Long Sequences and Neocompact Sets

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1 Introduction

Users of non-standard methods in mathematics have always been interested in the following question: what does non-standard analysis offer the mathematical community? The issues raised by this question are neverending and there is a whole spectrum of possible answers.

The paper [HK] offered an explanation from the point of view of logic, explaining how the principles used in the superstructure approach to nonstandard analysis are related to standard mathematical practice. One difficulty is that the mathematical community is not agreed on what "standard mathematical practice" is. [HK] used mathematical logic to provide a formal framework where these issues can be discussed.

The question posed above was approached from a different point of view in the series of papers beginning with [K1] and continuing with [FK1], [CK], [FK2], [K2] and [K3]. This series of papers develops the notion of a neometric space, and the whole program is explained in the survey paper [K6] in this volume. The approach may be intuitively described as follows. Start from a part of mathematics, probability theory, where nonstandard methods have clearly offered new insights and enriched the field with new and interesting results. Then isolate and present in "standard terms" those features of nonstandard practice that have made this success possible. The results appeared in [FK1] and [FK2] where the notions of neocompact sets and neometric families were presented, and the basic mathematical theory around these new concepts was developed.

A few words about these two papers will help to explain our reason for writing the present paper. In [FK1], entitled "Existence Theorems in Probability Theory", we developed a standard theory which captured the key elements from nonstandard analysis that made it possible to prove new existence theorems in stochastic analysis (see [AFHL], [K4] and [K5]). Using neocompact sets and neometric spaces we introduced a new class of probability spaces called "Rich Probability Spaces" and then proceeded to show that in those spaces the results obtained using nonstandard methods are true. The main new ingredient is that these results, in the new setting, are proved within standard mathematical practice. We just asked our readers to accept the existence of such spaces and then proceed to see what could be done with them.

In [FK2] we showed that rich spaces exist, a result that requires nonstandard analysis, and presented the theory of neometric spaces within the most general possible nonstandard framework, which we called the huge neometric family. There we explained how the properties of internal sets in nonstandard hulls give rise to neocompact sets and how the saturation property of the nonstandard universe translates into countable compactness for the neometric family.

A mathematician accustomed to working with nonstandard methods, and in particular within probability theory using liftings and standard parts, may be surprised by the way the results are presented in those papers. Our aim in this paper is to shed some light on the origins of our ideas. We are going to present "a nonstandard theory which explains the standard theory that came out of observing nonstandard practice." Moreover, the results here can be used as a translation tool between traditional nonstandard arguments and the new theory of neometric spaces.

The idea centers around a fundamental fact from nonstandard analysis: sequences indexed by $\mathbf{N}$, the natural numbers, can be extended to sequences indexed by ${}^*\mathbf{N}$, the hyperintegers. This is the reason for the name long sequence. This elementary procedure allows us to capture many important facts from nonstandard practice.

We shall refer to the survey paper [K6] in this volume for the definitions and basic facts concerning the general notion of a neometric family, and in particular the huge neometric family.

Long sequences are introduced in Section 2 of this paper, and the theory is developed further and applied to the huge neometric family in Section 3. Needless to say, we assume the reader is familiar with the superstructure approach to nonstandard analysis (see [L], [AFHL] and [C]). Acquaintance with [FK1] and [FK2] is highly desirable to get the complete picture of the subject.

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2 Long Sequences

One way to bring nonstandard analysis to bear in proofs by convergence is to use sequences indexed by the hyperintegers rather than the integers. We shall call such sequences **long sequences**. The paper [K2] made extensive use of long sequences, without using this name. In this section we develop some relations between long sequences and neometric spaces.

We fix an $\aleph_1$-saturated nonstandard universe and let $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ be its huge neometric family as defined in [K6].

