MODEL THEORY FOR REAL-VALUED STRUCTURES

H. JEROME KEISLER

ABSTRACT. We consider general structures where formulas have truth values in the real unit interval as in continuous model theory, but whose predicates and functions need not be uniformly continuous with respect to a distance predicate. Every general structure can be expanded to a pre-metric structure by adding a distance predicate that is a uniform limit of formulas. Moreover, that distance predicate is unique up to uniform equivalence. We use this to extend the central notions in the model theory of metric structures to general structures, and show that many model-theoretic results from the literature about metric structures have natural analogues for general structures.

CONTENTS

1. Introduction

We show that much of the model theory of metric structures carries over to general structures that still have truth values in $[0, 1]$. A general ($[0, 1]$-valued) structure is like a classical first-order structure, except that the predicate symbols have truth values in $[0, 1]$. It has a vocabulary consisting of predicate, function, and constant symbols. General structures are called models of $[0, 1]$-valued logic in [CK66], and are called $[0, 1]$-valued structures in [AH] and [Ca].

Continuous model theory, as developed in [BBHU], [BU], and [Fa], has been highly successful in the study of metric structures, which are more elaborate than general $[0, 1]$-valued structures. A metric (or pre-metric) structure has, in addition to a vocabulary, a metric signature that specifies a distance predicate symbol that is a metric (or pseudo-metric), and a modulus of uniform continuity for each predicate and function symbol. The notions of formula, truth value of a formula, and sentence, are the same for general structures as for pre-metric structures.

We show that almost all of the model-theoretic properties of metric structures in [BBHU] extend in a unique way (called the absolute version) to general structures, and have natural characterizations in terms of the general structure itself. We also show that almost all of the results about metric structures in [BBHU] imply analogous results about general structures. We get similar results for the infinitary continuous model theory in [Ea15], and the model theory for unbounded metric structures in [BY08]. We show, for instance, that the property of stability has an absolute version, and a general structure is stable if and only if it has a stable independence relation. One can readily find (e.g. using Theorem 4.10.1 below) examples of stable general structures that are quite different from the familiar examples of stable metric structures in [BBHU].

The formalism for metric structures in [BU] and [BBHU] is, to quote from [BU]. "an immediate generalization of classical first order logic, more natural and less technically involved than previous formalisms."[1] The main advantage of their formalism is that it is easy to describe a metric structure by specifying the universe, functions, constants, relations, and moduli of uniform continuity. Here we go a step further—the notion of a general structure is even less technically involved than the notion of a metric structure, since it does not require one to specify a metric signature giving a distance predicate and moduli of uniform continuity.

Formally, the key concepts that make things work in this paper are the notion of a pre-metric expansion of a theory (Definition 3.2.1), and the notion of an absolute version of a property of pre-metric structures (Definition 3.4.1).

Let T be a theory (or set of sentences) with vocabulary V. A pre-metric expansion of T is a metric theory T_e that makes every general model $\mathcal{M}$ of T into a pre-metric model $\mathcal{M}_e$ of T_e in the following way. T_e has a metric signature over $V \cup \{D\}$ with distance D. Moreover, there is a sequence $\langle d_m \rangle_{m \in \mathbb{N}}$ of formulas of V, called an approximate distance for T_e, such that for each general model $\mathcal{M}$ of T, $\mathcal{M}_e = (\mathcal{M}, D)$ with $D = \lim_{m \to \infty} d_m$.

By adapting arguments of Iovino [I94] and Ben Yaacov [BY05] to our setting, and using Theorem 4.23 of [BU], one can prove that every complete theory of

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[1]Previous formalisms included metric open Hausdorff cats in [BY03a] and Henson's Banach space logic in [He], [HI].

general structures with a countable vocabulary has a pre-metric expansion.[2] The Expansion Theorem 3.3.4 improves that by showing that every (not necessarily complete) theory T with a countable vocabulary has a pre-metric expansion in which each of the formulas $d_m(x, y)$ defines a pseudo-metric in T. We will see in Section 4 that the Expansion Theorem has far-reaching consequences. Proposition 3.3.7 shows that the pre-metric expansion of a theory is unique up to uniform equivalence (but far from unique).

We say that a property $\mathcal{P}$ of general structures with parameters is an absolute version of a property $\mathcal{Q}$ of pre-metric structures if whenever $\mathcal{M}$ is a general model of a theory T and T_e is a pre-metric expansion of T, $\mathcal{M}$ has property $\mathcal{P}$ if and only if $\mathcal{M}_e$ has property $\mathcal{Q}$. If $\mathcal{Q}$ has an absolute version, its absolute version is unique. We regard the absolute version of a property of pre-metric structures as the "right" extension of that property to general structures.

General structures correspond to first order structures without equality in the same way that metric structures with a distance predicate correspond to first order structures with equality. In first order model theory, each structure without equality can be expanded to a pre-structure with equality in a unique way, so there is very little difference between structures without equality and structures with equality.

In $[0, 1]$-valued model theory, every general structure can be expanded to a pre-metric structure by taking a pre-metric expansion. In some cases, such as Banach spaces, there is a natural distance predicate that can be regarded as the analogue of equality. But in other cases, a general structure is easily described, but every pre-metric expansion is complicated. In general, there may not be a natural distance, but since the pre-metric expansion is unique up to uniform equivalence, there will always be a natural notion of uniform convergence.

In continuous model theory, one can either focus on general structures with a vocabulary, or focus on metric structure with a metric signature. When the goal is to apply the methods of continuous model theory to a family of structures with a natural metric, it is better to focus on metric structures as in [BBHU]. But when the goal is to discover potential new applications of the results in continuous model theory, it may be better to focus on general structures.

In Section 2 we lay the groundwork by developing some basic model-theory for general structures, and also give a brief review of pre-metric structures. In Section 3 we introduce pre-metric expansions and prove their existence, and then introduce the notion of an absolute property. In Section 4 we extend a variety of deeper properties and results from metric structures to general structures.

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2. Basic Model Theory for General Structures

We assume that the reader is familiar with the model theory of metric structures as developed in the paper [BBHU], and we will freely use notation from that paper. Our focus in this section will be on the definitions. Only brief comments will be given on the proofs, which will be similar to the proofs of the corresponding classical results that can be found, for example, in [CK12], as well as to proofs in [BBHU],

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[2]In [BU], Theorem 4.23 was used to show that their formalism of continuous model theory is equivalent to the formalism of open Hausdorff cats in [BY03a].

and in the much earlier monograph [CK66] that treated metric structures with the discrete metric.

2.1. **General Structures.** The space of truth values will be $[0, 1]$, with 0 representing truth. A *vocabulary* V consists of a set of predicate symbols P of finite arity, a set of function symbols F of finite arity, and a set of constant symbols c. A *general* ($[0, 1]$-valued) structure $\mathcal{M}$ consists of a vocabulary V, a non-empty universe set M, an element $c^{\mathcal{M}} \in M$ for each constant symbol c, a mapping $P^{\mathcal{M}} \colon M^n \to [0, 1]$ for each predicate symbol P of arity n, and a mapping $F^{\mathcal{M}} \colon M^n \to M$ for each function symbol F of arity n. A general structure determines a vocabulary, but does not determine a metric signature.

The *formulas* are as in [BBHU], with the connectives[3] being all continuous functions from finite powers of $[0, 1]$ into $[0, 1]$, and the quantifiers $\sup_x, \inf_x$. The *truth value* of a formula $\varphi(\vec{x})$ at a tuple $\vec{a} \in M^{|\vec{x}|}$ in a general structure $\mathcal{M}$ is an element of $[0, 1]$ denoted by $\varphi^{\mathcal{M}}(\vec{a})$. It is defined in the usual way by induction on the complexity of formulas. The syntax and semantics of general structures described here are the same as in [AH], and are the same as the *restricted continuous logic* in [Ca], except that there the value 1 denotes truth.

We deviate slightly from [BBHU] by defining a *theory* to be a set of sentences (rather than a set of statements of the form $\varphi = 0$). Similarly, we define an *n-type* over a set of parameters A to be a set of formulas with free variables in $\vec{x} = \langle x_1, \ldots, x_n \rangle$ and parameters from A. A *general model*[4] of a theory T is a general structure $\mathcal{M}$ such that $\varphi^{\mathcal{M}} = 0$ for each $\varphi \in T$. An n-tuple $\vec{b}$ in a general structure $\mathcal{M}$ *satisfies* a formula $\varphi(\vec{x})$ in $\mathcal{M}$ if $\varphi^{\mathcal{M}}(\vec{b}) = 0$, and *satisfies*, or *realizes*, an n-type p if $\vec{b}$ satisfies every formula $\varphi(\vec{x}) \in p$.

Hereafter, $\mathcal{M}, \mathcal{N}$ will denote general structures with universe sets M, N and vocabulary V, and S, T, U will denote sets of sentences (that is, theories) with the vocabulary V.

The notions of substructure (denoted by $\subseteq$), elementary equivalence (denoted by $\equiv$), elementary substructure and extension (denoted by $\prec$ and $\succ$), and elementary chain are as defined as in [BBHU], but applied to general structures as well as metric structures. $\mathcal{M} \models T$ means that $\mathcal{M}$ is a general model of T, $\mathcal{M} \models \varphi(\vec{b})$ means that $\varphi^{\mathcal{M}}(\vec{b}) = 0$, and $T \models U$ means that every general model of T is a general model of U. T and U are *equivalent* if they have the same general models. The complete theory $\mathrm{Th}(\mathcal{M})$ of $\mathcal{M}$ is the set of all sentences true in $\mathcal{M}$. Thus $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{M}')$ if and only if $\mathcal{M} \equiv \mathcal{M}'$. We state two elementary results.

Fact 2.1.1. *(Downward Löwenheim-Skolem) For every general structure $\mathcal{M}$, there is a general structure $\mathcal{M}' \prec \mathcal{M}$ such that $|M'| \leq \aleph_0 + |V|$.*

Fact 2.1.2. *(Elementary Chain Theorem) The union of an elementary chain of general structures $\langle \mathcal{M}_\alpha \colon \alpha < \beta \rangle$ is an elementary extension of each $\mathcal{M}_\alpha$.*

By an *embedding* $h \colon \mathcal{M} \to \mathcal{N}$ we mean a function $h \colon M \to N$ such that $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for each constant symbol $c \in V$, and for every n and $\vec{a} \in M^n$, $h(F^{\mathcal{M}}(\vec{a})) = F^{\mathcal{N}}(h(\vec{a}))$ for every function symbol $F \in V$ of arity n, and $P^{\mathcal{M}}(\vec{a}) = P^{\mathcal{N}}(h(\vec{a}))$ for every predicate symbol $P \in V$ of arity n. We say that $\mathcal{M}$ is *embeddable in*

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[3] we consider the elements of $[0, 1]$ to be 0-ary connectives, so each $r \in [0, 1]$ is a formula.

[4] To avoid confusion, we will always write "general model" instead of just "model", because in the literature on continuous model theory, "model" is used to mean what we call "metric model".

$\mathcal{N}$ if there is an embedding $h \colon \mathcal{M} \to \mathcal{N}$. Note that the image of an embedding $h \colon \mathcal{M} \to \mathcal{N}$ is a substructure of $\mathcal{N}$. An *elementary embedding* $h \colon \mathcal{M} \prec \mathcal{N}$ is an embedding that preserves the truth value of every formula. We say that $\mathcal{M}$ is *elementarily embeddable* in $\mathcal{N}$ if there exists an $h \colon \mathcal{M} \prec \mathcal{N}$.

In [BBHU], the reduction of a pre-metric structure was defined by identifying elements that are at distance zero from each other. Some care is needed to choose the right notion of reduction for general structures. In first order logic without equality, the reduction of a structure is formed by identifying two elements that cannot be distinguished by atomic formulas. We do the analogous thing here for general structures.

Definition 2.1.3. For $a, b \in M$, we write $a \doteq^{\mathcal{M}} b$ if for every atomic formula $\varphi(x, \vec{z})$ and tuple $\vec{c} \in M^{|\vec{z}|}$, $\varphi^{\mathcal{M}}(a, \vec{c}) = \varphi^{\mathcal{M}}(b, \vec{c})$. $\mathcal{M}$ is *reduced* if whenever $a \doteq^{\mathcal{M}} b$ we have $a = b$.

The relation $\doteq^{\mathcal{M}}$ is a very old idea that goes back to Leibniz around 1840, and is called *Leibniz equality*.

Remark 2.1.4. *For any general structure $\mathcal{M}$ and $a, b \in M$, $a \doteq^{\mathcal{M}} b$ if and only if in $\mathcal{M}$, (a, b) satisfies the set of formulas*

$$\{\sup_{\vec{z}} |\varphi(a, \vec{z}) - \varphi(b, \vec{z})| \colon \varphi \text{ is atomic}\}.$$

The *reduction map* for $\mathcal{M}$ is the mapping that sends each element of M to its equivalence class under $\doteq^{\mathcal{M}}$. The *reduction* of the general structure $\mathcal{M}$ is the reduced structure $\mathcal{N}$ such that N is the set of equivalence classes of elements of M under $\doteq^{\mathcal{M}}$, and the reduction map for $\mathcal{M}$ is an embedding of $\mathcal{M}$ onto $\mathcal{N}$. We say that $\mathcal{M}, \mathcal{M}'$ are *isomorphic*, in symbols $\mathcal{M} \cong \mathcal{M}'$, if there is an embedding from the reduction of $\mathcal{M}$ onto the reduction of $\mathcal{M}'$. We write $h \colon \mathcal{M} \cong \mathcal{N}$ if $h \colon \mathcal{M} \to \mathcal{N}$, and for each $b \in N$ there exists $a \in M$ such that $h(a) \doteq^{\mathcal{N}} b$.

Remark 2.1.5.
- $\cong$ *is an equivalence relation on general structures.*
- *Every general structure is isomorphic to its reduction.*
- $\mathcal{M}, \mathcal{N}$ *are isomorphic if and only their reductions are isomorphic.*
- *If there is an embedding of $\mathcal{M}$ onto $\mathcal{N}$, then $\mathcal{M} \cong \mathcal{N}$.*
- $\mathcal{M} \cong \mathcal{N}$ *implies* $\mathcal{M} \equiv \mathcal{N}$.
- $\mathcal{M} \cong \mathcal{N}$ *if and only there exists h such that $h \colon \mathcal{M} \cong \mathcal{N}$.*

If $V^0 \subseteq V$, and $\mathcal{M}^0$ is obtained from $\mathcal{M}$ by forgetting every symbol of $V \setminus V^0$, we call $\mathcal{M}$ an *expansion* of $\mathcal{M}^0$ to V, and call $\mathcal{M}^0$ *the V^0-part* of $\mathcal{M}$.

Remark 2.1.6. *Suppose $V^0 \subseteq V$, and $\mathcal{M}^0$ is the V^0-part of $\mathcal{M}$. Then for every formula $\varphi(\vec{x})$ in the vocabulary V^0, and every tuple $\vec{b} \in M^{|\vec{x}|}$, we have $\varphi^{\mathcal{M}}(\vec{b}) = \varphi^{\mathcal{M}^0}(\vec{b})$.*

Remark 2.1.7. *If $\mathcal{M}^0$ is reduced and $\mathcal{M}$ is an expansion of $\mathcal{M}^0$, then $\mathcal{M}$ is reduced.*

Proof. For all $a, b \in M$, $a \doteq^{\mathcal{M}} b$ implies $a \doteq^{\mathcal{M}^0} b$. $\qquad\square$

2.2. **Ultraproducts.** The ultraproduct of an indexed family of general structures will be defined below as the reduction of the pre-ultraproduct, which is a general structure whose universe is the cartesian product. As we will see later, this will be a

direct generalization of the ultraproduct of metric structures as defined in [BBHU]. Recall that for any ultrafilter $\mathcal{D}$ over a set I and function $g\colon I \to [0,1]$, there is a unique value $r = \lim_{\mathcal{D}} g$ in $[0,1]$ such that for each neighborhood Y of r, the set of $i \in I$ such that $g(i) \in Y$ belongs to $\mathcal{D}$.

Definition 2.2.1. Let $\mathcal{D}$ be an ultrafilter over a set I and $\mathcal{M}_i$ be a general structure for each $i \in I$. The *pre-ultraproduct* $\prod^{\mathcal{D}} \mathcal{M}_i$ is the general structure $\mathcal{M}' = \prod^{\mathcal{D}} \mathcal{M}_i$ such that:

- $M' = \prod_{i \in I} M_i$, the cartesian product.
- For each constant symbol $c \in V$, $c^{\mathcal{M}'} = \langle c^{\mathcal{M}_i} \rangle_{i \in I}$.
- For each n-ary function symbol $G \in V$ and n-tuple $\vec{a}$ in M',
$$G^{\mathcal{M}'}(\vec{a}) = \langle G^{\mathcal{M}_i}(\vec{a}(i)) \rangle_{i \in I}.$$
- For each n-ary predicate symbol $P \in V$ and n-tuple $\vec{a}$ in M',
$$P^{\mathcal{M}'}(\vec{a}) = \lim_{\mathcal{D}} \langle P^{\mathcal{M}_i}(\vec{a}(i)) \rangle_{i \in I}.$$

The *ultraproduct* $\prod_{\mathcal{D}} \mathcal{M}_i$ is the reduction of the pre-ultraproduct $\prod^{\mathcal{D}} \mathcal{M}_i$. For each $a \in M'$ we also let $a_{\mathcal{D}}$ denote the equivalence class of a under $\doteq^{\mathcal{M}'}$.

The following fact is the analogue for general structures of the fundamental theorem of Łoś.

Fact 2.2.2. *Let $\mathcal{M}_i$ be a general structure for each $i \in I$, let $\mathcal{D}$ be an ultrafilter over I, and let $\mathcal{M} = \prod_{\mathcal{D}} \mathcal{M}_i$ be the ultraproduct. Then for each formula φ and tuple $\vec{b}$ in the cartesian product $\prod_{i \in I} M_i$,*
$$\varphi^{\mathcal{M}}(\vec{b}_{\mathcal{D}}) = \lim_{\mathcal{D}} \langle \varphi^{\mathcal{M}_i}(\vec{b}_i) \rangle_{i \in I}.$$

The special case of Fact 2.2.2 where each $\mathcal{M}_i$ has a symbol $=_i$ for the discrete metric was already stated and proved in [CK66]. The general case can be obtained from that special case by observing that if each $\mathcal{M}_i$ has vocabulary V and $\mathcal{N}_i = (\mathcal{M}_i, =_i)$, then $\prod_{\mathcal{D}} \mathcal{M}_i$ is the reduction of the V-part of $\prod_{\mathcal{D}} \mathcal{N}_i$. Alternatively, Fact 2.2.2 can be proved from scratch by induction on the complexity of formulas, as is done in first order logic.

If $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, the ultraproduct $\prod_{\mathcal{D}} \mathcal{M}_i$ is called the *ultrapower* of $\mathcal{M}$ modulo $\mathcal{D}$, and is denoted by $\mathcal{M}^I/\mathcal{D}$.

Corollary 2.2.3. *For each $\mathcal{M}$ and ultrafilter $\mathcal{D}$, $\mathcal{M}$ is elementarily embeddable into $\mathcal{M}^I/\mathcal{D}$.*

Corollary 2.2.4. *Suppose $\mathcal{M}_i \cong \mathcal{N}_i$ for each $i \in I$, and $\mathcal{D}$ is an ultrafilter over I. Then $\prod_{\mathcal{D}} \mathcal{M}_i \cong \prod_{\mathcal{D}} \mathcal{N}_i$.*

Proof. By Remark 2.1.5, for each $i \in I$ there is a map $h_i\colon \mathcal{M}_i \cong \mathcal{N}_i$. Let $h\colon \prod_{i \in I} M_i \to \prod_{i \in I} N_i$ be the mapping such that for $a \in \prod_{i \in I} M_i$, $h(a) = \langle h_i(a_i) \rangle_{i \in I}$. For each continuous formula $\varphi(\vec{v})$ and tuple $\vec{a} \in (\prod_{i \in I} M_i)^{|\vec{v}|}$, it follows from Fact 2.2.2 that
$$\varphi^{\prod^{\mathcal{D}} \mathcal{M}_i}(\vec{a}) = \lim_{\mathcal{D}}(\varphi^{\mathcal{M}_i}(\vec{a}_i)) = \lim_{\mathcal{D}}(\varphi^{\mathcal{N}_i}(h_i(\vec{a}_i))) = \varphi^{\prod^{\mathcal{D}} \mathcal{N}_i}(h(\vec{a})).$$

Therefore $h\colon \prod^{\mathcal{D}} \mathcal{M}_i \to \prod^{\mathcal{D}} \mathcal{N}_i$.

By Fact 2.2.2, if $b, c \in \prod_{i \in I} N_i$ and $b_i \doteq^{\mathcal{N}_i} c_i$ for each $i \in I$, then $b \doteq^{\prod^{\mathcal{D}} \mathcal{N}_i} c$. It follows that $h\colon \prod^{\mathcal{D}} \mathcal{M}_i \cong \prod^{\mathcal{D}} \mathcal{N}_i$, so by Remark 2.1.5, $\prod_{\mathcal{D}} \mathcal{M}_i \cong \prod_{\mathcal{D}} \mathcal{N}_i$. $\square$

The following is proved in the usual way, using Fact 2.2.2.

Fact 2.2.5. *(Compactness) If every finite subset of T has a general model, then T has a general model.*

2.3. **Definability and Types.** We write

- $r \dotplus s$ for $\min(r + s, 1)$,
- $r \dotminus s$ for $\max(r - s, 0)$,
- $r \dotleq s \dotleq t$ for $\max(r \dotminus s, s \dotminus t)$.

Thus $r \leq s$ if and only if $r \dotminus s = 0$. In the literature, $r \dotminus s$ is sometimes written $r \dot{-} s$. Note that for any general structure $\mathcal{M}$, the following are equivalent:

- $\mathcal{M} \models \sup_{\vec{x}}[\varphi(\vec{x}) \dotleq \psi(\vec{x}) \dotleq \theta(\vec{x})]$.
- $(\forall \vec{a} \in M^{|\vec{x}|})[\varphi^{\mathcal{M}}(\vec{a}) \leq \psi^{\mathcal{M}}(\vec{a})$ and $\psi^{\mathcal{M}}(\vec{a}) \leq \theta^{\mathcal{M}}(\vec{a})]$.

In what follows, all formulas mentioned are understood to be in the vocabulary of a theory T. Let $\vec{x}$ be a tuple of variables, and $\vec{y}$ be a finite or infinite sequence of variables, where all the symbols x_i and y_j are distinct. Given a formula $\theta(\vec{x}, \vec{y})$, we let $\sup_{\vec{y}} \theta(\vec{x}, \vec{y})$ denote the formula $\sup_{\vec{u}} \theta(\vec{x}, \vec{y})$ where $\vec{u}$ is the (necessarily finite) tuple of variables from $\vec{y}$ that occur freely in θ.

We say that a sequence $\langle \varphi_m(\vec{x}, \vec{y}) \rangle_{m \in \mathbb{N}}$ of formulas is *Cauchy* in T if for each $\varepsilon > 0$ there exists m such that for all $k \geq m$,

$$T \models \sup_{\vec{x}} \sup_{\vec{y}} |\varphi_m(\vec{x}, \vec{y}) - \varphi_k(\vec{x}, \vec{y})| \dotleq \varepsilon.$$

Cauchy in $\mathcal{M}$ means Cauchy in the complete theory $\mathrm{Th}(\mathcal{M})$.

If $\langle \varphi_m(\vec{x}, \vec{y}) \rangle_{m \in \mathbb{N}}$ is Cauchy in T, then for each general model $\mathcal{M}$ of T there is a unique mapping from $M^{|\vec{x}|} \times M^{|\vec{y}|}$ into $[0, 1]$, denoted by $[\lim \varphi_m]^{\mathcal{M}}$, such that

$$(\forall \vec{b} \in M^{|\vec{x}|})(\forall \vec{c} \in M^{|\vec{y}|})[\lim \varphi_m]^{\mathcal{M}}(\vec{b}, \vec{c}) = \lim_{m \to \infty} \varphi_m^{\mathcal{M}}(\vec{b}, \vec{c}).$$

We say that $\langle \varphi_m(\vec{x}, \vec{y}) \rangle_{m \in \mathbb{N}}$ is *exponentially Cauchy* in T if whenever $m \leq k$ we have

$$T \models \sup_{\vec{x}} \sup_{\vec{y}} |\varphi_m(\vec{x}, \vec{y}) - \varphi_k(\vec{x}, \vec{y})| \dotleq 2^{-m}.$$

Note that every exponentially Cauchy sequence of formulas in T is Cauchy, and every Cauchy sequence in T has an exponentially Cauchy subsequence.

Definition 2.3.1. We say that a mapping $\mathsf{P} \colon M^{|\vec{x}|} \times M^{|\vec{y}|} \to [0, 1]$ is *defined* by $\langle \varphi_m(\vec{x}, \vec{y}) \rangle_{m \in \mathbb{N}}$ in a general structure $\mathcal{M}$, and is *definable in* $\mathcal{M}$, if $\langle \varphi_m(\vec{x}, \vec{y}) \rangle_{m \in \mathbb{N}}$ is Cauchy in $\mathcal{M}$ and $\mathsf{P} = [\lim \varphi_m]^{\mathcal{M}}$.

Note that for each general structure $\mathcal{M}$,

$$\delta_{\mathcal{M}}(\varphi, \psi) := \sup_{\vec{x}} \sup_{\vec{y}} |\varphi(\vec{x}, \vec{y}) - \psi(\vec{x}, \vec{y})|^{\mathcal{M}}$$

is a pseudo-metric on the set of all formulas with free variables from $(\vec{x}, \vec{y})$, and the above definition says that $\langle \varphi_m(\vec{x}, \vec{y}) \rangle_{m \in \mathbb{N}}$ is Cauchy in T if and only if it is Cauchy with respect to $\delta_{\mathcal{M}}$ uniformly for all $\mathcal{M} \models T$.

We often consider the case where $\vec{y}$ is empty, or equivalently, where only finitely many of the variables in $\vec{y}$ actually occur in some $\varphi_m(\vec{x}, \vec{y})$. In that case, we have the notion of a Cauchy sequence of formulas $\langle \varphi_m(\vec{x}) \rangle_{m \in \mathbb{N}}$ and a definable mapping $\mathsf{P} \colon M^{|\vec{x}|} \to [0, 1]$ in $\mathcal{M}$.

