Maryanthe Malliaris and Saharon Shelah. *Cofinality spectrum problems in model theory, set theory and general topology.* ***J. Amer. Math. Soc.***, vol. 29 (2016), pp. 237-297.

Maryanthe Malliaris and Saharon Shelah. *Existence of optimal ultrafilters and the fundamental complexity of simple theories. To appear.* ***Advances in Mathematics***, arXiv:1404:2919[math.LO].

Maryanthe Malliaris and Saharon Shelah. *Keisler's order has infinitely many classes.* arXiv:1503.08341[math.LO].

In this ground-breaking series of papers, longstanding open problems in both model theory and set theory are solved. The authors indicate that these papers are part of an ongoing program. In this review we give a snapshot of the current state of the program.

For the past fifty years, one of the dominant themes in model theory has been Shelah's classification program, which seeks to classify first order theories according to how 'tame' the models of the theory are. The set of stable theories has been classified in a coherent way using the Morley rank and other concepts, and is well understood. A major goal of current research in model theory is to achieve a similar understanding of the unstable theories.

In the paper [H. Jerome Keisler, Ultraproducts which are not saturated, Journal of Symbolic Logic vol. 32 (1967), 23–46] I introduced a pre-ordering $\trianglelefteq$ on all complete first order theories, that measures the degree of reluctance of ultrapowers of models of a theory to be saturated. In that paper I posed the problem of determining the structure of this ordering, and asked whether it would give a fruitful classification of first order theories. This problem has turned out to be extraordinarily difficult, and little progress had been made until the recent work of Malliaris and Shelah.

The papers under review show that the ordering $\trianglelefteq$ gives a new and unexpected classification of the simple theories. They also solve a major problem in set theory that had been open since the 1940's by showing that $\mathfrak{p} = \mathfrak{t}$. It is quite surprising that this is actually a consequence of ZFC rather than an independence result, and that it comes out of a model-theoretic study of the $\trianglelefteq$ ordering.

Here is a brief review of the situation before about 2010. We use $T, U, \ldots$ for complete first order theories with countable signatures, D for an ultrafilter over an infinite set I, and λ for the cardinality of I. By the fundamental theorem of Łoś, every ultrapower of a model of T is a model of T. We say that D **saturates** T (or is **good** for T) if the ultrapower M^I/D is λ^+-saturated for every model M of T. Two important properties of filters are regularity and goodness. A filter F over I is **regular** if there is a set $E \subseteq F$ of cardinality λ such that each $i \in I$ belongs to only finitely many $e \in E$. F is **good** if every monotonic function $f : [\lambda]^{<\aleph_0} \to F$ has a multiplicative refinement $g : [\lambda]^{<\aleph_0} \to F$ (that is, $g(u) \subseteq f(u)$ and $g(u \cap v) = g(u) \cap g(v)$). In the early 1960's I introduced good filters, and showed that a regular ultrafilter D is good if and only if D saturates every T, and that the GCH implies that regular good ultrafilters exist. In 1972, Kunen proved the existence of regular good ultrafilters in ZFC.

In the 1967 paper referenced above, $T \trianglelefteq U$ is defined to mean that for every infinite cardinal λ, every regular ultrafilter over a set of power λ that saturates U also saturates T. $T \triangleleft U$ means $T \trianglelefteq U$ and not $U \trianglelefteq T$. The relation $\trianglelefteq$ is a pre-ordering of the class of all complete theories, and induces a partial ordering on the set of $\trianglelefteq$-equivalence classes of theories. The following results are from that paper. If D is regular and M^I/D is λ^+-saturated for some model of T, then D saturates T. There is a theory T that is $\trianglelefteq$-minimal in the sense that $T \trianglelefteq U$ for all U, and a theory U that is $\trianglelefteq$-maximal in the sense that $T \trianglelefteq U$ for all T. If T is $\trianglelefteq$-minimal, then T does not have the finite cover property (fcp) and is not $\trianglelefteq$-maximal.

Shelah proved the following results in the 1970's; details can be found in Chapter VI of the book [Saharon Shelah, Classification Theory, North-Holland 1990]. The minimal $\trianglelefteq$-equivalence class is the set all theories that do not have the fcp, and every such theory is stable. The next lowest $\trianglelefteq$-equivalence class is the set of stable theories with the fcp. If T is stable and U is unstable then $T \triangleleft U$. Every theory with the strict order property is $\trianglelefteq$-maximal. There was little further progress for the next several decades.