Definition 2.1 *A function $\langle x_n \rangle$ mapping $\mathbf{N}$ into a set S will be called a* **sequence** *in S, and a (possibly external) function $\langle x_J \rangle$ mapping ${}^*\mathbf{N}$ into S will be called a* **long sequence** *in S.*

As a warmup, before establishing the connection between long sequences and neocompact spaces, we prove some basic facts about sets of hyperintegers and long sequences. We frequently use the following consequences of ω_1-saturation (e.g. see [SB]):

Lemma 2.2 *(i) The infinite hyperintegers have coinitiality ω_1, that is, every countable set of elements of ${}^*\mathbf{N} - \mathbf{N}$ has an infinite lower bound.*

(ii) For every internal set S and every sequence $\langle X_n \rangle$ in S, there exists an internal long sequence $\langle Y_J \rangle$ in S such that $Y_n = X_n$ for all $n \in \mathbf{N}$. $\square$

Definition 2.3 *We say that a statement $\phi(J)$ holds a.e., or that $\phi(J)$ holds for* **all sufficiently small infinite** *J, if there is an infinite hyperinteger K such that $\phi(J)$ is true for all infinite hyperintegers $J \le K$.*

The following lemma is often used to verify that a statement holds a.e.

Lemma 2.4 *(i) (Overspill principle) If S is an internal subset of ${}^*\mathbf{N}$, then $J \in S$ a.e. if and only if $n \in S$ for all but finitely many $n \in \mathbf{N}$.*

(ii) (Countable completeness) The set of all $S \subset {}^*\mathbf{N}$ such that $J \in S$ a.e. is a countably complete filter.

Proof: (i) is in any book on nonstandard analysis. (ii) If $J \in S$ a.e. and $S \subset T$, then obviously $J \in T$ a.e. Suppose $J \in S_n$ a.e. for all $n \in \mathbf{N}$, and let $S = \bigcap_n S_n$. Then for each $n \in \mathbf{N}$ there is an infinite hyperinteger K_n such that $J \in S_n$ for all infinite $J \le K_n$. By ω_1-saturation there is an infinite hyperinteger K such that $K \le K_n$ for all $n \in \mathbf{N}$. Then $J \in S$ for all infinite $J \le K$, so $J \in S$ a.e. $\square$

The overspill principle will often be used in the following form.

Corollary 2.5 *Let $\langle X_J \rangle$ be an internal long sequence in ${}^*\mathbf{R}$ and let $b \in {}^*\mathbf{R}$.*

(i) $X_J \le b$ a.e. if and only if $X_n \le b$ for all but finitely many $n \in \mathbf{N}$.

(ii) $X_J \approx 0$ a.e. if and only if

$$\lim_{n\to\infty} st(X_n) = 0.$$

Proof: (i) Apply the overspill principle to the internal set $S = \{J \in {}^*\mathbf{N} : X_J \le b\}$. (ii) follows from (i) and countable completeness. $\square$

We now turn to long sequences in neometric spaces in the huge neometric family $(\mathbf{H}, \mathcal{B}, \mathcal{C})$. For the remainder of this paper, $\mathcal{M}$ and $\mathcal{N}$ will always belong to $\mathbf{H}$. The following is the key new concept we introduce in this paper.

Definition 2.6 *If $\langle x_J \rangle$ is a long sequence in $\mathcal{M}$ and $\langle X_J \rangle$ is an internal long sequence in $\bar{M}$ such that $x_J = {}^oX_J$ for all finite J and all sufficiently small infinite J, we say that $\langle X_J \rangle$* **lifts** *$\langle x_J \rangle$. By an $\mathcal{M}$-***sequence** *we shall mean a long sequence $\langle x_J \rangle$ in $\mathcal{M}$ which has a lifting. A (short) sequence $\langle x_n \rangle$ of elements of $\mathcal{M}$ will be said to be $\mathcal{M}$-***extendible** *if it is the restriction to $\mathbf{N}$ of some $\mathcal{M}$-sequence $\langle x_J \rangle$, and $\langle x_J \rangle$ will be called an $\mathcal{M}$-***extension** *of $\langle x_n \rangle$.*

By ω_1-saturation, for every sequence $\langle x_n \rangle$ in $\mathcal{M}$ there is an internal long sequence $\langle X_J \rangle$ such that X_n lifts x_n, and hence $X_n \in \mathrm{monad}(\mathcal{M})$ for each $n \in \mathbf{N}$. If in addition we have $X_J \in \mathrm{monad}(\mathcal{M})$ a.e., then the sequence $\langle x_n \rangle$ is $\mathcal{M}$-extendible and its $\mathcal{M}$-extension is the $\mathcal{M}$-sequence given by $x_J = {}^oX_J$ a.e. If $\langle x_n \rangle$ is an $\mathcal{M}$-extendible sequence, we use the convention that $\langle x_J \rangle$ denotes an $\mathcal{M}$-extension of $\langle x_n \rangle$.