We now introduce complete n-types. For each theory T and $n \in \mathbb{N}$, a *complete n-type over* T is an n-type $p(\vec{x})$ that is maximal with respect to being satisfiable in a general model of T. $S_n(T)$ denotes the set of all complete n-types over T. In particular, $S_0(T)$ is the set of all complete extensions of T. For each $p \in S_n(T)$ and formula $\varphi(\vec{x})$ with n free variables, we let $\varphi(\vec{x})^p$ be the unique r such that $|\varphi(\vec{x}) - r| \in p$. The *logic topology* on $S_n(T)$ is the topology whose closed sets are the sets of the form $\{p \in S_n(T) \colon \Gamma(\vec{x}) \subseteq p\}$ for some n-type $\Gamma(\vec{x})$. It follows from the Compactness Theorem that:

Fact 2.3.2. *For each theory T and $n \in \mathbb{N}$, the logic topology on $S_n(T)$ is compact.*

Given a general structure $\mathcal{M}$ and a set $A \subseteq M$, let $\mathcal{M}_A = (\mathcal{M}, a)_{a \in A}$. The *complete type* of an n-tuple $\vec{b}$ over A in $\mathcal{M}$ is the set $\mathrm{tp}_{\mathcal{M}}(\vec{b}/A)$ of all formulas satisfied by $\vec{b}$ in $\mathcal{M}_A$. The set $S_n(\mathrm{Th}(\mathcal{M}_A))$ of all complete n-types over A realized in models of $\mathrm{Th}(\mathcal{M}_A)$ is denoted by $S_n^{\mathcal{M}}(A)$. Thus the logic topology on $S_n^{\mathcal{M}}(A)$ is compact.

We say that a mapping $\mathsf{P}(\vec{x}, \vec{y})$ is *definable over* A in $\mathcal{M}$ if P is definable in $\mathcal{M}_A$.

The following lemma gives a relationship between $\mathsf{P}(\vec{x})$ being definable over a countable sequence of parameters, and $\mathsf{P}(\vec{x}, \vec{y})$ being definable without parameters.

Lemma 2.3.3. *Let $B \subseteq M$. A mapping $\mathsf{P}(\vec{x})$ is definable over B in $\mathcal{M}$ if and only if there is a countable sequence $\vec{b}$ of elements of B and an exponentially Cauchy sequence $\langle \varphi(\vec{x}, \vec{y}) \rangle$ of formulas in $\mathcal{M}$ such that for all $\vec{a} \in M^{|\vec{x}|}$ we have $\mathsf{P}(\vec{a}) = \mathsf{Q}(\vec{a}, \vec{b})$ where Q is the mapping defined by $\langle \varphi(\vec{x}, \vec{y}) \rangle$ in $\mathcal{M}$.*

Proof. It is clear that if there is such a sequence of formulas, then $\mathsf{P}(\vec{x})$ is definable over B in $\mathcal{M}$. Suppose $\mathsf{P}(\vec{x})$ is definable over B in $\mathcal{M}$, by a sequence $\langle \theta_m(\vec{x}) \rangle_{m \in \mathbb{N}}$ with parameters in B. By taking a subsequence if necessary, we may insure that for some sequence $\vec{b}$ of elements of B, each θ_m has parameters from $\vec{b}$, and for each m we have

$$(1) \qquad \mathcal{M} \models \sup_{\vec{x}} |\theta_m(\vec{x}, \vec{b}) - \theta_{m+1}(\vec{x}, \vec{b})| \leq 2^{-(m+1)}.$$

We now use the forced convergence trick in [BU] to find a new sequence of formulas $\varphi_m(\vec{x}, \vec{y})$ in such a way that

$$(2) \qquad \mathcal{M} \models \sup_{\vec{x}} \sup_{\vec{y}} |\varphi_m(\vec{x}, \vec{y}) - \varphi_{m+1}(\vec{x}, \vec{y})| \leq 2^{-(m+1)},$$

and that $\varphi_m^{\mathcal{M}}(\vec{a}, \vec{b}) = \theta_m^{\mathcal{M}}(\vec{a}, \vec{b})$ whenever $\vec{a} \in M^{|\vec{x}|}$ and $m \in \mathbb{N}$. To do that we inductively define $\varphi_0 = \theta_0$, and

$$\varphi_{m+1} = \max(\varphi_m - 2^{-(m+1)}, \min(\varphi_m + 2^{-(m+1)}, \theta_{m+1})).$$

Condition (2) implies that $\varphi_m(\vec{x}, \vec{y})$ is exponentially Cauchy in $\mathcal{M}$, and defines a mapping Q in $\mathcal{M}$ such that $\mathsf{P}(\vec{a}) = \mathsf{Q}(\vec{a}, \vec{b})$ for all $\vec{a} \in M^{|\vec{x}|}$. $\qquad \square$

The following result is the analogue for general structures of Theorem 9.9 of [BBHU].

Proposition 2.3.4. *Let $\mathcal{M}$ be a general structure, $A \subseteq M$, and $\mathsf{P} \colon M^n \to [0, 1]$. The following are equivalent:*

(i) P *is a definable predicate over A in $\mathcal{M}$.*

(ii) *There is a unique function* $\Psi \colon S_n^{\mathcal{M}}(A) \to [0,1]$ *such that* $\mathsf{P}(\vec{c}) = \Psi(\mathrm{tp}_{\mathcal{M}}(\vec{c}/A))$ *for all* $\vec{c} \in M^n$. *Furthermore,* Ψ *is continuous in the logic topology on* $S_n^{\mathcal{M}}(A)$

Proof. The proof that (ii) $\Rightarrow$ (i) is the same as the proof of Theorem 9.9 in [BBHU], which did not use a metric on $\mathcal{M}$.

(i) $\Rightarrow$ (ii): Suppose that P is defined by $\langle \varphi_m \rangle_{m \in \mathbb{N}}$ over A in $\mathcal{M}$. Then $\langle \varphi_m \rangle_{m \in \mathbb{N}}$ is Cauchy in $\mathrm{Th}(\mathcal{M}_A)$. There is a unique $\Psi \colon S_n^{\mathcal{M}}(A) \to [0,1]$ such that $\Psi(\mathrm{tp}_{\mathcal{M}}(\vec{c}/A)) = [\lim \varphi_m]^{\mathcal{M}_A}(\vec{c}) = \mathsf{P}(\vec{c})$ for each $\vec{c} \in M^n$. For each closed interval $I \subseteq [0,1]$, there is a n-type $\Gamma_I(\vec{x})$ with parameters in A such that for all $\vec{c} \in M^n$, $\Psi(\mathrm{tp}_{\mathcal{M}}(\vec{c}/A)) \in I$ if and only if $\mathcal{M}_A \models \Gamma_I(\vec{c})$. Therefore $\Psi^{-1}(I)$ is closed in $S_n^{\mathcal{M}}(A)$, so Ψ is continuous. $\square$

Remark 2.3.5. *Let* $A \subseteq M$ *and* $\vec{b}$ *in* M^n.
- *If* $\mathcal{N} \succ \mathcal{M}$ *then* $\mathrm{tp}_{\mathcal{M}}(\vec{b}/A) = \mathrm{tp}_{\mathcal{N}}(\vec{b}/A)$.
- $\varphi^{\mathcal{M}}(\vec{b}, A) = r$ *if and only if* $|\varphi(\vec{x}, A) - r|$ *belongs to* $\mathrm{tp}_{\mathcal{M}}(\vec{b}/A)$.

Using the compactness theorem, we have
- *If* $\mathcal{N} \succ \mathcal{M}$ *then* $S_n^{\mathcal{M}}(A) = S_n^{\mathcal{N}}(A)$.
- *There exists* $\mathcal{N} \succ \mathcal{M}$ *in which every type in* $\bigcup_n S_n^{\mathcal{M}}(A)$ *is realized.*

2.4. **Saturated and Special Structures.** For an infinite cardinal κ, we say that a general structure $\mathcal{M}$ is κ-*saturated* if for every set $A \subseteq M$ of cardinality $|A| < \kappa$, every set of formulas in the vocabulary of $\mathcal{M}$ with one free variable and parameters from A that is finitely satisfiable in $\mathcal{M}_A$ is satisfiable in $\mathcal{M}_A$.

Remark 2.4.1. *If* $\mathcal{M}$ *is* κ-*saturated,* $A \subseteq M$, *and* $|A| < \kappa$, *then every complete type* $p \in S_n(A)$ *is realized in* $\mathcal{M}$.

Remark 2.4.2. $\mathcal{M}$ *is* κ-*saturated if and only if the reduction of* $\mathcal{M}$ *is* κ-*saturated.*

Definition 2.4.3. By a *special cardinal* we mean a cardinal κ such that $2^{\lambda} \leq \kappa$ for all $\lambda < \kappa$. We say that $\mathcal{M}$ is *special* if $|M|$ is an uncountable special cardinal and $\mathcal{M}$ is the union of an elementary chain of structures $\langle \mathcal{M}_{\lambda} \colon \lambda < |M| \rangle$ such that each $\mathcal{M}_{\lambda}$ is λ^+-saturated. $\mathcal{M}$ is κ-*special* if κ is special and $\mathcal{M}$ is the reduction of a special structure of cardinality κ.

Note that every strong limit cardinal is special, and if $2^{\lambda} = \lambda^+$ then 2^{λ} is special. Thus every κ-special structure is reduced and has cardinality $\leq \kappa$. A κ-special structure may have cardinality less than κ. For example, every metric structure with a compact metric is κ-special for every special cardinal κ but has cardinality at most $2^{\aleph_0}$.

Remark 2.4.4. *If* $\mathcal{M}$ *is* κ-*special and* $V^0 \subseteq V$, *then the reduction of the* V^0-*part of* $\mathcal{M}$ *is* κ-*special.*

Remark 2.4.5. *If* κ *is an uncountable inaccessible cardinal, then* $\mathcal{M}$ *is* κ-*special if and only if* $\mathcal{M}$ *is a reduced* κ-*saturated structure of cardinality* κ.

Fact 2.4.6. *(Uniqueness Theorem for Special Models) If* T *is complete and* $\mathcal{M}, \mathcal{N}$ *are* κ-*special models of* T, *then* $\mathcal{M}$ *and* $\mathcal{N}$ *are isomorphic.*

An easy consequence is:

Fact 2.4.7. *Suppose* $\mathcal{M}$ *is* κ-*special,* $A \subseteq M$, *and* $|A| < $ *cofinality of* κ. *Then for all tuples* $\vec{b}, \vec{c}$ *in* M, $\mathrm{tp}_{\mathcal{M}}(\vec{b}/A) = \mathrm{tp}_{\mathcal{M}}(\vec{c}/A)$ *if and only if* $\mathcal{M}$ *has an automorphism that sends* $\vec{b}$ *to* $\vec{c}$ *and is the identity on* A.

The following result is proved using the Compactness, Downward Löwenheim-Skolem, and Elementary Chain Theorems.

Fact 2.4.8. *(Existence Theorem for Special Models) If κ is a special cardinal and $\aleph_0 + |V| < \kappa$, then every reduced structure $\mathcal{M}$ such that $|M| \leq \kappa$ has a κ-special elementary extension.*

The statement of Fact 2.4.8 above differs slightly from the corresponding result in [CK66], because we do not have a symbol for the equality relation here.

2.5. **Pre-metric Structures.** A *metric signature* L over V specifies a distinguished binary predicate symbol $d \in V$ for distance, and equips each predicate or function symbol $S \in V$ with a modulus of uniform continuity $\triangle_S \colon (0,1] \to (0,1]$ with respect to d.

In the literature on metric structures, one usually fixes a metric signature L once and for all, but here we focus on general structures that are not equipped with metric signatures, and often consider many metric signatures at the same time. For that reason, we will officially define a pre-metric structure to be pair consisting of a general structure and a metric signature.

A *pre-metric structure* $\mathcal{M}_+ = (\mathcal{M}, L)$ consists of a general structure $\mathcal{M}$ with vocabulary V, and a metric signature L over V, such that $d^{\mathcal{M}}$ is a pseudo-metric on M, and for each predicate symbol P and function symbol F of arity n, and for all $\vec{a}, \vec{b} \in M^n$ and $\varepsilon \in (0,1]$,

$$\max_{k \leq n} d^{\mathcal{M}}(a_k, b_k) < \triangle_P(\varepsilon) \Rightarrow |P^{\mathcal{M}}(\vec{a}) - P^{\mathcal{M}}(\vec{b})| \leq \varepsilon.$$

and

$$\max_{k \leq n} d^{\mathcal{M}}(a_k, b_k) < \triangle_F(\varepsilon) \Rightarrow d^{\mathcal{M}}(F^{\mathcal{M}}(\vec{a}), F^{\mathcal{M}}(\vec{b})) \leq \varepsilon.$$

A *metric structure* is a pre-metric structure $(\mathcal{M}, L)$ such that $(M, d^{\mathcal{M}})$ is a complete metric space. The paper [BBHU] emphasized metric structures, but in this paper we will focus more on pre-metric than metric structures.

Given a pre-metric structure $\mathcal{M}_+ = (\mathcal{M}, L)$, we will call $\mathcal{M}$ the *downgrade of* $\mathcal{M}_+$, and call $\mathcal{M}_+$ the *upgrade of* $\mathcal{M}$ to L. Note that two different pre-metric structures can have the same downgrade, because they may have different metric signatures.

We will slightly abuse notation and say that $\mathcal{M}$ *is a pre-metric (or metric) structure for L* when $(\mathcal{M}, L)$ is a pre-metric (or metric) structure. We say that a pre-metric structure $\mathcal{M}_+$ for L is κ-saturated if its downgrade is κ-saturated. Similarly for other properties of general structures, such as being reduced, or being an ultraproduct of a family of structures.

We remind the reader that every pre-metric structure for L has a unique completion up to isomorphism, that this completion is a metric structure for L, and that every pre-metric structure is elementarily embeddable in its completion.

Remark 2.5.1. *Every pre-metric structure for L that is reduced and $\aleph_1$-saturated is a metric structure for L.*

In [BBHU], when $\mathcal{M}_+$ is a metric structure, a mapping from M^n into $[0,1]$ that is uniformly continuous with respect to $d^{\mathcal{M}}$ is called a *predicate on $\mathcal{M}_+$*.

We will make frequent use of the following result from [BBHU].

Fact 2.5.2. *(Theorem 3.5 in [BBHU]) For every metric signature L and formula $\varphi(\vec{x})$ in the vocabulary of L, there is a function $\triangle_\varphi \colon (0,1] \to (0,1]$ that is a modulus of uniform continuity for the mapping $\varphi^{\mathcal{M}} \colon M^{|\vec{x}|} \to [0,1]$ in every pre-metric structure $\mathcal{M}_+$ for L.*

Fact 2.5.3. *(See [Ca], page 112, and Definition 2.4 in [AH].) For each metric signature L over V, there is a theory $\mathrm{met}(L)$ whose general models are exactly the downgrades of pre-metric structures for L. Each sentence in $\mathrm{met}(L)$ consists of finitely many $\sup$ quantifiers followed by a quantifier-free formula.*

The sentences in $\mathrm{met}(L)$ formally express that d is a pseudo-metric, and that the functions and predicates in V respect the moduli of uniform continuity for L.

Corollary 2.5.4. *Every ultraproduct of pre-metric structures for L is a pre-metric structure for L.*

Proof. By Facts 2.2.2 and 2.5.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A *metric theory* (T, L) consists of a metric signature L and a set T of sentences in the vocabulary of L such that $T \models \mathrm{met}(L)$. A pre-metric (or metric) *model* of a metric theory (T, L) is a pre-metric (or metric) structure $\mathcal{M}_+ = (\mathcal{M}, L)$ for L such that $\varphi^{\mathcal{M}} = 0$ for each sentence $\varphi \in T$.

When (T, L) is a metric theory, we also say that *T is a metric theory with signature L*. Note that for every metric theory (T, L), pre-metric structure $(\mathcal{M}, L)$, and continuous sentence φ, we have $T \models \mathcal{M}$ iff $(T, L) \models (\mathcal{M}, L)$, and $T \models \varphi$ iff $(T, L) \models \varphi$. Also, if $(\mathcal{M}, L)$ is a pre-metric model of a metric theory (T, L), then $\mathcal{M}$ is a general model of T.

Part (i) of the next lemma shows that when $\mathcal{M}_+$ is a metric structure, the definition of a definable predicate here agrees with the notion of a definable predicate in [BBHU].

Lemma 2.5.5.

    (i) *In every pre-metric structure $\mathcal{M}_+$ for L, every mapping that is definable in the sense of Definition 2.3.1 is uniformly continuous.*

    (ii) *For every metric theory T with signature L, and sequence of formulas $\langle \varphi_m(\vec{x}) \rangle_{m \in \mathbb{N}}$ that is Cauchy in T, there is a function $\triangle_{\langle \varphi \rangle}$ that is a modulus of uniform continuity for the definable predicate $[\lim \varphi_m]^{\mathcal{M}}$ in every pre-metric model $\mathcal{M}_+$ of T.*

Proof. (ii) trivially implies (i). We prove (ii). Given $\varepsilon > 0$, take the least m so that for all $n \geq m$,

$$T \models \sup_{\vec{x}} |\varphi_m(\vec{x}) - \varphi_n(\vec{x})| \leq \varepsilon/3.$$

Let $\triangle_{\langle \varphi \rangle}(\varepsilon) = \triangle_{\varphi_m}(\varepsilon/3)$. Suppose $\mathcal{M}_+$ is a pre-metric model of T, and $\max_{k \leq n} d^{\mathcal{M}}(a_k, b_k) < \triangle_{\langle \varphi \rangle}(\varepsilon)$. Then

$$|[\lim \varphi_m]^{\mathcal{M}}(\vec{a}) - [\lim \varphi_m]^{\mathcal{M}}(\vec{b})| \leq$$

$$|[\lim \varphi_m]^{\mathcal{M}}(\vec{a}) - \varphi_m^{\mathcal{M}}(\vec{a})| + |\varphi_m^{\mathcal{M}}(\vec{a}) - \varphi_m^{\mathcal{M}}(\vec{b})| + |\varphi_m^{\mathcal{M}}(\vec{b}) - [\lim \varphi_m]^{\mathcal{M}}(\vec{b})| \leq$$

$$\varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

$$\square$$

The next fact follows at once from Theorem 3.7 in [BBHU].

Fact 2.5.6. *If $\mathcal{M}_+$ is a pre-metric structure, $a \doteq^{\mathcal{M}} b$ if and only if $d^{\mathcal{M}}(a,b) = 0$.*

As mentioned above, in [BBHU], the notion of a reduction of a pre-metric structure $\mathcal{M}_+$ was defined as the structure obtained by identifying elements x, y when $d(x,y) = 0$, rather than when $x \doteq y$. So the reduction of a pre-metric structure as defined here is the same as the reduction of a pre-metric structure as defined in [BBHU]. It follows that the ultraproduct of a family of metric structures with signature L as defined here is the same as the ultraproduct of a family of metric structures with signature L as defined in [BBHU].

Fact 2.5.7. *(By Proposition 5.3 in [BBHU].) Every ultraproduct of metric structures for L is a metric structure for L.*

Corollary 2.5.8. *(Metric Compactness Theorem) If T is a metric theory with signature L, and every finite subset of T has a metric model, then T has a metric model.*

In [BBHU], the definitions of a pre-metric structure and of an ultraproduct were designed to insure that Fact 2.5.8 and the Metric Compactness Theorem hold. There were pitfalls to avoid in extending the notion of an ultraproduct to general structures. For example, if one adds a discontinuous predicate or function to a pre-metric structure and tries to define an ultraproduct by identifying elements at distance 0 from each other, the added predicate or function symbol would be undefined in the ultraproduct.

Such pitfalls were avoided here by defining the reduction of a general structure to be the general structure formed by identifying x with y when $x \doteq^{\mathcal{M}} y$. Using that notion of reduction, we obtained a well-defined notion of ultraproduct that coincides with the notion in [BBHU] for metric structures, satisfies the theorem of Łoś (Fact 2.2.2) in all cases, and leads to the Compactness Theorem for general structures.

2.6. **Some Variants of Continuous Model Theory.** In this subsection we discuss some cases from the literature where variants of continuous model theory have been developed in order to study special classes of general structures that share some features with pre-metric structures. In each of these cases, one can instead work with the model theory of general structures as developed here.

Example 2.6.1. (Infinitary Continuous Logic) Christopher Eagle, in [Ea14] and [Ea15], developed an infinitary logic $\mathcal{L}_{\omega_1\omega}$ that is analogous to continuous logic, but has much greater power of expression and fails to satisfy the compactness theorem. The $\mathcal{L}_{\omega_1\omega}$-formulas in a vocabulary V are built from the atomic formulas using the connectives and quantifiers of continuous logic and, in addition, the operations $\sup_m \varphi_m$ and $\inf_m \varphi_m$ whenever $\langle \varphi_m \rangle_{m \in \mathbb{M}}$ is a sequence of formulas in which only finitely many variables occur freely. Given an $\mathcal{L}_{\omega_1\omega}$-formula $\varphi(\vec{x})$ and a pre-metric structure $\mathcal{M}$, the truth value function $\varphi^{\mathcal{M}} \colon M^{|\vec{x}|} \to [0,1]$ is defined in [Ea15] as one would expect by induction on complexity.

We adopt here the same definition for the truth value $\varphi^{\mathcal{M}}$ of an $\mathcal{L}_{\omega_1\omega}$-formula in a general structure $\mathcal{M}$. Then $(\mathcal{M}, \varphi^{\mathcal{M}})$ is a general structure whose vocabulary has an extra $|\vec{x}|$-ary predicate symbol. However, even if $\mathcal{M}$ is a metric structure, $(\mathcal{M}, \varphi^{\mathcal{M}})$ is not necessarily a metric structure, and $\varphi^{\mathcal{M}}$ may even be discontinuous. We will revisit this concept in Subsection 4.4.

Example 2.6.2. (Geodesic Logic) The paper [Cho17] introduced a variant of continuous model theory, called *geodesic logic*, whose structures are pre-metric structures with extra functions that are possibly discontinuous. He used this to obtain approximate fixed point results and metastability results for certain discontinuous functions.

A *geodesic signature* G consists of a metric signature H over a vocabulary V, and a set Y of extra symbols for possibly discontinuous functions, including a set $\{L_t \colon t \in [0,1]\}$ of binary function symbols (the linear structure). A *geodesic structure* with signature G is a general structure in our present sense, that has a vocabulary $V \cup Y$ whose V-part is a pre-metric structure with signature H, that satisfies the following sentences for each $s, t \in [0,1]$:

$$(3) \qquad \sup_x \sup_y [|d(L_s(x,y), L_t(x,y)) - |s - t|d(x,y)| \leq 0],$$

$$(4) \qquad \sup_x \sup_y [d(L_t(x,y), L_{1-t}(y,c)) \leq 0].$$

Thus a geodesic structure is just a general model of the theory

$$T_G = \operatorname{met} H \cup \{(3),(4)\}.$$

For example, in our setting, Theorem 6.6 (a) of [Cho17] can be stated as follows:

Let $\lambda \in [0,1]$ and let G be a geodesic signature with distance predicate d, and an extra unary function symbol F. Let $\mathcal{M}$ be a general model of the theory

$$(5) \qquad T_G \cup \{\sup_x [d(Fx, FL_\lambda(x, Fx)) \stackrel{.}{\leq} d(x, Fx)]\}.$$

In $\mathcal{M}$, when $x_{n+1} = L_\lambda(x_n, Fx_n)$ for all n, we have $\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$.

The compactness theorem can then be used to get a uniform metastability bound across all general models of (5) (Theorem 6.6 (b) of [Cho17]).

Example 2.6.3. (Sorted Vocabularies) Many-sorted metric structures are prominent in the literature. Here we will work exclusively with general structures, as defined at the beginning of this section, but introduce the notions of a sorted vocabulary, and of a general structure that respects the sorts in that vocabulary. A general structure that respects sorts cannot be a pre-metric structure. We have the flexibility of starting with a general structure that respects sorts, and adding a new unsorted distance predicate to form a pre-metric structure that does not respect sorts. As we will see later, that flexibility will be useful when we consider such topics as unbounded metrics and imaginary elements.

A *sorted vocabulary* W consists of a set of sorts, sets of finitary predicate and function symbols, and a set of constant symbols. Each argument of a predicate or function symbol is equipped with a sort, and each function and constant symbol is equipped with a sort for its value. Moreover, W contains a unary predicate symbol $U_\mathbb{S}$ associated with each sort $\mathbb{S}$, and a constant symbol u (for "unsorted") that has no sort.

We say that a general structure $\mathcal{M}$ with vocabulary W *respects sorts* if:

- The universe M of $\mathcal{M}$ contains a family of pairwise disjoint non-empty universe sets $\mathbb{S}^{\mathcal{M}}$ corresponding to the sorts $\mathbb{S}$ of W.
- For each sort $\mathbb{S}$ of W and $a \in M$, $U_\mathbb{S}^{\mathcal{M}}(a) = 0$ when $a \in \mathbb{S}^{\mathcal{M}}$, and $U_\mathbb{S}^{\mathcal{M}}(a) = 1$ when $a \notin \mathbb{S}^{\mathcal{M}}$.

- For each k-ary predicate symbol P of W and $\vec{a} \in M^k$, $P^{\mathcal{M}}(\vec{a}) = 1$ when at least one argument is of the wrong sort.
- For each k-ary function symbol F of W with value sort $\mathbb{S}$ and $\vec{a} \in M^k$, $F^{\mathcal{M}}(\vec{a})$ belongs to $\mathbb{S}^{\mathcal{M}}$ when each argument is of the correct sort, and is $u^{\mathcal{M}}$ when at least one argument is of the wrong sort.
- The value $c^{\mathcal{M}}$ of each constant symbol c of sort $\mathbb{S}$ is an element of $\mathbb{S}^{\mathcal{M}}$.
- The constant $u^{\mathcal{M}}$ does not belong to any of the sets $\mathbb{S}^{\mathcal{M}}$.

The unsorted constant symbol u serves two purposes: It allows one to interpret each function symbol as a total function rather than as a partial function from a product of sorts to a sort, and it insures that each structure that respects sorts has an unsorted element.

Lemma 2.6.4. *Let W be a sorted vocabulary and let $\mathcal{M}$ be a general structure with vocabulary W.*

(i) *If $\mathcal{M}$ respects sorts, then the reduction of $\mathcal{M}$ also respects sorts.*

(ii) *If $\mathcal{M}$ is reduced and respects sorts, then $u^{\mathcal{M}}$ is the unique element of M that does not belong to $\mathbb{S}^{\mathcal{M}}$ for any sort $\mathbb{S}$ of W.*

(iii) *There is a set $\mathrm{rs}(W)$ of sentences such that for every reduced structure $\mathcal{M}$ with vocabulary W, $\mathcal{M}$ respects sorts if and only if $\mathcal{M} \models \mathrm{rs}(W)$.*

Proof. (i) is clear.

(ii): Let a be an element of M that does not belong to $\mathbb{S}^{\mathcal{M}}$ for any sort $\mathbb{S}$ of W. Then for any tuple $\vec{c}$ in M and atomic formula $\varphi(x, \vec{c})$ in which the variable x occurs, we have $\varphi^{\mathcal{M}}(a, \vec{c}) = 1$ and $\varphi^{\mathcal{M}}(u^{\mathcal{M}}, \vec{c}) = 1$. Thus $a \doteq^{\mathcal{M}} u^{\mathcal{M}}$, so (ii) holds.

(iii): Each of the requirements for respecting sorts can be expressed in $\mathcal{M}$ be a set of sentences. For example, the requirement that two sorts $\mathbb{S}_1$ and $\mathbb{S}_2$ are pairwise disjoint is expressed by the sentence

$$\sup_x (1 \stackrel{.}{\le} \max(U_{\mathbb{S}_1}(x), U_{\mathbb{S}_2}(x))).$$

By Remark 2.1.4, $F(x, \vec{y}) \doteq u$ can be expressed in $\mathcal{M}$ by a set of formulas. One can use that to show that when $\mathcal{M}$ is reduced, the requirement that $F^{\mathcal{M}}(x, \vec{y}) = u^{\mathcal{M}}$ when x does not have sort $\mathbb{S}$ can be expressed by a set of sentences. The other requirements are similar. With a bit more work, one can take each sentence in $\mathrm{rs}(W)$ to be a finite set of sup quantifiers followed by a quantifier-free formula. $\square$

Definition 2.6.5. Given a sorted vocabulary W, a *sorted metric signature L over W* consists of a distinguished distance predicate symbol $d_{\mathbb{S}}$ for each sort $\mathbb{S}$ of W, and a modulus of uniform continuity for each predicate and function symbol with respect to these distance predicates. By a *sorted metric (or pre-metric) structure* with the sorted signature L we mean a reduced general structure $\mathcal{M}$ with vocabulary W that respects sorts, such that in $\mathcal{M}$, each $d_{\mathbb{S}}$ is a complete metric (or metric) on $\mathbb{S}$, and each symbol of W satisfies the modulus of uniform continuity given by L with respect to these metrics.

We will return to sorted metric structures in Subsection 4.5.

Example 2.6.6. (Bounded Continuous Logic)

The version of continuous logic as defined in Section 2 of [BBHU], which we will here call *bounded continuous logic*, is slightly broader than the $[0, 1]$-valued version. (See also [BY09] and [Fa]). A *bounded vocabulary* is a vocabulary V equipped

with a bounded real interval $[r^P, s^P]$ where $r^P < s^P$, for each predicate symbol $P \in V$. The metric signatures, continuous formulas, and metric structures with the bounded vocabulary V are as in the $[0,1]$-valued case, except that the bounds of uniform continuity are maps from $(0, \infty)$ into $(0, \infty)$, the distinguished metric d is a metric with values in $[0, s^d]$, the logical connectives are continuous functions from $\mathbb{R}^n$ into $\mathbb{R}$, and the predicates P have truth values in the bounded interval $[r^P, s^P]$. The truth values of formulas in a bounded metric structure are defined by induction on complexity of formulas in the usual way, and the truth values are in $\mathbb{R}$. Many-sorted bounded metric structures are treated in a similar way.

One can regard $[0,1]$-valued continuous logic as the special case of bounded continuous logic where where each predicate symbol P, including the distinguished metric d, has the interval $[0,1]$. The model theory of $[0,1]$-valued metric structures carries over to bounded metric structures in a routine way. As pointed out in [BBHU], there is no real loss of generality in working with $[0,1]$-valued continuous logic, and for simplicity, [BBHU] develops things in detail only for the $[0,1]$-valued case.

One can convert any bounded metric structure $\mathcal{M}$ to a $[0,1]$-valued metric structure $\mathcal{M}^1$ by replacing each predicate $P^{\mathcal{M}}$ by the $[0,1]$-valued predicate $P^{\mathcal{M}^1}$ where

$$P^{\mathcal{M}^1}(\vec{v}) = (P^{\mathcal{M}}(\vec{v}) - r^P)/(s^P - r^P).$$

It then turns out that a predicate $P \colon [0,1]^n \to [0,1]$ is definable in $\mathcal{M}$ if and only if it is definable in $\mathcal{M}^1$. In that sense, the $[0,1]$-valued language has the same expressive power as the bounded language. But in many applications, the formulas in the bounded language will be easier to understand than the corresponding formulas in the $[0,1]$-valued language.

In keeping with the perspective of this paper, one can also consider bounded general structures with a given bounded vocabulary, by simply dropping the requirement that there is a distinguished metric and a signature giving moduli of uniform continuity. However, we retain the bounded vocabulary that equips each predicate symbol P with a bounded real interval. Formulas and truth values are defined in the same way for bounded general structures as they are for bounded metric structures in [BBHU].

Example 2.6.7. (Unbounded Continuous Logic) Many important mathematical structures, such as Banach spaces, valued fields, and C^*-algebras, are structures with an unbounded complete metric d, functions that are uniformly continuous with respect to d, predicates with truth values in $\mathbb{R}$ that are uniformly continuous with respect to d, and constant symbols. We will call such structures *unbounded metric structures*.

The inductive definition of the truth value of a formula in a bounded metric structure does not automatically carry over to unbounded metric structures, because the sup of a set of values may not exist in $\mathbb{R}$. In the literature, there are at least three approaches to the study of an unbounded metric structure $\mathcal{N}$ by applying the existing model theory to some metric structure that is associated with $\mathcal{N}$.

One approach is to add a constant symbol 0 for a distinguished element of $\mathcal{N}$, and look at the many-sorted bounded metric structure $\mathcal{M}$ that has a sort $\mathbb{S}_m$ for the closed m-ball around 0 for each $0 < m \in \mathbb{N}$, and, for each $0 < m < k$, a function i_{mk} of sort $\mathbb{S}_m \to \mathbb{S}_k$ for the inclusion map. For simplicity, suppose $\mathcal{N}$ has no function symbols. For every constant symbol c and n-ary predicate symbol P of $\mathcal{N}$, and every

sort $\mathbb{S}_m$, $\mathcal{M}$ will have a constant symbol c_m and an n-ary predicate symbol P_m of sort $\mathbb{S}_m$. The uniform continuity property of $\mathcal{N}$ guarantees that for each predicate symbol P of $\mathcal{N}$, and each $m > 0$, $P^{\mathcal{N}}$ maps the closed m-ball around 0 into a bounded interval $[r_m^P, s_m^P]$. For each predicate symbol P, the bounded vocabulary of $\mathcal{M}$ will be equipped with both the arity of P and the bounded interval $[r_m^P, s_m^P]$.

In [Fa], C^*-algebras are treated as many-sorted bounded metric structures as in the preceding paragraph. In order to make things work properly, axioms are explicitly added to guarantee that the inclusion map sends the sort $\mathbb{S}_m$ onto the closed m-ball in $\mathbb{S}_k$. In [BBHU], Sections 15 and 17, Hilbert spaces and Banach lattices are treated in a similar way (see also Remark 4.6 in [BU], and [BY09]). In those cases, in any metric model of the many-sorted theory, the inclusion map sends the sort $\mathbb{S}_m$ onto the closed m-ball around 0 in $\mathbb{S}_k$.

A second, and simpler, approach is to look at the single-sorted $[0,1]$-valued metric structure formed by restricting the universe of $\mathcal{N}$ to the unit ball around 0, and normalizing each predicate symbol so that it takes truth values in $[0,1]$. [BBHU] show that for Hilbert and L^p spaces, the many-sorted metric structure has essentially the same model-theoretic properties as the single-sorted structure on the unit ball, and are able to successfully simplify things by working exclusively with the unit ball.

However, as pointed out in [BY08] and [BY14], there are other cases, such as in non-archimedean valued fields, where neither of the above approaches is adequate. The many-sorted structure $\mathcal{M}$ may have definable predicates that would not be considered definable in the original structure $\mathcal{N}$. For example, if $m < k$, the distance $\mathrm{dist}(x, \mathsf{B})$ from a point $x \in \mathbb{S}_k$ to the set $\mathsf{B} = \mathrm{range}(i_{mk})$ might not be definable in $\mathcal{N}$, but is always definable in $\mathcal{M}$ by the formula $\inf_y d_k(x, i_{mk}(y))$.

To deal with that problem, Ben Yaacov [BY08] developed a variant of continuous logic with an unbounded metric and modified notions of formula and ultraproduct. That logic has a Lipschitz *gauge function* $\nu\colon \mathcal{N} \to [0, \infty)$ that is thought of as the size of an element, and determines the closed m-balls $B_m = \{x\colon \nu(x) \le m\}$. In most examples, the gauge $\nu(x)$ is $d(x, 0)$, the distance from x to a distinguished point 0. The universe of the modified pre-ultraproduct is the set of elements x of the usual pre-ultraproduct such that $\langle \nu_i(x_i)\rangle_{i \in I}$ is bounded. This means that the gauge of x is finite, which in most examples means that $d(x, 0)$ is finite. Using that logic, [BY08] converted an unbounded metric structure $\mathcal{N}$ into an ordinary $[0,1]$-valued metric structure $\mathcal{N}^\infty$ via his *emboundment* construction. The intuitive idea is to add a point at infinity to $\mathcal{N}$, and to add a new metric to form a metric structure $\mathcal{N}^\infty$ such that the theories of $\mathcal{N}$ and $\mathcal{N}^\infty$ have the same model-theoretic properties. The reason for doing that is to study the unbounded metric structure $\mathcal{N}$ by applying ordinary continuous model theory to $\mathcal{N}^\infty$.

The logic developed in [BY08] is in many ways equivalent to the logic initiated in 1976 by Henson [He], and developed further by Henson and Iovino [HI], and Dueñez and Iovino [DI]. That approach uses a similar notion of unbounded metric structure and modified ultraproduct as [BY08], but with a different notion of formula, and a semantics based on approximate truth. Another form of unbounded continuous logic, that uses a similar notion of unbounded metric structure and modified ultraproduct but a different notion of formula, is developed in [Lu].

The bounded continuous logic of [BBHU] is better suited for applications than either of the logics developed in [BY08] or in [HI] and [DI], because the notions

of formula and truth value are simpler and more natural, and the formulas are easier to understand, in [BBHU] than in the other approaches. For that reason, it desirable to find a way to use bounded continuous logic to study unbounded metric structures when possible.

We will return to unbounded metric structures in Subsection 4.6 below.

3. Turning General Structures into Metric Structures

In this section we define the key notion of a pre-metric expansion, and show that for every theory T there exists a pre-metric expansion of T.

3.1. Definitional Expansions.
In this subsection we introduce definitional expansions of a theory, and in the next subsection we will introduce pre-metric expansions as a special case.

Definition 3.1.1. Let T be a general theory in a vocabulary V, let D be a predicate symbol that may or may not belong to V, and let $V_D = V \cup \{D\}$. A *definitional expansion* of T over V_D is a theory T_e with vocabulary V_D such that for some sequence $\langle d \rangle = \langle d_m(\vec{x}) \rangle_{m \in \mathbb{N}}$ of formulas of V that is Cauchy in T:

   (i) For every general model $\mathcal{M}$ of T, $\mathcal{M}_e = (\mathcal{M}, [\lim d_m]^{\mathcal{M}})$ is the a unique expansion of $\mathcal{M}$ to a general model of T_e.
   (ii) Every general model of T_e is equal to $(\mathcal{M}, [\lim d_m]^{\mathcal{M}})$ where $\mathcal{M}$ is a general model of T.

We say that the sequence $\langle d \rangle$ *approximates* D in T_e. Note that if $\langle d \rangle$ approximates D in T_e, then so does every subsequence of $\langle d \rangle$. Therefore, for every definitional expansion T_e of T over V_D, there is an exponentially Cauchy sequence of formulas of V that approximates D in T_e.

Lemma 3.1.2. *(Axioms for T_e) Suppose T_e is a definitional expansion of T, and $\langle d_m(x, y) \rangle_{m \in \mathbb{N}}$ is exponentially Cauchy in T and approximates D in T_e. Let T'_e be the union of T and the set of sentences*

$$(6) \qquad \{\sup_x \sup_y (|d_m(x, y) - D(x, y)| \lesssim 2^{-m}) \colon m \in \mathbb{N}\}.$$

Then T_e is equivalent to T'_e.

Proof. Let $\mathcal{N}$ be a general structure with vocabulary V_D, and let $\mathcal{M}$ be the V-part of $\mathcal{N}$. Then $\mathcal{N}$ satisfies (6) if and only if $D^{\mathcal{N}} = [\lim d_m]^{\mathcal{M}}$. By 3.1.1 (i), every general model of T'_e is a general model of T_e. By 3.1.1 (ii), every general model of T_e is a general model of T'_e. $\square$

Remark 3.1.3. *Suppose T_e is a definitional expansion of T with vocabulary V_D.*
   (i) *For each sequence $\langle d_m \rangle_{m \in \mathbb{N}}$ of formulas of V that approximates D in T_e, and each general model $\mathcal{M}$ of T, $D^{\mathcal{M}_e} = [\lim d_m]^{\mathcal{M}}$ is defined by $\langle d \rangle$ in $\mathcal{M}$.*
   (ii) *If T_e, T_f are definitional expansions of T with vocabulary V_D, and there is a sequence of formulas of V that approximates D in both T_e and T_f, then T_e and T_f are equivalent.*
   (iii) *If $T \subseteq U$, then $U_e := T_e \cup U$ is a definitional expansion of U, every sequence that approximates D in T_e approximates D in U_e, and for every general model $\mathcal{M}$ of U, the definitional expansion $\mathcal{M}_e$ of $\mathcal{M}$ is the same for U_e as for T_e.*

(iv) *For each general model $\mathcal{M}$ of T, $\mathrm{Th}(\mathcal{M}_e)$ is a definitional expansion of $\mathrm{Th}(\mathcal{M})$.*

The next remark says that definitional expansions are preserved under the addition of constant symbols.

Remark 3.1.4. *Suppose $V' = V \cup C$ where C is a set of constant symbols. If T_e is a definitional expansion of the theory T with vocabulary V_D, then T_e is still a definitional expansion of T with vocabulary V_D'.*

Lemma 3.1.5. *Suppose T_e is a definitional expansion of a theory T, and $\mathcal{M}$ is a general model of T.*

  (i) $(\doteq^{\mathcal{M}}) = (\doteq^{\mathcal{M}_e})$.
  (ii) *$\mathcal{M}_e$ is reduced if and only if $\mathcal{M}$ is reduced.*
  (iii) *If $\mathcal{M}'$ is the reduction of $\mathcal{M}$, then $\mathcal{M}_e'$ is the reduction of $\mathcal{M}_e$.*
  (iv) *If $\mathcal{M}'$ is a general model of T and $h\colon \mathcal{M} \cong \mathcal{M}'$, then $h\colon \mathcal{M}_e \cong \mathcal{M}_e'$.*

Proof. Let $\langle d_m(\vec{x}) \rangle_{m \in \mathbb{N}}$ approximate D in T_d.

(i): It is trivial that $x \doteq^{\mathcal{M}_e} y$ implies $x \doteq^{\mathcal{M}} y$. Suppose $x \doteq^{\mathcal{M}} y$. Consider an atomic formula $D(\vec{\tau}(x, \vec{z}))$, where $\vec{\tau}$ is a tuple of terms, and none of the variables $x, y, \vec{z}$ occur freely in the any of the approximating formulas d_m. Then for all $\vec{z}$ in $\mathcal{M}_e$ we have

$$D(\vec{\tau}(x, \vec{z})) = \lim_{m \to \infty} d_m(\vec{\tau}(x, \vec{z})) = \lim_{m \to \infty} d_m(\vec{\tau}(y, \vec{z})) = D(\vec{\tau}(y, \vec{z})),$$

so $x \doteq^{\mathcal{M}_e} y$. Therefore $(\doteq^{\mathcal{M}_e}) = (\doteq^{\mathcal{M}})$.

(ii): Expansions of reduced structures are always reduced. Suppose $\mathcal{M}_e$ is reduced. By the proof of (ii), if $x \doteq^{\mathcal{M}} y$ then $x \doteq^{\mathcal{N}} y$, and since $\mathcal{M}_e$ is reduced, $x = y$. Hence $\mathcal{M}$ is reduced.

(iii) For each $x \in M$ let $x' \in M'$ be the equivalence class of x under $\doteq^{\mathcal{M}}$. Then

$$D^{\mathcal{M}_e}(\vec{x}) = \lim_{m \to \infty} d_m^{\mathcal{M}}(\vec{x}) = \lim_{m \to \infty} d_m^{\mathcal{M}'}(\vec{x}') = D^{\mathcal{M}_e'}(\vec{x}'),$$

so $\mathcal{M}_e'$ is the reduction of $\mathcal{M}_e$.

(iv): By (ii), we may assume that $\mathcal{M}, \mathcal{M}'$ are reduced. Then any isomorphism $h\colon \mathcal{M} \cong \mathcal{M}'$ sends $[\lim d_m]^{\mathcal{M}}$ to $[\lim d_m]^{\mathcal{M}'}$. $\qquad \square$

Proposition 3.1.6. *(Definitional expansions commute with ultraproducts.) Suppose T_e is a definitional expansion of T, $\mathcal{D}$ is an ultrafilter over a set I, and $\mathcal{M}_i \models T$ for each $i \in I$. Then $(\prod^{\mathcal{D}} \mathcal{M}_i)_e = \prod^{\mathcal{D}}((\mathcal{M}_i)_e)$, and $(\prod_{\mathcal{D}} \mathcal{M}_i)_e = \prod_{\mathcal{D}}((\mathcal{M}_i)_e)$.*

Proof. By Fact 2.2.2, $\prod^{\mathcal{D}} \mathcal{M}_i, \prod_{\mathcal{D}} \mathcal{M}_i$ are models of T, and $\prod^{\mathcal{D}}((\mathcal{M}_i)_e), \prod_{\mathcal{D}}((\mathcal{M}_i)_e)$ are models of T_e. $(\prod^{\mathcal{D}} \mathcal{M}_i)_e = \prod^{\mathcal{D}}((\mathcal{M}_i)_e)$ because $\prod^{\mathcal{D}} \mathcal{M}_i$ is the V-part of $\prod^{\mathcal{D}}((\mathcal{M}_i)_e)$. Therefore, by Lemma 3.1.5 (ii), $(\prod_{\mathcal{D}} \mathcal{M}_i)_e = \prod_{\mathcal{D}}((\mathcal{M}_i)_e)$. $\qquad \square$

3.2. **Pre-metric Expansions.** We assume hereafter that D is a binary predicate symbol.

Definition 3.2.1. Let T be a theory in a vocabulary V. We say that T_e is a pre-metric expansion of T (with signature L_e) if:

  (i) (T_e, L_e) is a metric theory whose signature L_e has vocabulary V_D and distance predicate D.

(ii) There is a Cauchy sequence $\langle d \rangle$ of formulas in T such that the general
models of T_e are exactly the structures of the form $\mathcal{M}_e = (\mathcal{M}, [\lim d_m]^{\mathcal{M}})$,
where $\mathcal{M}$ is a general model of T.

Since (T_e, L_e) is a metric theory, we have $T_e \models \mathrm{met}(L_e)$, and every general model
of T_e is the downgrade of a pre-metric structure with signature L_e. Condition (ii)
in the above definition just says that T_e is a definitional expansion of T. So each
of the results 3.1.2 – 3.1.6 hold for pre-metric expansions because they hold for all
definitional expansions.

We call $(\mathcal{M}_e, L_e)$ the *pre-metric expansion of* $\mathcal{M}$ for T_e, and call $\mathcal{M}$ the *non-
metric part* of $\mathcal{M}_e$. We abuse notation by using $\mathcal{M}_e$ to denote both the pre-metric
expansion $(\mathcal{M}_e, L_e)$ of $\mathcal{M}$, and its downgrade $\mathcal{M}_e$. We call a sequence $\langle d \rangle$ that
approximates D in T_e an *approximate distance for* T_e, and also for $\mathcal{M}_e$. Every
pre-metric expansion of T has approximate distances, but they are not unique.

Here are three rather trivial examples of pre-metric expansions.

Example 3.2.2. Suppose V has no predicate symbols. Then up to equivalence,
the unique pre-metric expansion of T is the theory $T_e = T \cup \{\sup_x \sup_y D(x, y) \dot{\le} 0\}$
in which the distance between any two elements is 0. Every reduced model of T or
of T_e is a one-element structure.

Example 3.2.3. Suppose the predicate symbol D already belongs to V, L is a
metric signature over V with distance predicate D, and $T \models \mathrm{met}(L)$. Let T_e be
the metric theory with signature L and the same set of sentences as T. Then T_e is
a pre-metric expansion of T, and for each general model $\mathcal{M}$ of T, the upgrade of $\mathcal{M}$
to L is the pre-metric expansion of $\mathcal{M}$ for T_e.

Example 3.2.4. If there is a formula $D(x, y)$ in the vocabulary V such that $D^{\mathcal{M}}$
is $\dot{=}^{\mathcal{M}}$ in every general model $\mathcal{M}$ of T, then T has a pre-metric expansion with the
distinguished distance D and the trivial moduli of uniform continuity. We call such
a pre-metric expansion *discrete*. A theory T has a discrete pre-metric expansion if
and only if T has a Cauchy sequence of formulas $\langle d_m(x, y) \rangle_{m \in \mathbb{N}}$ such that in every
reduced model $\mathcal{N}$ of T, $[\lim d_m]^{\mathcal{N}}$ is the discrete metric $=^{\mathcal{N}}$. (Hint: Take $d_m(x, y)$
to be exponentially Cauchy, so that d_2 is always within $1/4$ of the limit, and take
$D(x, y) = C(d_2(x, y))$ for an appropriate connective C.)

Note that a classical structure without equality (where every atomic formula
has truth value 0 or 1) will not necessarily have a discrete pre-metric expansion.
But by the Expansion Theorem below, it will have a pre-metric expansion if V is
countable.

*From this point on, except when we specify otherwise, every general structure
we consider will be understood to be a general structure whose vocabulary has at
most countably many predicate and function symbols. We let V be a vocabulary
that contains at most countably many predicate and function symbols (we make no
restriction on the number of constant symbols). We let D be a binary predicate
symbol and let $V_D = V \cup \{D\}$.*

Formulas in the vocabulary V will be called V-*formulas*, and general structures
for V will be called V-structures. Similarly for terms and sentences. Unless we say
otherwise, T will be a V-theory, that is, a set of V-sentences.

Definition 3.2.5. Let T be a complete V-theory. We call $\langle d \rangle = \langle d_m(x,y) \rangle_{m \in \mathbb{N}}$ an *exact distance* in T if

    (i) $\langle d \rangle$ is Cauchy in T.

    (ii) $[\lim d_m]^{\mathcal{M}}$ is a pseudo-metric in every general model of T.

    (iii) For each general model $\mathcal{M} \models T$ and V-formula $\varphi(\vec{x})$, the mapping $\varphi^{\mathcal{M}} \colon M^{|\vec{x}|} \to [0,1]$ is uniformly continuous in the pseudo-metric space $(M, [\lim d_m]^{\mathcal{M}})$.

Proposition 3.2.6. *Suppose T_e is a pre-metric expansion of a complete V-theory T with approximate distance $\langle d \rangle = \langle d_m(x,y) \rangle_{m \in \mathbb{N}}$. Then $\langle d \rangle$ is an exact distance in T.*

Proof. By the definition of a pre-metric expansion, Definition 3.2.5 (i) and (ii) hold, and for every $\mathcal{M} \models T$, $\mathcal{M}_e = (\mathcal{M}, [\lim d_m]^{\mathcal{M}})$ is a pre-metric structure with signature L_e. Then by Fact 2.5.2, (iii) holds. $\qquad\square$

We now state two results, Theorems 3.2.7 and 3.2.8, that follow from the proofs of results from [I94] in the context of Henson's Banach space model theory [He], and from [BU] in the context of open Hausdorff cats. Those results together imply that every complete theory has a pre-metric expansion. In order to convert the proofs from [I94] and [BU] to proofs of Theorems 3.2.7 and 3.2.8 in our present setting, we would need a long detour through positive bounded formulas and open Hausdorff cats. Instead, in the next subsection we will give a different and self-contained proof of a stronger result, that every (not necessarily complete) theory has a pre-metric expansion with an approximate distance $\langle d_m \rangle_{m \in \mathbb{N}}$ such that each d_m defines a pseudo-metric in every general model of T.

Theorem 3.2.7. *(By the proof of Proposition 53 of [I94], Theorem 5.1 of [I99], or Theorem 2.20 of [BY05].) Every complete V-theory T has an exact distance.*

Theorem 3.2.8. *(By Theorem 4.23 of [BU].) Let T be a complete V-theory. If T has an exact distance, then T has a pre-metric expansion.*

Note that by Proposition 3.2.6, every complete V-theory has a pre-metric expansion if and only if both Theorem 3.2.7 and Theorem 3.2.8 hold.

3.3. **The Expansion Theorem.** We will show that every V-theory T has a pre-metric expansion with an approximate distance $\langle d_m(x,y) \rangle_{m \in \mathbb{N}}$ such that each d_m defines a pseudo-metric in every general model of T. In Lemma 3.3.3 below we prove this in the special case that V has no function symbols. We will use that to prove the general result in Theorem 3.3.4. By Remark 3.1.3 (iii), it is enough to show that the empty set of sentences with vocabulary V has such a pre-metric expansion.

Definition 3.3.1. A formula $\varphi(\vec{x}, \vec{y})$ is *pseudo-metric in T* if $|\vec{x}| = |\vec{y}|$ and $\varphi^{\mathcal{M}}$ is a pseudo-metric on $M^{|\vec{x}|}$ for every general model $\mathcal{M} \models T$. By a *pseudo-metric approximate distance for* a pre-metric expansion T_e of T, we mean an approximate distance $\langle d_m(x,y) \rangle_{m \in \mathbb{N}}$ for T_e such that each d_m is a pseudo-metric formula in T.

Remark 3.3.2. *(i) If $\langle \varphi_m(\vec{x}, \vec{y}) \rangle_{m \in \mathbb{N}}$ is Cauchy in T and each formula φ_m is pseudo-metric in T, then $[\lim \varphi_m]^{\mathcal{M}}$ is a pseudo-metric on $M^{|\vec{x}|}$ for every $\mathcal{M} \models T$.*

    (ii) If $T \subseteq U$ and T_e is a pre-metric expansion of T, then $U_e = T_e \cup U$ is a pre-metric expansion of U with the same metric signature and approximate distance as T_e.

Proof. (i) is clear. (ii) follows from Remark 3.1.3 (iii) and part (i) above. $\quad\square$

Lemma 3.3.3. *If the vocabulary V has countably many predicate symbols and no function symbols, then every V-theory T has a pre-metric expansion with a pseudo-metric approximate distance.*

Proof. Let $x, y, u, z_1, z_2, \ldots$ be distinct variables. Arrange all atomic formulas with no constant symbols whose variables are among $u, z_1, z_2, \ldots$ in a countable list $\alpha_0, \alpha_1, \ldots$. For each m, let $\alpha_m(x, \vec{z})$ and $\alpha_m(y, \vec{z})$ be the formulas formed from $\alpha_m(u, \vec{z})$ by replacing u by x and y respectively. For each m, let $\beta_m(x, y)$ be the V-formula

$$\beta_m(x, y) = \sup_{\vec{z}} |\alpha_m(x, \vec{z}) - \alpha_m(y, \vec{z})|.$$

Let $d_0(x, y) = \beta_0(x, y)$, and for each $m > 0$ let $d_m(x, y)$ be the V-formula

$$d_m(x, y) = \max(d_{m-1}(x, y), 2^{-m}\beta_m(x, y)).$$

Then for every V-structure $\mathcal{M}$ and every m, $\beta_m^{\mathcal{M}}$ and $d_m^{\mathcal{M}}$ are pseudo-metrics on M. Moreover,

$$\models d_m(x, y) \lesssim d_{m+1}(x, y) \lesssim d_m(x, y) \dotplus 2^{-(m+1)},$$

so $\langle d \rangle := \langle d_m(x, y) \rangle_{m \in \mathbb{N}}$ is exponentially Cauchy in T. For every general model $\mathcal{M}$ of T, let $\mathcal{M}_e = (\mathcal{M}, D^{\mathcal{M}_e})$ where $D^{\mathcal{M}_e} = [\lim d_m]^{\mathcal{M}}$. Since each $d_m^{\mathcal{M}}$ is a pseudo-metric on M, the limit $D^{\mathcal{M}_e}$ is a pseudo-metric on M.

For every $\mathcal{M} \models T$, $\mathcal{M}_e$ is a general model of the V_D-theory

$$T_{\langle d \rangle} = \{\sup_x \sup_y [d_m(x, y) \lesssim D(x, y) \lesssim d_m(x, y) \dotplus 2^{-m}] : m \in \mathbb{N}\}.$$

We will find a metric signature L_e such that $T_e = T \cup T_{\langle d \rangle}$ is a pre-metric expansion of T with approximate distance $\langle d \rangle$. It suffices to specify a modulus of uniform continuity $\triangle_P$ for each k-ary predicate symbol $P \in V$ that is satisfied with respect to D in every general model of $T_{\langle d \rangle}$. For each $1 \le i \le k$, let $P(u, \vec{z})_i$ be the atomic formula obtained from $P(z_1, \ldots, z_k)$ by replacing z_i by u. Then $P(u, \vec{z})_i$ is α_{m_i} for some $m_i \in \mathbb{N}$. Let $m = \max(m_1, \ldots, m_k)$. For each $1 \le i \le k$, we have

$$T_{\langle d \rangle} \models \sup_{\vec{z}} |P(x, \vec{z})_i - P(y, \vec{z})_i| = \beta_{m_i}(x, y) \lesssim 2^m d_m(x, y) \lesssim 2^m D(x, y).$$

Therefore whenever $\vec{x}, \vec{y}$ differ only in the i-th argument we have

$$T_{\langle d \rangle} \models |P(\vec{x}) - P(\vec{y})| \lesssim 2^m D(x_i, y_i).$$

Since one can change any k-tuple $\vec{x}$ to $\vec{y}$ in k steps by changing one variable at a time, for every pair of k-tuples $\vec{x}, \vec{y}$ we have

$$T_{\langle d \rangle} \models |P(\vec{x}) - P(\vec{y})| \lesssim 2^m k \max(D(x_1, y_1), \ldots, D(x_k, y_k)).$$

It follows that P has modulus of uniform continuity $\triangle_P(\varepsilon) = 2^{-m}k^{-1}\varepsilon$ with respect to D in each general model of $T_{\langle d \rangle}$. Therefore $T_e = T \cup T_{\langle d \rangle}$ is a pre-metric expansion of T. $\quad\square$

In the general case that V contains function symbols, we must also specify a modulus of uniform continuity for each function symbol with respect to D. In a pre-metric structure with distance d, one can eliminate function symbols by using the formula $d(F(\vec{x}), y)$ for the graph of $F(\vec{x})$. This will not work in a general structure, because the formula $D(F(\vec{x}), y)$ for the graph of $F(\vec{x})$ is not a V-formula. We will circumvent that difficulty by an "atomic Morleyization", adding a new predicate symbol for each atomic V-formula.

Theorem 3.3.4. *(Expansion Theorem) For every vocabulary V with countably many predicate symbols and countably many function symbols, every V-theory T has a pre-metric expansion with a pseudo-metric approximate distance.*

Proof. By Remark 3.3.2 (ii), we may assume that T is the empty set of sentences. Let V' be the union of V and a k-ary predicate symbol P_α for each atomic V-formula $\alpha(\vec{x})$ with k variables and no constant symbols. Let T' be the set of V'-sentences $\sup_{\vec{x}} |P_\alpha(\vec{x}) - \alpha(\vec{x})|$ for each $\alpha(\vec{x})$. Then $V' \setminus V$ is a countable set of new predicate symbols, and every V-structure $\mathcal{M}$ has a unique expansion to a general model $\mathcal{M}'$ of T'. It can be shown by induction on complexity that every V'-formula $\varphi(\vec{x})$ is T'-equivalent to a V-formula $\varphi^0(\vec{x})$, and is also T'-equivalent to a V'-formula $\varphi''(\vec{x})$ that has no function symbols.

Let V'' be the set of all predicate and constant symbols in V', and for each V-structure $\mathcal{M}$ let $\mathcal{M}''$ be the V''-part of $\mathcal{M}'$. Let T'' be the set of all V''-sentences that hold in all general models of T', which is the same as the set of all V''-sentences that hold in $\mathcal{M}''$ for every V-structure $\mathcal{M}$. Note that $V'' \subseteq V'$, $T'' \subseteq T'$, and for each V'-formula $\varphi(\vec{x})$, the formula $\varphi''(\vec{x})$ defined in the previous paragraph is a V''-formula. By Lemma 3.3.3, T'' has a pre-metric expansion T''_e with a pseudo-metric approximate distance $\langle d \rangle = \langle d_m(x,y) \rangle_{m \in \mathbb{N}}$. Then for each V-structure $\mathcal{M}$, $\mathcal{M}''_e = (\mathcal{M}'', [\lim d_m]^{\mathcal{M}''})$.

We next find a metric signature L'_e over V'_D such that $T'_e = T''_e \cup T'$ with signature L'_e is a pre-metric expansion of T' with the same approximate distance. Since each $\mathcal{M}'$ is an expansion of $\mathcal{M}''$, $d_m^{\mathcal{M}'} = d_m^{\mathcal{M}''}$, so $[\lim d_m]^{\mathcal{M}'} = [\lim d_m]^{\mathcal{M}''}$. We must find a modulus of uniform continuity for each k-ary function symbol F in V. For each m, let $\theta_m(\vec{x}, y)$ be a V''-formula that is T'-equivalent to $d_m(F(\vec{x}), y)$. Then $\langle \theta_m(\vec{x}, y) \rangle_{m \in \mathbb{N}}$ is Cauchy in T'', and for each $\mathcal{M} \models T$ and all $(\vec{b}, c) \in M^{k+1}$,

$$D^{\mathcal{M}''_e}(F^{\mathcal{M}}(\vec{b}), c) = [\lim d_m]^{\mathcal{M}''}(F^{\mathcal{M}}(\vec{b}), c) = [\lim \theta_m]^{\mathcal{M}''}(\vec{b}, c) = [\lim \theta_m]^{\mathcal{M}''_e}(\vec{b}, c).$$

By Lemma 2.5.5, there is a function $\triangle_F$ that is a modulus of uniform continuity for $[\lim \theta_m]^{\mathcal{M}''_e}$ in the pre-metric structure $\mathcal{M}''_e$ for every general model $\mathcal{M}$ of T. We note that Proposition 9.23 of [BBHU] on definable functions holds for pre-metric structures as well as metric structures. Therefore in $\mathcal{M}''_e$, the function $F^{\mathcal{M}}$ is definable and has the same modulus of uniform continuity $\triangle_F$. This shows that $\mathcal{M}'_e = (\mathcal{M}', [\lim d_m]^{\mathcal{M}'})$ is a pre-metric structure with the metric signature L'_e that agrees with L''_e on V'' and gives F the modulus of uniform continuity $\triangle_F$.

Finally, we show that there is a pre-metric expansion T_e of the empty theory T with a pseudo-metric approximate distance. We take L_e to be the restriction of L'_e to V_D (so the symbols of V_D have the same moduli of uniform continuity in L_e as in L'_e). Then L_e is a metric signature with distance predicate D over V_D. For each m let $\widehat{d}_m(x, y)$ be a V-formula that is T'-equivalent to $d_m(x, y)$, and let $\langle \widehat{d} \rangle = \langle \widehat{d}_m(x, y) \rangle_{m \in \mathbb{N}}$. Since $\langle d \rangle$ is Cauchy in T' and each d_m is pseudo-metric in every general model of T', and every $\mathcal{M} \models T$ has a unique expansion to a general model $\mathcal{M}' \models T'$, $\langle \widehat{d} \rangle$ is Cauchy in T' and each $\widehat{d}_m$ is pseudo-metric in each general model of T. We let T_e be the union of T and a set of sentences saying that $D(x, y) = \lim_{m \to \infty} \widehat{d}_m(x, y)$ with signature L_e. Then T_e is a pre-metric expansion of T with pseudo-metric approximate distance $\langle \widehat{d} \rangle$. $\square$

Theorem 3.3.4 shows that a pre-metric expansion with a pseudo-metric approximate distance always exists, but the distance predicate built in the proof depends on an arbitrary enumeration of the atomic formulas, and may not be natural.

Corollary 3.3.5. *Every complete V-theory T has an exact distance $\langle d_m \rangle_{m \in \mathbb{N}}$ in which each $d_m^{\mathcal{M}}$ is a pseudo-metric for each $\mathcal{M} \models T$.*

Proof. By Theorem 3.3.4 and Proposition 3.2.6. $\qquad\square$

Question 3.3.6. *Does every pre-metric expansion of T have a pseudo-metric approximate distance?*

Added in October, 2020: With the permission of James E. Hanson, we report that he has shown that the answer to Question 3.3.6 is "Yes". If $\langle d_k \rangle$ is an approximate distance for T_e, then $\langle e_k \rangle$ is a pseudo-metric approximate distance for T_e, where $e_n(x, y) = \sup_x |d_k(x, z) - d_k(y, z)|$.

We now show that the pre-metric expansion of a V-structure $\mathcal{M}$ is unique up to a uniformly continuous homeomorphism.

Proposition 3.3.7. *Let T_e, T_f be pre-metric expansions of T. There is a function $\triangle(\cdot) \colon (0, 1] \to (0, 1]$ such that for every general model $\mathcal{M}$ of T, the identity function is a uniformly continuous homeomorphism from $(M, D^{\mathcal{M}_e})$ onto $(M, D^{\mathcal{M}_f})$ with modulus $\triangle(\cdot)$.*

Proof. We must find a function $\triangle(\cdot)$ that is a modulus of uniform continuity for $D^{\mathcal{M}_f}$ in the pre-metric structure $\mathcal{M}_e$ for each $\mathcal{M} \models T$. Let $\langle d_m \rangle_{m \in \mathbb{N}}$ be an approximate distance for f. For each $\mathcal{M} \models T$, $\mathcal{M}_e$ is a pre-metric structure with signature L_e. $\langle d_m \rangle_{m \in \mathbb{N}}$ is Cauchy in the V-theory T. T is also a V_D-theory, and $D^{\mathcal{M}_f} = [\lim d_m]^{\mathcal{M}} = [\lim d_m]^{\mathcal{M}_e}$. By Lemma 2.5.5 (ii), there is a function $\triangle(\cdot)$ with the required property. $\qquad\square$

Let us look at the pre-metric expansion process in the reverse direction. The general structure $\mathcal{M}$ can by obtained from the pre-metric expansion $\mathcal{M}_e$ by simplifying in two steps: first downgrade $\mathcal{M}_e$ to a general structure $\mathcal{N}$ by forgetting the metric signature, and then take the V-part of $\mathcal{N}$ by forgetting the distance to get the original general structure $\mathcal{M}$. $\mathcal{M}$ is the non-metric part of $\mathcal{M}_e$. We thus have two equivalence relations on the class of pre-metric structures with vocabulary V_D, the fine relation of having the same downgrade, and the coarse relation of having the same non-metric part.

Formally, the class of general structures is disjoint from the class of pre-metric structures because only the latter includes a metric signature. But intuitively, we regard $\mathcal{M}_e$ and the downgrade of $\mathcal{M}_e$ as the same structure presented in two different ways.

3.4. **Absoluteness.** In this rather philosophical subsection we develop a framework that allows one to apply known model-theoretic results about metric structures to general structures.

For the remainder of this paper, we let T be a V-theory, T_e be an arbitrary pre-metric expansion of T with signature L_e, $\mathcal{M}$ be an arbitrary general model of T, and $\mathcal{M}_e$ be the pre-metric expansion of $\mathcal{M}$ for T_e.

By a *general property* we mean a class of general structures that is preserved under automorphism (i.e., bijective embeddings), and by a *pre-metric property* we mean a class of pre-metric structures, again preserved under automorphisms. When

$\mathcal{P}$ is a property, we interchangeably use the phrases "$\mathcal{N}$ belongs to $\mathcal{P}$", "$\mathcal{N}$ has property $\mathcal{P}$", "$\mathcal{N}$ satisfies $\mathcal{P}$", and "$\mathcal{P}$ holds in $\mathcal{N}$".

Definition 3.4.1. An *absolute version* of a pre-metric property $\mathcal{Q}$ is a general property $\mathcal{P}$ such that whenever $\mathcal{M}_e$ is a pre-metric expansion of $\mathcal{M}$, $\mathcal{M}$ has property $\mathcal{P}$ if and only if $\mathcal{M}_e$ has property $\mathcal{Q}$.

The following corollary was stated in the Introduction.

Corollary 3.4.2. *Every pre-metric property $\mathcal{Q}$ has at most one absolute version.*

Proof. Suppose $\mathcal{P}$ and $\mathcal{P}'$ are absolute versions of $\mathcal{Q}$, and consider a general structure $\mathcal{M}$. By Theorem 3.3.4, $\mathcal{M}$ has a pre-metric expansion $\mathcal{M}_e$. Then the following are equivalent: $\mathcal{M}$ has property $\mathcal{P}$, $\mathcal{M}_e$ has property $\mathcal{Q}$, $\mathcal{M}$ has property $\mathcal{P}'$.   □

Definition 3.4.3. We say that a general property $\mathcal{P}$ is *absolute* if whenever $\mathcal{M}_e$ is a pre-metric expansion of $\mathcal{M}$, $\mathcal{M}$ has property $\mathcal{P}$ if and only if the downgrade of $\mathcal{M}_e$ has property $\mathcal{P}$.

Corollary 3.4.4. *Suppose $\mathcal{P}$ is an absolute version of $\mathcal{Q}$. Then $\mathcal{P}$ is absolute.*

Proof. By Example 3.2.3, each pre-metric structure $\mathcal{N}$ is a pre-metric expansion of the downgrade of $\mathcal{N}$. Therefore for each $\mathcal{M}$ and $\mathcal{M}_e$, the following are equivalent: $\mathcal{M}$ has property $\mathcal{P}$, $\mathcal{M}_e$ has property $\mathcal{Q}$, the downgrade of $\mathcal{M}_e$ has property $\mathcal{P}$.   □

By Lemma 3.1.5 (i) we have:

Proposition 3.4.5. *The property* [$\mathcal{M}$ is reduced] *in Definition 2.1.3 is absolute.*

In view of Remark 3.1.4, we often consider properties of a general structure $\mathcal{M}$ with additional parameters from M.

Proposition 3.4.6. *Let $p(\vec{x}, A)$ be a set of V-formulas with parameters from M. The property* [$p(\vec{x}, A)$ is realized in $\mathcal{M}_A$] *is absolute.*

Proof. Every pre-metric expansion of $\mathcal{M}$ is an expansion of $\mathcal{M}$. It follows from Remark 2.1.6, that if $\mathcal{M}'$ is an expansion of $\mathcal{M}$, $A \subseteq M$, and $\vec{b} \subseteq M$, then $\vec{b}$ realizes $p(\vec{x}, A)$ in $\mathcal{M}'_A$ if and only if $\vec{b}$ realizes $p(\vec{x}, A)$ in $\mathcal{M}_A$.   □

For the same reason, Proposition 3.4.6 also holds when $p(\vec{x}, A)$ is an infinitary $L_{\omega_1\omega}$-formula in the sense of [Ea15]

In view of Corollaries 3.4.2 and 3.4.4, we consider the absolute version of a pre-metric property $\mathcal{Q}$, if there is one, to be the "right" way to extend $\mathcal{Q}$ to all general structures. When we have a name for structures with a pre-metric property $\mathcal{Q}$, it will be convenient to use the same name for general structures that satisfy an absolute version of $\mathcal{Q}$. So we adopt the following convention.

Definition 3.4.7. A pre-metric property $\mathcal{Q}$ is said to be *absolute* if $\mathcal{Q}$ has an absolute version. If $\mathcal{Q}$ is an absolute pre-metric property, a general structure that satisfies the absolute version of $\mathcal{Q}$ will be called a *general structure with property $\mathcal{Q}$*.

Proposition 3.4.8. *For any pre-metric property $\mathcal{Q}$, the following are equivalent:*
  (i) *$\mathcal{Q}$ is absolute.*
  (ii) *The class of general structures with property $\mathcal{Q}$ is absolute.*
  (iii) *For every general structure $\mathcal{M}$ and all pre-metric expansions $\mathcal{M}_e, \mathcal{M}_f$ of $\mathcal{M}$, $\mathcal{M}_e$ has property $\mathcal{Q}$ if and only if $\mathcal{M}_f$ has property $\mathcal{Q}$.*

Proof. (i) and (ii) are equivalent by Corollary 3.4.4. Assume (i), so $\mathcal{Q}$ has an absolute version $\mathcal{P}$. Let $\mathcal{M}$ be a general structure and let $\mathcal{M}_e$ and $\mathcal{M}_f$ be two pre-metric expansions of $\mathcal{M}$. Then the following are equivalent: $\mathcal{M}_e$ has $\mathcal{Q}$, the downgrade of $\mathcal{M}_e$ has $\mathcal{P}$, $\mathcal{M}$ has $\mathcal{P}$, the downgrade of $\mathcal{M}_f$ has $\mathcal{P}$, $\mathcal{M}_f$ has $\mathcal{Q}$. Therefore (iii) holds.

Assume (iii). Let $\mathcal{P}$ be the property such that for each general structure $\mathcal{M}$, $\mathcal{P}$ holds for $\mathcal{M}$ if and only if $\mathcal{Q}$ holds for some pre-metric expansion of $\mathcal{M}$. By (iii), for each $\mathcal{M}$ and $\mathcal{M}_e$, $\mathcal{P}$ holds for $\mathcal{M}$ if and only if $\mathcal{Q}$ holds for $\mathcal{M}_e$. By Example 3.2.3, each pre-metric structure $\mathcal{N}$ is a pre-metric expansion of the downgrade of $\mathcal{N}$, so $\mathcal{Q}$ holds for $\mathcal{N}$ if and only if $\mathcal{P}$ holds for the downgrade of $\mathcal{N}$. In particular, for each $\mathcal{M}$ and $\mathcal{M}_e$, $\mathcal{Q}$ holds for $\mathcal{M}_e$ if and only if $\mathcal{P}$ holds for the downgrade of $\mathcal{M}_e$. It follows that $\mathcal{P}$ is an absolute version of $\mathcal{Q}$, so (i) holds.   □

Here are some examples of pre-metric properties that are *not* absolute.

Example 3.4.9.

- Let $0 < r \leq 1$. The property of being a pre-metric structure whose distinguished distance has diameter r is not absolute. Similarly for diameter $\leq r$, and for diameter $\geq r$.
- Let $d(x, y)$ be a V-formula that defines a pseudo-metric in every V-structure. The property of being a pre-metric structure $\mathcal{M}_e$ with distinguished distance D such that $\mathcal{M}_e$ satisfies $\sup_x \sup_y (d(x, y) \dot{\leq} D(x, y))$, is not absolute. (Hint: If $\mathcal{M}_e$ has an approximate distance $\langle d_m \rangle_{m \in \mathbb{N}}$, then $(\mathcal{M}, \max(\mathsf{D}, d^{\mathcal{M}}))$ is a pre-metric expansion of $\mathcal{M}$ with approximate distance $\langle \max(d_m, d) \rangle_{m \in \mathbb{N}}$ and the same signature L_e.)
- Say that a pre-metric structure is *Lipschitz* if every predicate and function symbol S has a modulus of uniform continuity $\triangle_S$ that is linear, that is, for each S there is a positive real ℓ such that $\triangle_S(x) = \ell x$ for all $x \in (0, 1]$. The property of being a Lipschitz pre-metric structure is not absolute. (Hint: Consider pre-metric expansions of the unit ball in an n-dimensional vector space over the reals.)

In the next section, we will show that many pre-metric properties $\mathcal{Q}$ in the literature have absolute versions, and also give necessary and sufficient conditions for $\mathcal{M}$ to be a general structure with property $\mathcal{Q}$. Note that the characterization of the absolute version of $\mathcal{Q}$ given by the proof of Proposition 3.4.8 mentions pre-metric expansions. In the results that follow, we will always give necessary and sufficient conditions for a general structure $\mathcal{M}$ to have property $\mathcal{Q}$ that are about $\mathcal{M}$ itself, rather than conditions that mention pre-metric expansions of $\mathcal{M}$. This is desirable for potential applications, because a general structure $\mathcal{M}$ may have a very simple description even though all its pre-metric expansions are complicated.

4. PROPERTIES OF GENERAL STRUCTURES

In this section we use absoluteness to extend known results about metric structures to general structures.

4.1. **Types in Pre-metric Expansions.**

Proposition 4.1.1. *If $\mathcal{M} \models T$ and $\mathcal{N} \equiv \mathcal{M}$, then $\mathcal{N}_e \equiv \mathcal{M}_e$.*

Proof. Let κ be a special cardinal such that $|V| + \aleph_0 < \kappa$. By the Existence Theorem for Special Models, there are κ-special V_D-structures $\mathcal{M}' \equiv \mathcal{M}_e$ and $\mathcal{N}' \equiv \mathcal{N}_e$. Let $\mathcal{M}'', \mathcal{N}''$ be the V-parts of $\mathcal{M}', \mathcal{N}'$ respectively. Then $\mathcal{M}'', \mathcal{N}'' \models T$, $\mathcal{M}' = \mathcal{M}''_e$ and $\mathcal{N}' = \mathcal{N}''_e$. By Remark 2.4.4, the reductions of $\mathcal{M}''$ and $\mathcal{N}''$ are κ-special. By the Uniqueness Theorem for Special Models, we have $\mathcal{M}'' \cong \mathcal{N}''$. By Lemma 3.1.5 (iii) we have $\mathcal{M}''_e \cong \mathcal{N}''_e$, or in other words, $\mathcal{M}' \cong \mathcal{N}'$. Therefore $\mathcal{N}_e \equiv \mathcal{N}' \equiv \mathcal{M}' \equiv \mathcal{M}_e$. $\square$

Corollary 4.1.2. *Suppose $A \subseteq M$ and $h \colon A \to N$. Then $(\mathcal{M}, a)_{a \in A} \equiv (\mathcal{N}, ha)_{a \in A}$ if and only if $(\mathcal{M}_e, a)_{a \in A} \equiv (\mathcal{N}_e, ha)_{a \in A}$*

Proof. By Remark 3.1.4 and Proposition 4.1.1. $\square$

Corollary 4.1.3. *$h \colon \mathcal{M} \prec \mathcal{N}$ if and only if $h \colon \mathcal{M}_e \prec \mathcal{N}_e$.*

Proof. By Corollary 4.1.2 with $A = M$. $\square$

Corollary 4.1.4. *The property of two tuples realizing the same type over A is absolute—$\mathrm{tp}_{\mathcal{M}}(\vec{b}/A) = \mathrm{tp}_{\mathcal{M}}(\vec{c}/A)$ if and only if $\mathrm{tp}_{\mathcal{M}_e}(\vec{b}/A) = \mathrm{tp}_{\mathcal{M}_e}(\vec{c}/A)$. Also, indiscernibility over A is absolute.*

Proof. Apply Corollary 4.1.2 with $\mathcal{N} = \mathcal{M}$ and where h is the identity on A and maps $\vec{b}$ to $\vec{c}$. $\square$

The next corollary shows that for each complete theory T, T_e has essentially the same complete types as T.

Corollary 4.1.5. *If $T = \mathrm{Th}(\mathcal{M})$ and $T_e = \mathrm{Th}(\mathcal{M}_e)$, there is a homeomorphism h from $S_n(T)$ onto $S_n(T_e)$ such that for each $\vec{c} \in M^n$, $h(\mathrm{tp}_{\mathcal{M}}(\vec{c})) = \mathrm{tp}_{\mathcal{M}_e}(\vec{c})$.*

Proof. This follows from Corollary 4.1.4. $\square$

Proposition 4.1.6. *Let κ be an infinite cardinal. The κ-saturation property is absolute—for each general model $\mathcal{M}$ of T, $\mathcal{M}$ is κ-saturated if and only if $\mathcal{M}_e$ is κ-saturated.*

Proof. We prove the non-trivial direction. Suppose $\mathcal{M}$ is κ-saturated. By Remark 2.4.2 and Lemma 3.1.5 (i), we may assume without loss of generality that $\mathcal{M}$ is reduced. Then $\mathcal{M}_e$ is reduced. Let $A \subseteq M$ with $|A| < \kappa$, and let $\Gamma(x)$ be a set of V_D-formulas that is finitely satisfiable in $(\mathcal{M}_e)_A$. By the Existence Theorem for Special Models, there is a reduced κ-saturated elementary extension $\mathcal{N}' \succ \mathcal{M}_e$. Then $\mathcal{N}'$ is equal to $\mathcal{N}_e$ where $\mathcal{N}$ is the V-part of $\mathcal{N}'$, and $\mathcal{N} \succ \mathcal{M}$. Since $\mathcal{N}_e$ is κ-saturated, some $c \in N$ satisfies $\Gamma(x)$ in $(\mathcal{N}_e)_A$. Since $\mathcal{M}$ is κ-saturated, there exists $b \in M$ such that $\mathrm{tp}_{\mathcal{M}}(b/A) = \mathrm{tp}_{\mathcal{N}}(c/A)$, so $\mathrm{tp}_{\mathcal{N}}(b/A) = \mathrm{tp}_{\mathcal{N}}(c/A)$. By Corollary 4.1.4, $\mathrm{tp}_{\mathcal{N}_e}(b/A) = \mathrm{tp}_{\mathcal{N}_e}(c/A)$. Therefore b satisfies $\Gamma(x)$ in $\mathcal{N}_e$, and hence also satisfies $\Gamma(x)$ in $\mathcal{M}_e$. $\square$

Corollary 4.1.7. *$\mathcal{M}$ is κ-special if and only if $\mathcal{M}_e$ is κ-special.*

Proof. By Corollary 4.1.3 and Proposition 4.1.6. $\square$

4.2. **Definable Predicates.** The notion of a definable predicate in $\mathcal{M}$ was introduced in Definition 2.3.1. In Proposition 4.2.2 below we will show that the general property [P is a definable predicate] is absolute. The next lemma will be used several times in this paper.

Lemma 4.2.1. *Let T_e be a pre-metric expansion of T. For every V_D-formula $\varphi(\vec{x})$, there is a Cauchy sequence of V-formulas $\langle \varphi_m(\vec{x}) \rangle_{m \in \mathbb{N}}$ in T such that for every general model $\mathcal{M}$ of T, $\varphi^{\mathcal{M}_e} = [\lim \varphi_m]^{\mathcal{M}}$. Hence $\varphi^{\mathcal{M}_e}$ is definable in $\mathcal{M}$.*

Proof. Let $\langle d \rangle = \langle d_m(u,v) \rangle_{m \in \mathbb{N}}$ be an approximate distance for T_e, such that no bound variable in $d_m(u,v)$ occurs in $\varphi(\vec{x})$. Let Ψ be the set of all subformulas of $\varphi(\vec{x})$. For every $\psi \in \Psi$, let ψ_m be the V-formula obtained by replacing every subformula of ψ of the form $D(\sigma, \tau)$ by $d_m(\sigma, \tau)$, where σ, τ are V-terms. It then follows by induction on complexity that for every $\psi \in \Psi$ we have:

$$\langle \psi_m^{\mathcal{M}} \rangle_{m \in \mathbb{N}} \text{ is Cauchy in } T \text{ and for each } \mathcal{M} \models T, \psi^{\mathcal{M}_e} = [\lim \psi_m]^{\mathcal{M}}.$$

In particular, this holds when $\psi = \varphi(\vec{x})$, as required. $\qquad \square$

Proposition 4.2.2. *Let $\mathsf{P} \colon M^k \to [0,1]$. The general property [$\mathsf{P}$ is a definable predicate] is absolute.*

Proof. It is clear that if P is definable in $\mathcal{M}$, then P is definable in $\mathcal{M}_e$. Suppose that P is definable in $\mathcal{M}_e$. Then $\mathsf{P} = [\lim \varphi_m]^{\mathcal{M}_e}$ for some sequence $\langle \varphi_m \rangle_{m \in \mathbb{N}}$ of V_D-formulas that is Cauchy in $\mathrm{Th}(\mathcal{M}_e)$. By Lemma 4.2.1, each $\varphi_m^{\mathcal{M}_e}$ is definable in $\mathcal{M}$, so for each m there is a V-formula $\psi_m(\vec{x})$ such that

$$(\forall \vec{b} \in M^k)|\psi_m^{\mathcal{M}}(\vec{b}) - \varphi_m^{\mathcal{M}_e}(\vec{b})| \leq 2^{-m}.$$

Then $\mathsf{P} = [\lim \varphi_m]^{\mathcal{M}_e} = [\lim \psi_m]^{\mathcal{M}}$, so P is definable in $\mathcal{M}$. $\qquad \square$

The following corollary shows that when we add countably many predicates that are definable in $\mathcal{M}$ to a pre-metric expansion of $\mathcal{M}$, we still have a pre-metric expansion.

Corollary 4.2.3. *Suppose $V' = V \cup W$ where W is a countable set of new predicate symbols, $\mathcal{M}$ is a V-structure, $\mathcal{M}_e$ is a pre-metric expansion of $\mathcal{M}$, and $\mathcal{M}' = (\mathcal{M}, P^{\mathcal{M}'})_{P \in W}$ is a V'-structure such that $P^{\mathcal{M}'}$ is a definable predicate in $\mathcal{M}$ for each $P \in W$. Then the V'_D-structure $(\mathcal{M}_e, P^{\mathcal{M}'})_{P \in W}$ is a pre-metric expansion of $\mathcal{M}'$.*

Proof. Let $\langle d \rangle = \langle d_m \rangle_{m \in \mathbb{N}}$ be an approximate distance for T_e. By Lemma 2.5.5, for each predicate symbol $P \in W$, $P^{\mathcal{M}'}$ is uniformly continuous with respect to $D^{\mathcal{M}_e}$ with some modulus of uniform continuity $\triangle_P$. Let L_f agree with L_e on V and give each new predicate symbol $P \in W$ the modulus of uniform continuity $\triangle_P$. Then

$$\mathcal{M}'_f := (\mathcal{M}', [\lim d_m]^{\mathcal{M}'}) = (\mathcal{M}', D^{\mathcal{M}_e}) = (\mathcal{M}_e, P^{\mathcal{M}'})_{P \in W}$$

is a pre-metric expansion of $\mathcal{M}'$ with signature L_f. $\qquad \square$

Given a set of parameters $A \subseteq M$, we say that a mapping $\mathsf{P} \colon M^k \to [0,1]$ is a *definable predicate over A* in $\mathcal{M}$ if P is a definable predicate in $\mathcal{M}_A$. In view of Remark 3.1.4, all of the results in this section hold for definable predicates over A in $\mathcal{M}$.

Definition 4.2.4. We say that a general structure $\mathcal{N}$ whose vocabulary W contains V *admits quantifier elimination over V* if for every W-formula $\varphi(\vec{x})$, $\varphi^{\mathcal{N}}$ is defined by a sequence of quantifier-free V-formulas in $\mathcal{N}$.

Note that admitting quantifier elimination over V is a stronger property than admitting quantifier elimination over V_D.

Corollary 4.2.5.

(i) $\mathcal{M}$ *admits quantifier elimination over* V *if and only if* $\mathcal{M}_e$ *admits quantifier elimination over* V.

(ii) *If* $D^{\mathcal{M}_e}(x,y)$ *is defined in* $\mathcal{M}_e$ *by a sequence of quantifier-free* V*-formulas, and* $\mathcal{M}_e$ *admits elimination of quantifiers over* V_D, *then* $\mathcal{M}$ *admits elimination of quantifiers over* V.

Proof. This follows easily from Lemma 4.2.1.   □

4.3. **Topological and Uniform Properties.** The following definition was given in [BBHU] for metric structures, but makes sense for all general structures.

Definition 4.3.1. A set $C \subseteq M^k$ is said to be *type-defined* by $\Phi(\vec{x})$ in a general structure $\mathcal{M}$, and that C is *type-definable in* $\mathcal{M}$, if $\Phi(\vec{x})$ is a set of formulas in the vocabulary of $\mathcal{M}$ with parameters in M, and

$$C = \{\vec{c} \in M^k \colon \mathcal{M} \models \Phi(\vec{c})\}.$$

Note that type definability in $\mathcal{M}$ is preserved under finite unions and arbitrary intersections.

Definition 4.3.2. Let $\mathcal{N}$ be a pre-metric structure with distinguished distance predicate d. A set $C \subseteq N^k$ is *closed* in $\mathcal{N}$ if it is closed with respect to the pseudo-metric $d(\vec{x}, \vec{y}) = \max_{i \leq k} d(x_i, y_i)$ on N^k.

Lemma 4.3.3. *Let* $\mathcal{N}$ *be a pre-metric structure. A set* $C \subseteq N^k$ *is closed in* $\mathcal{N}$ *if and only if* C *is type-definable in* $\mathcal{N}$.

Proof. Assume $C \subseteq N^k$ is closed in $\mathcal{N}$. For each $\vec{b} \in N^k \backslash C$ let $\varepsilon_{\vec{b}} = \inf_{\vec{c} \in C} \max_{i \leq k} d(b_i, c_i)$, so $\varepsilon_{\vec{b}} > 0$. Then C is type-defined in $\mathcal{N}$ by the set of formulas

$$\Phi(\vec{x}) = \{\varepsilon_{\vec{b}} \lesssim \max_{i \leq k} d(b_i, x_i) \colon \vec{b} \in N^k \setminus C\}.$$

Now suppose C is type-defined in $\mathcal{N}$ by some set of formulas $\Psi(\vec{x})$. By Fact 2.5.2, for each $\psi(\vec{x}) \in \Psi(\vec{x})$, the set $\{\vec{c} \in N^k \colon \mathcal{N} \models \psi(\vec{c})\}$ is closed in $\mathcal{N}$. Therefore C is closed in $\mathcal{N}$.   □

Proposition 4.3.4. *The property of a set* $C \subseteq M^k$ *being closed in* $\mathcal{M}$ *is absolute.* C *is closed in* $\mathcal{M}$ *if and only if* C *is type-definable in* $\mathcal{M}$.

Proof. By taking a subsequence if necessary, we can find an approximate distance $\langle d_m \rangle_{m \in \mathbb{N}}$ for T_e such that

$$T_e \models \sup_x \sup_y |d_m(x,y) - D(x,y)| \lesssim 2^{-m}$$

for each m. Let $C \subseteq M^k$. By Corollary 3.4.2 and Lemma 4.3.3, it suffices to prove that C is type-definable in $\mathcal{M}_e$ if and only if C is type-definable in $\mathcal{M}$. It is clear that if C is type-definable in $\mathcal{M}$ then C is type-definable in $\mathcal{M}_e$

Suppose C is type-defined in $\mathcal{M}_e$ by $\Phi(\vec{x})$. Let $\Psi(\vec{x})$ be the set of all V-formulas $\psi(\vec{x})$ with parameters in M such that $C \subseteq \psi^{\mathcal{M}_e}$, and let $B = \{\vec{b} \colon \mathcal{M}_e \models \Psi(\vec{b})\}$. Then $B = \{\vec{b} \colon \mathcal{M} \models \Psi(\vec{b})\}$, so B is type-definable in $\mathcal{M}$. To prove that C is type-definable in $\mathcal{M}$ we show that $B = C$. Clearly $C \subseteq B$. Let $\vec{b} \in B \setminus C$. By Lemma 4.3.3, C is closed in $\mathcal{M}_e$, so there is an $\varepsilon > 0$ such that $\varepsilon \leq \max_{i \leq k} D^{\mathcal{M}_e}(b_i, c_i)$ for

all $\vec{c} \in C$. Hence for each $m \in \mathbb{N}$, we have $\varepsilon \leq (\max_{i \leq k} d_m^{\mathcal{M}}(b_i, c_i) + 2^{-m})$ for all $\vec{c} \in C$. Therefore the V-formula

$$\varepsilon \dot{\leq} (\max_{i \leq k} d_m(b_i, x_i)) \dot{+} 2^{-m})$$

belongs to $\Psi(\vec{x})$. But then

$$\mathcal{M}_e \models (\varepsilon \dot{\leq} \max_{i \leq k} d_m(b_i, b_i)) \dot{+} 2^{-m}$$

for each $m \in \mathbb{N}$, which contradicts the fact that $D(x, y) = \lim_{m \to \infty} d_m(x, y)$ in $\mathcal{M}_e$. □

It follows that every pre-metric expansion $\mathcal{M}_e$ has the same topology as $\mathcal{M}$. Hence all topological properties of subsets $C \subseteq M^k$ or sequences in M^k are absolute. For example, the properties that C is dense, that C is compact, and that $\lim_{n \to \infty} c_n \dot{=} c$, are absolute. We let $\mathrm{cl}_{\mathcal{M}}(C)$ denote the closure of C in $\mathcal{M}$. Then $\mathrm{cl}_{\mathcal{M}}(C) = \mathrm{cl}_{\mathcal{M}_e}(C)$.

We now obtain absolute versions of properties related to uniform convergence.

Proposition 4.3.5. *The property $[\langle c_n \rangle_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{M}]$ is absolute. A sequence $\langle c_n \rangle_{n \in \mathbb{N}}$ is Cauchy in a general structure $\mathcal{M}$ if and only if $\langle c_n \rangle_{n \in \mathbb{N}}$ converges to some point in some elementary extension $\mathcal{M}' \succ \mathcal{M}$.*

Proof. It suffices to show that the following are equivalent:

(a) $\langle c_n \rangle_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{M}_e$.
(b) $\langle c_n \rangle_{n \in \mathbb{N}}$ converges in some elementary extension of $\mathcal{M}_e$.
(c) $\langle c_n \rangle_{n \in \mathbb{N}}$ converges in some elementary extension of $\mathcal{M}$.

It is clear that (a) $\Leftrightarrow$ (b).

(b) $\Rightarrow$ (c): Assume (b). Then there is a general structure $\mathcal{N}$ and a point $c \in N$ such that $\mathcal{N}_e \succ \mathcal{M}_e$ and $\lim_{n \to \infty} c_n \dot{=} c$ in $\mathcal{N}_e$. By Corollary 4.1.3, $\mathcal{N} \succ \mathcal{M}$. By absoluteness of convergence, $\lim_{n \to \infty} c_n \dot{=} c$ in $\mathcal{N}$, so (c) holds..

(c) $\Rightarrow$ (b): Suppose $\mathcal{N} \succ \mathcal{M}$, $c \in N$, and $\lim_{n \to \infty} c_n \dot{=} c$ in $\mathcal{N}$. Since convergence of a sequence is absolute, $\lim_{n \to \infty} c_n \dot{=} c$ in $\mathcal{N}_e$. By Corollary 4.1.3, $\mathcal{N}_e \succ \mathcal{M}_e$, so (b) holds. □

We say that a general structure $\mathcal{M}$ is *complete* if every pre-metric expansion $\mathcal{M}_e$ of $\mathcal{M}$ is a metric structure.

Corollary 4.3.6. *The property of being complete is absolute. A general structure $\mathcal{M}$ is complete if and only if $\mathcal{M}$ is reduced and every Cauchy sequence in $\mathcal{M}$ converges to some point in M.*

Corollary 4.3.7. *Every $\aleph_1$-saturated reduced structure is complete.*

Proof. Suppose $\mathcal{M}$ is reduced and $\aleph_1$-saturated. By Lemma 3.1.5 (ii) and Proposition 4.1.6, $\mathcal{M}_e$ is a reduced $\aleph_1$-saturated pre-metric structure, and hence is complete. Let $\langle c_n \rangle_{n \in \mathbb{N}}$ be Cauchy in $\mathcal{M}$. By Proposition 4.3.5, $\langle c_n \rangle_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{M}_e$. Therefore $\langle c_n \rangle_{n \in \mathbb{N}}$ converges to some point c in $\mathcal{M}_e$. By the absoluteness of convergence, $\langle c_n \rangle_{n \in \mathbb{N}}$ converges to some point c in $\mathcal{M}$. □

Recall that if $\mathcal{M}$ is a pre-metric structure, then a completion of $\mathcal{M}$ is a metric structure $\mathcal{N}$ such that the reduction of $\mathcal{M}$ is a dense elementary substructure of $\mathcal{N}$.

Definition 4.3.8. We say that a general structure $\mathcal{N}$ is a *completion* of a general structure $\mathcal{M}$ if $\mathcal{N}$ is complete, and the reduction of $\mathcal{M}$ is a dense elementary substructure of $\mathcal{N}$.

The elementary substructure requirement in the above definition ensures that the approximate distances are preserved when passing from $\mathcal{M}$ to $\mathcal{N}$.

Corollary 4.3.9. *Let $\mathcal{M}, \mathcal{N}$ be general models of T and let T_e be a pre-metric expansion of T. Then $\mathcal{N}$ is a completion of $\mathcal{M}$ if and only if $\mathcal{N}_e$ is a completion of $\mathcal{M}_e$.*

Proof. Let $\mathcal{M}'$ be the reduction of $\mathcal{M}$. By Lemma 3.1.5 (ii), $\mathcal{M}'_e$ is the reduction of $\mathcal{M}_e$. By Corollary 4.3.6, $\mathcal{N}_e$ is reduced and complete if and only if $\mathcal{N}$ is reduced and complete. By Proposition 4.3.4, $\mathcal{M}'_e$ is dense in $\mathcal{N}_e$ if and only if $\mathcal{M}'$ is dense in $\mathcal{N}$. By Corollary 4.1.3, $\mathcal{M}'_e \prec \mathcal{N}_e$ if and only if $\mathcal{M}' \prec \mathcal{N}$. $\square$

Proposition 4.3.10. *Every general structure $\mathcal{M}$ has a completion, which is unique up to an isomorphism that is the identity on the reduction of $\mathcal{M}$.*

Proof. By Theorem 3.3.4, there is a pre-metric expansion T_e of $\mathrm{Th}(\mathcal{M})$. Let $\mathcal{M}'$ be the reduction of $\mathcal{M}$. Then $\mathcal{M}_e$ is a pre-metric structure, $\mathcal{M}'_e$ is the reduction of $\mathcal{M}_e$, and there exists a completion $\mathcal{N}_1$ of $\mathcal{M}_e$ that is unique up to an isomorphism that is the identity on M'. Moreover, $\mathcal{N}_1 \succ \mathcal{M}'_e$. Then $\mathcal{N}_1 = \mathcal{N}_e$ where $\mathcal{N}$ is the V-part of $\mathcal{N}_1$. By Corollary 4.3.9, $\mathcal{N}$ is a completion of $\mathcal{M}$.

If $\mathcal{N}'$ is another completion of $\mathcal{M}$, then by Corollary 4.3.9, $\mathcal{N}'_e$ is a completion of $\mathcal{M}_e$, so there is an isomorphism $h \colon \mathcal{N}_e \cong \mathcal{N}'_e$ that is the identity on M'. Then $h \colon \mathcal{N} \cong \mathcal{N}'$, as required. $\square$

Let κ be an infinite cardinal. A complete metric theory T is κ-categorical if every two complete models of T of density character κ are isomorphic.

Corollary 4.3.11. *The property of having a κ-categorical complete theory is absolute. A complete general theory T is κ-categorical if and only if every two complete general models of T of density character κ are isomorphic.*

Proof. By Proposition 4.3.4 (closed is absolute), Corollary 4.3.6 (being complete is absolute), and Lemma 3.1.5 (iii) (being isomorphic is absolute). $\square$

4.4. **Infinitary Continuous Logic.** We return to the infinitary continuous formulas that were introduced in [Ea15] and discussed in Example 2.6.1. We assume in this subsection that $|V| \leq \aleph_0$. $\mathcal{L}_{\omega_1\omega}(V)$ denotes the set of all continuous $\mathcal{L}_{\omega_1\omega}$-formulas over the vocabulary V.

Lemma 4.4.1. *Let T_e be a pre-metric expansion of a V-theory T. For every formula $\psi(\vec{x}) \in \mathcal{L}_{\omega_1\omega}(V_D)$, there is a formula $\varphi(\vec{x}) \in \mathcal{L}_{\omega_1\omega}(V)$ such that $\psi^{\mathcal{M}_e} = \varphi^{\mathcal{M}}$ for every general V-model $\mathcal{M}$ of T.*

Proof. The pre-metric expansion T_e has an exponentially Cauchy approximate distance $\langle d_m(u,v) \rangle_{m \in \mathbb{N}}$ such that no bound variable in $d_m(u,v)$ occurs in $\psi(\vec{x})$ or in $\vec{z}$. Then for each m and pair of V-terms $\sigma(\vec{z}), \tau(\vec{z})$, $d_m(\sigma(\vec{z}), \tau(\vec{z}))$ is a V-formula, and for every general model $\mathcal{M}$ of T and tuple $\vec{z}$ in M we have

$$D^{\mathcal{M}_e}(\sigma(\vec{z}), \tau(\vec{z})) - 2^{-m} \leq d_m^{\mathcal{M}}(\sigma(\vec{z}), \tau(\vec{z})) \leq D^{\mathcal{M}_e}(\sigma(\vec{z}), \tau(\vec{z})) + 2^{-m}.$$

It follows that

$$D^{\mathcal{M}_e}(\sigma(\vec{z}), \tau(\vec{z})) = \inf_m \sup_k d_{m+k}^{\mathcal{M}}(\sigma(\vec{z}), \tau(\vec{z})) = \sup_m \inf_k d_{m+k}^{\mathcal{M}}(\sigma(\vec{z}), \tau(\vec{z})).$$

Let $\psi \in \mathcal{L}_{\omega_1\omega}(V_D)$, and let φ be the $\mathcal{L}_{\omega_1\omega}$-formula in the vocabulary V obtained by replacing each atomic subformula of ψ of the form $D(\sigma(\vec{z}), \tau(\vec{z}))$ by

$\inf_m \sup_k d_{m+k}(\sigma(\vec{z}), \tau(\vec{z}))$. It follows by induction on the complexity of ψ that $\psi^{\mathcal{M}_e} = \varphi^{\mathcal{M}}$ for every general V-model $\mathcal{M}$ of T. $\qquad\square$

We say that a mapping $\mathsf{P} \colon M^k \to [0,1]$ is $\mathcal{L}_{\omega_1\omega}$-*definable in* a general structure $\mathcal{M}$ if $\mathsf{P} = \varphi^{\mathcal{M}}$ for some $\mathcal{L}_{\omega_1\omega}$-formula $\varphi(\vec{x})$ with $|\vec{x}| = k$ in the vocabulary of $\mathcal{M}$. It is easily seen that if P is definable in $\mathcal{M}$ then P is $\mathcal{L}_{\omega_1\omega}$-definable in $\mathcal{M}$.

Proposition 4.4.2. *The property* $[\mathsf{P}$ *is* $\mathcal{L}_{\omega_1\omega}$-definable in $\mathcal{M}]$ *is absolute.*

Proof. Let $\mathcal{M}_e$ be a pre-metric expansion of $\mathcal{M}$. It is easily seen by induction on complexity that for each $\mathcal{L}_{\omega_1\omega}$-formula $\varphi(\vec{x})$ in the vocabulary V of $\mathcal{M}$ we have $\varphi^{\mathcal{M}} = \varphi^{\mathcal{M}_e}$, so $\mathcal{L}_{\omega_1\omega}$-definability in $\mathcal{M}$ implies $\mathcal{L}_{\omega_1\omega}$-definability in $\mathcal{M}_e$. Conversely, by Lemma 4.4.1, $\mathcal{L}_{\omega_1\omega}$-definability in $\mathcal{M}_e$ implies $\mathcal{L}_{\omega_1\omega}$-definability in $\mathcal{M}$. $\qquad\square$

We now generalize several results in [Ea15] from metric theories to general theories.

Proposition 4.4.3. *Let $\mathcal{M}$ be a separable complete V-structure. There is an $\mathcal{L}_{\omega_1\omega}$-sentence φ in the vocabulary V, called a* Scott sentence *of $\mathcal{M}$, such that for every separable complete V-structure $\mathcal{N}$, $\varphi^{\mathcal{N}} = 0$ if $\mathcal{M} \cong \mathcal{N}$ and $\varphi^{\mathcal{N}} = 1$ otherwise.*

Proof. By Theorem 3.2.1 of [Ea15] (which follows from [BDNT]), the result holds for metric structures. By the Expansion Theorem 3.3.4, the empty V-theory T has a pre-metric expansion T_e with signature L_e. By the preceding section, $\mathcal{M}_e$ is a separable complete pre-metric structure, and hence is a metric structure with signature L_e. Therefore there is an $\mathcal{L}_{\omega_1\omega}$-sentence ψ in the vocabulary V_D such that for every separable metric structure $\mathcal{N}'$ with signature L_e, $\psi^{\mathcal{N}'} = 0$ if $\mathcal{M}_e \cong \mathcal{N}'$, and $\psi^{\mathcal{N}'} = 1$ otherwise. By Lemma 4.4.1, there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ in the vocabulary V such that $\psi^{\mathcal{N}_e} = \varphi^{\mathcal{N}}$ for every V-structure $\mathcal{N}$. Moreover, every metric structure $\mathcal{N}'$ with signature L_e is a general model of T_e, and hence is equal to $\mathcal{N}_e$ where $\mathcal{N}$ is the V-part of $\mathcal{N}'$, and $\mathcal{N}$ is a separable complete V-structure. By Lemma 3.1.5 (iii), $\mathcal{M} \cong \mathcal{N}$ if and only if $\mathcal{M}_e \cong \mathcal{N}_e$. Therefore φ has the required property that for every separable complete V-structure $\mathcal{N}$, $\varphi^{\mathcal{N}} = 0$ if $\mathcal{M} \cong \mathcal{N}$ and $\varphi^{\mathcal{N}} = 1$ otherwise. $\qquad\square$

Proposition 4.4.4. *Let $\mathcal{M}$ be a separable complete general structure and let P be a mapping from M^n into $[0,1]$. The following are equivalent:*

(i) P *is $\mathcal{L}_{\omega_1\omega}$-definable in $\mathcal{M}$.*
(ii) P *is fixed by all automorphisms of $\mathcal{M}$.*

Proof. By Theorem 3.2.3 of [Ea15], the result holds for metric structures. By the preceding section, Lemma 3.1.5 (iii), and Proposition 4.4.2, both (i) and (ii) are absolute for complete separable general structures. $\qquad\square$

Proposition 4.4.7 below extends the Omitting Types Theorem 3.3.4 of [Ea15] to general theories T. The result in [Ea15] is about reduced pre-metric structures, which are pre-metric structures whose distinguished distance is a metric, but is not necessarily complete.

We first need the notion of a strong countable fragment of $\mathcal{L}_{\omega_1\omega}(V)$. By [BBHU], Section 6, we may fix a countable set $\mathcal{F}$ of connectives such that $\mathcal{F}$ is closed under composition and projections, and for each $n \in \mathbb{N}$ the set of n-ary connectives $C \in \mathcal{F}$ is uniformly dense in the set of all n-ary connectives. We also assume that $\mathcal{F}$ contains the connective $\dot{\leq}$ and each rational constant in $[0,1]$. Following [Ea15], by a *countable fragment* of $\mathcal{L}_{\omega_1\omega}(V)$ we mean a countable set $\mathcal{L} \subseteq \mathcal{L}_{\omega_1\omega}(V)$ that contains

the set of atomic formulas over V, and is closed under the connectives in $\mathcal{F}$, $\sup_x$, $\inf_x$, subformulas, and substituting terms for free variables. The smallest countable fragment of $\mathcal{L}_{\omega_1\omega}(V)$ is the set of all (finitary) V-formulas with connectives in $\mathcal{F}$.

Definition 4.4.5. We say that $\mathcal{L}$ is a *strong countable fragment of* $\mathcal{L}_{\omega_1\omega}(V)$ if there is a countable fragment $\mathcal{L}_D$ of $\mathcal{L}_{\omega_1\omega}(V_D)$ such that $\mathcal{L} = \mathcal{L}_D \cap \mathcal{L}_{\omega_1\omega}(V)$, and for each finitary V-formula $\theta(x, y)$ with connectives in $\mathcal{F}$ and pair of V-terms $\sigma(\vec{u}), \tau(\vec{v})$, $\mathcal{L}_D$ is closed under the operation of replacing each subformula of the form $D(\sigma(\vec{u}), \tau(\vec{v}))$ by $\theta(\sigma(\vec{u}), \tau(\vec{v}))$.

Clearly, every strong countable fragment of $\mathcal{L}_{\omega_1\omega}(V)$ is a countable fragment of $\mathcal{L}_{\omega_1\omega}(V)$. Also, the smallest countable fragment of $\mathcal{L}_{\omega_1\omega}(V)$ is a strong countable fragment of $\mathcal{L}_{\omega_1\omega}(V)$. It is easily seen that every countable subset of $\mathcal{L}_{\omega_1\omega}(V)$ is contained in a strong countable fragment of $\mathcal{L}_{\omega_1\omega}(V)$.

Definition 4.4.6. Let $\mathcal{L}$ be a countable fragment of $\mathcal{L}_{\omega_1\omega}(V)$, and let T be a set of sentences in $\mathcal{L}$. We say that a set $\Sigma(\vec{x}) \subseteq \mathcal{L}$ is *principal* over $(T, \mathcal{L})$ if there is a formula $\varphi(\vec{x}) \in \mathcal{L}$, an $|\vec{x}|$-tuple of V-terms $\vec{\tau}(\vec{y})$, and a rational $r \in (0, 1)$, such that:

- $T \cup \{\varphi(\vec{y})\}$ is satisfiable,
- $T \cup \{\varphi(\vec{y}) \lesssim r\} \models \Sigma(\vec{\tau}(\vec{y}))$.

Proposition 4.4.7. *Let $\mathcal{L}$ be a strong countable fragment of $\mathcal{L}_{\omega_1\omega}(V)$. Let T be a set of sentences in $\mathcal{L}$, and for each $n \in \mathbb{N}$, suppose $\Sigma_n(\vec{x}_n) \subseteq \mathcal{L}$ but $\Sigma_n(\vec{x}_n)$ is not principal over $(T, \mathcal{L})$. Then there is a reduced separable model of T in which none of the sets $\Sigma_n(\vec{x}_n)$ is realized.*

Proof. By hypothesis, $\mathcal{L} = \mathcal{L}_D \cap \mathcal{L}_{\omega_1\omega}(V)$ for some countable fragment $\mathcal{L}_D$ of $\mathcal{L}_{\omega_1\omega}(V_D)$ as in Definition 4.4.5. Let U_e be a pre-metric expansion of the empty V-theory U. Then every pre-metric model of $T \cup U_e$ is of the form $\mathcal{M}_e$ for some general model $\mathcal{M}$ of T.

Claim. For each $n \in \mathbb{N}$, $\Sigma_n(\vec{x}_n)$ is not principal over $(T \cup U_e, \mathcal{L}_D)$.

To prove this Claim, suppose that $\Sigma_n(\vec{x}_n)$ is principal over $(T \cup U_e, \mathcal{L}_D)$, witnessed by $\varphi(\vec{x}_n) \in \mathcal{L}_D$, a tuple of V-terms $\vec{\tau}(\vec{y})$, and a rational $r \in (0, 1)$. By approximating the connectives by connectives in $\mathcal{F}$ and taking a subsequence, we see that U_e has an exponentially Cauchy approximate distance $\langle d_m \rangle_{m \in \mathbb{N}}$ where each d_m is a finitary continuous formula that belongs to $\mathcal{L}$. Hence by Fact 2.5.3 and Lemma 3.1.2, we may take U_e to be a set of sentences of $\mathcal{L}_D$. For each $m \in \mathbb{N}$, let $\varphi_m(\vec{x})$ be the formula obtained from $\varphi(\vec{x})$ by replacing every subformula of φ of the form $D(\sigma_1(\vec{x}), \sigma_2(\vec{x}))$ by $d_m(\sigma_1(\vec{x}), \sigma_2(\vec{x}))$. Then $\varphi_m \in \mathcal{L}_D \cap \mathcal{L}_{\omega_1\omega}(V) = \mathcal{L}$. It follows by induction on the complexity of formulas that for each $\varepsilon > 0$ there is an $m \in \mathbb{N}$ such that for all $k \geq m$, $\mathcal{N} \models T_e$, and $\vec{b} \in N^{|\vec{x}|}$, $\varphi_m^{\mathcal{N}}(\vec{b})$ is within ε of $\varphi^{\mathcal{N}}(\vec{b})$. Taking $\varepsilon = r/2$, we obtain a formula $\varphi_m(\vec{x}) \in \mathcal{L}$ such that $\varphi_m(\vec{x}), \vec{\tau}(\vec{y})$, and $r/2$ witness that $\Sigma_n(\vec{x})$ is principal over $(T, \mathcal{L})$. This contradicts our hypothesis that $\Sigma_n(\vec{x}_n)$ is not principal over $(T, \mathcal{L})$, and proves the Claim.

Now, by Theorem 3.3.4 of [Ea15], there is a reduced separable pre-metric model $\mathcal{N}$ of $T \cup U_e$ in which none of the sets Σ_n is realized. Then the V-part of $\mathcal{N}$ is a reduced separable model of T in which none of the sets Σ_n is realized. $\square$

Since the set $\mathcal{L}_0(V)$ of all V-formulas built from connectives in $\mathcal{F}$ is a strong countable fragment of $L_{\omega_1\omega}(V)$, we have:

Corollary 4.4.8. *Let $T \subseteq \mathcal{L}_0(V)$, and for each $\in \mathbb{N}$, let $\Sigma_n(\vec{x}_n)$ be a non-principal subset of $\mathcal{L}_0(V)$ over $(T, \mathcal{L}_0(V))$. Then there is a reduced separable model of T in which none of the sets $\Sigma_n(\vec{x}_n)$ is realized.*

4.5. **Many-sorted Metric Structures.** In Definition 2.6.5 we defined a sorted metric structure $\mathcal{M}$ with sorted vocabulary W to be a reduced structure that respects the sorts of W and has a complete metric $d_{\mathbb{S}}$ on each sort $\mathbb{S}$. As a sorted vocabulary, W must have an unsorted constant symbol u and a unary predicate symbol $U_{\mathbb{S}}$ for each sort $\mathbb{S}$.

Definition 4.5.1. We say that $\mathrm{Th}(\mathcal{M})$, $\mathrm{Th}(\mathcal{M}_e)$ have *essentially the same types* if there is a homeomorphism h from $S_n(\mathrm{Th}(\mathcal{M}))$ onto $S_n(\mathrm{Th}(\mathcal{M}_e))$ such that for each $\vec{c} \in M^n$, $h(\mathrm{tp}_{\mathcal{M}}(\vec{c})) = \mathrm{tp}_{\mathcal{M}_e}(\vec{c})$.

Proposition 4.5.2. *Let W be a sorted vocabulary with sorts $\mathbb{S}_1, \mathbb{S}_2, \ldots$, and $\mathcal{M}$ be a sorted metric structure over W. Then any pre-metric expansion $\mathcal{M}_e = (\mathcal{M}, \mathsf{D})$ of $\mathcal{M}$ is a metric structure, and $\mathrm{Th}(\mathcal{M})$, $\mathrm{Th}(\mathcal{M}_e)$ have essentially the same types.*

Proof. Note that in $\mathcal{M}_e$, the new distance symbol D does not have sorts assigned to its arguments. Since $\mathcal{M}$ is reduced, $\mathcal{M}_e$ is a reduced pre-metric structure, so D is a metric. Moreover, $u^{\mathcal{M}}$ is the only sortless element by Lemma 2.6.4 (ii). We show that D is complete. Let $\langle d_m(x, y) \rangle_{m \in \mathbb{N}}$ be an approximate distance for $\mathcal{M}_e$. Suppose that $\langle a_k \rangle_{k \in \mathbb{N}}$ is Cauchy convergent in $\mathcal{M}_e$.

Case 1: For some sort $\mathbb{S}_n$, $a_k \in \mathbb{S}_n^{\mathcal{M}}$ for infinitely many k. By taking a subsequence, we may assume that $a_k \in \mathbb{S}_n^{\mathcal{M}}$ for all k. Since $\mathcal{M}_e$ is a pre-metric structure, $d_m^{\mathcal{M}}$ is uniformly continuous with respect to D, so $\langle a_k \rangle_{k \in \mathbb{N}}$ is Cauchy with respect to $d_m^{\mathcal{M}}$. $d_m^{\mathcal{M}}$ is a complete metric on $\mathbb{S}_m^{\mathcal{M}}$, so $\langle a_k \rangle_{k \in \mathbb{N}}$ converges to some point $b \in \mathbb{S}_m^{\mathcal{M}}$ with respect to $d_m^{\mathcal{M}}$. The set $\mathbb{S}_n^{\mathcal{M}}$ is defined by the formula $U_{\mathbb{S}_n}(x)$ in $\mathcal{M}$, and is therefore closed in $\mathcal{M}_e$. By Proposition 4.3.4, the topologies of $\mathcal{M}_e$ and $\mathcal{M}$ restricted to $\mathbb{S}_n^{\mathcal{M}}$ are the same. Since $\mathcal{M}$ is a sorted metric structure, that topology is the metric topology of $d_m^{\mathcal{M}}$ restricted to $\mathbb{S}^{\mathcal{M}}$. Therefore $\langle a_k \rangle_{k \in \mathbb{N}}$ converges to b with respect to D.

Case 2. For each sort $\mathbb{S}_n$, $a_k \notin \mathbb{S}_n^{\mathcal{M}}$ for all but finitely many k. Then $\langle a_k \rangle_{k \in \mathbb{N}}$ converges to some point b in the completion $\mathcal{M}'_e \succ \mathcal{M}_e$. For each n, $\mathbb{S}_n$ is open in $\mathcal{M}'_e$, so $b \notin \mathbb{S}_n^{\mathcal{M}'_e}$. Therefore $b = u^{\mathcal{M}'_e} = u^{\mathcal{M}_e}$. It follows that $\langle a_k \rangle_{k \in \mathbb{N}}$ converges to $u^{\mathcal{M}_e}$ with respect to D.

In both cases, $\langle a_k \rangle_{k \in \mathbb{N}}$ converges to a point with respect to D, so D is complete. The existence of the homeomorphism h follows from Corollary 4.1.5. $\square$

4.6. **Bounded and Unbounded Metric Structures.** Bounded vocabularies, metric structures, and general structures, were discussed in Example 2.6.6. Like the metric case, the model theory of $[0, 1]$-valued general structures carries over to bounded general structures in a routine way. When working with bounded continuous logic, we will freely use the same terminology that we use in the $[0, 1]$-valued case. In particular, the Expansion Theorem shows that every bounded general structure $\mathcal{M}$ has a pre-metric expansion $\mathcal{M}_e = (\mathcal{M}, \mathsf{D})$, which is a bounded pre-metric structure. By normalizing D, one can even get a pre-metric expansion where D has values in $[0, 1]$.

Unbounded metric structures were briefly discussed in Example 2.6.7. We mentioned three ways that an unbounded metric structure $\mathcal{N}$ has been treated in the literature using the model theory of metric structures. One way, as in [BBHU],

[Fa], and [BY09], was to look at the corresponding bounded many-sorted metric structure $\mathcal{M}$ with a sort for the closed m-ball around a distinguished constant 0 for each positive $m \in \mathbb{N}$. $\mathcal{M}$ can be viewed either as a many-sorted structure with a metric in each sort, or as a single sorted general structure with a unary predicate for each sort. By the Expansion Theorem, as a single-sorted general structure, $\mathcal{M}$ has a pre-metric expansion $\mathcal{M}_e$, and by Proposition 4.5.2, $\mathrm{Th}(\mathcal{M}_e)$ has essentially the same types as $\mathrm{Th}(\mathcal{M})$.

The Expansion Theorem opens up additional possibilities, which have not yet been explored in the literature, for using the model theory of metric structures to treat unbounded metric structures. It allows the flexibility to look at general structures that are built in some way from an unbounded metric structure, and then use the Expansion Theorem to get a related metric structure.

For instance, consider an unbounded metric structure $\mathcal{N}$ with a real-valued gauge predicate ν, as in the paper [BY08] that was discussed in Example 2.6.7 above. To study $\mathcal{N}$, one might look at the bounded general structure $\mathcal{N}'$ such that $\mathcal{N}'$ has the same universe and functions as $\mathcal{N}$ plus a point at ∞, and for each predicate P of $\mathcal{N}$ (including d and ν), and positive $m \in \mathbb{N}$, $\mathcal{N}'$ has the predicate P_m formed by truncating P at m, that is,

$$
P_m^{\mathcal{N}'}(\vec{v}) = \begin{cases} -m & \text{if } P^{\mathcal{N}}(\vec{v}) \le -m \\ m & \text{if } P^{\mathcal{N}}(\vec{v}) \ge m \\ P^{\mathcal{N}}(\vec{v}) & \text{otherwise.} \end{cases}
$$

This construction merely truncates the predicates of $\mathcal{N}$ rather than adding sorts for the closed m-balls. Again, $\mathcal{N}'$ will have a pre-metric expansion $\mathcal{N}'_e = (\mathcal{N}', \mathsf{D})$, and the theories of $\mathcal{N}'$ and $\mathcal{N}'_e$ will have essentially the same types. As usual, the distinguished metric D of $\mathcal{N}'_e$ may be wildly behaved and unnatural, but its existence does make the model theory of metric structures available for the study of the unbounded metric structure $\mathcal{N}$.

Note that for any x in $\mathcal{N}$, $\nu_m^{\mathcal{N}'}(x) = m$ whenever $m \le \nu^{\mathcal{N}}(x)$, and $\nu_m^{\mathcal{N}'}(x) = \nu^{\mathcal{N}}(x)$ whenever $m > \nu^{\mathcal{N}}(x)$. So in any model of $\mathrm{Th}(\mathcal{N}')$, $\lim_{m \to \infty} \nu_m(x)$ is finite if and only if $\nu_m(x)$ is eventually constant as $m \to \infty$.

Instead of looking at an arbitrary model of $\mathrm{Th}(\mathcal{N}')$, it may be useful to look at the substructure consisting of those elements y such that $\lim_{m \to \infty} \nu_m(y)$ is finite, plus the point at ∞. This corresponds to looking at the modified pre-ultraproduct of unbounded metric structures that was discussed in Example 2.6.7. The universe of the modified pre-ultraproduct $\mathcal{M} = \prod^{\mathcal{D}} \mathcal{N}_i$ is the set of elements y of the ordinary pre-ultraproduct $\mathcal{M}' = \prod^{\mathcal{D}} \mathcal{N}'_i$ such that the set $\{\nu^{\mathcal{N}_i}(y_i) \colon i \in I\}$ has a finite bound, plus the point at ∞.

To see the connection, one can check that for each element x of $\mathcal{M}'$, the following are equivalent, where $x =_{\mathcal{D}} y$ means $\{i \in I \colon x_i = y_i\} \in \mathcal{D}$:

- For some $y =_{\mathcal{D}} x$, y belongs to $\mathcal{M}$.
- For some $y =_{\mathcal{D}} x$, the set $\{\nu^{\mathcal{N}_i}(y_i) \colon i \in I\}$ has a finite bound.
- For some $y =_{\mathcal{D}} x$, the set $\{\lim_{m \to \infty} \nu_m^{\mathcal{N}'_i}(y_i) \colon i \in I\}$ has a finite bound.
- For some $y =_{\mathcal{D}} x$, $\lim_{m \to \infty} \nu_m^{\mathcal{M}'}(y)$ is finite.

4.7. **Imaginaries.** For each metric structure $\mathcal{M}$, [BU] (Section 5) introduces a sorted metric structure $\mathcal{M}^{eq}$ that has $\mathcal{M}$ as its home sort and infinitely many sorts

of imaginary elements. In a similar way, we will now introduce imaginary elements for any reduced general structure $\mathcal{M}$ with the vocabulary V.

In the following, we let T be a complete V-theory, $\vec{x}$ be a tuple of variables and $\vec{y}$ be a countable sequence of variables, and $\langle\varphi\rangle = \langle\varphi_m(\vec{x}, \vec{y})\rangle_{m\in\mathbb{N}}$ be an exponentially Cauchy sequence of formulas in T. Let $V^{\langle\varphi\rangle}$ be the two-sorted vocabulary obtained from V by adding a *home sort* $\mathbb{S}$, an *imaginary sort* $\mathbb{S}_i$, unary predicate symbols $U_{\mathbb{S}}$ for the home sort and $U_{\mathbb{S}_i}$ for the imaginary sort, a predicate symbol $d_{\langle\varphi\rangle}$ of sort $\mathbb{S}_i \times \mathbb{S}_i$, a predicate symbol $P_{\langle\varphi\rangle}$ of sort $\mathbb{S}^{|\vec{x}|} \times \mathbb{S}_i$, and the unsorted constant symbol u.

Intuitively, the imaginary elements will be equivalence classes of infinite sequences of parameters in the home sort, $P_{\langle\varphi\rangle}(\vec{x}, \vec{y})$ will be $\lim_{m\to\infty} \varphi_m(\vec{x}, \vec{y})$, and $d_{\langle\varphi\rangle}(\vec{y}, \vec{z})$ will be $\lim_{m\to\infty}(\sup_{\vec{x}} |\varphi_m(\vec{x}, \vec{y}) - \varphi_m(\vec{x}, \vec{z})|)$.

Definition 4.7.1. Let $T^{\langle\varphi\rangle}$ be the set of $V^{\langle\varphi\rangle}$-sentences

$$\sup_{wz}[|\sup_{\vec{x}} |P_{\langle\varphi\rangle}(\vec{x}, w) - P_{\langle\varphi\rangle}(\vec{x}, z)| - d_{\langle\varphi\rangle}(w, z)| \lesssim 0],$$

$$\{\sup_z \inf_{\vec{y}} \sup_{\vec{x}}[|\varphi_m(\vec{x}, \vec{y}) - P_{\langle\varphi\rangle}(\vec{x}, z)| \lesssim 2 \cdot 2^{-m} : m \in \mathbb{N}\},$$

$$\{\sup_{\vec{y}} \inf_z \sup_{\vec{x}}[|\varphi_m(\vec{x}, \vec{y}) - P_{\langle\varphi\rangle}(\vec{x}, z)| \lesssim 2 \cdot 2^{-m} : m \in \mathbb{N}\}.$$

$T^{\langle\varphi\rangle}$ is the natural analogue, for general structures, of the theory T_ψ that is defined in [BU].

Proposition 4.7.2. *For every reduced model $\mathcal{M}$ of T, there is, up to isomorphism, a unique reduced model $\mathcal{M}^{\langle\varphi\rangle}$ of $T \cup T^{\langle\varphi\rangle}$ that respects sorts in $V^{\langle\varphi\rangle}$ and agrees with $\mathcal{M}$ in the home sort.*

Proof. The argument is the same as in [BU], pages 29-30. $\square$

The elements of sort $\mathbb{S}_i$ in $\mathcal{M}^{\langle\varphi\rangle}$ are called *imaginary elements*, or *canonical parameters*. By Lemma 2.6.4, $u^{\mathcal{M}^{\langle\varphi\rangle}}$ is the only element without a sort.

Proposition 4.7.3. *For each reduced model $\mathcal{M}$ of T, $d_{\langle\varphi\rangle}$ is a metric on the imaginary sort in $\mathcal{M}^{\langle\varphi\rangle}$.*

Proof. We work in $\mathcal{M}^{\langle\varphi\rangle}$. It is clear from the definition that $d_{\langle\varphi\rangle}$ is the limit of a uniformly convergent sequence of pseudo-metrics on $\mathbb{S}_i$, and hence is itself a pseudo-metric on $\mathbb{S}_i$. Suppose that b, c have sort $\mathbb{S}_i$ and $d_{\langle\varphi\rangle}(b, c) = 0$. Since $\mathcal{M}^{\langle\varphi\rangle}$ is reduced, it suffices to show that $b \doteq c$. Let $\vec{z}$ be a tuple in $M^{\langle\varphi\rangle}$, and let $\alpha(b, \vec{z})$ be an atomic formula. If α begins with a predicate symbol other than $d_{\langle\varphi\rangle}$ or $P_{\langle\varphi\rangle}$ or $U_{\mathbb{S}_i}$, then b and c have the wrong sort, so $\alpha(b, \vec{z}) = \alpha(c, \vec{z}) = 1$.

Suppose α begins with the predicate symbol $d_{\langle\varphi\rangle}$, so $\alpha(b, \vec{z})$ has the form

$$d_{\langle\varphi\rangle}(\sigma(b, \vec{z}), \tau(b, \vec{z}))$$

where σ, τ are terms. No function or constant symbol in $V^{\langle\varphi\rangle}$ has value sort $\mathbb{S}_i$. So if either σ or τ starts with a function or constant symbol, then $\alpha(b, \vec{z}) = \alpha(c, \vec{z}) = 1$. The only other possibilities are that for some variable z_0, $\alpha(b, \vec{z})$ is either $d_{\langle\varphi\rangle}(b, b)$, $d_{\langle\varphi\rangle}(z_0, z_0)$, $d_{\langle\varphi\rangle}(b, z_0)$, or $d_{\langle\varphi\rangle}(z_0, b)$. In each of those cases, since $d_{\langle\varphi\rangle}(b, c) = 0$, we have $\alpha(b, \vec{z}) = \alpha(c, \vec{z})$.

If α begins with the predicate symbol $P_{\langle\varphi\rangle}$ or $U_{\mathbb{S}_i}$, then an argument that is similar to the preceding paragraph again shows that $\alpha(b, \vec{z}) = \alpha(c, \vec{z})$. Thus in all cases we have $\alpha(b, \vec{z}) = \alpha(b, \vec{z})$ and hence $b \doteq c$. $\square$

Proposition 4.7.4. *If $\mathcal{M}$ is κ-special, then $\mathcal{M}^{\langle\varphi\rangle}$ is κ-special. If in addition κ has uncountable cofinality, then $d_{\langle\varphi\rangle}$ is a complete metric on the imaginary sort.*

Proof. By the Existence Theorem for Special Models, there is a κ-special model $\mathcal{N}$ of $T\cup T^{\langle\varphi\rangle}$. It is easily checked that the V-part $\mathcal{M}'$ of $\mathcal{N}$ is κ-special, and $\mathcal{N}=\mathcal{M}'^{\langle\varphi\rangle}$. By the Uniqueness Theorem for Special Models, $\mathcal{M}'\cong\mathcal{M}$, so $\mathcal{M}'^{\langle\varphi\rangle}\cong\mathcal{M}^{\langle\varphi\rangle}$. If κ has uncountable cofinality, then $\mathcal{M}^{\langle\varphi\rangle}$ is $\aleph_1$-saturated, and as in Remark 2.5.1, it follows that $d_{\langle\varphi\rangle}$ is a complete metric on the imaginary sort. $\qquad\square$

With more work, one can show that if $\mathcal{M}$ is κ^+-saturated, then $\mathcal{M}^{\langle\varphi\rangle}$ is κ^+-saturated, but we will not need that fact here.

Definition 4.7.5. We say that two exponentially Cauchy sequences $\langle\varphi\rangle$ and $\langle\psi\rangle$ are *equivalent* in T if $[\lim\varphi_m]^{\mathcal{M}}=[\lim\psi_m]^{\mathcal{M}}$ for all general models $\mathcal{M}$ of T.

Remark 4.7.6. *If $\mathcal{M}$ is a reduced model of T and $\langle\varphi\rangle$ and $\langle\psi\rangle$ are equivalent exponentially Cauchy sequences in T, then $\mathcal{M}^{\langle\varphi\rangle}=\mathcal{M}^{\langle\psi\rangle}$, in the sense that they have the same universes and the identity function is an isomorphism between them.*

We now define the sorted vocabulary V^{eq} and structure $\mathcal{M}^{eq}$. Let Φ be a set of exponentially Cauchy sequences $\langle\varphi\rangle$ for $\mathcal{M}$ that contains exactly one member of each equivalence class. For each $\langle\varphi\rangle\in\Phi$, take a copy of $\mathcal{M}^{\langle\varphi\rangle}$ in such a way that the home sorts are all the same, and the imaginary sorts of $\mathcal{M}^{\langle\varphi\rangle}$ and $\mathcal{M}^{\langle\psi\rangle}$ are disjoint when $\langle\varphi\rangle\neq\langle\psi\rangle$. Denote the imaginary sort of $\mathcal{M}^{\langle\varphi\rangle}$ by $\mathbb{S}_i^{\langle\varphi\rangle}$.

Let V^{eq} be the union of the sorted vocabularies $\{V^{\langle\varphi\rangle}\colon\langle\varphi\rangle\in\Phi\}$. Finally, let $\mathcal{M}^{eq}$ be a V^{eq}-structure that respects sorts and is a common expansion of the structures $\{\mathcal{M}^{\langle\varphi\rangle}\colon\langle\varphi\rangle\in\Phi\}$.

Remark 4.7.7. *It follows from Remark 4.7.6 that for each reduced V-structure $\mathcal{M}$, $\mathcal{M}^{eq}$ is unique up to isomorphism.*

Note that V^{eq} may have uncountably many sorts and hence uncountably many predicate symbols. To avoid that difficulty, one can restrict things to countable sets of sorts. For any set $\Theta\subseteq\Phi$, let V^{Θ} and $\mathcal{M}^{\Theta}$ be the parts of V^{eq} and $\mathcal{M}^{eq}$ with only the main sort and the imaginary sorts $\mathbb{S}_i^{\langle\varphi\rangle}$ where $\langle\varphi\rangle\in\Theta$. Whenever Θ is countable, the vocabulary V^{Θ} has countably many predicate and function symbols.

The next result concerns pre-metric expansions of $\mathcal{M}$.

Proposition 4.7.8. *Suppose $\mathcal{M}$ is a reduced V-structure, and $\mathcal{M}_e=(\mathcal{M},\mathsf{D})$ is a pre-metric expansion of $\mathcal{M}$ with vocabulary V_D. Then $(V_D)^{eq}$ has the same sorts as V^{eq}, and $(\mathcal{M}_e)^{eq}$ is a sorted pre-metric structure whose signature has:*

- *the distinguished distance D in the home sort,*
- *the distinguished distance $d_{\langle\varphi\rangle}$ in each imaginary sort $\mathbb{S}_i^{\langle\varphi\rangle}$,*
- *the same modulus of uniform continuity as $\mathcal{M}_e$ has for each symbol of V,*
- *the modulus of uniform continuity for $P_{\langle\varphi\rangle}$ that $\mathcal{M}_e$ has for $[\lim\varphi_m]$,*
- *the trivial modulus of uniform continuity for $U_{\mathbb{S}}$ and each $U_{\mathbb{S}_i^{\langle\varphi\rangle}}$.*

The same holds with a set $\Theta\subseteq\Phi$ in place of eq.

Proof. Since every exponentially Cauchy sequence of formulas in T is also exponentially Cauchy in T_e, every sort of $\mathcal{M}^{eq}$ is also a sort of $(\mathcal{M}_e)^{eq}$. Also, for each $\langle\varphi\rangle\in\Phi$, $(\mathcal{M}_e)^{\langle\varphi\rangle}=(\mathcal{M}^{\langle\varphi\rangle},\mathsf{D})$ is the structure obtained by adding the predicate D in the home sort. If $\langle\varphi\rangle$ and $\langle\psi\rangle$ are equivalent in T then they are equivalent

in T_e. The proof of Lemma 4.2.1 shows that every exponentially Cauchy sequence $\langle \varphi \rangle$ in T_e is equivalent in T_e to an exponentially Cauchy sequence $\langle \psi \rangle$ in T, so by Remark 4.7.6 we have $(\mathcal{M}_e)^{\langle \varphi \rangle} = (\mathcal{M}_e)^{\langle \psi \rangle}$. It follows that $(V_e)^{eq}$ has the same sorts as V^{eq}, and $(\mathcal{M}_e)^{eq}$ with the signature described above is a pre-metric expansion of $\mathcal{M}^{eq}$. $\qquad \square$

Assume the hypotheses of Proposition 4.7.8, and let $\Theta \subseteq \Phi$. One can combine all the distinguished distances in $\mathcal{M}_e^\Theta$ into a single metric D^Θ in the following way. If x, y are in the home sort, $\mathsf{D}^\Theta(x, y) = \mathsf{D}(x, y)$. If x, y have the same imaginary sort $\mathbb{S}_i^{\langle \varphi \rangle}$, then $\mathsf{D}^\Theta(x, y) = d_{\langle \varphi \rangle}(x, y)$. If neither x nor y has a sort, then $\mathsf{D}^\Theta(x, y) = 0$. Otherwise, $\mathsf{D}^\Theta(x, y) = 1$. According to our definition, the structure $(\mathcal{M}^\Theta, \mathsf{D}^\Theta)$ does not respect sorts. However, $\mathcal{M}^\Theta$ is a general structure whose vocabulary is V^Θ, and $(\mathcal{M}^\Theta, \mathsf{D}^\Theta)$ is a V_D^Θ-structure.

Corollary 4.7.9. *Assume the hypotheses of Proposition 4.7.8, and $\Theta \subseteq \Phi$ is finite. Then $(\mathcal{M}^\Theta, \mathsf{D}^\Theta)$ is a pre-metric expansion of $\mathcal{M}^\Theta$.*

Proof. The vocabulary V^Θ has countably many predicate and function symbols. It is easily seen that there is a V^Θ-formula $\theta(x, y)$ that defines $\mathsf{D}^\Theta(x, y)$ in $\mathcal{M}^\Theta$. $\qquad \square$

Proposition 4.7.10. *If $\mathcal{M}$ is κ-special, then $\mathcal{M}^{eq}$ is κ^+-special, and $\mathcal{M}^\Theta$ is κ-special for each $\Theta \subseteq \Phi$.*

Proof. The proof is similar to the proof of Proposition 4.7.4, but with many imaginary sorts. $\qquad \square$

Corollary 4.7.11. *Suppose T_e is a pre-metric expansion of T, $\mathcal{M}$ is κ-special, and κ has uncountable cofinality. Then $(\mathcal{M}_e)^{eq}$ is a sorted metric structure, and $(\mathcal{M}_e)^\Theta$ is a sorted metric structure for each $\Theta \subseteq \Phi$.*

Proof. By Propositions 4.7.4, 4.7.10, and Remark 2.5.1. $\qquad \square$

4.8. **Definable Sets.** Recall that in [BBHU], in a metric structure $\mathcal{M}_e$, the *distance* between a k-tuple $\vec{x} \in M^k$ and a closed set $\mathsf{S} \subseteq M^k$ is the mapping

$$\operatorname{dist}^{\mathcal{M}_e}(\vec{x}, \mathsf{S}) = \inf \{ \max_{i \leq k} D^{\mathcal{M}_e}(x_i, y_i) \colon \vec{y} \in \mathsf{S} \},$$

and S is a *definable set* over A in $\mathcal{M}_e$ if S is a closed subset of M^k and $\operatorname{dist}^{\mathcal{M}_e}(\vec{x}, \mathsf{S})$ is a definable predicate over A in $\mathcal{M}_e$. Note that the empty set $\emptyset$ is always definable over $\emptyset$, and $\operatorname{dist}^{\mathcal{M}_e}(\vec{x}, \emptyset) = 1$.

The following result uses the fact that the Expansion Theorem 3.3.4 gives a pseudo-metric approximate distance.

Proposition 4.8.1. *The general property [$\mathcal{M}$ is complete and S is a definable set in $\mathcal{M}$ over A] is absolute. For each complete general structure $\mathcal{M}$, set $A \subseteq M$, and set $\mathsf{S} \subseteq M^k$, the following are equivalent:*

(a) S *is definable over A in $\mathcal{M}$.*
(b) S *is closed in $\mathcal{M}$, and for each V-formula $\varphi(\vec{x}, \vec{y})$, the mapping*

$$\operatorname{dist}_\varphi^{\mathcal{M}}(\vec{x}, \mathsf{S}) := \inf \{ \varphi^{\mathcal{M}}(\vec{x}, \vec{y}) \colon \vec{y} \in \mathsf{S} \}$$

is a definable predicate over A in $\mathcal{M}$.
(c) S *is closed in $\mathcal{M}$, and for each V-formula $\varphi(\vec{x}, \vec{y})$ that is pseudo-metric in $\operatorname{Th}(\mathcal{M})$, $\operatorname{dist}_\varphi^{\mathcal{M}}(\vec{x}, \mathsf{S})$ is a definable predicate over A in $\mathcal{M}$.*

Proof. We first consider an arbitrary pre-metric expansion $\mathcal{M}_e$ of $\mathcal{M}$. By Proposition 4.3.6, $\mathcal{M}$ is complete if and only if $\mathcal{M}_e$ is complete. By Proposition 4.3.4, a subset of M^k is closed in $\mathcal{M}_e$ if and only if it is closed in $\mathcal{M}$. Let (a')–(c') be the statements (a)–(c) with $\mathcal{M}_e$ in place of $\mathcal{M}$. By Theorem 9.17 of [BBHU], (a'), (b'), and (c') are equivalent. We now show that (b) is equivalent to (b').

(b') $\Rightarrow$ (b): Assume (b'). Let $\varphi(\vec{x}, \vec{y})$ be a V-formula. By (b'), $\mathrm{dist}_\varphi^{\mathcal{M}_e}$ is definable in $\mathcal{M}_e$ over A. $\mathrm{dist}_\varphi^{\mathcal{M}_e}$ is definable in $\mathcal{M}$ over A by Proposition 4.2.2. Since $\psi^{\mathcal{M}} = \psi^{\mathcal{M}_e}$ for every V-formula ψ, $\mathrm{dist}_\varphi^{\mathcal{M}_e} = \mathrm{dist}_\varphi^{\mathcal{M}}$. This proves (b).

(b) $\Rightarrow$ (b'): Assume (b). Let $\varphi_m(\vec{x}, \vec{y})$ be the V-formula $\max_{i \le k} d_m(x_i, y_i)$. By (b),

$$\mathrm{dist}_{\varphi_m}^{\mathcal{M}}(\vec{x}, \mathsf{S}) = \inf\{\max_{i \le k} d_m^{\mathcal{M}}(x_i, y_i) \colon \vec{y} \in \mathsf{S}\}$$

is a definable predicate over A in $\mathcal{M}$. Moreover, $\langle \varphi_m \rangle_{n \in \mathbb{N}}$ is Cauchy in T, and converges uniformly to $\mathrm{dist}^{\mathcal{M}_e}(\vec{x}, \mathsf{S})$. Therefore $\mathrm{dist}^{\mathcal{M}_e}(\vec{x}, \mathsf{S})$ is a definable predicate over A in $\mathcal{M}_e$, so (a') holds and hence (b') holds.

Since (b) is equivalent to (b') for all $\mathcal{M}_e$, the property [$\mathcal{M}$ is complete and (b)] is the absolute version of the property [$\mathcal{M}_e$ is complete and (b')]. The absolute version is unique and (a') is equivalent to (b'), so (a) is equivalent to (b). It is trivial that (b) implies (c).

(c) $\Rightarrow$ (b): Assume (c). By Theorem 3.3.4, there exists a pre-metric expansion $\mathcal{M}_f$ of $\mathcal{M}$ with a pseudo-metric approximate distance $\langle d_m(x, y) \rangle_{m \in \mathbb{N}}$. By hypothesis, each formula $d_m(x, y)$ is pseudo-metric in $\mathrm{Th}(\mathcal{M})$. Let $\varphi_m(\vec{x}, \vec{y})$ be the V-formula $\max_{i \le k} d_m(x_i, y_i)$, which is also pseudo-metric in $\mathrm{Th}(\mathcal{M})$. By (c),

$$\mathrm{dist}_{\varphi_m}^{\mathcal{M}}(\vec{x}, \mathsf{S}) = \inf\{\max_{i \le k} d_m^{\mathcal{M}}(x_i, y_i) \colon \vec{y} \in \mathsf{S}\}$$

is a definable predicate over A in $\mathcal{M}$. Moreover, $\langle \varphi_m \rangle_{m \in \mathbb{N}}$ is Cauchy in $\mathrm{Th}(\mathcal{M})$, and converges uniformly to $\mathrm{dist}^{\mathcal{M}_f}(\vec{x}, \mathsf{S})$. Therefore $\mathrm{dist}^{\mathcal{M}_f}(\vec{x}, \mathsf{S})$ is a definable predicate over A in $\mathcal{M}_f$. It follows that (a') holds for $\mathcal{M}_f$, and therefore (b) holds. $\square$

We now turn to the notions of definable and algebraic closure. The following definitions agree with the corresponding definitions in [BBHU] in the case that $\mathcal{M}$ is a metric structure.

Definition 4.8.2. Let $\mathcal{M}$ be a complete general structure. An element b belongs to the *definable closure of* A in $\mathcal{M}$, in symbols $b \in \mathrm{dcl}_{\mathcal{M}}(A)$, if the singleton $\{b\}$ is definable over A in $\mathcal{M}$. b belongs to the *algebraic closure of* A in $\mathcal{M}$, $b \in \mathrm{acl}_{\mathcal{M}}(A)$, if there is a compact set C in $\mathcal{M}$ such that $b \in C$ and C is definable over A in $\mathcal{M}$. A tuple $\vec{b}$ belongs to $\mathrm{dcl}_{\mathcal{M}}(A)$ or $\mathrm{acl}_{\mathcal{M}}(A)$ if each term b_i does.

One can use Proposition 4.8.1 to obtain necessary and sufficient conditions for $b \in \mathrm{dcl}_{\mathcal{M}}(A)$ and $b \in \mathrm{acl}_{\mathcal{M}}(A)$.

Corollary 4.8.3. *If $\mathcal{M}$ is a complete general structure, then* $\mathrm{dcl}_{\mathcal{M}}(A) = \mathrm{dcl}_{\mathcal{M}_e}(A)$ *and* $\mathrm{acl}_{\mathcal{M}}(A) = \mathrm{acl}_{\mathcal{M}_e}(A)$.

Proof. By Propositions 4.3.4 and 4.8.1. $\square$

Corollary 4.8.4. *Suppose $\mathcal{M}, \mathcal{N}$ are complete general structures and $A \subseteq \mathcal{M} \prec \mathcal{N}$. Then* $\mathrm{dcl}_{\mathcal{M}}(A) = \mathrm{dcl}_{\mathcal{N}}(A)$ *and* $\mathrm{acl}_{\mathcal{M}}(A) = \mathrm{acl}_{\mathcal{N}}(A)$.

Proof. By Proposition 4.3.6, $\mathcal{M}_e$ and $\mathcal{N}_e$ are metric structures. By Corollary 4.1.3 we have $\mathcal{M}_e \prec \mathcal{N}_e$. By Corollary 4.8.3 above and Corollary 10.5 of [BBHU], $\mathrm{dcl}_{\mathcal{M}}(A) = \mathrm{dcl}_{\mathcal{M}_e}(A) = \mathrm{dcl}_{\mathcal{N}_e}(A) = \mathrm{dcl}_{\mathcal{N}}(A)$, and similarly for acl. $\square$

Corollary 4.8.5. *Suppose $\mathcal{M}$ is a reduced $\aleph_1$-saturated general structure, $A \subseteq M$, and $b \in M$.*

  (i) $b \in \mathrm{dcl}_{\mathcal{M}}(A)$ *if and only if b is the only realization of* $\mathrm{tp}(b/A)$ *in $\mathcal{M}$.*
  (ii) $b \in \mathrm{acl}_{\mathcal{M}}(A)$ *if and only if the set of realizations of* $\mathrm{tp}(b/A)$ *is compact in $\mathcal{M}$.*

Proof. By Exercises 10.7 (4) and 10.8 (4) of [BBHU], (i) and (ii) hold with an $\aleph_1$-saturated metric structure $\mathcal{N}$ in place of $\mathcal{M}$. By Corollary 4.3.7, $\mathcal{M}$ is complete. By Proposition 4.1.6, $\mathcal{M}_e$ is an $\aleph_1$-saturated metric structure. By Corollary 4.8.3, $\mathrm{dcl}_{\mathcal{M}}(A) = \mathrm{dcl}_{\mathcal{M}_e}(A)$ and $\mathrm{acl}_{\mathcal{M}}(A) = \mathrm{acl}_{\mathcal{M}_e}(A)$. By Corollary 4.1.4, an element $c \in M$ realizes $\mathrm{tp}_{\mathcal{M}}(b/A)$ if and only if it realizes $\mathrm{tp}_{\mathcal{M}_e}(b/A)$. By Proposition 4.3.4, a subset of M is compact in $\mathcal{M}$ if and only if it is compact in $\mathcal{M}_e$. Therefore (i) and (ii) hold for $\mathcal{M}$. $\square$

The paper [EG] introduced the notion of a metric structure admitting weak elimination of finitary imaginaries. We now introduce the analogous notion for general complete structures.

An exponentially Cauchy sequence $\langle \varphi \rangle$ in $\mathcal{M}$ is called *finitary* if there is an $\ell \in \mathbb{N}$ such that for each $m \in \mathbb{N}$, φ_m has at most the free variables $(\vec{x}, y_0, \ldots, y_\ell)$. If $\langle \varphi \rangle$ is finitary, the elements of sort $\mathbb{S}_i^{\langle \varphi \rangle}$ in $\mathcal{M}^{\langle \varphi \rangle}$ are called *finitary imaginaries*.

Definition 4.8.6. Let $\mathcal{M}$ be a reduced $\aleph_1$-saturated general structure

  (i) $\mathcal{M}$ *admits elimination of finitary imaginaries* if for every finitary imaginary $b \in \mathcal{M}^{\langle \varphi \rangle}$, there is a finite tuple $\vec{c}$ from M such that in $\mathcal{M}^{\langle \varphi \rangle}$, $b \in \mathrm{dcl}(\vec{c})$ and $\vec{c} \in \mathrm{dcl}(b)$.
  (ii) $\mathcal{M}$ *admits weak elimination of finitary imaginaries* if for every finitary imaginary $b \in \mathcal{M}^{\langle \varphi \rangle}$, there is a finite tuple $\vec{c}$ from M such that in $\mathcal{M}^{\langle \varphi \rangle}$, $b \in \mathrm{dcl}(\vec{c})$ and $\vec{c} \in \mathrm{acl}(b)$.

The next result shows that the property of being reduced and admitting (or weakly admitting) elimination of finitary imaginaries is absolute.

Corollary 4.8.7. *For every reduced $\aleph_1$-saturated general structure $\mathcal{M}$ and pre-metric expansion $\mathcal{M}_e$ of $\mathcal{M}$, $\mathcal{M}_e$ admits (or weakly admits) elimination of finitary imaginaries if and only if $\mathcal{M}$ admits (or weakly admits) elimination of finitary imaginaries.*

Proof. By Corollaries 4.7.9 and 4.8.3. $\square$

4.9. **Stable Theories.** As is often done the literature on stable theories, we will work in a monster structure of inaccessible cardinality. The axioms of ZFC do not imply the existence of an inaccessible cardinal. However, one can avoid inaccessible cardinals by working in a universal domain, at the cost of some minor complications (see [BBHU]).

We assume hereafter that T is a complete V-theory, and that υ is an inaccessible cardinal greater than $|V| + \aleph_0$. By a monster structure *we mean a reduced υ-saturated structure of cardinality υ. We let $\mathbb{M}$ be a monster model of T, and let $\mathbb{M}_e$ be a pre-metric expansion of $\mathbb{M}$.*

For the rest of this paper we will work exclusively within $\mathbb{M}$ and $\mathbb{M}_e$. By a *small set* we mean a set of cardinality $< \upsilon$. A, B will always denote small subsets of $\mathbb{M}$.

Remark 4.9.1. *By Remark 2.4.5, $\mathbb{M}$ is υ-special. By the Uniqueness Theorem for Special Models, T has a unique monster model up to isomorphism. By Corollary 4.1.7, $\mathbb{M}_e$ is also monster structure. By Corollary 4.3.7, $\mathbb{M}$ and $\mathbb{M}_e$ are complete, so $\mathbb{M}_e$ is a metric structure.*

Let us recall the definition of a λ-stable metric theory in [BBHU], where λ is an infinite cardinal. For each small set A, the D-metric on the type space $S_1^{\mathbb{M}_e}(A)$ is defined by

$$D^{\mathbb{M}_e}(p, q) = \inf\{D^{\mathbb{M}_e}(b, c) \colon \mathrm{tp}_{\mathbb{M}_e}(b/A) = p, \mathrm{tp}_{\mathbb{M}_e}(c/A) = q\}.$$

The complete metric theory $\mathrm{Th}(\mathbb{M}_e)$ is λ-*stable* if for each A of cardinality $\leq \lambda$ there is a dense subset of $S_1^{\mathbb{M}_e}(A)$ of cardinality $\leq \lambda$ with respect to the D-metric. And $\mathrm{Th}(\mathbb{M}_e)$ is stable if it is λ-stable for some small cardinal λ. Here, it will be convenient to say that $\mathbb{M}_e$ is stable instead of saying that $\mathrm{Th}(\mathbb{M}_e)$ is stable. So, by a λ-stable metric structure we mean a monster metric structure whose complete metric theory is λ-stable.

Proposition 4.9.2. *The property of being λ-stable is absolute. $\mathbb{M}$ is λ-stable if and only if for each A of cardinality $\leq \lambda$ there is a set $B \subseteq S_1^{\mathbb{M}}(A)$ of cardinality $\leq \lambda$ such that the set $\{b \colon \mathrm{tp}_{\mathbb{M}}(b/A) \in B\}$ is dense in $\mathbb{M}$.*

Proof. By Corollary 4.1.4, for all elements b, c of $\mathbb{M}$, $\mathrm{tp}_{\mathbb{M}}(b/A) = \mathrm{tp}_{\mathbb{M}}(c/A)$ if and only if $\mathrm{tp}_{\mathbb{M}_e}(b/A) = tp_{\mathbb{M}_e}(c/A)$. By Proposition 4.3.4, a subset of $\mathbb{M}$ is dense in $\mathbb{M}$ if and only if it is dense in $\mathbb{M}_e$. So it suffices to prove that a set $B \subseteq S_1^{\mathbb{M}_e}(A)$ is dense in $S_1^{\mathbb{M}_e}(A)$ with respect to the D-metric if and only if the set $B' := \{b \colon \mathrm{tp}_{\mathbb{M}_e}(b/A) \in B\}$ is dense in $\mathbb{M}_e$. We prove the non-trivial direction here. Let $c \in \mathbb{M}_e$ and $p = \mathrm{tp}_{\mathbb{M}_e}(c/A)$. Since B is dense in $S_1^{\mathcal{M}_e}(A)$, there is a sequence $\langle p_n \rangle_{n \in \mathbb{N}}$ in B that converges to p in the D-metric. Since $\mathbb{M}_e$ is κ^+-saturated, for each n there exists b_n such that $\mathrm{tp}_{\mathbb{M}_e}(b_n/A) = p_n$ and $D^{\mathbb{M}_e}(b_n, c) = D^{\mathbb{M}_e}(p_n, p)$. Then $b_n \in B'$ for each n and $\lim_{n \to \infty} b_n = c$ in $\mathbb{M}_e$, so B' is dense in $\mathbb{M}_e$. $\square$

Corollary 4.9.3. *$\mathbb{M}$ is $\aleph_0$-stable if and only if $\mathbb{M}$ is λ-stable for all infinite λ.*

Proof. This follows from the corresponding result for metric structures (Remark 14.8 in [BBHU]) and the fact that being λ-stable is absolute (Proposition 4.9.2). $\square$

Exercise 4.9.4. *Suppose every predicate symbol and every function symbol in V is unary. Then T is $\aleph_0$-stable. Hint: Show that for all $b.c \in \mathbb{M}$ and all $A \subseteq \mathbb{M}$, $\mathrm{tp}_{\mathbb{M}}(b) = \mathrm{tp}_{\mathbb{M}}(c)$ if and only if $\mathrm{tp}_{\mathbb{M}}(b/A) = \mathrm{tp}_{\mathbb{M}}(c/A)$.*

Corollary 4.9.5. *Suppose $\lambda = \lambda^{\aleph_0}$. The following are equivalent:*
  (i) *T is stable.*
  (ii) *T is λ-stable.*
  (iii) *For every set A of cardinality $\leq \lambda$, $|S_1^{\mathbb{M}}(A)| \leq \lambda$.*

Proof. The equivalence of (i) and (ii) follows from the corresponding result for metric structures (Theorem 8.5 in [BU]) and the fact that λ-stability is absolute. It is trivial that (iii) implies (ii). Assume (ii). Suppose $A \subseteq \mathbb{M}$, and $|A| \leq \lambda$. By (ii) and Proposition 4.9.2, there is a set $B \subseteq S_1^{\mathbb{M}}(A)$ of cardinality $\leq \lambda$ such that the set $B' := \{b \colon \mathrm{tp}_{\mathbb{M}}(b/A) \in B\}$ is dense in $\mathbb{M}$. By Remark 2.4.1, every complete

type in $S_1^{\mathbb{M}}(A)$ is realized in $\mathbb{M}$. Therefore, since B' is dense in $\mathbb{M}$, B is dense in $S_1^{\mathbb{M}}(A)$. So $|S_1^{\mathbb{M}}(A)| \leq \lambda^{\aleph_0} = \lambda$, and (iii) follows. $\qquad\square$

In [BBHU], a *stable independence relation* on a monster metric structure $\mathbb{M} = \mathbb{M}_e$ is a ternary relation $A \underset{C}{\bigcup} B$ on small subsets of $\mathbb{M}$ that has the following properties[5].

- *Invariance under automorphisms of $\mathbb{M}$.*
- *Symmetry:* $A \underset{C}{\bigcup} B$ if and only if $B \underset{C}{\bigcup} A$.
- *Transitivity:* $A \underset{C}{\bigcup} BD$ if and only if $A \underset{C}{\bigcup} B$ and $A \underset{BC}{\bigcup} D$.
- *Finite character:* $A \underset{C}{\bigcup}$ if and only if $\vec{a} \underset{C}{\bigcup} B$ for all finite $\vec{a} \subseteq A$.
- *Full existence:* For all A, B, C there exists A' such that $(\mathbb{M}_{A'})_C \equiv (\mathbb{M}_A)_C$ and $A' \underset{C}{\bigcup} B$.
- *Strong local character:* For each finite $\vec{a}$, there exists $B_0 \subseteq B$ of cardinality $\leq |V| + \aleph_0$ with $\vec{a} \underset{B_0}{\bigcup} B$.
- *Stationarity:* For all small complete $\mathcal{M}^0 \prec \mathbb{M}$ and all small A, A', B, if
$$(\mathbb{M}_A)_{M^0} \equiv (\mathbb{M}_{A'})_{M^0}, \quad A \underset{M^0}{\bigcup} B, \quad A' \underset{M^0}{\bigcup} B,$$
  then
$$(\mathbb{M}_A)_{B \cup M^0} \equiv (\mathbb{M}_{A'})_{B \cup M^0}$$

We define a *stable independence relation* on a monster general structure $\mathbb{M}$ to be a ternary relation that satisfies the same seven properties with respect to $\mathbb{M}$.

Theorem 4.9.6.

(i) *A relation $\bigcup$ is a stable independence relation on $\mathbb{M}$ if and only if it is a stable independence relation on $\mathbb{M}_e$.*

(ii) *$\mathbb{M}$ is stable if and only if it has a stable independence relation, and also if and only if it has a unique stable independence relation.*

Proof. (i): Let $\bigcup$ be a ternary relation on small subsets of the universe of $\mathbb{M}$. We show that each of the properties listed above for a stable independence relation is absolute, that is, holds for $\mathbb{M}$ if and only if it holds for $\mathbb{M}_e$. It is trivial that Symmetry, Transitivity, Finite Character, and Local Character are absolute. By Lemma 3.1.5 (iii), Invariance under automorphisms is absolute. To show that Full Existence is absolute, use Corollary 4.1.4, which says that two tuples realize the same complete type over C in $\mathbb{M}$ if and only if they realize the same complete type over C in $\mathbb{M}_e$. To show that Stationarity is absolute, use Corollary 4.1.4 again and Corollary 4.1.3, which says that $\mathcal{M}^0 \prec \mathbb{M}$ if and only if $\mathcal{M}_e^0 \prec \mathbb{M}_e$.

(ii) follows immediately from (i) and Theorem 14.1 of [BBHU]. $\qquad\square$

We now consider the approach to stability theory via definable types.

Definition 4.9.7. We say that a complete k-type $\mathrm{tp}_{\mathbb{M}}(\vec{a}/B)$ is *definable over C* in $\mathbb{M}$ if for each V-formula $\varphi(\vec{x}, \vec{y})$ with parameters in C there is a mapping $\mathsf{Q}\colon M^{|\vec{y}|} \to [0, 1]$ that is definable over C in $\mathbb{M}$ such that for all $\vec{b} \in B^{|\vec{y}|}$ we have $\varphi^{\mathbb{M}}(\vec{a}, \vec{b}) = \mathsf{Q}(\vec{b})$.

In the case that $\mathbb{M} = \mathbb{M}_e$ (so $\mathbb{M}$ is a metric structure), the above definition is the same as the corresponding definition in [BBHU]. We now show that the general property [$\mathrm{tp}(\vec{a}/B)$ is definable over C] is absolute.

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[5]In naming these properties, we follow Adler [Ad], rather than [BBHU]

Proposition 4.9.8. *Let* $\vec{a} \in M$ *and* B, C *be small subsets of* M. $\mathrm{tp}_{\mathbb{M}}(\vec{a}/B)$ *is definable over* C *in* $\mathbb{M}$ *if and only if* $\mathrm{tp}_{\mathbb{M}_e}(\vec{a}/B)$ *is definable over* C *in* $\mathbb{M}_e$.

Proof. Suppose first that $\mathrm{tp}_{\mathbb{M}_e}(\vec{a}/B)$ is definable over C in $\mathbb{M}_e$. Let $\varphi(\vec{x}, \vec{y})$ be a V-formula with parameters in C. By hypothesis there is a mapping $\mathsf{Q} \colon M^{|\vec{y}|} \to [0, 1]$ that is definable over C in $\mathbb{M}_e$ such that for all $\vec{b} \in B^{|\vec{y}|}$ we have $\varphi^{\mathbb{M}_e}(\vec{a}, \vec{b}) = \mathsf{Q}(\vec{b})$. By Proposition 4.2.2, Q is definable over C in $\mathbb{M}$, and by Remark 2.1.6, $\varphi^{\mathbb{M}}(\vec{a}, \vec{b}) = \varphi^{\mathbb{M}_e}(\vec{a}, \vec{b}) = \mathsf{Q}(\vec{b})$, so $\mathrm{tp}_{\mathbb{M}}(\vec{a}/B)$ is definable over C in $\mathbb{M}$.

Now suppose that $\mathrm{tp}_{\mathbb{M}}(\vec{a}/B)$ is definable over C in $\mathbb{M}$. Let $\varphi(\vec{x}, \vec{y})$ be a V_D-formula with parameters in C. By Lemma 4.2.1, $\varphi^{\mathbb{M}_e}$ is definable in $\mathbb{M}$ over C. Then there is a sequence of V-formulas $\langle \varphi_m(\vec{x}, \vec{y}) \rangle_{m \in \mathbb{N}}$ with parameters in C such that for each $m \in \mathbb{N}$,

$$(\forall \vec{a} \in M^{|\vec{x}|})(\forall \vec{b} \in M^{|\vec{y}|})|\varphi_m^{\mathbb{M}}(\vec{a}, \vec{b}) - \varphi^{\mathbb{M}_e}(\vec{a}, \vec{b})| \leq 2^{-m}.$$

By hypothesis, for each $m \in \mathbb{N}$ there is a mapping $\mathsf{Q}_m \colon \mathbb{M}^{|\vec{y}|} \to [0, 1]$ definable over C in $\mathbb{M}$ such that for all $\vec{b} \in B^{|\vec{y}|}$ we have $\varphi_m^{\mathbb{M}}(\vec{a}, \vec{b}) = \mathsf{Q}_m(\vec{b})$. By Proposition 4.2.2, each Q_m is definable in $\mathbb{M}$ over C, so there is a sequence of V-formulas $\langle \psi_m(\vec{y}) \rangle_{m \in \mathbb{N}}$ with parameters in C such that

$$(\forall \vec{b} \in \mathbb{M}^{|\vec{y}|})(|\psi_m^{\mathbb{M}}(\vec{b}) - \mathsf{Q}_m(\vec{b})| \leq 2^{-m})$$

and

$$(\forall \vec{b} \in B^{|\vec{y}|})(|\psi_m^{\mathbb{M}}(\vec{b}) - \mathsf{Q}_m(\vec{b})| = |\psi_m^{\mathbb{M}}(\vec{b}) - \varphi_m^{\mathbb{M}}(\vec{a}, \vec{b})| \leq 2^{-m}).$$

Then

$$(\forall \vec{b} \in B^{|\vec{y}|})(|\psi_m^{\mathbb{M}}(\vec{b}) - \varphi^{\mathcal{M}_e}(\vec{a}, \vec{b})| \leq 2 \cdot 2^{-m}.$$

We now modify $\langle \psi_m(\vec{y}) \rangle_{m \in \mathbb{N}}$ to a sequence of V-formulas that is Cauchy in $\mathrm{Th}(\mathbb{M})$ using the forced convergence trick of [BU]. Note that for all $\vec{b} \in B^{|\vec{y}|}$,

$$|\psi_m^{\mathbb{M}}(\vec{b}) - \psi_{m+1}^{\mathbb{M}}(\vec{b})| \leq 3 \cdot 2^{-m}.$$

We inductively define $\theta_0 = \psi_0$, and

$$\theta_{m+1} = \max(\theta_m - 3 \cdot 2^{-m}, \min(\theta_m + 3 \cdot 2^{-m}, \psi_{m+1})).$$

Then $\theta_m^{\mathbb{M}}(\vec{b}) = \psi_m^{\mathbb{M}}(\vec{b})$ for all m and all $\vec{b} \in B^{|\vec{y}|}$. Moreover, $\langle \theta_m(\vec{y}) \rangle_{m \in \mathbb{N}}$ is a sequence of V-formulas with parameters in C such that

$$(\forall \vec{b})|\theta_m^{\mathbb{M}}(\vec{b}) - \theta_{m+1}^{\mathbb{M}}(\vec{b})| \leq 3 \cdot 2^{-m},$$

so $\langle \theta_m(\vec{y}) \rangle_{m \in \mathbb{N}}$ in Cauchy in $\mathrm{Th}(\mathbb{M})$. Therefore by Proposition 4.2.2, $\mathsf{Q} := [\lim \theta_m]^{\mathbb{M}}$ is a definable predicate over C in $\mathbb{M}_e$, and $\varphi^{\mathbb{M}_e}(\vec{a}, \vec{b}) = \mathsf{Q}(\vec{b})$ for all $\vec{b} \in B^{|\vec{y}|}$. Therefore $\mathrm{tp}_{\mathbb{M}_e}(\vec{a}/B)$ is definable over C in $\mathbb{M}_e$. $\qquad \square$

Definition 4.9.9. We say that a complete type $p = \mathrm{tp}_{\mathbb{M}}(\vec{a}/B)$ *does not fork over* C *in* $\mathbb{M}$ if p is definable over $\mathrm{acl}_{\mathbb{M}}(C)$.

In the case that $\mathbb{M} = \mathbb{M}_e$, the above definition is the same as the definition in [BBHU]. We now show that the property $[\mathrm{tp}(\vec{a}/B)$ does not fork over $C]$ is absolute.

Corollary 4.9.10. *For every* $a, B,$ *and* C, $\mathrm{tp}_{\mathbb{M}}(a/B)$ *does not fork over* C *in* $\mathbb{M}$ *if and only if* $\mathrm{tp}_{\mathbb{M}_e}(a/B)$ *does not fork over* C *in* $\mathbb{M}_e$.

Proof. By Proposition 4.9.8, $\mathrm{tp}_{\mathbb{M}}(\vec{a}/B)$ is definable over $\mathrm{acl}_{\mathbb{M}}(C)$ in $\mathbb{M}$ if and only if $\mathrm{tp}_{\mathbb{M}_e}(\vec{a}/B)$ is definable over $\mathrm{acl}_{\mathbb{M}}(C)$ in $\mathbb{M}_e$. We have $\mathrm{acl}_{\mathbb{M}}(C) = \mathrm{acl}_{\mathbb{M}_e}(C)$ by Corollary 4.8.3. $\qquad \square$

Corollary 4.9.11. $\mathbb{M}$ *is stable if and only if for every small* $\mathbb{N} \prec \mathbb{M}$ *and every tuple* $\vec{a}$ *of elements of* $\mathbb{M}$, $\mathrm{tp}_{\mathbb{M}}(\vec{a}/N)$ *is definable over* N *in* $\mathbb{M}$.

Proof. By Theorem 14.16 of [BBHU], the result holds with the monster metric structure $\mathbb{M}_e$ in place of $\mathbb{M}$. Corollary 4.1.3, $\mathbb{N} \prec \mathbb{M}$ if and only if $\mathbb{N}_e \prec \mathbb{M}_e$. By Proposition 4.9.8, $\mathrm{tp}_{\mathbb{M}}(\vec{a}/N)$ is definable over N in $\mathbb{M}$ if and only if $\mathrm{tp}_{\mathbb{M}_e}(\vec{a}/N)$ is definable over N in $\mathbb{M}_e$. By Proposition 4.9.2, $\mathbb{M}$ is stable if and only if $\mathbb{M}_e$ is stable, and the result follows. $\square$

4.10. **Building Stable Theories.** In both first order and continuous logic, stable structures are of particular interest because they are well-behaved and can be analyzed. The literature contains a wide variety of examples of stable first order structures, and several examples of stable metric structures in areas such as Banach spaces and probability algebras. We now present a way to build many examples of stable general structures from first order structures that are stable, or even stable for positive formulas. As in [HI] and [BY03a], we exploit a connection between $[0, 1]$-valued structures and positive formulas in first order structures. We need some definitions.

A first order formula is *positive* if it is built from atomic formulas using only quantifiers and the connectives $\wedge, \vee$. A first-order structure is *positively κ-saturated* if every set of positive formulas with the free variable x and fewer than κ parameters that is finitely satisfiable is satisfiable in the structure. Note that every κ-saturated first-order structure is positively κ-saturated. A *positive monster structure* is a positively υ-saturated first order structure of cardinality υ.

In a positive monster structure $\mathbb{K}$, the *complete positive type* of a k-tuple $\vec{b}$ over a set A is the set of all positive formulas $\varphi(\vec{x})$ with parameters in A satisfied by $\vec{b}$. We say that a first-order structure $\mathbb{K}$ is *positively λ-stable* if $\mathbb{K}$ is a positive monster structure and, for every set A of cardinality λ, the set of complete positive 1-types over A in $\mathbb{K}$ has cardinality $\leq \lambda$. In particular, every λ-stable first order monster structure is a positively λ-stable.

By a $\bigwedge$-*formula* we mean a finite or countable conjunction of positive formulas, possibly with parameters in $\mathbb{K}$. We also allow the empty conjunction, whose truth value is always true. Let $\mathbb{D}$ be the set of all dyadic rationals in $[0, 1]$. Hereafter we let q, r, s vary over $\mathbb{D}$. Let $\mathbb{J}$ be the set of all intervals of the form $[0, r]$ or $[r, 1]$ where $r \in \mathbb{D}$.

Let $\mathbb{K}$ be a first-order positive monster structure with universe M whose vocabulary W contains at least all the function and constant symbols of V. By a *positive interpretation of V in $\mathbb{K}$* we mean a function $\mathbf{I}$ that associates, with each k-ary predicate symbol $P \in V$ and interval $J \in \mathbb{J}$, a $\bigwedge$-formula $\mathbf{I}(P, J)$ in the vocabulary W with k free variables, such that whenever $r < s$ we have:

(a) $\mathbf{I}(P, [0, r])^{\mathbb{K}} \subseteq \mathbf{I}(P, [0, s])^{\mathbb{K}}$ and $\mathbf{I}(P, [r, 1])^{\mathbb{K}} \supseteq \mathbf{I}(P[s, 1])^{\mathbb{K}}$.
(b) $\mathbf{I}(P, [0, r])^{\mathbb{K}} \cap \mathbf{I}(P, [s, 1])^{\mathbb{K}} = \emptyset$.
(c) $\mathbf{I}(P, [r, 1])^{\mathbb{K}} \cup \mathbf{I}(P, [0, s])^{\mathbb{K}} = M^k$.

Theorem 4.10.1. *Suppose $\mathbf{I}$ is a positive interpretation of V in a positive monster structure $\mathbb{K}$. There is a unique V-structure $\mathbb{M} = \mathbf{I}(\mathbb{K})$ with universe M such that $\mathbb{M}$ agrees with $\mathbb{K}$ on all function and constant symbols in V, and for each k-ary predicate symbol $P \in V$, $r \in \mathbb{D}$, and $\vec{b} \in M^k$,*

$$P^{\mathbb{M}}(\vec{b}) \in [0, r] \text{ if and only if } \mathbb{K} \models \bigwedge_{s > r} \mathbf{I}(P, [0, s])(\vec{b})$$

and

$$P^{\mathbb{M}}(\vec{b}) \in [r, 1] \text{ if and only if } \mathbb{K} \models \bigwedge_{s < r} \mathbf{I}(P, [s, 1])(\vec{b})$$

(with the empty conjunction being the true sentence). Moreover, $\mathbb{M}$ is a monster structure, and for each λ, if $\mathbb{K}$ is positively λ-stable then $\mathbb{M}$ is λ-stable.

Theorem 4.10.1 shows that one can build a λ-stable general structure $\mathbb{M}$ by starting with a positively λ-stable monster structure $\mathbb{K}$ and taking any positive interpretation $\mathbf{I}$ of V in $\mathbb{K}$. In fact, it is not hard to show that if $\lambda = \lambda^{\aleph_0}$, then every λ-stable general structure $\mathbb{M}$ can be obtained in this way. Since λ-stability is absolute, one can then get a λ-stable metric structure by taking any pre-metric expansion $\mathbb{M}_e$.

Proof of Theorem 4.10.1. It is clear that there is at most one such $\mathbb{M}$. We first prove that such an $\mathbb{M}$ exists. Let

$$\mathbf{I}'(P, [0, r]) = \bigwedge_{s > r} \mathbf{I}(P, [0, s]), \qquad \mathbf{I}'(P, [r, 1]) = \bigwedge_{s < r} \mathbf{I}(P, [s, 1]).$$

Using the fact that $\mathbb{D}$ is dense, one can easily check that $\mathbf{I}'$ is a positive interpretation of V in $\mathbb{K}$, and that $\mathbf{I}'' = \mathbf{I}'$. For each k-ary $P \in V$ and $\vec{b} \in M^k$, let

$$X = X(P, \vec{b}) = \{r \colon \mathbb{K} \models \mathbf{I}'(P, [r, 1])(\vec{b})\},$$

$$Y = Y(P, \vec{b}) = \{s \colon \mathbb{K} \models \mathbf{I}'(P, [0, s])(\vec{b})\}.$$

By (c), X and Y are non-empty. Therefore $\sup X$ exists. We define $\mathbb{M}$ to be the V-structure with universe M that agrees with $\mathbb{K}$ on every function and constant symbol in V, and such that $P^{\mathbb{M}}(\vec{b}) = \sup X$ for each P and $\vec{b}$. We first show that:

(7)   For each P and $\vec{b} \in M^k$, $\sup X = P^{\mathbb{M}}(\vec{b}) = \inf Y$.

If $r \in X$ and $s \in Y$, then by (b), $\neg s < r$ and thus $r \leq s$. Hence $\sup X \leq \inf Y$. By (a), $q \leq r \in X \Rightarrow q \in X$, and $q \geq s \in Y \Rightarrow q \in Y$. If $\sup X < \inf Y$, then there is a $q \in \mathbb{D}$ with $\sup X < q < \inf Y$, which would contradict (c). This proves (7).

It follows that for all $r < 1$ in $\mathbb{D}$,

(8) $P^{\mathbb{M}}(\vec{b}) \in [0, r] \Leftrightarrow \inf Y \in [0, r] \Leftrightarrow (\forall s > r)\mathbb{K} \models \mathbf{I}'(P, [0, s])(\vec{b})$.

Then using $\mathbf{I}'' = \mathbf{I}'$, one can see that whenever $r < 1$,

$$P^{\mathbb{M}}(\vec{b}) \in [0, r] \Leftrightarrow P^{\mathcal{N}}_{[0,r]}(\vec{b}).$$

By (c), this also holds when $r = 1$. By a similar argument,

$$P^{\mathbb{M}}(\vec{b}) \in [r, 1] \Leftrightarrow \mathbb{K} \models \mathbf{I}'(P, [r, 1])(\vec{b})$$

for all $r \in \mathbb{D}$.

We next prove that $\mathbb{M}$ is υ-saturated, and thus is a monster structure. Note that any small set of $\bigwedge$-formulas with parameters in M that is finitely satisfiable in $\mathbb{K}$ is satisfiable in $\mathbb{K}$. Finite disjunctions and countable conjunctions of $\bigwedge$-formulas are logically equivalent to $\bigwedge$-formulas. Moreover, since $\mathbb{K}$ is $\aleph_1$-saturated, for every $\bigwedge$-formula $\Theta(\vec{x}, y)$, $(\exists y)\Theta$ and $(\forall y)\Theta$ are equivalent to $\bigwedge$-formulas in $\mathbb{K}$.

By a $\mathbb{D}$-*interval* we mean either the empty set or an interval $[r, s]$ where $0 \leq r \leq s \leq 1$ and $r, s \in \mathbb{D}$. A $\mathbb{D}$-*rectangle* is a finite cartesian product of $\mathbb{D}$-intervals. Note that for any continuous connective $C \colon [0, 1]^k \to [0, 1]$ and $\mathbb{D}$-interval $[r, s]$, $C^{-1}([r, s])$ is a countable intersection of finite unions of $\mathbb{D}$-rectangles.

Using the above two paragraphs, one can show by induction on the complexity of formulas that for every V-formula $\varphi(\vec{x})$ and $\mathbb{D}$-interval $[r, s]$, there is a $\bigwedge$-formula $\Theta(\vec{x})$ such that for all $\vec{b} \in M^k$, $\varphi^{\mathbb{M}}(\vec{b}) \in [r, s]$ if and only if $\mathbb{K} \models \Theta(\vec{b})$. For every small set A, every finitely satisfiable set of $\bigwedge$-formulas $\Theta(x)$ with parameters in A is satisfiable in $\mathbb{K}$. It follows that every finitely satisfiable set of V-formulas $\varphi(x)$ with parameters in A is satisfiable in $\mathbb{M}$, so $\mathbb{M}$ is a monster structure.

It also follows that if $\vec{b}, \vec{c}$ have the same positive type over A in $\mathbb{K}$, then $\mathrm{tp}_{\mathbb{M}}(\vec{b}/A) = \mathrm{tp}_{\mathbb{M}}(\vec{c}/A)$. Therefore if $\mathbb{K}$ is positively λ-stable, then $\mathbb{M}$ is λ-stable. $\qquad\square$

4.11. **Simple and Rosy Theories.** The notion of a simple theory was introduced in the context of cats in [BY03b]. In the literature (see [EG], for example), the definition of a simple complete metric theory is obtained by translating the definition of a simple cat in [BY03b] into the context of continuous model theory. Fact 4.11.2 below is a necessary and sufficient condition for a complete metric theory to be simple that is proved in [BY03b]. Using that fact, we will show here that the property of being simple is absolute. We need the relation $\equiv_C^{Ls}$ (for Lascar equivalence) from [BY03b].

Definition 4.11.1. In a monster general structure $\mathbb{M}$, we write $A \equiv_C^{Ls} B$ if A, B are small sequences of the same length and there exist finitely many sequences $A_1, \ldots, A_n$ such that $A = A_1, B = A_n$, and for each $k < n$, A_k and A_{k+1} both occur on some C-indiscernible sequence.

Fact 4.11.2. *(By Theorem 1.51 of [BY03b]) A complete metric theory is simple if and only if its monster metric model $\mathbb{M}$ has a ternary relation $A \underset{C}{\downarrow} B$ on small sets that has the following properties:*

- *Invariance under automorphisms of $\mathbb{M}$.*
- *Symmetry.*
- *Transitivity.*
- *Finite character.*
- *For every A and C, $A \underset{C}{\downarrow} C$.*
- *Local character: For every A there exists a small cardinal λ such that for every B there exists $B_0 \subseteq B$ with $B_0 \leq \lambda$ and $A \underset{B_0}{\downarrow} B$.*
- *Extension: If $A \underset{C}{\downarrow} B$ and $\widehat{B} \supseteq B$, there exists $A' \equiv_{BC} A$ such that $A' \underset{C}{\downarrow} \widehat{B}$.*
- *Independence theorem: Whenever $A_0 \underset{C}{\downarrow} A_1$, $B_0 \underset{C}{\downarrow} A_0$, $B_1 \underset{C}{\downarrow} A_1$, and $B_0 \equiv_C^{Ls} B_1$, there exists B such that $B \underset{C}{\downarrow} A_0 A_1$, $B \equiv_{CA_0}^{Ls} B_0$, and $B \equiv_{CA_1}^{Ls} B_1$.*

As in first order model theory, every stable metric theory is simple.

By a *simple metric structure* we mean a monster metric structure whose complete theory is simple. The following corollary shows that the property of being a simple metric structure has an absolute version. By Definition 3.4.7, this tells us that the right definition of a simple general structure is a monster general structure that satisfies that absolute version.

Corollary 4.11.3. *The property of being simple is absolute. A general monster structure $\mathbb{M}$ is simple if and only if there exists a ternary relation $A \underset{C}{\downarrow} B$ on small subsets of $\mathbb{M}$ that satisfies the properties listed in Fact 4.11.2.*

Proof. It suffices to show that for each ternary relation $\underset{\smile}{\downarrow}$ on the universe of $\mathbb{M}$, each of the condit6ions listed in Fact 4.11.2 is absolute, that is, it holds for $\mathbb{M}$ if and only if it holds for $\mathbb{M}_e$. We have already seen that Invariance under automorphisms is absolute. Absoluteness for the other conditions follow easily from Corollary 4.1.4, which says that the property $\text{tp}(\vec{b}/A) = \text{tp}(c/A)$ is absolute, and hence that indiscernibility over C is absolute.   □

The notion of a rosy metric theory is introduced by Ealy and Goldbring [EG].

Definition 4.11.4. A metric structure $\mathbb{M}$, or its complete metric theory $\text{Th}(\mathbb{M})$, is *real rosy* if $\mathbb{M}$ is a monster metric structure and has a ternary relation $\underset{\smile}{\downarrow}$ on small sets with the following properties:

- *Invariance under automorphisms of* $\mathbb{M}$.
- *Monotonicity:* If $A \underset{C}{\downarrow} B$, $A' \subseteq A$, and $B' \subseteq B$, then $A' \underset{C}{\downarrow} B'$.
- *Base monotonicity:* Suppose $C \in [D, B]$. If $A \underset{D}{\downarrow} B$, then $A \underset{C}{\downarrow} B$.
- *Transitivity.*
- *Normality:* $A \underset{C}{\downarrow} B$ implies $AC \underset{C}{\downarrow} B$.
- *Extension.*
- *Countable character:* $A \underset{C}{\downarrow} B$ if and only if $\vec{a} \underset{C}{\downarrow} B$ for all countable $\vec{a} \subseteq A$.
- *Local character.*
- *Anti-reflexivity:* $a \underset{B}{\downarrow} a$ implies $a \in \text{acl}_{\mathbb{M}}(B)$.

A ternary relation with the above properties is called a *strict independence relation.* Every strict independence relation also satisfies Symmetry, Full Existence, and $A \underset{C}{\downarrow} C$.

Corollary 4.11.5. *The property of being real rosy is absolute. A general monster structure $\mathbb{M}$ is real rosy if and only if it has a strict independence relation.*

Proof. It is enough to show that for each ternary relation $\underset{\smile}{\downarrow}$ on the universe M, each of the conditions in Definition 4.11.4 is absolute, that is, it holds for $\mathbb{M}$ if and only if it holds for $\mathbb{M}_e$. We have already observed that Invariance under automorphisms is absolute. The absoluteness of Anti-reflexivity follows from Corollaries 4.1.4 about types, and Corollary 4.8.3 about algebraic closure. It is trivial that the other properties for a strict independence relation are absolute.   □

By Proposition 4.7.10, if $\mathbb{M}$ is a monster general structure, then $\mathbb{M}^{eq}$ is a monster structure, and $\mathbb{M}^{\Theta}$ is a monster structure for each $\Theta \subseteq \Phi$.

Definition 4.11.6. A metric structure $\mathbb{M}$, or its theory $\text{Th}(\mathbb{M})$, is called *rosy* if $\mathbb{M}$ is a monster metric structure and $\mathbb{M}^{eq}$ is real rosy.

Lemma 4.11.7. *Let $\mathbb{M}$ be a monster general structure, and let $\mathbb{M}_e$ be a pre-metric expansion of $\mathbb{M}$.*

(i) *$(\mathbb{M}_e)^{eq}$ has the same sorts, and the same universe in each sort, as $\mathbb{M}^{eq}$,*

(ii) *If $\vec{a}, \vec{b}$ are tuples and C is a small set in $\mathbb{M}^{eq}$, then*

$$\text{tp}_{\mathbb{M}^{eq}}(\vec{a}/C) = \text{tp}_{\mathbb{M}^{eq}}(\vec{b}/C) \Leftrightarrow \text{tp}_{(\mathbb{M}_e)^{eq}}(\vec{a}/C) = \text{tp}_{(\mathbb{M}_e)^{eq}}(\vec{b}/C).$$

Proof. By Proposition 4.7.8, $(\mathbb{M}_e)^{eq}$ is a pre-metric expansion of $\mathbb{M}^{eq}$. Therefore (i) holds. The proof of Corollary 4.1.4 also works in this case, even though the vocabulary V^{eq} has uncountably many predicate symbols, and shows that (ii) holds.   □

Corollary 4.11.8. *The property of being rosy is absolute. A general monster structure $\mathbb{M}$ is rosy if and only if $\mathbb{M}^{eq}$ has a strict independence relation.*

Proof. By Lemma 4.11.7 (i), $\mathbb{M}^{eq}$ and $(\mathbb{M}_e)^{eq}$ have the same sorts and universe sets. Therefore they have the same ternary relations. It follows from Lemma 4.11.7 (i) that for each ternary relation $\underset{\smile}{\downarrow}$, each of the conditions in Definition 4.11.4 holds for $\mathbb{M}^{eq}$ if and only if it holds for $(\mathbb{M}_e)^{eq}$. $\qquad\square$

5. Conclusion

We have shown that every general structure with truth values in $[0,1]$ can be made into a metric structure by adding a distance predicate that is a uniform limit of pseudo-metric formulas, and completing the metric. We used that result to extend many notions and results from the class of metric structures to the class of general structures. Thus the model theory of metric structures is considerably more broadly applicable than it initially appears.

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University of Wisconsin-Madison, Department of Mathematics, Madison, WI 53706-
1388

Email address: `keisler@math.wisc.edu`