Iterated Dominance Revisited*

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Abstract

Epistemic justifications of solution concepts often refer to type structures that are sufficiently rich. One important notion of richness is that of a *complete type structure*, i.e., a type structure that induces all possible beliefs about types. For instance, it is often said that, in a complete type structure, the set of strategies consistent with rationality and common belief of rationality are the set of strategies that survive iterated dominance. This paper shows that this classic result is false, absent certain topological conditions on the type structure. In particular, it provides an example of a finite game and a complete type structure in which there is no state consistent with rationality and common belief of rationality. This arises because the complete type structure does not induce all hierarchies of beliefs—despite inducing all beliefs about types. This raises the question: Which beliefs does a complete type structure induce? We provide several positive results that speak to that question. However, we also show that, within ZFC, one cannot show that a complete structure induces all second-order beliefs.

JEL Codes C70, C72, C79, D81, D89

1 Introduction

Iterated deletion of strongly dominated strategies has a long tradition in game theory. Bernheim (1984) and Pearce (1984) asserted that—up to issues of correlation—the iteratively undominated (IU) strategies are the strategies consistent with rationality and common belief of rationality. In many ways, this step seems intuitive and obvious. Brandenburger and Dekel (1987) and Tan and Werlang (1988) are early treatments that provide a formal statement of this claim. (See, also, Battigalli and Siniscalchi (2002) and Arieli (2010), among many others.) Each of these papers provide epistemic conditions for IU.

This paper argues that, in rather subtle ways, we have an incomplete understanding of the epistemic conditions for IU. The gap in our understanding arises because we have failed to appreciate

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subtleties of an epistemic framework that has become standard in the literature. We give a series of results aimed at improving our understanding of that framework and, in turn, IU.

To provide foundations for IU, we must first specify a framework in which players reason about the strategies others play. Modern treatments follow Harsanyi (1967), by using a type structure to implicitly model players' beliefs. They assume that the type structure is **complete**—that it contains all possible beliefs. This is, in a sense, a requirement that the type structure is "rich." In Section 2, we review why a richness condition is crucial for an epistemic characterization of IU.

The main theorem shows that, for any non-trivial finite game, there exists a complete type structure in which no strategy is consistent with **rationality and common belief of rationality** (**RCBR**). (See Theorems 6.1-6.2.) This can occur because a complete type structure need not induce all possible hierarchies of beliefs—despite the fact that it induces all possible beliefs about types.[1] Put differently, when a type structure induces all possible hierarchies of beliefs, the IU strategies are consistent with RCBR and so there is some state at which there is RCBR. In fact, the same holds if the type structure induces all hierarchies of beliefs that arise from finite structures. (See Theorem 8.1.) But the same conclusion need not follow for a type structure that induces all possible beliefs about types.

This then raises the question: Which hierarchies of beliefs are induced by a complete type structure? We show that a complete structure induces all finite-order beliefs that can arise in countable type structures. Moreover, in economically relevant environments, they also induce finite-order beliefs that arise from (what we will call) atomic type structures. (See Proposition 9.2.) But, importantly, we stop short of showing that they induce all finite-order beliefs. In fact, we show a powerful negative result: Within ZFC, one cannot prove that complete type structures induce all second-order beliefs. (See Theorem 9.4.)

One message of this paper is that topological assumptions on the set of types implicitly impose substantive assumptions on players' reasoning. Typically, the literature assumes that the type sets are Polish (or, even, compact metrizable) and the belief maps are continuous.[2] When possible, this paper will refrain from making such assumptions with an aim at understanding the extent to which such assumptions involve substantive import.

The paper proceeds as follows. Section 2 reviews the literature and gives an informal preview of the results. Sections 3-5 provides the underlying framework and key epistemic conditions. Section 6 shows the main theorem: The existence of a complete type structure, where there is no RCBR state. Sections 7-8 are aimed at understanding this result. Section 7 describes how types induce hierarchies of beliefs. Section 8 shows that the RCBR predictions coincide with IU in a type structure that induces a sufficiently rich set of hierarchies of beliefs. The implication is that a complete type structure need not induce all hierarchies of beliefs. Section 9 addresses the question of which finite-order beliefs are induced by a complete type structure. Finally, Section 10 concludes

[1] The fact that such a complete structure can exist was not known prior to this paper.

[2] There is even such an implicit assumption in the so-called "topology-free" approach to type structures, see e.g., Heifetz and Samet (1998, 1999). When the underlying set of uncertainty is Polish, the literature typically constructs a "large type structure" whose type sets turn out to be Polish and belief maps turn out to be continuous.

with conceptual discussions. Proofs not in the main text can be found in the appendices.

2 Heuristic Treatment

The purpose of this section is two-fold. First, it reviews the known foundations for IU. In doing so, it provides a discussion of important philosophical and/or substantive underpinnings of extant results. Second, it provides an informal statement of the central results. While it postpones formal definitions and proofs to Sections 3–9, it references the formal counterparts to come.

2.1 The Framework

Let G^* be a two-player game, where Ann's strategy set is $S_a = \{U, M, D\}$ and Bob's strategy set is $S_b = \{L, C, R\}$. Figure 2.1 describes the payoffs. Observe that each of Ann's strategies is undominated. Bob's strategy R is the only dominated strategy. The IU strategy set is $\{U, M\} \times \{L, C\}$.

A \ B	L	C	R
U	4,4	1,1	0,0
M	1,1	5,5	0,0
D	0,1	0,1	6,0

Figure 2.1Game G

What are the implications of the requirement that each player is rational, each player believes the other player is rational, etc.? To say if a strategy is rational for Ann, we must specify what Ann believes about the strategy Bob employs. For instance, U is rational for Ann if she assigns probability 1 to L; but, it is irrational for Ann if she assigns probability 0 to L. Likewise, whether Ann believes Bob is rational depends on her belief both about what Bob plays and what he beliefs about her own play. (After all, whether a strategy is rational or irrational for Bob also depends on Bob's beliefs.) And so on. So, to talk about the implications of rationality and common belief of rationality, we need to enrich the description of the game—to describe Ann's and Bob's hierarchies of beliefs about the play of the game.

We follow the literature and use a **type structure** (Harsanyi, 1967) to implicitly model these hierarchies of beliefs. There are two ingredients: First, for each player c, there is a set of (**belief**) **types**. Second, for each player c, there is a **belief map**; the belief map β_c takes each type of a player c to a belief about the strategies and types of the other player. Figure 2.2 gives an example of a type structure $\mathcal{T}^*$ for the game G^*. In $\mathcal{T}^*$, there is one type of Ann, viz. t_a, and one type of Bob, viz. t_b. Type t_a assigns probability 1 to (L, t_b) and type t_b assigns probability 1 to (U, t_a). So, type t_a assigns probability 1 to "Bob plays L and assigns probability 1 to me playing U, etc."

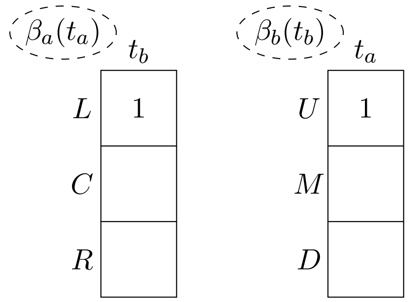

Figure 2.2Type Structure $\mathcal{T}^*$

Taken together, $(G^*, \mathcal{T}^*)$ (Figures 2.1-2.2) describe the strategic situation, i.e., the rules of the game, payoff functions, and the players' beliefs about the strategies played. This description is called an epistemic game. More concretely, an **epistemic game** consists of a game G and a type structure associated with G; that type structure implicitly describes hierarchies of beliefs about the strategies played. Note, the type structure is part of the definition of an epistemic game and so part of the description of the strategic situation.

The epistemic game induces a set of **states**, i.e., a set of strategy-type pairs for each player. A state describes how the game is played and what the players' believe. For instance, in $(G^*, \mathcal{T}^*)$, there is a state (M, t_a, R, t_b). At that state, Bob plays R and Bob believes "Ann plays U and holds the beliefs associated with t_a." Note, at this state, Bob chooses a dominated strategy. So, this can be seen as an 'irrational' state. We will focus on states that capture strategic reasoning. With this in mind, we restrict attention to states satisfying rationality and common belief of rationality.

Rationality is the requirement that a player maximize her subjective expected utility, given her belief about how the other plays the game. Thus, rationality is a property of a strategy-type pair. For the epistemic game associated with Figures 2.1-2.2, (s_a, t_a) is rational if s_a is a best response given the belief associated with t_a; since t_a assigns probability 1 to (L, t_b), (s_a, t_a) is rational if and only if $s_a = \mathrm{U}$. Likewise, (s_b, t_b) is rational if and only if $s_b = L$. Thus, the set of rational strategy-type pairs for Ann is

$$\mathrm{R}_a^1 = \{(\mathrm{U}, t_a)\}$$

and the set of rational strategy-type pairs for Bob is

$$\mathrm{R}_b^1 = \{(\mathrm{L}, t_b)\}.$$

Now turn to the requirement of belief in rationality. Since types are associated with beliefs, this is a requirement on types of a player. A type **believes** an event E if it assigns probability one to E. Here, $\beta_a(t_a)$ assigns probability one to R_b^1. So, the set of strategy-type pairs of Ann that are "rational and 'believe Bob is rational'" is $\mathrm{R}_a^2 = \mathrm{R}_a^1 = \{(\mathrm{U}, t_a)\}$. Likewise, the set of strategy-type pairs for Bob that are "rational and 'believe Ann is rational'" is $\mathrm{R}_b^2 = \mathrm{R}_b^1$.

Proceeding inductively, we can see that the set of strategy-type pairs for Ann consistent with **rationality and m$^{\text{th}}$-order belief of rationality** (**RmBR**) is $\mathrm{R}_a^{m+1} = \mathrm{R}_a^2$. And, likewise,

$\beta_a(t_a)$	t_b	u_b
L	1	
C		
R		

$\beta_a(u_a)$	t_b	u_b
L		
C		1
R		

$\beta_a(v_a)$	t_b	u_b
L		
C		
R	.5	.5

$\beta_b(t_b)$	t_a	u_a	v_a
U	1		
M			
D			

$\beta_b(u_b)$	t_a	u_a	v_a
U			
M		1	
D			

Figure 2.3Type Structure $\mathcal{T}^{**}$

$\mathrm{R}_b^{m+1} = \mathrm{R}_b^2$. Thus, the set of states at which there is **rationality and common belief of rationality** (**RCBR**) is $\mathrm{R}_a^\infty \times \mathrm{R}_b^\infty = \mathrm{R}_a^2 \times \mathrm{R}_b^2$. So, in the epistemic game $(G^*, \mathcal{T}^*)$, the prediction of RCBR is $\{\mathrm{U}\} \times \{\mathrm{L}\}$. This is a subset of the IU strategy set.

2.2 A Connection Between IU and RCBR Predictions

In the epistemic game $(G^*, \mathcal{T}^*)$ (Figures 2.1-2.2), the RCBR prediction was contained in the set of IU strategies. This is true more generally.

Result 2.1 (Brandenburger and Dekel, 1987) *Fix an epistemic game. The set of strategies consistent with RCBR is contained in the IU strategy set.*

The epistemic game in Figures 2.1-2.2 highlighted the fact that the RCBR prediction may be a strict subset of the IU strategies. It raises the question of whether—for a given game G—all the IU strategies are consistent with RCBR. Indeed they are. We can find a type structure so that the set of states consistent with RCBR is exactly the IU strategy set.

Result 2.2 (Brandenburger and Dekel, 1987) *Fix a game. There is an associated epistemic game so that the set of strategies consistent with RCBR is the IU strategy set.*

To illustrate Result 2.2, consider the epistemic game $(G^*, \mathcal{T}^{**})$, where $\mathcal{T}^{**}$ is as in Figure 2.3. Now Ann has three types and Bob has two types. Type t_a of Ann (resp. u_a, v_a) assigns probability 1 to Bob playing L (resp. C, R). So, the set of rational strategy-type pairs for Ann is

$$\mathrm{R}_a^1 = \{(\mathrm{U}, t_a), (\mathrm{M}, u_a), (\mathrm{D}, v_a)\}$$

and the set of rational strategy-type pairs for Bob is

$$\mathrm{R}_b^1 = \{(\mathrm{L}, t_b), (\mathrm{C}, u_b)\}.$$

Each of Ann's types t_a and u_a assign probability 1 to R_b^1; however, v_a does not. Thus, $\mathrm{R}_a^2 = \{(\mathrm{U}, t_a), (\mathrm{M}, u_a)\}$ is the set of strategy-type pairs at which Ann is rational and believes Bob is rational. Likewise, each type of Bob believes R_a^1; thus, $\mathrm{R}_b^2 = \mathrm{R}_b^1$. Continuing along these lines, the set of states at which there is RCBR is

$$\{(\mathrm{U}, t_a), (\mathrm{M}, u_a)\} \times \{(\mathrm{L}, t_b), (\mathrm{C}, u_b)\}.$$

Thus the RCBR predictions, namely $\{U, M\} \times \{L, C\}$, coincide with the set of IU strategies.

2.3 Interpretation

Taken together, Results 2.1-2.2 provide a basic connection between IU and the RCBR predictions: Fix a game G and its associated set of IU strategies, viz. S^∞. For any epistemic game associated with G, the set of RCBR predictions is contained in S^∞. Conversely, there is an epistemic game associated with G so that S^∞ is the set of RCBR predictions.

The interpretation of this result is more subtle than may, at first, be apparent. To understand why, note that, under the epistemic game theory approach, beliefs are part of the description of the strategic situation. When we write down an epistemic game $(G, \mathcal{T})$, we specify what beliefs players do versus do not consider possible. From the perspective of the players, other type structures are simply irrelevant: They may have types that the players do not themselves consider possible. Or they may not include types the players do consider possible. (Formally, each type structure represents an event E about hierarchies of beliefs that is commonly believed by the players. From the perspective of players who commonly believe an event E, type structures that capture common belief of $F \neq E$ are irrelevant. See Battigalli and Friedenberg (2012*a*) for a more complete discussion of the interpretation.)

With this in mind, from the perspective of the players themselves, some of the IU strategies may not be RCBR predictions. As such, Results 2.1-2.2 are best understood from the perspective of an analyst who does not know the players' beliefs. In that case, the analyst will seek the predictions of RCBR across all epistemic games associated with G. This set of predictions is exactly the set of IU strategies.

2.4 Justifying IU from the Perspective of Players

This paper is concerned with justifying IU from the perspective of the players themselves. In particular, it asks: Can the players themselves see *all* the IU strategies as the result of a certain thought process? If we have found a situation where that is indeed the case, we will say that we have found an **epistemic justification of IU**.[3]

The epistemic game in Figures 2.1-2.2 suggest that, to provide a positive answer, we will need to restrict the class of type structures we analyze. What is the desired restriction on type structures? To get at an answer, refer to the epistemic game given by Figures 2.1-2.2. Why did we not get all of the IU strategies? Note, there was no type of Ann that assigns strictly positive probability to Bob playing C. So there is no type of Ann for which M is a best response, i.e., M is inconsistent with rationality. Even if there were such a type of Ann, there is no type of Bob for which C is a best response. So, Ann cannot both assign positive probability to Bob playing C and believe that Bob is rational. And so on. This suggests that, to justify IU (from the players' perspective), we need a requirement that "players consider enough beliefs possible." Put differently, we need a requirement that the type structure is "rich."

Tan and Werlang (1988) studied a specific rich type structure. In particular, they restricted attention to the canonical construction of the so-called **universal type structure**, in Mertens and Zamir (1985). This is a particular construction of a type structure that induces all hierarchies of beliefs about the play of the game. The result is:

Result 2.3 *[Tan and Werlang, 1988] Fix an epistemic game associated with a universal type structure.*

(i) The set of strategies consistent with RmBR is the set of $(m+1)$-undominated strategies.

(ii) The set of strategies consistent with RCBR is the set of IU strategies.

Tan and Werlang stated the result without proof. In the course of attempting to prove the result, the literature identified properties of the canonical construction (of a universal type structure) that suffice for a proof. Thus, the literature was able to modify the richness condition—expanding it beyond the single universal type structure of Mertens and Zamir (1985).

The key property (due to Brandenburger, 2003) is completeness. A type structure is **complete** if the belief maps, viz. β_a and β_b, are onto. Thus, for every belief a player can hold (about the strategies and types of the other players), there is a type of the player which induces that belief. That is, the type structure induces all possible beliefs about types. Call an epistemic game complete if the associated type structure is complete. Now:

Result 2.4 *[Folk Result; Proposition 5.5] Fix a complete epistemic game.*

(i) The set of strategies consistent with RmBR is the set of $(m+1)$-undominated strategies.

(ii) If, in addition, the type sets are compact and the belief maps are continuous, then the set of strategies consistent with RCBR is the set of IU strategies.[4]

[3]It is important to note that, for the purpose of this paper, we use the phrase "epistemic justification" only if we have justified IU from the perspective of the players themselves. It is certainly reasonable to also use the phrase for a justification from the perspective of an analyst—i.e., for the situaiton discussed in Section 2.2. We use the term in a limited way; no confusion should result.

[4]This "folk result" is a special case of Proposition 6 in Battigalli and Siniscalchi (2002).

2.5 Technical Assumptions May Be Substantive Assumptions

Result 2.4(ii) says that we can justify the entire IU set as an output of players' reasoning: IU is an output of RCBR in a type structure that is complete with compact type spaces and continuous belief maps. So, compactness, continuity, completeness, and RCBR comprise so-called epistemic conditions for IU. The universal type structure is compact, continuous and complete.

How should these conditions be interpreted? RCBR is a condition on players' reasoning. Likewise, we can interpret the completeness condition as an output of players' reasoning. It is the condition that all possible beliefs (about types) are present—and so all beliefs (about types) are "considered" by the players. But, compact type sets and continuous belief maps are technical conditions. As technical conditions, they are not easily interpretable.

This raises the question: Why do we find compactness and continuity in the statement of Result 2.4(ii)? If the answer is "for technical convenience," then these requirements can be ignored. But, if the answer is "because they are necessary for the result," then they are integral components of the epistemic conditions for IU. In the latter case, arguably, we have not yet understood conditions on players' reasoning that give IU—precisely, because compactness and continuity restrict players' reasoning in a way that is not transparent.

