Quantifiers in Limits

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Abstract

The standard definition of $\lim_{z\to\infty} F(z) = \infty$ is an $\forall\exists\forall$ sentence. Mostowski showed that in the standard model of arithmetic, these quantifiers cannot be eliminated. But Abraham Robinson showed that in the nonstandard setting, this limit property for a standard function F is equivalent to the one quantifier statement that $F(z)$ is infinite for all infinite z. In general, the number of quantifier blocks needed to define the limit depends on the underlying structure $\mathcal{M}$ in which one is working. Given a structure $\mathcal{M}$ with an ordering, we add a new function symbol F to the vocabulary of $\mathcal{M}$ and ask for the minimum number of quantifier blocks needed to define the class of structures $(\mathcal{M}, F)$ in which $\lim_{z\to\infty} F(z) = \infty$ holds.

We show that the limit cannot be defined with fewer than three quantifier blocks when the underlying structure $\mathcal{M}$ is either countable, special, or an o-minimal expansion of the real ordered field. But there are structures $\mathcal{M}$ which are so powerful that the limit property for arbitrary functions can be defined in both two-quantifier forms.

1 Introduction

An important advantage of the nonstandard approach to elementary calculus is that it eliminates two quantifiers in the definition of a limit. For example, the standard definition of

$$\lim_{z\to\infty} F(z) = \infty$$

requires three quantifier blocks,

$$\forall x\, \exists y\, \forall z\, [y \le z \Rightarrow x \le F(z)].$$

Mostowski showed that in the standard model of arithmetic, these quantifiers cannot be eliminated. But Abraham Robinson showed that in the nonstandard setting, this limit property is equivalent to the one quantifier statement

$$\forall z\, [z \in I \Rightarrow F(z) \in I],$$

where F is a standard function and I is the set of infinite elements. Because of the quantifiers, beginning calculus students cannot follow the standard definition but have no trouble with the nonstandard definition. Since all the basic notions in the calculus depend on limits, students often find the nonstandard approach to the calculus to be easier to understand than the standard approach (see [Ke2], [Su]).

In general, the number of quantifier blocks needed to define the limit depends on the underlying structure $\mathcal{M}$ in which one is working. Given a structure $\mathcal{M}$ with an ordering, we add a new function symbol F to the vocabulary of $\mathcal{M}$ and ask for the minimum number of quantifier blocks needed to define the class of structures $(\mathcal{M}, F)$ in which $\lim_{z\to\infty} F(z) = \infty$ holds.

We show that in the standard setting the limit cannot be defined with fewer than three quantifier blocks when the underlying structure $\mathcal{M}$ is not too powerful. We obtain this result in the case that $\mathcal{M}$ is countable, and in the case that $\mathcal{M}$ is an o-minimal expansion of the real field $\mathcal{R} = (\mathbb{R}, \leq, +, \cdot)$.

As is usual in the literature, we consider the quantifier hierarchy which takes into account both n-quantifier forms. For each n, one can ask whether a property is Π_n, Σ_n, both Π_n and Σ_n (called Δ_n), or Boolean in Π_n. Each of Σ_n and Π_n implies Boolean in Π_n, which in turn implies Δ_{n+1}. The standard definition of limit is Π_3.

In Section 4 we will see that the limit property can never be Boolean in Π_1 sentences. However, if $\mathcal{M} = (\mathbb{R}, \leq, \mathbb{N}, g)$ where g maps $\mathbb{R}$ onto $\mathbb{R}^{\mathbb{N}}$, then the limit property is Δ_2. What happens here is that there is a standard definition of limit which uses one less quantifier block than the usual definition, but needs a function which codes sequences of real numbers by real numbers, and is therefore beyond the scope of an elementary calculus course.

In Section 5 we show that when $\mathcal{M}$ is countable, the limit property is not Σ_3. In Section 6 we prove that the limit property is not Σ_3 when $\mathcal{M}$ is the real ordering with an embedded structure with universe $\mathbb{N}$. In Section 7 we prove that the limit property is not Σ_3 when $\mathcal{M}$ is a saturated or special structure, even when one adds a predicate for the set of infinite elements. This shows that Robinson's result for standard functions does not carry over to arbitrary functions.

In Section 8 we consider infinitely long sentences. In an ordered structure $\mathcal{M}$ with universe set $\mathbb{R}$ and at least a constant symbol for each natural number, the limit property can be expressed naturally by a countable conjunction of countable disjunctions of Π_1 sentences. We show that the limit property cannot be expressed by a countable disjunction of countable conjunctions of Σ_1 sentences.

In Section 9 we prove our main result: If $\mathcal{M}$ is an o-minimal expansion of the real ordered field , then the limit property is not Boolean over Π_2. We leave open the question of whether one can improve this result by showing that the limit property is not Σ_3.

We also consider the similar but simpler property of a function being bounded. The standard definition of boundedness is Σ_2, or alternatively, a countable disjunction of universal sentences. We show that when $\mathcal{M}$ is countable, special, or an o-minimal expansion of the real ordered field, the boundedness property is not Π_2. When $\mathcal{M}$ has universe set $\mathbb{R}$, boundedness cannot be expressed by a countable conjunction of existential sentences.

2 Preliminaries

We introduce a general framework for the study of quantifiers required for defining limits.

We assume throughout that $\mathcal{M}$ is an ordered structure with no greatest element. That is, $\mathcal{M}$ is a first order structure for a vocabulary $L(\mathcal{M})$ that contains at least the order relation $\leq$, and $\leq$ is a linear ordering with no greatest element. We will consider structures $\mathcal{K} = (\mathcal{M}, f)$ where $f : \mathcal{M} \to \mathcal{M}$ is a unary function. The vocabulary of $(\mathcal{M}, f)$ is $L(\mathcal{M}) \cup \{F\}$ where F is an extra unary function symbol. For a formula $\varphi(\vec{x}, \vec{y})$ of $L(\mathcal{M}) \cup \{F\}$, the notation $\varphi(f, \vec{x}, \vec{y})$ means that $(\mathcal{M}, f) \models \varphi(\vec{x}, \vec{y})$.

We use the notation $|X|$ for the cardinality of X, $\equiv$ for elementary equivalence, $\prec$ for elementary substructure, and $\cong$ for isomorphic.

Quantifier-free formulas are called Π_0 formulas and also called Σ_0 formulas. A Π_{n+1} formula is a formula of the form $\forall \vec{x}\, \theta$ where θ is a Σ_n-formula. A Σ_{n+1} formula is one a formula of the form $\exists \vec{x}\, \theta$ where θ is a Π_n-formula. The negation of a Π_n formula is equivalent to a Σ_n formula, and vice versa. A formula is said to be **Boolean in** Π_n if it is built from Π_n formulas using $\wedge, \vee, \neg$. Any formula which is Boolean in Π_n is equivalent to both a Π_{n+1} formula and a Σ_{n+1} formula.

Definition 2.1 *In the language $L(\mathcal{M}) \cup \{F\}$, BDD is the Σ_2 sentence which says that f is bounded,*

$$BDD = \exists x \forall y\, F(y) \leq x.$$

LIM is the Π_3 sentence which says that $\lim_{x\to\infty} f(x) = \infty$,

$$LIM = \forall x \exists y \forall z [y \leq z \Rightarrow x \leq F(z)].$$

We will say that a sentence θ of $L(\mathcal{M}) \cup \{F\}$ is Π_n **over** $\mathcal{M}$ if there is a Π_n sentence of $L(\mathcal{M}) \cup \{F\}$ which is equivalent to θ in every structure $(\mathcal{M}, f)$. Similarly for Σ_n. We say that θ is Δ_n **over** $\mathcal{M}$ if it is both Π_n and Σ_n over $\mathcal{M}$. We will say that θ is **Boolean in** Π_n **over** $\mathcal{M}$, or B_n **over** $\mathcal{M}$, if there is a sentence of $L(\mathcal{M}) \cup \{F\}$ which is Boolean in Π_n and is equivalent to θ in every structure $(\mathcal{M}, f)$. With this terminology, there are two quantifier hierarchies of sentences over $\mathcal{M}$,

$$\Delta_1 \subset \Pi_1 \subset B_1 \subset \Delta_2 \subset \Pi_2 \subset B_2 \subset \Delta_3 \subset \Pi_3,$$

$$\Delta_1 \subset \Sigma_1 \subset B_1 \subset \Delta_2 \subset \Sigma_2 \subset B_2 \subset \Delta_3 \subset \Sigma_3.$$

By the **quantifier level** of a sentence over $\mathcal{M}$ we mean the lowest class in these hierarchies to which a sentence belongs over $\mathcal{M}$. Note that the level of a sentence over an expansion of $\mathcal{M}$ is at most its level over $\mathcal{M}$.

In this paper we will consider the following problem.

Problem *Find the quantifier level of BDD and LIM over a given structure $\mathcal{M}$.*

For any $\mathcal{M}$, BDD is at most Σ_2 and LIM is at most Π_3 over $\mathcal{M}$.

We remark that whenever $\mathcal{M}$ is an expansion of an ordered field, the limit property $\lim_{x\to 0} f(x) = c$ will be at the same quantifier level as LIM. This can be seen by the change of variables $z = 1/x$. Similar remarks can be made for other limit concepts in the calculus.

3 Results of Mostowski and Robinson

In order to compare the results in this paper to earlier results of Mostowski and Robinson, we define the quantifier level of a sentence over a structure $\mathcal{M}$ relative to a set $\mathcal{F} \subseteq \mathcal{M}^{\mathcal{M}}$, where $\mathcal{M}^{\mathcal{M}}$ is the set of all functions from $\mathcal{M}$ into $\mathcal{M}$. We way that a sentence θ of $L(\mathcal{M}) \cup \{F\}$ is Π_n **over $\mathcal{M}$ relative to** $\mathcal{F}$ if there is a Π_n sentence of $L(\mathcal{M}) \cup \{F\}$ which is equivalent to θ in every structure $(\mathcal{M}, f)$ with $f \in \mathcal{F}$. Similarly for Σ_n. Thus a sentence is Π_n over $\mathcal{M}$ in our original sense iff it is Π_n over $\mathcal{M}$ relative to $\mathcal{M}^{\mathcal{M}}$.

Note that if a sentence is Π_n over $\mathcal{M}$, then it is Π_n over $\mathcal{M}$ relative to every set $\mathcal{F} \subseteq \mathcal{M}^{\mathcal{M}}$, and similarly for Σ_n. Thus for ever $\mathcal{M}$ and $\mathcal{F}$, BDD is at most Σ_2 over $\mathcal{M}$ relative to $\mathcal{F}$, and LIM is at most Π_3 over $\mathcal{M}$ relative to $\mathcal{F}$.

