Rich and Saturated Adapted Spaces

H. Jerome Keisler

1 Introduction

In the paper [HK] the notions of an adapted distribution and of a saturated adapted probability space were introduced. The adapted distribution of a random variable on an adapted space (with values in a complete separable metric space) is the natural analogue of the distribution of a random variable on a probability space. An adapted space Ω is **saturated** if for any random variable x on Ω and pair of random variables $\bar{x}$ and $\bar{y}$ on another adapted space Γ such that x and $\bar{x}$ have the same adapted distribution, there is a random variable y on Ω such that (x, y) and $(\bar{x}, \bar{y})$ have the same adapted distribution. For stochastic differential equations and a wide variety of other existence problems, every existence theorem which holds on some adapted space holds on a saturated adapted space.

The paper [FK1] introduced a new method for proving existence theorems in probability theory, based on the notion of a neocompact set of random variables. A set of random variables on an adapted space is said to be basic if it is either compact or is the set of all random variables which are measurable at time t and whose law belongs to a compact set C of measures, for some t and C. The family of neocompact sets is the closure of the family of basic sets under finite unions and Cartesian products, countable intersections, existential projections, and "universal projections with respect to a nonempty basic set". An adapted space is said to be **rich** if the family of neocompact sets is countably compact. On a rich adapted space, the neocompact sets play a role analogous to the compact sets. They were used in the papers [FK1] and [CK] to prove a variety of optimization and existence theorems. The existence of rich adapted spaces for any linearly ordered set of times was proved in [FK2].

The purpose of this paper is to find the relationship between richness and saturation. Our main theorem is that richness and saturation are equivalent for adapted spaces with a countable set of times. For example, the two notions are equivalent for probability spaces, for discrete time adapted spaces, and for adapted spaces with dyadic rational times. We also show that for any rich adapted space with dyadic rational times, the associated right continuous adapted space with real times is saturated but cannot be rich.

Our proofs will use a "quantifier elimination" theorem from the paper [K5] which shows that in a rich or saturated space with a countable time set, the neocompact sets can be represented in a simple form. The paper [K5], which was aimed primarily at model theorists, introduced a very general notion called a law structure, which is an abstraction of the distribution and the adapted distribution in probability theory. This paper is aimed at probabilists, and applies the results of [K5] to probability spaces and adapted spaces.

In Section 2 we introduce the notion of a law mapping on a probability space Ω, which is a special case of the notion of a law structure from [K5]. We shall also state the results we need from [K5]. In Section 3 we prove some general results about law mappings. The rest of the paper deals with the particular law mappings which correspond to the distribution of a random variable and adapted distribution of a stochastic process. In Sections 4 through 8 we prove our main results showing that saturation is equivalent to richness for probability spaces, adapted spaces with finite time sets, and adapted spaces with infinite time sets. Finally, in Section 9 we prove that every rich adapted space with rational times induces a saturated right continuous adapted space with real times.

I wish to thank Sergio Fajardo and Siu-Ah Ng for helpful suggestions on this article. This research was supported in part by the National Science Foundation and the Vilas Trust Fund.

2 Law Mappings

In this section we introduce the notion of a law mapping on a probability space Ω, and state the theorems we shall need from the paper [K5].

Throughout this paper we let $K = (K, \tau)$, $M = (M, \rho)$, and $N = (N, \sigma)$ be complete separable metric spaces, and let $\Omega = (\Omega, P, \mathcal{G})$ be a probability space. For each complete separable M, we use the corresponding script letter $\mathcal{M}$ to denote the metric space $\mathcal{M} = (L^0(\Omega, M), \rho_0)$ of all equivalence classes of P-measurable functions from Ω into M. Here two functions are equivalent if they are equal P-almost surely, and ρ_0 is the metric of convergence in probability on $L^0(\Omega, M)$,

$$\rho_0(x, y) = \inf\{\varepsilon : P[\rho(x(\omega), y(\omega)) \le \varepsilon] \ge 1 - \varepsilon\}.$$

We shall let $\mathbf{M}_\Omega$ be the family of all the metric spaces $L^0(\Omega, M)$ where M is a complete separable metric space, so that $\mathcal{M}, \mathcal{N}, \mathcal{K}$ are arbitrary elements of $\mathbf{M}_\Omega$.

A Cartesian product $M \times N$ with the metric $\rho \times \sigma$ defined by

$$(\rho \times \sigma)((x, y), (\bar{x}, \bar{y})) = \max(\rho(x, \bar{x}), \sigma(y, \bar{y}))$$

is again a complete separable metric space. The metrics $\rho_0 \times \sigma_0$ and $(\rho \times \sigma)_0$ on $\mathcal{M} \times \mathcal{N} = L^0(\Omega, M \times N)$ are different, but determine the same topology. A similar remark holds for countable Cartesian products $\prod_n K_n$ with the metric $\tau = \prod_n \tau_n$ defined by

$$\tau(x, y) = \sum_n \min(1, \tau_n(x_n, y_n))/2^n.$$

A subset of a topological space Λ is **relatively compact** if it is contained in a compact subset of Λ. Recall that a topological space is first countable if every point has a countable neighborhood base. For example, every metrizable space is first countable. In a first countable space, a set is closed if and only if it contains the limit of any convergent sequence of points in the set.

For each continuous function $f : M \to N$, we let $\hat{f} : \mathcal{M} \to \mathcal{N}$ be the function defined by $(\hat{f}(x))(\omega) = f(x(\omega))$. $\hat{f}$ is continuous from $\mathcal{M}$ to $\mathcal{N}$.

The space of Borel probability measures on M with the Prohorov metric

$$d(\mu, \nu) = \inf\{\varepsilon : \mu(C) \le \nu(C^\varepsilon) + \varepsilon \text{ for all closed } C \subseteq M\}$$

is denoted by $Meas(M)$. It is again a complete separable metric space, and convergence in $Meas(M)$ is the same as weak convergence. Each random variable $x \in \mathcal{M}$ induces the measure $law(x) \in Meas(M)$ such that $(law(x))(S) = P[x^{-1}(S)]$ for each Borel $S \subseteq M$, and the function $law : \mathcal{M} \to Meas(M)$ is continuous.

Definition 2.1 *A* **law mapping** *on Ω is a pair (λ, Λ) which assigns to each $\mathcal{M} \in \mathbf{M}_\Omega$ a first countable Hausdorff space $\Lambda(\mathcal{M})$ and a continuous function $\lambda : \mathcal{M} \to \Lambda(\mathcal{M})$ such that:*

1. *For each $x, x_1, x_2, \ldots$ in $\mathcal{M}$, if $\lambda(x_n) \to \lambda(x)$ in $\Lambda(\mathcal{M})$, then law$(x_n) \to$ law(x) in Meas(M).*

2. *If $A \subseteq \mathcal{M}, B \subseteq \mathcal{N}$, and the images $\lambda(A)$ and $\lambda(B)$ are relatively compact in $\Lambda(\mathcal{M})$ and $\Lambda(\mathcal{N})$, then the image $\lambda(A \times B)$ is relatively compact in $\Lambda(\mathcal{M} \times \mathcal{N})$.*

3. *For each continuous function $f : M \to N$ there is a continuous function $\bar{f} : \lambda(\mathcal{M}) \to \lambda(\mathcal{N})$ such that the following diagram is commutative:*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\lambda"] \arrow[d, "f"'] & \lambda(\mathcal{M}) \arrow[d, "\bar{f}"] \\ \mathcal{N} \arrow[r, "\lambda"'] & \lambda(\mathcal{N}) \end{tikzcd}$$

Moreover, if f is a metric isometry of M onto N, then $\bar{f}$ is a homeomorphism from $\lambda(\mathcal{M})$ to $\lambda(\mathcal{N})$.

Since λ is continuous, convergence in probability implies convergence in λ. Condition (2.1.1) says that convergence in λ in turn implies convergence in *law*. One consequence of condition (2.1.1) is that $\lambda(x) = \lambda(y)$ implies *law*$(x) =$ *law*(y). Another consequence is that $\lambda(x, y) = \lambda(z, z)$ implies $x = y$.

In condition (2.1.3), $\lambda(\mathcal{M})$ denotes the image of $\mathcal{M}$ under the function λ, which is a possibly proper subspace of the space $\Lambda(\mathcal{M})$. (2.1.3) says that for each continuous $f : M \to N$, the function $\lambda(x) \mapsto \lambda(f(x))$ is well-defined and continuous, and is denoted by $\bar{f}$.

Condition (2.1.3) is often applied to projections. If f is the projection from $M \times N$ to M, then $\bar{f}$ sends $\lambda(x, y)$ to $\lambda(x)$ and is called the projection function from $\lambda(\mathcal{M} \times \mathcal{N})$ to $\lambda(\mathcal{M})$. There is an analogous projection function from $\lambda(\mathcal{M} \times \mathcal{N})$ to $\lambda(\mathcal{N})$. Combining the two projections, it follows that the function $\lambda(x, y) \mapsto (\lambda(x), \lambda(y))$ is a continuous surjection from $\lambda(\mathcal{M} \times \mathcal{N})$ to the product space $\lambda(\mathcal{M}) \times \lambda(\mathcal{N})$. In general, this function is many-one, and the space $\lambda(\mathcal{M} \times \mathcal{N})$ is more complicated than the product space $\lambda(\mathcal{M}) \times \lambda(\mathcal{N})$.

The canonical example of a law mapping on Ω, developed in Section 4, is the pair (λ, Λ) where $\Lambda(\mathcal{M})$ is the space *Meas*(M) and $\lambda(x) =$ *law*(x). In this case, $\lambda(x, y)$ is the law of the joint random variable (x, y), and $\lambda(x)$ and $\lambda(y)$ are the laws of the marginals x and y.

We shall usually suppress the target space Λ and speak of a law mapping λ. However, it should be kept in mind that a law mapping can be changed in an essential way by extending or restricting the space $\Lambda(\mathcal{M})$.

In the paper [K5] the notion of a law structure $(\mathbf{M}, \lambda, \Lambda)$ was introduced in the more general setting where the family $\mathbf{M}_\Omega$ is replaced by a family $\mathbf{M}$ of sets closed under finite Cartesian products. In that setting, the sets $X \in \mathbf{M}$ do not have metrics, the function λ is not required to be continuous, condition (2.1.1) is replaced by the weaker condition that $\lambda(x, y) = \lambda(z, z)$ implies $x = y$, and condition (2.1.3) is required only for the case that f is a projection from one finite Cartesian product to another.

Thus whenever (λ, Λ) is a law mapping on a probability space Ω, the triple $(\mathbf{M}_\Omega, \lambda, \Lambda)$ is a law structure in the sense of [K5].

We next introduce some properties of law mappings which were studied in [K5].

For each $x \in \mathcal{M}$ and each $C \subseteq \mathcal{N}$, let

$$\lambda(x, C) = \{\lambda(x, y) : y \in C\}, \lambda(C, x) = \{\lambda(y, x) : y \in C\}.$$

Definition 2.2 *Let λ be a law mapping on Ω.*

λ has the **back and forth property** *if for all $x, \bar{x} \in \mathcal{M}$ such that $\lambda(x) = \lambda(\bar{x})$, we have $\lambda(x, \mathcal{N}) = \lambda(\bar{x}, \mathcal{N})$ for all $\mathcal{N}$. That is, if $\lambda(x) = \lambda(\bar{x})$ then for every $y \in \mathcal{N}$ there exists $\bar{y} \in \mathcal{N}$ such that $\lambda(x, y) = \lambda(\bar{x}, \bar{y})$.*

λ is said to be **dense** *if whenever $x, \bar{x} \in \mathcal{M}$ and $\lambda(x) = \lambda(\bar{x}), \lambda(x, \mathcal{N})$ and $\lambda(\bar{x}, \mathcal{N})$ have the same closure in $\Lambda(\mathcal{M} \times \mathcal{N})$.*

λ is said to be **closed** *if $\lambda(\mathcal{M})$ is closed in $\Lambda(\mathcal{M})$ for all $\mathcal{M} \in \mathbf{M}_\Omega$.*

λ has the **Skorokhod property** *if for every $x \in \mathcal{M}$ and sequence c_n which converges to $\lambda(x)$ in $\lambda(\mathcal{M})$, there exists a sequence x_n in $\mathcal{M}$ such that $\lambda(x_n) = c_n$ for each n and x_n converges to x in $\mathcal{M}$.*

We shall see in Section 4 that the Skorokhod property is closely related to the Skorokhod representation theorem. The next proposition shows that the Skorokhod property for a law mapping is equivalent to a condition which does not mention the metric on $\mathcal{M}$ and was called the "strong open mapping property" in [K5].

Proposition 2.3 *Let λ be a law mapping on Ω. Then λ has the Skorokhod property if and only if for each $\mathcal{M}$ and each $y \in \mathcal{N}$, the projection from $\lambda(\mathcal{M} \times \mathcal{N})$ to $\lambda(\mathcal{M})$ restricted to $\lambda(\mathcal{M}, y)$ is open.*

Proof: The second condition is equivalent to the following:

(1) For each $x \in \mathcal{M}, y \in \mathcal{N}$, and sequence c_n converging to $\lambda(x)$ in $\lambda(\mathcal{M})$, there is a sequence x_n in $\mathcal{M}$ such that $\lambda(x_n) = c_n$ for all sufficiently large n, and $\lambda(x_n, y)$ converges to $\lambda(x, y)$ in $\lambda(\mathcal{M} \times \mathcal{N})$.

The Skorokhod property implies (1) because if $x_n \to x$ in $\mathcal{M}$ then $(x_n, y) \to (x, y)$ in $\mathcal{M} \times \mathcal{N}$, and by the continuity of λ, $\lambda(x_n, y) \to \lambda(x, y)$.

For the converse, assume (1). Let $c_n \to \lambda(x)$ in $\lambda(\mathcal{M})$. By (1) there exist x_n in $\mathcal{M}$ such that $\lambda(x_n) = c_n$ for all n, and $\lambda(x_n, x) \to \lambda(x, x)$. By (2.1.1) we have $law(x_n, x) \to law(x, x)$ in $Meas(M \times M)$. Therefore $x_n \to x$ in $\mathcal{M}$, and the Skorokhod property is proved. □

We state a result from [K5].

Proposition 2.4 *A law mapping λ on Ω has the back and forth property if and only if*

(i) λ is and dense, and

(ii) Whenever $\lambda(x, y_n)$ converges to $\lambda(\bar{x}, \bar{y})$ in $\lambda(\mathcal{M} \times \mathcal{N})$, there exists $y \in \mathcal{N}$ such that $\lambda(x, y) = \lambda(\bar{x}, \bar{y})$. □

Condition (ii) in the above proposition is called **completeness** in [HK] and [K5].

We now review the notions of a basic set and a basic section from [K5]. Basic sections play a central role in the study of neocompact sets.

Definition 2.5 *A set $B \subseteq \mathcal{M}$ is* **basic** *for a law mapping λ on Ω if B is of the form*

$$B = \{x \in \mathcal{M} : \lambda(x) \in \hat{B}\}$$

for some compact subset $\hat{B}$ of $\Lambda(\mathcal{M})$.

Let $z \in \mathcal{K}$. A set $C \subseteq \mathcal{M}$ is called a **basic section for λ with parameter** *z on Ω if C has the form*

$$C = \{x \in \mathcal{M} : \lambda(x, z) \in \hat{C}\}$$

for some compact subset $\hat{C}$ of $\Lambda(\mathcal{M} \times \mathcal{K})$.

We say that a family $\mathcal{C}$ of sets is **countably compact** *if every decreasing chain $C_0 \supset C_1 \supset \cdots$ of nonempty sets in $\mathcal{C}$ has a nonempty intersection $\bigcap_n C_n$.*

Every basic section for λ is closed in $\mathcal{M}$, because the function λ is continuous. The following proposition and theorem on basic sections were proved in [K5].

Proposition 2.6 *Let λ be a law mapping on Ω.*

(i) For every $z \in \mathcal{K}$, every basic set for λ is a basic section for λ with parameter z.

(ii) Let $y \in \mathcal{N}$ and $z \in \mathcal{K}$. Every basic section $B \subseteq \mathcal{M}$ for λ with parameter y is a basic section for λ with parameter (y, z).

(iii) If $A \subseteq \mathcal{M}$ and $B \subseteq \mathcal{M}$ are basic sections for λ, then $A \cap B$ and $A \cup B$ are basic sections for λ.

(iv) For each $\mathcal{M} \in \mathbf{M}_\Omega$, every finite subset $A = \{x_1, \dots, x_m\}$ of $\mathcal{M}$ is a basic section for λ with parameter $z = (x_1, \dots, x_m)$ in the Cartesian power $\mathcal{K} = \mathcal{M}^m$.

(v) Suppose λ is closed and has the back and forth property. Let $A \subseteq \mathcal{M}$ be a basic section for λ with parameter z. Then for each $y \in \mathcal{N}$, the set $B = \lambda(A, y)$ is compact in $\Lambda(\mathcal{M} \times \mathcal{N})$. $\square$

Theorem 2.7 *([K1, Theorem 4.9 and Corollary 4.10]).*

(i) A law mapping λ on Ω is closed if and only if the family of basic sets $B \subseteq \mathcal{M}$ for λ is countably compact.

(ii) A law mapping λ on Ω is closed and has the back and forth property if and only if it is dense and for each $z \in \mathcal{K}$, the family of basic sections $B \subseteq \mathcal{M}$ for λ with parameter z is countably compact. $\square$

We now introduce the notion of a neocompact set over a family of sets $\mathcal{A}$, which corresponds to the notion of a neocompact formula over $\mathcal{A}$ in [K5]. We shall then state a quantifier elimination theorem from [K5], which shows that the neocompact sets can be represented in a simple form.

Definition 2.8 *For each $\mathcal{M} \in \mathbf{M}_\Omega$, let $\mathcal{A}(\mathcal{M})$ be a family of subsets of $\mathcal{M}$. A* **neocompact set** *over $\mathcal{A}$ is a set which is built using the following rules.*

(a) Every set in $\mathcal{A}(\mathcal{M})$ is neocompact over $\mathcal{A}$.

(b) The union of two neocompact subsets of $\mathcal{M}$ over $\mathcal{A}$ is neocompact over $\mathcal{A}$.

(c) If $A \subseteq \mathcal{M}$ and $B \subseteq \mathcal{N}$ are neocompact over $\mathcal{A}$, then $A \times B$ is neocompact over $\mathcal{A}$.

(d) If $\langle A_n : n \in \mathbf{N} \rangle$ is a countable sequence of neocompact subsets of $\mathcal{M}$ over $\mathcal{A}$, then the intersection $\bigcap_n A_n$ is a neocompact set over $\mathcal{A}$.

(e) If $A \subseteq \mathcal{M} \times \mathcal{N}$ is neocompact over $\mathcal{A}$, then the existential projection

$$\{x \in \mathcal{M} : (\exists y \in \mathcal{N})(x, y) \in A\}$$

is neocompact over $\mathcal{A}$, and the analogous rule holds for each factor in a finite Cartesian product.

(f) If $A \subseteq \mathcal{M} \times \mathcal{N}$ is neocompact over $\mathcal{A}$ and $\emptyset \neq C \in \mathcal{A}(\mathcal{N})$, then the universal projection

$$\{x \in \mathcal{M} : (\forall y \in C)(x, y) \in A\}$$

is neocompact over $\mathcal{A}$, and the analogous rule holds for each factor in a finite Cartesian product.

In [FK1] and [FK2], the family of neocompact sets over $\mathcal{A}$ is called the neocompact family generated by $(\mathbf{M}_\Omega, \mathcal{A})$.

A function $f : \mathcal{M} \to \mathcal{N}$ is **neocontinuous over** $\mathcal{A}$ if the graph of $f|C$ is neocompact over $\mathcal{A}$ for each neocompact set $C \subseteq \mathcal{M}$ over $\mathcal{A}$.

Let $z \in \mathcal{K}$. We say that a set $C \subseteq \mathcal{M}$ is a **basic section over** $\mathcal{A}$ **with parameter** z if C has the form

$$C = \{x \in \mathcal{M} : (x, z) \in D\} \tag{1}$$

for some $D \in \mathcal{A}(\mathcal{M} \times \mathcal{K})$, and that C is a **neocompact section over** $\mathcal{A}$ **with parameter** z if C has the form (2) for some neocompact set $D \subseteq \mathcal{M} \times \mathcal{K}$ over $\mathcal{A}$.