In Section 6, we show that these seemingly technical requirements cannot be dispensed with. The result will apply to a wide class of games, called **non-trivial games**. For now, note, the game in Figure 2.1 is non-trivial. Moreover, games that have both no dominant strategy and no ties will be a special case of non-trivial games.

Result 2.5 *[Theorem 6.1] For each non-trivial game, there is an associated complete epistemic game so that the following holds:*

(i) The set of strategies consistent with RmBR is the set of $(m+1)$-undominated strategies.

(ii) There is no state at which there is RCBR.

To better understand Result 2.5, return to the game in Figure 2.1. The result says that there is an associated complete epistemic game, in which there is no state at which there is RCBR. So, while at first glace, a complete type structure may appear rich—in so far as it contains all possible beliefs about types—it may not be rich enough to deliver the IU strategy set (or even some state at which there is RCBR). Let us review why such a complete structure can exists.

Note, for the game in Figure 2.1, iterated dominance gives:

	Set of Strategies that Survive
Round 1	$\{U, M, D\} \times \{L, C\}$
Round 2	$\{U, M\} \times \{L, C\}$
Round 3	$\{U, M\} \times \{L, C\}$
Round 4	$\dots$

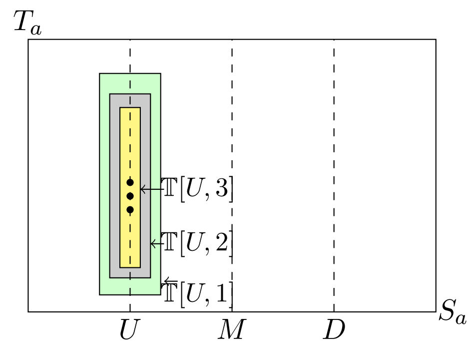

Figure 2.4

Now, refer to Result 2.4(i): It says that a strategy s_a survives m rounds of deletion if and only if (in a complete type structure) the set of types t_a so that (s_a, t_a) satisfies "rationality and $(m-1)^{th}$-order belief of rationality" is nonempty. Write $\mathbb{T}[s_a, m]$ for this set. Note, in our example then, for any complete type structure, we have:

$$\begin{array}{ccc} \mathbb{T}[U,1] \neq \emptyset & \mathbb{T}[M,1] \neq \emptyset & \mathbb{T}[D,1] \neq \emptyset \\ \mathbb{T}[U,2] \neq \emptyset & \mathbb{T}[M,2] \neq \emptyset & \mathbb{T}[D,2] = \emptyset \\ \mathbb{T}[U,3] \neq \emptyset & \mathbb{T}[M,3] \neq \emptyset & \mathbb{T}[D,3] = \emptyset \\ \cdots & \cdots & \cdots \end{array}$$

The question we then have is: If a strategy s_a survives IU, is it the case that there is a type t_a so that t_a believes "Bob is rational," "Bob is rational and 'Bob believes I am rational,'" etc.? This is equivalent to, if a strategy s_a survives IU, is it the case that there is a type t_a so that $t_a \in \mathbb{T}[s_a, 1], \mathbb{T}[s_a, 2], \ldots$? That is, if s_a survives IU, is $\bigcap_m \mathbb{T}[s_a, m]$ nonempty?

Refer to Figure 2.4. The strategy U survives IU and so we are looking for a type in $\bigcap_m \mathbb{T}[U, m]$. The sets $\mathbb{T}[U, m]$ are shrinking, i.e., $\mathbb{T}[U, m+1] \subseteq \mathbb{T}[U, m]$. In fact, because the type structure is complete, the sets $\mathbb{T}[U, m]$ must be strictly shrinking, i.e., $\mathbb{T}[U, m+1] \subsetneq \mathbb{T}[U, m]$.[5] In principle, then, we might have that the intersection is empty.

Result 2.4(ii) tells us that, when the type sets are compact and the belief maps are continuous, we must have a non-empty intersection—the reason this is the case is that, then, the sets $\mathbb{T}[U, m]$ are closed subsets of a compact set. But, Result 2.5(ii) says that we can construct some complete type structure so that the intersection of these sets is empty. In particular, we construct a complete type structure that has Polish (but not compact) type sets and continuous belief maps, but for which the intersection is empty. (This will be Theorem 6.1 below.) As a by-product, we also get an example of a complete type structure that has compact types sets and discontinuous belief maps, but for which the intersection is empty. (This is Theorem 6.2 below.)

[5]This is not particular to the example—it must hold in any non-trivial game. This is essentially the proof of Proposition 3.1 in Friedenberg (2010).

2.6 Revisiting the Epistemic Conditions for IU

Return to Result 2.4: For any complete epistemic game, the set of strategies consistent with RmBR is the set of $(m+1)$-undominated strategies. The implication is that a complete type structure induces a rich enough set of beliefs from the perspective of delivering the $(m+1)$-undominated strategies. However, Result 2.5 indicates that it need not induce a rich enough set of beliefs from the perspective of delivering the IU strategies. This raises two interrelated questions: First, how should we interpret this lack of richness of a complete structure? And, second, what richness condition would be sufficient from the perspective of delivering the IU strategies?

To address this question, let us return to what is known about the relationship between complete type structures and the hierarchies of beliefs that they induce.

Result 2.6 (Friedenberg, 2010; Theorem 3.1) *Let $\mathcal{T}$ be a complete type structure.*

- *(i) If $\mathcal{T}$ has Polish type sets, then $\mathcal{T}$ induces all finite-order beliefs.*
- *(ii) If $\mathcal{T}$ has compact type sets and continuous belief mappings, then $\mathcal{T}$ induces all hierarchies of beliefs.*

Result 2.4(i) is typically shown for a complete epistemic game with Polish type sets.[6] In fact, Result 2.5(i) constructs a complete type structure with Polish type sets. So, by Result 2.6(i), each of these results are concerned with type structures that induce all finite-order beliefs. Result 2.4(ii) says that completeness, compactness and continuity deliver IU and Result 2.6(ii) says that those structures also induce all hierarchies of beliefs.

This leads to two natural conjectures that address our questions: First, if an epistemic game induces all hierarchies of beliefs, then the RCBR predictions coincide with IU. Second, the complete type structure constructed for Result 2.5 does not induce all hierarchies of beliefs. Both conjectures will be true. In fact, we will show a stronger result:

Result 2.7 *[Theorem 8.1] Fix an epistemic game $(G, \mathcal{T})$.*

- *(i) Suppose $\mathcal{T}$ induces all finite-order beliefs that are induced by finite type structures. Then, the set of strategies consistent with RmBR is the set of $(m+1)$-undominated strategies.*
- *(ii) Suppose $\mathcal{T}$ induces all hierarchies of beliefs that are induced by finite type structures. Then the set of strategies consistent with RCBR is the set of IU strategies.*

Results 2.5-2.6-2.7 imply that there exists a complete type structure that induces all finite-order beliefs, but does not induce all hierarchies of beliefs that arise from finite type structures. (See Proposition 8.3.) As an implication, there exists a complete type structure that induces all finite-order beliefs, but does not induce all hierarchies of beliefs. So, while type structures implicitly model hierarchies of beliefs, properties of type structures (e.g., "all possible beliefs about types")

[6] Though, an implication of Proposition 5.5 to come is that the Polish assumption is not necessary.

may not transfer over, in a natural way, to properties of hierarchies of beliefs (e.g., "all hierarchies of beliefs").

This raises the question: Which hierarchies of beliefs are induced by a complete type structure (i.e., beyond the specific type structure constructed for Result 2.5)? We address a somewhat weaker question: Which finite-order beliefs are induced by a complete type structure? We show that a complete type structure necessarily induces finite-order beliefs associated with countable type structures. (See Theorem 9.2.) However, importantly, we stop short of showing that it induces all finite-order beliefs. Instead, we establish a negative result: One cannot prove in ZFC that every complete continuous type structure induces all possible finite-order beliefs. (See Theorem 9.4.) While there exists complete continuous type structures that do induce all possible finite-order beliefs, within ZFC, one cannot rule out the possibility that there are complete continuous type structures that do not induce all possible finite-order beliefs.

3 The Game

We begin with mathematical preliminaries used throughout the paper. Given a metrizable set Ω, we endow Ω with the Borel sigma-algebra. The set of Borel probability measures on Ω is $\mathcal{P}(\Omega)$; endow $\mathcal{P}(\Omega)$ with the topology of weak convergence. Endow the product of metrizable spaces with the product topology. Given metrizable sets Ω and Φ and a measurable map $f : \Omega \to \Phi$, define $\underline{f} : \mathcal{P}(\Omega) \to \mathcal{P}(\Phi)$ so that $\underline{f}(\mu)$ is the image measure of f under μ.

Throughout the paper, we fix a finite two-player simultaneous-move game $G = (S_a, S_b, \pi_a, \pi_b)$. The players are a and b.[7] Write c for an arbitrary player from a, b and write d for the other player. Player c has a finite strategy set S_c and a payoff function $\pi_c : S_c \times S_d \to \mathbb{R}$. Extend π_c to $\mathcal{P}(S_c) \times \mathcal{P}(S_d)$ in the usual way, i.e. $\pi_c(\sigma_c, \sigma_d) = \sum_{(s_c,s_d)\in S_c\times S_d} \sigma_c(s_c)\sigma_d(s_d)\pi_c(s_c, s_d)$.

Definition 3.1 *Fix $Y_a \times Y_b \subseteq S_a \times S_b$. A strategy $s_c \in Y_c$ is **strongly dominated with respect to** $Y_c \times Y_d$ if there exists $\sigma_c \in \mathcal{P}(S_c)$ with $\sigma_c(Y_c) = 1$ and $\pi_c(\sigma_c, s_d) > \pi_c(s_c, s_d)$ for every $s_d \in Y_d$. Otherwise, say s_c is **undominated with respect to** $Y_c \times Y_d$.*

Note the convention: If $Y_c \neq \emptyset$ but $Y_d = \emptyset$, then each $s_c \in Y_c$ is strongly dominated with respect to $Y_c \times Y_d$.

Set $S_c^0 = S_c$ and inductively define

$$S_c^{m+1} = \{s_c \in S_c^m : s_c \text{ is undominated with respect to } S_c^m \times S_d^m\}.$$

The set $S_a^m \times S_b^m$ is the set of **m-undominated** strategy profiles. Define $S_c^\infty = \bigcap_{m=1}^\infty S_c^m$. The set $S_a^\infty \times S_b^\infty$ is the set of **iteratively undominated** (**IU**) strategy profiles. Note, the set of IU strategy profiles is non-empty.

[7] The restriction to two players is immaterial, up to issues of correlation.

4 Type Structures and Epistemic Games

Following Section 2.1, we model the players hierarchies of beliefs implicitly, via a type structure. This section first defines a type structure for an abstract set of parameters that the players may be uncertain about. It then applies that definition to capture the type structure of interest.

Abstract Type Structures We are concerned with situations where each player c is uncertain about a set of parameters. Let X_c be a metrizable **parameter set** for player c. Call X_c **non-degenerate** if $|X_c| \geq 2$.

Definition 4.1 *An* (X_a, X_b)***-based type structure*** *is some* $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$*, where*

(i) T_c *is a metrizable* ***type set*** *for player c, and*

(ii) $\beta_c : T_c \to \mathcal{P}(X_c \times T_d)$ *is a measurable* ***belief map*** *for player c.*

Section 7 formalizes how an (X_a, X_b)-based type structure induces hierarchies of beliefs about (X_a, X_b). Refer to $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$ as a "type structure," when the underlying set of parameters (X_a, X_b) is clear from the context.

We will be interested in type structures that satisfy certain properties.

Definition 4.2 *Fix an* (X_a, X_b)*-based type structure, viz.* $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$*.*

(i) Say $\mathcal{T}$ *is* ***compact*** *(resp.* ***Polish****) if* T_a *and* T_b *are compact (resp. Polish).*

(ii) Say $\mathcal{T}$ *is* ***continuous*** *if* β_a *and* β_b *are continuous.*

(iii) Say $\mathcal{T}$ *is* ***complete*** *if* β_a *and* β_b *are onto.*

Definition 4.2 provides conditions that a type structure may or may not satisfy. The last condition—i.e., completeness—is a substantive requirement. It says that, for every belief $\mu_c \in \mathcal{P}(X_c \times T_d)$ that a player may hold, there is a type of the player that holds that belief. Thus, it is a requirement that the type structure is "rich." (Completeness is due to Brandenburger, 2003.) The other conditions—i.e., compactness, Polishness, and continuity—appear to be technical requirements. A main goal of this paper is to show that, in conjunction with completness, these technical requirements have substantive import.

Mertens and Zamir (1985), Brandenburger and Dekel (1993), and Heifetz and Samet (1998) each provided canonical constructions of a so-called (X_a, X_b)-based "universal type structure." When X_a, X_b are compact (resp. Polish), these universal type structures are each compact (resp. Polish), continuous, and complete. But there are other type structures—i.e., type structures that differ from these constructions—that are also compact (resp. Polish), continuous, and complete.

Epistemic Games We are concerned with the situation in which each player faces uncertainty about the strategy the other player employs. So, player a's (resp. b's) basic set of uncertainty is $X_a = S_b$ (resp. $X_b = S_a$). With this in mind:

Definition 4.3 *An **epistemic game** is a pair $(G, \mathcal{T})$ where $\mathcal{T}$ is a (S_b, S_a)-based type structure.*

Because the game G is fixed throughout the paper, we often conflate an epistemic game with its associated type structure.

5 Epistemic Conditions: RCBR

An epistemic game $(G, \mathcal{T})$ induces a set of states—namely, $S_a \times T_a \times S_b \times T_b$. That is, a **state** is a quadruple (s_a, t_a, s_b, t_b) that describes the play of the game and the players' beliefs. We focus on the states that satisfy rationality and common belief of rationality, as defined below.

Say s_c is **optimal under** $\sigma_c \in \mathcal{P}(S_d)$ if $\pi_c(s_c, \sigma_d) \geq \pi_c(r_c, \sigma_d)$ for all $r_c \in S_c$.

Definition 5.1 *A strategy-type pair (s_c, t_c) is **rational** if s_c is optimal under $\text{marg}_{S_d}\beta_c(t_c)$.*

Definition 5.2 *Say a type t_c **believes** $E_d \subseteq S_d \times T_d$ if E_d is Borel and $\beta_c(t_c)(E_d) = 1$.*

Write

$$\mathrm{B}_c(E_d) = S_c \times \{t_c : t_c \text{ believes } E_d\},$$

for the set of strategy-type pairs that believe E_d.

Set $\mathrm{R}_c^0 = S_c \times T_c$ and take R_c^1 for the set of rational strategy-type pairs of player c. For each $m \geq 1$, set $\mathrm{R}_c^{m+1} = \mathrm{R}_c^m \cap \mathrm{B}_c(\mathrm{R}_d^m)$. Likewise, set $\mathrm{R}_c^\infty = \bigcap_m \mathrm{R}_c^m$.

Definition 5.3 *The sets of states at which there is **rationality and $\mathbf{m^{th}}$-order belief of rationality (RmBR)** is $\mathrm{R}_a^{m+1} \times \mathrm{R}_b^{m+1}$. The sets of states at which there is **rationality and common belief of rationality (RCBR)** is $\mathrm{R}_a^\infty \times \mathrm{R}_b^\infty$.*

Note, the set of states at which there is RCBR—i.e., $\mathrm{R}_a^\infty \times \mathrm{R}_b^\infty$—depends on the type structure. (The examples in Figures 2.2-2.3 illustrated this point.)

We are interested in the RCBR predictions, i.e., $\text{proj}_{S_a}\mathrm{R}_a^\infty \times \text{proj}_{S_b}\mathrm{R}_b^\infty$. As Result 2.1 stated, these predictions are necessarily a subset of the IU strategy set. Lemma 5.4 formalizes that result.[8]

Lemma 5.4 *Fix an epistemic game $(G, \mathcal{T})$.*

(i) For each m, $\text{proj}_{S_a}\mathrm{R}_a^m \times \text{proj}_{S_b}\mathrm{R}_b^m \subseteq S_a^m \times S_b^m$.

(ii) $\text{proj}_{S_a}\mathrm{R}_a^\infty \times \text{proj}_{S_b}\mathrm{R}_b^\infty \subseteq S_a^\infty \times S_b^\infty$.

[8]Proofs not found in the main text can be found in the appendices.

The example in Figure 2.2 illustrated that, for a given epistemic game, the RCBR predictions may be a strict subset of the IU strategy set. But, as Result 2.4 stated, the RCBR predictions coincide with IU when the epistemic game has a complete, compact and continuous type structure.

Proposition 5.5 (Folk-Result) *Fix an epistemic game $(G, \mathcal{T})$ where $\mathcal{T}$ is complete.*

(i) For each $m \geq 1$, $\mathrm{proj}_{S_a} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m$.

(ii) If $\mathcal{T}$ is compact and continuous, then $\mathrm{proj}_{S_a} \mathrm{R}_a^\infty \times \mathrm{proj}_{S_b} \mathrm{R}_b^\infty = S_a^\infty \times S_b^\infty$.

Part (i) says that, in a complete type structure, the set of strategies consistent with RmBR is the set of strategies that survive $(m+1)$ rounds of eliminating dominated strategies. Part (ii) says that if, in addition, the complete type structure is compact and continuous, then the set of strategies consistent with RCBR is the set of IU strategies. A goal of this paper is to explore whether compactness and continuity are required for the conclusion of part (ii). The next section shows that, indeed, they are.

6 A Negative Result

Call G **non-trivial** if, for each player c and each strategy s_c, there is some $\sigma_d \in \mathcal{P}(S_d)$ so that s_c is not optimal under σ_d. Note, G is non-trivial if and only if no player has a very weakly dominant strategy (Marx and Swinkels, 1997). If G is non-trivial, S_a and S_b are both non-degenerate. When G is non-trivial, we have two negative results that each establish Result 2.5.

Theorem 6.1 *Let G be non-trivial. There exists a Polish, continuous, and complete (S_b, S_a)-based type structure $\mathcal{T}$ so that*

(i) For each $m \geq 0$, $\mathrm{proj}_{S_a} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m$.

(ii) $\mathrm{R}_a^\infty = \emptyset$ and $\mathrm{R}_b^\infty = \emptyset$.