The next proposition shows that the notion of an $\mathcal{M}$-extendible sequence is a generalization of the notion of a convergent sequence.

Proposition 2.7 *(i) If*

$$\lim_{n\to\infty} x_n = b$$

in $\mathcal{M}$, then $\langle x_n \rangle$ is $\mathcal{M}$-extendible and $\langle x_J \rangle = b$ a.e.

(ii) If $\langle x_n \rangle$ is $\mathcal{M}$-extendible and $\langle y_n \rangle$ is a sequence in $\mathcal{M}$ such that

$$\lim_{n\to\infty} \rho(x_n, y_n) = 0,$$

then $\langle y_n \rangle$ is $\mathcal{M}$-extendible.

Proof: (i) Let X_n lift x_n, let Y lift b, and extend $\langle X_n \rangle$ to a long sequence $\langle X_J \rangle$. By overspill, $X_J \approx Y$ and hence ${}^oX_J = b \in \mathcal{M}$ a.e. Therefore $\langle x_J \rangle = \langle {}^oX_J \rangle$ is an $\mathcal{M}$-extension of $\langle x_n \rangle$ and $x_J = b$ a.e.

(ii) Let $\langle x_J \rangle$ be an $\mathcal{M}$-extension of $\langle x_n \rangle$ and let $\langle X_J \rangle$ lift $\langle x_J \rangle$. For each $n \in \mathbf{N}$, let Y_n lift y_n, and by ω_1-saturation let $\langle Y_J \rangle$ be an internal long sequence in $\bar{M}$ extending $\langle Y_n \rangle$. Then

$$\lim_{n \to \infty} {}^o\bar{\rho}(X_n, Y_n) = 0,$$

so by overspill, $X_J \approx Y_J$ a.e. It follows that $\langle Y_J \rangle$ lifts an $\mathcal{M}$-sequence $\langle y_J \rangle$ which extends $\langle y_n \rangle$, whence $\langle y_n \rangle$ is $\mathcal{M}$-extendible. $\Box$

Given a product $\mathcal{M} \times \mathcal{N}$ of two spaces $\mathcal{M}, \mathcal{N} \in \mathbf{H}$, $\langle z_J \rangle$ is an $(\mathcal{M} \times \mathcal{N})$-sequence if and only if there is an $\mathcal{M}$-sequence $\langle x_J \rangle$ and an $\mathcal{N}$-sequence $\langle y_J \rangle$ such that $z_J = (x_J, y_J)$ a.e. Thus if $\langle x_n \rangle$ is $\mathcal{M}$-extendible and $\langle y_n \rangle$ is $\mathcal{N}$-extendible, then the sequence of pairs $\langle z_n \rangle = \langle (x_n, y_n) \rangle$ is $\mathcal{M} \times \mathcal{N}$-extendible, and $z_J = (x_J, y_J)$ a.e.

The following shows that the $\mathcal{M}$-extension of a sequence is unique a.e.

Proposition 2.8 *(i) Let $\langle x_J \rangle$ and $\langle y_J \rangle$ be $\mathcal{M}$-sequences. Then*

$$\lim_{n \to \infty} \rho(x_n, y_n) = 0$$

if and only if $x_J = y_J$ a.e.

(ii) (Uniqueness of the $\mathcal{M}$-extension) Let $\langle x_J \rangle$ and $\langle y_J \rangle$ be two $\mathcal{M}$-extensions of the same sequence $\langle x_n \rangle$ in $\mathcal{M}$. Then $x_J = y_J$ a.e.