Thus C is a basic section for λ as previously defined if and only if C is a basic section over the family of basic sets for λ.

It is obvious that every neocompact set over $\mathcal{A}$ is a neocompact section over $\mathcal{A}$. The following proposition is a converse.

Proposition 2.9 *([FK1, Proposition 3.6]). Suppose that for each $\mathcal{M}$, every finite subset of $\mathcal{M}$ belongs to $\mathcal{A}(\mathcal{M})$. Then every neocompact section over $\mathcal{A}$ is a neocompact set over $\mathcal{A}$.* $\square$

The next theorem was proved in [K5, Theorem 6.5 and Corollary 6.6].

Theorem 2.10 *(Quantifier Elimination for Neocompact Formulas) Let λ be a closed law mapping on Ω, and let $\mathcal{A}(\mathcal{M})$ be the family of basic subsets of $\mathcal{M}$ for λ. The following are equivalent.*

(i) λ has the back and forth and Skorokhod properties.

(ii) Each neocompact set over $\mathcal{A}$ is basic for λ.

(iii) Each neocompact section over $\mathcal{A}$ with parameter z is a basic section over $\mathcal{A}$ with parameter z. $\square$

Corollary 2.11 *Let λ be a closed law mapping with the back and forth and Skorokhod properties. Let $\mathcal{B}(\mathcal{M})$ be the family of subsets of $\mathcal{M}$ which are either finite or basic for λ. Then a set is neocompact over $\mathcal{B}$ if and only if it is a basic section for λ.*

Proof: By Proposition 2.9, every basic section for λ is neocompact over $\mathcal{B}$. By the Quantifier Elimination Theorem and Proposition 2.6, the family of basic sections for λ is closed under the rules (a)–(f) with respect to $\mathcal{B}(\mathcal{M})$. $\square$

The existential quantifier step of the proof of the Quantifier Elimination Theorem 2.10 used the following result (Theorem 5.2 in [K5]), which will be useful in its own right in this paper.

Theorem 2.12 *Let λ be a closed law mapping. The following are equivalent.*

(i) λ has the back and forth property.

(ii) For every basic set $A \subseteq \mathcal{M} \times \mathcal{N}$ for λ, the set

$$B = \{x \in \mathcal{M} : (\exists y \in \mathcal{N})(x, y) \in A\} \tag{2}$$

is basic for λ.

The implication (ii) $\Rightarrow$ (i) holds for all law mappings.

3 Basic Sections and Neocompact Sets

In this section we shall prove some additional results about basic sections for law mappings. Throughout this section we assume that λ is a law mapping on a probability space Ω.

Lemma 3.1 *For each countable sequence $\langle C_n \rangle$ of basic sections for λ in $\mathcal{M}$, there is a single space $\mathcal{K} \in \mathbf{M}_\Omega$ and $z \in \mathcal{K}$ such that each C_n is a basic section for λ with parameter z.*

Proof: We have

$$C_n = \{x \in \mathcal{M} : \lambda(x, z_n) \in D_n\}$$

where $z_n \in \mathcal{K}_n \in \mathbf{M}_\Omega$ and D_n is compact in $\Lambda(\mathcal{M} \times \mathcal{K}_n)$. Let $\mathcal{K}$ be the product space $\mathcal{K} = \prod_n \mathcal{K}_n$. Let $z = \langle z_n \rangle \in \mathcal{K}$. Fix $n \in \mathbf{N}$. Let $\hat{C}_n$ be the closure of the set

$$\{\lambda(u, z) : u \in \mathcal{M} \text{ and } \lambda(u, z_n) \in D_n\}.$$

$\hat{C}_n$ is relatively compact and hence compact by (2.1.3) and (2.1.2). We claim that

$$C_n = \{x \in \mathcal{M} : \lambda(x, z) \in \hat{C}_n\}.$$

By the definition of $\hat{C}_n, C_n$ is contained in the right side. Suppose $\lambda(x, z) \in \hat{C}_n$. Then

$$\lambda(x, z) = \lim_{k \to \infty} \lambda(x_k, z)$$

for some sequence x_k in $\mathcal{M}$ with $\lambda(x_k, z_n) \in D_n$. By (2.1.3),

$$\lambda(x, z_n) = \lim_{k \to \infty} \lambda(x_k, z_n),$$

so $\lambda(x, z_n) \in D_n$ and $x \in C_n$. This proves the claim and shows that C_n is a basic section for λ with parameter z. $\square$

Corollary 3.2 *For each $\mathcal{M}$ the family of basic sections for λ in $\mathcal{M}$ is closed under countable intersections.*

Proof: This follows from the preceding lemma and the equation

$$\bigcap_n \{x \in \mathcal{M} : \lambda(x,z) \in D_n\} = \{x \in \mathcal{M} : \lambda(x,z) \in \bigcap_n D_n\} \ \Box$$

Corollary 3.3 *If λ is closed and has the back and forth property, then for each $\mathcal{M} \in \mathbf{M}_\Omega$ the family of basic sections for λ in $\mathcal{M}$ is countably compact.*

Proof: By Theorem 2.7 and Lemma 3.1. $\Box$

We shall call a set $B \subseteq \mathcal{M}$ **basic/compact** for λ if it is either a basic subset of $\mathcal{M}$ for λ or is a compact subset of $\mathcal{M}$. We say that C is **neocompact for** λ if C is neocompact over the family of basic/compact sets for λ. By Proposition 2.9, every neocompact section over the basic/compact sets for λ is a neocompact set for λ.

Proposition 3.4 *Each compact set $C \subseteq \mathcal{M}$ is a basic section for λ. Moreover, a set $A \subseteq \mathcal{M}$ is a basic section for λ if and only if A is a basic section over the basic/compact sets for λ.*

Proof: Suppose C is nonempty, and choose a countable sequence $z = \langle z_n \rangle$ which is dense in C. Then z belongs to the countable Cartesian power $\mathcal{K} = \mathcal{M}^{\mathbf{N}}$, and $\mathcal{K} \in \mathbf{M}_\Omega$. Since λ is continuous on $\mathcal{M} \times \mathcal{K}$, the set

$$D = \{\lambda(x,z) : x \in C\}$$

is compact in $\Lambda(\mathcal{M} \times \mathcal{K})$. It suffices to show that

$$C = \{x \in \mathcal{M} : \lambda(x,z) \in D\}. \tag{3}$$

Clearly C is contained in the right side. Let $\lambda(x,z) \in D$. Then $\lambda(x,z) = \lambda(y,z)$ for some $y \in C$. Therefore some subsequence of z_n converges to y in $\mathcal{M}$. To simplify notation suppose that $\lim_{n\to\infty} z_n = y$. By (2.1.3), we have $\lambda(x,z_n) = \lambda(y,z_n)$ for each n. Moreover,

$$\lim_{n\to\infty} \lambda(x,z_n) = \lambda(x,y)$$

and

$$\lim_{n\to\infty} \lambda(y,z_n) = \lambda(y,y).$$

Therefore $\lambda(x,y) = \lambda(y,y)$. Thus by (2.1.1), we have $x = y \in C$. This proves (2). $\Box$

In many of the applications of neocompact sets in [FK1], the compact sets were included in the initial family $\mathcal{A}(\mathcal{M})$ which was used as the starting point in building the neocompact sets. Proposition 3.4 shows that every compact set is a basic section for λ. We now complete the picture by showing that the family of basic sections for λ is closed under universal projections with respect to a compact set.

Proposition 3.5 *Let $C \subseteq \mathcal{N}$ be a nonempty compact set. If A is a basic section in $\mathcal{M} \times \mathcal{N}$ for λ then the set*

$$B = \{x \in \mathcal{M} : (\forall y \in C)(x, y) \in A\}$$

is also a basic section for λ.

Proof: Let $\{y_n : n \in \mathbf{N}\}$ be a countable dense subset of C. The set A has the form

$$A = \{(x, y) \in \mathcal{M} \times \mathcal{N} : \lambda(x, y, z) \in \hat{A}\}$$

for some $z \in \mathcal{K} \in \mathbf{M}_\Omega$ and some compact set $\hat{A}$. For each n, the set

$$B_n = \{x \in \mathcal{M} : \lambda(x, y_n, z) \in \hat{A}\} = \{x \in \mathcal{M} : (x, y_n) \in A\}$$

is a basic section for λ. By Corollary 3.2, the intersection $\bigcap_n B_n$ is a basic section for λ. We show that $B = \bigcap_n B_n$. It is obvious that $B \subseteq \bigcap_n B_n$. Suppose $x \in \bigcap_n B_n$. Let $y \in C$. Then some sequence $\langle y_{k_n} \rangle$ of $\langle y_n \rangle$ converges to y. For each n we have $\lambda(x, y_{k_n}, z) \in \hat{A}$. Since $\hat{A}$ is compact and λ is continuous, we have $\lambda(x, y, z) \in \hat{A}$, so $(x, y) \in A$ and $x \in B$. $\square$

Theorem 3.6 *Suppose λ has the back and forth and Skorokhod properties. Then a set is neocompact for λ if and only if it is a basic section for λ. Moreover, for each $\mathcal{M}$ the family of neocompact subsets of $\mathcal{M}$ for λ is countably compact.*

Proof: By the Quantifier Elimination Theorem 2.10 and Propositions 3.4 and 3.5 the family of basic sections for λ is closed under the rules (a)–(f) where $\mathcal{A}(\mathcal{M})$ is the family of basic/compact sets. Countable compactness follows from Corollary 3.3. $\square$

For the remainder of this section we let λ_k be a sequence of law mappings on Ω. For each $k \in \mathbf{N}$, we shall let $\vec{\lambda}_k$ be the finite product

$$\vec{\lambda}_k = \langle \lambda_0, \ldots, \lambda_k \rangle,$$

and let $\vec{\Lambda}_k(\mathcal{M})$ be the topological product $\Lambda_0(\mathcal{M}) \times \cdots \times \Lambda_k(\mathcal{M})$.

We also let λ be the countable product where $\lambda(x) = \langle \lambda_k(x) : k \in \mathbf{N} \rangle$, and let $\Lambda(\mathcal{M})$ be the countable topological product $\prod_k \Lambda_k(\mathcal{M})$.

We state two more results which are proved in [K5].

Proposition 3.7 *([K1, Proposition 7.1 and Lemma 7.3]).*

(i) For each k, $(\vec{\lambda}_k, \vec{\Lambda}_k)$ is a law mapping on Ω, and (λ, Λ) is a law mapping on Ω.

(ii) Suppose that $\vec{\lambda}_k$ is a dense law mapping for each $k \in \mathbf{N}$. Then λ is a dense law mapping.

(iii) A set B is basic for λ if and only if $B = \bigcap_k B_k$ for some sequence of basic sets B_k for $\vec{\lambda}_k$. $\Box$

Theorem 3.8 *([K1, Theorem 7.7 and Corollary 7.8]). Suppose that λ is closed and has the back and forth property, and for each k, $\vec{\lambda}_k$ is closed and has the back and forth and Skorokhod properties. Let $\mathcal{A}_k(\mathcal{M})$ be the family of all basic subsets of $\mathcal{M}$ for $\vec{\lambda}_k$, and let $\mathcal{A}(\mathcal{M}) = \bigcup_k \mathcal{A}_k(\mathcal{M})$. Then every neocompact set over $\mathcal{A}$ is basic for λ, and every neocompact section over $\mathcal{A}$ is a basic section for λ.* $\Box$

We now improve Theorem 3.8 by replacing the basic sets by the basic/compact sets.

Theorem 3.9 *Suppose that λ is closed and has the back and forth property, and for each k, $\vec{\lambda}_k$ is closed, and has the back and forth and Skorokhod properties. Let $\mathcal{B}_k(\mathcal{M})$ be the family of all basic/compact subsets of $\mathcal{M}$ for $\vec{\lambda}_k$, and let $\mathcal{B}(\mathcal{M}) = \bigcup_k \mathcal{B}_k(\mathcal{M})$. Then a set is neocompact over $\mathcal{B}$ if and only if it is a basic section for λ. Moreover, for each $\mathcal{M}$ the family of neocompact subsets of $\mathcal{M}$ over $\mathcal{B}$ is countably compact.*

Proof: By Proposition 2.9, every basic section for λ is neocompact over $\mathcal{B}$. By Theorem 3.8, the family of basic sections for λ is closed under the operations (a)–(e) and under universal projections with respect to basic sets for $\vec{\lambda}_k$. By Proposition 3.5, for each $k \in \mathbf{N}$ the family of basic sections for $\vec{\lambda}_k$ is closed under universal projections with respect to compact sets. By Proposition 3.7 (ii), every basic section for λ is the intersection of a descending chain of basic sections B_k for $\vec{\lambda}_k$, and it follows that the family of basic sections for λ is closed under universal projections with respect to compact sets. This shows that every neocompact set over $\mathcal{B}$ is a basic section for λ. Countable compactness follows from Corollary 3.3. $\Box$

4 Probability Spaces

In this section we study the law mapping $(law, Meas)$ for an atomless probability space Ω. We shall see that this law mapping has the back and forth property if and only if Ω is rich. A measure space $(\Gamma, \mathcal{G}, Q)$ with $0 < Q(\Gamma) < \infty$ is said to be atomless if for each set $S \in \mathcal{G}$ of measure $Q(S) > 0$ and each positive $r < Q(S)$ there is a subset $U \subseteq S$ in $\mathcal{G}$ such that $Q(U) = r$. Note that if $\Omega = (\Omega, \mathcal{F}, P)$ is an atomless probability space and $P(\Gamma) > 0$ then the restriction of Ω to Γ is an atomless measure space.

The following formula due to Strassen (see [EK, Theorem 1.2 on p. 96]) characterizes the Prohorov metric in terms of the metric of convergence in probability when Ω is an atomless probability space.

Lemma 4.1 *Suppose Ω is atomless and $\mathcal{M} \in \mathbf{M}_\Omega$. Then for all $b, c \in Meas(M)$,*

$$d(b,c) = inf\{\rho_0(x,y) : x, y \in \mathcal{M}, law(x) = b, law(y) = c\}.$$

The next proposition rephrases some well known facts in our framework.

Proposition 4.2 *Let Ω be an atomless probability space. Then law is a closed law mapping on Ω.*

Proof: By Proposition 4.1, $law : \mathcal{M} \to Meas(M)$ is uniformly continuous, and in fact, $d(law(x), law(y)) \leq \rho_0(x,y)$. Conditions (2.1.1) and (2.1.3) are easily checked, and condition (2.1.2) follows from the characterization of relative compactness given by Prohorov's theorem. Thus *law* is a law mapping. It is well known that a probability space Ω is atomless if and only if *law* maps $\mathcal{M}$ onto $Meas(M)$ for each M. Therefore $(law, Meas)$ is closed on Ω. $\square$

We now recall the notion of a saturated probability space from [HK].

Definition 4.3 *We say that $\Omega = (\Omega, P, \mathcal{G})$ is a* **saturated probability space** *if for every probability space Γ and all complete separable metric spaces M and N, if*

$$\bar{x} \in L^0(\Gamma, M), \bar{y} \in L^0(\Gamma, N), x \in \mathcal{M},$$

and $law(x) = law(\bar{x})$, then there exists $y \in \mathcal{N}$ such that $law(x,y) = law(\bar{x}, \bar{y})$.

It is easily seen that every saturated probability space is atomless and has the back and forth property. It was shown in [HK] that uncountable powers of $[0,1]$ and atomless Loeb probability spaces are saturated, and thus that saturated probability spaces exist.

It is well known that the set of **simple functions** (functions with finite range) is dense in each $\mathcal{M} \in \mathbf{M}_\Omega$. Every atomless probability space satisfies the special case of saturation where x is a simple function.

Proposition 4.4 *Let Ω be an atomless probability space, and let Γ be another probability space. Then for every simple $x \in \mathcal{M}$ and every pair of random variables $(\bar{x}, \bar{y}) \in L^0(\Gamma, M \times N)$ such that $law(\bar{x}) = law(x)$, there exists $y \in \mathcal{N}$ such that $law(x, y) = law(\bar{x}, \bar{y})$.*

Proof: Let $\Omega = (\Omega, \mathcal{F}, P)$ and $\Gamma = (\Gamma, \mathcal{G}, Q)$. Let $\{m_1, \ldots, m_k\}$ be the range of x, and let $A_j = \{\omega : x(\omega) = m_j\}$ and $B_j = \{\gamma : \bar{x}(\gamma) = m_j\}$. We may assume without loss of generality that $P(A_j) > 0$ for each j. Let $\bar{y}_j$ be the restriction of $\bar{y}$ to the set B_j. Since the restriction of Ω to A_j is atomless and $Q(B_j) = P(A_j)$, there is a random variable y_j on A_j such that $law(y_j) = law(\bar{y}_j)$. Now take $y \in \mathcal{N}$ such that $y(\omega) = y_j(\omega)$ whenever $\omega \in A_j$ for $j = 1, \ldots, n$. Then $law(x, y) = law(\bar{x}, \bar{y})$. $\square$

The next proposition applies the Skorokhod representation theorem in probability theory, and is the reason for our use of the name "Skorokhod property".

Proposition 4.5 *If Ω is an atomless probability space, then law has the Skorokhod property on Ω.*

Proof: Let $x \in \mathcal{M}$, and let c_n be a sequence converging to $law(x)$ in $Meas(M)$. We must find a sequence x_n converging to x in $\mathcal{M}$ such that $law(x_n) = c_n$ for all n.

The Skorokhod representation theorem says that on some probability space Γ there are random variables $z_n, z \in L^0(\Gamma, M)$ such that $law(z_n) = c_n$ for all n, $law(z) = law(x)$, and $z_n \to z$ almost surely (see [EK, p. 102]). Let Δ be a saturated probability space. Then there are random variables y_n, y in $L^0(\Delta, M)$ such that

$$law(y, \langle y_n \rangle) = law(z, \langle z_n \rangle).$$

It follows that $y_n \to y$ in $L^0(\Delta, M)$. Let $u_n, n \in \mathbf{N}$ be a sequence of simple random variables converging to x in $\mathcal{M}$. Since Δ is saturated there is a sequence $v_n, n \in \mathbf{N}$ in $L^0(\Delta, M)$ such that

$$law(y, \langle v_n \rangle) = law(x, \langle u_n \rangle).$$

By the preceding proposition, for each n there exists $x_n \in \mathcal{M}$ such that $law(u_n, x_n) = law(v_n, y_n)$. Then $law(x_n) = c_n$ for all n. Since y_n and v_n both converge to y, we have $\rho_0(y_n, v_n) \to 0$, and thus $\rho_0(x_n, u_n) \to 0$. Therefore $x_n \to x$ in $\mathcal{M}$. $\square$

Proposition 4.6 *Let Ω be an atomless probability space. Then law is dense on Ω.*

Proof: Let $x, \bar{x} \in \mathcal{M}$ with $law(x) = law(\bar{x})$, and let $y \in \mathcal{N}$. Let $\langle x_n \rangle$ be a sequence of simple functions converging to x in $\mathcal{M}$. Then $law(x_n) \to law(\bar{x})$. By the Skorokhod property there is a sequence $\langle \bar{x}_n \rangle$ converging to $\bar{x}$ in $\mathcal{M}$ such that

$law(\bar{x}_n) = law(x_n)$ for each n. Then for each n, $\bar{x}_n$ is simple. By Proposition 4.4 there exists $\bar{y}_n \in \mathcal{N}$ such that $law(\bar{x}_n, \bar{y}_n) = law(x_n, y)$. We have $(x_n, y) \to (x, y)$ in $\mathcal{M} \times \mathcal{N}$, and therefore $law(\bar{x}_n, \bar{y}_n) \to law(x, y)$. Moreover, since $\bar{x}_n \to \bar{x}$ in $\mathcal{M}$, $d(law(\bar{x}_n, \bar{y}_n), law(\bar{x}, \bar{y}_n)) \to 0$. Therefore $law(\bar{x}, \bar{y}_n) \to law(x, y)$, so $law(\bar{x}, \mathcal{N})$ is dense in the closure of $law(x, \mathcal{N})$. $\square$

We now review the notion of a rich probability space from [FK1]. We shall see that richness and saturation are equivalent.