Theorem 6.2 *Let G be non-trivial. There exists a compact and complete (S_b, S_a)-based type structure $\mathcal{T}$ so that*

(i) For each $m \geq 0$, $\mathrm{proj}_{S_a} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m$.

(ii) $\mathrm{R}_a^\infty = \emptyset$ and $\mathrm{R}_b^\infty = \emptyset$.

The two theorems differ in the topological properties of the associated complete type structure. Theorem 6.1 constructs a complete type structure that is continuous. Theorem 6.2 constructs a complete type structure that is compact. In both cases, Part (i) coincides with Proposition 5.5(i). But, Part (ii) is different. We first prove Theorem 6.1 and then discuss how Theorem 6.2 can be derived as (almost) a corollary. (Some technical steps in the proof of Theorem 6.2 are stated as lemmas; see Appendix C for their proofs.)

Proof of Theorem 6.1. Recall, the idea from Section 2. The set $\mathbb{T}[s_c, m]$ was the set of types t_c so that $(s_c, t_c) \in \mathrm{R}_c^m$. When G is non-trivial and the type structure is complete, the sets $\mathbb{T}[s_c, m]$ are shrinking, i.e., $\mathbb{T}[s_c, m+1] \subsetneq \mathbb{T}[s_c, m]$. The goal is to construct a complete type structure so that the intersection of these sets is empty.

To construct this type structure, it will be useful to express the m-undominated strategies as an output of an **m^th^-order best response maps**: For each $m \geq 0$, let

$$\mathrm{BR}_c^{m+1} : \{\sigma_d \in \mathcal{P}(S_d) : \sigma_d(S_d^m) = 1\} \to 2^{S_c} \setminus \{\emptyset\}.$$

map each probability measure $\sigma_d \in \mathcal{P}(S_d)$ with $\sigma_d(S_d^m) = 1$ into the set of strategies that are optimal under that measure. We refer to an element of the range of BR_c^m as an **$\mathbf{m^{th}}$-order best response set**. Write $\mathbb{S}_c^m$ for the range of the m^{th}-order best response map, i.e., the collection of all m^{th}-order best response sets. Back to Figure 2.1. There, $\mathbb{S}_a^1$ is the set of all non-empty subsets of S_a and, for each $m \geq 2$, $\mathbb{S}_a^m = \{\{U\}, \{M\}, \{U, M\}\}$. Note the following properties of $\mathbb{S}_c^m$:

Properties 6.3

(i) If $Q_c \in \mathbb{S}_c^m$, then $Q_c \subseteq S_c^m$

(ii) If $s_c \in S_c^m$, then there exists some $Q_c \in \mathbb{S}_c^m$ so that $s_c \in Q_c$.

Conditions (i)-(ii) say that the sets $\mathbb{S}_c^m$ characterize the set of m-undominated strategies. That is, the union over the sets in $\mathbb{S}_c^m$ is the set of m-undominated strategies.

In constructing the type structure for Theorem 6.1, we will "match up" subsets of types with m^{th}-order best response sets. Specifically, for each $Q_c \in \mathbb{S}_c^m$, we will have a subset of types $\mathrm{T}[Q_c, m]$. We will later show that we can choose the belief maps so that $\mathrm{T}[Q_c, m]$ is the set of all types t_c so that (i) the set of strategies optimal under $\mathrm{marg}_{S_d}\beta_c(t_c)$ is Q_c and (ii) t_c believes R_d^{m-1}.

Let $T_a = T_b = \mathbb{N}^{\mathbb{N}}$. So, each element of T_a (resp. T_b) is some $(n_0, n_1, n_2, \ldots)$, where each n_k is a natural number. Take as an open basis of $\mathbb{N}^{\mathbb{N}}$ the family of "cones," i.e.,

$$\{\{(n_0, n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}} : (n_0, \ldots, n_k) = (o_0, \ldots, o_k)\} : k \in \mathbb{N} \text{ and } o_0, \ldots, o_k \in \mathbb{N}\}.$$

With this, each T_c is Polish, but not compact.

It will be useful to index the sets in the range of the 1^{st}-order best response map, i.e., setting $\mathbb{S}_c^1 = \{Q_{c,0}, \ldots, Q_{c,K}\}$. Then, for each integer k with $K > k \geq 0$ set

$$\mathrm{T}[Q_{c,k}, 1] = \{(n_0, n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}} : n_0 = k\}$$

and set

$$\mathrm{T}[Q_{c,K}, 1] = \{(n_0, n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}} : n_0 \geq K\}.$$

Figure 6.1 illustrates the sets $\mathrm{T}[\{U\}, 1]$ and $\mathrm{T}[\{D\}, 1]$, for the game in Figure 2.1. In the illustration, each type $t_a \in \mathbb{N}^{\mathbb{N}}$ corresponds to a path that contains one point in each row. The set $\mathbb{S}_a^1$ is

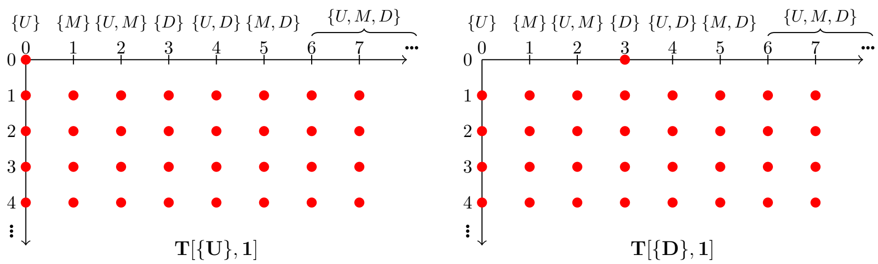

Figure 6.1

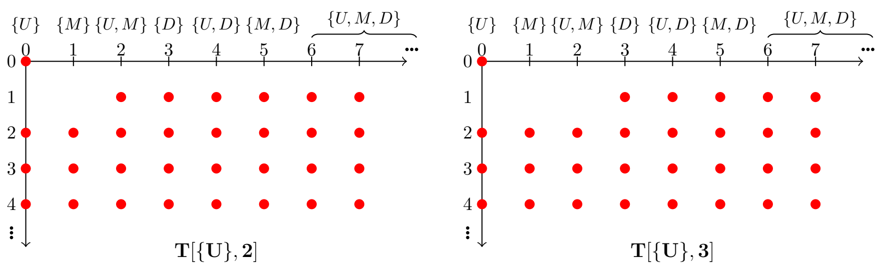

Figure 6.2

the set of non-empty subsets of S_a. List elements of $\mathbb{S}_a^1$ according to the following order:

$$\{U\}, \{M\}, \{U, M\}, \{D\}, \{U, D\}, \{M, D\}, \{U, M, D\}$$

and note that, in the illustration, the heading of the k^{th} column is the k^{th} set $Q_{a,k}$ in this list. For example, $\{D\} = Q_{a,3}$ is the heading of column 3. So, $\mathrm{T}\left[\{D\}, 1\right]$ is the set of all points in $n \in \mathbb{N}^{\mathbb{N}}$ such that $n_0 = 3$, and corresponds to the set of all paths that select one red dot in each row. Thus, the 0^{th} row of a type can be seen as "tracking an associated 1^{st}-order best response set."

Next, for each $m \geq 1$, set

$$\mathrm{T}\left[Q_{c,k}, m+1\right] = \begin{cases} \{(n_0, n_1, n_2, \ldots) \in \mathrm{T}\left[Q_{c,k}, m\right] : n_1 \geq m+1\} & \text{if } Q_{c,k} \in \mathbb{S}_c^m \\ \emptyset & \text{if } Q_{c,k} \notin \mathbb{S}_c^m. \end{cases}$$

So, referring to our example, $\mathrm{T}\left[\{D\}, m\right] = \emptyset$, for all $m \geq 2$. Figure 6.2 illustrates the sets $\mathrm{T}\left[\{U\}, 2\right]$ and $\mathrm{T}\left[\{U\}, 3\right]$. Note, then, the first row can be seen as "tracking whether a 1^{st}-order best response set is an m^{th}-order best response set," for $m \geq 2$.

We have the following properties:

Properties 6.4 *For each $Q_c \in \mathbb{S}_c^1$, the sets $\mathrm{T}\left[Q_c, 1\right], \mathrm{T}\left[Q_c, 2\right], \ldots$ satisfy the following:*

(i) $\mathrm{T}\left[Q_c, m+1\right] \subseteq \mathrm{T}\left[Q_c, m\right]$.

(ii) $\bigcap_m \mathrm{T}\left[Q_c, m\right] = \emptyset$.

(iii) $\mathrm{T}[Q_c, m]$ *is clopen in* T_c.

(iv) If $\mathrm{T}[Q_c, m] \neq \emptyset$, $\mathrm{T}[Q_c, m]$ *is a topological space homeomorphic to* $\mathbb{N}^{\mathbb{N}}$.

(v) If $\mathrm{T}[Q_c, m] \neq \emptyset$, $\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1]$ *is a topological space homeomorphic to* $\mathbb{N}^{\mathbb{N}}$.

Let $P_c^0 = S_c \times T_c$ and, for each $m \geq 1$, define

$$P_c^m = \bigcup \{Q_c \times \mathrm{T}[Q_c, m] : Q_c \in \mathbb{S}_c^m\}$$

Thus, P_c^m tracks the m^{th}-order best response sets in the space of strategies-cross-types, by pairing each set $Q_c \in \mathbb{S}_c^m$ with the types $\mathrm{T}[Q_c, m]$. As such:

Lemma 6.5 *For each* $m \geq 0$, $\mathrm{proj}_{S_a} P_a^m \times \mathrm{proj}_{S_b} P_b^m = S_a^m \times S_b^m$.

Proof. For $m = 0$ the claim is immediate. Take $m \geq 1$. If $s_c \in \mathrm{proj}_{S_c} P_c^m$, then there exists $Q_c \in \mathbb{S}_c^m$ so that $s_c \in Q_c$. By Property 6.3(i), $s_c \in S_c^m$. Conversely, fix $s_c \in S_c^m$. Then, by Property 6.3(ii), there exists $Q_c \in \mathbb{S}_c^m$ so that $s_c \in Q_c$. It follows that $s_c \in \mathrm{proj}_{S_c} P_c^m$. ■

We will construct β_c so that, for each $m \geq 0$, $\mathrm{T}[Q_c, m+1]$ maps onto the set of all probability measures $\psi_d \in \mathcal{P}(S_d \times T_d)$ with $\psi_d(P_d^m) = 1$ and $Q_c = \mathrm{BR}_c^1(\mathrm{marg}_{S_d} \psi_d)$. To be more precise, extend the m^{th}-order best response maps so that its domain is the space of strategies cross types: Specifically, for each $m \geq 0$ set

$$\mathbb{BR}_c^{m+1} : \{\psi_d \in \mathcal{P}(S_d \times T_d) : \psi_d(P_d^m) = 1\} \to 2^{S_c} \setminus \{\emptyset\}$$

so that $\mathbb{BR}_c^{m+1}(\psi_d)$ is the set of all strategies that are optimal under $\mathrm{marg}_{S_d} \psi_d$. As such, $(\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is the set of probability measures ψ_d so that (i) $\psi_d(P_d^m) = 1$, and (ii) Q_c is the set of strategies optimal under $\mathrm{marg}_{S_d} \psi_d$. The following Lemma formalizes the plan set forth at the start of this paragraph:

Lemma 6.6 *There exists a continuous onto mapping* β_c *so that, for each* $m \geq 1$ *and each* $Q_c \in \mathbb{S}_c^1$,

$$(\beta_c)^{-1}((\mathbb{BR}_c^m)^{-1}(Q_c)) = \mathrm{T}[Q_c, m]. \tag{1}$$

Lemma 6.6 generates a complete and continuous type structure that satisfies a particular property. (Below, we will return to prove the Lemma.) Because the type structure satisfies this property, it generates two important consequences, formulated by Lemmata 6.7-6.8 below. First, in this type structure, the set of states at which there is R$(m-1)$BR coincides with $P_a^m \times P_b^m$; as a consequence, the predictions of R$(m-1)$BR are the m-undominated strategies. Second, the set of states at which there is RCBR is empty.

Lemma 6.7 *Suppose* β_a *and* β_b *are such that, for each* $m \geq 1$ *and each* $Q_c \in \mathbb{S}_c^1$, *Equation* (1) *holds. Then, for each* $m \geq 1$:

(i) $\mathrm{R}_a^m \times \mathrm{R}_b^m = P_a^m \times P_b^m$, *and*

(ii) $\mathrm{proj}_{S_a}\mathrm{R}_a^m \times \mathrm{proj}_{S_b}\mathrm{R}_b^m = S_a^m \times S_b^m$.

Proof. Part (ii) follows from part (i) and Lemma 6.5. So, we focus on part (i). The proof is by induction on m. For $m = 0$ the claim is immediate. Assume the result holds for $m \geq 1$. We will show it also holds for $m + 1$.

Suppose $(s_c, t_c) \in P_c^{m+1}$. Then, there exists some $Q_c \in \mathbb{S}_c^{m+1}$ so that $(s_c, t_c) \in Q_c \times \mathrm{T}[Q_c, m+1]$. By Equation (1), $Q_c = \mathbb{BR}_c^{m+1}(\beta_c(t_c))$. Since $s_c \in Q_c$, it follows that s_c is optimal under $\mathrm{marg}_{S_d}\beta_c(t_c)$. Moreover, since $\beta_c(t_c)$ is in the domain of $\mathbb{BR}_c^{m+1}$, it follows from the induction hypothesis that $\beta_c(t_c)$ believes $P_d^m = \mathrm{R}_d^m$. Moreover, by the induction hypothesis and Property 6.4(iii), each $\mathrm{R}_d^1, \ldots, \mathrm{R}_d^m$ is Borel. Thus, t_c believes each $\mathrm{R}_d^1, \ldots, \mathrm{R}_d^m$ and so $(s_c, t_c) \in \mathrm{R}_c^{m+1}$.

Conversely, suppose $(s_c, t_c) \in \mathrm{R}_c^{m+1}$. Then, using the induction hypothesis, $\beta_c(t_c)$ believes $\mathrm{R}_d^m = P_d^m$. It follows that $s_c \in \mathrm{BR}_c^{m+1}(\mathrm{marg}_{S_d}\beta_c(t_c))$. Write $Q_c = \mathrm{BR}^c_{m+1}(\mathrm{marg}_{S_d}\beta_c(t_c))$ and note that $s_c \in Q_c \in \mathbb{S}_c^{m+1}$. Moreover, $t_c \in (\beta_c)^{-1}((\mathbb{BR}_c^{m+1})^{-1}(Q_c))$ and so, by Equation (1), $t_c \in \mathrm{T}[Q_c, m+1]$. That is, $(s_c, t_c) \in Q_c \times \mathrm{T}[Q_c, m+1] \subseteq P_c^{m+1}$, as required. ■

Lemma 6.8 *Suppose β_a and β_b are such that, for each $m \geq 1$ and each $Q_c \in \mathbb{S}_c^1$, Equation (1) holds. Then $\mathrm{R}_a^\infty \times \mathrm{R}_b^\infty = \emptyset$.*

Proof. For each $Q_c \subseteq S_c$, $\bigcap_m T[Q_c, m] = \emptyset$. Thus, $\bigcap_m P_c^m = \emptyset$. So, by Lemma 6.7(i), $\mathrm{R}_c^\infty = \emptyset$. ■

To place Lemma 6.8 in the context of Section 2, observe that (by Lemma 6.7(i)) $\mathbb{T}[s_c, m]$ is the union over all $T[Q_c, m]$ with $s_c \in Q_c$. By construction, for each Q_c, $\bigcap_m T[Q_c, m] = \emptyset$. So, $\bigcap_m \mathbb{T}[s_c, m] = \emptyset$. Now we return to prove Lemma 6.6.

Proof of Lemma 6.6. Consider the collection of sets of types

$$\mathcal{T}_c = \{\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1] : Q_c \in \mathbb{S}_c^1 \text{ and } m \geq 1\}$$

and the collection of sets of probabilities

$$\mathcal{P}_c = \{(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c) : Q_c \in \mathbb{S}_c^1 \text{ and } m \geq 1\}.$$

Note, $\mathcal{T}_c$ partitions T_c into pairwise disjoint sets and $\mathcal{P}_c$ partitions $\mathcal{P}(S_d \times T_d)$ into pairwise disjoint sets. (See Lemmata C.1-C.2.) Step 1 shows that we can find a continuous mapping from each element of $\mathcal{T}_c$ to its "matching set" in $\mathcal{P}_c$. Step 2 uses the fact that these collections form a partition to construct a continuous onto map β_c satisfying Equation (1).

Step 1: Fix $Q_c \in \mathbb{S}_c^1$ and some $m \geq 1$. Note, $\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1] \neq \emptyset$ if and only if $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c) \neq \emptyset$. (See Lemma C.6, which makes use of the fact that the game is nontrivial.) Moreover, by Properties 6.4(iii)-(v), if $\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1] \neq \emptyset$, then

$\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1]$ is a topological space homeomorphic to $\mathbb{N}^{\mathbb{N}}$ that is clopen in T_c. Similarly, $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is a Polish space. (See Lemma C.4.) So, if $\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1] \neq \emptyset$, there is a continuous mapping, viz. $\beta_c^{[Q_c,m]}$, from $\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1]$ onto $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$. (See Kechris, 1995, Theorem 7.9)

Step 2: Take β_c to be the union of the maps $\beta_c^{[Q_c,m]}$. Note, this is well-defined since $\mathcal{T}_c$ partitions T_c. Using the fact that $\mathcal{P}_c$ partitions $\mathcal{P}(S_d \times T_d)$ and each of the maps $\beta_c^{[Q_c,m]}$ is onto, it follows that the map β_c is onto. Again using the fact that $\mathcal{P}_c$ partitions $\mathcal{P}(S_d \times T_d)$, it follows that Equation (1) is satisfied. Finally, continuity follows from Lemma C.7. ■

Proof of Theorem 6.2. By Theorem 6.1, there exists a complete type structure

$$\mathcal{T}^* = (S_b, S_a, T_a^*, T_b^*, \beta_a^*, \beta_b^*),$$

so that T_a^*, T_b^* are Polish, β_a^*, β_b^* are continuous, and the sets $R_a^{*,m}, R_b^{*,m}$ satisfy conditions (i)-(ii) of Theorem 6.1. Note, T_a^* and T_b^* are uncountable: Since the game is non-trivial, each S_c is non-degenerate. As such, any complete type structure has uncountable type sets.