In the paper [Mo2], Mostowski showed that several limit concepts, including BDD and LIM, have the highest possible quantifier level over the standard model of arithmetic. In fact, over this particular structure, he obtains the stronger result that BDD and LIM have the highest possible quantifier level relative to the set of primitive recursive functions.

Theorem 3.1 *(Mostowski [Mo2]). Let $\mathcal{F}$ be the set of all primitive recursive functions, and let $\mathcal{N}$ be the standard model of arithmetic with a symbol for every function in $\mathcal{F}$.*

(i) BDD is not Π_2 over $\mathcal{N}$ relative to $\mathcal{F}$.

(ii) LIM is not Σ_3 over $\mathcal{N}$ relative to $\mathcal{F}$.

The proof of Theorem 2 in [Mo2] shows that for every Σ_2 formula $\theta(y)$ in $L(\mathcal{N})$ there is a primitive recursive function $g(x, y)$ such that $\theta(y)$ defines the set of y for which $g(\cdot, y)$ is bounded. By the Arithmetical Hierarchy Theorem of Kleene [Kl] and Mostowski [Mo1], θ may be taken to be Σ_2 but not Π_2 over $\mathcal{N}$, and (i) follows.

Similarly, the proof of Theorem 3 in [Mo2] shows that for every Π_3 formula $\psi(y)$ in $L(\mathcal{N})$ there is a primitive recursive function $h(x, y)$ such that $\psi(y)$ defines the set of y for which $\lim_{x\to\infty} h(x, y) = \infty$, and (ii) follows by taking ψ to be Π_3 but not Σ_3 over $\mathcal{N}$.

Abraham Robinson's characterization of infinite limits with one universal quantifier uses an elementary extension of $\mathcal{M}$.

Definition 3.2 *In an elementary extension ${}^*\mathcal{M}$ of $\mathcal{M}$, an element is* ***infinite*** *(over $\mathcal{M}$) if it is greater than every element of $\mathcal{M}$. By a* ***hyperextension*** *of $\mathcal{M}$ we mean a structure $({}^*\mathcal{M}, I)$ where ${}^*\mathcal{M}$ is an elementary extension of $\mathcal{M}$ with at least one infinite element, and I is a unary predicate for the set of* *infinite elements. A function ${}^*f : {}^*\mathcal{M} \to {}^*\mathcal{M}$ is **standard** if there is a function $f : \mathcal{M} \to \mathcal{M}$ such that $({}^*\mathcal{M}, {}^*f)$ is an elementary extension of $(\mathcal{M}, f)$.*

Theorem 3.3 *(A. Robinson) Let $({}^*\mathcal{M}, I)$ be a hyperextension of $\mathcal{M}$. Then BDD and LIM are Π_1 over $({}^*\mathcal{M}, I)$ relative to the set standard functions.*

In fact, Robinson shows that for every standard function ${}^*f : {}^*\mathcal{M} \to {}^*\mathcal{M}$, $({}^*\mathcal{M}, {}^*f)$ satisfies

$$BDD \Leftrightarrow \forall x[F(x) = 0 \Rightarrow \neg I(x)],$$

$$LIM \Leftrightarrow \forall x[I(x) \Rightarrow I(F(x))].$$

In the nonstandard treatment of elementary calculus, one works in a hyperextension of the system $\mathbb{R}$ of real numbers or the system $\mathbb{N}$ of natural numbers. The extra predicate I for infinite elements eliminates one quantifier block in the definition of boundedness and two quantifier blocks in the definition of limit. The question we address here is whether one can also eliminate these quantifiers in the original first order language $L(\mathcal{M}) \cup \{F\}$. That is, are there any "quantifier shortcuts" for the statements BDD or LIM in $\mathcal{M}$? This question is completely standard in nature, but is motivated by Robinson's results in nonstandard analysis.

4 Cases with Low Quantifier Level

In this section we show that when the underlying structure $\mathcal{M}$ is sufficiently powerful, the properties BDD and LIM are equivalent to sentences in both two quantifier forms. All that is needed is a symbol for a particular function which is first order definable in the real field with a predicate for the natural numbers. This gives a warning that some restrictions are needed on $\mathcal{M}$ in order to show that BDD and LIM are higher than Δ_2 over $\mathcal{M}$. We also show that BDD and LIM can never be B_1 over $\mathcal{M}$.

Theorem 4.1 *Let $\mathcal{M} = (\mathbb{R}, \leq, \mathbb{N}, g)$ be the ordered set of real numbers with a predicate for the natural numbers and a function $g : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ such that $x \mapsto g(x, \cdot)$ maps $\mathbb{R}$ onto $\mathbb{R}^{\mathbb{N}}$, that is, for each $y \in \mathbb{R}^{\mathbb{N}}$ there exists $x \in \mathbb{R}$ such that $y = g(x, \cdot)$.*
(i) BDD is Δ_2 over $\mathcal{M}$.
(ii) LIM is Δ_2 over $\mathcal{M}$.

Proof. The language $L(\mathcal{M})$ has the vocabulary $\{\mathbb{N}, \leq, G\}$.

(i) BDD is itself a Σ_2 sentence. In every structure $(\mathcal{M}, f)$, the negation of BDD is equivalent to the Σ_2 sentence

$$\exists x(\forall n \in \mathbb{N})\, n \leq F(G(x, n)).$$

(ii) In every structure $(\mathcal{M}, f)$, LIM is equivalent to the Σ_2 sentence

$$\exists x(\forall n \in \mathbb{N})\forall y[G(x, n) \leq y \Rightarrow n \leq F(y)],$$

and the negation of LIM is equivalent to the Σ_2 sentence

$$(\exists m \in \mathbb{N})\exists x(\forall n \in \mathbb{N})[n \leq G(x,n) \wedge F(G(x,n)) \leq m].$$

■

It is well known that in the above theorem, the function g can be taken to be definable by a first order formula in the structure $(\mathbb{R}, \leq, +, \cdot, \mathbb{N})$. One way to do this is to let $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a definable pairing function, and for irrational x, let $h(x, \cdot) \in \mathbb{N}^{\mathbb{N}}$ be the continued fraction expansion of x, and take $g(x,n) = z$ iff $\forall m\, h(z,m) = h(x, \pi(m,n))$.

We now show that for an arbitrary $\mathcal{M}$, LIM and BDD cannot be Boolean in Π_1.

Theorem 4.2 *Let $\mathcal{M}$ be an ordered structure with no greatest element.*
(i) BDD is not Boolean in Π_1 over $\mathcal{M}$.
(ii) LIM is not Boolean in Π_1 over $\mathcal{M}$.

Proof. Let φ be a sentence of $L(\mathcal{M}) \cup \{F\}$ which is Boolean in Π_1. Since conjunctions and disjunctions of Π_1 sentences are equivalent to Π_1 sentences, and similarly for Σ_1, φ is equivalent to a sentence

$$(\alpha_1 \vee \beta_1) \wedge \cdots \wedge (\alpha_n \vee \beta_n)$$

where each α_i is Σ_1 and each β_i is Π_1. We may assume that α_i and β_i have the form

$$\alpha_i = \exists \vec{x}\, \exists \vec{y}\, [F(\vec{x}) = \vec{y} \wedge \overline{\alpha}_i(\vec{x}, \vec{y})],$$
$$\beta_i = \forall \vec{x}\, \forall \vec{y}\, [F(\vec{x}) = \vec{y} \Rightarrow \overline{\beta}_i(\vec{x}, \vec{y})]$$

where $\overline{\alpha}_i$ and $\overline{\beta}_i$ are quantifier-free formulas of $L(\mathcal{M})$. (This can be proved by induction on the number of occurrences of F).

Let us say that a pair of tuples $(\vec{a}, \vec{b})$ in $\mathcal{M}$ decides φ if for any $f, g : \mathcal{M} \to \mathcal{M}$ such that $f(\vec{a}) = g(\vec{a}) = \vec{b}$, φ holds in either both or neither of the models $(\mathcal{M}, f)$, $(\mathcal{M}, g)$.

Claim: There is a pair $(\vec{a}, \vec{b})$ which decides φ.

Proof of Claim: The proof is by induction on n. The claim is trivial for $n = 0$, where the empty conjunction is taken to be always true. Suppose $n > 0$ and the claim holds for $n - 1$. Then there is a pair $(\vec{a}, \vec{b})$ which decides

$$(\alpha_1 \vee \beta_1) \wedge \cdots \wedge (\alpha_{n-1} \vee \beta_{n-1}).$$

A pair $(\vec{c}, \vec{d})$ is called compatible with $(\vec{a}, \vec{b})$ if $b_i = d_j$ whenever $a_i = c_j$, that is, there exists a function f with $f(\vec{a}, \vec{c}) = (\vec{b}, \vec{d})$.

Case 1. There is a pair $(\vec{c}, \vec{d})$ compatible with $(\vec{a}, \vec{b})$ such that $\overline{\alpha}_n(\vec{c}, \vec{d})$. Then $f(\vec{c}) = \vec{d}$ implies α_n.

Case 2. Case 1 fails but there is a pair $(\vec{c}, \vec{d})$ compatible with $(\vec{a}, \vec{b})$ such that $\neg\overline{\beta}_n(\vec{c}, \vec{d})$. Then $f(\vec{a}, \vec{c}) = (\vec{b}, \vec{d})$ implies $\neg(\alpha_n \vee \beta_n)$.

Case 3. For every pair $(\vec{c}, \vec{d})$ compatible with $(\vec{a}, \vec{b})$,

$$\neg\overline{\alpha}_n(\vec{c}, \vec{d}) \wedge \overline{\beta}_n(\vec{c}, \vec{d}).$$

Then $f(\vec{a}) = \vec{b}$ implies β_n.

In each case, $(\vec{a} \cup \vec{c}, \vec{b} \cup \vec{d})$ decides φ, completing the induction.

Now let $(\vec{a}, \vec{b})$ decide φ. There are functions f, g such that $f(\vec{a}) = g(\vec{a}) = \vec{b}$, but BDD holds in $(\mathcal{M}, f)$ and fails in $(\mathcal{M}, g)$. Therefore φ cannot be equivalent to BDD over $\mathcal{M}$. A similar argument holds for LIM. ■

5 Countable Structures

In this section we consider the case that the universe of $\mathcal{M}$ is countable. In that case, we apply Mostowski's Theorem 3.1 to show that BDD and LIM have the highest possible quantifier level. We first observe that Mostowski's proof of Theorem 3.1 also gives a relativized form of the result.

Theorem 5.1 *Let α be a finite tuple of finitary functions on $\mathbb{N}$, let $\mathcal{F}(\alpha)$ be the set of all functions which are primitive recursive in α, and let $\mathcal{N}$ be the standard model of arithmetic with an extra symbol for each function in the set $\mathcal{F}(\alpha)$.*

(i) BDD is not Π_2 over $\mathcal{N}$ relative to $\mathcal{F}(\alpha)$.

(ii) LIM is not Σ_3 over $\mathcal{N}$ relative to $\mathcal{F}(\alpha)$.

Corollary 5.2 *Let $\mathcal{N}$ be an expansion of the standard model of arithmetic with the natural ordering $<$.*

(i) BDD is not Π_2 over $\mathcal{N}$.

(ii) LIM is not Σ_3 over $\mathcal{N}$.

Theorem 5.3 *Suppose $\mathcal{M}$ is an ordered structure with no greatest element and the universe of $\mathcal{M}$ is countable.*

(i) BDD is not Π_2 over $\mathcal{M}$.

(ii) LIM is not Σ_3 over $\mathcal{M}$.

Proof. We may assume that the vocabulary of $\mathcal{M}$ is finite, since only finitely many symbols occur in a Σ_3 formula. We may also assume that the universe of $\mathcal{M}$ is the set $\mathbb{N}$ of natural numbers. Let $<_\mathcal{M}$ be the ordering of $\mathcal{M}$ (which may be different from the natural order $<$ of $\mathbb{N}$). Since $\mathcal{M}$ has no greatest element, there is a function $h : \mathbb{N} \to \mathbb{N}$ such that $h(n) <_\mathcal{M} h(n+1)$ for each n, and $\forall x \exists n\, x \leq_\mathcal{M} h(n)$. For each x let $\lambda(x)$ be the least n such that $x \leq_\mathcal{M} h(n)$.

We prove (ii). The proof of (i) is similar. We observe that for each function $g : \mathbb{N} \to \mathbb{N}$, $\lim_{x\to\infty} h(g(\lambda(x))) = \infty$ with respect to $<_\mathcal{M}$ if and only if $\lim_{x\to\infty} g(x) = \infty$ with respect to $<$. If LIM is not Σ_3 over an expansion of $\mathcal{M}$, then it is not Σ_3 over $\mathcal{M}$. We may therefore assume that $\mathcal{M}$ has symbols for $<, +, \cdot, h$, and λ.

Now suppose that LIM is Σ_3 over $\mathcal{M}$, that is, there is a Σ_3 sentence $\exists\vec{x}\,\forall\vec{y}\,\exists\vec{z}\,\varphi(\vec{x}, \vec{y}, \vec{z})$ of $L(\mathcal{M}) \cup \{F\}$ which is equivalent to LIM in all structures $(\mathcal{M}, f)$. To complete the proof we will get a contradiction.

Let $\psi(\vec{x}, \vec{y}, \vec{z})$ be the formula obtained from $\varphi(\vec{x}, \vec{y}, \vec{z})$ by replacing each term $F(u)$ by $h(F(\lambda(u)))$. By our observation above, for any function $g : \mathbb{N} \to \mathbb{N}$, we have $\lim_{x\to\infty} g(x) = \infty$ with respect to $<$ if and only if $(\mathcal{M}, g)$ satisfies the Σ_3 sentence $\exists\vec{x}\,\forall\vec{y}\,\exists\vec{z}\,\psi(\vec{x}, \vec{y}, \vec{z})$. This contradicts Corollary 5.2 and completes the proof.

We also give a second argument, which will be useful later. Instead of Corollary 5.2, this argument uses the following analogous fact from descriptive set theory (see [Kec, Exercise 23.2):

In $\mathbb{N}^{\mathbb{N}}$, the set L of functions g with $\lim_{x\to\infty} g(x) = \infty$ is $\mathbf{\Pi}_3$ but not $\mathbf{\Sigma}_3$ in the Borel hierarchy.

We have

$$L = \bigcup_{\vec{x}\in\mathbb{N}} \bigcap_{\vec{y}\in\mathbb{N}} \bigcup_{\vec{z}\in\mathbb{N}} \{g : (\mathcal{M}, g) \models \psi(\vec{x}, \vec{y}, \vec{z})\}).$$

For each $(\vec{x}, \vec{y}, \vec{z})$, the set $\{g : (\mathcal{M}, g) \models \psi(\vec{x}, \vec{y}, \vec{z})\}$ depends only on finitely many values of g and thus is clopen in $\mathbb{N}^{\mathbb{N}}$. It follows that L is $\mathbf{\Sigma}_3$ in $\mathbb{N}^{\mathbb{N}}$, contrary to the preceding paragraph. ■

The next corollary follows by the Löwenheim-Skolem theorem.

Corollary 5.4 *Let $\mathcal{M}$ be an ordered structure with no greatest element.*

(i) There is no Π_2 sentence of $L(\mathcal{M}) \cup \{F\}$ which is equivalent to BDD in all models of $Th(\mathcal{M})$.

(ii) There is no Σ_3 sentence of $L(\mathcal{M}) \cup \{F\}$ which is equivalent to LIM in all models of $Th(\mathcal{M})$.

A result in this direction was previously obtained by Kathleen Sullivan in [Su]. She showed that there is no Σ_2 sentence of $L(\mathcal{M}) \cup \{F\}$, and no Π_2 sentence of $L(\mathcal{M}) \cup \{F\}$, which is equivalent to LIM in all models of $Th(\mathcal{M})$.

6 The Real Line

The sentence LIM is a Π_3 sentence in the vocabulary with the single relation symbol $\leq$. It is therefore natural to ask whether LIM is Σ_3 over the real ordering $(\mathbb{R}, \leq)$. In this section we show that the answer is no. Given a structure $\mathcal{N}$ with universe $\mathbb{N}$, we let $(\mathbb{R}, \leq, \mathcal{N})$ be the structure formed by adding to $(\mathbb{R}, \leq)$ a symbol for $\mathbb{N}$ and the relations of $\mathcal{N}$.

Theorem 6.1 *Let $\mathcal{N}$ be a structure with universe $\mathbb{N}$.*

(i) BDD is not Π_2 over $(\mathbb{R}, \leq, \mathcal{N})$.

(ii) LIM is not Σ_3 over $(\mathbb{R}, \leq, \mathcal{N})$.

Proof. We prove (ii). The proof of (i) is similar. We may assume that $\mathcal{N}$ has a countable vocabulary. Let $\mathbb{Q}$ be the set of rational numbers, and let $\lambda(x) = \min\{n \in \mathbb{N} : x \leq n\}$.

Claim 1: $(\mathbb{Q}, \leq, \mathcal{N})$ is an elementary substructure of $(\mathbb{R}, \leq, \mathcal{N})$.

Proof of Claim 1. We note that for any two increasing n-tuples $\vec{a}, \vec{b}$ in $\mathbb{R}$ such that $\vec{a} \cap \mathbb{N} = \vec{b} \cap \mathbb{N}$ and $\lambda(\vec{a}) = \lambda(\vec{b})$, there is an automorphism of $(\mathbb{R}, \leq, \mathcal{N})$ which sends $\vec{a}$ to $\vec{b}$. Claim 1 now follows from the Tarski-Vaught test ([CK], Proposition 3.1.2).

Define the function $\alpha : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{R}}$ by $(\alpha(g))(x) = g(\lambda(x))$, and let $\beta(g)$ be the restriction of $\alpha(g)$ to $\mathbb{Q}$.

Claim 2: For each function $g \in \mathbb{N}^{\mathbb{N}}$, $(\mathbb{Q}, \leq, \mathcal{N}, \beta(g))$ is an elementary substructure of $(\mathbb{R}, \leq, \mathcal{N}, \alpha(g))$.

To see this, note that by Claim 1, $(\mathbb{Q}, \leq, (\mathcal{N}, g))$ is an elementary substructure of $(\mathbb{R}, \leq, (\mathcal{N}, g))$. It is easily seen that λ is definable in $(\mathbb{R}, \leq, \mathbb{N})$, and hence $\alpha(g)$ is definable in $(\mathbb{R}, \leq, (\mathcal{N}, g))$. Claim 2 follows.

Let L be the set of $g \in \mathbb{N}^{\mathbb{N}}$ such that $(\mathcal{M}, \alpha(g)) \models LIM$. We note that $\lim_{n\to\infty} g(n) = \infty$ in $\mathbb{N}^{\mathbb{N}}$ if and only if $g \in L$, and hence that L is not $\mathbf{\Sigma_3}$ in $\mathbb{N}^{\mathbb{N}}$. Now suppose that LIM is Σ_3 over $(\mathbb{R}, \leq, \mathcal{N})$, and take a Σ_3 sentence

$$\theta = \exists \vec{x}\, \forall \vec{y}\, \exists \vec{z}\, \varphi(\vec{x}, \vec{y}, \vec{z})$$

of $L(\mathbb{R}, \leq, \mathcal{N}) \cup \{F\}$ which is equivalent to LIM in all structures $(\mathbb{R}, \leq, \mathcal{N}, f)$. By Claim 2, for each $g \in \mathbb{N}^{\mathbb{N}}$, $(\mathbb{R}, \leq, \mathcal{N}, \alpha(g))$ satisfies θ if and only if $(\mathbb{Q}, \leq, \mathcal{N}, \beta(g))$ satisfies θ. Therefore

$$L = \bigcup_{\vec{x}\in\mathbb{Q}} \bigcap_{\vec{y}\in\mathbb{Q}} \bigcup_{\vec{z}\in\mathbb{Q}} (\{g : \varphi(\beta(g), \vec{x}, \vec{y}, \vec{z})\}).$$

For each $(\vec{x}, \vec{y}, \vec{z})$, the set $(\{g : \varphi(\beta(g), \vec{x}, \vec{y}, \vec{z})\})$ depends only on finitely many values of g and thus is clopen in $\mathbb{N}^{\mathbb{N}}$. Since $\mathbb{Q}$ is countable, it follows that L is $\mathbf{\Sigma}_3$ in $\mathbb{N}^{\mathbb{N}}$. This contradiction proves (ii). ∎

We remark that the above theorem also holds, with the same proof, when $\mathbb{R}$ is replaced by any dense linear ordering with a cofinal copy of $\mathbb{N}$.

7 Saturated and Special Structures

In this section we show that BDD and LIM have the highest possible quantifier level if the underlying structure $\mathcal{M}$ is saturated, or more generally, special. This happens even if one adds a symbol for the set I of infinite elements to the vocabulary. Thus Robinson's Theorem 3.3 for standard functions cannot be extended to the set of all functions on a special structure.

We recall some basic facts (See [CK], Chapter 5). By definition, a structure $\mathcal{M}$ is **κ-saturated** if every set of fewer than κ formulas with parameters in $\mathcal{M}$ which is finitely satisfiable in $\mathcal{M}$ is satisfiable in $\mathcal{M}$. $\mathcal{M}$ is **saturated** if it is $|\mathcal{M}|$-saturated. $\mathcal{M}$ is **special** if it is the union of an elementary chain of λ^+-saturated structures where λ ranges over all cardinals less than $|\mathcal{M}|$.

Let us call a cardinal κ **nice** if $\kappa = \sum_{\lambda<\kappa} 2^\lambda$. For example, every inaccessible cardinal is nice, every strong limit cardinal is nice, and every cardinal $\lambda^+ = 2^\lambda$ is nice.

Facts 7.1 *(i) If $\omega \le |\mathcal{M}| < \kappa$ and κ is nice then $\mathcal{M}$ has a special elementary extension of cardinality κ.*

(ii) Any two elementarily equivalent special models of the same cardinality are isomorphic.

(iii) Reducts of special models are special.

(iv) If $\mathcal{M}$ is special, then $(\mathcal{M}, \vec{a})$ is special for each finite tuple $\vec{a}$ in $\mathcal{M}$.

Theorem 7.2 *Suppose $\mathcal{M}$ is an ordered structure with no greatest element, $\mathcal{M}$ is special, $|\mathcal{M}|$ is nice, and $|L(\mathcal{M})| < |\mathcal{M}|$.*

(i) BDD is not Π_2 over $\mathcal{M}$.

(ii) LIM is not Σ_3 over $\mathcal{M}$.

Proof. We prove (ii). The proof of (i) is similar. Let T be the set of all Σ_3 consequences of $Th(\mathcal{M}) \cup \{LIM\}$. Consider a finite subset $T_0 \subseteq T$. The conjunction of T_0 is equivalent to a Σ_3 sentence θ, and $Th(\mathcal{M}) \cup \{LIM\} \models \theta$. By Corollary 5.4, we cannot have $Th(\mathcal{M}) \cup \{\theta\} \models LIM$, and therefore we cannot have $Th(\mathcal{M}) \cup T_0 \models LIM$. By the Compactness Theorem, there is a model $(\mathcal{M}_0, g)$ of $Th(\mathcal{M}) \cup T \cup \{\neg LIM\}$. Now let U be the set of all Π_3 sentences which hold in $(\mathcal{M}_0, g)$. By the Compactness Theorem again, there is a model $(\mathcal{M}_1, f)$ of $Th(\mathcal{M}) \cup U \cup \{LIM\}$. Then every Σ_3-sentence holding in $(\mathcal{M}_1, f)$ holds in $(\mathcal{M}_0, g)$. Let $\kappa = |\mathcal{M}|$. Since $|L(\mathcal{M})| < \kappa$ and κ is nice, it follows from Fact (i) that we may take $(\mathcal{M}_0, g)$ and $(\mathcal{M}_1, f)$ to be special models of cardinality κ. By Fact (ii), $\mathcal{M}_0$ and $\mathcal{M}_1$ are isomorphic to $\mathcal{M}$, so we may take $\mathcal{M}_1 = \mathcal{M}_0 = \mathcal{M}$. Then LIM holds in $(\mathcal{M}, f)$ and fails in $(\mathcal{M}, g)$, but every Σ_3 sentence holding in $(\mathcal{M}, f)$ holds in $(\mathcal{M}, g)$. This proves (ii). The proof of (i) is similar. ∎

Given a hyperextension $({}^*\mathcal{M}, I)$ of $\mathcal{M}$, we will say that a function $g : {}^*\mathcal{M} \to {}^*\mathcal{M}$ is **standard** if $({}^*\mathcal{M}, g)$ is an elementary extension of $(\mathcal{M}, f)$, or in other words, if $({}^*\mathcal{M}, g, I)$ is a hyperextension of $(\mathcal{M}, f)$. Robinson's Theorem 3.3 shows that BDD and LIM are Π_1 for standard functions over any hyperextension $({}^*\mathcal{M}, I)$ of $\mathcal{M}$.

Our next theorem will show that Robinson's result does not carry over to nonstandard functions. In fact, we will show that Theorem 7.2 holds even when a symbol for the set I of infinite elements is added to the vocabulary. Thus for nonstandard functions, one cannot lower the quantifier level of BDD or LIM by adding a symbol for I.

Lemma 7.3 *Let $\mathcal{M}$ be an ordered structure with no greatest element, such that every element of $\mathcal{M}$ is a constant symbol of $L(\mathcal{M})$. Suppose that $({}^*\mathcal{M}, I)$ a hyperextension of $\mathcal{M}$ of cardinality $\kappa = |{}^*\mathcal{M}|$ such that $|L(\mathcal{M})| < \kappa$, κ is nice, $({}^*\mathcal{M}, f, g)$ is special, and every Π_n sentence of $L(\mathcal{M}) \cup \{F\}$ true in $({}^*\mathcal{M}, f)$ is true in $({}^*\mathcal{M}, g)$. Then every Π_n sentence of $L(\mathcal{M}) \cup \{F, I\}$ true in $({}^*\mathcal{M}, f, I)$ is true in $({}^*\mathcal{M}, g, I)$. Similarly for Σ_n.*

Proof. The result for Σ_n follows from the result for Π_n by interchanging f and g. Let $\kappa = |{}^*\mathcal{M}|$. By the Keisler Sandwich Theorem ([CK], Proposition

5.2.7 and Exercise 5.2.7), there is a chain

$$(\mathcal{M}_0, h_0) \subseteq (\mathcal{M}_1, h_1) \subseteq \cdots (\mathcal{M}_n, h_n)$$

such that

$$(\mathcal{M}_0, h_0) \equiv (\mathcal{M}, f), \quad (\mathcal{M}_1, h_1) \equiv (\mathcal{M}, g),$$

and for each $m < n-1$,

$$(\mathcal{M}_m, h_m) \prec (\mathcal{M}_{m+2}, h_{m+2}).$$

By Fact (i) we may take each $(\mathcal{M}_m, h_m)$ to be a special structure of cardinality κ. For each $m \leq n$ let I_m be the set of elements of $\mathcal{M}_m$ greater than each element of $\mathcal{M}$. We then have

$$(\mathcal{M}_0, h_0, I_0) \subseteq (\mathcal{M}_1, h_1, I_1) \subseteq \cdots (\mathcal{M}_n, h_n, I_n).$$

Since each element of $\mathcal{M}$ has a constant symbol in $L(\mathcal{M})$, it follows from Fact (i) that

$$(\mathcal{M}_0, h_0, I_0) \cong ({}^*\mathcal{M}, f, I) \text{ and } (\mathcal{M}_1, h_1, I_1) \cong ({}^*\mathcal{M}, g, I).$$

By remark (iv), $(\mathcal{M}_m, h_m, \vec{a})$ is still special for each m and finite tuple $\vec{a}$ in $\mathcal{M}_m$. Then by Fact (ii), for each $m < n-1$ and finite tuple $\vec{a}$ in $\mathcal{M}_m$, there is an isomorphism from $(\mathcal{M}_m, h_m)$ onto $(\mathcal{M}_{m+2}, h_{m+2})$ which fixes $\vec{a}$ and each element of $\mathcal{M}$. This isomorphism also sends I_m onto I_{m+2}, so

$$(\mathcal{M}_m, h_m, I_m, \vec{a}) \cong (\mathcal{M}_{m+2}, h_{m+2}, I_{m+2}, \vec{a}).$$

Using the Tarski-Vaught criterion for elementary extensions ([CK], Proposition 3.1.2), it follows that

$$(\mathcal{M}_m, h_m, I_m) \prec (\mathcal{M}_{m+2}, h_{m+2}, I_{m+2}).$$

By the Keisler Sandwich Theorem again, every Π_n-sentence true in $(\mathcal{M}_0, h_0, I_0)$ is true in $(\mathcal{M}_1, h_1, I_1)$, and hence every Π_n-sentence true in $({}^*\mathcal{M}, f, I)$ is true in $({}^*\mathcal{M}, g, I)$. ∎

Theorem 7.4 *Let $\mathcal{M}$ be an ordered structure with no greatest element, and let $({}^*\mathcal{M}, I)$ be a hyperextension of $\mathcal{M}$ such that ${}^*\mathcal{M}$ is special and $|{}^*\mathcal{M}|$ is uncountable and nice.*

(i) BDD is not Π_2 over $({}^*\mathcal{M}, I)$.

(ii) LIM is not Σ_3 over $({}^*\mathcal{M}, I)$.

Proof. We prove (ii). The proof of (i) is similar. By the proof of Theorem 7.2, there are functions f, g such that $({}^*\mathcal{M}, f)$ and $({}^*\mathcal{M}, g)$ are special, LIM holds in $({}^*\mathcal{M}, f)$ and fails in $({}^*\mathcal{M}, g)$, and every Σ_3 sentence true in $({}^*\mathcal{M}, f)$ is true in $({}^*\mathcal{M}, g)$. By Facts (i) and (ii), we may take f and g so that $({}^*\mathcal{M}, f, g)$ is special. Then by the preceding lemma, every Σ_3 sentence true in $({}^*\mathcal{M}, f, I)$ is true in $({}^*\mathcal{M}, g, I)$. Therefore LIM is not Σ_3 over $({}^*\mathcal{M}, I)$. ∎

8 Infinitely Long Sentences

In this section we consider the quantifier levels of BDD and LIM in the infinitary logic $L_{\omega_1\omega}$ formed by adding countable conjunctions and disjunctions to first order logic. See [Ke1] for a treatment of the model theory of this logic.

Definition 8.1 *Let Q be a set of sentences of $L_{\omega_1\omega}$ in the vocabulary $L(\mathcal{M})$. $\bigwedge Q$ denotes the set of countable conjunctions of sentences in Q. $\bigvee Q$ denotes the set of countable disjunctions of sentences in Q. BQ denotes the set of finite Boolean combinations of sentences in Q.*

For example, $\bigvee\bigwedge B\bigwedge\Pi_1$ is the set of sentences of the form $\bigvee_m\bigwedge_n\theta_{mn}$ where each θ_{mn} is a finite Boolean combination of countable conjunctions of universal first order sentences.

We say that $\mathcal{M}$ has **cofinality** ω if $\mathcal{M}$ has a countable increasing sequence $a_0, a_1, \ldots$ which is unbounded, that is, $\forall x \bigvee_n x \le a_n$. Every countable $\mathcal{M}$ and every structure $\mathcal{M}$ with universe $\mathbb{R}$ has cofinality ω. In this section our main focus will be uncountable $\mathcal{M}$ with cofinality ω.

Suppose that $\mathcal{M}$ has universe $\mathbb{R}$ and a constant symbol for each natural number. Then BDD is $\bigvee\Pi_1$ over $\mathcal{M}$, because

$$(\mathcal{M}, f) \models LIM \Leftrightarrow \bigvee_n \forall z\, F(z) \le n.$$

LIM is $\bigwedge\bigvee\Pi_1$ over $\mathcal{M}$, because

$$(\mathcal{M}, f) \models LIM \Leftrightarrow \bigwedge_m \bigvee_n \forall z\,[n \le z \Rightarrow m \le F(z)].$$

LIM is also $\bigwedge\Sigma_2$ over $\mathcal{M}$, because

$$(\mathcal{M}, f) \models LIM \Leftrightarrow \bigwedge_m \exists y \forall z\,[y \le z \Rightarrow m \le F(z)].$$

The next theorem shows that if $\mathcal{M}$ has cofinality ω, then BDD is not $\bigwedge\Sigma_1$ over $\mathcal{M}$ and LIM is not $\bigvee\bigwedge\Sigma_1$ over $\mathcal{M}$. Thus if the outer quantifiers $\exists$ and $\forall$ are replaced by $\bigvee$ and $\bigwedge$, then Theorem 5.3 holds for uncountable structures $\mathcal{M}$ of cofinality ω. In fact, we get a stronger result with $B\bigwedge\Pi_1$ in place of Σ_1.

Theorem 8.2 *Suppose that $\mathcal{M}$ has cofinality ω.*
(i) BDD is not $\bigwedge B\bigwedge\Pi_1$ over $\mathcal{M}$.
(ii) LIM is not $\bigvee\bigwedge B\bigwedge\Pi_1$ over $\mathcal{M}$.

Proof. We prove (ii). The proof of (i) is similar.

Let a_n be an unbounded strictly increasing sequence in $\mathcal{M}$. Since each sentence of $L_{\omega_1\omega}$ has countably many symbols, we may assume without loss of generality that the vocabulary $L(\mathcal{M})$ is countable. We may also assume that $L(\mathcal{M})$ has a constant symbol, say $\mathbf{n}$, for each a_n, and a symbol for the function

$\lambda(x) = \min\{a_n : x \leq a_n\}$. As in the proof of Theorem 5.3, we let $\alpha : \mathbb{N}^{\mathbb{N}} \to \mathcal{M}^{\mathcal{M}}$ be the function defined by $(\alpha(g))(x) = g(\lambda(x))$, and let L be the set of $g \in \mathbb{N}^{\mathbb{N}}$ such that $(\mathcal{M}, \alpha(g)) \models LIM$.

Claim: For each first order quantifier-free formula $\varphi(F, \vec{x})$ in the vocabulary $L(\mathcal{M}) \cup \{F\}$, the set

$$\{g \in \mathbb{N}^{\mathbb{N}} : (\mathcal{M}, \alpha(g)) \models \exists \vec{z}\, \varphi(F, \vec{z})\}$$

is $\mathbf{\Sigma}_1$ in the Borel hierarchy.

Proof of Claim: We may assume without loss of generality that $\exists \vec{z}\, \varphi(F, \vec{z})$ has the form

$$\exists \vec{x} \exists \vec{y}\, [F(\vec{x}) = \vec{y} \wedge \psi(\vec{x}, \vec{y})]$$

where $\psi(\vec{x}, \vec{y})$ is a first order quantifier-free formula of $L(\mathcal{M})$. Since $(\alpha(g))(x) = (\alpha(g))(\lambda(x))$ for all g and x, $(\mathcal{M}, \alpha(g))$ satisfies $\exists \vec{z}\, \varphi(F, \vec{z})$ if and only if it satisfies

$$\bigvee_{\vec{\mathbf{p}}} \bigvee_{\vec{\mathbf{q}}} [F(\vec{\mathbf{p}}) = \vec{\mathbf{q}} \wedge \exists \vec{x}\, [\lambda(\vec{x}) = \vec{\mathbf{p}} \wedge \psi(\vec{x}, \vec{\mathbf{q}})]].$$

Let

$$U = \{(\vec{\mathbf{p}}, \vec{\mathbf{q}}) : \mathcal{M} \models \exists \vec{x}\, [\lambda(\vec{x}) = \vec{\mathbf{p}} \wedge \psi(\vec{x}, \vec{\mathbf{q}})]\}.$$

Then $(\mathcal{M}, \alpha(g))$ satisfies $\exists \vec{z}\, \varphi(F, \vec{z})$ if and only if $g(\vec{\mathbf{p}}) = \vec{\mathbf{q}}$ for some $(\vec{\mathbf{p}}, \vec{\mathbf{q}}) \in U$. For each pair $(\vec{\mathbf{p}}, \vec{\mathbf{q}})$, the set $\{g : g(\vec{\mathbf{p}}) = \vec{\mathbf{q}}\}$ is clopen in $\mathbb{N}^{\mathbb{N}}$, so the union of these sets over U is $\mathbf{\Sigma}_1$ in the Borel hierarchy, as required.

Now suppose to the contrary that there is a $\bigvee \bigwedge B \bigwedge \Pi_1$ sentence

$$\theta = \bigvee_m \bigwedge_n \varphi_{mn}(F)$$

which is equivalent to LIM in all models $(\mathcal{M}, f)$, where $\varphi_{mn}(F)$ is $B \bigwedge \Pi_1$ in the vocabulary $L(\mathcal{M}) \cup \{F\}$.

We observe that any finite conjunction or disjunction of $\bigwedge \Pi_1$ sentences is equivalent to a $\bigwedge \Pi_1$ sentence, and the negation of a $\bigwedge \Pi_1$ sentence is equivalent to a $\bigvee \Sigma_1$ sentence. It follows that θ is equivalent to a sentence

$$\bigvee_m \bigwedge_n \left[\bigwedge_p \forall \vec{z} \varphi_{mnp}(F, \vec{z}) \vee \bigvee_q \exists \vec{z}\, \psi_{mnq}(F, \vec{z}) \right]$$

where each $\varphi_{mnp}(F, \vec{z})$ and $\psi_{mnq}(F, \vec{z})$ is a first order quantifier-free formula of $L(\mathcal{M}) \cup \{F\}$.

Using the claim, it follows that

$$L = \bigcup_m \bigcap_n \left(\bigcap_p A_{mnp} \cup \bigcup_q B_{mnq} \right)$$

where each A_{mnp} and B_{mnq} is clopen in $\mathbb{N}^{\mathbb{N}}$. By renumbering and rearranging, we can get

$$L = \bigcup_m \bigcap_n \bigcup_q (A_{mnq} \cup B_{mnq}).$$

Therefore L is $\mathbf{\Sigma}_3$ in $\mathbb{N}^{\mathbb{N}}$. This is a contradiction, and proves (ii). ■

9 O-minimal Structures

In this section we show that BDD has the highest possible quantifier level, and that LIM is not Boolean in Π_2, in the case that $\mathcal{M}$ is an o-minimal expansion of the ordered field of real numbers. We leave open the question whether LIM can be Σ_3. We will use two lemmas from [FM] concerning fast and indiscernible functions.

An ordered structure is **o-minimal** if every set definable with parameters is a finite union of intervals and points. There is an extensive literature on the subject (e.g. see [VDD1], [VDD2]). An example of an o-minimal structure is the ordered field of reals with analytic functions restricted to the unit cube and with the (unrestricted) exponential function. It is easily seen that if $\mathcal{M}$ is o-minimal, then every ordered reduct of $\mathcal{M}$ is o-minimal, and the expansion of $\mathcal{M}$ formed by adding a symbol for each definable relation is o-minimal.

Throughout this section we let $\mathcal{M}$ be an o-minimal expansion of the real ordered field $\mathcal{R} = (\mathbb{R}, \leq, +, \cdot)$. "Definable" will always mean first order definable in $\mathcal{M}$ with parameters from $\mathcal{M}$.

When working with o-minimal structures, one often restricts attention to definable functions. It is easily seen that LIM is Π_2 over an o-minimal structure $\mathcal{M}$ relative to the set of definable functions, since LIM fails for a definable function f if and only if $\exists x \exists y \forall z [x \leq z \Rightarrow f(z) \leq y]$. However, here we consider arbitrary functions on $\mathcal{M}$.

Definition 9.1 *A strictly increasing sequence s of positive integers is $\mathcal{M}$-**fast** if for each definable function $f : \mathbb{R} \to \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $f(s(k)) < s(k+1)$ for all $k > N$. For convenience we also require that $s(0) = 0$.*

Lemma 9.2 *([FM], 3.3) If $L(\mathcal{M})$ is countable then there exists an $\mathcal{M}$-fast sequence.*

Given $\vec{u}, \vec{v} \in \mathbb{N}^n$ and $N \in \mathbb{N}$, we write $\vec{u} =_N \vec{v}$ if $\min(u_i, N) = \min(v_i, N)$ for $i = 1, \ldots, n$, and $\vec{u} \sim \vec{v}$ if $\vec{u}, \vec{v}$ are order isomorphic, that is, $u_i \leq u_j$ iff $v_i \leq v_j$ for $i, j = 1, \ldots, n$. We write $\vec{u} \sim_N \vec{v}$ if $\vec{u} =_N \vec{v}$ and $\vec{u} \sim \vec{v}$. Note that $\sim_N$ is an equivalence relation on $\mathbb{N}^n$ with finitely many classes.

Given an $\mathcal{M}$-fast sequence s and a tuple $\vec{u} \in \mathbb{N}^n$, we write $s(\vec{u})$ for $(s(u_1), \ldots, s(u_n))$.

Lemma 9.3 *([FM], 3.6). Suppose s is $\mathcal{M}$-fast, $h : \mathbb{R}^n \to \mathbb{R}$ is definable, and $g : \mathbb{N}^n \to \mathbb{R}$ is the function given by $\vec{u} \mapsto h(s(\vec{u}))$. Then there exists $N \in \mathbb{N}$ such that*

$$g(\vec{u}) \leq g(\vec{v}) \Leftrightarrow g(\vec{w}) \leq g(\vec{z})$$

whenever $\vec{u} =_N \vec{v}$ and $(\vec{u}, \vec{v}) \sim_N (\vec{w}, \vec{z})$.

In [FM], the function g is called **almost indiscernible**. We will call N an **indiscernibility bound** for h.

Corollary 9.4 *Suppose s is $\mathcal{M}$-fast, $A \subseteq \mathbb{R}^n$ is definable, and*

$$B = \{\vec{u} \in \mathbb{N}^n : A(s(\vec{u}))\}.$$

There exists $N \in \mathbb{N}$ such that $B(\vec{u}) \Leftrightarrow B(\vec{v})$ whenever $\vec{u}, \vec{v} \in \mathbb{N}^n$ and $\vec{u} \sim_N \vec{v}$.

Proof. The characteristic function h of A is definable. Let N be an indiscernibility bound for h, and let g be as in Lemma 9.3. Suppose $\vec{u}, \vec{v} \in \mathbb{N}^n$ and $\vec{u} \sim_N \vec{v}$. Let $k \in \mathbb{N}$ be greater than N and each u_i and v_i. By replacing the u_i by $k + u_i$ whenever $N \leq u_i$, we obtain a tuple $\vec{w}$ such that $\vec{w} \sim_N \vec{u}$ and $(\vec{u}, \vec{w}) \sim_N (\vec{v}, \vec{w})$. By Lemma 9.3, $g(\vec{u}) \leq g(\vec{w})$ iff $g(\vec{v}) \leq g(\vec{w})$, and $g(\vec{u}) \geq g(\vec{w})$ iff $g(\vec{v}) \geq g(\vec{w})$. It follows that $g(\vec{u}) = g(\vec{v})$, so $B(\vec{u}) \Leftrightarrow B(\vec{v})$. ■

By an indiscernibility bound for a formula ψ of $L(\mathcal{M})$ with parameters from $\mathcal{M}$ we mean an indiscernibility bound for the characteristic function of the set defined by ψ. It is clear that if N is an indiscernibility bound for ψ, then any $M \geq N$ is also an indiscernibility bound for ψ.

Theorem 9.5 *Suppose that $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R}, \leq, +, \cdot)$. Then BDD is not Π_2 over $\mathcal{M}$.*

Proof. We may assume that $\mathcal{M}$ has a countable vocabulary. By Lemma 9.2 there is an $\mathcal{M}$-fast sequence s. Suppose to the contrary that there is a Π_2 sentence $\forall \vec{x} \varphi(F, \vec{x})$ of $L(\mathcal{M}) \cup \{F\}$ which is equivalent to BDD in all structures $(\mathcal{M}, f)$. We may assume that $\forall \vec{x} \varphi(F, \vec{x})$ has the form

$$\forall \vec{x}\, \exists \vec{y}\, \exists \vec{z}\, [F(\vec{x}, \vec{y}) = \vec{z} \wedge \theta(\vec{x}, \vec{y}, \vec{z})]$$

where θ is a quantifier-free formula in which F does not occur, $|\vec{z}| = |\vec{x}| + |\vec{y}| = j + k$, and

$$F(\vec{x}, \vec{y}) = (F(x_1), \ldots, F(x_j), F(y_1), \ldots, F(y_k)).$$

(This can be proved by induction on the number of occurrences of F in φ). Let $f : \mathbb{R} \to \mathbb{R}$ be the function such that $f(x) = x$ if $x = s(4i)$ for some $i \in \mathbb{N}$, and $f(x) = 0$ otherwise. For each $m \in \mathbb{N}$ let $f_m : \mathbb{R} \to \mathbb{R}$ be the function such that $f_m(x) = f(x)$ for $x \leq s(m)$ and $f_m(x) = 0$ for $x > s(m)$. Then each f_m is bounded but f is unbounded. Therefore $\forall \vec{x}\, \varphi(f_m, \vec{x})$ holds for each $m \in \mathbb{N}$, but $\forall \vec{x} \varphi(f, \vec{x})$ fails. (Recall that $\forall \vec{x}\, \varphi(f, \vec{x})$ means that $(\mathcal{M}, f)$ satisfies $\forall \vec{x}\, \varphi(F, \vec{x})$.) Hence there exists a tuple $\vec{a}$ such that $\neg \varphi(f, \vec{a})$.

The notation $\vec{w}_1 < \vec{w} < \vec{w}_2$ means that each coordinate of $\vec{w}$ is strictly between each coordinate of $\vec{w}_1$ and $\vec{w}_2$, that is, $\vec{w}$ is in the open box with vertices $\vec{w}_1$ and $\vec{w}_2$. We will deal with variables which are outside the range of s by putting them into an open box with vertices in the range of s.

By Corollary 9.4, there is an $N \in \mathbb{N}$ which is an indiscernibility bound for each formula of the form

$$\exists \vec{w}\, [\vec{w}_1 < \vec{w} < \vec{w}_2 \wedge \theta(\vec{a}, \vec{y}, \vec{z})]$$

where $\vec{w} \subseteq \vec{y}$. We also take N so that $s(N) > \max(\vec{a})$.

By hypothesis, we have

$$\exists \vec{y}\, \exists \vec{z}\, [f_N(\vec{a}, \vec{y}) = \vec{z} \wedge \theta(\vec{a}, \vec{y}, \vec{z})].$$

Choose $\vec{b}$ in $\mathcal{M}$ such that

$$\exists \vec{z}\,[f_N(\vec{a},\vec{b}) = \vec{z} \wedge \theta(\vec{a},\vec{b},\vec{z})].$$

Then $\theta(\vec{a},\vec{b},f_N(\vec{a},\vec{b}))$.

We wish to find $\vec{e}$ in $\mathcal{M}$ such that $\theta(\vec{a},\vec{e},f(\vec{a},\vec{e}))$. This will show that $\varphi(f,\vec{a})$, contrary to hypothesis, and complete the proof that BDD is not Π_2 over $\mathcal{M}$. We cannot simply take $\vec{e} = \vec{b}$, because b could have a coordinate b_i such that $f(b_i) \neq f_N(b_i)$. This happens when $b_i = s(4m)$ where $4m > N$, so that $f(b_i) = b_i$ but $f_N(b_i) = 0$.

Claim: For each $\vec{u} \in \mathbb{N}^n$ there exists $\vec{v} \in \mathbb{N}^n$ such that $\vec{v} \sim_N \vec{u}$, $f(s(\vec{v})) = f_N(s(\vec{v}))$, and if $u_j = u_i + 1$ then $v_j = v_i + 1$.

Proof of Claim: Take $\vec{v}$ such that $\vec{v} \sim_N \vec{u}$, and v_i is not a multiple of 4 when $v_i > N$, and $v_j = v_i + 1$ when $u_j = u_i + 1$. Then $v_i = u_i$ below N, and $f(s(v_i)) = f(s(u_i)) = f_N(s(v_i)) = f_N(s(u_i)) = 0$ when $u_i > N$. It follows that $f(s(\vec{v})) = f_N(s(\vec{v}))$.

We may rearrange $\vec{b}$ into a k-tuple $(\vec{c},\vec{d})$ such that the terms of $\vec{c}$ belong to the range of s and the terms of $\vec{d}$ do not. Then $f_N(\vec{d}) = \vec{0}$ is a tuple of 0's. Let $(\vec{t},\vec{w})$ be the corresponding rearrangement of $\vec{y}$. Then $|\vec{c}| = |\vec{t}| = n$. Let $\vec{d}_1,\vec{d}_2$ be the vertices of the smallest open box containing $\vec{d}$ with coordinates in the range of s. That is, $\vec{d}_1 < \vec{d} < \vec{d}_2$ and the coordinates of $\vec{d}_1,\vec{d}_2$ are consecutive in the range of s. Then

$$\exists \vec{w}\,[\vec{d}_1 < \vec{w} < \vec{d}_2 \wedge \theta(\vec{a},\vec{c},\vec{w},f_N(\vec{a},\vec{c}),\vec{0})]$$

holds in $\mathcal{M}$. We have $(\vec{c},\vec{d}_1,\vec{d}_2) = s(\vec{u})$ for some $\vec{u} \in \mathbb{N}^n$. Take $\vec{v}$ as in the claim and let $(\vec{e}_0,\vec{d}_3,\vec{d}_4) = s(\vec{v})$. Then by indiscernibility,

$$\exists \vec{w}\,[\vec{d}_3 < \vec{w} < \vec{d}_4 \wedge \theta(\vec{a},\vec{e}_0,\vec{w},f_N(\vec{a},\vec{e}_0),\vec{0})].$$

Note that $s(N) > \max(\vec{a})$, so $f(\vec{a}) = f_N(\vec{a})$. Now let $\vec{e} = (\vec{e}_0,\vec{e}_1)$ where $\vec{e}_1$ is a witness for $\vec{w}$ in the above formula. Since the coordinates of $\vec{d}_3$ and $\vec{d}_4$ are consecutive in the range of s, the coordinates of $\vec{e}_1$ must be outside the range of s, where f and f_N are 0. It follows that $f(\vec{a},\vec{e}) = f_N(\vec{a},\vec{e})$. Therefore $\theta(\vec{a},\vec{e},f(\vec{a},\vec{e}))$, and hence BDD is not Π_2 over $\mathcal{M}$. ■

Corollary 9.6 *Suppose that $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R},\leq,+,\cdot)$. Then*
(i) BDD is not $\bigwedge\Pi_2$ over $\mathcal{M}$.
(ii) LIM is not $\bigwedge\Pi_2$ over $\mathcal{M}$.

Proof. This follows from the proof of Theorem 9.5. For (i), we suppose that there is a $\bigwedge\Pi_2$ sentence $\bigwedge_i \forall\vec{x}\varphi_i(F,\vec{x})$ which is equivalent to BDD in all structures $(\mathcal{M},f)$, choose $i \in \mathbb{N}$ such that $\forall\vec{x}\,\varphi_i(f,\vec{x})$ fails, and get a contradiction as in the proof of Theorem 9.5.

For part (ii), we first observe that all the functions $f : \mathcal{M} \to \mathcal{M}$ used in the proof of (i) satisfy the Π_1 sentence

$$\forall x\,[F(x) = x \vee F(x) = 0].$$

Call this sentence $\beta(F)$. It follows that $\beta(F) \wedge BDD(F)$ is not $\bigwedge \Pi_2$ over $\mathcal{M}$. We also note that

$$\beta(F) \Leftrightarrow \beta(x - F(x))$$

and

$$\beta(f) \Rightarrow [BDD(F) \Leftrightarrow LIM(x - F(x))].$$

Now suppose that $LIM(F)$ is $\bigwedge \Pi_2$ over $\mathcal{M}$. Then $\beta(F) \Rightarrow LIM(x - F(x))$ is also $\bigwedge \Pi_2$ over $\mathcal{M}$. Hence $\beta(F) \wedge BDD(F)$ is $\bigwedge \Pi_2$ over $\mathcal{M}$, a contradiction. ■

Theorem 9.7 *Suppose that $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R}, \leq, +, \cdot)$. Then LIM is not $\bigvee B_2$ over $\mathcal{M}$.*

Proof. We may assume that $\mathcal{M}$ has a countable vocabulary. By Lemma 9.2 there is an $\mathcal{M}$-fast sequence s. Suppose to the contrary that there is a $\bigvee B_2$ sentence ψ which is equivalent to LIM in all structures $(\mathcal{M}, f)$. Then $\neg\psi$ is equivalent to a $\bigwedge B_2$ sentence. Since finite conjunctions and disjunctions of Π_2 sentences are equivalent to Π_2 sentences, and similarly for Σ_2, $\neg\psi$ is equivalent to a sentence

$$\bigwedge_m (\alpha_m \vee \beta_m)$$

where each α_m is Σ_2 and each β_m is Π_2. We may assume that α_m and β_m have the form

$$\alpha_m = \exists\vec{x}\,\forall\vec{y}\,\forall\vec{z}\,[F(\vec{x}, \vec{y}) = \vec{z} \Rightarrow \overline{\alpha}_m(\vec{x}, \vec{y}, \vec{z})],$$

$$\beta_m = \forall\vec{x}\,\exists\vec{y}\,\exists\vec{z}\,[F(\vec{x}, \vec{y}) = \vec{z} \wedge \overline{\beta}_m(\vec{x}, \vec{y}, \vec{z})]$$

where $\overline{\alpha}_m$ and $\overline{\beta}_m$ are quantifier-free formulas of $L(\mathcal{M})$.

We now define a family of functions from $\mathbb{N}$ into $\mathbb{N}$ which we will later use to build functions from $\mathbb{R}$ into $\mathbb{R}$. We begin with a sequence $h : \mathbb{N} \to \mathbb{N}$ with the following properties:

(a) For each $m \in \mathbb{N}$, $h(m) \leq m$ and $h(m)$ is either 0 or a power of 2.
(b) Each power of 2 occurs infinitely often in the sequence h,
(c) If $h(m) > 0$, them m is halfway between two powers of 2.

Note that $h(i)$ is "usually" 0, and that whenever $i < j$ and $h(i) > 0$, $h(j) > 0$, we have $h(h(i)) = 0$ and $2i \leq j$. For each finite or infinite sequence of natural numbers σ, let h_σ be the function obtained from h by putting $h_\sigma(i) = 0$ if $h(i) = 2^n$ and $i > \sigma(n)$, and putting $h_\sigma(i) = h(i)$ otherwise. Thus when n is in the domain of σ, $h_\sigma^{-1}\{2^n\}$ is the finite set $h^{-1}\{2^n\} \cap \{0, \ldots, \sigma(n)\}$. When σ is finite and n is outside its domain, $h_\sigma^{-1}\{2^n\}$ is the infinite set $h^{-1}\{2^n\}$.

For each σ, define the function $f_\sigma : \mathbb{R} \to \mathbb{R}$ by putting $f_\sigma(s(i)) = s(h_\sigma(i))$ if $h_\sigma(i) > 0$, and $f_\sigma(x) = x$ for all other x. Note that for each infinite sequence σ we have $\lim_{x\to\infty} f_\sigma(x) = \infty$, so LIM holds in $(\mathcal{M}, f_\sigma)$. But for each finite sequence σ we have $\liminf_{x\to\infty} f_\sigma(x) < \infty$, so LIM fails in $(\mathcal{M}, f_\sigma)$.

We will now build an infinite sequence $\sigma = (\sigma(0), \sigma(1), \ldots)$ such that $(\mathcal{M}, f_\sigma)$ satisfies $\neg\psi$. This will give us the desired contradiction, since $(\mathcal{M}, f_\sigma)$ satisfies LIM. We will simultaneously build a sequence of tuples $\vec{a}_m, m \in \mathbb{N}$, and a strictly increasing "growth sequence" $g(0) < g(1) < \cdots$. We will form an increasing chain

$$\sigma[0] \subset \sigma[1] \subset \cdots$$

of finite sequences, and take σ to be their union. This chain will have the property that each term of $\sigma[m] \setminus \sigma[m-1]$ will be $\geq g(m-1)$. Whenever possible, $\sigma[m]$ will be chosen so that α_m holds in $(\mathcal{M}, f_{\sigma[m]})$, and $\vec{a}_m$ will be a witness for the initial existential quantifiers of α_m.

For convenience we let $\sigma[-1]$ denote the empty sequence and put $g(-1) = 1$. Suppose $m \in \mathbb{N}$ and we already have $\sigma[i]$, $\vec{a}_i$, and $g(i)$ for each $i < m$. We have two cases.

Case 1. There is a finite sequence $\eta \supset \sigma[m-1]$ such that α_m holds in $(\mathcal{M}, f_\eta)$, and each term of $\eta \setminus \sigma[m-1]$ is $\geq g(m-1)$. In this case we take $\sigma[m]$ to be such an η, and take $\vec{a}_m$ to be a tuple in $\mathbb{R}$ which witnesses the initial existential quantifiers of α_m in $(\mathcal{M}, f_{\sigma[m]})$, that is,

$$\forall \vec{y} \forall \vec{z} \, [f_{\sigma[m]}(\vec{a}_m, \vec{y}) = \vec{z} \Rightarrow \overline{\alpha}_m(\vec{a}_m, \vec{y}, \vec{z})].$$

Case 2. Otherwise. In this case we take $\sigma[m]$ to be an arbitrary finite sequence such that $\sigma[m] \supset \sigma[m-1]$ and each term of $\sigma[m] \setminus \sigma[m-1]$ is $\geq g(m-1)$, and let $\vec{a}_m$ be an arbitrary tuple in $\mathbb{R}$.

We now define $g(m)$. By Corollary 9.4, there is an $N \in \mathbb{N}$ which is an indiscernibility bound for each formula of the form

$$\forall \vec{w} [\vec{w}_1 < \vec{w} < \vec{w}_2 \Rightarrow \overline{\alpha}_m(\vec{a}_m, \vec{y}_1, \vec{w}, \vec{z}_1, \vec{w})]$$

where $\vec{w} \subseteq \vec{y}$, $\vec{y}_1$ is the part of $\vec{y}$ outside $\vec{w}$, and $\vec{z}_1$ is the corresponding part of $\vec{z}$.

Let p be the number of variables in the sentences $\alpha_i, \beta_i, i \leq m$. Take $g(m)$ so that:

(d) $g(m) \geq N$, $g(m) \geq 2p$, $g(m) > g(m-1)$, $g(m) > \max(\sigma[m])$,
(e) $s(g(m)) > \max(\vec{a}_m)$,
(f) The condition stated before Claim 2.

Finally, we define σ to be the union $\sigma = \bigcup_m \sigma[m]$. To complete the proof we prove two claims, Claim 1 concerning α_m and Claim 2 concerning β_m. Condition (f) will not be used in Claim 1, but will be needed later for Claim 2.

Claim 1: Suppose α_m holds in $(\mathcal{M}, f_{\sigma[m]})$ (Case 1 above). Then α_m holds in $(\mathcal{M}, f_\sigma)$.

Proof of Claim 1: We show that

$$\forall \vec{y} \forall \vec{z} \, [f_\sigma(\vec{a}_m, \vec{y}) = \vec{z} \Rightarrow \overline{\alpha}_m(\vec{a}_m, \vec{y}, \vec{z})].$$

Suppose not. Then there is a tuple $\vec{b}$ in $\mathcal{M}$ such that

$$\neg \forall \vec{z} \, [f_\sigma(\vec{a}_m, \vec{b}) = \vec{z} \Rightarrow \overline{\alpha}_m(\vec{a}_i, \vec{b}, \vec{z})].$$

As in the proof of Theorem 9.5, we may rearrange $\vec{b}$ into a k-tuple $(\vec{c}, \vec{d})$ such that the terms of $\vec{c}$ belong to the range of s and the terms of $\vec{d}$ do not. Then $f_\sigma(\vec{c})$ is also in the range of s, and $f_\sigma(\vec{d}) = \vec{d}$. Let $(\vec{t}, \vec{w})$ be the corresponding rearrangement of $\vec{y}$. Let $\vec{d}_1, \vec{d}_2$ be the vertices of the smallest open box containing $\vec{d}$ with coordinates in the range of s. Then

$$\neg\forall\vec{w}\,[\vec{d}_1 < \vec{w} < \vec{d}_2 \Rightarrow \overline{\alpha}_m(\vec{a}_m, \vec{c}, \vec{w}, f_\sigma(\vec{a}_m, \vec{c}), \vec{w})].$$

Let $\vec{u}$ be the sequence in $\mathbb{N}$ such that $(\vec{c}, \vec{d}_1, \vec{d}_2) = s(\vec{u})$. Since the terms of σ beyond $\sigma[m]$ are greater than $g(m)$, we have $h_\sigma(i) = h_{\sigma[m]}(i)$ for all $i \le g(m)$ and $f_\sigma(x) = f_{\sigma[m]}(x)$ for all $x \le s(g(m))$. For $i > g(m)$ we have either $h_\sigma(i) = h_{\sigma[m]}(i)$, or $h_\sigma(i) = 0$ and $h_{\sigma[m]}(i) > 0$. Recall that $g(m) \ge 2p \ge 2|\vec{u}|$, where p is the number of variables in $\alpha_m \vee \beta_m$. It follows that the sequence $h_{\sigma[m]}$ has enough zeros to insure that there is a sequence $\vec{v}$ in $\mathbb{N}^p$ such that:

(g) For each i, $(\vec{u}, 2^i) \sim_{g(m)} (\vec{v}, 2^i)$,
(h) $v_i = u_i$ whenever $h_\sigma(i) > 0$,
(i) If $u_i = u_i + 1$ then $v_i = v_i + 1$,
(j) $h_{\sigma[m]}(\vec{v}) = h_\sigma(\vec{u})$.

Therefore $(\vec{v}, h_{\sigma[m]}(\vec{v})) \sim_{g(m)} (\vec{u}, h_\sigma(\vec{u}))$. Let $(\vec{e}_0, \vec{d}_3, \vec{d}_4) = s(\vec{v})$. Then by (d), (e), (g), and indiscernibility, we have

$$\neg\forall\vec{w}\,[\vec{d}_3 < \vec{w} < \vec{d}_4 \Rightarrow \overline{\alpha}_m(\vec{a}_m, \vec{e}_0, \vec{w}, f_{\sigma[m]}(\vec{a}_m, \vec{e}_0), \vec{w})].$$

We may therefore extend $\vec{e}_0$ to a tuple $\vec{e} = (\vec{e}_0, \vec{e}_1)$ such that

$$\vec{d}_3 < \vec{e}_1 < \vec{d}_4 \wedge \neg\,\overline{\alpha}_m(\vec{a}_m, \vec{e}, f_{\sigma[m]}(\vec{a}_m, \vec{e}_0), \vec{e}_1).$$

The coordinates of $\vec{d}_1$ and $\vec{d}_2$ are consecutive in the range of s, so by (i) the coordinates of $\vec{d}_1$ and $\vec{d}_2$ are consecutive in the range of s. Therefore the coordinates of $\vec{e}_1$ are outside the range of s, so $f_{\sigma[m]}(\vec{e}_1) = \vec{e}_1$. Then

$$\neg\,\overline{\alpha}_m(\vec{a}_m, \vec{e}, f_{\sigma[m]}(\vec{a}_m, \vec{e}).$$

This contradicts the fact that

$$\forall\vec{y}\forall\vec{z}\,[f_{\sigma[m]}(\vec{a}_i, \vec{y}) = \vec{z} \Rightarrow \overline{\alpha}_m(\vec{a}_m, \vec{y}, \vec{z})].$$

and completes the proof of Claim 1.

We now state the postponed condition (f) for the growth sequence g. Recall that p is the number of variables in the sentences $\alpha_i, \beta_i, i \le m$.

(f) For each $\vec{u} \in \mathbb{N}^p$ there exists $\vec{v} \in \mathbb{N}^p$ with the following properties:
(f1) $\max(\vec{v}) < g(m)$,
(f2) If $u_j = u_i + 1$ then $v_j = v_i + 1$,
(f3) $(\vec{v}, h_{\sigma[m]}(\vec{v})) \sim_m (\vec{u}, h_{\sigma[m]}(\vec{u}))$.

There is a $g(m)$ with these properties because the equivalence relation $\sim_m$ has only finitely many classes.

Claim 2: The sentence $\neg\psi$ holds in $(\mathcal{M}, f_\sigma)$.

Proof of Claim 2. We must show that for each $m \in \mathbb{N}$, $\alpha_m \vee \beta_m$ holds in $(\mathcal{M}, f_\sigma)$. Since LIM fails in $(\mathcal{M}, f_{\sigma[m]})$, $\alpha_m \vee \beta_m$ holds in $(\mathcal{M}, f_{\sigma[m]})$. In Case 1 above, by definition α_m holds in $(\mathcal{M}, f_{\sigma[m]})$, and by Claim 1, α_m holds in $(\mathcal{M}, f_\sigma)$.

In Case 2, α_m fails in $(\mathcal{M}, f_{\sigma[r]})$ for each $r \geq m$, and therefore β_m holds in $(\mathcal{M}, f_{\sigma[r]})$ for each $r \geq m$. In this case we prove that β_m holds in $(\mathcal{M}, f_\sigma)$. We fix a tuple $\vec{a}$ in $\mathcal{M}$ and prove

$$\exists\vec{y}\,\exists\vec{z}\,[f_\sigma(\vec{a},\vec{y}) = \vec{z} \wedge \overline{\beta}_m(\vec{a},\vec{y},\vec{z})].$$

By Corollary 9.4 there is an $M \in \mathbb{N}$ which is an indiscernibility bound for each formula of the form

$$\exists\vec{w}[\vec{w}_1 < \vec{w} < \vec{w}_2 \wedge \overline{\beta}_m(\vec{a},\vec{y}_1,\vec{w},\vec{z}_1,\vec{w})]$$

where $\vec{w} \subseteq \vec{y}$, $\vec{y}_1$ is the part of $\vec{y}$ outside $\vec{w}$, and $\vec{z}_1$ is the corresponding part of $\vec{z}$.

Take r large enough so that $r \geq m$ and $r \geq M$, and $s(r) > \max(\vec{a})$. Since β_m holds in $(\mathcal{M}, f_{\sigma[r]})$, we have

$$\exists\vec{y}\,\exists\vec{z}\,[f_{\sigma[r]}(\vec{a},\vec{y}) = \vec{z} \wedge \overline{\beta}_m(\vec{a},\vec{y},\vec{z})].$$

We may therefore choose $\vec{b}$ in $\mathcal{M}$ so that

$$\overline{\beta}_m(\vec{a},\vec{b},f_{\sigma[r]}(\vec{a},\vec{b})).$$

As in the proof of Theorem 9.5, we may rearrange $\vec{b}$ into a tuple $(\vec{c},\vec{d})$ such that the terms of $\vec{c}$ belong to the range of s and the terms of $\vec{d}$ do not. Then $f_{\sigma[r]}(\vec{d}) = \vec{d}$. Let $(\vec{t},\vec{w})$ be the corresponding rearrangement of $\vec{y}$, and let $\vec{d}_1, \vec{d}_2$ be the vertices of the smallest open box containing $\vec{d}$ with coordinates in the range of s. Then

$$\exists\vec{w}\,[\vec{d}_1 < \vec{w} < \vec{d}_2 \wedge \overline{\beta}_m(\vec{a},\vec{c},\vec{w},f_{\sigma[r]}(\vec{a},\vec{c}),\vec{w})].$$

Let $\vec{u}$ be the tuple in $\mathbb{N}$ such that $s(\vec{u}) = (\vec{c},\vec{d}_1,\vec{d}_2)$. Then there is a tuple $\vec{v}$ in $\mathbb{N}$ such that conditions (f1)–(f3) hold with r in place of m. Here p is the number of variables in $\alpha_i, \beta_i, i \leq r$, so β_m has at most p variables, and $\vec{u}$ can be taken with length p.

Let $s(\vec{v}) = (\vec{e}_0,\vec{d}_3,\vec{d}_4)$. Since $r \geq M$, by (f3) and indiscernibility we have

$$\exists\vec{w}\,[\vec{d}_3 < \vec{w} < \vec{d}_4 \wedge \overline{\beta}_m(\vec{a},\vec{e}_0,\vec{w},f_{\sigma[r]}(\vec{a},\vec{e}_0),\vec{w})].$$

Take $\vec{e}_1$ in $\mathcal{M}$ such that

$$\vec{d}_3 < \vec{e}_1 < \vec{d}_4 \wedge \overline{\beta}_m(\vec{a},\vec{e}_0,\vec{e}_1,f_{\sigma[r]}(\vec{a},\vec{e}_0),\vec{e}_1).$$

By (f2), every coordinate of $\vec{e}_1$ is outside the range of s, so $f_{\sigma[r]}(\vec{e}_1) = \vec{e}_1$. Putting $\vec{e} = (\vec{e}_0, \vec{e}_1)$, we have

$$\overline{\beta}_m(\vec{a}, \vec{e}, f_{\sigma[r]}(\vec{a}, \vec{e})).$$

It remains to prove that $f_\sigma(\vec{a}, \vec{e}) = f_{\sigma[r]}(\vec{a}, \vec{e})$. Suppose $i \leq g(r)$ and $h_{\sigma[r]}(i) = 2^j$. If j is in the domain of $\sigma[r]$, then $h_\sigma(i) = 2^j$ because $\sigma(j) = \sigma[r](j)$. Otherwise $\sigma(j) \geq g(r)$, hence $i \leq g(r) < \sigma(j)$ and $h(i) = 2^j$, and again $h_\sigma(i) = 2^j$. If $h_{\sigma[r]}(k) = 0$ then $h_\sigma(k) = 0$. Therefore $h_\sigma(k) = h_{\sigma[r]}(k)$ for all $k \leq g(r)$.

By (f1), $\max(\vec{v}) < g(r)$. Therefore $h_\sigma(\vec{v}) = h_{\sigma[r]}(\vec{v})$, and hence $f_\sigma(\vec{v}) = f_{\sigma[r]}(\vec{v})$. Since we also have $\max(\vec{a}) < s(r)$, it follows that $f_\sigma(\vec{a}, \vec{e}) = f_{\sigma[r]}(\vec{a}, \vec{e})$. Therefore

$$\overline{\beta}_m(\vec{a}, \vec{e}, f_\sigma(\vec{a}, \vec{e})),$$

and we have the required formula

$$\exists \vec{y}\, \exists \vec{z}\, [f_\sigma(\vec{a}, \vec{y}) = \vec{z} \wedge \overline{\beta}_m(\vec{a}, \vec{y}, \vec{z})].$$

This establishes Claim 2 and completes the proof. ■

We conclude with a problem which we state as a conjecture.

Conjecture 9.8 *Suppose that $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R}, \leq, +, \cdot)$. Then LIM is not Σ_3 over $\mathcal{M}$.*

10 Conclusion

Given an ordered structure $\mathcal{M}$, the quantifier level of a sentence θ of $L(\mathcal{M}) \cup \{F\}$ over $\mathcal{M}$ is the lowest class in the hierarchies $\Delta_1 \subset \Pi_1 \subset B_1 \subset \Delta_2 \subset \Pi_2 \ldots$ and $\Delta_1 \subset \Sigma_1 \subset B_1 \subset \Delta_2 \subset \Sigma_2 \ldots$ which contains a sentence equivalent to θ in all structures $(\mathcal{M}, f)$. We investigate the quantifier levels of the Σ_2 sentence BDD, which says that f is bounded, and the Π_3 sentence LIM, which says that $\lim_{z \to \infty} f(z) = \infty$, over a given ordered structure $\mathcal{M}$. This work is motivated by Mostowski's result that BDD is not Π_2 and LIM is not Σ_3 relative to the primitive recursive functions over the standard model of arithmetic, and Abraham Robinson's result which characterizes BDD and LIM for standard functions by Π_1 sentences in a language with an added predicate for the set of infinite elements.

We show that BDD and LIM can never be B_1 over a structure $\mathcal{M}$, but if $\mathcal{M}$ is an expansion of the real ordered field with a symbol for $\mathbb{N}$ and each definable function, then BDD and LIM are at the lowest possible level Δ_2 over $\mathcal{M}$. We show that BDD is at its highest possible level, Σ_2 but not Π_2, and that LIM is at its highest possible level, Π_3 but not Σ_3, in the following cases: $\mathcal{M}$ is countable, $\mathcal{M}$ is the real ordering with an embedded structure on the natural numbers, and $\mathcal{M}$ is special of nice cardinality with an extra predicate for the infinite elements.

When $\mathcal{M}$ has universe $\mathbb{R}$, we obtain analogous results with the outer quantifiers replaced by countable disjunctions and conjunctions. In that case we show that BDD cannot be expressed by a countable conjunction of existential sentences, and LIM cannot be expressed by a countable disjunction of countable conjunctions of existential sentences.

The most interesting case is where $\mathcal{M}$ is an o-minimal expansion of the real ordered field. In that case we show that BDD is at the maximum level, and that LIM is not B_2. Moreover, BDD and LIM are not $\bigwedge \Pi_2$, and LIM is not $\bigvee B_2$. We leave open the question of whether LIM is at its maximum level, not Σ_3, in that case.

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