Definition 4.7 *Ω is said to be a* **rich probability space** *if Ω is atomless and for each $\mathcal{M} \in \mathbf{M}_\Omega$, the family of neocompact subsets of $\mathcal{M}$ for law is countably compact.*

From the previous sections, a set B is basic for *law* in $\mathcal{M}$ if it is of the form

$$\{x \in \mathcal{M} : law(x) \in C\}$$

for some compact set $C \subseteq Meas(M)$, and is a basic section for *law* in $\mathcal{M}$ if it is of the form

$$\{x \in \mathcal{M} : law(x, z) \in D\}$$

for some compact set $D \subseteq Meas(M \times N)$ and some $z \in \mathcal{N}$.

Theorem 4.8 *Let Ω be an atomless probability space. The following are equivalent.*

(i) Ω is saturated.

(ii) law has the back and forth property on Ω.

(iii) Ω is rich.

(iv) For each $\mathcal{M} \in \mathbf{M}_\Omega$ the family of basic sections for law in $\mathcal{M}$ is countably compact.

(v) For each $\mathcal{M}, \mathcal{N} \in \mathbf{M}_\Omega$ and basic relation C for law in $\mathcal{M} \times \mathcal{N}$, the existential projection

$$\{x : \exists y (x, y) \in C\}$$

is basic for law.

Proof: By Propositions 4.2 and 4.6, *law* is closed and dense on Ω.

We first prove that (i) and (ii) are equivalent. It is easily seen that (i) implies (ii). To prove (ii) implies (i), assume (ii) and let $\bar{x} \in L^0(\Gamma, M), \bar{y} \in L^0(\Gamma, N), x \in \mathcal{M}$, and $law(x) = law(\bar{x})$. Since Ω is atomless there exists $(x', y') \in \mathcal{M} \times \mathcal{N}$ such that $law(x', y') = law(\bar{x}, \bar{y})$. Then $law(x') = law(x)$, and by the back and forth property for *law* on Ω there exists $y \in \mathcal{N}$ such that $law(x, y) = law(\bar{x}, \bar{y})$.

Next we assume (i) and prove (iii). By Proposition 4.5, *law* has the Skorokhod property on Ω. Then Ω is rich by Theorem 3.6, so (iii) holds.

Since every basic section for *law* is neocompact, (iii) implies (iv).

Assume (iv). Since *law* is dense on Ω, it has the back and forth property on Ω by Theorem 2.7. Thus (iv) implies (ii). (v) is equivalent to (ii) by Theorem 2.12. $\square$

We conclude this section with some examples arising in probability spaces which are not rich. By an **ordinary** probability space we shall mean a probability space of the form $(\Gamma, \mu, \mathcal{G})$ where Γ is a complete separable metric space and μ is the completion of a Borel probability measure on the family of Borel sets $\mathcal{G}$ in Γ. These spaces are the ones most commonly used in the literature. It is shown in [FK1] shows that no ordinary probability space is rich.

Consider an atomless ordinary probability space $(\Gamma, \mu, \mathcal{G})$, and let $\mathcal{B} = \{O_n : n \in \mathbf{N}\}$ be a countable open basis for Γ. We say that a measurable set A is **independent** of a family of sets $\mathbf{S}$ in $(\Gamma, \mu, \mathcal{G})$ if

$$\mu(A \cap B) = \mu(A)\mu(B) \text{ for all } B \in \mathbf{S}.$$

The same terminology is applied to families of characteristic functions of sets.

For each n, let x_n be the characteristic function of O_n, considered as a random variable on Γ with values in the two-element space $\{0,1\}$. Every measurable set in Γ can be approximated by sets in the basis $\mathcal{B}$, and therefore no set of measure one half in Γ can be independent of $\{x_n : n \in \mathbf{N}\}$.

Example 4.9 *For each n, let C_n be the set of all z on Γ such that z is the characteristic function of a set of measure $1/2$ and z is independent of $x_1, \ldots, x_n$. Then C_n is a decreasing chain of nonempty neocompact subsets of $L^0(\Gamma, \{0,1\})$ for law, but $\bigcap_n C_n$ is empty. In fact, each C_n is a basic section for law. This shows that Γ is not rich.*

Example 4.10 *Now consider the product space $\Gamma \times \Gamma$, and let $\bar{x}_n(\gamma_1, \gamma_2) = x_n(\gamma_1)$. Let u be the characteristic function of a set of measure one half in Γ, and let $\bar{y}(\gamma_1, \gamma_2) = u(\gamma_2)$ on $\Gamma \times \Gamma$. Then $law(\langle x_n \rangle) = law(\langle \bar{x}_n \rangle)$, but there is no y on Γ such that $law(\langle x_n \rangle, y) = law(\langle \bar{x}_n \rangle, \bar{y})$. This shows directly that Γ is not saturated. The example can by modified by taking $\bar{x}_n, \bar{y}$ on Γ itself, giving a direct example of the failure of the back and forth property on Γ.*

Example 4.11 *Let C be the set of all pairs (x, y) such that $x = \langle x_n \rangle$ is a sequence of characteristic functions of sets, and y is the characteristic function of a set of measure $1/2$ which is independent of the the family $\{x_n : n \in \mathbf{N}\}$. Then C is a basic relation for law on Γ. However, the existential projection $D = \{x : \exists y(\langle x_n \rangle, y) \in C\}$ is not closed in $L^0(\Gamma, \{0,1\})$ and therefore cannot even be a basic section for law. In fact, if $x = \langle x_1, x_2, \ldots, x_n, \ldots \rangle \notin D$ and $z_k = \langle x_1, \ldots, x_k, 0, 0, \ldots \rangle$, then $z_k \in D$ and $z_k \to x$.*

5 Adapted Spaces with Finite Time Sets

We now apply our results to adapted probability spaces with finite time sets. In this and the next two sections we shall introduce law mappings for these adapted spaces, and prove that in this setting saturation is again equivalent to richness.

For the next four sections of this paper (through Section 8), we shall take $\Lambda(\mathcal{M})$ to be the space

$$\Lambda(\mathcal{M}) = \mathbf{R}^{\mathbf{N}} \times Meas(M)$$

with the product metric. Whenever we introduce a law mapping, it will be understood that the target space is this particular space $\Lambda(\mathcal{M})$.

Let $\mathbf{T}$ be a finite set of nonnegative real numbers. By a **$\mathbf{T}$-adapted (probability) space** we mean a structure

$$\Omega_{\mathbf{T}} = (\Omega, P, \mathcal{G}_\infty, \mathcal{G}_t)_{t \in \mathbf{T}}$$

where $(\Omega, P, \mathcal{G}_\infty)$ is a probability space, $\mathcal{G}_t$ is a σ-subalgebra of $\mathcal{G}_\infty$ for each $t \in \mathbf{T}$, and $\mathcal{G}_s \subseteq \mathcal{G}_t$ whenever $s \leq t$ in $\mathbf{T}$.

Throughout this section we let $\Omega_{\mathbf{T}}$ be a $\mathbf{T}$-adapted probability space and let $\mathcal{M} \in \mathbf{M}_\Omega$. $\mathcal{R}$ is the metric space $L^0(\Omega, \mathbf{R})$.

We now recall the notion of an adapted function, which was introduced in [HK].

Definition 5.1 *The class of* **$\mathbf{T}$-adapted functions** *on $\mathcal{M}$ is the least class of functions from $\mathcal{M}$ into $\mathcal{R}$ such that:*

(i) For each bounded continuous function $\phi : M \to \mathbf{R}$, the function $(\hat{\phi}(x))(\omega) = \phi(x(\omega))$ is a $\mathbf{T}$-adapted function;

(ii) If $f_1, \ldots, f_m$ are $\mathbf{T}$-adapted functions on $\mathcal{M}$ and $g : \mathbf{R}^m \to \mathbf{R}$ is continuous, then $h(x) = g(f_1(x), \ldots, f_m(x))$ is a $\mathbf{T}$-adapted function;

(iii) If f is a $\mathbf{T}$-adapted function and $t \in \mathbf{T}$, then $g(x)(\omega) = E[f(x)|\mathcal{G}_t](\omega)$ is a $\mathbf{T}$-adapted function.

Observe that each $\mathbf{T}$-adapted function on $\mathcal{M}$ is uniformly bounded, so the expected value $E[f(x)]$ is defined and finite for every $\mathbf{T}$-adapted function f and every $x \in \mathcal{M}$. Two processes $x, y \in \mathcal{M}$ are said to have the **same adapted distribution** if $E[f(x)] = E[f(y)]$ for all $\mathbf{T}$-adapted functions f.

We shall now take advantage of the separability of M to choose a countable set of $\mathbf{T}$-adapted functions on $\mathcal{M}$ which is dense in an appropriate sense.

A set Ψ of bounded functions $f : M \to \mathbf{R}$ is said to be **bounded pointwise dense** if every bounded Borel function $g : M \to \mathbf{R}$ belongs to the closure of Ψ under pointwise convergence of uniformly bounded sequences of functions.

Suppose Ψ is bounded pointwise dense. Then Ψ separates points in M, that is, if $u \neq v$ in M then $\psi(u) \neq \psi(v)$ for some $\psi \in \Psi$. Moreover, Ψ is separating in $Meas(M)$, that is, if $b \neq c$ in $Meas(M)$ then $\int \psi \, db \neq \int \psi \, dc$ for some $\psi \in \Psi$.

For each complete separable metric space M, there exists a countable set $\Phi(M)$ of bounded continuous functions $\phi : M \to \mathbf{R}$ which is bounded pointwise dense (see [EK, Proposition 4.2]). For each M, we shall choose such a set $\Phi(M)$ once and for all.

Definition 5.2 *The class of* **T-adapted functions** *built from* $\Phi(M)$ *is the least class of* $\mathbf{T}$*-adapted functions on* $\mathcal{M}$ *such that:*

(i′) For each function $\phi \in \Phi(M)$*, the corresponding function* $\hat{\phi} : \mathcal{M} \to \mathcal{R}$ *is a* $\mathbf{T}$*-adapted function built from* $\Phi(M)$*;*

(ii′) If $f_1, \ldots, f_m$ *are* $\mathbf{T}$*-adapted functions built from* $\Phi(M)$ *and* $p : \mathbf{R}^m \to \mathbf{R}$ *is a polynomial with rational coefficients, then* $h(x) = p(f_1(x), \ldots, f_m(x))$ *is a* $\mathbf{T}$*-adapted function built from* $\Phi(M)$*;*

(iii′) If f *is a* $\mathbf{T}$*-adapted function built from* $\Phi(M)$ *and* $t \in \mathbf{T}$*, then* $g(x)(\omega) = E[f(x)|\mathcal{G}_t](\omega)$ *is a* $\mathbf{T}$*-adapted function built from* $\Phi(M)$*.*

There are only countably many $\mathbf{T}$-adapted functions built from $\Phi(M)$. Let us arrange them in a list $\langle \beta_k : k \in \mathbf{N} \rangle$. We now define the $\mathbf{T}$-adapted law function.

Definition 5.3 *By the* **T-adapted law** *of a random variable* $x \in \mathcal{M}$ *we mean the pair*

$$law_{\mathbf{T}}(x) = (\langle E[\beta_k(x)] : k \in \mathbf{N} \rangle, law(x))$$

in the space $\mathbf{R}^{\mathbf{N}} \times Meas(M)$*.*

The reader may wonder why the second coordinate $law(x)$ is needed. One reason is to insure that condition (2.1.1) holds. Another reason is to insure that $law_{\mathbf{T}}$ is closed on Ω. The image of the first term $\langle E[\beta_k(x)] : k \in \mathbf{N} \rangle$ is almost never closed in $\mathbf{R}^{\mathbf{N}}$, but we shall see in Proposition 6.7 that for all "atomless" $\mathbf{T}$-adapted spaces the image of $law_{\mathbf{T}}$ on $\mathcal{M}$ is a closed subset of $\mathbf{R}^{\mathbf{N}} \times Meas(M)$. The second coordinate of $law_{\mathbf{T}}$ will also be needed in Lemma 7.8, which is used in the proof that saturated adapted spaces are rich.

The $\mathbf{T}$-adapted law function is defined for every $\mathbf{T}$-adapted space $\Omega_{\mathbf{T}}$, and it will sometimes be useful to compare $law_{\mathbf{T}}(x)$ and $law_{\mathbf{T}}(y)$ where x and y are random variables on two different $\mathbf{T}$-adapted spaces $\Omega_{\mathbf{T}}$ and $\Gamma_{\mathbf{T}}$.

Note that for any $x \in \mathcal{M}$ and sequence x_n in $\mathcal{M}$, if $\mathbf{S} \subseteq \mathbf{T}$ and

$$\lim_{n \to \infty} law_{\mathbf{T}}(x_n) = law_{\mathbf{T}}(x),$$

then

$$\lim_{n\to\infty} law_{\mathbf{S}}(x_n) = law_{\mathbf{S}}(x).$$

We now give a series of lemmas which we shall use to show that $law_{\mathbf{T}}$ is a law mapping on Ω. The next lemma shows that the expected value of each $\mathbf{T}$-adapted function depends continuously on the $\mathbf{T}$-adapted law.

Lemma 5.4 *Let x, $x_n \in \mathcal{M}$. The following are equivalent:*
(i) $\lim_{n\to\infty} law_{\mathbf{T}}(x_n) = law_{\mathbf{T}}(x)$.
(ii) $\lim_{n\to\infty} law(x_n) = law(x)$ *and* $\lim_{n\to\infty} E[f(x_n)] = E[f(x)]$ *for every* $\mathbf{T}$-*adapted function f on $\mathcal{M}$.*

Proof: This follows from [HK, Theorem 2.26]. $\square$

Lemma 5.5 *Let f be a $\mathbf{T}$-adapted function on $\mathcal{M}$. Then f is continuous from $\mathcal{M}$ into $\mathcal{R}$, and uniformly continuous on $law^{-1}(C)$ for each compact set $C \subseteq Meas(M)$.*

Proof: We argue by induction on the steps used in constructing f. We begin with the basis step. Let C be a compact subset of $Meas(M)$ and let $\phi : M \to \mathbf{R}$ be continuous and bounded by β. Let $\varepsilon > 0$. By Prohorov's theorem, there is a compact set $D \subseteq M$ such that $\mu(D) \geq 1 - \varepsilon/(2\beta)$ for each $\mu \in C$. Since D is compact there exists $\delta > 0$ such that $|\phi(b) - \phi(c)| < \varepsilon$ whenever $b \in D, c \in M$, and $\rho(b, c) < \delta$. Taking $\delta < \varepsilon/(2\beta)$, we see that whenever $x, y \in law^{-1}(C)$ and $\rho_0(x, y) < \delta$, $\hat{\phi}(x)$ is within ε of $\hat{\phi}(y)$ in the metric of convergence in probability. Thus $\hat{\phi}$ is uniformly continuous on $law^{-1}(C)$. The steps of the induction from $f(x)$ to $E[f(x)|\mathcal{G}_t]$, and from $f_1(x), \ldots, f_m(x)$ to $g(f_1(x), \ldots, f_m(x))$, are routine. $\square$

Corollary 5.6 *The function $law_{\mathbf{T}}$ is continuous from $\mathcal{M}$ into $\mathbf{R}^{\mathbf{N}} \times Meas(M)$, and uniformly continuous on $law^{-1}(C)$ for each compact set $C \subseteq Meas(M)$.* $\square$

Lemma 5.7 *For each set $A \subseteq \mathcal{M}$, the set $law_{\mathbf{T}}(A) \subseteq \mathbf{R}^{\mathbf{N}} \times Meas(M)$ is relatively compact if and only if the set $law(A) \subseteq Meas(M)$ is relatively compact.*

Proof: If $law_{\mathbf{T}}(A) \subseteq C$ where C is compact then $law(A)$ is contained in the compact set $\pi(C)$ where π is the projection map $\pi : \mathbf{R}^{\mathbf{N}} \times Meas(M) \to Meas(M)$. Suppose $law(A) \subseteq D$ where D is compact. For each adapted function $f, E[f(x)]$ is bounded uniformly in x. Then by the Tychonoff product theorem, $law_{\mathbf{T}}(\mathcal{M}) \subseteq B \times Meas(M)$ for some compact set $B \subseteq \mathbf{R}^{\mathbf{N}}$. Therefore $law_{\mathbf{T}}(A)$ is contained in the compact set $B \times D$. $\square$

We now show that $law_{\mathbf{T}}$ is a law mapping.

Proposition 5.8 *For any* $\mathbf{T}$*-adapted space* $\Omega_{\mathbf{T}}$*,* $law_{\mathbf{T}}$ *is a law mapping.*

Proof: The continuity of $law_{\mathbf{T}}$ was established in Corollary 5.6. Condition (2.1.1) holds because $law_{\mathbf{T}}(x)$ is a pair whose second coordinate is $law(x)$.

Suppose $A \subseteq \mathcal{M}$, $B \subseteq \mathcal{N}$, and $law_{\mathbf{T}}(A)$ and $law_{\mathbf{T}}(B)$ are relatively compact. By Lemma 5.7, $law(A)$ and $law(B)$ are relatively compact. It follows, e.g. by Prohorov's theorem, that $law(A \times B)$ is relatively compact. Then by Lemma 5.7, $law_{\mathbf{T}}(A \times B)$ is relatively compact. This proves (2.1.2) for $law_{\mathbf{T}}$.

To prove condition (2.1.3), let $h : M \to N$ be continuous. For any adapted function f on $\mathcal{M}$, the function $g(x) = f(\hat{h}(x))$ is an adapted function on $\mathcal{M}$. Suppose that $x, y \in \mathcal{M}$ and $law_{\mathbf{T}}(x) = law_{\mathbf{T}}(y)$. By Lemma 5.4, we have $E[f(\hat{h}(x))] = E[f(\hat{h}(y))]$. Moreover, $law(\hat{h}(x)) = law(\hat{h}(y))$. Therefore $law_{\mathbf{T}}(\hat{h}(x)) = law_{\mathbf{T}}(\hat{h}(y))$. This shows that the function $\bar{h}(law_{\mathbf{T}}(x)) = law_{\mathbf{T}}(\hat{h}(x))$ is well-defined. Another application of Lemma 5.4 shows that $\bar{h}$ is continuous, so (2.1.3) holds. □

6 Atomless Adapted Spaces

In this section we introduce atomless $\mathbf{T}$-adapted spaces, and show that such spaces have natural law mappings which are closed on Ω. In the next section we shall see, using the notion of a saturated $\mathbf{T}$-adapted space, that these law mappings are also dense and have the Skorokhod property. The notion of an atomless $\mathbf{T}$-adapted spaces is taken from [HK]. For notational convenience we let 0 be the least element of $\mathbf{T}$.

Definition 6.1 *([HK]) Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be* σ*-subalgebras of* $\mathcal{G}_\infty$ *with* $\mathcal{E} \subseteq \mathcal{F}$*.* $\mathcal{F}$ *is said to be* **atomless** *over* $\mathcal{E}$ *if for every* $U \in \mathcal{F}$ *of positive probability, there is a set* $V \subseteq U$ *in* $\mathcal{F}$ *such that*

$$0 < P[V|\mathcal{E}] < P[U|\mathcal{E}]$$

on a set of positive probability.

A $\mathbf{T}$*-adapted space* $\Omega_{\mathbf{T}}$ *is said to be* **atomless** *if* $\mathcal{G}_0$ *is atomless over the trivial* σ*-algebra and* $\mathcal{G}_t$ *is atomless over* $\mathcal{G}_s$ *whenever* $s < t$ *in* $\mathbf{T} \cup \{\infty\}$*.*

We now introduce a $\mathbf{T}$-adapted analogue of a simple function x such that $P[x = r]$ is rational for each $r \in \mathcal{M}$. This notion will be useful in analyzing the law mapping $law_{\mathbf{T}}$.

A finite algebra $\mathcal{E}$ of subsets of Ω will be called **uniform** if each atom of $\mathcal{E}$ has the same measure.

