{k \in \mathbb{N}} C_k$ is neocompact, and $C \subseteq \mathbb{A}$. By Proposition 9.9, $S_t C = \bigcap_{k \in \mathbb{N}} S_t C_k$. Then $x \in S_t C \subseteq S_t \mathbb{A}$. It follows that $\mathbb{A} = S_t \mathbb{A}$.

Now let \mathcal{O} be a neoopen set in X such that $\mathbb{A} \subseteq \mathcal{O}$. To prove that $S_t D \subseteq \mathcal{O}$ eventually for each bounded set $D \subseteq X$ it suffices to show that $S_t E \subseteq \mathcal{O}$ eventually. Suppose not. Then there is a sequence $\{t_k\}$ such that $\lim_{k \rightarrow \infty} t_k = \infty$ and $S_{t_k} E \setminus \mathcal{O} \neq \emptyset$ for each $k \in \mathbb{N}$. Choose $x_k \in S_{t_k} E \setminus \mathcal{O}$. For each $k \in \mathbb{N}$ let $B_k = \{x_n : n \geq k\}$. By (c) and Lemma 5.16, there is a neocompact set C_k

such that $B_k \subseteq C_k \subseteq \mathbb{A} \cup B_k$. Since \mathcal{O} is neoopen in X and $B_k \cap \mathcal{O} = \emptyset$, we may take C_k disjoint from \mathcal{O} . We may also take the sets C_k to be a decreasing chain. By countable compactness, the intersection $C = \bigcap_{k \in \mathbb{N}} C_k$ is nonempty. But $C \subseteq \mathbb{A}$ and $C \cap \mathcal{O} = \emptyset$, contradicting $\mathbb{A} \subseteq \mathcal{O}$. This shows that \mathbb{A} is a neo-attractor for S on X .

Finally, the hypotheses (a)–(c) also hold for any subflow $S \upharpoonright Z$ such that $\text{monad}(Z)$ is countably determined. ■

Here is a sufficient condition for \mathbb{A} being a neo-attractor which will be used in the next section.

Corollary 5.18 *Let S be a neo-semiflow on X . Assume conditions (a) and (b) of Theorem 5.17, and*

(c') There is a bounded absorbing set E for S and a decreasing chain of internal sets $C_n, n \in \mathbb{N}$ such that $\bigcap_n C_n \subseteq \text{monad}(X)$, and for each $n \in \mathbb{N}$, $S_t(E) \subseteq {}^\circ C_n$ eventually.

Then Theorem 5.17 (c) holds for E , so $\mathbb{A} = \bigcap_{u \in [0, \infty)} S_u E$ is a neo-attractor for S on X .

Proof. Suppose $\lim_{n \rightarrow \infty} t_n = \infty$, $B = \{x_1, x_2, \dots\} \subseteq E$, and $x_n \in S_{t_n} E$ for each $n \in \mathbb{N}$. We may assume without loss of generality that $C_1 = M$. For each k let n_k be the greatest $n \leq k$ such that $S_{t_k}(E) \subseteq {}^\circ C_n$. Then $\lim_{k \rightarrow \infty} n_k = \infty$. Take $\mathbf{x}_k \in C_{n_k}$ such that ${}^\circ \mathbf{x}_k = x_k$. Let $B = \{\mathbf{x}_k : k \in \mathbb{N}\}$. Then $B \setminus C_n$ is finite, so $D_n = C_n \cup B$ is a decreasing chain of internal sets with

$$B \subseteq \bigcap_{n \in \mathbb{N}} D_n \subseteq \text{monad}(X).$$

Then $D = {}^\circ \bigcap_{n \in \mathbb{N}} D_n$ is neocompact and $B \subseteq D \subseteq X$. ■

6 Process attractors for 3D stochastic Navier-Stokes equations

In the paper [14] a set X of solutions to the stochastic Navier–Stokes equations (14) was defined and it was shown that there is a set $\mathbb{A} \subseteq X$ that is an attractor in some sense — but falling short of the requirements of the definition in Part I for deterministic settings (Definition 2.2). In particular \mathbb{A} is not compact. The main result of the present paper is that \mathbb{A} is however a *neo-attractor* — which seems to be the strongest kind of notion likely to be applicable in this framework. Here we outline the results of [14] in preparation for the next section, where we will see how they fit into the framework of neometric spaces.

The equations considered are

$$du = [-\nu Au - B(u) + f(u)]dt + g(u)dw_t \quad (15)$$

in the functional formulation so that

$$u : [0, \infty) \times \Omega \rightarrow \mathbf{H}$$

and for simplicity (though it is not essential) the process w is taken to be a 1-dimensional Wiener process. The only conditions imposed on the time-independent coefficients f, g are as follows, where c_0, d_1, d_2 are positive real constants.

(H1) $f : \mathbf{H} \rightarrow \mathbf{H}$ and $|f(u)| \leq c_0 + d_1|u|$.

(H2) $g : \mathbf{H} \rightarrow \mathbf{H}$ and $|g(u)| \leq c_0 + d_2|u|$.

(H3) f and g are continuous.

(H4) $2d_1 + 3d_2^2 < 2\nu\lambda_1$.

The general theory of stochastic Navier–Stokes equations expounded in [8] shows that the equation (15) can be solved with only the assumptions (H1)–(H3). The additional growth restriction (H4) on f, g is needed here to obtain the attractor.

6.1 The space Ω and the semiflow

The particular space Ω that we use is a filtered Loeb space similar to that used in [8] for the construction of solutions to the stochastic Navier–Stokes equations. Loeb spaces constitute a special class of probability spaces that are very rich – in a sense that can be made precise (see for example [22]). The richness is needed to be able to solve the general stochastic Navier–Stokes equations in dimension 3, and it was needed in [14] to show that the single space Ω has solutions to (15) with the same (prescribed) Wiener process w_t for any random initial condition.

For the rest of the paper we fix the following adapted Loeb space. Set $\Omega = {}^*(C_0(\mathbb{R}))$, the internal space of * continuous functions $\omega : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$ with $\omega(0) = 0$, and let Q be the internal * Wiener measure on Ω .

Thus the canonical process

$$W(\tau, \omega) = \omega(\tau)$$

is a two-sided * Wiener process under Q . This gives the internal filtered probability space

$$\bar{\Omega} = (\Omega, \mathcal{G}, (\mathcal{G}_\tau)_{\tau \in {}^*\mathbb{R}}, Q),$$

where $\mathcal{G}_\tau = {}^*\sigma(\{\mathcal{W}(\tau') : \tau' \leq \tau\})$ and $\mathcal{G} = \bigvee_{\tau \in {}^*\mathbb{R}} \mathcal{G}_\tau$.

A family of internal measure preserving maps $\Theta_\tau : \Omega \rightarrow \Omega$ is defined for $\tau \in {}^*\mathbb{R}$ by

$$(\Theta_\tau(\omega))(\sigma) = \omega(\sigma - \tau) - \omega(-\tau).$$

That is, Θ_τ is a shift of the path ω to the right by τ and then adjusted to be 0 at 0.

Now let $P = Q_L$ be the Loeb measure obtained from Q with the corresponding Loeb σ -algebra $\mathcal{F} = L(\mathcal{G})$, giving the Loeb probability space $(\Omega, L(\mathcal{G}), Q_L) = (\Omega, \mathcal{F}, P)$, and denote the P -null sets by \mathcal{N} .

Definition 6.1

(a) The filtered probability space $\mathbf{\Omega}$ is

$$\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P),$$

where the right continuous filtration $(\mathcal{F}_t)_{t \geq 0}$ is defined by

$$\mathcal{F}_t = \bigcap_{t < \tau} \sigma(\mathcal{G}_\tau) \vee \mathcal{N}.$$

(b) The Wiener process $w(t, \omega)$ on $\mathbf{\Omega}$ is defined by

$$w(t, \omega) = {}^\circ W(t, \omega). \tag{16}$$

(c) The family of measure preserving transformations $(\theta_t)_{t \geq 0}$ is given by

$$\theta_t = \Theta_t.$$

That is, the restriction of the family (Θ_τ) to non-negative standard times.

It is well known that (16) defines an almost surely continuous Wiener process on $\mathbf{\Omega}$. It is clear that the family θ_t satisfies conditions $(\theta 1)$, $(\theta 2)$, $(\theta 3)$.

With the space $\mathbf{\Omega}$ and the family $(\theta_t)_{t \geq 0}$ now fixed, the semiflow of processes S_t defined by Definition 4.4 is also fixed for the rest of the paper.

6.2 Solutions to the stochastic Navier–Stokes equations

We define below a particular class X of weak solutions to the stochastic Navier–Stokes equations (15). Each element u of X is an adapted stochastic process (with $u(t, \omega) \in \mathbf{H}$ for all t, ω). The properties required for membership of X are among those that can be deduced heuristically from (15) using elementary stochastic calculus. The definition incorporates some *truncation functions* ψ_n, φ_n , defined from the following real C^2 function $\psi : [0, \infty) \rightarrow [0, 1]$ which is designed to be concave on $[0, 1]$ and constant (with value 1) on $[1, \infty)$.

Definition 6.2 (Truncation functions) (a)

$$\psi(x) = \begin{cases} (x - 1)^3 + 1 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

(b) For each $n \in \mathbb{N}$,

$$\psi_n(x) = \psi(x/n^2).$$

(c) For each $n \in \mathbb{N}$ a function $\varphi_n(u)$ is defined for u in any Hilbert space (finite or infinite dimensional) by

$$\varphi_n(u) = u^2 \psi_n(u^2),$$

where we write u^2 to mean $|u|^2$ to ease the notation.

The way the functions φ_n figure in the definition of X is to guarantee a kind of uniform integrability of solutions in the time variable, which crops up in the paper [14] in the guise of S-integrability from Loeb integration theory: an internal random vector $\mathbf{u}(\omega)$, $|\mathbf{u}|^2$ is S-integrable if and only if ${}^\circ\mathbb{E}(\varphi_n(\mathbf{u})) \rightarrow 0$ as $n \rightarrow \infty$.

Now for the definition of the class of solutions X .

Definition 6.3 (a) *Given positive real constants $k_1, k_2, k_3, \alpha, \beta$, denote by X the class of adapted stochastic processes $u : (0, \infty) \times \Omega \rightarrow \mathbf{H}$ with the following properties.*

(X1) *For a.a. ω the path $u(\cdot, \omega)$ belongs to the following spaces:*

$$L_{\text{loc}}^\infty(0, \infty; \mathbf{H}) \cap L_{\text{loc}}^2[0, \infty; \mathbf{H}) \cap L_{\text{loc}}^2(0, \infty; \mathbf{V}) \cap C(0, \infty; \mathbf{H}_{\text{weak}}).$$

(X2) *For all $t_1 \geq t_0 > 0$*

$$u(t_1) = u(t_0) + \int_{t_0}^{t_1} [-\nu Au(t) - B(u(t)) + f(u(t))]dt + \int_{t_0}^{t_1} g(u(t))dw_t.$$

(X3) *For a.a. $t_0 > 0$ and all $t_1 \geq t_0$,*

$$\mathbb{E}(|u(t_1)|^2) \leq \mathbb{E}(|u(t_0)|^2) \exp(-k_1(t_1 - t_0)) + k_2. \quad (17)$$

(X4) *For a.a. $t_0 > 0$ and all $t_1 \geq t_0$,*

$$\mathbb{E} \left(\sup_{t_0 \leq s \leq t_1} |u(s)|^2 + \int_{t_0}^{t_1} \|u(s)\|^2 ds \right) \leq \alpha \mathbb{E}(|u(t_0)|^2) + \beta(t_1 - t_0). \quad (18)$$

(X5) *For a.a. $t_0 > 0$ and all $t_1 \geq t_0$ and $n \in \mathbb{N}$,*

$$\mathbb{E}(\varphi_n(u(t_1))) \leq \mathbb{E}(\varphi_n(u(t_0)) \exp(-k_3(t_1 - t_0))) + n^{-\frac{1}{2}}(\alpha \mathbb{E}(|u(t_0)|^2) + \beta). \quad (19)$$

(X6) $\mathbb{E} \int_0^1 |u(t)|^2 dt < \infty$.

(b) *Denote by X_k the set of $u \in X$ with*

(X6_k) $\mathbb{E} \int_0^1 |u(t)|^2 dt \leq k$.

Remarks

1. The class X depends on the constants $k_1, k_2, k_3, \alpha, \beta$; there is a natural choice of these – see Theorem 6.10 below.

2. The sets X_k increase with k .

3. The above conditions tell us nothing about $u(t, \omega)$ at $t = 0$ and there may be a singularity there. In this sense the class X is a class of *generalized weak solutions* to the stochastic Navier–Stokes equations (cf. [26], p.12).

4. The meaning of “loc” in the path properties (X1) is as follows: $L^p_{\text{loc}}(0, \infty)$ means $L^p[1/n, n]$ for all n , whereas $L^p_{\text{loc}}[0, \infty)$ means $L^p[0, n]$ for all n .

5. The conditions (X5) follow naturally from the Foias equation for the stochastic Navier–Stokes equations (see [5]), which may be deduced heuristically from the equation (15). The choice of the functions φ_n makes (X5) a uniform integrability condition for $|u(t)|^2$ on any $[t_0, \infty)$.

6. The semiflow S_t maps X into X .

The paths of any solution $u \in X$ lie in the space $M = L^2(0, \infty; \mathbf{H}; \mu)$ where $\mu(dt) = \exp(-t)dt$, defined in Section 3.1. Thus

$$X \subseteq L^2(\Omega, M).$$

The following lemma relates the sets X_k to bounded sets.

Lemma 6.4 ([14] Lemma 4.2) *If $B \subseteq X$, then B is bounded in $L^2(\Omega, M)$ if and only if $B \subseteq X_k$ for some $k \in \mathbb{N}$.*

The notion of an attractor that was discussed in [14] is as follows. Here we call it a *process attractor* to avoid confusion with the notion of Part I (Definition 2.2). It involves laws of processes, and needs the following definitions concerning the laws of solutions viewed as probability distributions on the space of paths M .

Definition 6.5 *Let $u(t, \omega)$ be a process with paths in M .*

(a) *law(u) is the probability law on M induced by u ; i.e.*

$$\text{law}(u)(Z) = \mathbb{P}(u(\cdot, \omega) \in Z)$$

for Borel $Z \subseteq M$.

(b) *law $_w$ (u) = law(u, w), the probability law induced on $M \times C_0$ by the pair of processes $(u(t, \omega), w(t, \omega))$, where $C_0 = C_0[0, \infty)$.*

For the space of probability laws $\mathcal{M}_1(S)$ on a separable metric space S a fundamental metric is the Prohorov metric, which we denote by d_0 ; this makes $\mathcal{M}_1(S)$ separable. Here we are thinking of $S = M$ and $S = M \times C_0$.