Proof: (i) Let $\langle X_J \rangle$ lift $\langle x_J \rangle$ and $\langle Y_J \rangle$ lift $\langle y_J \rangle$. By overspill, the following are equivalent:

$$\lim_{n \to \infty} \rho(x_n, y_n) = 0.$$

$$\lim_{n \to \infty} {}^o\bar{\rho}(X_n, Y_n) = 0.$$

$$\bar{\rho}(X_J, Y_J) \approx 0 \, a.e.$$

$$x_J = y_J \, a.e.$$

(ii) is a special case of (i). $\Box$

Proposition 2.9 *If $\langle x_n \rangle$ is $\mathcal{M}$-extendible, then for each $c \in \mathcal{M}$, the sequence $\langle \rho(x_n, c) \rangle$ is bounded in $\mathbf{R}$.*

Proof: Let $\langle X_J \rangle$ be a lifting of an $\mathcal{M}$ extension $\langle x_J \rangle$ of $\langle x_n \rangle$. Suppose $\rho(x_n, c)$ is not bounded. Let $\bar{c}$ lift c. Then for each $k \in \mathbf{N}$ there are arbitrarily large $n \in \mathbf{N}$ such that $\rho(x_n, c) > k$, and hence $\bar{\rho}(X_n, \bar{c}) \geq k$. By overspill, for each infinite $K \in^* \mathbf{N}$ there is an infinite $J \leq K$ such that $\bar{\rho}(X_J, \bar{c})$ is infinite. This contradicts the hypothesis that $\langle X_J \rangle$ is a lifting of $\langle x_J \rangle$. $\Box$

3 The Huge Neometric Family

We now give conditions for neocompactness, neoclosedness, neocontinuity, and neoseparability in the huge family $\mathbf{H}$ in terms of long sequences. The first proposition is crucial.

Proposition 3.1 *Let $\langle x_J \rangle$ be an $\mathcal{M}$-sequence. Then for all sufficiently small infinite K, the set $\{x_J : J \leq K\}$ is basic in $\mathcal{M}$ and the set*

$$\{x_J : J \leq K \textit{ and } J \textit{ is infinite } \}$$

is neocompact in $\mathcal{M}$.

Proof: Let $\langle X_J \rangle$ lift $\langle x_J \rangle$ with respect to $\mathcal{M}$. Then for all sufficiently small infinite K, ${}^o X_J = x_J$ for all $J \leq K$. For any such K, let

$$B = \{X_J : J \leq K\}, C = \{x_J : J \leq K\},$$

$$D = \{x_J : J \leq K \text{ and } J \text{ is infinite } \}.$$

Then B is internal and $C = {}^o B$, so C is basic. Moreover,

$$D = {}^o(\bigcap_n (B - \{X_m : m \leq n\})),$$

so D is neocompact. $\square$

We need the notion of a countably determined set, which was introduced by Henson [He] and played an important role in [K2].

Definition 3.2 *A set $D \subset \bar{M}$ is* **countably determined** *if there is a countable sequence $\langle D_n \rangle$ of internal subsets of $\bar{M}$ such that D is an infinite Boolean combination of the D_n's. Equivalently, there is a countable sequence $\langle D_n \rangle$ of internal subsets of $\bar{M}$ and a set S of subsets of $\mathbf{N}$ such that*

$$D = \bigcup_{F \in S} (\bigcap_{n \in F} D_n). \tag{1}$$

In fact, this representation can be chosen so that for any distinct $F, G \in S$, the intersections $\bigcap_{n \in F} D_n$ and $\bigcap_{n \in G} D_n$ are disjoint from each other.

Note that every internal, Π_1^0, and Σ_1^0 set is countably determined.

Theorem 3.3 *A set $C \subset \mathcal{M}$ is neocompact if and only if*

(a) The monad of C is countably determined, and

(b) Every (short) sequence $\langle x_n \rangle$ in C has an $\mathcal{M}$-extension to a long sequence $\langle x_J \rangle$ in C.

Proof: First assume that C is neocompact. By Basic Fact 2.3 in [K6], there exists a sequence of internal sets $\langle C_n \rangle$ such that

$$\text{monad}(C) = \bigcap_m ((C_m)^{1/m}).$$

So the monad of C is countably determined. Let $\langle x_n \rangle$ be a (short) sequence in C. $\langle x_n \rangle$ has a lifting $\langle X_J \rangle$. For each n and m, $X_n \in (C_m)^{1/m}$. By overspill and countable completeness,

$$X_J \in \bigcap_m ((C_m)^{1/m}) \; a.e.,$$

so $\langle X_J \rangle$ lifts an $\mathcal{M}$-extension $\langle x_J \rangle$ in C.

Now assume that the monad of C is countably determined and that every (short) sequence $\langle x_n \rangle$ in C has an $\mathcal{M}$-extension to a long sequence $\langle x_J \rangle$ in C. Then the monad of C can be represented in the form (1) with any two distinct intersections being disjoint. We claim that

$$\text{monad}(C) = \bigcap \{B : \text{ for some finite } s \subset \mathbf{N}, B = (\bigcup_{n \in s} C_n) \supset \text{monad}(C)\}.$$

This will show that the monad of C is a Π_1^0 set, and hence that C is neocompact. Clearly monad(C) is included in the right side. Suppose X belongs to the right side. Let $G = \{n \in \mathbf{N} : X \notin C_n\}$. Since X belongs to the right side, for each n there exists

$$Y_n \in \text{monad}(C) - \bigcup \{C_k : n \geq k \in G\}.$$

Let $y_n = {}^oY_n$. Then $\langle y_n \rangle$ is a (short) sequence in C, so by hypothesis it has an $\mathcal{M}$-extension $\langle Y_J \rangle$ in C. We have $Y_J \in \text{monad}(C)$ for all J. By overspill,

$$Y_J \notin \bigcup \{C_k : k \in G\} \; a.e.$$

Take an infinite J with this property. Then for some $F \in S$, $Y_J \in \bigcap_{n \in F} C_n$. Moreover, for all $n \in F$, $Y_J \in C_n$ and hence $n \notin G$ and $X \in C_n$. Therefore

$$X \in \bigcap_{n \in F} C_n \subset \text{monad}(C).$$

This proves our claim and completes the proof. $\square$

Just as Basic Fact 2.3 in [K6] shows that the monad of a neocompact set is countably determined, Basic Fact 2.4 in [K6] shows that the monad of a neoseparable set is countably determined.

The following corollary is a good illustration of how our neometric theory is closely related to the classical theory of metric spaces. Notice what happens if you replace "$\mathcal{M}$-extendible" by "relatively compact".

Corollary 3.4 *A sequence $\langle x_n \rangle$ in $\mathcal{M}$ is $\mathcal{M}$-extendible if and only if there is a neocompact set $C \subset \mathcal{M}$ such that $x_n \in C$ for all $n \in \mathbf{N}$.*

Proof: If $\langle x_n \rangle$ has an $\mathcal{M}$-extension $\langle x_J \rangle$, then by Proposition 3.1 the set $C = \{x_J : J \le K\}$ is neocompact for some infinite K, and $\{x_n : n \in \mathbf{N}\} \subset C \subset \mathcal{M}$. If C is neocompact and $\{x_n : n \in \mathbf{N}\} \subset C \subset \mathcal{M}$, then $\langle x_n \rangle$ is $\mathcal{M}$-extendible by Theorem 3.3. $\square$

In applications of long sequences, it is important to know which sequences are $\mathcal{M}$-extendible. We can use Corollary 3.4 to characterize the $\mathcal{M}$-extendible sequences in various particular neometric spaces which have been studied in [K1], [FK1] and [FK2]. The next example characterizes the extendible sequences in a nonstandard hull.

Example 3.5 *A sequence $\langle x_n \rangle$ is $\mathcal{H}(\bar{M}, c)$-extendible if and only if $\rho(x_n, d)$ is bounded where $d \in \mathcal{H}(\bar{M}, c)$.*

Proof: By Proposition 2.9, for any $\mathcal{H}(\bar{M}, c)$-extendible sequence $\langle x_n \rangle$ and any $d \in \mathcal{H}(\bar{M}, c)$, the sequence $\langle \rho(x_n, d) \rangle$ is bounded. Suppose $\langle x_n \rangle$ is a sequence in $\mathcal{H}(\bar{M}, c)$ such that $\rho(x_n, d)$ has a finite bound b. Each x_n belongs to the closed ball $B = \{y \in \mathcal{H}(\bar{M}, c) : \rho(y, d) \le b\}$. B is the standard part of an internal set and is therefore basic in $\mathcal{H}(\bar{M}, c)$. Thus by Corollary 3.4, $\langle x_n \rangle$ is $\mathcal{H}(\bar{M}, c)$-extendible. $\square$

Let's now consider standard neometric spaces. We shall see that the only $\mathcal{M}$-extendible sequences on a standard neometric space $\mathcal{M} \in \mathbf{S}$ are the trivial ones, that is, the relatively compact sequences.

By definition, a sequence $\langle x_n \rangle$ in a complete metric space $\mathcal{M}$ is **relatively compact** if there is a compact set $C \subset \mathcal{M}$ which contains each x_n, or equivalently, every subsequence of $\langle x_n \rangle$ has a convergent subsequence. Thus a sequence in Euclidian space $\mathbf{R}^m$ is relatively compact if and only if it is bounded. By Corollary 3.4, in every neometric space $\mathcal{M}$, every relatively compact sequence is $\mathcal{M}$-extendible.

Example 3.6 *Let $\mathcal{M}$ be a standard neometric space. Then a sequence $\langle x_n \rangle$ in $\mathcal{M}$ is $\mathcal{M}$-extendible if and only if $\langle x_n \rangle$ is relatively compact.*

Proof: This follows from Corollary 3.4 and Basic Fact 2.5 in [K6]. $\square$

For the following examples let Ω be a Loeb probability space. The paper [K1] gave characterizations of the $\mathcal{M}$-extendible sequences when M is a complete separable metric space and $\mathcal{M}$ is either the space $L^0(\Omega, M)$ of Loeb measurable functions with the metric of convergence of probability or the space $L^p(\Omega, M)$ where $p \in [1, \infty)$. In fact, these results were the original inspiration for the long sequences approach to neometric spaces.

If $x \in L^0(\Omega, M)$, the Borel probability measure on M induced by x is denoted by $\mathrm{law}(x)$. The space of all Borel probability measures on M with the Prohorov metric is denoted by $\mathrm{Meas}(M)$. (See, for example [EK]).

Example 3.7 *([K1], Theorem 3.2 and Lemmas 7.2 and 7.4). Let Ω be a Loeb probability space and M be a complete separable metric space.*

(i) A sequence $\langle x_n \rangle$ is $L^0(\Omega, M)$-extendible if and only if $\langle law(x_n) \rangle$ is relatively compact in $Meas(M)$.

(ii) Let $p \in [1, \infty)$. A sequence $\langle x_n \rangle$ is $L^p(\Omega, M)$-extendible if and only if $\langle law(x_n) \rangle$ is relatively compact in $Meas(M)$ and $(\rho(x_n(\cdot), a))^p$ is uniformly integrable for each $a \in M$.

The next result gives another characterization of neocompact sets in the case that $\mathcal{M}$ is neoseparable.

Proposition 3.8 *Suppose $\mathcal{M}$ is neoseparable. A set $C \subset \mathcal{M}$ is neocompact if and only if C is neoclosed in $\mathcal{M}$ and (b) of Theorem 3.3 holds, that is, every sequence $\langle x_n \rangle$ in C has an $\mathcal{M}$-extension $\langle x_J \rangle$ in C.*

Proof: Neocompactness implies neoclosed and (b) by Basic Fact 1.1 in [K6] and Theorem 3.3. Assume that (b) holds and that C is neoclosed but not neocompact. By Basic Fact 2.6 in [K6], C has a countable covering $\{O_n : n \in \mathbf{N}\}$ by neoopen sets in $\mathcal{M}$ which has no finite subcover. Let $C_n = C - (\bigcup_{k \leq n} O_k)$. Then $\langle C_n \rangle$ is a decreasing chain of nonempty neoclosed sets in $\mathcal{M}$, and $\bigcap_n C_n$ is empty. Choose $x_n \in C_n$. Then $x_n \in C$, and by (b) we can extend $\langle x_n \rangle$ to a long sequence $\langle x_J \rangle$ in C. By Proposition 3.1 we may choose an infinite K so that the set $S = \{x_J : J \leq K\}$ is basic. Then $\langle S \cap C_n \rangle$ is a decreasing chain of neocompact subsets of C. By countable compactness of the huge neometric family, the intersection

$$\bigcap_n (S \cap C_n) = S \cap (\bigcap_n C_n)$$

is nonempty, and this is a contradiction. $\square$

Theorem 3.9 *Let $C \subset \mathcal{M}$. If C is neoclosed in $\mathcal{M}$ then*

(c) For every $\mathcal{M}$-sequence $\langle x_J \rangle$ such that $x_n \in C$ for all $n \in \mathbf{N}$, we have $x_J \in C$ a.e.

If the monad of C is countably determined, then C is neoclosed in $\mathcal{M}$ if and only if this condition holds.

Proof: Suppose first that C is neoclosed in $\mathcal{M}$. Let $\langle x_n \rangle$ be a sequence in C and $\langle x_J \rangle$ be an $\mathcal{M}$-extension of $\langle x_n \rangle$. By Proposition 3.1 there is an infinite K such that the set $D = \{x_J : J \leq K\}$ is basic in $\mathcal{M}$. Then $C \cap D$ is neocompact in $\mathcal{M}$, and $x_n \in C \cap D$ for all $n \in \mathbf{N}$. By Theorem 3.3, $\langle x_n \rangle$ has an $\mathcal{M}$-extension to a long sequence $\langle y_J \rangle$ in $C \cap D$. By uniqueness of the $\mathcal{M}$-extension, $y_J = x_J$ a.e. Then $x_J \in C$ a.e., and (c) is proved.

Now suppose that the monad of C is countably determined and (c) holds. Let D be neocompact in $\mathcal{M}$. Then the monad of D is countably determined. Since

$$\mathrm{monad}(C \cap D) = \mathrm{monad}(C) \cap \mathrm{monad}(D),$$

monad$(C \cap D)$ is countably determined. Let $\langle x_n \rangle$ be a sequence in $C \cap D$. By Theorem 3.3, $\langle x_n \rangle$ has an $\mathcal{M}$-extension to a long sequence $\langle x_J \rangle$ in D. By condition (c), $x_J \in C$ a.e. Then $\langle x_J \rangle$ is a long sequence in $C \cap D$. We have shown that conditions (a) and (b) of Theorem 3.3 hold for $C \cap D$. By Theorem 3.3, $C \cap D$ is neocompact in $\mathcal{M}$, so C is neoclosed in $\mathcal{M}$. $\square$

Corollary 3.10 *Suppose $C \subset \mathcal{M}$, C is neoclosed, and $\langle x_J \rangle$ is an $\mathcal{M}$-sequence such that*

$$\lim_{n \to \infty} \rho(x_n, C) = 0.$$

Then $x_J \in C$ a.e.

Proof: For each $n \in \mathbf{N}$ we may choose $y_n \in C$ such that $\rho(x_n, y_n) \leq 2\rho(x_n, C)$. Then

$$\lim_{n \to \infty} \rho(x_n, y_n) = 0,$$

so by Propositions 2.7 and 2.8, $\langle y_n \rangle$ is $\mathcal{M}$-extendible and $x_J = y_J$ a.e. Theorem 3.9 shows that $y_J \in C$ a.e., and therefore $x_J \in C$ a.e. $\square$

Observe that from the above theorem it follows right away that every neoclosed set in the huge neometric family is closed. Basic Fact 1.7 in [K6] says that this is true in all neometric families. Now, let's take a look at a characterization of neocontinuity in terms of $\mathcal{M}$-sequences.

Theorem 3.11 *Let $C \subset \mathcal{M}$ be neoclosed, and let $f : C \to \mathcal{N}$. If f is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$, then*

(d) For any $\mathcal{M}$-sequence $\langle x_J \rangle$ in C, $\langle f(x_J) \rangle$ is an $\mathcal{N}$-sequence.

If the monad of the graph of f is countably determined, then f is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$ if and only if this condition holds.

Proof: This generalizes a result from [K2]. Suppose first that f is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$. Let $\langle x_J \rangle$ be an $\mathcal{M}$-sequence in C. By Proposition 3.1 there is an infinite K such that the set $D = \{x_J : J \leq K\}$ is basic in $\mathcal{M}$. By Basic Fact 2.7 in [K6] there is an internal function F such that ${}^oF(X) = f({}^oX)$ for all $X \in \text{monad}(D)$. Since $\langle X_J \rangle$ and F are internal, $\langle F(X_J) \rangle$ is internal. For all $J \leq K$, ${}^oF(X_J) = f({}^oX_J) = f(x_J)$. Therefore $\langle F(X_J) \rangle$ is a lifting of $\langle f(x_J) \rangle$ with respect to $\mathcal{N}$, so $\langle f(x_J) \rangle$ is an $\mathcal{N}$-sequence. This proves the first half.

Now suppose the monad of the graph of f is countably determined, and assume (d). Let $D \subset C$ be neocompact. Then monad(D) is countably determined, so

$$\text{monad}(f|D) = \text{monad}(f) \cap (\text{monad}(D) \times \bar{N})$$

is countably determined. Let $\langle x_n, f(x_n) \rangle$ be a sequence in the graph of $f|D$. By Theorem 3.3, $\langle x_n \rangle$ has an $\mathcal{M}$-extension $\langle x_J \rangle$ in D. By hypothesis, $\langle f(x_J) \rangle$ is an $\mathcal{N}$-sequence. Then $\langle x_J, f(x_J) \rangle$ is an $(\mathcal{M} \times \mathcal{N})$-extension of $\langle x_n, f(x_n) \rangle$ in $f|D$. Therefore by Theorem 3.9, $f|D$ is neocompact in $\mathcal{M} \times \mathcal{N}$, so f is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$. $\square$

Corollary 3.12 *Let $C \subset \mathcal{M}$, and let $f : C \to \mathcal{N}$ be neocontinuous from $\mathcal{M}$ to $\mathcal{N}$. If a sequence $\langle x_n \rangle$ in C is $\mathcal{M}$-extendible, then $\langle f(x_n) \rangle$ is $\mathcal{N}$-extendible to an $\mathcal{N}$-sequence $\langle y_J \rangle$, and $f(x_J) = y_J$ a.e.* $\square$

Finally, we give a necessary condition for neoseparability in terms of long sequences. An open question is whether this condition, together with the condition that the monad of the set is countably determined, is sufficient for neoseparability.

Proposition 3.13 *Let C be neoseparable in $\mathcal{M}$, and let $\langle x_J \rangle$ be an $\mathcal{M}$-sequence such that $x_J \in C$ a.e. Then for each $k \in \mathbf{N}$, $x_n \in C^{1/k}$ for all but finitely many $n \in \mathbf{N}$.*

Proof: Let monad$(C) = \bigcap_n \bigcup_m (C_m)^{1/n}$. Suppose that there is a $k \in \mathbf{N}$ and an infinite subset $p \subset \mathbf{N}$ such that $x_n \notin C^{1/k}$ for all $n \in p$. By taking a subsequence, we may assume without loss of generality that $x_n \notin C^{1/k}$ for all $n \in \mathbf{N}$. Let $\langle X_J \rangle$ lift $\langle x_n \rangle$. Then $X_J \in \bigcap_n \bigcup_m (C_m)^{1/n}$ a.e., and hence $X_J \in \bigcup_m (C_m)^{1/k}$ a.e. However, for all $n \in \mathbf{N}$ we have $X_n \notin \bigcup_m (C_m)^{1/k}$ because ${}^o(\bigcup_m (C_m)^{1/k}) \subset C^{1/k}$. Then for each $n, m \in \mathbf{N}$ we have $X_n \notin (C_m)^{1/k}$, and by overspill, $X_J \notin (C_m)^{1/k}$ a.e. By countable completeness, $X_J \notin \bigcup_m (C_m)^{1/k}$ a.e., contrary to our previous assumption. $\square$

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