Let $\mathbf{T} = \{t_1, \ldots, t_k\}$, and put $t_k + 1 = \infty$. By a **T-partition** of Ω we shall mean a sequence $\mathcal{E} = \langle \mathcal{E}_t, t \in \mathbf{T} \cup \{\infty\} \rangle$ of finite algebras such that $\mathcal{E}_t \subseteq \mathcal{G}_t$, and for each $j \leq k, \mathcal{E}_{t_{j+1}}$ is generated by $\mathcal{E}_{t_j}$ and a uniform finite algebra whose atoms are independent of $\mathcal{G}_{t_j}$. (A set $S \subseteq \Omega$ is independent of $\mathcal{G}_s$ if the conditional probability $P[x|\mathcal{G}_s]$ is constant). We say that a random variable $x \in \mathcal{M}$ is $\mathcal{E}$-**measurable** if it is $\mathcal{E}_\infty$-measurable, and $\mathbf{T}$-**simple** if x is $\mathcal{E}$-measurable for some $\mathbf{T}$-partition $\mathcal{E}$. Note that if (x, y) is $\mathbf{T}$-simple, then both x and y are $\mathbf{T}$-simple, but the converse does not hold in general. Also, if (x, y) and (x, z) are both $\mathbf{T}$-simple, then (x, y, z) is $\mathbf{T}$-simple.

Given a $\mathbf{T}$-partition $\mathcal{E}$ of Ω, let $\Omega/\mathcal{E}_t$ be the set of all $\mathcal{E}_t$-atoms and let $\Omega/\mathcal{E}$ be the set of all $\mathcal{E}_\infty$-atoms. Two $\mathbf{T}$-partitions $\mathcal{E}$ of Ω and $\mathcal{F}$ of Γ are **equivalent** if there is a bijection $h : \Omega/\mathcal{E} \to \Gamma/\mathcal{F}$ such that $h(\Omega/\mathcal{E}_t) = \Gamma/\mathcal{F}_t$ for each $t \in \mathbf{T}$, and h is called an **isomorphism** from $\mathcal{E}$ to $\mathcal{F}$. If x is $\mathcal{E}$-measurable and h is an isomorphism from $\mathcal{E}$ to $\mathcal{F}, h(x)$ is the $\mathcal{F}$-measurable function y such that $y(h(\omega)) = x(\omega)$ for each $\omega \in \Omega/\mathcal{E}$.

The following lemma can be proved by an inductive argument using the results in Maharam [M]. It is a strengthening of the fact that every random variable can be approximated by simple random variables.

Lemma 6.2 *Let $\Omega_\mathbf{T}$ be atomless.*

(i) Let $\Gamma_\mathbf{T}$ be a $\mathbf{T}$-adapted space. For each $\mathbf{T}$-partition $\mathcal{F}$ of Γ there is an equivalent $\mathbf{T}$-partition $\mathcal{E}$ of Ω.

(ii) Let f be a $\mathbf{T}$-adapted function. If $\mathcal{E}$ is a $\mathbf{T}$-partition of Ω and $x \in \mathcal{M}$ is $\mathcal{E}$-measurable, then $f(x)$ is $\mathcal{E}$-measurable, and for each $t \in \mathbf{T}$, $E[f(x)|\mathcal{G}_t]$ is $\mathcal{E}_t$-measurable. If x is $\mathbf{T}$-simple then $f(x)$ is $\mathbf{T}$-simple.

(iii) Let $\mathcal{E}$ be a $\mathbf{T}$-partition of Ω and let x be $\mathcal{E}$-measurable. Then $law_\mathbf{T}(x) = law_\mathbf{T}(y)$ if and only if $y = h(x)$ for some $\mathbf{T}$-partition $\mathcal{F}$ and isomorphism $h : \mathcal{E} \to \mathcal{F}$.

(iv) For every $\mathcal{M}$, the set of $\mathbf{T}$-simple random variables is dense in $\mathcal{M}$. In fact, for each $\mathbf{T}$-simple $y \in \mathcal{N}$, the set of $x \in \mathcal{M}$ such that (x, y) is $\mathbf{T}$-simple is dense in $\mathcal{M}$. $\square$

Corollary 6.3 *Let $\Omega_\mathbf{T}$ be atomless, and let $x \in L^0(\Gamma, M)$ be $\mathbf{T}$-simple on some other $\mathbf{T}$-adapted space $\Gamma_\mathbf{T}$. Then there is a $\mathbf{T}$-simple $y \in L^0(\Omega, M)$ such that $law_\mathbf{T}(x) = law_\mathbf{T}(y)$.*

Proof: Let x be $\mathcal{E}$-measurable where $\mathcal{E}$ is a $\mathbf{T}$-partition of Γ. By Lemma 6.2 (i) there is an equivalent $\mathbf{T}$-partition $\mathcal{F}$ of Ω and an isomorphism h from $\mathcal{E}$ to $\mathcal{F}$. By Lemma 6.2 (iii), $y = h(x)$ is $\mathbf{T}$-simple and $law_\mathbf{T}(x) = law_\mathbf{T}(y)$. $\square$

Here is a back and forth property for $\mathbf{T}$-simple random variables.

Corollary 6.4 *Let $\Omega_\mathbf{T}$ be atomless. For each $\mathbf{T}$-simple $(x, y) \in \mathcal{M} \times \mathcal{N}$ and $\mathbf{T}$-simple $\bar{x} \in \mathcal{M}$ such that $law_\mathbf{T}(\bar{x}) = law_\mathbf{T}(x)$, there exists $\bar{y} \in \mathcal{N}$ such that $(\bar{x}, \bar{y})$ is $\mathbf{T}$-simple and $law_\mathbf{T}(\bar{x}, \bar{y}) = law_\mathbf{T}(x, y)$.*

Proof: Let (x, y) be $\mathcal{E}$-measurable where $\mathcal{E}$ is a $\mathbf{T}$-partition. Since $law_{\mathbf{T}}(\bar{x}) = law_{\mathbf{T}}(x)$, there is a $\mathbf{T}$-partition $\bar{\mathcal{E}}$ of Ω and an isomorphism $h : \mathcal{E} \to \bar{\mathcal{E}}$ such that $\bar{x} = h(x)$. Then $\bar{y} = h(y)$ has the required properties. $\square$

The following lemma is a consequence of Lemma 4.1 for atomless probability spaces. It will be used here as the first step in an inductive argument for $\mathbf{T}$-adapted spaces.

Lemma 6.5 *Suppose Ω is an atomless probability space, $M \subseteq \mathbf{R}^n$ is compact, and $\varepsilon > 0$. There exists $\delta > 0$ and a finite set of polynomials $p_1, \ldots, p_m$ in n variables with rational coefficients such that for every uniform finite algebra $\mathcal{E} \subseteq \mathcal{G}$, and all $\mathcal{E}$-measurable $x, y \in \mathcal{M}$ such that*

$$|E[p_i(x)] - E[p_i(y)]| < \delta \textit{ for } i = 1, \ldots, m, \tag{4}$$

there is a permutation h of the atoms of $\mathcal{E}$ such that $\rho_0(x, h(y)) < \varepsilon$. $\square$

Proof: By the compactness of M, there exist $\delta > 0$ and $p_1, \ldots, p_m$ such that whenever $x, y \in \mathcal{M}$ and equation (4) holds, we have

$$d(law(x), law(y)) < \varepsilon.$$

Suppose $x, y \in \mathcal{M}$ satisfy equation (4). By Lemma 4.1, there exist $\bar{x}, \bar{y} \in \mathcal{M}$ such that

$$law(\bar{x}) = law(x),\ law(\bar{y}) = law(y),$$

and

$$\rho_0(\bar{x}, \bar{y}) < \varepsilon.$$

Now suppose that $\mathcal{E} \subseteq \mathcal{G}$ is a uniform finite algebra with k atoms $E_1, \ldots, E_k$ and that x and y are $\mathcal{E}$-measurable. Then x is simple, and by Lemma 4.3 we may take $\bar{x} = x$. To complete the proof it suffices to show that $\bar{y}$ may also be taken to be $\mathcal{E}$-measurable.

Consider the set $A = \{z \in \mathcal{M} : law(z) = law(y)\}$. Each $z \in A$ is determined by a uniform $\mathcal{F} \subseteq \mathcal{G}$ with k atoms, and an ordering of its atoms $F_1, \ldots, F_k$. The joint distribution $law(x, z)$ is determined by the $k \times k$ matrix $p(x, z) = (P[E_i \cap F_j])$. The set $p(x, A)$ of all such matrices is a convex polyhedron of dimension k^2, whose vertices are permutation matrices corresponding to $\mathcal{E}$-measurable processes of the form $z = h(y)$. For each $\alpha > 0$, the probability that $\rho(x(\omega), z(\omega)) \geq \alpha$ depends linearly on the matrix $p(x, z)$. Therefore this probability takes its minimum at a vertex of $p(x, A)$. It follows that the set of distances $\{\rho_0(x, z) : z \in A\}$ has a minimum at a point $\rho_0(x, \bar{y})$ where $\bar{y} \in A$ is $\mathcal{E}$-measurable. $\square$

Lemma 6.6 *Let $\Omega_{\mathbf{T}}$ be atomless, let $\mathcal{M} \in \mathbf{M}_{\Omega}$, and let $D \subseteq Meas(M)$ be compact. For each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in law^{-1}(D)$, (x, y) is $\mathbf{T}$-simple, and $law_{\mathbf{T}}(y)$ is within δ of $law_{\mathbf{T}}(x)$, there exists z such that (y, z) is $\mathbf{T}$-simple, $law_{\mathbf{T}}(z) = law_{\mathbf{T}}(y)$, and $\rho_0(x, z) < \varepsilon$.*

Proof: Let $\varepsilon > 0$ and $D \subseteq Meas(M)$ be compact. By Prohorov's theorem there is a compact set $C \subseteq M$ such that $P[x(\omega) \in C] > 1 - \varepsilon/3$ whenever $law(x) \in D$. Since $\Phi(M)$ separates points in M and each $\phi_n \in \Phi(M)$ is bounded and continuous, there exists $\delta_1 > 0$ and a finite subset $\{\phi_1, \ldots, \phi_m\} \subseteq \Phi(M)$ such that $\delta_1 < \varepsilon/3$ and whenever $b, c \in C$,

$$|\phi_i(b) - \phi_i(c)| < \delta_1 \text{ for } i = 1, \ldots, m \text{ implies } \rho(b, c) < \varepsilon. \tag{5}$$

Using Lemma 6.5 inductively for each $t \in \mathbf{T}$ in increasing order, we can find $\delta > 0$ and finitely many adapted functions $f_1, \ldots, f_k$ built from $\{\phi_1, \ldots, \phi_m\}$ such that for every $\mathbf{T}$-partition $\mathcal{E}$ of Ω and $\mathcal{E}$-measurable $x, y \in \mathcal{M}$, if

$$|E[f_i(x)] - E[f_i(y)]| < \delta \text{ for } i = 1, \ldots, k, \tag{6}$$

there exists an automorphism h of $\mathcal{E}$ such that

$$\langle \phi_1(x), \ldots, \phi_m(x) \rangle \text{ is within } \delta_1 \text{ of } \langle \phi_1(h(y)), \ldots, \phi_m(h(y)) \rangle \tag{7}$$

in the metric of convergence in probability in $\mathbf{R}^m$.

Suppose $x, y \in law^{-1}(D)$, (x, y) is $\mathbf{T}$-simple, and $law_{\mathbf{T}}(y)$ is within δ of $law_{\mathbf{T}}(x)$. Then (6) holds, and both x and y are $\mathcal{E}$-measurable for some $\mathbf{T}$-partition $\mathcal{E}$ of Ω. Thus (7) holds for some automorphism h of $\mathcal{E}$. Let $z = h(y)$. Then (x, z) is $\mathbf{T}$-simple and $law_{\mathbf{T}}(y) = law_{\mathbf{T}}(z)$. By (5) and (7), we have

$$x(\omega) \in C, z(\omega) \in C, \text{ and } \rho(x(\omega), z(\omega)) < \varepsilon$$

with probability at least $1 - (\varepsilon/3 + \varepsilon/3 + \varepsilon/3)$. Therefore $\rho_0(x, z) < \varepsilon$. □

We are now ready to show that $law_{\mathbf{T}}$ is closed on Ω when $\Omega_{\mathbf{T}}$ is atomless. The proof will take advantage of the fact that the ordinary law was tacked on as a second coordinate to the $law_{\mathbf{T}}$ function.

Proposition 6.7 *For every atomless $\mathbf{T}$-adapted space $\Omega_{\mathbf{T}}$, $law_{\mathbf{T}}$ is closed on Ω.*

Proof: Let (b, c) be a point in the closure of $law_{\mathbf{T}}(\mathcal{M})$ in $\mathbf{R}^{\mathbf{N}} \times Meas(M)$. We must find an $x \in \mathcal{M}$ such that $law_{\mathbf{T}}(x) = (b, c)$. By Lemma 6.2 (iv) there is a sequence $x_n \in \mathcal{M}$ such that $law_{\mathbf{T}}(x_n)$ converges to (b, c) and $(x_1, \ldots, x_n)$ is $\mathbf{T}$-simple for each

n. Then $law(x_n)$ converges to c, and the set $D = \{c\} \cup \{law(x_n) : n \in \mathbf{N}\}$ is compact. Let δ_n be the δ corresponding to D and $\varepsilon = 2^{-n}$ in Lemma 6.6. Then x_n has a subsequence y_n such that $law_{\mathbf{T}}(y_n)$ is within $\delta_n/2$ of (b, c), and hence $law_{\mathbf{T}}(y_{n+1})$ within δ_n of $law_{\mathbf{T}}(y_n)$, for each n. By Lemma 6.6 there is a sequence $z_n \in \mathcal{M}$ such that (y_n, z_n) is $\mathbf{T}$-simple, $law_{\mathbf{T}}(z_n) = law_{\mathbf{T}}(y_n)$, and $\rho_0(z_n, z_{n+1}) \leq 2^{-n}$ for each n. Then the limit $z = \lim_{n\to\infty} z_n$ exists in $\mathcal{M}$ and $law_{\mathbf{T}}(z) = (b, c)$. $\square$

Proposition 6.8 *A* $\mathbf{T}$*-adapted space* $\Omega_{\mathbf{T}}$ *is atomless if and only if it is* **universal***, that is, for every random variable* x *on some other* $\mathbf{T}$*-adapted space* $\Gamma_{\mathbf{T}}$ *there exists* $y \in \mathcal{M}$ *such that* $law_{\mathbf{T}}(y) = law_{\mathbf{T}}(x)$*.*

Proof: Atomless implies universal by Corollary 6.3 and Proposition 6.7. Universal implies atomless by [HK, Lemma 4.4 (iv)]. $\square$

7 Richness and Saturation

In this section we shall introduce the notion of a saturated $\mathbf{T}$-adapted space from [HK], and a notion of a rich $\mathbf{T}$-adapted space which is analogous to the rich continuous time adapted spaces from [FK1].

Definition 7.1 *A* $\mathbf{T}$*-adapted space* $\Omega_{\mathbf{T}}$ *is* **saturated** *if for every* $\mathbf{T}$*-adapted space* $\Gamma_{\mathbf{T}}$ *and all complete separable metric spaces* M *and* N*, if*

$$\bar{x} \in L^0(\Gamma, M), \bar{y} \in L^0(\Gamma, N), x \in \mathcal{M},$$

and $law_{\mathbf{T}}(x) = law_{\mathbf{T}}(\bar{x})$*, then there exists* $y \in \mathcal{N}$ *such that* $law_{\mathbf{T}}(x, y) = law_{\mathbf{T}}(\bar{x}, \bar{y})$*.*

It is shown in [HK] that saturated $\mathbf{T}$-adapted spaces exist. It is obvious that $law_{\mathbf{T}}$ has the back and forth property for every saturated $\mathbf{T}$-adapted space $\Omega_{\mathbf{T}}$. Let us now prove that such spaces also have the Skorokhod property.

Our next order of business is to prove that when $\Omega_{\mathbf{T}}$ is atomless, $law_{\mathbf{T}}$ has the Skorokhod property and is dense on Ω. Our arguments will parallel the corresponding methods for ordinary probability spaces in Section 4. The next result is a weak saturation property which holds for all atomless $\mathbf{T}$-adapted spaces.

Proposition 7.2 *Let* $\Omega_{\mathbf{T}}$ *be atomless, and let* $\Gamma_{\mathbf{T}}$ *be another* $\mathbf{T}$*-adapted space. Then for every* $\mathbf{T}$*-simple* $x \in \mathcal{M}$ *and every pair of random variables* $(\bar{x}, \bar{y}) \in L^0(\Gamma, M \times N)$ *such that* $law_{\mathbf{T}}(\bar{x}) = law_{\mathbf{T}}(x)$*, there exists* $y \in \mathcal{N}$ *such that* $law_{\mathbf{T}}(x, y) = law_{\mathbf{T}}(\bar{x}, \bar{y})$*.*

Proof: Let x be $\mathcal{E}$-measurable for some $\mathbf{T}$-partition $\mathcal{E}$ of $\Omega_{\mathbf{T}}$. Then $\bar{x}$ is $\mathcal{F}$-measurable for some $\mathbf{T}$-partition $\mathcal{F}$ of $\Gamma_{\mathbf{T}}$ which is equivalent to $\mathcal{E}$. By Lemma 6.2, we have $\bar{x} = h(x)$ for some isomorphism $h : \mathcal{E} \to \mathcal{F}$. For each $A \in \mathcal{E}_\infty$, form the $\mathbf{T}$-adapted space $\Omega_{A,\mathbf{T}} = (A, \mathcal{G}_{A,t}, P_A)$ where $P_A(U) = P(A \cap U)$ and $\mathcal{G}_{A,t} = \{A \cap U : U \in \mathcal{G}_t\}$. Define $\Gamma_{A,\mathbf{T}}$ similarly. Then $\Omega_{A,\mathbf{T}}$ is an atomless adapted measure space, and the measures of A in $\Omega_{A,\mathbf{T}}$ and of $h(A)$ in $\Gamma_{h(A),\mathbf{T}}$ are finite and equal. By Proposition 6.8, $\Omega_{A,\mathbf{T}}$ is universal, so there exists y_A on $\Omega_{A,\mathbf{T}}$ with the same $\mathbf{T}$-adapted law as the restriction of $\bar{y}$ to $h(A)$ in $\Gamma_{h(A),\mathbf{T}}$. Let $y \in \mathcal{N}$ be the random variable whose restriction to each $A \in \mathcal{E}_\infty$ is y_A. Then $law_{\mathbf{T}}(x, y) = law_{\mathbf{T}}(\bar{x}, \bar{y})$. $\square$

Hoover obtained a generalization of the Skorokhod representation theorem for $\mathbf{T}$-adapted spaces in [H1, Corollary 10.2], which shows in our terminology that atomless $\mathbf{T}$-adapted spaces with the Skorokhod property exist. We now improve that result by showing that all atomless $\mathbf{T}$-adapted spaces have the Skorokhod property.

Proposition 7.3 *Let $\Omega_{\mathbf{T}}$ be an atomless $\mathbf{T}$-adapted space. Then $law_{\mathbf{T}}$ has the Skorokhod property on Ω.*

Proof: We first prove the result in the case that $\Omega_{\mathbf{T}}$ is a saturated $\mathbf{T}$-adapted space, and then prove the general case.