Because of these early results, people suspected that the ordering $\trianglelefteq$ would be coarse, and that there may only be a small finite number of $\trianglelefteq$-equivalence classes. The recent papers by Malliaris and Shelah contain the following results about $\trianglelefteq$, which completely change the picture.

(1) There is a minimum $\trianglelefteq$-equivalence class of unstable theories. It contains the theory T_{rg} of the random graph.
(2) If T is not low or not simple, then $T_{rg} \triangleleft T$. Thus every theory that is $\trianglelefteq$-equivalent to T_{rg} is low, simple, and unstable.
(3) There is an infinite sequence of theories $(U_0, U_1, \dots, U_n, \dots)$ such that

$$T_{rg} \triangleleft \dots \triangleleft U_n \triangleleft \dots \triangleleft U_1 \triangleleft U_0,$$

and for each n, every theory that is $\trianglelefteq$-equivalent to U_n is low and simple.
(4) Suppose there is a supercompact cardinal. If T is simple and $U \trianglelefteq T$, then U is simple.
(5) There is a minimum $\trianglelefteq$-equivalence class of non-simple theories. It contains the theory T^*_{feq}, which is the model completion of the theory of an infinite family of independent parametrized equivalence relations.
(6) Every SOP_2 theory is $\trianglelefteq$-maximal.

It follows from (1)–(3) that the set of $\trianglelefteq$-equivalence classes of simple unstable theories has a $\trianglelefteq$-minimum element, and is infinite and not well-ordered by $\trianglelefteq$.

Result (1) is from the paper [Malliaris, Hypergraph sequences as a tool for saturation of ultrapowers, Journal of Symbolic Logic vol. 77 (2012), pp. 195-223]. Result (2) is from the paper [Malliaris and Shelah, A dividing line within simple unstable theories, Advances in Mathematics vol. 249 (2013), pp. 250-288]. Result (3) is partly in the second and partly in the third paper under review. (4) is in the second paper under review. (5) and (6) are in the first paper under review.

In order to obtain the results (1)–(6), Malliaris and Shelah introduced new model-theoretic notions involving the realization and omitting of types, and new set-theoretic properties of ultrafilters. The latter are often "refinement properties"—weakenings of goodness that require only that certain monotonic functions have multiplicative refinements. Powerful methods were developed to construct ultrafilters, and to relate properties of ultrafilters to properties of ultrapowers. The following paragraphs will describe some of these methods in broad terms, skipping many details. From now on, D will always denote a regular ultrafilter.

A basic method in the proofs of results (2)–(4) is the construction of a regular ultrafilter over a set I from a (not necessarily regular) ultrafilter on a complete Boolean algebra $\mathfrak{B}$. Let D_* be an ultrafilter on $\mathfrak{B}$. Say that D is **built from** D_* if there is a surjective homomorphism $j\colon \mathcal{P}(I) \to \mathfrak{B}$ such that $j^{-1}(\{1_{\mathfrak{B}}\})$ is a good regular filter over I, and $D = j^{-1}(D_*)$. A general result, called "separation of variables", shows that an ultrafilter D_* on $\mathfrak{B}$ has a refinement property called **moral for** T if and only if every D built from D_* saturates T. This is useful because one often has more freedom in constructing ultrafilters on $\mathfrak{B}$ than over I.

Given cardinal numbers $\theta \le \mu < \lambda$, $\mathfrak{B}_{2^\lambda,\mu,\theta}$ is the completion of the Boolean algebra freely generated by elements x_f where f is a function whose domain is a subset of 2^λ of size $< \theta$, subject to the conditions that $x_f \le x_g$ when $f \subseteq g$, and $x_f \cap x_g = 0$ when

f, g are incompatible. It is shown that for every ultrafilter D_* on $\mathfrak{B}_{2^\lambda,\mu,\theta}$, there is a surjective j such that $j^{-1}(\{1_{\mathfrak{B}}\})$ is a good regular filter over I, and hence there exists a regular D that is built from D_*