We now construct a new type structure. The type sets are given by $T_a^{**} = T_b^{**} = \{0,1\}^{\mathbb{N}}$. Note, each T_c^{**} is a compact metric space. Observe that, since T_c^* and T_c^{**} are both uncountable Polish spaces, they each have the cardinality of the continuum (Kechris, 1995, Corollary 6.5). So, by the Borel isomorphism theorem (see Kechris, 1995, Theorem 15.6), there is a Borel bijection $\alpha_c : T_c^* \to T_c^{**}$. Write $\mathrm{id}_d : S_d \to S_d$ for the identity map. Note that $\underline{\mathrm{id}_d \times \alpha_d} : \mathcal{P}(S_d \times T_d^*) \to \mathcal{P}(S_d \times T_d^{**})$ is a Borel bijection. (See Lemmata A.1-A.2.) Let $\beta_c^{**} = \underline{(\mathrm{id}_d \times \alpha_d)} \circ \beta_c^* \circ \alpha_c^{-1}$. It follows that $\mathcal{T}^{**} = (S_b, S_a; T_a^{**}, T_b^{**}; \beta_a^{**}, \beta_b^{**})$ is a compact and complete type structure.

Write $R_c^{**,m}$ for the set of strategy-type pairs of c at which, in $\mathcal{T}^{**}$, there is R$(m-1)$BR. By the construction and induction,

$$(\mathrm{id}_c \times \alpha_c)^{-1}(R_c^{**,m}) = R_c^{*,m} \qquad \text{and} \qquad (\mathrm{id}_c \times \alpha_c)(R_c^{*,m}) = R_c^{**,m}.$$

Thus, the sets $R_a^{**,m}, R_b^{**,m}$ also satisfy conditions (i)-(ii) of Theorem 6.2.

7 Type Structures and Hierarchies of Beliefs

Proposition 5.5 says that, in a complete, compact, and continuous type structure, the predictions of RCBR coincide with the IU strategy set. Theorems 6.1-6.2 say that, if either compactness or continuity is dropped, the conclusion may not hold. In fact, there may be no RCBR predictions. Section 8 will show that this occurs because the constructed complete type structure does not induce all hierarchies of beliefs or, even, all hierarchies of beliefs induced by finite type structures. To get there, we must first specify a relationship between type structures and hierarchies of beliefs. This is the goal of the current section.

Fix parameter sets (X_a, X_b). For each player c, set $Z_c^1 = X_c$ and inductively define $Z_c^{m+1} = Z_c^m \times \mathcal{P}(Z_d^m)$. The set Z_c^m is the **m$^{\text{th}}$-order space of uncertainty for** c. So, an m^{th}-order belief for c will be an element of $\mathcal{P}(Z_c^m)$. Observe that Z_c^m (resp. $\mathcal{P}(Z_c^m)$) is metrizable.

Fix a type structure $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$. To illustrate how $\mathcal{T}$ induces hierarchies of beliefs about (X_a, X_b), we will first inductively define auxiliary maps $\rho_c^m : X_c \times T_d \to Z_c^m$ so that

$$\rho_c^1(x_c, t_d) = x_c \qquad \text{and} \qquad \rho_c^{m+1}(x_c, t_d) = (\rho_c^m(x_c, t_d), \underline{\rho}_c^m \circ \beta_c).$$

(Recall from page 11: $\underline{\rho}_c^m$ maps each $\mu_c \in \mathcal{P}(X_c \times T_d)$ to the image of μ_c under ρ_c^m.) We use these maps to define the m-th order belief mapping $\delta_c^m : T_c \to \mathcal{P}(Z_c^m)$ as $\delta_c^m = \underline{\rho}_c^m \circ \beta_c$. Type t_c's **m$^{\text{th}}$-order belief** is $\delta_c^m(t_c)$. Define the hierarchies of beliefs mapping $\delta_c : T_c \to \prod_{m=1}^{\infty} \mathcal{P}(Z_c^m)$ so that, for each t_c, $\delta_c(t_c) = (\delta_c^1(t_c), \delta_c^2(t_c), \ldots)$. Type t_c's **hierarchy of beliefs** is $\delta_c(t_c)$.

We now make two remarks: First, we point out why the maps above are well-defined. Second, we illustrate why, conceptually, the mapping δ_c captures hierarchies of beliefs.

Remark 7.1 *Lemma A.1 inductively establishes that each $\rho_c^m : X_c \times T_d \to Z_c^m$ and $\underline{\rho}_c^m \circ \beta_c : T_c \to \mathcal{P}(Z_c^m)$ is measurable. So these maps are well-defined. As such, each δ_c^m is well-defined.*

Remark 7.2 *To better illustrate these definitions, let us show how δ_c^1 is computed. Note that $\underline{\rho}_c^1 \circ \beta_c : T_c \to \mathcal{P}(Z_c^1)$, i.e., $\underline{\rho}_c^1 \circ \beta_c$ maps each type to a probability measure on the 1^{st}-order space of uncertainty for c, viz. $Z_c^1 = X_c$. So, for each type t_c and each Borel set $E_c \subseteq Z_c^1 = X_c$,*

$$((\underline{\rho}_c^1 \circ \beta_c)(t_c))(E_c) = (\beta_c(t_c))((\rho_c^1)^{-1}(E_c)) = \beta_c(t_c)(E_c \times T_d).$$

By setting $\delta_c^1(t_c) = (\underline{\rho}_c^1 \circ \beta_c)(t_c)$, we specify t_c's first order belief; in particular, we have $\delta_c^1(t_c)(E_c) = \beta_c(t_c))(E_c \times T_d)$, as desired.

We will be interested in type structures that induce a rich set of hierarchies of beliefs. With this in mind, we will think about mapping one type structure to a second type structure, so that we preserve hierarchies of beliefs.

Definition 7.3 *Fix two (X_a, X_b)-based type structures, viz. $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$ and $\mathcal{T}^* = (X_a, X_b; T_a^*, T_b^*; \beta_a^*, \beta_b^*)$.*

*(i) Say that $\mathcal{T}^*$ is **finitely terminal for** $\mathcal{T}$ if, for each m and each type $t_c \in T_c$, there is a type $t_c^* \in T_c^*$ with $(\delta_c^{*,1}(t_c^*), \ldots, \delta_c^{*,m}(t_c^*)) = (\delta_c^1(t_c), \ldots, \delta_c^m(t_c))$.*

*(ii) Say that $\mathcal{T}^*$ is **terminal for** $\mathcal{T}$ if, for each type $t_c \in T_c$, there is a type $t_c^* \in T_c^*$ with $\delta_c^*(t_c^*) = \delta_c(t_c)$.*

Definition 7.3 says that the type structure $\mathcal{T}^*$ is finitely terminal for $\mathcal{T}$ if, for each type t_c in $\mathcal{T}$ and each m, there is a type t_c^* in $\mathcal{T}^*$ whose hierarchy agrees with t_c up to level m. (Note, here, t_c^* can depend both on the t_c and m.) More informally, $\mathcal{T}^*$ is finitely terminal for $\mathcal{T}$ if it induces all finite-order beliefs associated with types in $\mathcal{T}$. It is terminal for $\mathcal{T}$ if it induces all hierarchies of beliefs associated with types in $\mathcal{T}$.

We will be interested in a type structure $\mathcal{T}^*$ that is finitely terminal (resp. terminal) for class(es) of type structures. Definitions 7.4-7.5 below capture two prominent cases.

Definition 7.4 *Call an (X_a, X_b)-based type structure $\mathcal{T}^*$* ***finitely terminal*** *(resp.* ***terminal****) if it is finitely terminal (resp. terminal) for each (X_a, X_b)-based type structure $\mathcal{T}$.*

Definition 7.5 *Call an (X_a, X_b)-based type structure $\mathcal{T}^*$* ***finitely terminal for all finite structures*** *(resp.* ***terminal for all finite structures****) if it is finitely terminal (resp. terminal) for each (X_a, X_b)-type structure $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$ with T_a and T_b finite.*

Definition 7.4 says that, if $\mathcal{T}^*$ is finitely terminal (resp. terminal), then it induces all finite-order beliefs (resp. hierarchies of beliefs) associated with some type structure.[9] Definition 7.5 says that, if $\mathcal{T}^*$ is finitely terminal for all finite structures (resp. terminal for all finite structures), then it induces all finite-order beliefs (resp. hierarchies of beliefs) associated with type structures that have finite type sets. (Finite type structures capture the idea that there is a finite event (in the space of hierarchies of beliefs) that is commonly believed by the players.)

Note, if a type structure is terminal then it is finitely terminal; but, Proposition 8.3 (to come) will show that there is a finitely terminal type structure that is not terminal. The universal type structures constructed in Mertens and Zamir (1985), Brandenburger and Dekel (1993), and Heifetz and Samet (1998) are each terminal.

8 A Positive Result

This section demonstrates that, if the RCBR predictions differ from IU, then it must be because the type structure is not terminal. In particular, we will see that, if the RCBR predictions differ from IU, it is not terminal for all finite type structures—i.e., it does not induce all hierarchies of beliefs induced by finite type structures. To show this, we will show that, if a type structure is terminal for all finite type structures, the RCBR predictions coincide with the IU strategy set. Specifically:

Theorem 8.1 *Fix an epistemic game $(G, \mathcal{T})$.*

(i) If $\mathcal{T}$ is finitely terminal for all finite structures, then $\text{proj}_{S_a}\mathrm{R}_a^m \times \text{proj}_{S_b}\mathrm{R}_b^m = S_a^m \times S_b^m$ for each $m \geq 0$.

(ii) If $\mathcal{T}$ is terminal for all finite structures, then $\text{proj}_{S_a}\mathrm{R}_a^\infty \times \text{proj}_{S_b}\mathrm{R}_b^\infty = S_a^\infty \times S_b^\infty$.

Theorem 8.1 formalizes Result 2.7. It provides a novel epistemic justification for IU. Part (i) says that, if the type structure induces all finite-order beliefs associated with finite type structures,

[9] Here we use the phrase "terminal" in the spirit of Böge and Eisele's (1979) original usage. Some authors reserve the phrase "terminal" for a type structure that satisfies a stronger embedding property. (See, e.g., Meier, 2006.) No confusion should result.

then, for each m, the RmBR predictions are characterized by the $(m+1)$-undominated strategies. Part (ii) says that, if the type structure induces all hierarchies of beliefs associated with finite type structures, then the RCBR predictions are characterized by the IU strategies. (See Section 10.D for an important discussion of the interpretation.)

A terminal type structure is terminal for all finite structures. As a consequence, we have:

Corollary 8.2 *Fix an epistemic game $(G, \mathcal{T})$.*

(i) If $\mathcal{T}$ is finitely terminal, then $\text{proj}_{S_a} \mathrm{R}_a^m \times \text{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m$, for each $m \geq 0$.

(ii) If $\mathcal{T}$ is terminal, then $\text{proj}_{S_a} \mathrm{R}_a^\infty \times \text{proj}_{S_b} \mathrm{R}_b^\infty = S_a^\infty \times S_b^\infty$.

Note, Proposition 5.5(ii) is a special case of Corollary 8.2(ii). (Because the game is finite, a complete, compact, and continuous type structure is terminal. See Friedenberg 2010.) We will later see that Proposition 5.5(i) is a special case of 8.1(i). In particular, a complete type structure is finitely terminal for all finite structures. See Theorem 9.2 to come.

Taken together, Theorems 6.1-8.1 imply that there are complete type structures that are finitely terminal but not terminal for all finite structures. So, in particular, there are complete structures that induce all finite-order beliefs but do not induce all hierarchies of beliefs.

Proposition 8.3 *Suppose G is such that S_a and S_b are non-degenerate. Then there exists an (S_b, S_a)-based complete type structure $\mathcal{T}$ that is finitely terminal but not terminal for all finite structures.*

Proof. Since S_a and S_b are non-degenerate, there is a non-trivial game G' with strategy sets S_a and S_b. (Note, G may or may not be non-trivial itself.) So, by Theorem 6.1, there exists an (S_b, S_a)-based complete type structure $\mathcal{T} = (S_b, S_a; T_a, T_b; \beta_a, \beta_b)$ so that (i) T_a, T_b are Polish and (ii) $\mathrm{R}_a^\infty \times \mathrm{R}_b^\infty = \emptyset$. Since T_a, T_b are Polish, $\mathcal{T}$ is finitely terminal. (See Theorem 3.1 in Friedenberg, 2010.) Since $\mathrm{R}_a^\infty \times \mathrm{R}_b^\infty = \emptyset$, $\mathcal{T}$ is not terminal for all finite structures. (See Corollary 8.2.) ■

An implication of Proposition 8.3 is that, for any game G with S_a and S_b are non-degenerate, there exists an (S_b, S_a)-based complete type structure $\mathcal{T}$ that is finitely terminal but not terminal.

Remark 8.4 *Proposition 8.3 holds more generally. In particular:*

> *If X_a, X_b are Polish and non-degenerate, there exists an (X_a, X_b)-based complete type structure that is finitely terminal but not terminal for all finite structures.*

This is shown as Proposition D.5 in the Appendix.

9 Completeness and Finite Terminality

Proposition 5.5(i) stated that, in a complete type structure, the predictions of RmBR coincide with the $(m+1)$-undominated strategies. This section begins by providing a rationale for that result: A complete type structure is necessarily finitely terminal for all finite type structures. Theorem 9.2 will show that a complete type structure is finitely terminal for a broader set of type structures. However, we stop short of showing that a complete type structure is finitely terminal. Instead, Theorem 9.4 establishes a negative result: A complete type structure need not induce all finite-order beliefs—in fact, it need not even induce all second order beliefs.

It will be useful to begin with mathematical preliminaries. First, write $\mathfrak{c}$ for the cardinality of the continuum; so $|[0,1]| = \mathfrak{c}$. Next, we record some basic facts about probability measures on a metrizable space Ω. An **atom** of μ is a Borel set E with $\mu(E) > 0$ and $\mu(F) = \mu(E)$ for each Borel $F \subseteq E$. Call a probability measure $\mu \in \mathcal{P}(\Omega)$ **atomic** if every Borel set of positive measure contains an atom. If some countable set has measure one, then μ is atomic. If Ω is Polish, μ is atomic if and only if there is a countable set of measure one. If μ has countable support (i.e., there is a countable closed set of measure one), then μ is atomic; but there may also be atomic measures with uncountable support.

Definition 9.1 *Fix a type structure* $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$.

(i) Call $\mathcal{T}$ ***countable*** *if* T_a *and* T_b *are at most countable.*

(ii) Call $\mathcal{T}$ ***atomic*** *if, for each* c*, each type* $t_c \in T_c$*, and each* m*,* $\delta_c^m(t_c)$ *is atomic.*

Note, every type structure with finite type sets is countable. Moreover, if X_a, X_b are at most countable, then a countable type structure is atomic.[10] (See Corollary E.3.) Also note that a type structure may be atomic even if its beliefs are not atomic—that is, even if it is not the case that each $\beta_c(t_c)$ is atomic.[11]

9.1 Positive Result

In the spirit of Definition 7.5, an (X_a, X_b)-based type structure $\mathcal{T}^*$ **finitely terminal for all countable** (resp. **atomic**) **structures** if it is finitely terminal for each countable (resp. atomic) (X_a, X_b)-type structure $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$.

Theorem 9.2 *Fix a complete* (X_a, X_b)*-based type structure* $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$.

(i) The type structure $\mathcal{T}$ *is finitely terminal for all countable structures.*

[10] This is not true more generally. For instance, take $X_a = X_b = [0,1]$, $T_a = \{t_a\}$, $T_b = \{t_b\}$, and $\beta_a(t_a)$ non-atomic. Then $\delta_a^1(t_a)$ is not atomic.

[11] For instance, take $X_a = X_b = \{x\}$, $T_a = \{t_a\}$, $T_b = [0,1]$, and $\beta_a(t_a)$ non-atomic. Here, $\delta_a^1(t_a)$ is atomic.

(ii) If X_a and X_b have cardinality at most $\mathfrak{c}$, then the type structure $\mathcal{T}$ is finitely terminal for all atomic structures.[12]

Corollary 9.3 *If X_a and X_b are Polish, then every (X_a, X_b)-based complete type structure is finitely terminal for all atomic structures.*

Proof. By Theorem 9.2(ii) and the fact that every Polish space has cardinality at most $\mathfrak{c}$. ∎

These results raise the question: *Do Theorem 9.2 and Corollary 9.3 hold if we replace "finitely terminal" with "terminal"?* We do not know and leave it as an open question.

9.2 Negative Result

ZFC (Zermelo-Fraenkel set theory with choice) is the default axiom system in most of the mathematical literature. Our negative is a result about what cannot be proved in ZFC.

To state the result, it will be convenient to introduce the following terminology: Call $\mathcal{T}^*$ **2-terminal** if, for each (X_a, X_b)-based type structure $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$, each player c, and each $t_c \in T_c$, there is a type $t_c^* \in T_c^*$ so that $\delta_c^{*,2}(t_c^*) = \delta_c^2(t_c)$. So, informally, $\mathcal{T}^*$ is 2-terminal if it induces all second-order beliefs.

Theorem 9.4 *Suppose X_a and X_b are non-degenerate and have cardinality at most $\mathfrak{c}$. If ZFC is consistent, then one cannot prove in ZFC that every complete continuous (X_a, X_b)-based type structure is finitely terminal, or even 2-terminal.*

So, in particular, if X_a, X_b are non-degenerate Polish spaces and ZFC is consistent, then one cannot prove in ZFC that every complete continuous (X_a, X_b)-based type structure is finitely terminal.

Here, we provide a sketch of the proof. First, we show that any atomic type structure is not 2-terminal. We then use that fact to show that one cannot prove in ZFC that every complete continuous (X_a, X_b)-based type structure is finitely terminal.