Suppose that x_n is a sequence in $\mathcal{M}$, $x \in \mathcal{M}$, and $law_{\mathbf{T}}(x_n)$ converges to $law_{\mathbf{T}}(x)$. $law_{\mathbf{T}}$ is closed by Proposition 6.7. By Lemma 6.2 (iv), there are sequences $y_k, z_{k,n}$ in $\mathcal{M}$ such that the pair $(y_k, z_{k,n})$ is $\mathbf{T}$-simple for each k, n, $y_k \to x$ in probability, and $z_{k,n} \to x_n$ in probability for each n. Let D be the compact set

$$\{law(y_k), law(x), law(z_{k,n}), law(x_n) : k, n \in \mathbf{N}\}.$$

For each k and n, let $C_{k,n}$ be the set

$$C_{k,n} = \{u \in \mathcal{M} : law_{\mathbf{T}}(u) \in \{law_{\mathbf{T}}(z_{m,n}) : k \leq m\} \cup \{law_{\mathbf{T}}(x_n)\}\}.$$

This set is basic for $law_{\mathbf{T}}$. Consider an $\varepsilon > 0$. For each k and n, the set

$$B_{k,n} = C_{k,n} \cap \{u \in \mathcal{M} : \rho_0(u, x) \leq \varepsilon\}$$

is a basic section for $law_{\mathbf{T}}$ with parameter x. For each n, for all sufficiently large k we have

$$d(law_{\mathbf{T}}(z_{k,n}), law_{\mathbf{T}}(y_k)) < 2d(law_{\mathbf{T}}(x_n), law_{\mathbf{T}}(x)).$$

Applying Lemma 6.6 with the above compact set D, we see that for all sufficiently large n, $\rho_0(y_n, x) < \varepsilon/2$ and for all sufficiently large k there exists $u \in \mathcal{M}$ such that

$$law_{\mathbf{T}}(u) = law_{\mathbf{T}}(z_{k,n}) \text{ and } \rho_0(u, y_k) < \varepsilon/2.$$

Then $\rho_0(u,x) \le \varepsilon$, and thus $u \in B_{k,n}$. Therefore for all sufficiently large n the sets $B_{k,n}$ form a decreasing chain of nonempty sets as $k \to \infty$. We assume at this point that $\Omega_{\mathbf{T}}$ is saturated, so that it has the back and forth property. By Corollary 3.3, the family of basic sections in $\mathcal{M}$ for $law_{\mathbf{T}}$ is countably compact. Therefore for all sufficiently large n there exists $u_n \in \bigcap_k B_{k,n}$. Then $law_{\mathbf{T}}(u_n) = law_{\mathbf{T}}(x_n)$ and $\rho_0(u_n,x) \le \varepsilon$. Letting $\varepsilon \to 0$, we obtain a sequence v_n in $\mathcal{M}$ such that $law_{\mathbf{T}}(v_n) = law_{\mathbf{T}}(x_n)$ for each n and $v_n \to x$ in $\mathcal{M}$. This proves the result in the case that $\Omega_{\mathbf{T}}$ is saturated.

We now prove the general case. Let $x \in \mathcal{M}$, and let c_n be a sequence converging to $law_{\mathbf{T}}(x)$ in $Meas(M)$. We must find a sequence x_n converging to x in $\mathcal{M}$ such that $law_{\mathbf{T}}(x_n) = c_n$ for all n. Let $\Gamma_{\mathbf{T}}$ be a saturated adapted space. By the preceding paragraph, there are random variables $z_n, z \in L^0(\Gamma, M)$ such that $law_{\mathbf{T}}(z_n) = c_n$ for all n, $law_{\mathbf{T}}(z) = law_{\mathbf{T}}(x)$, and $z_n \to z$ in $L^0(\Gamma, M)$. Let $u_n, n \in \mathbf{N}$ be a sequence of $\mathbf{T}$-simple random variables converging to x in $\mathcal{M}$. Since $\Gamma_{\mathbf{T}}$ is saturated there is a sequence $v_n, n \in \mathbf{N}$ in $L^0(\Gamma, M)$ such that

$$law_{\mathbf{T}}(y, \langle v_n \rangle) = law_{\mathbf{T}}(x, \langle u_n \rangle).$$

By the preceding proposition, for each n there exists $x_n \in \mathcal{M}$ such that $law_{\mathbf{T}}(u_n, x_n) = law_{\mathbf{T}}(v_n, y_n)$. Then $law_{\mathbf{T}}(x_n) = c_n$ for all n. Since y_n and v_n both converge to y, we have $\rho_0(x_n, u_n) \to 0$, and thus $x_n \to x$ in $\mathcal{M}$. $\square$

Proposition 7.4 *For every atomless $\mathbf{T}$-adapted space $\Omega_{\mathbf{T}}$, $law_{\mathbf{T}}$ is dense on Ω.*

Proof: The argument is the same as the proof of Proposition 4.6, but using $\mathbf{T}$-simple processes and Proposition 7.2 instead of Proposition 4.4. $\square$

We now turn to the notion of a rich $\mathbf{T}$-adapted space. We shall consider two different families of basic sets, the family $\mathcal{B}_{\Omega_{\mathbf{T}}}$ of basic/compact sets for $law_{\mathbf{T}}$, and a simpler family $\mathcal{A}_{\Omega_{\mathbf{T}}}$ which is defined in one step from the ordinary law function $law(x)$ and the notion of a $\mathcal{G}_t$-measurable function. Using this simpler family $\mathcal{A}_{\Omega_{\mathbf{T}}}$, we were able to define rich adapted spaces in [FK1] without introducing the complicated adapted law function $law_{\mathbf{T}}$.

Definition 7.5 *For each $\mathcal{M} \in \mathbf{M}_\Omega$, let $\mathcal{A}_{\Omega_{\mathbf{T}}}(\mathcal{M})$ and $\mathcal{B}_{\Omega_{\mathbf{T}}}(\mathcal{M})$ be the following families of subsets of $\mathcal{M}$.*

$A \in \mathcal{A}_{\Omega_{\mathbf{T}}}(\mathcal{M})$ iff A is compact or

$$A = \{x \in \mathcal{M} : x \text{ is } \mathcal{G}_t - \textit{measurable and } law(x) \in D\}$$

for some compact set $D \subseteq Meas(M)$ and some $t \in \mathbf{T} \cup \{\infty\}$.

$A \in \mathcal{B}_{\Omega_{\mathbf{T}}}(\mathcal{M})$ iff A is basic/compact for $law_{\mathbf{T}}$, that is, A is compact or

$$A = \{x \in \mathcal{M} : law_{\mathbf{T}}(x) \in C\}$$

for some compact set $C \subseteq \mathbf{R}^{\mathbf{N}} \times Meas(M)$.

Recall that by Proposition 3.6, the family of basic sections over $\mathcal{B}_{\Omega_{\mathbf{T}}}$ is the same as the family of basic sections for $law_{\mathbf{T}}$.

Definition 7.6 *A $\mathbf{T}$-adapted space $\Omega_{\mathbf{T}}$ is said to be* **rich** *if $\Omega_{\mathbf{T}}$ is atomless and for each $\mathcal{M}$ the family of neocompact subsets of $\mathcal{M}$ over $\mathcal{A}_{\Omega_{\mathbf{T}}}$ is countably compact.*

We need the following result from [FK1].

Lemma 7.7 *Suppose $\Omega_{\mathbf{T}}$ is rich. The function $law(\cdot)$ is neocontinuous over $\mathcal{A}_{\Omega_{\mathbf{T}}}$, and each $\mathbf{T}$-adapted function on $\mathcal{M}$ is neocontinuous over $\mathcal{A}_{\Omega_{\mathbf{T}}}$.*

Proof: This follows from [FK1, Proposition 5.12 and Theorem 7.6]. $\square$

The next lemma is another key point where we need the ordinary law function as the second coordinate of the $\mathbf{T}$-adapted law function.

Lemma 7.8 *Let $\Omega_{\mathbf{T}}$ be a $\mathbf{T}$-adapted space, and let $\mathcal{M} \in \mathbf{M}_{\Omega}$.*
(i) $\mathcal{A}_{\Omega_{\mathbf{T}}}(\mathcal{M}) \subseteq \mathcal{B}_{\Omega_{\mathbf{T}}}(\mathcal{M})$.
(ii) If $\Omega_{\mathbf{T}}$ is rich, then every basic section for $law_{\mathbf{T}}$ is neocompact over $\mathcal{A}_{\Omega_{\mathbf{T}}}$.

Proof: (i) Let $A \in \mathcal{A}_{\Omega_{\mathbf{T}}}(\mathcal{M})$. If A is compact then $A \in \mathcal{B}_{\Omega_{\mathbf{T}}}(\mathcal{M})$ by definition. The other possibility is that A is a set of the form

$$A = \{x \in \mathcal{M} : x \text{ is } \mathcal{G}_t - \text{measurable and } law(x) \in D\}$$

for some $t \in \mathbf{T} \cup \{\infty\}$ and some compact $D \subseteq Meas(M)$. By Lemma 5.7 the set $law_{\mathbf{T}}(law^{-1}(D))$ has a compact closure $C \subseteq \mathbf{R}^{\mathbf{N}} \times D$. We have $law_{\mathbf{T}}(x) \in C$ if and only if $law(x) \in D$, so $law_{\mathbf{T}}{}^{-1}(C) = law^{-1}(D)$.

In the case $t = \infty$, we have $A = law^{-1}(D)$, and hence $A = law_{\mathbf{T}}{}^{-1}(C) \in \mathcal{B}_{\Omega_{\mathbf{T}}}(\mathcal{M})$. Now suppose $t \in \mathbf{T}$. Then an $x \in \mathcal{M}$ is $\mathcal{G}_t$-measurable if and only if

$$E[|\hat{\phi}(x) - E[\hat{\phi}(x)|\mathcal{G}_t]|] = 0$$

for every $\phi \in \Phi(M)$. Since

$$|\hat{\phi}(x) - E[\hat{\phi}(x)|\mathcal{G}_t]|$$

is a $\mathbf{T}$-adapted function for each ϕ, it follows that

$$A = law_{\mathbf{T}}{}^{-1}(B) \cap law^{-1}(D) = law_{\mathbf{T}}{}^{-1}(B \cap C)$$

for some closed set $B \subseteq \mathbf{R}^{\mathbf{N}} \times Meas(M)$, and again $A \in \mathcal{B}_{\Omega_{\mathbf{T}}}(\mathcal{M})$.

(ii) Suppose $\Omega_{\mathbf{T}}$ is rich. Since countable intersections of neocompact sets are neocompact, it follows from Lemma 7.7 that the $law_{\mathbf{T}}$ function is neocontinuous over $\mathcal{A}_{\Omega_{\mathbf{T}}}$, where we take $law_{\mathbf{T}}(x)$ to be a random variable with a constant value in $\mathbf{R}^{\mathbf{N}} \times Meas(M)$. Let A be a basic section in $\mathcal{M}$ for $law_{\mathbf{T}}$. Then for some $\mathcal{K}$, A is a section of a set in $\mathcal{B}_{\Omega_{\mathbf{T}}}(\mathcal{M} \times \mathcal{K})$. Suppose that A is not already compact. Then A has the form

$$A = \{x \in \mathcal{M} : law_{\mathbf{T}}(x, z) \in \hat{A}\}$$

for some $z \in \mathcal{K}$ and some compact set $\hat{A}$ in $\mathbf{R}^{\mathbf{N}} \times Meas(M \times K)$. Then by [FK1, Proposition 3.9], $A \cap B$ is neocompact over $\mathcal{A}_{\Omega_{\mathbf{T}}}$ for each neocompact set $B \subseteq \mathcal{M}$ over $\mathcal{A}_{\Omega_{\mathbf{T}}}$. The projection $\hat{D}$ of $\hat{A}$ into $Meas(M \times K)$ is compact, so A is contained in the neocompact set

$$D = \{x \in \mathcal{M} : law(x, z) \in \hat{D}\}$$

over $\mathcal{A}_{\Omega_{\mathbf{T}}}$. Therefore A is neocompact over $\mathcal{A}_{\Omega_{\mathbf{T}}}$. $\square$

Theorem 7.9 *Let $\Omega_{\mathbf{T}}$ be an atomless $\mathbf{T}$-adapted space. The following are equivalent.*

(i) $\Omega_{\mathbf{T}}$ is saturated.

(ii) $law_{\mathbf{T}}$ has the back and forth property.

(iii) $\Omega_{\mathbf{T}}$ is rich.

(iv) For each $\mathcal{M}$ the family of basic sections for $law_{\mathbf{T}}$ is countably compact.

(v) For each basic relation C in $\mathcal{M} \times \mathcal{N}$ for $law_{\mathbf{T}}$, the existential projection

$$\{x : \exists y (x, y) \in C\}$$

is basic for $law_{\mathbf{T}}$.

Proof: $law_{\mathbf{T}}$ is closed by Proposition 6.7, dense by Proposition 7.4, and has the Skorokhod property by Proposition 7.3.

The equivalence of (i) and (ii) is proved exactly as in Theorem 4.8, using the fact that $\Omega_{\mathbf{T}}$ is universal by Proposition 6.8. (v) is equivalent to (ii) by Theorem 2.12.

We assume (i) and prove (iii). The family of neocompact sets over $\mathcal{B}_{\Omega_{\mathbf{T}}}$ is countably compact by Theorem 3.6. Since $\mathcal{A}_{\Omega_{\mathbf{T}}}(\mathcal{M}) \subseteq \mathcal{B}_{\Omega_{\mathbf{T}}}(\mathcal{M})$, every neocompact set over$\mathcal{A}_{\Omega_{\mathbf{T}}}$ is neocompact over $\mathcal{B}_{\Omega_{\mathbf{T}}}$. Thus (iii) holds.

(iii) implies (iv) by Lemma 7.8 (ii), and (iv) implies (ii) by Theorem 2.7. $\square$

Since saturated $\mathbf{T}$-adapted spaces exist by [HK, Lemma 5.7], it follows that rich $\mathbf{T}$-adapted spaces exist. The following corollary gives four characterizations of the neocompact sets for a rich adapted space. It follows from Lemma 7.8 and the proof of Theorem 7.9.

Corollary 7.10 *Let $\Omega_{\mathbf{T}}$ be a rich $\mathbf{T}$-adapted space. The following four families of subsets of $\mathcal{M}$ are the same.*

(i) The family of neocompact sets over $\mathcal{A}_{\Omega_{\mathbf{T}}}$.

(ii) The family of neocompact sets over $\mathcal{B}_{\Omega_{\mathbf{T}}}$.

(iii) The family of neocompact sets for $law_{\mathbf{T}}$.

(iv) The family of sets which are intersections of a set of the form

$$\{x \in \mathcal{M} : law(x, z) \in C\}$$

and countably many sets of the form

$$\{x \in \mathcal{M} : E[f_n(x, z)] \in D_n\}$$

where each f_n is a $\mathbf{T}$-adapted function, $z \in \mathcal{N}$, C is compact in $Meas(M \times N)$, and each D_n is compact in $\mathbf{R}$. $\square$

This follows from Lemma 7.8 and the proof of Theorem 7.9. Condition (iv) gives a characterization of the neocompact sets directly in terms of adapted functions rather than in terms of the $law_{\mathbf{T}}$ function.

8 Adapted Spaces with Infinite Time Sets

The paper [FK1] introduced rich adapted spaces with times indexed by the dyadic rationals. Each adapted space with times indexed by the dyadic rationals has an associated right continuous adapted space with times indexed by the nonnegative reals. Neocompact sets were applied to prove several optimization and existence theorems for such spaces.

In this section we shall consider adapted spaces with times in an arbitrary linearly ordered set, and apply our results on law mappings to such spaces. This general approach will include the natural special cases of adapted spaces with times indexed by the natural numbers (discrete time), by the dyadic rationals, and by the nonnegative reals. It is known from [FK2] and [FK1] that rich adapted spaces exist for every linearly ordered time set, but rich adapted spaces with right continuous filtrations on the reals never exist.

There are two cases where an adapted space induces a law mapping in a natural way. The first case, where the set of times is countable, is treated in this section. In this case, saturation is equivalent to richness. The second case, where the times are nonnegative reals and the adapted space is right continuous, is treated in the next section. We shall see that the right continuous adapted space which is associated with a rich adapted space on the dyadic rationals is saturated and satisfies a weak form of richness.

Let $\langle \mathbf{L}, \leq \rangle$ be a linearly ordered set. For convenience we assume that $\mathbf{L}$ contains a least element 0, and use the convention that $t < \infty$ for all $t \in \mathbf{L}$. By an **$\mathbf{L}$-adapted space** we mean a structure

$$\Omega_{\mathbf{L}} = (\Omega, P, \mathcal{G}_\infty, \mathcal{G}_t)_{t \in \mathbf{L}}$$

such that $(\Omega, P, \mathcal{G}_\infty)$ is a complete probability space, $\mathcal{G}_t$ is a σ-subalgebra of $\mathcal{G}_\infty$ for each $t \in \mathbf{L}$, and $\mathcal{G}_s \subseteq \mathcal{G}_t$ whenever $s < t$ in $\mathbf{L}$. We shall write $\mathcal{G} = \mathcal{G}_\infty$, so that $(\Omega, P, \mathcal{G})$ is the probability space associated with the adapted space Ω.

For each finite subset $\mathbf{T} \subseteq \mathbf{L}$, each $\mathbf{L}$-adapted space $\Omega_{\mathbf{L}}$ has a corresponding $\mathbf{T}$-adapted space

$$\Omega_{\mathbf{T}} = (\Omega, P, \mathcal{G}_\infty, \mathcal{G}_t)_{t \in \mathbf{T}}.$$

Definition 8.1 *Let $\Omega_{\mathbf{L}}$ be an $\mathbf{L}$-adapted space. We say that f is an* **adapted function** *on $\mathcal{M}$ for $\Omega_{\mathbf{L}}$ if f is a $\mathbf{T}$-adapted function for the corresponding $\mathbf{T}$-adapted space $\Omega_{\mathbf{T}}$ for some finite subset $\mathbf{T} \subseteq \mathbf{L}$.*

An $\mathbf{L}$-adapted space $\Omega_{\mathbf{L}}$ is **atomless** if $\mathcal{G}_0$ is atomless over the trivial σ-algebra, and $\mathcal{G}_t$ is atomless over $\mathcal{G}_s$ whenever $s < t \in \mathbf{L} \cup \{\infty\}$. Note that $\Omega_{\mathbf{L}}$ is atomless if and only if $\Omega_{\mathbf{T}}$ is atomless for each finite $\mathbf{T} \subseteq \mathbf{L}$.

We now define families of basic sets $\mathcal{A}_{\Omega_{\mathbf{L}}}$ and $\mathcal{B}_{\Omega_{\mathbf{L}}}$ which generalize the families $\mathcal{A}_{\Omega_{\mathbf{T}}}$ and $\mathcal{B}_{\Omega_{\mathbf{T}}}$ introduced in the preceding section, and review the notion of a rich $\mathbf{L}$-adapted space from [FK1] and [FK2].

Definition 8.2 *Let $\mathcal{M} \in \mathbf{M}_\Omega$. We define*

$$\mathcal{A}_{\Omega_{\mathbf{L}}}(\mathcal{M}) = \bigcup \{\mathcal{A}_{\Omega_{\mathbf{T}}}(\mathcal{M}) : \mathbf{T} \subseteq \mathbf{L} \text{ and } \mathbf{T} \text{ is finite}\}.$$

That is, $A \in \mathcal{A}_{\Omega_{\mathbf{L}}}(\mathcal{M})$ if A is either compact or of the form

$$C = \{x \in \mathcal{M} : x \text{ is } \mathcal{G}_t - \text{measurable and } law(x) \in D\}$$

for some $t \in \mathbf{L} \cup \{\infty\}$ and some compact set $D \subseteq Meas(M)$.

$\Omega_{\mathbf{L}}$ is a **rich** *$\mathbf{L}$-adapted space if $\Omega_{\mathbf{L}}$ is atomless and for each $\mathcal{M} \in \mathbf{M}_\Omega$, the family of neocompact subsets of $\mathcal{M}$ over $\mathcal{A}_{\Omega_{\mathbf{L}}}$ is countably compact.*

We let

$$\mathcal{B}_{\Omega_{\mathbf{L}}}(\mathcal{M}) = \bigcup \{\mathcal{B}_{\Omega_{\mathbf{T}}}(\mathcal{M}) : \mathbf{T} \subseteq \mathbf{L} \text{ and } \mathbf{T} \text{ is finite}\}.$$

Thus $C \in \mathcal{B}_{\Omega_{\mathbf{L}}}(\mathcal{M})$ if and only if C is basic/compact for $law_{\mathbf{T}}$ for some finite $\mathbf{T} \subseteq \mathbf{L}$.

We shall need the following existence result from [FK2].