(2) is related to a refinement property that has been around since the 1960's: D is OK if every monotone function $f : [\lambda]^{<\aleph_0} \to D$, such that $f(u) = f(v)$ whenever $|u| = |v|$, has a multiplicative refinement $g : [\lambda]^{<\aleph_0} \to D$. In the above (2012) paper, Malliaris introduced a refinement property called **flexibility**, which was later found to be equivalent to being OK. She showed that if D saturates a theory that is not low or not simple, then D is flexible. Let $\omega \le \mu < \lambda \le 2^\mu$ and let $\mathfrak{B} = \mathfrak{B}_{2^\lambda,\mu,\omega}$. In the above (2013) paper, it is shown that no ultrafilter D that is built from an ultrafilter on $\mathfrak{B}$ is flexible, and that there is an ultrafilter D_* in $\mathfrak{B}$ that is moral for T_{rg}. Hence any D that is built from D_* saturates T_{rg}, and any theory that is saturated by D is low and simple. Then for any theory U that is not low or not simple, not $T_{rg} \trianglelefteq U$. We also have $T_{rg} \trianglelefteq U$ by (1), so $T_{rg} \triangleleft U$.

(3) Let $T_{m,k}$ be the model completion of the theory with one symmetric irreflexive $(k+1)$-ary relation with no complete subgraphs on $m+1$ vertices. Hrushovski showed that $T_{m,k}$ is a low simple theory when $m > k \ge 2$. For each n, let U_n be the disjoint union of the theories $T_{k+1,k}$ for $k \ge 2n+2$. It is shown that (3) holds for these particular theories U_n.

To do this, the notion of a **perfect ultrafilter** is introduced. Being (λ, μ)-perfect is a refinement property of an ultrafilter D_* on the complete Boolean algebra $\mathfrak{B} = \mathfrak{B}_{2^\lambda,\mu,\omega}$. Suppose α is an ordinal, $2 \le k < \ell$, $\mu = \aleph_\alpha$, and $\lambda = \aleph_{\alpha+\ell}$. The following results are proved. There exists a (λ, μ)-perfect ultrafilter D_* on $\mathfrak{B}$. For every (λ, μ)-perfect ultrafilter D_* on $\mathfrak{B}$ and every $m > \ell$, D_* is moral for $T_{m+1,m}$. But for every ultrafilter D_* on $\mathfrak{B}$, D_* is not moral for $T_{k+1,k}$. So taking $\ell = k+1$, we get D_* that is moral for $T_{m+1,m}$ for all $m > k+1$, but not for $T_{k+1,k}$. Then any D that is built from D_* will saturate $T_{m+1,m}$ for all $m > k+1$, but will not saturate $T_{k+1,k}$. From the proof of (2) above, every theory that D saturates will be low and simple. It is easily seen that a regular ultrafilter will saturate U_n if and only if it saturates $T_{k+1,k}$ for every $k \ge 2n+2$. The result (3) follows.

(4) Assume that σ is an uncountable supercompact cardinal. It is shown that there is a regular ultrafilter D that saturates the simple theories and only the simple theories. This easily implies (4). D will be built from a *σ-complete* ultrafilter D_* on $\mathfrak{B}_{2^\lambda,\mu,\sigma}$. This is rather unexpected because a σ-complete ultrafilter can never be regular. We saw from the proof of (2) that an ultrafilter that is built from an ultrafilter on $\mathfrak{B}_{2^\lambda,\mu,\omega}$ cannot be flexible, and hence cannot saturate a simple theory that is not low. This obstacle is avoided here by assuming that there is a supercompact cardinal σ and working with $\mathfrak{B}_{2^\lambda,\mu,\sigma}$.

Now assume that μ, λ are finite successor cardinals of σ with $\mu < \lambda$. (The papers under review used a more general setting, carrying along a quadruple $\lambda \ge \mu \ge \theta \ge \sigma$ of cardinals satisfying a condition called suitability). Let $\mathfrak{B} = \mathfrak{B}_{2^\lambda,\mu,\sigma}$. The model-theoretic property of a theory being (λ, μ, σ)**-explicitly simple** is introduced. It gets weaker as μ increases. Intuitively, μ is a bound on the complexity of amalgamation. A key result is that when $\lambda = \mu^+$, (λ, μ, σ)-explicit simplicity is equivalent to simplicity.