There is a natural projection mapping $\pi : \mathcal{M}_1(M \times C_0) \rightarrow \mathcal{M}_1(M)$ defined by

$$\pi(\lambda)(Z) = \lambda(Z \times C_0).$$

In the current situation the laws on the space M that we are interested are laws of L^2 random variables, so it is appropriate to define a stronger metric to reflect this.

Definition 6.6 (a) $\mathcal{M}_{1,2}(M) = \{\mu \in \mathcal{M}_1(M) : \mathbb{E}_\mu(|u|^2) < \infty\}$.

(b) the metric d on $\mathcal{M}_{1,2}(M)$ is defined by

$$d(\mu_1, \mu_2) = d_0(\mu_1, \mu_2) + |\mathbb{E}_{\mu_1}(|u|^2) - \mathbb{E}_{\mu_2}(|u|^2)|.$$

(c) $\mathcal{M}_{1,2}(M \times C_0) = \{\lambda \in \mathcal{M}_1(M \times C_0) : \pi(\lambda) \in \mathcal{M}_{1,2}(M)\}$.

(d) the metric d on $\mathcal{M}_{1,2}(M \times C_0)$ is defined by

$$d(\lambda_1, \lambda_2) = d_0(\lambda_1, \lambda_2) + |\mathbb{E}_{\mu_1}(|u|^2) - \mathbb{E}_{\mu_2}(|u|^2)|,$$

where $\mu_i = \pi(\lambda_i)$ ($i = 1, 2$).

The following lemma is easy to check (where ρ is the L^2 metric on $L^2(\Omega, M)$).

Lemma 6.7 (a) The function law maps $L^2(\Omega, M)$ into $\mathcal{M}_{1,2}(M)$ and is continuous with respect to the metrics ρ and d .

(b) The function law $_w$ maps $L^2(\Omega, M)$ into $\mathcal{M}_{1,2}(M \times C_0)$ and is continuous with respect to the metrics ρ and d .

(c) The mapping $\pi : \mathcal{M}_1(M \times C_0) \rightarrow \mathcal{M}_1(M)$ defined by

$$\pi(\lambda)(Z) = \lambda(Z \times C_0)$$

is continuous with respect to the metric d .

Using these notions we have:

Definition 6.8 (a) A set of laws $\mathcal{A} \subseteq \text{law}_w(X)$ is a **law-attractor** for the semiflow S_t on X if:

(i) (**Invariance**) $\widehat{S}_t \mathcal{A} = \mathcal{A}$ for all $t \geq 0$.

(ii) (**Attraction**) For any open set $\mathcal{O} \supset \mathcal{A}$ and d -bounded set $\mathcal{B} \subseteq \text{law}_w(X)$,

$$\widehat{S}_t \mathcal{B} \subseteq \mathcal{O}$$

eventually (i.e. for some $t_0 = t_0(\mathcal{O}, \mathcal{B})$, this holds for all $t \geq t_0$).

(iii) (**Compactness**) \mathcal{A} is compact in the metric d .

(b) A **process attractor** for the semiflow S_t on X is a set of processes $\mathbb{A} \subseteq X$ such that:

(i) $\text{law}_w(\mathbb{A})$ is a law-attractor (in particular $\text{law}_w(\mathbb{A})$ is compact in the metric d , and so \mathbb{A} is bounded).

(ii) (**Invariance**) $S_t \mathbb{A} = \mathbb{A}$ for all $t \geq 0$.

(iii) (**Attraction**) For any bounded set $B \subseteq X$ and compact set $K \subseteq L^2(\Omega, M)$,

$$\underline{\lim}_{t \rightarrow \infty} \rho(S_t B, K) \geq \rho(A, K).$$

(iv) \mathbb{A} is closed in the space $L^2(\Omega, M)$.

A smaller class $\hat{X} \subseteq X$ of solutions defined at 0 is as follows.

Definition 6.9 Denote by \hat{X} the class of stochastic processes $u : [0, \infty) \times \Omega \rightarrow \mathbf{H}$ with $u \in X$ (that is, the restriction of u to $(0, \infty)$ lies in X) with the following additional properties:

($\hat{X}1$) For a.a. ω , the path $u(\cdot, \omega)$ is in

$$L_{\text{loc}}^\infty[0, \infty; \mathbf{H}] \cap L_{\text{loc}}^2[0, \infty; \mathbf{H}] \cap L_{\text{loc}}^2[0, \infty; \mathbf{V}] \cap C[0, \infty; \mathbf{H}_{\text{weak}}].$$

($\hat{X}2$) For all $t_1 \geq t_0 \geq 0$,

$$u(t_1) = u(t_0) + \int_{t_0}^{t_1} [-\nu Au(t) - B(u(t)) + f(u(t))] dt + g(u(t)) dw_t.$$

($\hat{X}3$) $\mathbb{E}(|u(t)|^2)$ is bounded on $[0, \infty)$.

Note that ($\hat{X}1$) implies (X1), ($\hat{X}2$) implies (X2), and ($\hat{X}3$) implies (X6). (In [14] the symbol Y was used for the subspace \hat{X} .)

The main theorem of the paper [14] is:

Theorem 6.10 ([14] **Theorems 4.3, 9.12**) *There are constants $k_1, k_2, k_3, \alpha, \beta$ (which are given explicitly in terms of the parameters c_0, d_1, d_2, λ_1 and ν of the system) such that*

- (a) *for every L^2 \mathcal{F}_0 -measurable initial condition there is a solution $u \in \hat{X}$;*
- (b) *there is a process attractor \mathbb{A} for the semiflow S_t on X and $\mathbb{A} \subset \hat{X}$.*

7 Neo-attractors for 3D stochastic Navier-Stokes equations

In order to show that the set \mathbb{A} given by Theorem 6.10 is in fact a neo-attractor we will prove that the conditions of Corollary 5.18 hold for the set X , the semiflow $(S_t)_{t \geq 0}$ and an absorbing set $E \subseteq X$ in the context of $L^2(\Omega, M)$ as a neometric space. For this it is necessary to give some properties of the neometric spaces $L^p(\Omega, M)$, and then to describe a little more of the detail of the paper [14].

7.1 The neometric spaces $L^p(\Omega, M)$

The definitions we review here are applicable to any complete separable metric space (M, d) , although we will only need them for the specific space $M = L^2(0, \infty; \mathbf{H}; \mu)$ that figures in previous section (and defined in Section 3.1). Among the most important examples of neometric spaces are the spaces $L^p(\Omega, M)$, where $p \in \{0\} \cup [1, \infty)$. The parent of the space $\mathcal{M} = L^p(\Omega, M)$ is

$$\mathbf{M} = (SL^0(\Omega, M), \bar{\rho}_p)$$

where $SL^0(\Omega, M)$ is the set of all $*$ -measurable functions $\mathbf{x} : \Omega \rightarrow *M$,

$$\bar{\rho}_0(\mathbf{x}, \mathbf{y}) = * \inf \{ \varepsilon : Q[*d(\mathbf{x}(\omega), \mathbf{y}(\omega)) \geq \varepsilon] \leq \varepsilon \},$$

and when $p \geq 1$,

$$\bar{\rho}_p(\mathbf{x}, \mathbf{y}) = \left[\int (*d(\mathbf{x}(\omega), \mathbf{y}(\omega)))^p dQ \right]^{1/p}.$$

It is convenient that for all p , the parent spaces are $*$ -metric spaces on the same set $SL^0(\Omega, M)$, even though the $*$ -metrics $\bar{\rho}_p$ are different. We have

$$\begin{aligned} \text{ns}(\mathcal{M}) &= \text{ns}^p(\Omega, M) \\ &= \{ \mathbf{x} \in \text{ns}^0(\Omega, M) : (*d(\mathbf{x}(\omega), z))^p \text{ is S-integrable for all } z \in M \} \end{aligned}$$

where

$$\text{ns}^0(\Omega, M) = \{ \mathbf{x} \in SL^0(\Omega, M) : \mathbf{x}(\omega) \in \text{ns}(M) \text{ for a.a. } \omega \}.$$

In the following proposition we list some facts we need about these neometric spaces.

Proposition 7.1 *Let M, N be complete separable metric spaces and let $\mathcal{M}_1(M)$ be the space of Borel probability measures on M with the Prohorov metric.*

(a) (**[18], Propositions 5.7 and 6.10.**) *For each $p \in \{0\} \cup [1, \infty)$, the law function is neocontinuous from $L^p(\Omega, M)$ to $\mathcal{M}_1(M)$.*

(b) (**[18], Theorem 5.14.**) *For each compact set C in $\mathcal{M}_1(M)$, the set*

$$\{ x \in L^0(\Omega, M) : \text{law}(x) \in C \}$$

is neocompact in $L^0(\Omega, M)$.

(c) (**[18], Theorem 6.7.**) *If $p \in [1, \infty)$, a set $C \subseteq L^p(\Omega, M)$ is neocompact in $L^p(\Omega, M)$ if and only if it is neocompact in $L^0(\Omega, M)$ and uniformly p -integrable.*

(d) (**[17], Proposition 9.4**) *For each $p \in \{0\} \cup [1, \infty)$, the set of all adapted stochastic processes is neoclosed in $L^p(\Omega, M)$.*

(e) (**[17], Lemma 5.19.**) *For every continuous function $f : M \rightarrow N$, the function $x(\cdot) \rightarrow f(x(\cdot))$ is neocontinuous from $L^0(\Omega, M)$ to $L^0(\Omega, N)$.*

From now on all discussion is in the context of the neometric space $\mathcal{M} = L^2(\Omega, M)$ where $M = L^2(0, \infty; \mathbf{H}; \mu)$ as in Section 6.

We now show that $S = (S_t)_{t \geq 0}$ is a neo-semiflow on $X \subset \mathcal{M} = L^2(\Omega, M)$. In fact, it is a neocontinuous semiflow on X as well as on the larger neometric space $\mathcal{M} = L^2(\Omega, M)$. To do this we define an internal semiflow that represents S . The semiflow S_t on X extends naturally to give

$$S : [0, \infty) \times L^2(\Omega, M) \rightarrow L^2(\Omega, M)$$

which has a natural internal counterpart

$$\mathbb{T} : {}^*[0, \infty) \times SL^0(\Omega, M) \rightarrow SL^0(\Omega, M)$$

given by

$$(\mathbb{T}_\tau \mathbf{u})(\sigma, \omega) = \mathbf{u}(\sigma + \tau, \Theta_\tau \omega)$$

The next proposition is routine, where we define

$$\text{NS} = \text{ns}^2(\Omega, M) = \text{ns}(\mathcal{M})$$

with $\mathcal{M} = L^2(\Omega, M)$.

Proposition 7.2 *For finite $\tau > 0$ and for $\mathbf{u} \in \text{NS}$*

$${}^\circ(\mathbb{T}_\tau \mathbf{u}) = S_{\circ\tau} {}^\circ \mathbf{u}. \quad (20)$$

Thus S is a neocontinuous semiflow on \mathcal{M} and on X .

Proof. Neocontinuity follows from Proposition 9.8 ■

7.2 Internal approximate solutions

In [14] the solutions $u \in X$ are represented by internal *approximate* solutions living in the hyperfinite dimensional space $\mathbf{H}_N \subset {}^*\mathbf{H}$. These are carried on the internal filtered probability space $\bar{\Omega} = (\Omega, \mathcal{G}, (\mathcal{G}_\tau)_{\tau \geq 0}, Q)$. The following extracts from [14] the key properties of approximate solutions.

Definition 7.3 (a) *For each $k \in \mathbb{N}$ and $n \in {}^*\mathbb{N}$ the internal set $\mathbf{X}_{k,n}$ is a set of * -adapted (with respect to $(\mathcal{G}_\tau)_{\tau \geq 0}$) processes*

$$\mathbf{u} : {}^*[0, \infty) \times \Omega \rightarrow \mathbf{H}_N$$

such that

(X1) $\mathbf{u}_\tau(\omega)$ has paths * a.s. in *M and $\mathbf{u} \in {}^*L^2(\Omega, {}^*M)$ i.e.

$$\mathbb{E} \left(\int_0^{*\infty} |\mathbf{u}_\tau(\omega)|^2 \exp(-\tau) d\tau \right) < {}^*\infty.$$

(X2_n) \mathbf{u} is a $\frac{1}{n}$ -approximate solution (that is, \mathbf{u} has properties that approximate the conditions (X2)-(X5) of Definition 6.3 to order $\frac{1}{n}$)

(X3_{k,n}) \mathbf{u} is bounded by $k + \frac{1}{n}$; that is

$$\mathbb{E} \int_0^1 |\mathbf{u}_\tau|^2 d\tau \leq k + \frac{1}{n}$$

(b) For each $k \in \mathbb{N}$, define

$$\mathbf{X}_k = \bigcap_{n \in \mathbb{N}} \mathbf{X}_{k,n}.$$

(c) The set of internal approximate solutions is

$$\mathbf{X} = \bigcup_{k \in \mathbb{N}} \mathbf{X}_k.$$

The importance of \mathbf{X} lies in the following result from [14].

Theorem 7.4 ([14] **Theorem 9.2**) (a) For each $k \in \mathbb{N}$,

$${}^\circ(\mathbf{X}_k \cap \text{NS}) = X_k,$$

and hence

(b)

$${}^\circ(\mathbf{X} \cap \text{NS}) = X.$$

Corollary 7.5 (a) For each $k \in \mathbb{N}$, the monad of X_k is a Π_1^0 set. The monad of X is countably determined, and in fact is a countable union of Π_1^0 sets.

(b) Each X_k is neoclosed;

(c) X is neoclosed

Proof. (a) For each $k, n \in \mathbb{N}$, $(\mathbf{X}_{k,n})^{1/n}$ is internal, and $\text{monad}(X_k) = \bigcap_{n \in \mathbb{N}} ((\mathbf{X}_{k,n})^{1/n})$. We also have $\text{monad}(X) = \bigcup_{k \in \mathbb{N}} \text{monad}(X_k)$.

(b) By Proposition 9.6 using (a).

(c) Let D be neocompact (in the neometric space $\mathcal{M} = L^2(\Omega, M)$); then D is bounded and so $D \cap X \subseteq X_k$ for some k so that $D \cap X = D \cap X_k$ is neocompact. ■

The internal semiflow \mathbb{T}_τ has the following properties when restricted to \mathbf{X} .

Proposition 7.6 ([14] **Lemma 8.3**) For finite $\tau > 0$:

(a) $\mathbb{T}_\tau \mathbf{X} \subseteq \mathbf{X}$.

(b) $\mathbb{T}_\tau(\mathbf{X} \cap \text{NS}) \subseteq \mathbf{X} \cap \text{NS}$.

Next note that there is an S -absorbing set for the internal semiflow T_τ on X .

Lemma 7.7 ([14] Lemma 8.4) *There is $k_0 \in \mathbb{N}$ such that X_{k_0} is S -absorbing. That is, for each $k \in \mathbb{N}$ there is an $r(k) \in \mathbb{N}$ such that*

$$T_\tau X_k \subseteq X_{k_0}$$

for all finite $\tau \geq r(k)$. In fact we may take any $k_0 > k_2$.

Now make the following definitions.

Definition 7.8 (a) $E = X_{k_0}$;
 (b) $E_n = X_{k_0, n}$ (so that $E = \bigcap_{n \in \mathbb{N}} E_n$);
 (c) $E = X_{k_0}$

Theorem 7.9 E is a bounded neoclosed absorbing set for S on X .

Proof. E is an absorbing set by Proposition 7.2, Theorem 7.4, Proposition 7.6 and Lemma 7.7. E is bounded by Lemma 6.4. E is neoclosed by Proposition 9.6, since $E = {}^\circ(E \cap NS)$ and E is a Π_1^0 set. ■

Now let

$$\mathbb{A} = \bigcap_{t \geq 0} S_t E$$

The main theorem of [14] (see Theorem 6.10 above) shows that \mathbb{A} is the process attractor for the semiflow S_t on X . To see that it is a neoattractor we show that the conditions of Corollary 5.18 are fulfilled.

7.3 \mathbb{A} is a neoattractor

Most of the conditions of Corollary 5.18 follow from the previous section. For the remaining conditions of we continue with some details from [14] From the S -absorbing set E define the following set C , called the S -attractor for the internal semiflow T_τ on X .

Definition 7.10 Define sets C and C_n (for $n \in \mathbb{N}$) as follows.

- (a) $C_n = \bigcap_{0 \leq \tau \leq n} T_\tau E_n$.
- (b) $C = \bigcap_{n \in \mathbb{N}} C_n$.

(Note: The sets C_n are those denoted \widehat{C}_n in [14].)

Proposition 7.11 (a) The sets C_n are internal and decreasing.

(b) \mathbf{C} is a Π_1^0 set.

(c) $\mathbf{C} \subseteq \mathbf{E}$.

Now comes a key result of [14], which uses the uniform integrability condition on solutions in X that is coded up by the truncation functions.

Theorem 7.12 ([14] **Theorem 9.4(d)**) *\mathbf{C} is nearstandard; that is, $\mathbf{C} \subseteq \text{NS}$.*

Finally we have

Proposition 7.13 *For each $n \in \mathbb{N}$*

$$S_t(E) \subseteq {}^\circ(\mathbf{C}_n)$$

eventually.

Proof. Since \mathbf{E} is \mathbf{S} -absorbing there is finite r_0 with $\mathbf{T}_\tau \mathbf{E} \subseteq \mathbf{E}$ for finite $\tau \geq r_0$; then for any finite $\tau \geq 0$ we have

$$\mathbf{T}_{n+r_0+\tau} \mathbf{E} = \mathbf{T}_n \mathbf{T}_{r_0+\tau} \mathbf{E} \subseteq \mathbf{T}_n \mathbf{E} \subseteq \mathbf{C}_n.$$

So by Proposition 7.2 and Theorem 7.4, if $t \geq n + r_0$ then

$$S_t E \subseteq {}^\circ(\mathbf{C}_n)$$

■

Gathering this together we have

Theorem 7.14 *\mathbb{A} is a neo-attractor for the the neocontinuous semiflow $(S_t)_{t \geq 0}$ on the set X of solutions to the stochastic Navier–Stokes equations (15).*

Proof. We verify the hypotheses of Corollary 5.18. Hypotheses (a) and (b) hold by Corollary 7.5 and Proposition 7.2. E is a bounded absorbing set by Theorem 7.9. The sets \mathbf{C}_n form a decreasing chain of internal sets by Proposition 7.11, and $\mathbf{C} \subseteq \text{monad}(X)$ by Proposition 7.11 (c) and Theorem 7.12. Finally, $S_t(E) \subseteq {}^\circ \mathbf{C}_n$ eventually by Proposition 7.13. The result now follows from Corollary 5.18. ■

We remark that in [14], the set \mathbb{A} is defined as $\mathbb{A} = {}^\circ \mathbf{C}$, and it is then proved that \mathbb{A} is a process attractor and is equal to $\bigcap_{t \geq 0} S_t E$.

Below (Proposition 7.16) we show that this result is a genuine strengthening of the main result of [14]. Close examination of the results of [14] would show that the full strength of the results of section 5.2 is not essential to prove Theorem 7.14. However, the following result does not follow from the results of [14] and gives a much stronger result than proved there.

The following result shows that one still gets a neo-attractor after adding additional inequalities that are preserved under the semiflow. It gives a scheme for proving the existence of neoattractors for systems of stochastic Navier–Stokes equations with additional specialised properties.

Theorem 7.15 *Any neo-subflow $S \upharpoonright Y$ of S such that the monad of Y is countably determined has a neo-attractor.*

Proof. By Theorem 5.17 and Corollary 5.18. ■

Finally, we show that the notion of neo-attractor for the stochastic Navier–Stokes equations is stronger than the notion of a process attractor given in [14] and described above, because the compactness and attraction properties required of a process attractor are special cases of the more general properties possessed by a neo-attractor. Thus Theorem 7.14 improves the main result of [14].

Proposition 7.16 *Suppose \mathbb{A} is a neo-attractor for S on X . Then \mathbb{A} is a process attractor for S on X .*

Proof. The function $\text{law}_w(\cdot) : \mathcal{M} \rightarrow \mathcal{M}_{1,2}$ is neocontinuous, so $\mathcal{A} = \text{law}_w(\mathbb{A})$ is neocompact in $\mathcal{M}_{1,2}$. Since $\mathcal{M}_{1,2}$ is separable, \mathcal{A} is compact in the metric d . The invariance of \mathcal{A} follows at once from the invariance of \mathbb{A} . For any open set \mathcal{O} in $\mathcal{M}_{1,2}$, $O = \text{law}_w^{-1}(\mathcal{O}) \cap X$ is neopen in X . If $\mathcal{A} \subseteq \mathcal{O}$, then $\mathbb{A} \subseteq O$. Then for each bounded set $\mathcal{B} \subseteq \text{law}_w(X)$, $B = X \cap \text{law}_w^{-1}(\mathcal{B})$ is bounded in X , and $S_t B \subseteq O$ eventually, so $\widehat{S}_t(\mathcal{B}) \subseteq \mathcal{O}$ eventually. This shows that \mathcal{A} is a law-attractor for S on X .

For each compact set $K \subseteq \mathcal{M}$ and each $r \in (0, \infty)$, K is neocompact, so $K^{\leq r}$ is neoclosed in \mathcal{M} and its complement $O_r = \{x \in X : \rho(x, K) > r\}$ is neopen in X . If $\rho(\mathbb{A}, K) > r$, then $\mathbb{A} \subseteq O_r$, and thus $S_t B \subseteq O_r$ eventually for each bounded $B \subseteq X$. It follows that $\liminf_{t \rightarrow \infty} \rho(S_t B, K) \geq \rho(\mathbb{A}, K)$. This shows that \mathbb{A} has the attraction property of a process attractor (Definition 6.8). Finally, \mathbb{A} is closed by Proposition 9.5. ■

7.4 Two-sided solutions

In [14] it is proved that the attractor (now shown to be a neo-attractor) is characterised as the restriction to nonnegative times of the set of all bounded two-sided solutions (that is, solutions defined for all time negative and positive). If \bar{X} is the set of all such solutions to the stochastic Navier-Stokes equations we have:

Theorem 7.17 ([14] **Theorem 10.3**)

$$\mathbb{A} = \bar{X} \upharpoonright [0, \infty)$$

We conclude by noting that

Theorem 7.18 *\bar{X} is neo-compact.*

Proof. It is shown in [14] that $\bar{X} = {}^\circ\bar{X}$ where \bar{X} is the set of internal approximate two-sided solutions; moreover \bar{X} is a Π_1^0 set and all members of \bar{X} are nearstandard. We refer the reader to [14] for details. ■

8 Appendix 1: Nonstandard preliminaries

We work in an \aleph_1 -saturated nonstandard universe that contains a nonstandard extension $^*\mathcal{J}$ for every mathematical object \mathcal{J} involved in our theory. In particular we have $^*\mathbb{R}, ^*\mathbb{N}, ^*\mathbf{H}, ^*M, ^*C_0(\mathbb{R}), ^*\text{Wiener measure}$, etc. The Appendix of [14] provides a brief introduction to those parts of nonstandard analysis that are needed there and the reader is referred to that paper. Below we mention the most important ideas needed for this paper; for further details see [13] for example or any of the standard references [1],[2],[8],[21] or [23].

Here is a brief description;

8.1 The nonstandard universe

We start with a base set \mathbb{B} which contains all the standard objects involved in our discussion. In particular, \mathbb{B} should contain the set of reals \mathbb{R} and the linear space \mathbf{H} . The following *superstructure over* \mathbb{B} , denoted by $\mathbb{V} = V(\mathbb{B})$, is an adequate (standard) mathematical universe for our purposes (where $\mathcal{P}(A)$ denotes the set of all subsets of a set A):

$$\begin{aligned} V_0(\mathbb{B}) &= \mathbb{B} \\ V_{n+1}(\mathbb{B}) &= V_n(\mathbb{B}) \cup \mathcal{P}(V_n(\mathbb{B})), \quad n \in \mathbb{N} \end{aligned}$$

and

$$\mathbb{V} = V(\mathbb{B}) = \bigcup_{n \in \mathbb{N}} V_n(\mathbb{B}).$$

Next, we use the ultrapower construction to build the nonstandard extension $^*\mathbb{B} \supset \mathbb{B}$, and at the same time construct a mapping $^* : V(\mathbb{B}) \rightarrow V(^*\mathbb{B})$ which associates to each set $A \in \mathbb{V}$ a nonstandard counterpart $^*A \in V(^*\mathbb{B})$. At level 0, we simply have $^*b = b$ for each $b \in \mathbb{B}$. At level 1, for each $A \subset \mathbb{B}$ we have $A \subset ^*A \subset ^*\mathbb{B}$, with $^*A \setminus A$ consisting of “ideal” or “nonstandard” elements. For example $^*\mathbb{N} \setminus \mathbb{N}$ consists of infinite (hyper)natural numbers.

In general, for each set $A \in \mathbb{V}$, the mapping * maps A injectively into *A . So even for mathematical objects³ \mathcal{J} at higher levels, $^*\mathcal{J}$ can be regarded as an extension of \mathcal{J} .

The resulting *nonstandard universe* is the collection

$$^*\mathbb{V} = \{x : x \in ^*A \text{ for some } A \in \mathbb{V}\}$$

consisting of all members of nonstandard counterparts of sets in \mathbb{V} . Although $^*\mathbb{V} \subset V(^*\mathbb{B})$, it is crucial to realize that $^*\mathbb{V}$ is **not** the same as $V(^*\mathbb{B})$. Sets in $^*\mathbb{V}$ are known as *internal sets*; a set is *external* if it is not internal.

The key property of the nonstandard universe that makes it tractable is the Transfer Principle which indicates precisely which properties of the superstructure \mathbb{V} are inherited by $^*\mathbb{V}$.

³We are taking the approach that every mathematical object is actually a set.

Theorem 8.1 (The Transfer Principle) *Suppose that φ is a bounded quantifier statement. Then φ holds in \mathbb{V} if and only if ${}^*\varphi$ holds in ${}^*\mathbb{V}$.*

A *bounded quantifier statement (bqs)* is simply a statement of mathematics that can be written in such a way that all quantifiers range over a prescribed set. That is, we have subclauses such as $\forall x \in A$ and $\exists y \in B$ but not unbounded quantifiers such as $\forall x$ and $\exists y$. Most quantifiers in mathematical practice are bounded (often only implicitly in exposition). A *bqs* φ may also contain fixed sets \mathcal{M} from \mathbb{V} , which will be replaced in ${}^*\varphi$ by ${}^*\mathcal{M}$.

Members of internal sets are internal (this follows easily from the construction) and since the sets ${}^*\mathcal{M}$ are also internal, it follows that the information we obtain from the Transfer Principle is entirely about *internal sets*.

It is possible (and quite convenient) to take an axiomatic approach to ${}^*\mathbb{V}$, which simply postulates the existence of a set ${}^*\mathbb{V}$ and a mapping ${}^* : \mathbb{V} \rightarrow {}^*\mathbb{V}$ that obeys the Transfer Principle. For most purposes (and certainly the construction of Loeb measures) the further assumption of \aleph_1 -saturation is needed — a property that comes with the ultrapower construction.

8.2 \aleph_1 -saturation

Definition 8.2 A nonstandard universe ${}^*\mathbb{V}$ is said to be \aleph_1 -saturated if the following holds:

if $(A_m)_{m \in \mathbb{N}}$ is a countable decreasing sequence of *internal sets* with each $A_m \neq \emptyset$, then $\bigcap_{m \in \mathbb{N}} A_m \neq \emptyset$.

Theorem 8.3 *A nonstandard universe ${}^*\mathbb{V}$ constructed as a countable ultrapower is \aleph_1 -saturated.*

\aleph_1 -saturation is a kind of compactness property that is essential for the Loeb measure construction which plays a central role in this paper.

The basic fact is that for each (internal) * probability space $\Omega = (\Omega, \mathcal{G}, Q)$, the finitely additive probability measure ${}^\circ Q : \mathcal{G} \rightarrow \mathbb{R}$ has a σ -additive extension. (This is an important consequence of \aleph_1 -saturation.) The unique completion of this σ -additive extension is called the **corresponding Loeb space**, and is denoted by $(\Omega, \mathcal{G}_L, Q_L)$. For convenience, we assume here that Ω is * countably additive, although most of the general theory carries over to the * finitely additive case.

The theory of the Loeb measure and Loeb integration is assumed in this paper (see [8, 12, 13] for example).

8.3 Standard parts

Given a standard Hausdorff space $S \subset \mathbb{B}$, we have $S \subseteq {}^*S$. If $x \in S$ and $\mathfrak{x} \in {}^*S$, we say that x is the *standard part of* \mathfrak{x} , in symbols $x = {}^\circ \mathfrak{x}$ or $\mathfrak{x} \approx x$, if

$x \in {}^*O$ for every open neighborhood O of x . Since S is Hausdorff, each $x \in {}^*S$ has at most one standard part. An element $x \in {}^*S$ is said to be *near-standard*, in symbols $x \in \text{ns}(S)$, if X has a standard part (in S). Thus the standard part function maps $\text{ns}(S)$ onto S and is the identity on S . The *standard part* of a set $B \subseteq \text{ns}(S)$ is the set ${}^\circ B = \{{}^\circ x : x \in B\}$. Here is a useful immediate consequence of the definition of standard part.

Remark 8.4 Suppose $x = {}^\circ x$ in a Hausdorff space S .

- (a) If O is open in S , $x \in O$ implies $x \in {}^*O$.
- (b) If C is closed in S , $x \in {}^*C$ implies $x \in C$.

In the particular case of a standard metric space (S, ρ) , $x = {}^\circ x$ if and only if ${}^*\rho(x, x) \approx 0$, and two points $x, y \in {}^*S$ are said to be *infinitely close*, in symbols $x \approx y$, if ${}^*\rho(x, y) \approx 0$.

The following fundamental result is one of the keys to the power of non-standard analysis.

Theorem 8.5 Suppose S is a Hausdorff space and $C \subseteq S$. Then

- (a) C is compact if and only if ${}^*C \subseteq \text{ns}(C)$
- (b) C is relatively compact if and only if ${}^*C \subseteq \text{ns}(S)$

This criterion will often be used in conjunction with the fact that for a metric space compactness is equivalent to sequential compactness, as follows.

Theorem 8.6 Suppose S is a metric space and $C \subseteq S$. Then C is compact if and only if for every sequence (x_n) in C , $x_N \in \text{ns}(C)$ for every infinite N .

The nonstandard criterion for continuity is very intuitive.

Theorem 8.7 Let $f : S_1 \rightarrow S_2$ where S_i are topological spaces. Then f is continuous if and only if

$${}^*f(x) \approx {}^*f(y) \text{ whenever } x \approx y$$

The book [8] gives information about the standard part mapping for various topologies on the standard set \mathbf{H} . The most important are as follows. Here, ${}^*\mathbf{H}$ has an internal * basis $\{{}^*e_n\}_{n \in {}^*\mathbb{N}}$, and we write $E_n = {}^*e_n$. Thus for each $N \in {}^*\mathbb{N}$, $\mathbf{H}_N = {}^*\text{span}\{E_1, \dots, E_N\} \subseteq {}^*\mathbf{H}$. We also write $u_{(n)} = (u, e_n)$ for $u \in \mathbf{H}$ and $u_{(n)} = (u, E_n)$ for $u \in {}^*\mathbf{H}$.

Lemma 8.8 Let $u \in {}^*\mathbf{H}$. Then:

- (a) If $|u| < \infty$ (i.e. $|u|$ is finite) then u is weakly nearstandard in \mathbf{H} , and the weak standard part $u = \text{st}_{\text{weak}}(u)$ is defined by

$$u_{(n)} = {}^\circ(u_{(n)}), n \in \mathbb{N}.$$

(b) If \mathbf{u} is nearstandard in the strong topology of \mathbf{H} then $|\mathbf{u}| < \infty$ and

$$\text{st}_{\text{weak}}(\mathbf{u}) = \text{st}(\mathbf{u}).$$

(c) If $\|\mathbf{u}\| < \infty$ then \mathbf{u} is (strongly) nearstandard in \mathbf{H} .

In view of the consistency (b) above we use ${}^\circ\mathbf{u}$ to denote the standard part of \mathbf{u} whenever $|\mathbf{u}|$ is finite.

Near-standard points and standard parts also appear in the more general setting of an internal $*$ metric space, and in particular the space $SL^0(\Omega)$ of $*$ random variables on a $*$ measure space Ω . This is the starting point of the theory of neometric spaces, which is outlined in Section 5 and applied in Section 7.

9 Appendix 2: Neometric spaces

We give here a brief summary of those parts of the theory of neometric spaces that we need, as developed in the papers [17] and [18]. The fundamental definitions have been given in Section 5 so we do not repeat them here.

We will require the following consequence of \aleph_1 -saturation for countably determined sets.

Lemma 9.1 ([20]) *Projections of countably determined sets in \mathbf{M}^2 are countably determined in \mathbf{M} .*

The next two results give basic facts about neocompact sets and a theorem that is useful for proving results about neocompact sets, which follows from ([18] Corollary 3.8) and ([19] Proposition 3.1, Theorem 3.3).

Proposition 9.2 (a) *Every compact set is neocompact.*

(b) *If C is neocompact in $\mathcal{M} \times \mathcal{N}$, then the projection*

$$D = \{x \in \mathcal{M} : (\exists y \in \mathcal{N})(x, y) \in C\}$$

is neocompact in \mathcal{M} .

(c) *If C is neocompact in $\mathcal{M} \times \mathcal{N}$ and K is a nonempty compact subset of \mathcal{N} , then*

$$B = \{x \in \mathcal{M} : (\forall y \in K)(x, y) \in C\}$$

is neocompact in \mathcal{M} .

Theorem 9.3 *The following are equivalent for a set $C \subseteq \mathcal{M}$:*

(a) *C is neocompact.*

(b) *The monad of C is a Π_1^0 set.*

(c) *The monad of C is countably determined, and each countable subset of C is contained in a neocompact subset of C .*

Proof. It is obvious that (b) implies (a). To prove (a) implies (b), we observe that if $C = \circ \bigcap_{n \in \mathbb{N}} A_n$ where each A_n is internal, then $\text{monad}(C) = \bigcap_{n \in \mathbb{N}} ((A_n)^{1/n})$.

Clearly (a) and (b) implies (c), so it remains to prove that (c) implies (b). Assume (c). Then $\text{monad}(C)$ is an infinite Boolean combination of sets from a countable sequence B_1, B_2, \dots of internal subsets of M . Let us write $x \equiv_n y$ if x, y belong to the same sets among B_1, \dots, B_n , and let

$$D_n = \{x : (\exists y \in \text{monad}(C)) x \equiv_n y\}.$$

Then each D_n is internal because it is the finite union of \equiv_n equivalence classes. It suffices to prove that

$$\text{monad}(C) = \bigcap_{n \in \mathbb{N}} D_n.$$

It is clear that the left side is contained in the right side. Suppose $x \in \bigcap_{n \in \mathbb{N}} D_n$. Then for each $n \in \mathbb{N}$ there exists $y_n \in \text{monad}(C)$ such that $y_n \equiv_n x$. By hypothesis there is a neocompact set A such that

$$\{\circ y_n : n \in \mathbb{N}\} \subseteq A \subseteq C.$$

Since (a) implies (b), there is a decreasing chain A_n of internal sets such that $\text{monad}(A) = \bigcap_{n \in \mathbb{N}} A_n$. Then $y_n \in A_n$ for each $n \in \mathbb{N}$. By \aleph_1 -saturation there exists y such that $y \in A_n$ and $y \equiv_n x$ for all $n \in \mathbb{N}$. Then $y \in \text{monad}(A) \subseteq \text{monad}(C)$. Since $\text{monad}(C)$ is an infinite Boolean combination of the sets B_n , it follows that $x \in \text{monad}(C)$. ■

An important consequence is the following fact, which says that neocompact sets behave like compact sets. It can often be used as a shortcut in place of overspill.

Theorem 9.4 (Countable Compactness) *If B_m is a decreasing chain of nonempty neocompact sets, then $\bigcap_m B_m$ is a nonempty neocompact set.*

Proof. Let $B = \bigcap_m B_m$. By Lemma 9.3, for each m , $\text{monad}(B_m)$ is equal to a nonempty Π_1^0 set $\bigcap_n C_{mn}$. It is easily seen that

$$\bigcap_m \left(\bigcap_n C_{mn} \right) = \bigcap_m (\text{monad}(B_m)) = \text{monad}(B),$$

so B is neocompact. For each k there is a point x_k which belongs to C_{mn} for all $m, n < k$. By \aleph_1 -saturation, there is a point x that belongs to C_{mn} for all $m, n \in \mathbb{N}$. Therefore $\circ x \in B$, so B is nonempty. ■

Relationships between closed, neoclosed and neocompact are given next.

Proposition 9.5 (a) ([17], Lemma 4.6.) *Every neocompact set is neoclosed and bounded in \mathcal{M} .*

(b) ([17], Proposition 4.5.) *Every neoclosed set is closed.*

Thus, whereas neocompact is weaker than compact (Proposition 9.2(a)), neoclosed is stronger than closed.

For a set $D \subseteq M$, let ${}^\circ D = \{{}^\circ x : x \in D\}$. Neoclosed sets can often be found using the following proposition.

Proposition 9.6 ([18], Proposition 4.1 and Lemma 3.7) *Let $D_n, n \in \mathbb{N}$ be a decreasing chain of internal subsets of M and let D be the Π_1^0 set $D = \bigcap_n D_n$. Then*

(a) ${}^\circ D \cap \mathcal{M}$ is neoclosed in \mathcal{M} .

(b)

$${}^\circ D \cap \mathcal{M} = \bigcap_n {}^\circ(D_n) \cap \mathcal{M} = \bigcap_n {}^\circ((D_n)^{\leq 1/n}) \cap \mathcal{M}.$$

Here are the basic properties of neocontinuous functions.

Proposition 9.7 (See [17]) (a) *Every neocontinuous function is continuous.*

(b) *If \mathcal{M} is separable then every continuous function $f : \mathcal{M} \rightarrow \mathcal{N}$ is neocontinuous.*

(c) *Compositions of neocontinuous functions are neocontinuous.*

(d) *Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be neocontinuous.*

(i) *If C is neocompact in \mathcal{M} , then $f(C)$ is neocompact in \mathcal{N} .*

(ii) *If D is neoclosed in \mathcal{N} , then $f^{-1}(D)$ is neoclosed in \mathcal{M} .*

Neocontinuous functions can often be built using the next proposition.

Proposition 9.8 ([18], Theorems 4.16 and 4.18) . *A function $f : \mathcal{M} \rightarrow \mathcal{N}$ is neocontinuous if and only if for each neocompact set C in \mathcal{M} , there is an internal function $F : M \rightarrow N$ such that for all $x \in \text{monad}(C)$, $F(x) \in \text{monad}(\mathcal{N})$ and ${}^\circ(F(x)) = f({}^\circ x)$.*

As examples using this proposition we have the distance and projection functions noted earlier.

We will need the following consequence of countable compactness.

Proposition 9.9 *Suppose $f : \mathcal{M} \rightarrow \mathcal{N}$ is neocontinuous and $\{C_n\}$ is a countable decreasing chain of neocompact sets in \mathcal{M} . Then*

$$f\left(\bigcap_{n \in \mathbb{N}} C_n\right) = \bigcap_{n \in \mathbb{N}} f(C_n).$$

Proof. We prove the nontrivial direction. Let $y \in \bigcap_{n \in \mathbb{N}} f(C_n)$ and let $D = f^{-1}\{y\}$. The set $\{y\}$ is neocompact since it is compact, so D is neoclosed by Proposition 9.7. Then $D \cap C_n$ is a decreasing chain of nonempty neocompact sets. By countable compactness, there exists $x \in \bigcap_{n \in \mathbb{N}} (D \cap C_n)$. Thus $x \in \bigcap_{n \in \mathbb{N}} C_n$ and $f(x) = y$, as required. ■

References

- [1] S. Albeverio, J.-E. Fenstad, R. Høegh-Krohn, and T.L. Lindstrøm, *Non-standard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, New York 1986.
- [2] L.Arkerud, N.J.Cutland, and C.W.Henson, Eds. *Nonstandard Analysis, Theory and Applications*, NATO ASI Series, Series C: Mathematics and Physical Sciences Vol.493, Kluwer Academic Publishers, Dordrecht — Boston — London 1997.
- [3] M. Capiński and N.J. Cutland, A simple proof of existence of weak and statistical solutions of Navier–Stokes equations, *Proc. Royal. Soc. London A* **436** (1992), 1-11.
- [4] M. Capiński & N.J. Cutland, Stochastic Navier-Stokes equations, *Applicandae Mathematicae* **25**(1991), 59-85.
- [5] M. Capiński & N.J. Cutland, Statistical solutions of stochastic Navier-Stokes equations, *Indiana University Math Journal* **43**(1994), 927-940.
- [6] M. Capiński & N.J. Cutland, Attractors for three-dimensional Navier–Stokes equations, *Proc. Roy. Soc. Lond. Ser.A*, **453**(1997), 2413-2426.
- [7] M. Capiński & N.J. Cutland, Measure attractors for stochastic Navier–Stokes equations, *Electronic J. Prob.* **3**(1998), Paper 8, 1-15.
- [8] M.Capiński & N.J.Cutland, *Nonstandard Methods for Stochastic Fluid Mechanics*, Series on Advances in Mathematics for Applied Sciences, Vol 27, World Scientific, Singapore 1995.
- [9] M. Capiński & N.J. Cutland, Existence of global stochastic flow and attractors for Navier–Stokes equations, *Probab. Theory Relat. Fields* **115**(1999), 121-151.
- [10] H. Crauel and F. Flandoli, Attractors for random dynamical systems, *Probability Theory and Related Fields* **100** (1994), 365–393.
- [11] N.J. Cutland (ed.), *Nonstandard Analysis and its Applications*, Cambridge University Press, Cambridge 1988.
- [12] N.J. Cutland, Loeb measure theory, in *Developments in Nonstandard Mathematics*, Eds. N.J.Cutland, F.Oliveira, V.Neves, J.Sousa-Pinto, Pitman Research Notes in Mathematics Series Vol. **336**, Longman 1995, 151-177.
- [13] N.J. Cutland, *Loeb Measures in Practice: Recent Advances*, Lecture Notes in Mathematics **1751**(2000), Springer.

- [14] N.J. Cutland and H.J. Keisler, Global attractors for 3D stochastic Navier-Stokes equations, to appear in *J. Dynamics and Diff. Equations*.
- [15] F. Flandoli and B. Schmalfuss, Random attractors for the 3D stochastic Navier–Stokes equation with multiplicative white noise, *Stochastic & Stochastics Reports*, **59**(1996), 21-45.
- [16] F. Flandoli and B. Schmalfuss, Weak solutions and attractors for three-dimensional Navier–Stokes equations with nonregular force, *J. Dynamics and Differential Equations*, **11**(1999), 355-398.
- [17] S. Fajardo and H.J. Keisler, Existence theorems in probability theory, *Advances in Mathematics*, **120**(1996), 191-257.
- [18] S. Fajardo and H.J.Keisler, Neometric spaces, *Advances in Mathematics*, **118**(1996), 134-175.
- [19] S. Fajardo and H.J.Keisler, Long Sequences and Neocompact Sets, in *Developments in Nonstandard Analysis*, (N. Cutland *et.al.*, Eds., pp. 251-260, Longman, Harlow/New York 1995.
- [20] C.W. Henson, Analytic sets, Baire sets, and the standard part map. *Canadian Journal of Mathematics* **31** (1979), pp. 663-672.
- [21] A.E.Hurd and P.A.Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press, New York 1985.
- [22] H. J. Keisler, Rich and saturated adapted spaces, *Advances in Mathematics*, **128**(1997), 242-288.
- [23] T.L.Lindstrøm, An invitation to nonstandard analysis, in [11], 1-105.
- [24] D.A.Ross, Loeb measure and probability, in *Nonstandard Analysis: Theory and Applications*, Eds. L.O. Arkeryd, N.J. Cutland, C.W. Henson, NATO ASI Series C: Vol **493**, Kluwer, Dordrecht, 1997, 91-120.
- [25] B. Schmallfuss, Measure attractors of the stochastic Navier–Stokes equation, Bremen Report No.258, 1991.
- [26] G.R. Sell, Global attractors for the three-dimensional Navier–Stokes equations, *J.Dynamics and Diff. Equations* **8**(1996), 1-33.
- [27] R.Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, 1983.
- [28] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, 1988.