Theorem 8.3 *For every linearly ordered set* $\mathbf{L}$*, rich* $\mathbf{L}$*-adapted spaces exist.* $\square$

In fact, it is proved in [FK2, Theorem 5.15] that every atomless Loeb $\mathbf{L}$-adapted space is rich. The Loeb adapted spaces are constructed using methods from non-standard analysis, and have been used extensively in the literature to prove existence theorems in probability theory (e.g. see [AFHL] or [K5]).

Lemma 8.4 *Let* $\Omega_{\mathbf{L}}$ *be an* $\mathbf{L}$*-adapted space. A set is neocompact over* $\mathcal{A}_{\Omega_{\mathbf{L}}}$ *if and only if it is neocompact over* $\mathcal{A}_{\Omega_{\mathbf{K}}}$ *for some finite or countable* $\mathbf{K} \subseteq \mathbf{L}$*. A similar result holds for* $\mathcal{B}_{\Omega_{\mathbf{L}}}$*. Thus if* $\Omega_{\mathbf{L}}$ *is rich then the corresponding* $\mathbf{K}$*-adapted space* $\Omega_{\mathbf{K}}$ *is rich for every* $\mathbf{K} \subseteq \mathbf{L}$*. Conversely, if* $\Omega_{\mathbf{K}}$ *is rich for every countable* $\mathbf{K} \subseteq \mathbf{L}$ *then* $\Omega_{\mathbf{L}}$ *is rich.*

Proof: The family of sets which are neocompact over $\mathcal{A}_{\Omega_{\mathbf{K}}}$ for some finite or countable $\mathbf{K} \subseteq \mathbf{L}$ is closed under the operations (a)-(f), and hence is the same as the family of neocompact sets over $\mathcal{A}_{\Omega_{\mathbf{L}}}$. Similarly for $\mathcal{B}_{\Omega_{\mathbf{L}}}$. $\square$

Using the preceding lemma, we have an analogue of Lemma 7.8.

Lemma 8.5 *Let* $\Omega_{\mathbf{L}}$ *be an* $\mathbf{L}$*-adapted space, and let* $\mathcal{M} \in \mathbf{M}_{\Omega}$*.*

(i) $\mathcal{A}_{\Omega_{\mathbf{L}}}(\mathcal{M}) \subseteq \mathcal{B}_{\Omega_{\mathbf{L}}}(\mathcal{M})$*.*

(ii) If $\Omega_{\mathbf{L}}$ *is rich, then every basic section over* $\mathcal{B}_{\Omega_{\mathbf{L}}}$ *is neocompact over* $\mathcal{A}_{\Omega_{\mathbf{L}}}$*.* $\square$

Now let $\mathbf{B}$ be the set of nonnegative dyadic rationals. Because of Lemma 8.4 and the fact that every countable linearly ordered set can be embedded in $\mathbf{B}$, we shall concentrate on $\mathbf{B}$-adapted spaces. For each $k \in \mathbf{N}$, let $\mathbf{B}_k$ be the finite set of multiples of 2^{-k} in the interval $[0, 2^k]$. Then $\mathbf{B} = \bigcup_k \mathbf{B}_k$. Each $\mathbf{B}$-adapted probability space $\Omega_{\mathbf{B}}$ has a corresponding $\mathbf{B}_k$-adapted space

$$\Omega_k = (\Omega, P, \mathcal{G}_\infty, \mathcal{G}_t)_{t \in \mathbf{B}_k}.$$

We have not defined a law function corresponding to an arbitrary $\mathbf{L}$-adapted space $\Omega_{\mathbf{L}}$. We shall now take advantage of the countability of the set $\mathbf{B}$ of dyadic rationals to introduce a law function corresponding to a $\mathbf{B}$-adapted space $\Omega_{\mathbf{B}}$. By the k^{th}**-adapted law** of a random variable $x \in \mathcal{M}$ we mean the function

$$law_k(x) = law_{\mathbf{B}_k}(x),$$

where $law_{\mathbf{B}_k}$ is the $\mathbf{B}_k$-adapted law function introduced in Section 5.

In order to fit these finite adapted law functions into an infinite product as in Section 2, for $k \geq 1$ we let $\beta_{k,n}, n \in \mathbf{N}$, be a list of all the adapted functions for $\Omega_{\mathcal{B}_k}$ built from Φ which are not adapted functions for $\Omega_{\mathbf{B}_{k-1}}$. Let

$$\lambda_0(x) = law(x), \lambda_k(x) = \langle E[\beta_{k,n}(x)] : n \in \mathbf{N}\rangle.$$

Then the finite product $\vec{\lambda}_k(x)$ is the k^{th} adapted law

$$law_k(x) = \langle \lambda_1(x), \ldots, \lambda_k(x), law(x)\rangle.$$

(We put $\lambda_0(x) = law(x)$ last in the sequence to conform with our practice in the preceding sections). The **adapted law** of x is the infinite product

$$law_{\mathbf{B}}(x) = \langle \lambda_1(x), \ldots, \lambda_k(x), \ldots, law(x)\rangle.$$

Both $law_k(x)$ and $law_{\mathbf{B}}(x)$ take values in $\mathbf{R}^{\mathbf{N}} \times Meas(M)$.

All of the lemmas 5.4 through 5.8 hold for $\mathbf{B}$ in place of $\mathbf{T}$. In each case, the result for $\mathbf{B}$ is an easy consequence of the result for $\mathbf{T}$.

Proposition 8.6 *(i) $law_{\mathbf{B}}$ is a law mapping.*

(ii) If $\Omega_{\mathbf{B}}$ is atomless then $law_{\mathbf{B}}$ is dense.

(iii) If $\Omega_{\mathbf{B}}$ is atomless then for any other $\mathbf{B}$-adapted space $\Gamma_{\mathbf{B}}$ and any $x \in L^0(\Gamma, M)$, $law_{\mathbf{B}}(x)$ is in the closure of $law_{\mathbf{B}}(\mathcal{M})$.

Proof: (i) follows from Propositions 3.7 (i) and 5.9. (ii) follows from Propositions 3.7 (ii) and 7.4.

We now prove (iii). For each k, Ω_k is an atomless $\mathbf{B}_k$-adapted space, and by Proposition 6.8 there exists $y_k \in \mathcal{M}$ such that $law_k(y_k) = law_k(x)$. Then

$$d(law_{\mathbf{B}}(y_k), law_{\mathbf{B}}(x)) \leq 2^{-k},$$

and thus $law_{\mathbf{B}}(y_k) \to law_{\mathbf{B}}(x)$ in $\Lambda(\mathcal{M})$. $\square$

Proposition 8.7 *Suppose $\Omega_{\mathbf{B}}$ be atomless.*

(i) If $law_{\mathbf{B}}$ has the back and forth property on Ω then law_k has the back and forth property on Ω for each $k \in \mathbf{N}$.

(ii) If $law_{\mathbf{B}}$ is closed on Ω then law_k is closed on Ω for each $k \in \mathbf{N}$.

Proof: (i) Assume that $law_{\mathbf{B}}$ has the back and forth property on Ω. Suppose that $law_k(x, y_n)$ converges to $law_k(\bar{x}, \bar{y})$ as $n \to \infty$. Let

$$A = \{(x_n, y) : n \in \mathbf{N}\}.$$

Then $law_k(A)$ is relatively compact. By Lemma 5.7, the sets $law(A)$ and $law_{\mathbf{B}}(A)$ are relatively compact. Therefore there is a subsequence $\langle y_m \rangle$ of $\langle y_n \rangle$ such that $law_{\mathbf{B}}(x, y_m)$ converges to a point $law_{\mathbf{B}}(x', y')$ as $m \to \infty$. By Proposition 2.4 there exists y such that $law_{\mathbf{B}}(x, y_m)$ converges to $law_{\mathbf{B}}(x, y)$. Then $law_k(x, y_m)$ converges to $law_k(x, y)$, so $law_k(x, y) = law_k(\bar{x}, \bar{y})$. Thus by Proposition 2.4, law_k has the back and forth property.

(ii) Assume that $law_{\mathbf{B}}$ is closed. Suppose that

$$\lim_{n \to \infty} law_k(x_n) = c.$$

By Lemma 5.7, $law_{\mathbf{B}}(\{x_n : n \in \mathbf{N}\})$ is relatively compact. Since $law_{\mathbf{B}}$ is closed, there is a subsequence $\langle x_m \rangle$ of $\langle x_n \rangle$ such that $law_{\mathbf{B}}(x_m)$ converges to $law_{\mathbf{B}}(x)$ for some $x \in \mathcal{M}$. Then $law_k(x_m)$ converges to $law_k(x)$, and hence $law_k(x) = c$. This shows that law_k is closed. $\square$

It will be convenient to take $\mathbf{B}_0$ to be the empty set and to identify law_0 with the ordinary law function on the probability space Ω.

Definition 8.8 *Following our earlier pattern, a* $\mathbf{B}$*-adapted space* $\Omega_{\mathbf{B}}$ *is* **saturated** *if for every* $\mathbf{B}$*-adapted space* $\Gamma_{\mathbf{B}}$ *and all complete separable metric spaces* M *and* N*, if*

$$\bar{x} \in L^0(\Gamma, M), \bar{y} \in L^0(\Gamma, N), x \in \mathcal{M},$$

and $law_{\mathbf{B}}(x) = law_{\mathbf{B}}(\bar{x})$*, then there exists* $y \in \mathcal{N}$ *such that* $law_{\mathbf{B}}(x, y) = law_{\mathbf{B}}(\bar{x}, \bar{y})$*.*

A $\mathbf{B}$*-adapted space* $\Omega_{\mathbf{B}}$ *is* **universal** *if for every* $\mathbf{B}$*-adapted space* $\Gamma_{\mathbf{B}}$ *and complete separable metric space* M*, for each* $\bar{x} \in L^0(\Gamma, M)$ *there exists* $x \in \mathcal{M}$ *such that* $law_{\mathbf{B}}(x) = law_{\mathbf{B}}(\bar{x})$*.*

It is clear that every saturated $\mathbf{B}$-adapted space is universal and has the back and forth property. The converse also holds, as one can see from the next theorem, which characterizes universal $\mathbf{B}$-adapted spaces.

Theorem 8.9 *A* $\mathbf{B}$*-adapted space* $\Omega_{\mathbf{B}}$ *is universal if and only if it is atomless and* $law_{\mathbf{B}}$ *is closed on* Ω*.*

Proof: Suppose $\Omega_{\mathbf{B}}$ is atomless and $law_{\mathbf{B}}$ is closed on Ω. Let $\Gamma_{\mathbf{B}}$ be another $\mathbf{B}$-adapted space and let $x \in L^0(\Gamma, M)$. By Proposition 8.6 (iii), $law_{\mathbf{B}}(x)$ is in the closure of $law_{\mathbf{B}}(\mathcal{M})$. Since $law_{\mathbf{B}}$ is closed on Ω, there exists $y \in \mathcal{M}$ such that $law_{\mathbf{B}}(y) = law_{\mathbf{B}}(x)$. Thus $\Omega_{\mathbf{B}}$ is universal.

For the converse, suppose $\Omega_{\mathbf{B}}$ is universal. Then for each k, Ω_k is universal, and hence is atomless by Proposition 6.8. It follows that $\Omega_{\mathbf{B}}$ is atomless. Suppose $x_n \in \mathcal{M}$ and $law_{\mathbf{B}}(x_n) \to c$ in $\Lambda(\mathcal{M})$. By Theorem 8.3, there exists a rich $\mathbf{B}$-adapted space $\Gamma_{\mathbf{B}}$. Then $\Gamma_{\mathbf{B}}$ is atomless. By Proposition 8.6 (ii), each $law_{\mathbf{B}}(x_n)$ and hence c belongs to the closure of $law_{\mathbf{B}}(L^0(\Gamma, M))$. Therefore there is a sequence $\langle y_n \rangle$ in $L^0(\Gamma, M)$ such that $law_{\mathbf{B}}(y_n) \to c$. Then the sets

$$B_k = \{law_{\mathbf{B}}(y_n) : n \geq k\} \cup \{c\}$$

form a decreasing chain of compact sets, and their inverse images

$$C_k = \{z \in L^0(\Gamma, M) : law_{\mathbf{B}}(z) \in B_k\}$$

form a decreasing chain of nonempty basic sets for $law_{\mathbf{B}}$ on Γ. Since $\Gamma_{\mathbf{B}}$ is rich, each of these sets is neocompact and by Lemma 8.5, and their intersection is nonempty. Thus there exists $y \in L^0(\Gamma, M))$ such that $law_{\mathbf{B}}(y) = c$. Since $\Omega_{\mathbf{B}}$ is universal, there exists $x \in \mathcal{M}$ with $law_{\mathbf{B}}(x) = c$. This shows that $law_{\mathbf{B}}$ is closed on Ω. $\square$

We now prove our main theorem on $\mathbf{B}$-adapted spaces.

Theorem 8.10 *Let $\Omega_{\mathbf{B}}$ be an atomless $\mathbf{B}$-adapted space. The following are equivalent.*

(i) $\Omega_{\mathbf{B}}$ is saturated.

(ii) $law_{\mathbf{B}}$ is closed and has the back and forth property.

(iii) $\Omega_{\mathbf{B}}$ is rich.

(iv) For each $\mathcal{M} \in \mathbf{M}_\Omega$, the family of basic sections in $\mathcal{M}$ for $law_{\mathbf{B}}$ is countably compact.

(v) For each basic relation C in $\mathcal{M} \times \mathcal{N}$ for $law_{\mathbf{B}}$, the existential projection

$$\{x : \exists y (x, y) \in C\}$$

is basic for $law_{\mathbf{B}}$.

Proof: $law_{\mathbf{B}}$ is dense on Ω by Proposition 8.6.

Assume (i). Clearly, $\Omega_{\mathbf{B}}$ is universal and $law_{\mathbf{B}}$ has the back and forth property on Ω. By Theorem 8.9, $law_{\mathbf{B}}$ is closed on Ω, so (ii) holds.

Using Theorem 8.9 and the argument in the proof of Theorem 4.8, we see that (ii) implies (i). As in our previous results, (v) is equivalent to (ii) by Theorem 2.12.

We now assume (ii) and prove (iii). Let $k \in \mathbf{N}$. By Proposition 8.7, law_k has the back and forth property and is closed. By Theorem 7.9, the corresponding $\mathbf{B}_k$-adapted space Ω_k is saturated. By Proposition 7.3, law_k has the Skorokhod property for each $k \in \mathbf{N}$. By Theorem 3.9, the family of basic sections for $law_{\mathbf{B}}$ is countably compact for each $\mathcal{M}$, and every neocompact set over $\mathcal{B}_{\Omega_{\mathbf{B}}}$ is a basic section for $law_{\mathbf{B}}$. By Lemma 8.5 (i), every neocompact set over $\mathcal{A}_{\Omega_{\mathbf{B}}}$ is a basic section for $law_{\mathbf{B}}$, and (iii) holds.

We now assume (iii), that $\Omega_{\mathbf{B}}$ is rich, and prove (iv). By Lemma 8.4, Ω_k is rich for each $k \in \mathbf{N}$. By Proposition 3.7 (iii), every basic section C for $law_{\mathbf{B}}$ is an intersection of a chain of basic sections C_k for law_k, and by Lemma 7.8 (ii), each C_k is neocompact over $\mathcal{A}_{\Omega_{\mathbf{B}}}$. It follows that the family of basic sections for $law_{\mathbf{B}}$ is countably compact, so (iv) holds.

Finally, (iv) implies (ii) by Theorem 2.7. □

It is natural to ask whether the above theorem can be improved to show that the family of neocompact sets for $law_{\mathbf{B}}$ is countably compact when $\Omega_{\mathbf{B}}$ is rich. The following negative result shows that this can never happen.

Proposition 8.11 *Let $\Omega_{\mathbf{B}}$ be a universal $\mathbf{B}$-adapted space, and let $M = \{0,1\}$. Then:*

(i) The function $law_{\mathbf{B}}$ does not have the Skorokhod property over Ω.

(ii) The family of neocompact subsets of $\mathcal{M}$ for $law_{\mathbf{B}}$ on Ω is not countably compact.

(iii) There is a basic set $C \subseteq \mathcal{M} \times \mathcal{M}$ for $law_{\mathbf{B}}$ and a nonempty basic set $C \subseteq \mathcal{M}$ for $law_{\mathbf{B}}$ such that the set $\{x : (\forall y \in \mathcal{M})(x,y) \in C\}$ is not basic for $law_{\mathbf{B}}$.

Proof: By Theorem 8.9, $\Omega_{\mathbf{B}}$ is atomless and $law_{\mathbf{B}}$ is closed on Ω. For each $n \in \mathbf{N}$ let $t_n = 1 - 2^{-n}$. By a result of Maharam [M2, p. 146], for each n there is a set $S_n \in \mathcal{G}_{t_{n+1}}$ of measure 1/2 which is independent of $\mathcal{G}_{t_n}$. Let $x_n \in \mathcal{M}$ be the characteristic function of S_n. If $k \leq n$, $s \in \mathbf{B}_k$, and $\phi : M \to \mathbf{R}$, then $E[\hat{\phi}(x_n)|\mathcal{G}_s]$ has the constant value $(\phi(0) + \phi(1))/2$ when $s \leq t_n$, and $E[\hat{\phi}(x_n)|\mathcal{G}_s] = \hat{\phi}(x_n)$ when $s > t_n$. It follows that for each k we have $law_k(x_m) = law_k(x_n)$ whenever $k \leq m$, $k \leq n$. Therefore the sequence $b_n = law_{\mathbf{B}}(x_n)$ is a Cauchy sequence and hence converges to a limit $b_\infty \in \Lambda(\mathcal{M})$. Since $law_{\mathbf{B}}$ is closed on Ω, there exists $x \in \mathcal{M}$ such that $law_{\mathbf{B}}(x) = b_\infty$. We observe that whenever $law_{\mathbf{B}}(y) = b_m$, $law_{\mathbf{B}}(z) = b_n$, and $m \neq n$ in $\mathbf{N} \cup \{\infty\}$, y is independent of z and hence $\rho_0(y,z) = 1/2$. Thus there cannot be a sequence $y_n \in \mathcal{M}$ such that $y_n \to x_\infty$ but $law_{\mathbf{B}}(y_n) = b_n$ for each n. This proves (i).

Let $B_m = \{y : law_{\mathbf{B}}(y) \in \{b_0, \ldots, b_m, b_\infty\}\}$ and $B = \bigcup_m B_m$. Then each of the sets B_m and B is basic for $law_{\mathbf{B}}$ and nonempty. Moreover, for each $m \in \mathbf{N}$ the set

$$C_m = \{(y, z) \in B_m \times B : \rho_0(y, z) \geq 1/2\}$$

is basic for $law_{\mathbf{B}}$. Let

$$D_m = \{z \in \mathcal{M} : (\forall y \in B_m)(y, z) \in C_m\}.$$

Then D_m is a decreasing chain of neocompact sets for $law_{\mathbf{B}}$ on Ω. We have $x_n \in D_m$ whenever $m < n \in \mathbf{N}$, so each D_m is nonempty. However, the intersection $\bigcap_m D_m$ is empty, because if $z \in \bigcap_m D_m$ then $z \in B_m$ for some m, and we would have $(z, z) \in C_m$ and $\rho_0(z, z) \geq 1/2$. This proves (ii).

To prove (iii), we show that at least one of the sets D_m is not basic for $law_{\mathbf{B}}$. Suppose to the contrary that each D_m is basic for $law_{\mathbf{B}}$. None of the sets D_m can be compact, because then D_n would be compact for all $n > m$ and the intersection could not be empty. Thus for each m, $D_m = \{y : law_{\mathbf{B}}(y) \in E_m\}$ for some compact set E_m. Since $x_n \in D_m$, we have $b_n \in E_m$ whenever $m < n$. Recalling that $b_n \to law_{\mathbf{B}}(x_\infty)$, we see that $law_{\mathbf{B}}(x_\infty) \in E_m$, and hence $x_\infty \in D_m$, for each m. This contradicts the fact that $\bigcap_m D_m$ is empty, and proves (iii). $\square$

The preceding proof also works in the same way when we use a decreasing sequence of times instead of an increasing sequence of times.

The following question about the Skorokhod property remains open.

Question 8.12 *Is there an atomless* $\mathbf{B}$*-adapted space* $\Omega_{\mathbf{B}}$ *such that* $law_{\mathbf{B}}$ *has the Skorokhod property on* Ω*?*

By Proposition 8.11, $law_{\mathbf{B}}$ cannot be both closed and have the Skorokhod property on Ω.

We now return to an arbitrary linearly ordered time set $\mathbf{L}$. If $\mathbf{L}$ is countable, we may represent $\mathbf{L}$ as the union of a countable chain of finite subsets $\mathbf{L}_k$, and define $law_{\mathbf{L}}$ in the same way as $law_{\mathbf{B}}$. Then all of the results of this section hold for $\Omega_{\mathbf{L}}$ as well as for $\Omega_{\mathbf{B}}$. (Proposition 8.11 holds whenever $\mathbf{L}$ is infinite). The following corollary holds even for uncountable $\mathbf{L}$, and is a generalization of Corollary 7.10.

Corollary 8.13 *Let* $\Omega_{\mathbf{L}}$ *be a rich* $\mathbf{L}$*-adapted space. The following four families of subsets of* $\mathcal{M}$ *are the same.*

(i) The family of neocompact sets over $\mathcal{A}_{\Omega_{\mathbf{L}}}$.

(ii) The family of neocompact sets over $\mathcal{B}_{\Omega_{\mathbf{L}}}$.

(iii) The family of countable intersections of basic sections over $\mathcal{B}_{\Omega_{\mathbf{L}}}$.

(iv) The family of sets which are intersections of a set of the form

$$\{x \in \mathcal{M} : law(x, z) \in C\}$$

and countably many sets of the form

$$\{x \in \mathcal{M} : E[f_n(x, z)] \in D_n\}$$

where each f_n is an $\mathbf{L}$-adapted function, $z \in \mathcal{N}$, C is compact in $Meas(M \times N)$, and each D_n is compact in $\mathbf{R}$.

(v) If $\mathbf{L}$ is countable, then the family of basic sections for $law_{\mathbf{L}}$ on Ω is equal to the families (i)-(iv). $\square$

Proof: The proof of Theorem 8.10 gives the result in the case that $\mathbf{L} = \mathbf{B}$. The general case now follows by Lemma 8.4 and Corollary 7.10. $\square$

9 Right Continuous Adapted Spaces

We now consider continuous time adapted spaces. As is usual in the literature, we restrict our attention to the case where the filtration is complete and right continuous. By a **right continuous adapted space** we mean a structure

$$\Omega_{\mathbf{R}} = (\Omega, P, \mathcal{F}_\infty, \mathcal{F}_t)_{t \in \mathbf{R}_+}$$

where $\mathbf{R}_+ = [0, \infty)$, $\mathcal{F}_t$ is a $\mathcal{F}_\infty$-complete σ-algebra, and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for each $t \in \mathbf{R}_+$. Similarly, a **right continuous B-adapted space** is a structure $(\Omega, P, \mathcal{F}_\infty, \mathcal{F}_t)_{t \in \mathbf{B}}$ where $\mathcal{F}_t$ is $\mathcal{F}_\infty$-complete and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for each $t \in \mathbf{B}$.

Each $\mathbf{B}$-adapted space $\Omega_{\mathbf{B}}$ has a corresponding right continuous adapted space $\Omega_{\mathbf{R}}$ where $\mathcal{F}_\infty = \mathcal{G}_\infty$ and $\mathcal{F}_t$ is the $\mathcal{G}_\infty$-completion of the σ-algebra $\bigcap_{s>t} \mathcal{G}_s$. The filtration $\mathcal{G}_t$ for $\Omega_{\mathbf{B}}$ is not necessary right continuous, and thus is not uniquely determined by the filtration $\mathcal{F}_t$ for $\Omega_{\mathbf{R}}$.

Each right continuous adapted space $\Omega_{\mathbf{R}}$ has a corresponding right continuous $\mathbf{B}$-adapted space

$$\Omega^{rt}_{\mathbf{B}} = (\Omega, P, \mathcal{F}_\infty, \mathcal{F}_t)_{t \in \mathbf{B}}.$$

The law function for $\Omega^{rt}_{\mathbf{B}}$ will be denoted by $law^{rt}_{\mathbf{B}}$.

If we start with a given $\mathbf{B}$-adapted space $\Omega_{\mathbf{B}}$, then $\Omega^{rt}_{\mathbf{B}}$ will denote the right continuous $\mathbf{B}$-adapted space obtained from the right continuous adapted space $\Omega_{\mathbf{R}}$ corresponding to $\Omega_{\mathbf{B}}$. Throughout this section we shall always assume that $\Omega_{\mathbf{B}}$, $\Omega_{\mathbf{R}}$, and $\Omega^{rt}_{\mathbf{B}}$ are related in this way. Note that $\Omega_{\mathbf{R}}$ is atomless if and only if $\Omega_{\mathbf{B}}$ is atomless, and also if and only if $\Omega^{rt}_{\mathbf{B}}$ is atomless.

We begin with a negative result. This result is an improvement of [FK1, Example 7.7], which showed that a right continuous $\mathbf{B}$-adapted space cannot be rich, and hence cannot be saturated in the sense of the preceding section.

Proposition 9.1 *Let $\Omega_{\mathbf{B}}$ be atomless.*
(i) The law mapping $law^{rt}_{\mathbf{B}}$ is not closed.
(ii) Neither $\Omega^{rt}_{\mathbf{B}}$ nor $\Omega_{\mathbf{R}}$ is rich in the sense of Definition 8.2.
(iii) If $law_{\mathbf{B}}$ is closed, then $\mathcal{F}_t$ is atomless over $\mathcal{G}_t$ for each $t \in \mathbf{B}$.

Proof: (i) There exist $x_n \in L^0(\Omega, \{0,1\})$ such that $E[x_n] = 1/2$ and x_n is $\mathcal{F}_{t+1/n}$-measurable but independent of $\mathcal{F}_t$. Then $law^{rt}_{\mathbf{B}}(x_n)$ converges in $\mathbf{R}^{\mathbf{N}} \times Meas(\{0,1\})$ but there is no $x \in L^0(\Omega, \{0,1\})$ such that $law^{rt}_{\mathbf{B}}(x_n)$ converges to $law^{rt}_{\mathbf{B}}(x)$. Therefore $law^{rt}_{\mathbf{B}}$ is not closed.

(ii) By (i) and Theorem 8.9, $\Omega^{rt}_{\mathbf{B}}$ is not rich. Since $\mathcal{A}_{\Omega^{rt}_{\mathbf{B}}} \subseteq \mathcal{A}_{\Omega_{\mathbf{R}}}$, $\Omega_{\mathbf{R}}$ is also not rich.

(iii) Suppose $law_{\mathbf{B}}$ is closed, and let x_n be as in the proof of part (i). By taking a subsequence we may assume without loss of generality that $law_{\mathbf{B}}(x_n)$ converges to some $c \in \mathbf{R}^{\mathbf{N}} \times Meas(\{0,1\})$. Then there is an $x \in L^0(\Omega, \{0,1\})$ such that $law_{\mathbf{B}}(x) = c$. For each $s > t$, x_n is $\mathcal{G}_s$-measurable for all sufficiently large n, and hence x is $\mathcal{G}_s$-measurable. Therefore x is $\mathcal{F}_t$-measurable. However, for each n we have $E[x_n|\mathcal{G}_t] = 1/2$ almost surely, and therefore $E[x|\mathcal{G}_t] = 1/2$ almost surely. It follows that $\mathcal{F}_t$ is atomless over $\mathcal{G}_t$. $\square$

Combining the above proposition with Theorems 8.8 and 8.9, we see that right continuous $\mathbf{B}$-adapted spaces are never saturated or even universal in the sense of the preceding section. To get around this difficulty, we use the notions of saturation and universality from [HK], which compare a right continuous adapted space with other right continuous adapted spaces rather than with arbitrary adapted spaces.

Definition 9.2 *Two random variables $x, y \in \mathcal{M}$ are* **adapted equivalent** *on $\mathbf{R}$, in symbols $x \equiv y$, if $E[f(x)] = E[f(y)]$ for every adapted function f on $\mathcal{M}$ for $\Omega_{\mathbf{R}}$. This notion can also be applied to random variables on two different right continuous adapted spaces. $\Omega_{\mathbf{R}}$ is* **universal** *if for every other right continuous adapted space $\Gamma_{\mathbf{R}}$ and random variable $\bar{x} \in L^0(\Gamma, M)$ there exists $x \in \mathcal{M}$ such that $x \equiv \bar{x}$. $\Omega_{\mathbf{R}}$ is* **saturated** *if for every other right continuous adapted space $\Gamma_{\mathbf{R}}$, if $\bar{x} \in L^0(\Gamma, M), \bar{y} \in L^0(\Gamma, N), x \in \mathcal{M}$, and $x \equiv \bar{x}$, then there exists $y \in \mathcal{N}$ such that $(x, y) \equiv (\bar{x}, \bar{y})$.*

The following result is proved in [HK, Corollary 2.13].

Proposition 9.3 *Let $\Omega_{\mathbf{R}}$ be a right continuous adapted space, $\mathcal{M} \in \mathbf{M}_\Omega$, and $x, y \in \mathcal{M}$. Then $x \equiv y$ if and only if $law^{rt}_{\mathbf{B}}(x) = law^{rt}_{\mathbf{B}}(y)$.* $\square$

The proof of the above proposition also yields the following result.

Proposition 9.4 *Let $\Omega_{\mathbf{B}}$ be an atomless $\mathbf{B}$-adapted space and $x, y \in \mathcal{M}$. If $law_{\mathbf{B}}(x) = law_{\mathbf{B}}(y)$ then $x \equiv y$ in $\Omega_{\mathbf{R}}$.* $\square$

Example 9.5 *The converse of the above proposition is false. If $law_{\mathbf{B}}$ is closed, there exist $x, y \in L^0(\Omega, \{0,1\})$ such that $x \equiv y$ but $law_{\mathbf{B}}(x) \neq law_{\mathbf{B}}(y)$.*

To see this, note first that by Proposition 9.1, $\mathcal{F}_0$ is atomless over $\mathcal{G}_0$. Take x to be the characteristic function of a $\mathcal{G}_0$-measurable set of measure 1/2, and take y to be the characteristic function of an $\mathcal{F}_0$-measurable set of measure 1/2 which is independent of $\mathcal{G}_0$. Then $x \equiv y$, but $E[(E[x|\mathcal{G}_0])^2] = 1/2$ and $E[(E[y|\mathcal{G}_0])^2] = 1/4$. This shows that $law_{\mathbf{B}}(x) \neq law_{\mathbf{B}}(y)$. $\square$

We next wish to show that richness for $\Omega_{\mathbf{B}}$ implies saturation for $\Omega_{\mathbf{R}}$. In order to do this we shall need a law mapping $(law_{\mathbf{R}}, \Lambda)$ such that:

$$x \equiv y \text{ if and only if } law_{\mathbf{R}}(x) = law_{\mathbf{R}}(y) \tag{8}$$

and

$$\text{For each } \mathcal{M}, \text{ the function } law_{\mathbf{R}} \text{ is neocontinuous over } \mathcal{A}_{\Omega_{\mathbf{B}}}. \tag{9}$$

By Proposition 9.3, the function $law_{\mathbf{B}}^{rt}$ obtained from a rich $\mathbf{B}$-adapted space $\Omega_{\mathbf{B}}$ has property (8). However, [FK1, Example 7.7] shows that $law_{\mathbf{B}}^{rt}$ cannot have property (9). To build a law mapping with both properties (8) and (9) we shall introduce the notion of a conditional process from [HK], which is an adapted function with variable times.

We let ν be the Borel probability measure on $\mathbf{R}_+$ with exponential density, so that $\nu([s,t]) = e^{-s} - e^{-t}$. $L^0(\mathbf{R}_+, \mathbf{R})$ will denote the space of measurable functions $y : \mathbf{R}_+ \to \mathbf{R}$ with the metric of convergence in probability with respect to ν. It is a complete separable metric space. An n-**fold** stochastic process on Ω is a random variable on Ω with values in the complete separable metric space $L^{0,n} = L^0((\mathbf{R}_+)^n, \mathbf{R})$. We shall let $\mathcal{L}^{0,n} = L^0(\Omega, L^{0,n})$ be the space of all n-fold stochastic process on Ω with the metric of convergence in probability.

Definition 9.6 *The class of* **conditional processes** *on $\mathcal{M}$ for a right continuous adapted space $\Omega_{\mathbf{R}}$ is the least class of functions from $\mathcal{M}$ into $\mathcal{L}^{0,n}$ such that:*

(i) For each bounded continuous function $\phi : M \to \mathbf{R}$, the function $(\hat{\phi}(x))(\omega) = \phi(x(\omega))$ is a 0-fold conditional process on $\mathcal{M}$.

(ii) If $f_1, \ldots, f_m$ are n-fold conditional processes on $\mathcal{M}$ and $g : \mathbf{R}^m \to \mathbf{R}$ is continuous, then $h(x) = g(f_1(x), \ldots, f_m(x))$ is an n-fold conditional process on $\mathcal{M}$.

(iii) If f is an n-fold conditional process on $\mathcal{M}$ and $\vec{t}$ varies over $(\mathbf{R}_+)^n$ then

$$(g(x)(\omega))(\vec{t}, s) = E[(f(x))(\vec{t})|\mathcal{F}_s](\omega)$$

is an $(n+1)$-fold conditional process on $\mathcal{M}$.

Each conditional process on $\mathcal{M}$ is uniformly bounded. For each $x \in \mathcal{M}$ and n-fold conditional process f on $\mathcal{M}$, we define the **expected path** $E[f(x)](\vec{t}) \in L^{0,n}$ by

$$E[f(x)](\vec{t}) = E[(f(x)(\cdot))(\vec{t})].$$

We shall now define the right continuous law function $law_{\mathbf{R}}$. As in the case of adapted functions, we choose a countable set of conditional processes on $\mathcal{M}$ which is dense in an appropriate sense. Recall that $\Phi(M)$ is a countable set of bounded continuous functions $f : M \to \mathbf{R}$ which is bounded pointwise dense. The class of conditional processes built from $\Phi(M)$ is defined in the natural way analogous to the class of adapted functions built from $\Phi(M)$, and is a countable set which we arrange in a list $\langle \gamma_0, \gamma_1, \ldots \rangle$. Each γ_n is a $j(n)$-fold conditional process for some $j(n) \in \mathbf{N}$. For each $\mathcal{M}$, the target space will be the product

$$\Lambda(\mathcal{M}) = \prod \{L^{0,j(n)} : n \in \mathbf{N}\} \times Meas(M).$$

(An alternative, would have been to use the space of right continuous functions with left limits from $(\mathbf{R}_+)^{j(n)}$ into $\mathbf{R}$ with the Skorokhod topology in place of $L^{0,j(n)}$).

Definition 9.7 *Let $\Omega_{\mathbf{R}}$ be a right continuous adapted space and $x \in \mathcal{M}$. The* **continuous time adapted law** *of x is the pair*

$$law_{\mathbf{R}}(x) = (\langle E[\gamma_0(x)](\cdot), E[\gamma_1(x)](\cdot), \ldots \rangle, law(x))$$

in the metric space $\Lambda(\mathcal{M})$.

Proposition 9.8 *$x \equiv y$ on $\mathbf{R}$ if and only if $law_{\mathbf{R}}(x) = law_{\mathbf{R}}(y)$.*

Proof: This follows from [HK, Corollary 2.15]. □

The proof of the next proposition is analogous to the proof of the corresponding result for the discrete time law function $law_{\mathbf{T}}$.

Proposition 9.9 *(i) The $law_{\mathbf{R}}$ function is continuous on $\mathcal{M}$, and uniformly continuous on $law^{-1}(C)$ for each compact set $C \subseteq Meas(M)$.*

(ii) For each set $A \subseteq \mathcal{M}$, $law_{\mathbf{R}}(A)$ is relatively compact if and only $law(A)$ is relatively compact.

(iii) $law_{\mathbf{R}}$ is a law mapping.

(iv) If $\Omega_{\mathbf{B}}$ is rich then the $law_{\mathbf{R}}$ function is neocontinuous over $\mathcal{A}_{\Omega_{\mathbf{B}}}$ for each $\mathcal{M}$.

Proof: Part (iv) follows from [FK1, Theorem 9.7]. □

One way of approximating the adapted space $\Omega_{\mathbf{R}}$ is to approximate the filtration $\mathcal{F}_t$ by a finite step function. Let $\mathcal{F}_{t,k} = \mathcal{F}_s$ where s is the least element of $\mathbf{B}_k \cup \{\infty\}$ such that $s \geq t$, let $\Omega_{\mathbf{R},k}$ be the right continuous adapted space $(\Omega, P, \mathcal{F}_\infty, \mathcal{F}_{t,k})_{t \in \mathbf{R}_+}$, and let $law_{\mathbf{R},k}$ be the right continuous adapted law for $\Omega_{\mathbf{R},k}$. By Proposition 9.9, $law_{\mathbf{R},k}$ is a law mapping. Thus $\mathcal{F}_{t,k}$ is a step function with steps in $\mathbf{B}_k$. For each conditional process f for $\Omega_{\mathbf{R}}$, let f_k be the corresponding conditional process for $\Omega_{\mathbf{R},k}$. The paths of each n-fold conditional process $f_k(x)$ for $\Omega_{\mathbf{R},k}$ are n-fold step functions which are constant on the interior of each cube with vertices in $(\mathbf{B}_k)^n$. Moreover, the value of the conditional process $f_k(x)$ at $\vec{t}$ is equal to the value of the adapted function $f(x)(\vec{s})$ at the greatest $\vec{s} \leq \vec{t}$ in $(\mathbf{B}_k)^n$.

Note that $law^{rt}_{\mathbf{B}_k}(x) = law^{rt}_{\mathbf{B}_k}(y)$ if and only if $law_{\mathbf{R},k}(x) = law_{\mathbf{R},k}(y)$. In fact, the mapping $law^{rt}_{\mathbf{B}_k}(x) \mapsto law_{\mathbf{R},k}(x)$ is a topological homeomorphism from $law^{rt}_{\mathbf{B}_k}(\mathcal{M})$ to $law_{\mathbf{R},k}(\mathcal{M})$. However, this map does not preserve the metrics.

We shall use the upcrossing inequality to prove that $law_{\mathbf{R},k}(x)$ converges to $law_{\mathbf{R}}(x)$ uniformly in $\Omega_{\mathbf{R}}$ and x, and then show that $law_{\mathbf{R}}$ is dense.

Lemma 9.10 *For each complete separable M and each $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that for all $k \geq m$, all right continuous adapted spaces $\Omega_{\mathbf{R}}$, and every $x \in \mathcal{M}$, $law_{\mathbf{R},k}(x)$ is within ε of $law_{\mathbf{R}}(x)$.*

Proof: It suffices to prove that for each conditional process f and $\varepsilon > 0$ there exists $m(\varepsilon, f) \in \mathbf{N}$ such that for all $k \geq m(\varepsilon, f)$, all $\Omega_{\mathbf{R}}$, and all $x \in \mathcal{M}$, $f_k(x)$ is within ε of $f(x)$ in $\mathcal{L}^{0,n}$. We do this by induction on the formation of f. The main difficulty is in the conditional expectation step. Assume the result holds for $f(x)(\vec{t})$ and let

$$g(x)(\vec{t}, s) = E[f(x)(\vec{t})|\mathcal{F}_s].$$

Let b be a uniform bound for f. For each $\delta > 0, k \geq m(\delta, f)$, and $x \in \mathcal{M}$, the set U_k of all $\vec{t} \in (\mathbf{R}_+)^n$ such that $f_k(x)(\vec{t})$ is within δ of $f(x)(\vec{t})$ in $\mathcal{R}$ has ν^n-measure at least $1 - \delta$. Then for each $\vec{t} \in U_k$,

$$E[|f_k(x)(\vec{t}) - f(x)(\vec{t})|] \leq \delta(1 + b).$$

For each $(\vec{t}, s) \in (\mathbf{B}_k)^{n+1}$,

$$g_k(x)(\vec{t}, s) = E[f_k(x)(\vec{t})|\mathcal{F}_s].$$

Let $h_k = g_k - g$. Then for each $\vec{t}$, $\langle h_k(x)(\vec{t}, s) : s \in \mathbf{B}_k \rangle$ is a martingale for $\Omega_{\mathbf{R},k}$. By the maximal inequality for martingales, for each $\vec{t} \in U_k$ the set

$$V_k(\vec{t}) = \{\omega \in \Omega : sup\{|h_k(x)(\vec{t}, s)| : s \in \mathbf{B}_k\} \leq \sqrt{\delta}\}$$

has measure at least $1-\sqrt{\delta}(1+b)$.