(λ, μ, σ)**-optimality** is a refinement property for σ-complete ultrafilters on $\mathfrak{B}$ that is a large-cardinal analogue of being (λ, μ)-perfect. The following results are proved. Every (λ, μ, σ)-optimal ultrafilter is moral for every (λ, μ, σ)-explicitly simple theory. There exists a σ-complete filter D^0_* on $\mathfrak{B}$ generated by μ^+ sets such that no σ-complete ultrafilter $D_* \supseteq D^0_*$ is moral for a non-simple theory. Every such D^0_* can be extended to a (λ, μ, σ)-optimal ultrafilter $D_* \supseteq D^0_*$. Hence for any theory T, D_* is moral for T if and only if T is simple. So any D that is built from D_* will be as required.

(5) is related to another refinement property of ultrafilters, called goodness for equality. D is said to be **good for equality** if for every set A and every set $X \subseteq A^I$ of power $|X| \leq \lambda$, there exists a mapping $d : X \to D$ such that for all $x, y \in X$, if the set $\{i \in d(x) \cap d(y) \colon x(i) = y(i)\}$ is non-empty then it belongs to D. In the (2012) paper mentioned above, Malliaris showed that if D saturates a theory that is not simple and is not SOP_2, then D is good for equality, and that if D is good for equality then D saturates T^*_{feq}. The result (6) implies that if D saturates an SOP_2 theory then D is good and hence also saturates T^*_{feq}. Thus (6) implies (5).

(6) In the proof of (6), ultrapowers of the model $\langle \mathbb{N}, < \rangle$ play a key role. Such ultrapowers were also key in Shelah's discovery of the two lowest $\trianglelefteq$-equivalence classes. A linear ordering $\langle X, < \rangle$ has a (κ_1, κ_2)-cut if κ_1, κ_2 are regular cardinals and X can be partitioned into two sets Y, Z such that $y < z$ for all $y \in Y$ and $z \in Z$, Y has cofinality κ_1, and Z has coinitiality κ_2. The **cut spectrum** of D below a cardinal μ is the set

$$C(D, \mu) = \{(\kappa_1, \kappa_2) \colon \kappa_1 + \kappa_2 < \mu \text{ and } \langle \mathbb{N}, < \rangle^I / D \text{ has a } (\kappa_1, \kappa_2)\text{- cut}\}.$$

Ultrapowers of trees also play a key role. Here a tree $\mathcal{T}$ is a set of sequences partially ordered by initial segment that is well-ordered below any element of $\mathcal{T}$. D is said to **have μ-treetops** if for every tree $\mathcal{T}$, every strictly increasing sequence in $\mathcal{T}^I/D$ of length $< \mu$ has an upper bound.

Note that if D is good then D has λ^+-treetops. Since $Th(\langle \mathbb{N}, < \rangle)$ is $\trianglelefteq$-maximal, it is easy to see that D is good iff $C(D, \lambda^+) = \emptyset$. It is shown that if D has λ^+-treetops then $C(D, \lambda^+) = \emptyset$, so D is good iff D has λ^+-treetops. The idea for doing this as follows. It is clear that there is a greatest cardinal $\mathfrak{p}_D$ such that $C(D, \mathfrak{p}_D) = \emptyset$, and a greatest cardinal $\mathfrak{t}_D$ such that D has $\mathfrak{t}_D$-treetops. Then $\mathfrak{p}_D \geq \mathfrak{t}_D$ if and only if $C(D, \mathfrak{t}_D) = \emptyset$. The main step is to show that $C(D, \mathfrak{t}_D) = \emptyset$ for every D. Then $\mathfrak{p}_D \geq \mathfrak{t}_D$. So if D has λ^+-treetops, then $\lambda^+ \leq \mathfrak{t}_D \leq \mathfrak{p}_D$ and hence $C(D, \lambda^+) = \emptyset$. As explained later, $\mathfrak{p}_D$ and $\mathfrak{t}_D$ are related to the cardinals $\mathfrak{p}$ and $\mathfrak{t}$.

It is also shown that if D saturates some SOP_2 theory, then D has λ^+-treetops. The result (6) follows.

There is an extensive discussion of the converse of (6), which is open:

(7) Conjecture: Every $\trianglelefteq$-maximal theory is SOP_2.