Lemma 9.5 *If either X_a or X_b is non-degenerate, then no atomic (X_a, X_b)-based type structure is 2-terminal.*

Proof. Suppose X_a is non-degenerate and fix $x_a \neq y_a$ in X_a. We will construct an (X_a, X_b)-based type structure $\mathcal{T}$ and use that type structure to show that no atomic (X_a, X_b)-based type structure is 2-terminal.

Take T_a to be the Polish space $[0, 1]$ and take $T_b = \{t_b\}$. Let $\beta_a(t_a)(\{(x_a, t_b)\}) = t_a$ and $\beta_a(t_a)(\{(y_a, t_b)\}) = 1 - t_a$. Fix $x_b \in X_b$ and let $\beta_b(t_b)$ be the measure μ on $X_b \times [0, 1]$ such that $\mu(\{x_b\} \times [u, v]) = (v - u)$ for each $0 \leq u \leq v \leq 1$.

[12]The Appendix shows a stronger version of this result, where the hypothesis that $|X_a|, |X_b| \leq \mathfrak{c}$ is replaced by a much weaker hypothesis.

Note, for each $t_a \in T_a$, $\delta_a^1(t_a)$ is the measure with support $\{x_a, y_a\}$ that gives $\{x_a\}$ measure t_a. Moreover, $\rho_b^2(x_b, t_a) = (x_b, \delta_a^1(t_a))$. Therefore,

$$(\rho_b^2)^{-1}(\{(x_b, \delta_a^1(t_a))\}) = \{(x_b, t_a)\}.$$

It follows that, for each $z \in Z_b^2 = X_b \times \mathcal{P}(X_a)$, $(\rho_b^2)^{-1}(\{z\})$ has cardinality at most 1 and, so, has measure 0. With this,

$$\delta_b^2(t_b)(\{z\}) = \beta_b(t_b)((\rho_b^2)^{-1}(\{z\})) = 0.$$

Hence $\delta_b^2(t_b)$ is not atomic.

Fix an (X_a, X_b)-based atomic type structure $\mathcal{T}^*$. Observe that, for each $t_b^* \in T_b^*$, $\delta_b^{*,2}(t_b^*)$ is atomic. So $\delta_b^{*,2}(t_b^*) \neq \delta_b^2(t_b)$. Hence $\mathcal{T}^*$ is not 2-terminal for $\mathcal{T}$, and hence not 2-terminal. ■

In light of Lemma 9.5, it suffices to show the following: In ZFC, one cannot rule out the possibility that there is a complete atomic type structure. That is, in ZFC, one cannot prove that no complete type structure is atomic. To show this, we will make use of the fact that the following set-theoretic property is unprovable in ZFC.

Definition 9.6 *Say that $\mathfrak{c}$ is **atomlessly measurable** if the Lebesgue measure on $[0,1]$ can be extended to a probability measure on $([0,1], 2^{[0,1]})$.*

As explained in Fremlin (2008), if $\mathfrak{c}$ is atomlessly measurable then it is very large in the sense that there are many weakly inaccessible cardinals below $\mathfrak{c}$.

Lemma 9.7 (Fremlin 2008)

(i) If ZFC is consistent, then one cannot prove in ZFC that $\mathfrak{c}$ is atomlessly measurable.

(ii) If $\mathfrak{c}$ is not atomlessly measurable and X has cardinality $\mathfrak{c}$, then every probability measure on $(X, 2^X)$ has a countable set of measure one.

An implication of part (ii) is the following (shown in the appendix):

Lemma 9.8 *Suppose X_a, X_b are discrete topological spaces of cardinality $\leq \mathfrak{c}$. If $\mathfrak{c}$ is not atomlessly measurable, then there is a complete continuous atomic (X_a, X_b)-based type structure.*

Proof of Theorem 9.4. Assume ZFC is consistent. Let R be the statement that every complete continuous type structure is 2-terminal, and let S be the statement that $\mathfrak{c}$ is atomlessly measurable. We wish to show that R is not provable in ZFC. By the deduction theorem in logic, if R is provable in ZFC and the implication $R \Rightarrow S$ is provable in ZFC, then S is provable in ZFC. But by Lemma 9.7(i), S is not provable in ZFC. So it suffices show that $R \Rightarrow S$ is provable in ZFC.

We now work in ZFC and prove $R \Rightarrow S$. Suppose R holds. Let X_a, X_b be discrete topological spaces of cardinality $\mathfrak{c}$. Suppose $\mathcal{T}$ is a complete continuous (X_a, X_b)-based type structure. By R,

$\mathcal{T}$ is 2-terminal. Then by Lemma 9.5, $\mathcal{T}$ is not atomic. Hence there is no complete continuous atomic (X_a, X_b)-based type structure. Therefore, by Lemma 9.8, S holds. ■

Theorem 9.4 raises the question: *Can one prove, in ZFC, that there exists a complete continuous type structure that is* not *finitely terminal?* We don't know the answer and leave it as an open question.

10 Discussion

10.A Justifying IU From the Perspective of the Players We pointed out that Results 2.1-2.2 draw a connection between the RCBR predictions and IU. We argued that the connection was best understood from the perspective of an analyst that does not know the players' type structure and, so, seeks predictions across all type structures. In this paper, we follow a tradition in the literature and seek to justify RCBR from the perspective of the players' themselves.

We view our study of IU as a baseline—a first step in providing foundations for other dominance concepts. Two notable concepts that have received substantial attention are foundations for extensive-form rationalizability (due to Pearce 1984) and iterated admissibility (i.e., iterated deletion of weakly dominated strategies). The known foundations for these concepts rely on suitably "rich" type structures. (See Battigalli and Siniscalchi, 2002, Brandenburger, Friedenberg and Keisler, 2008, Lee, 2016.) In particular, an analyst that applies the given epistemic conditions across all type structures may well have new predictions (i.e., predictions inconsistent with extensive-form rationalizability or iterated admissibility). (See Battigalli and Siniscalchi, 2002, Battigalli and Friedenberg, 2012*b*, and Brandenburger, Friedenberg and Keisler, 2008.) In fact, recent work by Catonini and De Vito (2018) uses Theorem 6.1 to prove a negative result on weak dominance. Moreover, in the spirit of our Theorem 8.1, Catonini and De Vito (2018)-Catonini and De Vito (2019) also prove positive results by studying "richness" conditions associated with terminality.

It is worth noting that continuity plays an important role in the literature on weak dominance. In particular, compare the output of Theorem 10.1 in Brandenburger, Friedenberg and Keisler (2008) versus the output of Theorems 3.2-3.4 in Keisler and Lee (2011). There is a striking difference: In the former case, there is no prediction whereas in the latter case the output is the set of strategies that survive iterated weak dominance (i.e., maximal simultaneous deletion of weakly dominated strategies). Yet, the input is remarkably similar, differing only based on a complete and continuous type structure versus a complete and discontinuous type structure. This raises the question: How do these type structures differ in terms of players' reasoning (i.e., hierarchies of beliefs)? This is an open question. The hope is that the ideas here are a step toward answering this question.

10.B Revisiting Epistemic Justifications for IU In Section 2.4, we pointed out that an epistemic justification for IU requires a "richness" requirement—i.e., that the players consider enough beliefs possible. We followed the literature and focused on the case where the richness condition is associated with either a universal type structure or a complete type structure.

One might ask whether there is not an alternate route given by Result 2.2. That result tells us that, for a given game and IU set thereof, we can tailor the type structure to be "rich enough," and, thereby, get the IU strategies as an output of RCBR. Yet, this does not give a satisfactory answer to the question at hand. To understand why, note that to get the type structure (as constructed by Brandenburger and Dekel, 1987), we begin with a particular game and use properties of the IU set to construct the type structure in question. In particular, the constructed type structure will typically be different for two games with the same strategy set but different payoff functions. In fact, it can be different for two games, with the same IU set, but whose payoff functions differ on the IU set. These differences in construction rely on differences in how the IU concept is applied to different games. Thus, if we took this richness condition to be the type structure constructed in Result 2.2, we would be justifying IU by making reference to IU itself. More informally, we would be rigging the assumptions to deliver the desired conclusions.[13] For this reason, we follow the literature and do not take this route.

10.C Topologies and Substantive Assumptions about Reasoning This paper highlights the fact that topological assumptions on the type structure may implicitly impose important substantive assumptions on players' reasoning. This message is reminiscent of—but distinct from—the goal of the so-called "topology-free approach to type structures." (See, e.g., Heifetz and Samet, 1998, 1999 and subsequent work.) To better understand the message here, it will be useful to contrast it with that literature.

That literature begins with a primitive set of uncertainty, e.g., X_a and X_b, that may or may not be a topological space. It then explicitly imposes a sigma-algebra on the set of beliefs on this set, e.g., on $\mathcal{P}(X_a)$ and $\mathcal{P}(X_b)$. The sigma-algebra on $\mathcal{P}(X_b)$ reflects the second-order events that Ann can reason about; for instance, Ann can reason about events of the form "Bob assigns probability at least p to $E_b \subseteq X_b$." In the specific case where X_b is Polish, the sigma-algebra (on $\mathcal{P}(X_b)$) coincides with the Borel sigma-algebra.[14] This, in turn, has implications for constructing type structures. For instance, absent topological assumptions on (X_a, X_b), there may be a hierarchy of beliefs that cannot be induced by any type structure. (See, Heifetz and Samet, 1999.) However, even in that case, there is a type structure that is terminal (i.e., that can induce any hierarchy of beliefs that is induced in some type structure). (See Heifetz and Samet, 1998.)

Here, we are primarily concerned with primitive sets of uncertainty that are topological—e.g., X_a, X_b are non-degenerate finite sets endowed with the discrete topology. Instead of using an explicit model of (X_a, X_b)-based hierarchies of beliefs, we begin with the type structure model. Standard measurability requirements alone are sufficient for type structures to induce hierarchies

[13]Notice, if we allowed ourselves to justify a solution concept in this way, we would decide (based on intuition) which behavior is "desirable" for a given game, and tailor an epistemic condition for that game so that by definition it delivers the "desirable" behavior. There would be no discipline to the epistemic programme.

[14]Provided $\mathcal{P}(X_b)$ is endowed with the weak topology.

of beliefs about (X_a, X_b). It is not clear that the presence or absence of topological assumptions on the *type structure* should have meaning. However, the results here show that they do—at least, in conjunction with completeness: Complete type structures need not induce all hierarchies of beliefs; however, they do, if the type structure is compact and continuous. Moreover, within ZFC, it cannot be proven that every complete type structure induce all finite-order beliefs; however, they do, if we focus on type structures that have analytic type sets. (See Friedenberg, 2010.)

To sum up, this paper focuses on the type structure model itself and shows that adding topological assumptions to that model implicitly impose substantive assumptions about players' reasoning. So, in particular, two seemingly "equivalent" models may, in fact, induce different hierarchies of beliefs. The results serve as a warning to the analyst who uses the type structure model, in that seemingly irrelevant modeling assumptions may have important implications. With this in mind, the message here is more closely related to that in Brandenburger and Keisler (2006), Friedenberg (2010), Friedenberg and Meier (2010), and Kets (2010).

10.D Terminal for all Finite Structures and Foundations for IU Theorem 8.1 provides an epistemic justification for IU based on the concept of "terminal for all finite structures." The universal type structure is a special case of a type structure that is terminal for all finite structures. Thus, Result 2.3 (Tan and Werlang, 1988) is a special case of Theorem 8.1.

Does there exist an (X_a, X_b)-based type structure that is terminal for all finite structures but which is *not* the universal type structure? If the answer were "no," then the epistemic justification provided by Theorem 8.1 would coincide with those provided by Result 2.3 (Tan and Werlang, 1988). Corollary F.3 shows this is not the case: For any (finite) game G, there exists an associated type structure that is terminal for all finite structures but not terminal for all countable structures—*a fortiori* not terminal.

The proof constructs a type structure that is terminal for all finite structures. The construction is analogous to constructions in Heifetz and Samet 1998 and Yildiz 2015, amongst others. It rules out all hierarchies of beliefs that are not contained in a finite-belief closed subset of the universal type structure. So, in particular, it rules out hierarchies of beliefs that are associated with the so-called staircase construction—i.e., where, for each $m \geq 1$, there is some event that is m^{th}-order belief but not $(m+1)^{th}$-order belief.[15] Thus, in a sense, each type in the constructed structure has a hierarchy of belief that is "determined" by some finite-order belief.

10.E Relationship to Dufwenberg and Stegeman (2002) The negative result is reminiscent of—but distinct from—a negative result in Dufwenberg and Stegeman (2002). They provide an example of an infinite game, with a compact metric strategy space and a discontinuous payoff function. That discontinuity leads to the m-undominated strategy sets, viz. S_c^m, to be half-closed intervals and the intersection of those sets to be empty. Here, we begin with a finite strategy set, so there are no discontinuities in the payoff functions. Thus, there is an iteratively undominated

[15] See Geanakoplos and Polemarchakis, 1982, Rubinstein, 1989, Aumann and Brandenburger, 1995, and Kajii and Morris, 1997 for prominent examples of such type structures.

strategy s_c^*. Theorem (ii) looks at a given complete type structure with compact metric type spaces and discontinuous belief functions and asks whether s_c^* is consistent with RCBR. The answer is no. The key is that, within that type structure, the sets $\mathbb{T}[s_c^*, m]$—i.e., the set of types t_c so that (s_c^*, t_c) is consistent with R$(m-1)$BR—have empty intersection. This can only occur because those sets are not closed.

Appendix A Mathematical Preliminaries

Lemma A.1 *Let Ω and Φ be metrizable and $f : \Omega \to \Phi$ be measurable. Then $\underline{f}$ is measurable.*

Proof. Note, an open sub-basis for $\mathcal{P}(\Phi)$ is given by the family of sets of the form

$$U(\overline{\nu}, G, \varepsilon) = \{\nu \in \mathcal{P}(\Phi) : \nu(G) > \overline{\nu}(G) - \varepsilon\},$$

for $\overline{\nu} \in \mathcal{P}(\Omega)$, G open in Φ, and $\varepsilon > 0$. (See Billingsley, 1968, page 236.) It suffices to show that, for each set $U = U(\overline{\nu}, G, \varepsilon)$ in this open sub-basis, $\underline{f}^{-1}(U)$ is Borel in $\mathcal{P}(\Omega)$.

Fix $U = U(\overline{\nu}, G, \varepsilon)$. Let $r = \overline{\nu}(G) - \varepsilon$ and note that $f^{-1}(G)$ is Borel in Ω. With this, $\mu \in \underline{f}^{-1}(U)$ if and only if $\underline{f}(\mu) \in U$ if and only if $\underline{f}(\mu)(G) > r$ if and only if $\mu(f^{-1}(G)) > r$. So, by Lemma 15.16 in Aliprantis and Border (2007), $\underline{f}^{-1}(U)$ is Borel in $\mathcal{P}(\Omega)$. ■

Lemma A.2 *Let Ω and Φ be Polish and $f : \Omega \to \Phi$ be measurable and bijective. Then $\underline{f}$ is bijective.*

Proof. The follows from the proof of Theorem 14.14 (2)-(3) in Aliprantis and Border (2007). ■

Appendix B Proofs for Section 5

Lemma 5.4 and Proposition 5.5 are well-known in environments with stronger topological structure. We show here that the stronger topological structure is not important.

Remark B.1 *A strategy $s_c \in Y_c$ is undominated given $Y_a \times Y_b$ if and only if there exists some $\sigma_d \in \mathcal{P}(S_d)$ with (i) $\pi_c(s_c, \sigma_d) \geq \pi_c(r_c, \sigma_d)$, for all $r_c \in Y_c$, and (ii) $\sigma_d(Y_d) = 1$.*

Proof of Lemma 5.4. Part (ii) is immediate from part (i). We show part (i) by induction on m.

$\mathbf{m = 1}:$ If $(s_c, t_c) \in \mathrm{R}_c^1$ then s_c is optimal under $\mathrm{marg}_{S_d}\beta_c(t_c)$. So, by Remark B.1, s_c is undominated.

$\mathbf{m \geq 2}:$ Assume the result for m. If $(s_c, t_c) \in \mathrm{R}_c^{m+1}$ then s_c is optimal under $\mathrm{marg}_{S_d}\beta_c(t_c)$ and $\beta_c(\mathrm{R}_d^m) = 1$. It follows that $\mathrm{marg}_{S_d}\beta_c(t_c)(\mathrm{proj}_{S_d}\mathrm{R}_d^m) = 1$. So, by the induction hypothesis, $\mathrm{marg}_{S_d}\beta_c(t_c)(S_d^m) = 1$. Now the claim follows from Remark B.1. ■

Proof of Proposition 5.5. Part (i) follows from Theorem 8.1(i) and Theorem 9.2 (i). Part (ii) follows from Corollary 8.2(ii) and Friedenberg (2010). ■

Appendix C Proofs for Section 6

The following lemmas were used in the proof of Lemma 6.6 , which was in turn used in the proof of Theorem 6.2. We follow the notation introduced in Section 6.

Lemma C.1 *The collection*

$$\{\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1] : Q_c \in \mathbb{S}_c^1 \textit{ and } m \geq 1\}$$

partitions T_c.

Proof. Clearly, elements of the collection are pairwise disjoint. It suffices to show that each t_c is contained in $\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1]$, for some $Q_c \in \mathbb{S}_c^1$ and some $m \geq 1$. Certainly, each $t_c \in \mathrm{T}[Q_c, 1]$, for some Q_c. Moreover, using the fact that $\bigcap_m \mathrm{T}[Q_c, m] = \emptyset$ (Property 6.4(ii)), it follows that there is some m^* so that $t_c \notin \mathrm{T}[Q_c, m^*]$ and, so, there exists some m so that $t_c \in \mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1]$. ■

Lemma C.2 *The collection of sets of probabilities*

$$\{(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c) : Q_c \in \mathbb{S}_c^1 \textit{ and } m \geq 1\}.$$

partitions $\mathcal{P}(S_d \times T_d)$.

Proof. Clearly, elements of the collection are pairwise disjoint. It suffices to show that each ψ_d is contained in $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ for some $Q_c \in \mathbb{S}_c^1$ and some $m \geq 1$. To show this, note that each $\psi_d \in (\mathbb{BR}_c^1)^{-1}(Q_c)$ for some Q_c. Moreover, using Property 6.4(ii), it can be seen that $\bigcap_m P_d^m = \emptyset$. So, no ψ_d can believe each P_d^m, i.e., there must be some m^* so that $\psi_d \notin (\mathbb{BR}_c^{m^*})^{-1}(Q_c)$. So, for each ψ_d, there exists some m with ψ_d contained in $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$. ■

Lemma C.3 *Fix some* $Q_c \in \mathbb{S}_c^1$ *and some* $m \geq 1$. *The set* $(\mathbb{BR}_c^m)^{-1}(Q_c)$ *is closed.*

Proof. Note,

$$(\mathbb{BR}_c^m)^{-1}(Q_c) = \{\psi_d : \mathbb{BR}_c^1(\psi_d) = Q_c\} \cap \{\psi_d : \psi_d(P_c^{m-1}) = 1\}.$$

A standard application of the Portmanteau Theorem (Kechris, 1995, Theorem 17.20(i)-(ii)) gives that the set $\{\psi_d : \mathbb{BR}_c^1(\psi_d) = Q_c\}$ is closed. It follows from Property 6.4(iii) that P_c^{m-1} is clopen. So, the Portmanteau Theorem (Kechris, 1995, Theorem 17.20(i)-(v)) gives that the set $\{\psi_d : \psi_d(P_c^{m-1}) = 1\}$ is closed. This establishes the result. ■

Lemma C.4 *Fix* $Q_c \in \mathbb{S}_c^1$ *and* $m \geq 1$. *The set* $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ *is a Polish space.*

Proof. It suffices to show that $(\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is a G_δ set, i.e., a countable intersection of open sets. If so, then the claim follows from Kechris (1995, Theorem 3.11).

Note,

$$(\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}(Q_c) = (\mathbb{BR}_c^m)^{-1}(Q_c) \cap [\mathcal{P}(S_d \times T_d)\backslash\{\psi_d : \psi_d(P_c^m) = 1\}].$$

By Lemma C.3 and Proposition 3.7 in Kechris (1995), $(\mathbb{BR}_c^m)^{-1}(Q_c)$ is a G_δ set. The Portmanteau Theorem (Kechris, 1995, Theorem 17.20(i)-(v)) gives that the set $\{\psi_d : \psi_d(P_c^m) = 1\}$ is closed, i.e., $\mathcal{P}(S_d \times T_d)\backslash\{\psi_d : \psi_d(P_c^m) = 1\}$ is open. So, $(\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is a countable intersection of open sets, as required. ■

Lemma C.5 *For each $m \geq 0$ and each $s_c \in S_c^m$, there exists a type t_c so that $(s_c, t_c) \in P_c^m \backslash P_c^{m+1}$.*

Proof. Throughout, fix $s_c \in S_c^m$. We break the proof into two cases.

$\mathbf{m = 0}$: Since the game is non-trivial, there exists some σ_d so that s_c is not optimal under σ_d, i.e., $s_c \notin \mathrm{BR}_c^1(\sigma_d)$. For any $t_c \in \mathrm{T}[\mathrm{BR}_c^1(\sigma_d), 1]$, $(s_c, t_c) \in P_c^0 \backslash P_c^1$.

$\mathbf{m \geq 1}$: Using Property 6.3, there exists some $Q_c \in \mathbb{S}_c^m$ so that $s_c \in Q_c$. It follows from the construction that there is some $t_c \in \mathrm{T}[Q_c, m]\backslash \mathrm{T}[Q_c, m+1]$. So, $(s_c, t_c) \in P_c^m \backslash P_c^{m+1}$. ■

Lemma C.6 *For each $Q_c \in \mathbb{S}_c^1$ and m, $\mathrm{T}[Q_c, m]\backslash \mathrm{T}[Q_c, m+1]$ is nonempty if and only if $(\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}$ is nonempty.*

Proof. Note, by construction, $\mathrm{T}[Q_c, m]\backslash \mathrm{T}[Q_c, m+1]$ is nonempty if and only if $\mathrm{T}[Q_c, m]$ is nonempty or, equivalently, if and only if $Q_c \in \mathbb{S}_c^m$. So, it suffices to show that $Q_c \in \mathbb{S}_c^m$ if and only if $(\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is nonempty.

First, suppose that $Q_c \in \mathbb{S}_c^m$. Then, there exists some $\sigma_d \in \mathcal{P}(S_d)$ so that $\mathrm{BR}_c^1(\sigma_d) = Q_c$ and $\sigma_d(S_d^{m-1}) = 1$. We will show that we can find some ψ_d with $\mathbb{BR}_c^m(\psi_d) = Q_c$ but $\psi_d(P_d^m) \neq 1$. This will establish the result.

Note, by Lemma C.5, there exists a mapping $g : S_d \to T_d$ so that, for each $s_d \in S_d^m$, $g(s_d) \in P_d^{m-1}\backslash P_d^m$. Construct ψ_d so that $\psi_d(\{s_d, g(s_d)\}) = \sigma_d(\{s_d\})$, for all s_d. Then $\mathrm{marg}_{S_d}\psi_d = \sigma_d$, $\psi_d(P_d^{m-1}) = 1$, and $\psi_d(P_d^m) \neq 1$, as required.

Now suppose that $\psi_d \in (\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}(Q_c)$. Then, $\mathrm{BR}_c^1(\mathrm{marg}_{S_d}\psi_d) = Q_c$ and $\psi_d(Q_d^{m-1}) = 1$. Using Lemma 6.5, $\mathrm{marg}_{S_d}\psi_d(S_d^{m-1}) = 1$. This establishes that $Q_c \in \mathbb{S}_c^m$. ■

Lemma C.7 *Fix Polish spaces Ω and Φ. Let $\{\Omega^1, \Omega^2, \ldots\}$ be a partition of Ω and $\{\Phi^1, \Phi^2, \ldots\}$ be a partition of Φ so that each Ω^n and Φ^n is clopen. Suppose, for each n, there is a continuous map $f^n : \Omega^n \to \Phi^n$. Let $f : \Omega \to \Phi$ be such that $f(\omega) = f^n(\omega)$ if $\omega \in \Omega^n$. Then, f is continuous.*

Proof. Since Ω is Polish, it suffices to show that, if ω_k converges to ω in Ω, then $f(\omega_k)$ converges to $f(\omega)$ in Φ. To show this, note that there exists some n so that so that $\omega \in \Omega^n$ and $f(\omega) \in \Phi^n$. Since Ω^n is clopen, $\omega_k \in \Omega^n$, for k large. So, by construction, $f(\omega_k) \in \Phi^n$ for k sufficiently large. Using the fact that f is continuous on Ω^n, it follows that $f(\omega_k)$ converges to $f(\omega)$, as required. ■

Appendix D Proofs for Section 8

This Appendix first proves Theorem 8.1 and then turns to prove Remark 8.4.

D.1 Proof of Theorem 8.1.

Let us give the idea of the proof. We begin with an argument from Brandenburger and Dekel (1987): There is an epistemic game $(G, \mathcal{T})$ with a finite type structure $\mathcal{T}$ so that the RCBR prediction is the set of IU strategies.[16] (See Lemma D.1.) The key is that the RmBR set is preserved when going from $\mathcal{T}$ to any type structure $\mathcal{T}^*$ that is finitely terminal for $\mathcal{T}$. (See Lemma D.2.) Taken together, this implies that, if a type structure is finitely terminal (resp. terminal) for all finite structures, then the RmBR (resp. RCBR) prediction contains the IU strategy set. (See Lemma D.4.)

Lemma D.1 *There exists an (S_b, S_a)-based type structure $\mathcal{T}$ with $|T_a| \leq |S_a|$ and $|T_b| \leq |S_b|$ so that*

(i) For each m, $\text{proj}_{S_a}\text{R}_a^m \times \text{proj}_{S_b}\text{R}_b^m = S_a^m \times S_b^m$.

(ii) $\text{proj}_{S_a}\text{R}_a^\infty \times \text{proj}_{S_b}\text{R}_b^\infty = S_a^\infty \times S_b^\infty$.

To prove Lemma D.1, it will be useful to introduce some notation. Because the game is finite, we can find an $M < \infty$ so that $S_a^M \times S_b^M = S_a^{M-1} \times S_b^{M-1} = S_a^\infty \times S_b^\infty$. Fix one such M. For each $s_c \in S_c^1$, let

$$\overline{m}(s_c) = \begin{cases} m & \text{if } s_c \in S_c^m \backslash S_c^{m+1} \\ M & \text{if } s_c \in S_c^\infty. \end{cases}$$

Now construct a mapping $p_c : S_c^1 \to \mathcal{P}(S_d)$ so that, for each $s_c \in S_c^1$, (i) s_c is optimal under $p_c(s_c)$, and (ii) $p_c(s_c)$ assigns probability 1 to $S_d^{\overline{m}(s_c)-1}$. (See Remark B.1.) Note, two features of this construction. First, if $s_c \in S_c^2$, $p_c(s_c)(S_d^1) = 1$. Second, if $s_c \in S_c^\infty$, then $p_c(s_c)(S_d^\infty) = p_c(s_c)(S_d^{M-1}) = 1$.

Proof of Lemma D.1. Construct a type structure $\mathcal{T}$ as follows: Take each $T_c = S_c^1$. For each $s_c \in T_c = S_c^1$, let $\beta_c(s_c) \in \mathcal{P}(S_d \times S_d^1)$ so that (i) for each $s_d \in S_d^1$, $\beta_c(s_c)(s_d, s_d) = p_c(s_c)(s_d)$, and (ii) for each $s_d \notin S_d^1$, $\beta_c(s_c)(\{s_d\} \times S_d^1) = p_c(s_c)(s_d)$.

It suffices to show the following:

(i) If $s_c \in S_c^m$, then $(s_c, s_c) \in R_c^m$.

(ii) If $(s_c, t_c) \in R_c^m$, then $s_c \in S_c^m$.

[16] The result of Brandenburger and Dekel (1987) pertains to "common knowledge of rationality" and not to RCBR, as stated here. The proof of Lemma D.1 follows the proof of Theorem 10.1(ii) in Brandenburger and Friedenberg (2008).

The proof is by induction on m.

$\mathbf{m=1}$: Part (i) is immediate. For Part (ii), fix $(s_c, t_c) \in R_c^1$. By construction, s_c is optimal under $\operatorname{marg}_{S_d}\beta_c(t_c) = p_c(t_c)$. So $s_c \in S_c^1$.

$\mathbf{m \geq 2}$: To show Part (i), fix $s_c \in S_c^{m+1}$. By the induction hypothesis, $(s_c, s_c) \in R_c^m$. So, it suffices to show that $s_c \in T_c$ believes R_d^{m-1}. Toward that end note that $p_c(s_c)(S_d^{m-1}) = 1$. So, by construction, $\beta_c(s_c)(s_d, t_d) > 0$ if and only if $s_d = t_d$. Then, by the induction hypothesis, $\beta_c(s_c)(R_d^{m-1}) = 1$.

To show Part (ii), fix $(s_c, t_c) \in R_c^{m+1}$. By construction, s_c is optimal under $\operatorname{marg}_{S_d}\beta_c(t_c) = p_c(t_c)$. Moreover, since $\beta_c(t_c)(R_d^m) = 1$, the induction hypothesis gives that $p_c(t_c)(S_d^m) = 1$. Thus, $p_c(t_c)(S_d^n) = 1$ for each $n = 0, \ldots, m$. From this, $s_c \in S_c^{m+1}$. ■

Lemma D.2 *Fix (S_b, S_a)-based type structures*

$$\mathcal{T} = (S_b, S_a; T_a, T_b; \beta_a, \beta_b) \quad \textit{and} \quad \mathcal{T}^* = (S_b, S_a; T_a^*, T_b^*; \beta_a^*, \beta_b^*),$$

with T_a, T_b are finite. If $\delta_c^m(t_c) = \delta_c^{*,m}(t_c^*)$ and $(s_c, t_c) \in R_c^m$, then $(s_c, t_c^*) \in R_c^{*,m}$.

In the proof of Lemma D.2, we will make use of the following:

Remark D.3 *Types induce so-called coherent hierarchies of beliefs. In particular, for each $t_c \in T_c$ and each $m \geq n \geq 1$, $\operatorname{marg}_{Z_c^n}\delta_c^m(t_c) = \delta_c^n(t_c)$.*

Proof of Lemma D.2. It suffices to show the following:

(i) If $\delta_c^m(t_c) = \delta_c^{*,m}(t_c^*)$, then $(s_c, t_c) \in R_c^m$ only if $(s_c, t_c^*) \in R_c^{*,m}$.

(ii) $(\rho_d^{*,m+1})^{-1}(\rho_d^{m+1}(\mathrm{R}_c^m)) \subseteq \mathrm{R}_c^{*,m}$.

The proof is by induction on m.

$\mathbf{m = 1}$: Begin with part (i). Suppose $\delta_c^1(t_c) = \delta_c^{*,1}(t_c^*)$. Note, $\delta_c^1(t_c) = \operatorname{marg}_{S_d}\beta_c(t_c)$ and $\delta_c^{*,1}(t_c^*) = \operatorname{marg}_{S_d}\beta_c^*(t_c^*)$. So, $(s_c, t_c) \in \mathrm{R}_c^1$ if and only if $(s_c, t_c^*) \in \mathrm{R}_c^{*,1}$.

For part (ii), fix $(s_c, t_c^*) \in (\rho_d^{*,2})^{-1}(\rho_d^2(\mathrm{R}_c^1))$. Then, there exists some t_c such that $(s_c, t_c) \in \mathrm{R}_c^1$ and $\rho_d^{*,2}(s_c, t_c^*) = \rho_d^2(s_c, t_c)$. So, $\delta_c^{*,1}(t_c^*) = \delta_c^1(t_c)$. Then, by part (i) (established for $m = 1$), $(s_c, t_c^*) \in \mathrm{R}_c^{*,1}$.

$\mathbf{m \geq 2}$: Assume the result holds for m and we will show that it also holds for $m+1$.

To prove part (i), suppose $\delta_c^{m+1}(t_c) = \delta_c^{*,m+1}(t_c^*)$ and $(s_c, t_c) \in \mathrm{R}_c^{m+1}$. By Remark D.3, $\delta_c^m(t_c) = \delta_c^{*,m}(t_c^*)$. So, by part (i) of the induction hypothesis, $(s_c, t_c^*) \in \mathrm{R}_c^{*,m}$. As such, it suffices to show that t_c^* believes $\mathrm{R}_d^{*,m}$.

Since $(s_c, t_c) \in \mathrm{R}_c^{m+1}$, $\beta_c(t_c)(\mathrm{R}_d^m) = 1$. Note, $\mathrm{R}_d^m \subseteq S_d \times T_d$ is finite and, so, $\rho_c^{m+1}(\mathrm{R}_d^m)$ is Borel. As such,

$$\delta_c^{m+1}(t_c)(\rho_c^{m+1}(\mathrm{R}_d^m)) = \beta_c(t_c)((\rho_c^{m+1})^{-1}(\rho_c^{m+1}(\mathrm{R}_d^m))) \geq \beta_c(t_c)(\mathrm{R}_d^m) = 1.$$

Using the fact that $\delta_c^{*,m+1}(t_c^*) = \delta_c^{m+1}(t_c)$,

$$\delta_c^{*,m+1}(t_c^*)(\rho_c^{m+1}(\mathrm{R}_d^m)) = 1.$$

So,

$$\beta_c^*(t_c^*)((\rho_c^{*,m+1})^{-1}(\rho_c^{m+1}(\mathrm{R}_d^m))) = 1.$$

By part (ii) of the induction hypothesis, $(\rho_c^{*,m+1})^{-1}(\rho_c^{m+1}(\mathrm{R}_d^m)) \subseteq \mathrm{R}_d^{*,m}$, so that $\beta_c^*(t_c^*)(\mathrm{R}_d^{*,m}) = 1$, as desired.

To prove part (ii), fix $(s_c, t_c^*) \in (\rho_d^{*,m+2})^{-1}(\rho_d^{m+2}(\mathrm{R}_c^{m+1}))$. There exists some $t_c \in T_c$ so that $(s_c, t_c) \in \mathrm{R}_c^{m+1}$ and $\rho_d^{*,m+2}(s_c, t_c^*) = \rho_d^{m+2}(s_c, t_c)$. So, $\delta_c^{*,m+1}(t_c^*) = \delta_c^{m+1}(t_c)$. By part (i) established for $m+1$, $(s_c, t_c^*) \in \mathrm{R}_c^{*,m+1}$. ■

Lemma D.4 *Fix an epistemic game $(G, \mathcal{T}^*)$, where $\mathcal{T}^*$ is finitely terminal for all finite structures.*

(i) For each m, $S_a^m \times S_b^m \subseteq \mathrm{proj}_{S_a} \mathrm{R}_a^{*,m} \times \mathrm{proj}_{S_b} \mathrm{R}_b^{*,m}$.

(ii) If $\mathcal{T}^*$ is terminal for all finite structures, then $S_a^\infty \times S_b^\infty \subseteq \mathrm{proj}_{S_a} \mathrm{R}_a^{*,\infty} \times \mathrm{proj}_{S_b} \mathrm{R}_b^{*,\infty}$.

Proof. Suppose $\mathcal{T}^* = (S_b, S_a; T_a^*, T_b^*, \beta_a^*, \beta_b^*)$ is finitely terminal for all finite type structures. Observe, by Lemma D.1, there is a type structure $\mathcal{T} = (S_b, S_a; T_a, T_b, \beta_a, \beta_b)$ with finite type spaces so that

- For each m, $\mathrm{proj}_{S_a} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m$.
- $\mathrm{proj}_{S_a} \mathrm{R}_a^\infty \times \mathrm{proj}_{S_b} \mathrm{R}_b^\infty = S_a^\infty \times S_b^\infty$.

Part (i): Fix $s_c \in S_c^m$. There exists some $t_c \in T_c$ so that $(s_c, t_c) \in \mathrm{R}_c^m$. Since T_a and T_b are finite and $\mathcal{T}^*$ is finitely terminal for all finite structures, there is some $t_c^* \in T_c^*$ with $\delta_c^{*,m}(t_c^*) = \delta_c^m(t_c)$. By Lemma D.2, $(s_c, t_c^*) \in \mathrm{R}_c^{*,m}$.

Part (ii): Suppose $\mathcal{T}^*$ is terminal for all finite structures. Fix $s_c \in S_c^\infty$. There exists some $t_c \in T_c$ so that $(s_c, t_c) \in \mathrm{R}_c^\infty$. Since T_a and T_b are finite and $\mathcal{T}^*$ is terminal for all finite structures, then there exists $t_c^* \in T_c^*$ with $\delta_c^*(t_c^*) = \delta_c(t_c)$. By Lemma D.2, $(s_c, t_c^*) \in \mathrm{R}_c^{*,\infty}$. ■

Proof of Theorem 8.1. Immediate from Lemmas 5.4 and D.4. ■

D.2 Complete and Not Finitely Terminal

We now turn to prove the claim in Remark 8.4:

Proposition D.5 *Suppose X_a and X_b are Polish and non-degenerate. Then there exists an (X_a, X_b)-based Polish complete type structure $\mathcal{T}$ that is finitely terminal but not terminal for all finite structures.*

Proof. We begin by constructing a game G: Let $S_a = X_b$ and $S_b = X_a$. Since each X_d is non-degenerate, there are distinct strategies $s'_c, s''_c \in S_c = X_d$. Construct π_c as follows:

$$\pi_c(s_c, s_d) = \begin{cases} 1 & \text{if } (s_c, s_d) \in \{(s'_c, s''_d), (s''_c, s'_d)\} \\ 2 & \text{if } (s_c, s_d) \in (\{s'_c, s''_c\} \times S_d) \backslash \{(s'_c, s''_d), (s''_c, s'_d)\} \\ 0 & \text{otherwise.} \end{cases}$$

Note, for each c and each $m \geq 1$, $S_c^m = \{s'_c, s''_c\}$ and $\mathbb{S}_c^m = \{\{s'_c\}, \{s''_c\}, \{s'_c, s''_c\}\}$.

Since, for $m \geq 1$, the sets S_c^m and $\mathbb{S}_c^m$ are each finite, there exists a a Polish and complete type structure $\mathcal{T} = (S_b, S_a; T_b, T_b; \beta_a, \beta_b)$ so that, in the epistemic game $(G, \mathcal{T})$, $\mathrm{R}_a^\infty \times \mathrm{R}_b^\infty = \emptyset$. (This amounts to repeating the proof of Theorem 6.1 line-by-line.) By Theorem 3.1 in Friedenberg (2010), $\mathcal{T}$ is finitely terminal. It remains to show that $\mathcal{T}$ is not terminal for all finite structures.

To do so, we show: If $(G, \mathcal{T}^*)$ is an epistemic game where $\mathcal{T}^*$ is terminal for all finite structures, then $\mathrm{R}_a^{*,\infty} \times \mathrm{R}_b^{*,\infty} \neq \emptyset$. For that, we note that there exists an epistemic game $(G, \hat{\mathcal{T}}) = (S_b, S_a; \hat{T}_b, \hat{T}_b; \hat{\beta}_a, \hat{\beta}_b)$ with $|\hat{T}_a| = |S_a^1| = 2$ and $|\hat{T}_b| = |S_b^1| = 2$ so that $\hat{\mathrm{R}}_a^\infty \times \hat{\mathrm{R}}_b^\infty = S_a^\infty \times S_b^\infty$. (Repeat the proof of Lemma D.1 which takes each $T_c = S_c^1$.) Then the proof of Lemma D.4 applies and we obtain that $S_a^\infty \times S_b^\infty \subseteq \mathrm{R}_a^{*,\infty} \times \mathrm{R}_b^{*,\infty}$. ■

Appendix E Proofs for Section 9

We divide this appendix into three parts. First, we record a mathematical fact that will be useful. Second, we show the positive result mentioned in the main text; in fact, we generalize the result. Third, we complete the proof of the negative result.

E.1 Preliminary Results

We begin with preliminary results that will be of use throughout this Appendix.

Lemma E.1 *If $\beta_c(t_c)$ has a countable set of measure one then, for each m, $\delta_c^m(t_c)$ is atomic.*

Proof. Let $t_c \in T_c$ so that there is a countable set $E \subseteq X_c \times T_d$ with $\beta_c(t_c)(E) = 1$. Then, the image $\rho_c^m(E)$ is countable and $E \subseteq (\rho_c^m)^{-1}(\rho_c^m(E))$. So,

$$\delta_c^m(t_c)(\rho_c^m(E)) = \beta_c(t_c)((\rho_c^m)^{-1}(\rho_c^m(E))) \geq \beta_c(t_c)(E) = 1.$$

As such, $\delta_c^m(t_c)$ is atomic. ■

Corollary E.2 *Let $\mathcal{T}$ be a type structure such that, for each c and each t_c, $\beta_c(t_c)$ has a countable set of measure one. Then $\mathcal{T}$ is atomic.*

Corollary E.3 *If X_a, X_b are at most countable, then every countable (X_a, X_b)-based type structure $\mathcal{T}$ is atomic.*

We will need the following concept:

Definition E.4 *Fix two (X_a, X_b)-based type structures, viz.*

$$\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b) \text{ and } \mathcal{T}^* = (X_a, X_b; T_a^*, T_b^*; \beta_a^*, \beta_b^*).$$

*Say (τ_a, τ_b) is a **type morphism** from $\mathcal{T}$ to $\mathcal{T}^*$ if each $\tau_c : T_c \to T_c^*$ is a measurable map with*

$$(\underline{\mathrm{id}_c \times \tau_d}) \circ \beta_c = \beta_c^* \times \tau_c,$$

where $\mathrm{id}_c \times \tau_d : X_c \times T_d \to X_c \times T_d^*$ satisfies $(\mathrm{id}_c \times \tau_d)(x_c, t_d) = (x_c, \tau_d(t_d))$.

Lemma E.5 *Fix two (X_a, X_b)-based type structures, viz. $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$ and $\mathcal{T}^* = (X_a, X_b; T_a^*, T_b^*; \beta_a^*, \beta_b^*)$. If (τ_a, τ_b) is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$, then $\delta_c(t_c) = \delta_c^*(\tau(t_c))$ for each $t_c \in T_c$.*

Lemma E.5 is standard (see, e.g., Heifetz and Samet, 1998, Proposition 5.1) and so the proof is omitted.[17]

E.2 Positive Result

Lemma E.6 *Fix (X_a, X_b)-based type structures $\mathcal{T}$ and $\mathcal{T}^*$. If $\delta_d^m(t_d) = \delta_d^{*,m}(t_d^*)$ then, for each $x_c \in X_c$, $\rho_c^{m+1}(x_c, t_d) = \rho_c^{m+1}(x_c, t_d^*)$.*

Proof. Let $\delta_d^m(t_d) = \delta_d^{*,m}(t_d^*)$ and note, by Remark D.3, $(\delta_c^1(t_c), \ldots, \delta_c^m(t_c)) = (\delta_c^{*,1}(t_c^*), \ldots, \delta_c^{*,m}(t_c^*))$. Now observe that

$$\begin{aligned} \rho_c^{m+1}(x_c, t_d) &= (\rho_c^m(x_c, t_d), \delta_d^m(t_d)) \\ &= (x_c, \delta_d^1(t_d), \ldots, \delta_d^m(t_d)) \\ &= (x_c, \delta_d^{*,1}(t_d^*), \ldots, \delta_d^{*,m}(t_d^*)) = \rho_c^{*,(m+1)}(x_c, t_d^*), \end{aligned}$$

as required. ■

[17] Heifetz and Samet (1998) define hierarchies of beliefs somewhat differently than here. That said, their proof can be replicated in this formalism.

In the proof below, we will apply Lemma E.6 both to the case where $\mathcal{T}$ and $\mathcal{T}^*$ are distinct type structures and to the case where they are the same type structure.

Proof of Theorem 9.2(i). Fix (X_a, X_b)-based type structures $\mathcal{T}$ and $\mathcal{T}^*$, where $\mathcal{T}$ is countable and $\mathcal{T}^*$ is complete. We will show that, for each $t_c \in T_c$, there exists $t_c^* \in T_c^*$ so that $\delta_c^{*,m}(t_c^*) = \delta_c^m(t_c)$. Then, by Remark D.3, $(\delta_c^{*,1}(t_c^*), \ldots, \delta_c^{*,m}(t_c^*)) = (\delta_c^1(t_c), \ldots, \delta_c^m(t_c))$, as required. The proof is by induction on m.

$\mathbf{m = 1}:$ Fix a type $t_c \in T_c$. By completeness, there exists a type $t_c^* \in T_c^*$ so that $\operatorname{marg}_{X_c}\beta_c^*(t_c^*) = \operatorname{marg}_{X_c}\beta_c(t_c)$. It follows that for each Borel $E \subseteq Z_c^1 = X_c$,

$$\delta_c^{*,1}(t_c^*)(E) = \operatorname{marg}_{X_c}\beta_c^*(t_c^*)(E) = \operatorname{marg}_{X_c}\beta_c(t_c)(E) = \delta_c^1(t_c)(E),$$

as required.

$\mathbf{m \geq 2}:$ Assume the result holds for m. By the induction hypothesis, there is a mapping $\tau_d^m : T_d \to T_d^*$ so that $\delta_d^{*,m}(\tau_d^m(t_d)) = \delta_d^m(t_d)$. By Lemma E.6, for each $x_c \in X_c$, $\rho_c^{m+1}(x_c, t_d) = \rho_c^{*,(m+1)}(x_c, \tau_d^m(t_d))$. Write $[t_d] := (\tau_d^m)^{-1}(\{\tau_d^m(t_d)\})$. (So, each $t_d' \in [t_d]$ induces the same m^{th}-order belief as t_d.) Write $\hat{T}_d = \{[t_d] : t_d \in T_d\}$.

Fix a type t_c. Construct $\psi \in \mathcal{P}(X_d \times T_d^*)$ to be the image measure of $\mathrm{id} \times \tau_d^m$ under $\beta_c(t_c)$. By completeness, there exists a type $t_c^* \in T_c^*$ with $\beta_c^*(t_c^*) = \psi$. We will show that $\delta_c^{*,m+1}(t_c^*) = \delta_c^{m+1}(t_c)$.

Fix some Borel $G \subseteq Z_c^{m+1}$. Since T_d is countable,

$$\begin{aligned}\delta_c^{m+1}(t_c)(G) &= \beta_c(t_c)((\rho_c^{m+1})^{-1}(G)) \\ &= \sum_{t_d \in T_d} \beta_c(t_c)(F_c[t_d] \times \{t_d\}),\end{aligned}$$

where $F_c[t_d] := (\rho_c^{m+1})^{-1}(G) \cap (X_c \times \{t_d\})$. Observe that, if $\tau_d^m(t_d) = \tau_d^m(t_d')$, then $F_c[t_d] = F_c[t_d']$; this follows from Lemma E.6. As such,

$$\delta_c^{m+1}(t_c)(G) = \sum_{[t_d] \in \hat{T}_d} \beta_c(t_c)(F_c[t_d] \times [t_d]).$$

By construction,

$$\delta_c^{m+1}(t_c)(G) = \sum_{[t_d] \in \hat{T}_d} \beta_c(t_c)(F_c[t_d] \times [t_d]) = \sum_{t_d \in T_d} \psi_c(F_c[t_d] \times \{\tau_d^k(t_d)\}).$$

Applying Lemma E.6 again, $F_c[t_d] = (\rho_c^{*,m+1})^{-1}(G) \cap (X_c \times \{\tau_d^k(t_d)\})$. Thus,

$$\begin{aligned}
\delta_c^{m+1}(t_c)(G) &= \sum_{t_d \in T_d} \psi_c(F_c[t_d] \times \{\tau_d^k(t_d)\}) \\
&= \psi_c((\rho_c^{*,m+1})^{-1}(G) \cap (X_c \times \tau_d^k(T_d))) \\
&= \psi_c((\rho_c^{*,m+1})^{-1}(G)) \\
&= \delta_c^{*,m+1}(t_c^*)(G),
\end{aligned}$$

as desired. ■

We will prove a stronger version of Theorem 9.2(ii). To do so, we will need to introduce the following terminology.

Definition E.7 *Call a cardinal κ is* ***large*** *if κ is uncountable and there is a set X and a probability measure ν on $(X, 2^X)$ such that*

(i) $|X| = \kappa$,

(ii) $\nu(E) \in \{0, 1\}$ *for each subset E of X, and*

(iii) $\nu(F) = 0$ *for each finite subset F of X.*

Call a cardinal λ ***small*** *if λ is not large.*

Theorem E.8 *Fix a complete (X_a, X_b)-based type structure $\mathcal{T}$. If $|X_a|$ and $|X_b|$ are small, $\mathcal{T}$ is finitely terminal for all atomic structures.*

Note, $\mathfrak{c}$ is small (Fremlin, 2008) and so Theorem E.8 implies Theorem 9.2(ii). More generally, it cannot be proved in ZFC that large cardinals exist—but if they do exist, the first large cardinal is much greater than $\mathfrak{c}$, and even much greater than the first uncountable inaccessible cardinal. (See Fremlin 2008.)

Remark E.9 *In the literature, a cardinal κ is called measurable if a set X of cardinality κ has a κ-additive probability measure ν on $(X, 2^X)$ such that Definition E.7 (i)–(iii) hold. The existence of an uncountable measurable cardinal cannot be proved in ZFC. A cardinal κ is large in the sense of Definition E.7 if and only if κ is $\geq$ the first uncountable measurable cardinal. So κ is small if and only if there is no uncountable measurable cardinal below κ.*

Say $\mathcal{T}$ is **finitely terminal for all countable atomic structures** if it is finitely terminal for all countable structures and it is finitely terminal for all atomic structures. To show Theorem E.8, it suffices to show the following:

Lemma E.10 *Fix an (X_a, X_b)-based type structure $\mathcal{T}$. If $|X_a|$ and $|X_b|$ are small, then $\mathcal{T}$ is finitely terminal for all atomic structures if and only if $\mathcal{T}$ is finitely terminal for all countable atomic structures.*

Proof of Theorem E.8. The result follows from Theorem 9.2(i) and Lemma E.10. ■

With the above in mind, it suffices to show Lemma E.10. To do so, it will be useful to begin with the following:

Lemma E.11 *(See Fremlin (2008)).*

- $\mathfrak{c}$ *is small.*
- *If* κ *is small and* $\lambda \leq \kappa$*, then* λ *is small.*
- *If* κ *is small, then* $\mathfrak{c}^\kappa$ *is small.*

Lemma E.12 *Suppose* $|X_a|$ *and* $|X_b|$ *are small. Then, for each* c *and each* m*,* $|Z_c^m|$ *is small.*

Proof. The proof is by induction on m. The case of $m = 1$ is immediate from the fact that $|X_a|$ and $|X_b|$ are small. Assume the result holds for m. Then, by the induction hypothesis, $|Z_a^m|$ and $|Z_b^m|$ are small. Each probability measure $\mu \in \mathcal{P}(Z_c^m)$ is a mapping from a subset of the power set of Z_c^m into $[0, 1]$; so $|\mathcal{P}(Z_c^m)| \leq \mathfrak{c}^\kappa$, where $\kappa = 2^{|Z_c^m|} \leq \mathfrak{c}^{|Z_c^m|}$. By Lemma E.11, κ is small, so $|\mathcal{P}(Z_c^m)|$ is small. Thus,

$$|Z_c^{m+1}| = |Z_c^m \times \mathcal{P}(Z_d^m)| = \max(|Z_c^m|, |\mathcal{P}(Z_d^m)|),$$

and so $|Z_c^{m+1}|$ is also small. ■

Lemma E.13 *Let* Ω *be a metrizable space. The following are equivalent:*

(i) Every discrete subset of Ω *has small cardinality.*

(ii) For every $\mu \in \mathcal{P}(\Omega)$*, every atom of* μ *contains a point mass of* μ*.*

(iii) If $\mu \in \mathcal{P}(\Omega)$*,* μ *is atomic if and only if* $\mu(D) = 1$ *for some countable* $D \subseteq \Omega$*.*

Proof. It is easily seen that (iii) implies (ii). It is also easily seen that the reverse direction of (iii) always holds: if $\mu(D) = 1$ for some countable $D \subseteq \Omega$ then μ is atomic.

(ii) implies (i): Assume that (i) fails, so there is a discrete subset E of Ω of large cardinality. Then there is a Borel probability measure μ on Ω such that $\mu(F) = 0$ for every finite $F \subseteq E$, $\mu(E) = 1$, and E is an atom of μ. Then E is an atom that does not contain a point mass, so (ii) fails.

(i) implies (iii): Suppose (i) holds. Let μ be an atomic Borel probability measure on Ω. Then μ has a separable support. (See Billingsley, 1968, Theorem 2, page 235.) This means that there is a closed separable set $C \subseteq \Omega$ such that $\mu(C) = 1$. Let D be a countable dense subset of C, and for each $n > 0$ let D_n be the union of all $\frac{1}{n}$-balls centered at elements of D. Then $C \subseteq D_n$ for each $n > 0$, and $\bigcap_n D_n = D$. Therefore $\mu(D) = 1$. ■

Corollary E.14 *If the cardinality of Ω is small, then $\mu \in \mathcal{P}(\Omega)$ is atomic if and only if $\mu(D) = 1$ for some countable $D \subseteq \Omega$.*

To prove Lemma E.10, we will take the union of pairwise disjoint type structures.

Definition E.15 *A family of (X_a, X_b)-based type structures $\{\mathcal{T}^i : i \in I\}$ is* ***pairwise disjoint*** *if, for each player c and each $i, j \in I$, T_c^i is disjoint from T_c^j.*

Definition E.16 *Let $\{\mathcal{T}^i : i \in I\}$ be a pairwise disjoint family of countable (X_a, X_b)-based type structures. The* ***disjoint union****, viz. $\mathcal{T}^* = \bigsqcup_{i \in I} \mathcal{T}^i$, is some $\mathcal{T}^* = (X_a, X_b; T_a^*, T_b^*; \beta_a^*, \beta_b^*)$ so that:*

(i) $T_c^* = \bigcup_{i \in I} T_c^i$;

(ii) T_c^* *has the discrete topology; and*

(iii) $\beta_c^* : T_c^* \to \mathcal{P}(X_c \times T_d^*)$ *is such that, for each $i \in I$, $t_c^i \in T_c^i$, and Borel set $E \subseteq X_c \times T_d^i$,* $(\beta_c^*(t_c^i))(E) = (\beta_c^i(t_c^i))(E)$.

So defined, $\mathcal{T}^* = \bigsqcup_{i \in I} \mathcal{T}^i$ is itself a type structure.[18]

Lemma E.17 *Let $\{\mathcal{T}^i : i \in I\}$ be a family of pairwise disjoint family of countable (X_a, X_b)-based type structures and let $\mathcal{T}^* = \bigsqcup_{i \in I} \mathcal{T}^i$ be the disjoint union of $\{\mathcal{T}^i : i \in I\}$.*

(i) For each $i \in I$ and each $t_c \in T_c^i$, $\delta_c^i(t_c) = \delta_c^*(t_c)$.

(ii) If the index set I is countable and $\mathcal{T}^i$ is atomic for each $i \in I$, then $\mathcal{T}^*$ is countable.

Proof. Part (ii) follows immediately from part (i). With this in mind, we show part (i). To do so, fix some $\mathcal{T}^i = (X_a, X_b; T_a^i, T_b^i; \beta_a^i, \beta_b^i)$ and write $\mathrm{id}_c^i : T_c^i \to T_c^*$ for the identity maps. Note, $(\mathrm{id}_a^i, \mathrm{id}_b^i)$ is a type morphism from $\mathcal{T}^i$ to $\mathcal{T}^*$. Then the claim follows from Lemma E.5. ■

Lemma E.18 *Fix an atomic (X_a, X_b)-based type structure $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$, where $|X_a|$ and $|X_b|$ are small. For each type $t_c \in T_c$ and each m, there exists a countable atomic (X_a, X_b)-based type structure $\overline{\mathcal{T}} = (X_a, X_b; \overline{T}_a, \overline{T}_b; \overline{\beta}_a, \overline{\beta}_b)$ and a type $\bar{t}_c \in \overline{T}_c$ so that $\overline{\delta}_c^m(\bar{t}_c) = \delta_c^m(t_c)$.*

Proof. The proof is by induction on m.

[18] The same would hold if we replaced the requirement that each $\mathcal{T}^i$ is countable with the requirement that each $\mathcal{T}^i$ has type sets endowed with the discrete topology.

$\mathbf{m=1}$: Fix some type $t_c \in T_c$. Construct $\overline{\mathcal{T}}$ as follows. Take $\overline{T}_c = \{\bar{t}_c\}$ and $\overline{T}_d \neq \emptyset$ finite. Set $\overline{\beta}_c(\bar{t}_c)$ so that, for each Borel $E_c \subseteq X_c$,

$$\overline{\beta}_c(\bar{t}_c)(E_c \times \overline{T}_d) = \beta_c(t_c)(E_c \times T_d).$$

Choose $\overline{\beta}_d$ so that each $\overline{\beta}_d(\bar{t}_d)$ is atomic.

Note, for each Borel $E_c \subseteq X_c$

$$\overline{\delta}_c^1(\bar{t}_c)(E_c) = \overline{\beta}_c(\bar{t}_c)(E_c \times \overline{T}_d) = \beta_c(t_c)(E_c \times T_d) = \delta_c^1(t_c)(E_c).$$

So $\overline{\delta}_c^1(\bar{t}_c) = \delta_c^1(t_c)$. Moreover, since $\mathcal{T}$ is atomic, $\overline{\delta}_c^1(\bar{t}_c) = \delta_c^1(t_c)$ is atomic. By construction, each $\overline{\beta}_d(\bar{t}_d)$ is atomic and, so, by Lemma E.1, each $\overline{\delta}_d^1(\bar{t}_d)$ is atomic.

$\mathbf{m \geq 2}$: Assume the result holds for each player, each type in $\mathcal{T}$, and m. We will show that the same holds for $m+1$.

Fix a type $t_c \in T_c$ and note, by assumption, $\delta_c^{m+1}(t_c)$ is atomic. Since $|X_a|, |X_a|$ are small, $|Z_c^{m+1}|$ is small and, so, each discrete subset of Z_c^{m+1} is small. So, by Lemma E.13, there is a finite or countable set of distinct points $E \subseteq Z_c^{m+1}$ so that (i) $\delta_c^{m+1}(t_c)(E) = 1$, and (ii) $\delta_c^{m+1}(t_c)(\{z\}) > 0$ for each $z \in E$. Note, E is the set of point masses of $\delta_c^m(t_c)$ and so depends on both t_c and m. It will be convenient to describe E as

$$E = \{z^k : k \in K\}$$

for some finite or countable index set K.[19]

For each $k \in K$,

$$\alpha^k := \delta_c^{m+1}(t_c)(\{z^k\}) = \beta_c(t_c)((\rho_c^{m+1})^{-1}(\{z^k\}) > 0.$$

So, for each $k \in K$ there exists a point $(x_c^k, t_d^k) \in X_c \times T_d$ with $\rho_c^{m+1}(x_c^k, t_d^k) = z^k$. By the induction hypothesis, for each $k \in K$, there is a countable atomic type structure

$$\overline{\mathcal{T}}^k = (X_a, X_b; \overline{T}_a^k, \overline{T}_b^k; \overline{\beta}_a^k, \overline{\beta}_b^k)$$

and a type $\bar{t}_d^k \in \overline{T}_d^k$ with $\overline{\delta}_d^{k,m}(\bar{t}_d^k) = \delta_d^m(t_d)$. By renaming points, we can take the family of type structures $\{\overline{\mathcal{T}}^k : k \in K\}$ to be pairwise disjoint. Thus, we can construct the disjoint union $\mathcal{T}^* = \bigsqcup_k \overline{\mathcal{T}}^k$. By Lemma E.17(ii), $\mathcal{T}^*$ is a countable atomic type structure. Note, by construction, we choose the list of $\bar{t}_d^k, k \in K$ to be distinct (i.e., even if two indices are associated with the same types in $\mathcal{T}$) and so they are distinct types in T_d^*.

Construct a new type structure, viz. $\overline{\mathcal{T}}$, as follows: Take a new point $\bar{t}_c \notin T_c^*$ and set $\overline{T}_c = T_c^* \cup \{\bar{t}_c\}$. Set $\overline{T}_d = T_d^*$. Endow $\overline{T}_c$ and $\overline{T}_d$ with the discrete topology. Choose the maps $\overline{\beta}_c$ and $\overline{\beta}_d$ so that the identity maps form a type morphism from $\overline{\mathcal{T}}$ to $\mathcal{T}^*$. For $\bar{t}_c$, let $\overline{\beta}_c(\bar{t}_c)$ be an atomic

[19] The superscript just denotes an enumeration and does not refer to $(m+1)$.

probability measure on $X_c \times \overline{T}_d$ such that

$$\overline{\beta}_c(\bar{t}_c)(\{(x_c^k, \bar{t}_d^k)\}) = \alpha^k$$

for each $k \in K$. By Lemma E.1, each $\overline{\delta}_c^m(\bar{t}_c)$ is atomic. Since $\mathcal{T}^*$ is countable and atomic and each $\overline{\delta}_c^m(\bar{t}_c)$ is atomic, it follows that $\overline{\mathcal{T}}$ is countable and atomic.

It remains to show that $\overline{\delta}_c^{m+1}(\bar{t}_c) = \delta_c^{m+1}(t_c)$: Note, since the identity maps are a type morphism from $\overline{\mathcal{T}}$ to $\mathcal{T}^*$, we have

$$\overline{\delta}_d^m(\bar{t}_d^k) = \overline{\delta}_d^{k,m}(\bar{t}_d^k) = \delta_d^m(t_d^k).$$

So,

$$\overline{\rho}_c^{m+1}(x_c^k, \bar{t}_d^k) = \rho_c^{m+1}(x_c^k, t_t^k) = z^k.$$

It follows that

$$\overline{\delta}_c^{m+1}(\bar{t}_c)(\{z^k\}) = \overline{\beta}_c(\bar{t}_c)((\overline{\rho}_c^{m+1})^{-1}(\{z^k\})) \geq \overline{\beta}_c(\bar{t}_c)(\{(x_c^k, \bar{t}_d^k)\}) = \alpha^k.$$

Using the fact that $\sum_k \alpha^k = 1$, it follows that $\overline{\delta}_c^{m+1}(\bar{t}_c)(\{z^k\}) = \alpha^k$ for each $k \in K$. So, $\overline{\delta}_c^{m+1}(\bar{t}_c) = \delta_c^{m+1}(t_c)$, as desired. ■

Proof of Lemma E.10. Fix an (X_a, X_b)-based type structure, viz. $\mathcal{T}^*$, that is finitely terminal for all countable atomic structures. Let $\mathcal{T}$ be an atomic (X_a, X_b). By Lemma E.18, $\mathcal{T}^*$ is finitely terminal for $\mathcal{T}$. ■

E.3 Negative Result

Proof of Lemma 9.8. Let $T_a = T_b = [0, 1]$ and endow T_a and T_b with the discrete topology. Then for each player c, $X_c \times T_d$ is discrete and has cardinality $\mathfrak{c}$. By Lemma 9.7(ii), each probability measure in $\mathcal{P}(X_c \times T_d)$ has a countable set of measure one, and thus is atomic and is determined by a countable sequence of elements of $X_c \times T_d \times [0, 1]$, corresponding to a sequence of points and measures of points. Therefore, $\mathcal{P}(X_c \times T_d)$ has cardinality $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. Hence there are bijective functions β_c from T_c onto $\mathcal{P}(X_c \times T_d)$. Let $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$. Since each T_c is discrete, each β_c is continuous; as such, $\mathcal{T}$ is a continuous type structure. Since each β_c is onto, $\mathcal{T}$ is a complete type structure.

It remains to show that $\mathcal{T}$ is atomic. By the preceding paragraph, for each $\beta_c(t_c)$, there is a countable set $E \subseteq X_c \times T_d$ such that $\beta_c(t_c)(E) = 1$. So, by Corollary E.2, $\mathcal{T}$ is atomic. ■

Appendix F Variants of Terminality: Existence

Heifetz and Samet's (1998) construction of a so-called universal type structure shows that there exists an (X_a, X_b)-based type structure that is terminal. Theorem 8.1 raises the question: Does there exist an (X_a, X_b)-based type structure that is terminal for all finite structures but *not* the universal type structure? If the answer were "no," then the epistemic conditions provided by Theorem 8.1 would coincide with those provided by Tan and Werlang (1988) (Result 2.3). This appendix shows that this is not the case. In particular, when (X_a, X_b) are finite, there exists an (X_a, X_b)-based type structure that is not terminal but is terminal for all finite type structures. To state the result, it will be useful to introduce some terminology.

Definition F.1 *Fix a type structure* $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$.

(i) Call $\mathcal{T}$ ***finite*** *if* T_a *and* T_b *are finite.*

(ii) Call $\mathcal{T}$ ***simple*** *if, for each* c*, each type* $t_c \in T_c$*, and each* m*,* $\delta_c^m(t_c)$ *has finite support.*

Note that a probability measure has finite support if and only if some finite set has measure one. Every probability measure with finite support is atomic, so every simple type structure is atomic.

In the spirit of Definition 7.5, call an (X_a, X_b)-based type structure $\mathcal{T}^*$ **finitely terminal for all simple** (resp. **finite simple**) **structures** if it is finitely terminal for each simple (resp. finite and simple) (X_a, X_b)-type structure $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$.

Proposition F.2 *Fix* (X_a, X_b).

(i) There is a simple (X_a, X_b)*-based type structure that is terminal for all finite structures.*

(ii) If $|X_a|$ *and* $|X_b|$ *are small, there is an atomic* (X_a, X_b)*-based type structure that is terminal for all countable structures.*

From this, we can conclude:

Corollary F.3 *Fix* (X_a, X_b).

(i) There is an (X_a, X_b)*-based type structure that is terminal for all finite structures but not terminal for all countable structures.*

(ii) If $|X_a|$ *and* $|X_b|$ *are small, there is an* (X_a, X_b)*-based type structure that is terminal for all countable structures but not terminal.*

Corollary F.3 follows from Proposition F.2 and the fact that there exists some type structure that induces a countable support (resp. full support) second-order belief. (This can be constructed or taken to follow from Heifetz and Samet, 1999.)

To prove Proposition F.2, we begin with a preliminary result:

Proposition F.4 *Fix* (X_a, X_b).

(i) There is a simple (X_a, X_b)*-based type structure that is terminal for all finite simple structures.*

(ii) There is an atomic (X_a, X_b)*-based type structure that is terminal for all countable atomic structures.*

The idea of the proof is clear: Construct an (X_a, X_b)-based type structure that is the disjoint union of all countable atomic (X_a, X_b)-based type structures. But this does not work because the class of all countable atomic type structures is a proper class. Instead, we construct a type structure that is the disjoint union of a set of countable atomic type structures that contains a copy of each countable atomic type structure.

Proof of Proposition F.4. Let $\mathcal{N} = \{\mathcal{T}^i : i \in I\}$ be the set of all finite simple (resp. countable atomic) (X_a, X_b)-based type structures $\mathcal{T}^i = (X_a, X_b; T_a^i, T_b^i; \beta_a^i, \beta_b^i)$ whose type spaces are subsets of $\mathbb{N}$ and are endowed with the discrete topology. This set of type structures is not pairwise disjoint, but we can replace it by a pairwise disjoint set in the following way: For each $\mathcal{T}^i \in \mathcal{N}$, let

$$\hat{\mathcal{T}}^i = (X_a, X_b; \hat{T}_a^i, \hat{T}_b^i; \hat{\beta}_a^i, \hat{\beta}_b^i),$$

where $\hat{T}_c^i = T_c^i \times \{i\}$ and $\hat{\beta}_c^i(E \times \{i\}) = \beta_c^i(E)$ for each Borel $E \subseteq X_c \times T_d^i$. Write $\hat{\mathcal{N}}$ for the set of all such $\hat{\mathcal{T}}^i$. Note that $\hat{\mathcal{N}} = \{\hat{\mathcal{T}}^i : i \in I\}$ is a pairwise disjoint set of countable atomic (X_a, X_b)-based type structures. Moreover, for each $\hat{\mathcal{T}}^i \in \hat{\mathcal{N}}$ (resp. $\mathcal{T}^i \in \mathcal{N}$), there is a type morphism from $\hat{\mathcal{T}}^i$ to $\mathcal{T}^i$ (resp. $\mathcal{T}^i$ to $\hat{\mathcal{T}}^i$).

Let $\mathcal{T}^* = \bigsqcup_{i \in I} \hat{\mathcal{T}}^i$. Note that, for each $(t_c, i) \in \hat{T}_c^i \subseteq T_c^*$ (resp. $t_c \in T_c^i$), $\delta_c^*((t_c, i)) = \hat{\delta}_c^i((t_c, i)) = \delta_c^i(t_c)$. (This follows from Lemma E.17(i) and Lemma E.5.) Since each $\mathcal{T}^i$ is simple (resp. atomic), $\mathcal{T}^*$ is simple (resp. atomic). Moreover, $\mathcal{T}^*$ is terminal for each $\mathcal{T}^i \in \mathcal{N}$.

Finally, fix an (X_a, X_b)-based type structure $\mathcal{T}$ that is finite and simple (resp. countable and atomic), but necessarily in $\mathcal{N}$. There exists a type morphism from $\mathcal{T}$ to some $\mathcal{T}^i \in \mathcal{N}$. By Lemma E.5, $\mathcal{T}^i$ is terminal for $\mathcal{T}$. As such, $\mathcal{T}^*$ is terminal for $\mathcal{T}$. ■

Proposition F.2(ii) follows immediately from Proposition F.4(ii) and Lemma E.10. To show Proposition F.2(i), we need the following analogue of Lemma E.10.

Lemma F.5 *An (X_a, X_b)-based type structure is finitely terminal for all simple structures if and only if it is finitely terminal for all finite simple structures.*

The remainder of this Appendix to devoted to showing Lemma F.5.

Lemma F.6 *If $\beta_c(t_c)$ has a finite set of measure one then, for each m, $\delta_c^m(t_c)$ is simple.*

Proof. Let $t_c \in T_c$ so that there is a finite set $E \subseteq X_c \times T_d$ with $\beta_c(t_c)(E) = 1$. Then, the image $\rho_c^m(E)$ is finite and $E \subseteq (\rho_c^m)^{-1}(\rho_c^m(E))$. So,

$$\delta_c^m(t_c)(\rho_c^m(E)) = \beta_c(t_c)((\rho_c^m)^{-1}(\rho_c^m(E))) \geq \beta_c(t_c)(E) = 1.$$

Hence, $\delta_c^m(t_c)$ is simple. ■

Lemma F.7 *Fix a finite index set I and a family of pairwise disjoint family of finite and simple (X_a, X_b)-based type structures $\{\mathcal{T}^i : i \in I\}$. Then, for each $i \in I$ and each $t_c \in T_c^i$, $\delta_c^i(t_c) = \delta_c^*(t_c)$.*

Proof. Immediate from Lemma E.17(i) ■

Lemma F.8 *Fix a simple (X_a, X_b)-based type structure $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$. For each type $t_c \in T_c$ and each m, there exists a finite and simple (X_a, X_b)-based type structure $\overline{\mathcal{T}} = (X_a, X_b; \overline{T}_a, \overline{T}_b; \overline{\beta}_a, \overline{\beta}_b)$ and a type $\bar{t}_c \in \overline{T}_c$ so that $\overline{\delta}_c^m(\bar{t}_c) = \delta_c^m(t_c)$.*

Proof. The proof is the same as the proof of Lemma E.18 except that the word "countable" is replaced with "finite," "atomic" is replaced with "simple," the index set K if finite, Lemma E.1 is replaced with Lemma F.6 and Lemma E.17 is replaced with Lemma F.7. Note, in this case, we need not require that $|X_c|$ is small: If $\mathcal{T}$ is simple, there is necessarily a finite set of points in Z_c^m that gets probability one under $\delta_c^m(t_c)$, irrespective of whether or not $|Z_c^m|$ is small. ■

Proof of Lemma F.5. Suppose $\mathcal{T}$ is an (X_a, X_b)-based type structure that is finitely terminal for all finite simple type structures. If $\mathcal{T}^*$ be an (X_a, X_b)-based simple type structure, then $\mathcal{T}$ is finitely terminal for $\mathcal{T}^*$. (See Lemma F.8.) ■

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