Let $\varepsilon > 0$ and take δ so that $\sqrt{\delta}(1+b) \leq \varepsilon/2$. Divide $[-b,b]$ into a finite set I of intervals of length $\varepsilon/8$ such that the center of one interval is the starting point of the next. The set I has cardinality $16b/\varepsilon$. By the upcrossing inequality (see [B2, p. 489]), for each $\vec{t}$ and k, the expected number of upcrossings of an interval of length $\varepsilon/8$ by either $g_k(x)(\vec{t},u)$ or $g(x)(\vec{t},u)$ for $u \in \mathbf{R}_+$ is at most $16b/\varepsilon$.

Suppose $\vec{t} \in U_k$ and $\omega \in V_k(\vec{t})$. Then

$$|h_k(x)(\vec{t},u)(\omega)| \leq \varepsilon/2$$

for all $u \in \mathbf{B}_k$. Let

$$W_k(\omega,\vec{t}) = \{u \in \mathbf{R}_+ : |h_k(x)(\vec{t},u)(\omega)| \geq \varepsilon\}.$$

If $s < s' \in \mathbf{B}_k$ and $W_k(\omega,\vec{t})$ meets $[s,s']$, then either $g_k(x)(\vec{t},v)(\omega)$ or $g(x)(\vec{t},v)(\omega)$ must have an upcrossing of one of the intervals in I while $v \in [s,s']$. Therefore there exists $m(\varepsilon,g) \in \mathbf{N}$ depending only on ε and g such that for all $k \geq m(\varepsilon,g)$ and $\vec{t} \in U_k$, the set of $\omega \in V_k(\vec{t})$ such that $\nu(W_k(\omega,\vec{t})) \geq \varepsilon$ has measure $\leq \varepsilon/2$. Then for each $\vec{t} \in U_k$ the set of $\omega \in \Omega$ such that $\nu(W_k(\omega,\vec{t})) < \varepsilon$ has measure at least $1-\varepsilon$. Since U_k has measure at least $1-\delta$, it follows that for all $\Omega_{\mathbf{R}}$ and all $x \in \mathcal{M}$, $g_k(x)$ is within ε of $g(x)$ in $\mathcal{L}^{0,n+1}$. $\square$

Proposition 9.11 *If $\Omega_{\mathbf{R}}$ is an atomless right continuous adapted space, then $law_{\mathbf{R}}$ is dense.*

Proof: By Proposition 7.4, $law^{rt}_{\mathbf{B}_k}$ is dense for each k. It follows that $law_{\mathbf{R},k}$ is dense for each k. Suppose $x,\bar{x} \in \mathcal{M}, \bar{y} \in \mathcal{N}$, and $law_{\mathbf{R}}(x) = law_{\mathbf{R}}(\bar{x})$. By Propositions 9.3 and 9.8, $law_{\mathbf{R},k}(x) = law_{\mathbf{R},k}(\bar{x})$ for each $k \in \mathbf{N}$. Let $\varepsilon > 0$. For each k we may choose $y_k \in \mathcal{N}$ such that $law_{\mathbf{R},k}(x,y_k)$ is within ε of $law_{\mathbf{R},k}(\bar{x},\bar{y})$. By Lemma 9.10, $law_{\mathbf{R},k}(x,y_k)$ is within ε of $law_{\mathbf{R}}(x,y_k)$ and $law_{\mathbf{R},k}(\bar{x},\bar{y})$ is within ε of $law_{\mathbf{R}}(\bar{x},\bar{y})$ for all sufficiently large k. Therefore $law_{\mathbf{R}}(x,y_k)$ is within 3ε of $law_{\mathbf{R}}(\bar{x},\bar{y})$, so $law_{\mathbf{R}}$ is dense. $\square$

Corollary 9.12 *Let $\Omega_{\mathbf{R}}$ be an atomless right continuous adapted space. The following are equivalent:*

(i) $law_{\mathbf{R}}$ has the back and forth property;

(ii) $law^{rt}_{\mathbf{B}}$ has the back and forth property.

Moreover, $\Omega_{\mathbf{R}}$ is saturated if and only if $\Omega_{\mathbf{R}}$ is universal and one of the above conditions (i)-(ii) holds.

Proof: $law_{\mathbf{R}}$ is dense by Proposition 9.11, and $law^{rt}_{\mathbf{B}}$ is dense by Proposition 8.5. (i) is equivalent to (ii) by Propositions 9.8 and 9.3. It follows from Proposition 9.8 that $\Omega_{\mathbf{R}}$ is saturated if and only if $\Omega_{\mathbf{R}}$ is universal and (i) holds. $\square$

Another proof of Corollary 9.12 is given by the proof of [HK, Theorem 5.2]. As pointed out by Hoover [H2], the statement of [HK, Theorem 5.2] was incorrect. Corollary 9.12 gives a corrected formulation of the result.

For any countable set $\mathbf{L} \subseteq \mathbf{R}_+$, one can define $\Omega^{rt}_{\mathbf{L}}$ and $law^{rt}_{\mathbf{L}}$ in the same way as we defined $\Omega^{rt}_{\mathbf{B}}$ and $law^{rt}_{\mathbf{B}}$. Then Corollary 9.12 also holds for any countable dense set $\mathbf{L} \subseteq \mathbf{R}_+$. Hoover [H3] proved that if $\Omega_{\mathbf{R}}$ is saturated then $law^{rt}_{\mathbf{L}}$ has the back and forth property for every countable $\mathbf{L} \subseteq \mathbf{R}_+$.

Proposition 9.13 *Let $\Omega_{\mathbf{R}}$ be an atomless right continuous adapted space. Then for each other right continuous adapted space $\Gamma_{\mathbf{R}}$ and each $M, law_{\mathbf{R}}(\mathcal{M})$ is dense in $law_{\mathbf{R}}(L^0(\Gamma, M))$.*

Proof: Let $x \in L^0(\Gamma, M)$. By Proposition 6.8, for each $k \in \mathbf{N}$ we may choose $y_k \in \mathcal{M}$ such that $law_{\mathbf{R},k}(y_k) = law_{\mathbf{R},k}(x)$. By Lemma 9.10, $law_{\mathbf{R}}(y_k)$ converges to $law_{\mathbf{R}}(x)$. $\square$

Theorem 9.14 *A right continuous adapted space $\Omega_{\mathbf{R}}$ is universal if and only if $\Omega_{\mathbf{R}}$ is atomless and $law_{\mathbf{R}}$ is closed.*

Proof: Suppose first that $\Omega_{\mathbf{R}}$ is universal. Then for each k, $\Omega_{\mathbf{R},k}$ is universal. By Proposition 6.8, $\Omega_{\mathbf{R},k}$ is atomless, so $\Omega_{\mathbf{R}}$ is atomless. Let x_n be a sequence in $\mathcal{M}$ such that $law_{\mathbf{B}}(x_n)$ converges to a point $c \in \Lambda(\mathcal{M})$. Let $\Gamma_{\mathbf{B}}$ be a rich $\mathbf{B}$-adapted space. Then $\Gamma_{\mathbf{R}}$ is atomless, so by Proposition 9.13 there is a sequence y_n in $L^0(\Gamma, M)$ such that $law_{\mathbf{R}}(y_n)$ converges to c. Then for each n, the set

$$C_n = \{c\} \cup \{law_{\mathbf{R}}(y_m) : n \leq m\}$$

is compact. Since $law_{\mathbf{R}}$ is neocontinuous over $\mathcal{A}_{\Gamma_{\mathbf{B}}}$, the sets $law_{\mathbf{R}}{}^{-1}(C_n), n \in \mathbf{N}$, form a decreasing chain of nonempty neocompact sets. By countable compactness, the intersection of this chain is nonempty, so there exists $y \in L^0(\Gamma, M)$ such that $law_{\mathbf{R}}(y) = c$. Since $\Omega_{\mathbf{R}}$ is universal, there exists $x \in \mathcal{M}$ such that $law_{\mathbf{R}}(x) = c$, so $law_{\mathbf{R}}$ is closed.

Now suppose $\Omega_{\mathbf{R}}$ is atomless and $law_{\mathbf{R}}$ is closed. Let $\Gamma_{\mathbf{R}}$ be another right continuous adapted space and let $y \in L^0(\Gamma, M)$. By Proposition 9.13 there is a sequence $x_n in \mathcal{M}$ such that $law_{\mathbf{R}}(x_n)$ converges to $law_{\mathbf{R}}(y)$. Since $law_{\mathbf{R}}$ is closed, there exists $x \in \mathcal{M}$ with $law_{\mathbf{R}}(x) = law_{\mathbf{R}}(y)$, so $\Omega_{\mathbf{R}}$ is universal. $\square$

Theorem 9.15 *If $\Omega_{\mathbf{B}}$ is a rich $\mathbf{B}$-adapted space, then $\Omega_{\mathbf{R}}$ is saturated.*

Proof: Since $\Omega_{\mathbf{B}}$ is rich, $\Omega_{\mathbf{R}}$ is atomless. By Proposition 9.9 (iv), $law_{\mathbf{R}}$ is neocontinuous over $\mathcal{A}_{\Omega_{\mathbf{B}}}$. Then each basic section for $law_{\mathbf{R}}$ is neocompact over $\mathcal{A}_{\Omega_{\mathbf{B}}}$, and thus the family of basic sections in $\mathcal{M}$ for $law_{\mathbf{R}}$ is countably compact. $law_{\mathbf{R}}$ is dense by Proposition 9.11. By Theorem 2.7, $law_{\mathbf{R}}$ is closed and has the back and forth property. By Theorem 9.14, $\Omega_{\mathbf{R}}$ is universal, and hence by Corollary 9.12, $\Omega_{\mathbf{R}}$ is saturated. $\square$

Since rich $\mathbf{B}$-adapted spaces can never have right continuous filtrations, the converse of the above theorem is false. That is, there are spaces $\Omega_{\mathbf{B}}$ such that $\Omega_{\mathbf{R}}$ is saturated but $\Omega_{\mathbf{B}}$ is not rich. The following related question is open.

Question 9.16 *If $\Omega_{\mathbf{R}}$ is a saturated right continuous adapted space, does there exist a rich $\mathbf{B}$-adapted space $\Gamma_{\mathbf{B}}$ such that $\Gamma_{\mathbf{R}} = \Omega_{\mathbf{R}}$?*

The following negative result can be proved by the same construction that was used in the proof of Proposition 8.11 in the preceding section.

Proposition 9.17 *Let $\Omega_{\mathbf{R}}$ be a universal right continuous $\mathbf{R}$-adapted space, and let $M = \{0, 1\}$. Then:*

(i) The function $law_{\mathbf{R}}$ does not have the Skorokhod property on Ω.

(ii) The family of neocompact subsets of $\mathcal{M}$ for $law_{\mathbf{R}}$ on Ω is not countably compact.

(iii) There is a basic set $C \subseteq \mathcal{M} \times \mathcal{M}$ for $law_{\mathbf{R}}$ and a nonempty basic set $C \subseteq \mathcal{M}$ for $law_{\mathbf{R}}$ such that the set $\{x : (\forall y \in \mathcal{M})(x, y) \in C\}$ is not basic for $law_{\mathbf{R}}$. $\square$

The following question is analogous to Question 8.12 for $\mathbf{B}$-adapted spaces.

Question 9.18 *Is there an atomless right continuous adapted space $\Omega_{\mathbf{R}}$ such that $law_{\mathbf{R}}$ has the Skorokhod property on Ω?*

We shall now give a characterization of saturated right continuous adapted spaces $\Omega_{\mathbf{R}}$ by a weaker analogue of richness which does not depend on the Skorokhod property.

Let us call a set $C \subseteq \mathcal{M}$ **existentially definable over $\mathcal{A}$** if C is built from sets in $\mathcal{A}(\mathcal{M})$ using only the rules (a)–(e), that is, without the universal projection rule (f). The following weak quantifier elimination theorem is a consequence of Theorem 2.12 and is proved in [K5].

Theorem 9.19 *(Existential Quantifier Elimination) Let λ be a closed law mapping. Let $\mathcal{A}(\mathcal{M})$ be the family of basic subsets of $\mathcal{M}$ for λ. The following are equivalent.*

(i) λ has the back and forth property.

(ii) Each existentially definable set over $\mathcal{A}$ is basic for λ. $\square$

The next result shows that saturation is equivalent to the analogue of richness for existentially definable sets.

Theorem 9.20 *Let $\Omega_{\mathbf{R}}$ be an atomless right continuous adapted space. The following are equivalent.*

(i) $\Omega_{\mathbf{R}}$ is saturated.

(ii) $law_{\mathbf{R}}$ is closed and has the back and forth property.

(iii) For each $\mathcal{M} \in \mathbf{M}_{\Omega}$, the family of basic sections in $\mathcal{M}$ for the law mapping $law_{\mathbf{R}}$ is countably compact.

(iv) For each $\mathcal{M} \in \mathbf{M}_{\Omega}$, the family of subsets of $\mathcal{M}$ which are existentially definable over the basic/compact sets for $law_{\mathbf{R}}$ is countably compact.

(v) $law_{\mathbf{R}}$ is closed and for every basic relation $C \subseteq \mathcal{M} \times \mathcal{N}$ for $law_{\mathbf{R}}$ the set $\{x : \exists y(x, y) \in C\}$ is basic for $law_{\mathbf{R}}$.

Proof: We first prove that (i) is equivalent to (ii). Assume (i). Then $\Omega_{\mathbf{R}}$ is universal, and $law_{\mathbf{R}}$ is closed by Theorem 9.14. $law_{\mathbf{R}}$ has the back and forth property by Proposition 9.9. This proves that (i) implies (ii). Now assume (ii). $\Omega_{\mathbf{R}}$ is universal by Theorem 9.14. Therefore $\Omega_{\mathbf{R}}$ is saturated, and thus (ii) implies (i).

We now prove that (ii) is equivalent to (iii). (ii) implies (iii) by Theorem 2.7. Assume (iii). By Proposition 9.11, $law_{\mathbf{R}}$ is dense. Then (ii) follows by Theorem 2.7. (ii) is equivalent to (v) by Theorem 9.19. Finally, we prove that (iii) is equivalent to (iv). Assume (iii). $law_{\mathbf{R}}$ is dense by Proposition 9.11, and has the back and forth property by Theorem 2.7. By Theorem 9.19, the family of basic sections is closed under the operations (b)–(e), and (iv) follows. The implication from (iv) to (iii) is trivial. $\square$

The main advantage of using a rich adapted space $\Omega_{\mathbf{B}}$ instead of a saturated right continuous adapted space $\Omega_{\mathbf{R}}$ is that we can use a rich adapted space to prove existence theorems without introducing the adapted law function (as in the paper [FK1]). Most of the applications of rich adapted spaces in the paper [FK1] use neocompact sets which are constructed using the universal projection rule (f), and thus require a rich space $\Omega_{\mathbf{B}}$ rather than merely a saturated right continuous space $\Omega_{\mathbf{R}}$. In particular, this applies to most applications involving conditional expectations or stochastic integrals. However, some of the applications, such as the result that every continuous process has a closest Brownian motion in the metric of convergence in probability ([FK1, Corollary 12.2]), do not depend on the universal projection rule (f) and thus hold for any saturated right continuous space $\Omega_{\mathbf{R}}$.

The results in this section can be applied to adapted Loeb spaces. Let $\Omega_{\mathbf{B}}$ be an atomless Loeb **B**-adapted space. By [FK2, Theorem 5.15], $\Omega_{\mathbf{B}}$ is rich. Then $\Omega_{\mathbf{R}}$ is saturated by Theorem 9.15, and thus $\Omega_{\mathbf{R}}$ satisfies conditions (i)-(iv) in Theorem 9.19. A similar result was proved in [HK, Theorem 4.2]. (In that proof, formula (4.2.8) was stated without adequate justification. This gap can be corrected using Lemma 9.10 of this paper). The adapted Loeb spaces have a particularly nice subcollection, the **hyperfinite adapted spaces**. It is shown in [K3] that for each hyperfinite adapted space $\Omega_{\mathbf{B}}$, the associated right continuous adapted space $\Omega_{\mathbf{R}}$ is universal and **homogeneous**. That is, for every pair of random variables $x, y \in \mathcal{M}, x \equiv y$ if and only if there is a measure preserving bijection $h : \Omega \to \Omega$ such that $y(\omega) = h(x(\omega))$ almost surely, and $h(\mathcal{F}_t) = \mathcal{F}_t$ for all $t \in \mathbf{R}_+$. It is easily seen that every universal homogeneous space $\Omega_{\mathbf{R}}$ is saturated

The result in [K3] shows that for each hyperfinite adapted space $\Omega_{\mathbf{B}}$, the associated right continuous adapted space $\Omega_{\mathbf{R}}$ is saturated.

References

[AFHL] S. Albeverio, J. E. Fenstad, R. Hoegh-Krohn, and T. Lindstrøm. Nonstandard Methods in Stochastic Analysis and Mathematical Physics. Academic Press, New York (1986).

[B1] P. Billingsley. Convergence of Probability Measures. Wiley (1968).

[B2] P. Billingsley. Probability and Measure. Wiley (1986).

[FK1] S. Fajardo and H. J. Keisler. Existence Theorems in Probability Theory. To appear.

[FK2] S. Fajardo and H. J. Keisler. Neometric Spaces. Advances in Mathematics 118 (1996), pp. 134-175.

[HK] D. N. Hoover and H. J. Keisler. Adapted Probability Distributions. Trans. Amer. Math. Soc. 286 (1984), 159-201.

[H1] D. N. Hoover. Convergence in Distribution and Skorokhod Convergence for the General Theory of Processes. Prob. Theory and Related Fields 89 (1991), 239-259.

[H2] D. N. Hoover. Extending Probability Spaces and Adapted Distribution. Preprint, 1990.

[H3] D. N. Hoover. Extending Probability Spaces and Adapted Distribution. Pp. 560-574 in Séminaire de Probabilités XXVI, ed. by J. Azéma, P.A.Meyer, and M. Yor, Lecture Notes in Math. 1526, 1992.

[K1] H. J. Keisler. An Infinitesimal Approach to Stochastic Analysis. Memoirs Amer. Math. Soc. 297 (1984).

[K2] H. J. Keisler. Probability Quantifiers. Model Theoretic Languages, Springer-Verlag, pages 509-556 in Model Theoretic Logics, edited by J. Barwise and S. Feferman, 1985.

[K3] H. J. Keisler. Hyperfinite Models of Adapted Probability Logic, Ann. Pure and Applied Logic 31 (1986), 71-86.

[K4] H. J. Keisler. From Discrete to Continuous Time. Ann. Pure and Applied Logic 52 (1991), 99-141.

[K5] H. J. Keisler. Quantifier Elimination for Neocompact Sets. To appear.

[K6] H. J. Keisler. A Neometric Survey. Pp. 233-250 in Developments in Nonstandard Analysis, ed. by N. Cutland et al, Longman 1995.

[M1] D. Maharam. On Homogeneous Measure Algebras. Proc. Nat. Acad. Sci. U.S.A. 28 (1942), 108-111.

[M2] D. Maharam. Decompositions of Measure Algebras and Spaces. Trans. Amer. Math. Soc. 69 (1950), 142-160.

University of Wisconsin, Madison WI. keisler@math.wisc.edu

Mathematical Subject Classification, 1991:
Primary: 03C10, 03C90
Secondary: 54E35, 60G07, 60B10, 60E05

Keywords: elimination, quantifier, adapted, metric space, neocompact, neometric