T^*_{feq} is not SOP_2, by [Shelah and Usvyatsev, More on SOP_1 and SOP_2, Annals of Pure and Applied Logic 155 (2008), 16–31]. So Conjecture (7) would imply that T^*_{feq} is not $\trianglelefteq$-maximal, and hence that there are at least two $\trianglelefteq$-equivalence classes of non-simple theories. As evidence for Conjecture (7), it is shown in the forthcoming paper [Malliaris and Shelah, Model theoretic applications of cofinality spectrum problems, arXiv:1503.08338.math.LO] that under the GCH, T is SOP_2 if and only if T is maximal with respect to a related ordering $\trianglelefteq^*$.

We now turn to the set-theoretic result $\mathfrak{p} = \mathfrak{t}$. We first review the classical definitions of $\mathfrak{p}$ and $\mathfrak{t}$. We write $A \subseteq^* B$ if $A \setminus B$ is finite. The **pseudo-intersection number** $\mathfrak{p}$ is the least cardinal of a subset $X \subseteq [\mathbb{N}]^{\aleph_0}$ such that the intersection of every finite subset of X is infinite, but there is no infinite set A such that $A \subseteq^* B$ for all $B \in X$. The **tower number** $\mathfrak{t}$ is the least cardinal of a subset $X \subseteq [\mathbb{N}]^{\aleph_0}$ such that X is linearly ordered by $\subseteq^*$ but there is no infinite set A such that $A \subseteq^* B$ for all $B \in X$. It is easily seen that $\mathfrak{p} \leq \mathfrak{t} \leq 2^{\aleph_0}$.

The idea for proving that $\mathfrak{p} = \mathfrak{t}$ is to show that an analogue of $C(D, \mathfrak{t}_D) = \emptyset$ holds in a more general setting, called a **cofinality spectrum problem** (CSP). A CSP is a tuple

$$s = (M, M_1, M^+, M_1^+, \Delta)$$

such that $M^+ \prec M_1^+$, M, M_1 are reducts of M^+, M_1^+, Δ is a set of formulas with parameters in M_1 that define discrete linear orderings with first and last elements, and (M_1^+, Δ) has enough set theory for trees in a natural sense. An example is the ultrapower CSP, in which

$$M = \langle \mathbb{N}, < \rangle, M_1 = M^I/D, M^+ = \langle \mathcal{H}(\theta), \in \rangle, M_1^+ = (M^+)^I/D$$

for some sufficiently large cardinal θ.

Given a CSP s, the cut spectrum $C(s, \mu)$ and the cardinals $\mathfrak{p}_s, \mathfrak{t}_s$ are defined in the expected way. The general theorem is that $C(s, \mathfrak{t}_s) = \emptyset$ for every CSP s.

The equation $\mathfrak{p} = \mathfrak{t}$ is proved as follows. Let G be a generic subset of $([\mathbb{N}]^{\aleph_0}, \supseteq^*)$. In the generic model $V[G]$ of set theory, one can find a CSP s such that

$$\mathfrak{t} \leq \mathfrak{t}_s, M = M^+ = (\mathcal{H}(\aleph_1), \in), \text{ and } M_1 = M_1^+ = M^\omega/G \text{ is the generic ultrapower.}$$

Therefore in $V[G]$, $C(s, \mathfrak{t}) = \emptyset$. It is shown that in $V[G]$, $\mathfrak{p} < \mathfrak{t}$ implies that $\mathfrak{p}_s \leq \mathfrak{p}$, and hence $\mathfrak{p}_s \leq \mathfrak{t}$. This contradicts $C(s, \mathfrak{t}) = \emptyset$, so $\mathfrak{p} = \mathfrak{t}$ holds in $V[G]$. Finally, this implies that $\mathfrak{p} = \mathfrak{t}$ holds in V.

The methods developed in these papers are likely to stimulate more research in model theory and set theory. An enticing possibility is that the general results on cofinality spectrum problems will have broader applications. Some definitions in the papers are quite complicated, and there will be a search for simpler alternatives. Many questions about the $\trianglelefteq$-ordering remain open. The results on the $\trianglelefteq$-ordering and on explicit simplicity will re-open work on the classification of simple theories.

Keisler, H. Jerome

Department of Mathematics,University of Wisconsin, Madison, WI, USA. keisler@math.wisc.edu.

Entry for the Table of Contents: