

USING ULTRAPOWERS TO COMPARE CONTINUOUS STRUCTURES

H. JEROME KEISLER

ABSTRACT. In 1967 the author introduced a pre-ordering of all first order complete theories where T is lower than U if it is easier for an ultrapower of a model of T than an ultrapower of a model of U to be saturated. In a long series of recent papers, Malliaris and Shelah showed that this pre-ordering is very rich and gives a useful way of classifying simple theories. In this paper we investigate the analogous pre-ordering in continuous model theory.

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1. INTRODUCTION

The results in this paper bring together two research projects that began in the 1960's, the so-called "Keisler order" \trianglelefteq , and continuous model theory. Both of these projects were mostly dormant for four decades, and then suddenly blossomed into active and highly successful programs during the last decade.

In the paper [Ke67], the relation \trianglelefteq was introduced and proposed as a way of classifying first order theories¹ with possibly different countable vocabularies. Informally, $T \trianglelefteq U$ if it is easier for an ultrapower of a model of T than an ultrapower of a model of U to be saturated. Formally, $T \trianglelefteq U$ means that for every cardinal λ , every regular ultrafilter \mathcal{D} over a set of cardinality λ , and every model \mathcal{M} of T and \mathcal{N} of U , if the ultrapower $\mathcal{N}_{\mathcal{D}}$ is λ^+ -saturated then so is the ultrapower $\mathcal{M}_{\mathcal{D}}$. T and U are \trianglelefteq -equivalent if $T \trianglelefteq U$ and $U \trianglelefteq T$. The relation \trianglelefteq is obviously transitive. It is shown in [Ke67] that $T \trianglelefteq T$, so \trianglelefteq is a pre-ordering on the set of all first order

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¹In this paper, "theory" always means "complete theory".

theories. In a remarkable series of recent papers, Malliaris and Shelah clarified the behavior of \trianglelefteq on first order theories. They showed that \trianglelefteq gives a useful classification of simple theories, that is so rich that it reveals 2^{\aleph_0} kinds of simple theories with countable vocabularies.

The modern treatment of continuous model theory deals with metric structures. A surprisingly large part of first order model theory can be generalized to the continuous case. Many notions and results in first order model theory have analogues for metric structures, including ultrapowers and κ -saturated structures. In recent years, the model theory of metric structures has had exciting mathematical applications in analysis.

In this paper we study the analogue of the relation \trianglelefteq on the class of metric theories (complete theories of metric structures). Some care is needed in our terminology, because we will be comparing metric theories with possibly different vocabularies.

Metric structures are defined as follows. By a vocabulary V we mean a countable set of constant, function, predicate symbols. A continuous formula in V is built from atomic formulas using continuous functions on the real unit interval $[0, 1]$ as connectives, and the quantifiers \inf and \sup . A *real-valued structure with vocabulary V* (briefly, a *V -structure*) is like a first order structure except that the formulas take truth values in $[0, 1]$, with 0 representing truth. A *metric signature L* over V specifies a distinguished binary predicate symbol $d \in V$, called the distance predicate of L , and a uniform continuity bound for each function and predicate symbol in V . An *L -metric structure* is a V -structure \mathcal{M} in which the distance predicate of L is a complete metric and all the function and predicate symbols of V respect the bounds of uniform continuity. First order structures with equality are just L -metric structures where the distance predicate of L is the discrete metric and all atomic formulas have truth values in $\{0, 1\}$. We define a *metric structure* to be a real-valued structure \mathcal{M} that is an L -metric structure for some metric signature L .

We next define what we mean by a metric theory. By a *V -sentence* we mean a continuous sentence in the vocabulary V . Given an L -metric structure \mathcal{M} where L is a metric signature over a vocabulary V , the *theory of \mathcal{M}* is the set $\text{Th}(\mathcal{M})$ of all V -sentences that have truth value 0 in \mathcal{M} . We say that T is a *metric theory* if $T = \text{Th}(\mathcal{M})$ for some metric structure \mathcal{M} . If \mathcal{M} is a metric structure and $T = \text{Th}(\mathcal{M})$, we call \mathcal{M} a *metric model* of T .

We say that a regular ultrafilter \mathcal{D} over a set I of cardinality λ *saturates* a metric structure \mathcal{M} if the ultrapower $\mathcal{M}_{\mathcal{D}}$ is λ^+ -saturated. For two metric theories T, U with possibly different vocabularies, we write $T \trianglelefteq U$ if for every metric model \mathcal{M} of T and \mathcal{N} of U , every regular ultrafilter that saturates \mathcal{N} saturates \mathcal{M} . We will prove the following results.

- A. Theorem 3.5: For any metric theory T (with a countable vocabulary), we have $T \trianglelefteq T$. This means that for any two metric models \mathcal{M}, \mathcal{N} of T , \mathcal{M} and \mathcal{N} are saturated by the same regular ultrafilters. It follows that \trianglelefteq is a pre-ordering (reflexive and transitive), and that $T \trianglelefteq U$ if and only if there exist metric models \mathcal{M} of T and \mathcal{N} of U such that every ultrafilter that saturates \mathcal{N} saturates \mathcal{M} .

Note that from the set of sentences $\text{Th}(\mathcal{M})$ one can recover the vocabulary V but not the metric signature L . In fact, it is possible for a metric theory T in the sense

of this paper to have many metric models with different distance predicates in V . By Result A, one can and should view \trianglelefteq as a relation between metric theories.

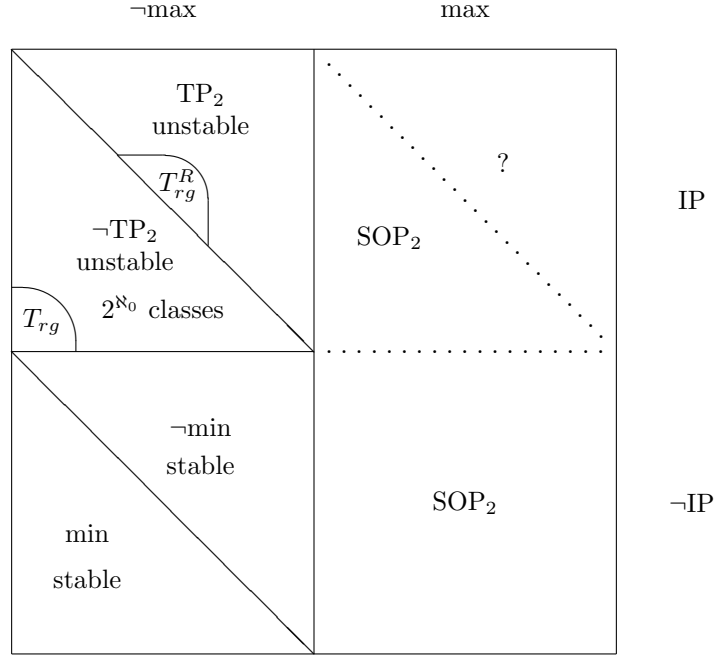
- B. Theorem 6.22: For metric theories T and U , if T is stable and U is unstable then $T \triangleleft U$.
- C. Theorem 7.12: There are just two \trianglelefteq -equivalence classes of stable metric theories, the \trianglelefteq -minimal theories and the others.
- D. Theorem 8.6: If T satisfies the metric analogue of the property SOP_2 then T is \trianglelefteq -maximal among all metric theories.
- E. Theorem 9.7: The first order theory T_{rg} of the random graph is \trianglelefteq -minimal among all unstable metric theories.
- F. Let T_{rg}^R be the randomization of T_{rg} . (A model of T_{rg}^R is built from a model of T_{rg} by replacing elements by random elements). Theorem 10.16: T_{rg}^R is \trianglelefteq -minimal among all metric theories with TP_2 , the tree property of the second kind. Moreover, $T_{rg} \triangleleft T_{rg}^R$, and T_{rg}^R is \trianglelefteq -equivalent to the first order theory called T_{feq}^* .

These results are summarized in Figure 1. The increasing direction in the pre-ordering is toward the upper right. The lower left quadrant of Figure 1 has the stable theories, the right half has the \trianglelefteq -maximal theories, and the top half has the theories with the independence property (IP). Every unstable theory has either IP or SOP_2 . It is open whether or not T_{rg}^R is \trianglelefteq -maximal, but Figure 1 is drawn as if it is not. The results A–C were announced in the author’s plenary lecture at the March, 2020 meeting of the Association for Symbolic Logic, which was cancelled by the covid-19 pandemic.

Historically, continuous model theory for real-valued structures was introduced by C. C. Chang and the author in 1966 in the monograph [CK66]. Four decades later, the 2008 paper [BBHU08] introduced the modern treatment of continuous model theory for metric structures. In [CK66] the real-valued structures had an equality predicate. In the current treatment, the real-valued structures do not have equality, and are the continuous analogue of first order structures without equality. Metric structures are the continuous analogue of first order structures with equality, with the distance predicate playing the role of equality.

The ultrapower construction for first order structures was introduced by Łoś [Lo55]. Frayne, Morel and Scott [FMS62] used ultrapowers modulo regular ultrafilters. The first order analogue of the result A appeared in Keisler [Ke67], and the first order analogues of B and C appeared in Shelah [Sh78]. For the next three decades, there was no further progress on the relation \trianglelefteq on first order theories until a breakthrough by Malliaris in [Ma09]. In that paper, Malliaris introduced notions, such as the distribution of a set of formulas in an ultrapower, that were key to subsequent work on the \trianglelefteq -order. The main result in [Ma09] showed that an ultrapower of a first order structure modulo a regular \mathcal{D} is λ^+ -saturated if, for every first order formula φ , the ultrapower is λ^+ -saturated for φ -types.

The first order analogues of the results D, E, and F appeared in the following papers. D: Malliaris and Shelah [MS16a]; E: Malliaris [Ma12]; and F: Malliaris and Shelah [MS13]. Those results show that the relation \trianglelefteq on first order theories also looks like Figure 1 (with T_{feq}^* instead of T_{rg}^R). The proofs of the first order results required the construction of ultrafilters with properties like being good. Kunen’s method of constructing ultrafilters via independent families of sets played a key role. Good ultrafilters were introduced in [Ke64] and constructed assuming the

FIGURE 1. The \leq order on metric theories

generalized continuum hypothesis (GCH). In 1972, Kunen [Ku72] used independent families of sets to construct good ultrafilters without the GCH.

The work of Malliaris and Shelah involved two parallel lines of research: the construction of ultrafilters with special set-theoretic properties, and the interplay between the \leq relation and model-theoretic properties of first order theories.

The results in this paper are obtained by extending the second of the above lines of research to metric theories. Along the way we obtain other results about metric structures that may be of interest in their own right. Stability and indiscernible sequences in metric theories have been studied extensively in many papers (see Section 6 below). However, as by-products of his proof of the results B and C above for first order theories, Shelah in [Sh78] proved many additional results about indiscernible sequences and the finite cover property, which are generalized to the continuous case here. Metric theories with the property TP_2 and the independence property have been studied in the paper [BY13]. The notion of an SOP_2 metric theory here may be new.

In the continuous model theory literature, a metric signature is usually held fixed and the ultraproduct is introduced for families of metric structures with the same metric signature. To prove the results in this paper we will need a more general ultraproduct construction, for families of real-valued structures with the same vocabulary. That leads us to consider the \leq relation between theories of arbitrary real-valued structures, rather than just between metric theories.

It turns out that the pre-ordering \leq is essentially the same in that case. For a real-valued structure \mathcal{M} with vocabulary V , $\text{Th}(\mathcal{M})$ is again defined as the set of all V -sentences with truth value 0 in \mathcal{M} , and we call $\text{Th}(\mathcal{M})$ a continuous theory with vocabulary V . We have

$$\{\text{first order theories}\} \subsetneq \{\text{metric theories}\} \subsetneq \{\text{continuous theories}\}.$$

We define the relation $T \leq U$ for continuous theories in the same way as for metric theories. Theorem 3.6 below is a stronger version of result A above, showing that $T \leq T$ for every continuous theory T , so any two real-valued models \mathcal{M}, \mathcal{N} of T (metric or not) are saturated by the same regular ultrafilters. Theorem 5.1 below will show that every continuous theory is \leq -equivalent to a metric theory. We may therefore identify the \leq -equivalence classes of continuous theories with the \leq -equivalence classes of metric theories in a way that preserves the \leq relation.

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2. PRELIMINARIES ON CONTINUOUS MODEL THEORY

We refer to [BBHU08] and [BU09] for background material on metric and pre-metric structures. See the Introduction above for the notion of a vocabulary V and a metric signature L over V .

Convention 2.1. *Throughout this paper, V will denote a countable vocabulary. To insure that the collection of all vocabularies is a set rather than a proper class, we also require that the elements of a vocabulary are hereditarily finite sets.*

By a *real-valued structure* \mathcal{M} with vocabulary V and universe set M (briefly, a V -structure) we mean an object that has a non-empty universe set M , and interpretations:

- $c^{\mathcal{M}} \in M$ for each constant symbol $c \in V$,
- $F^{\mathcal{M}} : M^n \rightarrow M$ for each n -ary function symbol $F \in V$,
- $P^{\mathcal{M}} : M^n \rightarrow [0, 1]$ for each n -ary predicate symbol $P \in V$.

We say that \mathcal{M} is a real-valued structure if \mathcal{M} is a V -structure for some V .

Let L be a metric signature over V with distance predicate $d \in V$. An L -premetric structure is a V -structure \mathcal{M} such that the interpretation $d^{\mathcal{M}}$ is a pseudo-metric on M , and for each predicate or function symbol $S \in V$, the interpretation $S^{\mathcal{M}}$ respects the modulus of uniform continuity given by L . An L -metric structure is an L -premetric structure such that the interpretation $d^{\mathcal{M}}$ is a complete metric on M . We say that \mathcal{M} is a *premetric structure* if there exists a metric signature L such that \mathcal{M} is an L -premetric structure. Similarly for metric structures.²

The notion of an atomic formula is the same as in first order logic. We assume familiarity with the notions of a continuous formula, a continuous sentence, and the truth value $\varphi^{\mathcal{M}}(\vec{a})$ of a continuous formula φ in a metric structure \mathcal{M} at a tuple \vec{a} in M . By a V -formula we mean a continuous formula built from symbols in V . The notion of truth value of a V -formula is exactly the same for arbitrary real-valued structures as for metric structures. We write $\varphi(\vec{v})$ to indicate that all the free variables of φ are in \vec{v} , and use similar notation for sets of formulas. A set $\Phi(\vec{v})$ of

²In [BBHU08], premetric structures are called “prestructures”, and metric structures are called “structures”. In [BU09], vocabularies are called “non-metric signatures”. In [Ke23], real-valued structures are called “general structures”.

V -formulas is said to be *satisfiable in* \mathcal{M} if there exist a $|\vec{v}|$ -tuple \vec{a} in M such that $\varphi^{\mathcal{M}}(\vec{a}) = 0$ for all $\varphi(\vec{v}) \in \Phi(\vec{v})$. A mapping $h: \mathcal{M} \rightarrow \mathcal{N}$ is an *elementary embedding* if for every tuple \vec{a} in M and every V -formula $\varphi(\vec{v})$ we have $\varphi^{\mathcal{M}}(\vec{a}) = \varphi^{\mathcal{N}}(h(\vec{a}))$. The elementary equivalence relation $\mathcal{M} \equiv \mathcal{N}$ and the elementary extension relation $\mathcal{M} \prec \mathcal{N}$ are defined as in [BBHU08], and are applied to arbitrary V -structures as well as to metric structures.

When a vocabulary V is clear from the context, V -formulas will simply be called formulas.

Given a V -structure \mathcal{M} and a (possibly uncountable) set $A \subseteq M$, we let $V_A = V \cup \{c_a: a \in A\}$ be the set formed by adding to V a new constant symbol c_a for each element $a \in A$. \mathcal{M}_A denotes the structure obtained from \mathcal{M} by interpreting c_a by a for each $a \in A$. The constant symbols $c_a, a \in A$, are called *parameters from* A , or *parameters from* \mathcal{M} . By a formula with parameters from A we mean a formula obtained from a formula without parameters by replacing some of the free variables by parameters in A . We say that a set Γ of formulas with parameters from A is *satisfiable in* \mathcal{M} if Γ is satisfiable in \mathcal{M}_A , and *finitely satisfiable in* \mathcal{M} if every finite subset of Γ is satisfiable in \mathcal{M}_A .

Recall from the Introduction that for a V -structure \mathcal{M} , the theory of \mathcal{M} is the set $\text{Th}(\mathcal{M})$ of all V -sentences φ such that $\varphi^{\mathcal{M}} = 0$. Thus $\text{Th}(\mathcal{M}) = \text{Th}(\mathcal{N})$ if and only if $\mathcal{M} \equiv \mathcal{N}$. Also, T is a *continuous theory* if $T = \text{Th}(\mathcal{M})$ for some real-valued structure \mathcal{M} , and T is a *metric theory* if $T = \text{Th}(\mathcal{M})$ for some metric structure \mathcal{M} .

Remark 2.2. *For each metric signature L with distance predicate d , there is a set of V -sentences that expresses the fact that d is a pseudo-metric and each function and predicate has the required modulus of uniform continuity.*

Fact 2.3. ³ *For every L -premetric structure \mathcal{M} , there is an L -metric structure \mathcal{N} , the completion of \mathcal{M} , such that $\mathcal{N} \equiv \mathcal{M}$, so $\text{Th}(\mathcal{M})$ is an L -metric theory.*

Corollary 2.4. *A V -theory T is an L -metric theory if and only if T contains the set of sentences mentioned in Remark 2.2.*

We say that \mathcal{M} is a *model* of a set of sentences U , in symbols $\mathcal{M} \models U$, if $\varphi^{\mathcal{M}} = 0$ for all $\varphi \in U$. Thus \mathcal{M} is a model of $\text{Th}(\mathcal{M})$. If $T = \text{Th}(\mathcal{M})$ and $\mathcal{M} \models U$, we also write $T \models U$. We write $\mathcal{M} \models \varphi$, or $\mathcal{M} \models \varphi(\vec{a}) = 0$, if $\varphi^{\mathcal{M}}(\vec{a}) = 0$. We write $\mathcal{M} \models \varphi(\vec{a}) \leq r$ if $\varphi^{\mathcal{M}}(\vec{a}) \leq r$.

Corollary 2.5. *If T is an L -metric theory, then every model of T is an L -premetric structure.*

Proof. By Remark 2.2. □

Fact 2.6. *Suppose \mathcal{M} is an L -premetric structure and L has distance predicate d . For any continuous formula $\varphi(\vec{v})$ in the vocabulary of \mathcal{M} , $\varphi^{\mathcal{M}}$ is a mapping from $M^{|\vec{v}|}$ into $[0, 1]$ that is uniformly continuous with respect to $(M, d^{\mathcal{M}})$.*

Definition 2.7. We say that a V -structure \mathcal{M} is κ -saturated if for every set $A \subseteq M$ of cardinality $|A| < \kappa$, every set of V -formulas with parameters from A that is finitely satisfiable in \mathcal{M} is satisfiable in \mathcal{M} .

By a *strict continuous formula* we mean a formula built from atomic formulas in finitely many steps using only the quantifiers sup, inf , and the connectives $0, 1, \text{min}, \text{max}, \cdot/2, \div$ (where $x \div y = \max(x - y, 0)$).

³When we state a fact without giving a reference, the result can be found in [BBHU08].

Remark 2.8.

- $|x - y| = \max(x \dot{-} y, y \dot{-} x)$.
- $x \dot{-} y = 0$ if and only if $x \leq y$.
- $\mathcal{M} \models \theta \leq r$ if and only if $\mathcal{M} \models \theta \dot{-} r$.
- We sometimes use the alternate notation $\theta \leq \psi$ for the formula $\theta \dot{-} \psi$.

Remark 2.9. *The set of strict V -formulas is countable.*

Fact 2.10. *(See Theorem 6.3 in [BBHU08]). For each V -formula $\varphi(\vec{x})$ and $\varepsilon > 0$ there is a strict V -formula $\theta(\vec{x})$ such that $|\varphi^{\mathcal{M}}(\vec{x}) - \theta^{\mathcal{M}}(\vec{x})| < \varepsilon$ for every \vec{x} in every V -structure \mathcal{M} .*

Lemma 2.11. *If $\kappa > \aleph_0$, then a V -structure \mathcal{M} is κ -saturated if and only if every set of fewer than κ formulas with parameters from M that is finitely satisfiable in \mathcal{M} is satisfiable in \mathcal{M} .*

Proof. By Remark 2.9, there are countably many strict V -formulas without parameters. Hence there are $|A| + \aleph_0$ strict V -formulas with parameters from A . So if $|A| < \kappa$ then there are fewer than κ strict V -formulas with parameters from A . The result now follows from Fact 2.10. \square

In [BBHU08], the ultraproduct of an indexed family of L -metric structures with the same metric signature L modulo an ultrafilter is defined, and it is proved to again be an L -metric structure. In this paper, we will need the slightly more general notion of an ultraproduct of an indexed family of V -structures with the same vocabulary V . We will use the following notion of reduction of V -structures from the paper [Ke23].

Definition 2.12. For $a, b \in M$, we write $a \dot{=}^{\mathcal{M}} b$ if for every atomic formula $\varphi(x, \vec{z})$ and tuple $\vec{c} \in M^{|\vec{z}|}$, $\varphi^{\mathcal{M}}(a, \vec{c}) = \varphi^{\mathcal{M}}(b, \vec{c})$. A V -structure \mathcal{M} is *reduced* if whenever $a \dot{=}^{\mathcal{M}} b$ we have $a = b$.

The relation $\dot{=}^{\mathcal{M}}$ goes back to Leibniz around 1840, The *reduction map* for \mathcal{M} is the mapping that sends each element of M to its equivalence class under $\dot{=}^{\mathcal{M}}$. The *reduction* of the V -structure \mathcal{M} is the reduced V -structure \mathcal{N} such that N is the set of equivalence classes of elements of M under $\dot{=}^{\mathcal{M}}$, and the reduction map for \mathcal{M} is an elementary embedding of \mathcal{M} onto \mathcal{N} . We say that $\mathcal{M}, \mathcal{M}'$ are *isomorphic*, in symbols $\mathcal{M} \cong \mathcal{M}'$, if there is an elementary embedding from the reduction of \mathcal{M} onto the reduction of \mathcal{M}' . The following fact gives a useful characterization of reduced V -structures \mathcal{M} in the case that \mathcal{M} is a premetric structure.

Fact 2.13. *(See Theorem 3.7 in [BBHU08]). Let \mathcal{M} be an L -premetric structure and let d be the distance predicate of L . \mathcal{M} is reduced if and only if for all $x, y \in M$, if $d^{\mathcal{M}}(x, y) = 0$ then $x = y$.*

Remark 2.14.

- \cong is an equivalence relation on V -structures.
- Every V -structure is isomorphic to its reduction.
- If there is an elementary embedding of \mathcal{M} onto \mathcal{N} , then $\mathcal{M} \cong \mathcal{N}$.
- $\mathcal{M} \cong \mathcal{N}$ implies $\mathcal{M} \equiv \mathcal{N}$.
- A V -structure is κ -saturated if and only if its reduction is κ -saturated.

Remark 2.15. *Every reduced \aleph_1 -saturated L -premetric structure \mathcal{M} is an L -metric structure.*

Proof. By Fact 2.13, $(M, d^{\mathcal{M}})$ is a metric space. Completeness of the metric follows easily from \aleph_1 -saturation. \square

Convention 2.16. λ will always denote an infinite cardinal, and I will always be a set of cardinality λ . When working with ultraproducts, we will use the notation $\langle r[t] \rangle_{t \in I}$ for a function r with domain I .

This convention will help keep things straight in the ultraproduct construction, by using t as a variable over I and leaving the symbol i as a variable ranging over the natural numbers.

Definition 2.17. Let \mathcal{D} be an ultrafilter over I . For each function $r: I \rightarrow [0, 1]$, $\lim_{\mathcal{D}} r$ is the unique element $s \in [0, 1]$ such that for each real $\varepsilon > 0$, the set $\{t \in I: |r[t] - s| < \varepsilon\}$ belongs to \mathcal{D} .

The following definition is taken from [Ke23].

Definition 2.18. Let \mathcal{D} be an ultrafilter over a set I and \mathcal{M}_t be a V -structure for each $t \in I$. The *pre-ultraproduct* $\prod^{\mathcal{D}} \mathcal{M}_t$ is the V -structure $\mathcal{M}' = \prod^{\mathcal{D}} \mathcal{M}_t$ such that:

- $M' = \prod_{t \in I} M_t$, the cartesian product.
- For each constant symbol $c \in V$, $c^{\mathcal{M}'} = \langle c^{\mathcal{M}_t} \rangle_{t \in I}$.
- For each n -ary function symbol $G \in V$ and n -tuple $\vec{a} = \langle \vec{a}[t] \rangle_{t \in I}$ in M' ,

$$G^{\mathcal{M}'}(\vec{a}) = \langle G^{\mathcal{M}_t}(\vec{a}[t]) \rangle_{t \in I}.$$

- For each n -ary predicate symbol $P \in V$ and n -tuple \vec{a} in M' ,

$$P^{\mathcal{M}'}(\vec{a}) = \lim_{\mathcal{D}} \langle P^{\mathcal{M}_t}(\vec{a}[t]) \rangle_{t \in I}.$$

The *ultraproduct* $\prod_{\mathcal{D}} \mathcal{M}_t$ is the reduction of the pre-ultraproduct $\prod^{\mathcal{D}} \mathcal{M}_t$. For each $a \in M'$ we also let $a_{\mathcal{D}}$ denote the equivalence class of a under $\dot{=}^{\mathcal{M}'}$.

Remark 2.19. It follows from Fact 2.13 that if \mathcal{M}_i is an L -metric structure with the same L for each $i \in I$, then $\prod_{\mathcal{D}} \mathcal{M}_t$ is exactly the ultraproduct of the metric structures \mathcal{M}_i as defined in [BBHU08].

The next fact is the analogue for V -structures of the fundamental theorem of Łoś. It is proved in the same way as the corresponding result for metric structures, Theorem 5.4 in [BBHU08].

Fact 2.20. (*Łoś Theorem*). Let \mathcal{M}_t be a V -structure for each $t \in I$, let \mathcal{D} be an ultrafilter over I , and let $\mathcal{M} = \prod_{\mathcal{D}} \mathcal{M}_t$ be the ultraproduct. Then for each formula φ and tuple \vec{b} in the cartesian product $\prod_{t \in I} M_t$,

$$\varphi^{\mathcal{M}}(\vec{b}_{\mathcal{D}}) = \lim_{\mathcal{D}} \langle \varphi^{\mathcal{M}_t}(\vec{b}[t]) \rangle_{t \in I}.$$

Corollary 2.21. Suppose \mathcal{D} is an ultrafilter over I , and for each $t \in I$, $\mathcal{M}_t, \mathcal{N}_t$ are V -structures.

- (i) If $\mathcal{M}_t \cong \mathcal{N}_t$ for each $t \in I$, then $\prod_{\mathcal{D}} \mathcal{M}_t \cong \prod_{\mathcal{D}} \mathcal{N}_t$.
- (ii) If $\mathcal{M}_t \equiv \mathcal{N}_t$ for each $t \in I$, then $\prod_{\mathcal{D}} \mathcal{M}_t \equiv \prod_{\mathcal{D}} \mathcal{N}_t$.

Fact 2.22. Every ultraproduct of L -metric structures with the same L is an L -metric structure.

One of the main reasons that metric signatures were introduced in the definition of a metric structure was to insure that Fact 2.22 holds.

In the case that all the \mathcal{M}_t are the same, $\mathcal{M}_t = \mathcal{M}$ for all $t \in I$, the pre-ultraproduct and ultraproduct are called the *pre-ultrapower* and *ultrapower* respectively, and are denoted by $\mathcal{M}^{\mathcal{D}}$ and $\mathcal{M}_{\mathcal{D}}$.

Corollary 2.23. *For every V -structure \mathcal{M} and ultrafilter \mathcal{D} over I , the diagonal embedding $a \mapsto I \times \{a\}$ is an elementary embedding of \mathcal{M} into $\mathcal{M}^{\mathcal{D}}$.*

Definition 2.24. An ultrafilter \mathcal{D} over I is said to be *regular* if there is a subset $\mathcal{X} \subseteq \mathcal{D}$ of cardinality λ such that each $t \in I$ belongs to only finitely many $X \in \mathcal{X}$. We also say that \mathcal{X} *regularizes* \mathcal{D} .

Fact 2.25. (*[FMS62]*) *For each infinite set I there exist regular ultrafilters over I .*

Hint: If I is the set of all finite subsets of J , let $\mathcal{X} = \{X_j : j \in J\}$ where $X_j = \{Y \subseteq I : \{j\} \subseteq Y\}$.

Remark 2.26. *Let \mathcal{D} is a regular ultrafilter over I .*

- (i) *If \mathcal{M}_t is a V -structure for each $t \in I$, then the ultraproduct $\prod_{\mathcal{D}} \mathcal{M}_t$ is \aleph_1 -saturated.*
- (ii) *If \mathcal{M}_i is an L -premetric structure for each $i \in I$, then the ultraproduct $\prod_{\mathcal{D}} \mathcal{M}_t$ is an L -metric structure.*

Proof. (i) Theorem 6.1.1 of [CK12] gives the corresponding result in first order logic. It is routine to modify the proof in [CK12] to obtain the result stated here.

(ii) By definition, $\prod_{\mathcal{D}} \mathcal{M}_t$ is a reduced structure. By Remark 2.2 and the Łoś theorem, $\prod_{\mathcal{D}} \mathcal{M}_t$ is an L -premetric structure. By Remark 2.15, $\prod_{\mathcal{D}} \mathcal{M}_t$ is \aleph_1 -saturated. Then by (i) above, $\prod_{\mathcal{D}} \mathcal{M}_t$ is an L -metric structure. \square

By a *two-valued structure*, or a *first order structure without equality*, we mean a V -structure \mathcal{M} such that for every n -ary predicate symbol $P \in V$, $P^{\mathcal{M}} : M^n \rightarrow \{0, 1\}$. By a *first order structure* we mean a two-valued L -metric structure \mathcal{M} such that:

- L has a distance predicate d and the trivial modulus of uniform continuity for each symbol in V .
- $d^{\mathcal{M}}(x, y)$ is the discrete metric; $d^{\mathcal{M}}(x, y) = 0$ if $x = y$, and $d^{\mathcal{M}}(x, y) = 1$ otherwise.

By a *first order formula* we mean a formula that is built from atomic formulas using only the quantifiers \sup, \inf , and the connectives $0, 1, \min, \max, 1 \div x$. Note that every first order formula is a strict continuous formula. A first order formula is traditionally written with the notation $\forall, \exists, \top, \perp, \vee, \wedge, \neg$ instead of $\sup, \inf, 0, 1, \min, \max, 1 \div x$. First order structures and formulas are sometimes called classical structures and formulas.

Fact 2.27. (*[FMS62]. See Proposition 4.3.7 of [CK12].*) *If \mathcal{D} is a regular ultrafilter over I and \mathcal{M} is an infinite first order structure, then $\mathcal{M}_{\mathcal{D}}$ has cardinality $|M|^{|I|}$.*

The following lemma is well-known and easily checked, but is stated explicitly here for completeness.

Lemma 2.28. *Let \mathcal{M} be a two-valued structure.*

- (i) *For every first order formula $\varphi(\vec{v})$ and $|\vec{v}|$ -tuple \vec{a} in \mathcal{M} , $\varphi^{\mathcal{M}}(\vec{a}) \in \{0, 1\}$.*

- (ii) Suppose (\mathcal{M}, \vec{a}) and (\mathcal{M}, \vec{b}) satisfy the same first order sentences. Then (\mathcal{M}, \vec{a}) and (\mathcal{M}, \vec{b}) satisfy the same continuous sentences, that is, $(\mathcal{M}, \vec{a}) \equiv (\mathcal{M}, \vec{b})$ as V -structures.
- (iii) Suppose \mathcal{M} is a first order structure that is κ -saturated in the sense of first order model theory. Then \mathcal{M} is κ -saturated as a V -structure.

If \mathcal{M} is a first order structure with vocabulary V , we abuse notation by letting $\text{Th}(\mathcal{M})$ denote the set of all first order V -sentences true in \mathcal{M} , rather than the set of continuous V -sentences true in \mathcal{M} . By a *first order theory* we mean the set $\text{Th}(\mathcal{M})$ where \mathcal{M} is a first order structure.

3. COMPARING TWO CONTINUOUS THEORIES

Convention 3.1. For the rest of this paper, \mathcal{D} will always denote a regular ultrafilter over a set I of cardinality λ .

We now define the relation \trianglelefteq on the set of all continuous theories, which contains the set of all metric theories. The definitions will be the same as for in first order model theory, but using the continuous notions of ultrapower and saturation.

Definition 3.2. Let \mathcal{M} be a V -structure. We say that \mathcal{D} *saturates* \mathcal{M} if the ultrapower $\mathcal{M}_{\mathcal{D}}$ is λ^+ -saturated (or equivalently, the pre-ultrapower $\mathcal{M}^{\mathcal{D}}$ is λ^+ -saturated).

Given a continuous theory T , we say that \mathcal{D} *saturates* T , in symbols $\mathcal{D} \in \text{Sat}(T)$, if \mathcal{D} saturates every model of T .

Definition 3.3. Given continuous theories T, U with possibly different vocabularies, we write $T \trianglelefteq U$ if $\text{Sat}(U) \subseteq \text{Sat}(T)$.

We write $T \triangleleft U$ if $T \trianglelefteq U$ but not $U \trianglelefteq T$.

We say that T, U are \trianglelefteq -equivalent, in symbols $T \trianglelefteq\triangleright U$, if $T \trianglelefteq U$ and $U \trianglelefteq T$.

Remark 3.4. $T \trianglelefteq\triangleright U$ if and only if $\text{Sat}(T) = \text{Sat}(U)$.

By Lemma 2.28, for first order theories, the relations $T \trianglelefteq U$ and $T \trianglelefteq\triangleright U$ are the same as in [Ke67].

Note that every ultraproduct of V -structures is reduced. By definition, the reduction map is an elementary embedding from the pre-ultraproduct onto the ultraproduct. We will usually work with the pre-ultraproduct rather than with the ultraproduct. One reason for this choice is that pre-ultrapowers of V -structures commute with expansions. If \mathcal{M} is a V -structure and W is a subset of V , the W -part⁴ of \mathcal{M} is the W -structure \mathcal{N} with the same universe as \mathcal{M} such that for each symbol $S \in W$, $S^{\mathcal{N}} = S^{\mathcal{M}}$. If \mathcal{N} is the W -part of \mathcal{M} , we say that \mathcal{M} is an *expansion* of \mathcal{N} . In that case, the pre-ultrapower $\mathcal{N}^{\mathcal{D}}$ is exactly the W -part of the pre-ultrapower $\mathcal{M}^{\mathcal{D}}$, while the ultrapower $\mathcal{N}_{\mathcal{D}}$ is only the reduction of the W -part of the ultrapower $\mathcal{M}_{\mathcal{D}}$. One reason that ultraproducts of V -structures will be useful in studying the \trianglelefteq relation on metric theories is that it allows us to take ultraproducts of arbitrary expansions of metric structures to a larger vocabulary.

In the next section we will prove the following theorem, which generalizes a result in [Ke67] from first order structures to real-valued structures.

⁴The W -part is called the W -reduct in first order model theory (see [CK12]). We say W -part here because the word reduction is used in a different way in continuous model theory.

Theorem 3.5. *For any continuous theory T , and hence for every metric theory T , we have $T \trianglelefteq T$.*

Corollary 3.6. *For any two models \mathcal{M}, \mathcal{N} of a continuous theory T , a regular ultrafilter \mathcal{D} saturates \mathcal{M} if and only if \mathcal{D} saturates \mathcal{N} . Thus \mathcal{D} saturates T if and only if \mathcal{D} saturates some model of T .*

It is obvious that the relation \trianglelefteq is transitive, so it follows that \trianglelefteq is a pre-ordering on the set of all continuous theories.

Definition 3.7. A continuous theory T is \trianglelefteq -minimal if $T \trianglelefteq U$ for every continuous theory U . A continuous theory U is \trianglelefteq -maximal if $T \trianglelefteq U$ for every continuous theory T . We define $\text{Sat}(T) = \text{Sat}(\mathcal{M})$ where \mathcal{M} is a model of T .

Note that any two \trianglelefteq -minimal theories are \trianglelefteq -equivalent, and any two \trianglelefteq -maximal theories are \trianglelefteq -equivalent.

We recall some definitions from [Ke64].

Definition 3.8. Let \mathcal{D} be an ultrafilter over I . For any set J , let $\mathcal{P}_{\aleph_0}(I)$ be the set of finite subsets of J . A mapping $\delta: \mathcal{P}_{\aleph_0}(J) \rightarrow \mathcal{D}$ is called *monotone* if $\delta(u) \supseteq \delta(v)$ whenever $u \subseteq v \in \mathcal{P}_{\aleph_0}(J)$, and is called *multiplicative* if $\delta(u \cup v) = \delta(u) \cap \delta(v)$ whenever $u, v \in \mathcal{P}_{\aleph_0}(J)$. We say that a mapping $\delta': \mathcal{P}_{\aleph_0}(I) \rightarrow \mathcal{D}$ *refines* δ if $\delta'(u) \subseteq \delta(u)$ for all $u \in \mathcal{P}_{\aleph_0}(J)$. \mathcal{D} is said to be *good* if whenever $|J| \leq |I|$, every monotone function $\delta: \mathcal{P}_{\aleph_0}(J) \rightarrow \mathcal{D}$ has a multiplicative refinement.

Fact 3.9. (Kunen [Ku72] in ZFC, [Ke64] under the GCH) *For every I , there exists a good ultrafilter over I .*

Lemma 3.10. ([Ke67] for the first order case).

- (i) *There exist \trianglelefteq -minimal theories. T is \trianglelefteq -minimal if and only if $\text{Sat}(T)$ is the class of all regular ultrafilters.*
- (ii) *There exist \trianglelefteq -maximal theories. T is \trianglelefteq -maximal if and only if $\text{Sat}(T)$ is the class of all good ultrafilters.*

Proof. (i): By Fact 2.27, for an infinite set M , $|M_{\mathcal{D}}| = |M^I| \geq |I|^+$. Therefore the first order theory of an infinite set with only the equality predicate is \trianglelefteq -minimal, so (i) holds.

(ii): In [Ke67], it is proved that every \mathcal{D} that saturates (for example) $(\mathcal{P}_{\aleph_0}(\mathbb{N}), \subseteq)$ or $(\mathbb{N}, +, \times)$ is good. In [Ke64], it is proved (in ZFC) that an ultrafilter \mathcal{D} is good if and only if it saturates every first order structure. The same proof works for real-valued structures, and (ii) follows. \square

Corollary 3.11. (i) *A first order theory is \trianglelefteq -minimal if and only if it is \trianglelefteq -minimal among first order theories.*

(ii) *A first order theory is \trianglelefteq -maximal if and only if it is \trianglelefteq -maximal among first order theories.*

Corollary 3.12. ([Ke67] for the first order case). *If T is \trianglelefteq -minimal and U is \trianglelefteq -maximal, then $T \triangleleft U$.*

Proof. If I is uncountable, there is a regular ultrafilter over I that is not good. \square

Let us take a brief look at the big picture. Let \mathbb{F}, \mathbb{M} , and \mathbb{C} be the sets of \trianglelefteq -equivalence classes of first order theories, metric theories, and continuous theories respectively. Then each of $(\mathbb{F}, \trianglelefteq)$, $(\mathbb{M}, \trianglelefteq)$, and $(\mathbb{C}, \trianglelefteq)$ is a partial ordering. By

Lemma 3.10, $(\mathbb{M}, \trianglelefteq)$ has a \trianglelefteq -minimal element $\min_{\mathbb{M}}$ and a \trianglelefteq -maximal element $\max_{\mathbb{M}}$, and $\min_{\mathbb{M}} \triangleleft \max_{\mathbb{M}}$. Similarly for \mathbb{C} and \mathbb{F} .

Let $\tau: \mathbb{F} \rightarrow \mathbb{M}$ be the mapping that sends the equivalence class of $Th(\mathcal{M})$ in \mathbb{F} to the equivalence class of $Th(\mathcal{M})$ in \mathbb{M} for every first order structure \mathcal{M} .

Corollary 3.13. *τ is an isomorphism from $(\mathbb{F}, \trianglelefteq)$ onto a substructure of $(\mathbb{M}, \trianglelefteq)$. Moreover, $\tau(\min_{\mathbb{F}}) = \min_{\mathbb{M}}$, and $\tau(\max_{\mathbb{F}}) = \max_{\mathbb{M}}$.*

Proof. By Corollary 3.11. □

We have not been able to answer the following important question.

Question 3.14. *Is every metric theory \trianglelefteq -equivalent to a first order theory? Equivalently, is the mapping $\tau: \mathbb{F} \rightarrow \mathbb{M}$ onto?*

The answer is “no” if and only if there are “new” \trianglelefteq -equivalence classes of metric theories that do not correspond to \trianglelefteq -equivalence classes of first order theories.

We will see later, in Theorem 5.1, that every \trianglelefteq -equivalence class in \mathbb{C} contains a metric theory. It follows that the mapping that sends the equivalence class of each metric theory in \mathbb{M} its equivalence class in \mathbb{C} is an isomorphism from $(\mathbb{M}, \trianglelefteq)$ onto $(\mathbb{C}, \trianglelefteq)$. So we regard the partial ordering $(\mathbb{C}, \trianglelefteq)$ as the same as $(\mathbb{M}, \trianglelefteq)$. Beginning in Section 6, we will investigate the structure of the partial ordering $(\mathbb{M}, \trianglelefteq)$.

One can also turn things around and use theories to compare ultrafilters instead of using ultraproducts to compare theories.

Definition 3.15. $\text{Sat}_{\mathbb{M}}(\mathcal{D})$ is the set of $H \in \mathbb{M}$ such that \mathcal{D} saturates some, or equivalently every, $T \in H$. $\text{Sat}_{\mathbb{F}}(\mathcal{D})$ is defined similarly with \mathbb{F} in place of \mathbb{M} .

Corollary 3.16. (i) *For all $H \in \mathbb{F}$, $H \in \text{Sat}_{\mathbb{F}}(\mathcal{D}) \Leftrightarrow \tau(H) \in \text{Sat}_{\mathbb{M}}(\mathcal{D})$.*

(ii) $\{\mathcal{D}: \text{Sat}_{\mathbb{F}}(\mathcal{D}) = \mathbb{F}\} = \{\mathcal{D}: \text{Sat}_{\mathbb{M}}(\mathcal{D}) = \mathbb{M}\} = \{\mathcal{D}: \mathcal{D} \text{ is good}\}$.

Proof. (i) is trivial. (ii) follows from Lemma 3.10 and Corollary 3.11. □

We now review some results of Malliaris and Shelah about first order theories. In $(\mathbb{F}, \trianglelefteq)$, the lowest three equivalence classes are, in increasing order, the minimal class, the class of stable non-minimal first order theories, and the equivalence class rg of the theory T_{rg} of the random graph. rg is the minimal class of unstable first order theories. The maximal class contains the class of SOP_2 first order theories, and Malliaris and Shelah conjecture that the maximal class is exactly the class of SOP_2 first order theories. An first order theory is simple if and only if it has neither SOP_2 nor TP_2 (and T_{rg} is simple). There are continuum many distinct equivalence classes of simple first order theories. There is an first order theory T_{feq}^* that is minimal among TP_2 theories, and hence minimal among non-simple first order theories. Also, $T_{rg} \triangleleft T_{feq}^*$. It is open whether or not the T_{feq}^* is maximal.

As mentioned in the Introduction, these first order results show that the partial ordering $(\mathbb{F}, \trianglelefteq)$ looks like Figure 1. In this paper we will show that the same picture applies to the partial ordering $(\mathbb{M}, \trianglelefteq)$. To do so, we will generalize several notions and results from first order model theory to continuous model theory.

4. ELEMENTARY EQUIVALENT STRUCTURES ARE EQUALLY SATURATABLE

In this section we will prove Theorem 3.5 above. The proof will be longer than the proof of the corresponding first order result in [Ke67], because continuous logic does not have negation, and the inf quantifier may only have approximate witnesses

in a real-valued structure. We will first prove two lemmas, and use those lemmas to prove a stronger version of Theorem 3.5 that concerns ultraproducts, rather than ultrapowers.

In what follows, \mathcal{M}, \mathcal{N} will be V -structures.

Lemma 4.1. *Suppose $\mathcal{M} \equiv \mathcal{N}$. Let $\Psi(\vec{v})$ be a finite set of formulas with the free variables \vec{v} . For every $n \in \mathbb{N}$ and every $|\vec{v}|$ -tuple \vec{a} in \mathcal{M} there is a $|\vec{v}|$ -tuple \vec{b} in \mathcal{N} such that for every $\psi \in \Psi$, $\psi^{\mathcal{M}}(\vec{a})$ is within 2^{-n} of $\psi^{\mathcal{N}}(\vec{b})$.*

Proof. Let $\vec{a} \in \mathcal{M}^{|\vec{v}|}$. For each $\psi \in \Psi$, let r_ψ be a dyadic rational that is within $2^{-(n+1)}$ of $\psi^{\mathcal{M}}(\vec{a})$. Since Ψ is finite,

$$\max_{\psi \in \Psi} |\psi(\vec{v}) - r_\psi| \leq 2^{-(n+2)}$$

is a formula that is satisfied by \vec{a} in \mathcal{M} . Then by Remark 2.8,

$$\mathcal{M} \models (\inf_{\vec{v}}) \max_{\psi \in \Psi} |\psi(\vec{v}) - r_\psi| \leq 2^{-(n+2)},$$

and since $\mathcal{M} \equiv \mathcal{N}$,

$$\mathcal{N} \models (\inf_{\vec{v}}) \max_{\psi \in \Psi} |\psi(\vec{v}) - r_\psi| \leq 2^{-(n+2)}.$$

Therefore there is a $|\vec{v}|$ -tuple \vec{b} in \mathcal{N} such that

$$\mathcal{N} \models \max_{\psi \in \Psi} |\psi(\vec{b}) - r_\psi| \leq 2^{-(n+1)},$$

and hence

$$|\psi^{\mathcal{M}}(\vec{a}) - \psi^{\mathcal{N}}(\vec{b})| \leq 2^{-n}.$$

for every $\psi \in \Psi$. □

Corollary 4.2. *Suppose $\mathcal{M} \equiv \mathcal{N}$, and \mathcal{N} is \aleph_1 -saturated. Then for every $\vec{a} \in \mathcal{M}^{|\vec{v}|}$ there exists $\vec{b} \in \mathcal{N}^{|\vec{v}|}$ such that $(\mathcal{N}, \vec{b}) \equiv (\mathcal{M}, \vec{a})$.*

Lemma 4.3. *Suppose $\mathcal{M} \equiv \mathcal{N}$. Let $\Psi(\vec{u}, \vec{v})$ be a finite set of formulas with the free variables \vec{u}, \vec{v} . For every $n \in \mathbb{N}$ and every $|\vec{v}|$ -tuple \vec{a} in \mathcal{M} there is a $|\vec{v}|$ -tuple \vec{b} in \mathcal{N} such that*

- (i) *For every $\vec{c} \in \mathcal{M}^{|\vec{u}|}$ there exists $\vec{e} \in \mathcal{N}^{|\vec{u}|}$ such that for every $\psi \in \Psi$, $\psi^{\mathcal{M}}(\vec{c}, \vec{a})$ is within 2^{-n} of $\psi^{\mathcal{N}}(\vec{e}, \vec{b})$.*
- (ii) *For every $\vec{e} \in \mathcal{N}^{|\vec{u}|}$ there exists $\vec{c} \in \mathcal{M}^{|\vec{u}|}$ such that for every $\psi \in \Psi$, $\psi^{\mathcal{M}}(\vec{c}, \vec{a})$ is within 2^{-n} of $\psi^{\mathcal{N}}(\vec{e}, \vec{b})$.*

Proof. Let (z_1, \dots, z_k) be a tuple of disjoint $|\vec{u}|$ -tuples of variables that are also disjoint from \vec{u} and \vec{v} . In \mathcal{M} , we say that an assignment of (z_1, \dots, z_k) is 2^{-n} -dense over \vec{a} if for every $|\vec{u}|$ -tuple \vec{c} there is an $\ell \leq k$ such that for every $\psi \in \Psi$, $\psi(\vec{c}, \vec{a})$ is within 2^{-n} of $\psi(z_\ell, \vec{a})$. It is clear that in \mathcal{M} , for any \vec{a} and any $n \in \mathbb{N}$, there exists an assignment (z_1, \dots, z_k) that is 2^{-n} -dense over \vec{a} .

Let $\theta(z_1, \dots, z_k, \vec{v})$ be the formula

$$(\sup_{\vec{u}}) \min_{\ell \leq k} \max_{\psi \in \Psi} (|\psi(\vec{u}, \vec{v}) - \psi(z_\ell, \vec{v})|).$$

Note that in \mathcal{M} , (z_1, \dots, z_k) is 2^{-n} -dense over \vec{a} if and only if

$$\mathcal{M} \models \theta(z_1, \dots, z_k, \vec{a}) \leq 2^{-n}.$$

Now fix a $|\vec{v}|$ -tuple \vec{a} in \mathcal{M} . Suppose $z = (z_1, \dots, z_k)$ is $2^{-(n+2)}$ -dense over \vec{a} in \mathcal{M} . Since $\mathcal{N} \equiv \mathcal{M}$ and θ is a formula, it follows from Lemma 4.1 that there exist tuples $(w_1, \dots, w_k, \vec{b})$ in \mathcal{N} such that

- (1) (w_1, \dots, w_k) is $2^{-(n+1)}$ -dense over \vec{b} in \mathcal{N} .
- (2) For each $\psi \in \Psi$ and $1 \leq \ell \leq k$, $\psi^{\mathcal{N}}(w_\ell, \vec{b})$ is within $2^{-(n+1)}$ of $\psi^{\mathcal{M}}(z_\ell, \vec{a})$.

Proof of (i): Let $\vec{c} \in \mathcal{M}^{|\vec{a}|}$. By (1), there is an $\ell \in \{1, \dots, k\}$ such that for every $\psi \in \Psi$, $\psi^{\mathcal{M}}(z_\ell, \vec{a})$ is within $2^{-(n+2)}$ of $\psi^{\mathcal{M}}(\vec{c}, \vec{a})$. Take $\vec{e} = z_\ell$. By (2), for each $\psi \in \Psi$, $\psi^{\mathcal{N}}(\vec{e}, \vec{b})$ is within $2^{-(n+1)}$ of $\psi^{\mathcal{M}}(z_\ell, \vec{a})$. Therefore $\psi^{\mathcal{N}}(\vec{e}, \vec{b})$ is within 2^{-n} of $\psi^{\mathcal{M}}(\vec{c}, \vec{a})$, as required.

Proof of (ii): Let $\vec{e} \in \mathcal{N}^{|\vec{a}|}$. By (1), there is an $\ell \in \{1, \dots, k\}$ such that for every $\psi \in \Psi$, $\psi^{\mathcal{N}}(w_\ell, \vec{b})$ is within $2^{-(n+1)}$ of $\psi^{\mathcal{N}}(\vec{e}, \vec{b})$. Take $\vec{c} = z_\ell$. By (2), for each $\psi \in \Psi$, $\psi^{\mathcal{M}}(\vec{c}, \vec{a})$ is within $2^{-(n+1)}$ of $\psi^{\mathcal{N}}(w_\ell, \vec{b})$. Therefore $\psi^{\mathcal{M}}(\vec{c}, \vec{a})$ is within 2^{-n} of $\psi^{\mathcal{N}}(\vec{e}, \vec{b})$. \square

The next result, Theorem 4.4, is the stronger form of Theorem 3.5 that we promised.

Theorem 4.4. *Suppose \mathcal{D} is a regular ultrafilter over a set I of infinite cardinality λ , $\{\mathcal{M}_t : t \in I\}$ and $\{\mathcal{N}_t : t \in I\}$ are families of real-valued structures with the same vocabulary, and $\mathcal{M}_t \equiv \mathcal{N}_t$ for each $t \in I$. Then $\prod^{\mathcal{D}} \mathcal{M}_t$ is λ^+ -saturated if and only if $\prod^{\mathcal{D}} \mathcal{N}_t$ is λ^+ -saturated.*

Our proof of Theorem 4.4 will use a continuous analogue of the notion of a distribution from Malliaris [Ma12]. Distributions give a useful criterion for a set of formulas with parameters to be satisfiable in a regular ultraproduct. In what follows, we let $\mathcal{M}' = \prod^{\mathcal{D}} \mathcal{M}_t$ be a pre-ultraproduct, A be a set of parameters in \mathcal{M}' of cardinality $|A| \leq \lambda$, and $\Gamma = \Gamma(\vec{x}, A)$ be a set of at most λ continuous formulas in the parameters A . For each $t \in I$ and $a \in A$, $a[t]$ will be the corresponding element of \mathcal{M}_t , and $A[t] = \{a[t] : a \in A\}$.

Definition 4.5. We define the approximation Γ^{ap} of Γ to be the set of formulas

$$\Gamma^{ap} = \Gamma^{ap}(\vec{x}, A) = \{\psi(\vec{x}, A) \div 2^{-n} : \psi(\vec{x}, A) \in \Gamma, n \in \mathbb{N}\}.$$

Remark 4.6. *A tuple \vec{c} satisfies Γ in \mathcal{M}' if and only if \vec{c} satisfies Γ^{ap} in \mathcal{M}' . Γ is finitely satisfiable in \mathcal{M}' if and only if Γ^{ap} is finitely satisfiable in \mathcal{M}' .*

Proof. By Fact 2.26, \mathcal{M}' is \aleph_1 -saturated. \square

Definition 4.7. A *distribution* of Γ in \mathcal{M}' is a monotone mapping $\delta : \mathcal{P}_{\aleph_0}(\Gamma^{ap}) \rightarrow \mathcal{D}$ such that:

- (a) $\delta : \mathcal{P}_{\aleph_0}(\Gamma^{ap}) \rightarrow \mathcal{X}$ for some \mathcal{X} that regularizes \mathcal{D} .
- (b) For each $\Psi \in \mathcal{P}_{\aleph_0}(\Gamma^{ap})$ and $t \in \delta(\Psi)$,

$$\mathcal{M}_t \models (\inf_{\vec{x}}) \max_{\psi \in \Psi} \psi(\vec{x}, A[t]) = 0.$$

For each $t \in I$, we define

$$\Gamma(\delta, t) = \{\psi \in \Gamma^{ap} : t \in \delta(\{\psi\})\}.$$

A distribution δ of Γ in \mathcal{M}' is *accurate* if

(c) For all $\Psi \in \mathcal{P}_{\aleph_0}(\Gamma^{ap})$,

$$\delta(\Psi) = \left\{ t \in \left(\bigcap_{\psi \in \Psi} \delta(\{\psi\}) \right) : \mathcal{M}_t \models (\inf_{\vec{x}}) \max_{\psi \in \Psi} \psi(\vec{x}, A[t]) = 0 \right\}.$$

If \mathcal{M} is a first order structure and Γ is a set of first order formulas, the notion of a distribution is the same as above except that Γ^{ap} is replaced by Γ , and $(\inf_{\vec{x}}) \max_{\psi \in \Psi} \psi(\vec{x}, A[t]) = 0$ is replaced by $(\exists \vec{x}) \bigwedge_{\psi \in \Psi} \psi(\vec{x}, A[t])$.

Remark 4.8. Let δ be a distribution of Γ in \mathcal{M}' , and let $\Psi \in \mathcal{P}_{\aleph_0}(\Gamma^{ap})$.

- (i) For each $t \in I$, $\Gamma(\delta, t)$ is finite.
- (ii) $\delta(\Psi) \subseteq \{t \in I : \Psi \subseteq \Gamma(\delta, t)\} \in \mathcal{D}$.
- (iii) If δ is multiplicative, then $\delta(\Psi) = \{t \in I : \Psi \subseteq \Gamma(\delta, t)\}$ and δ is accurate.
- (iv) We think of $\Gamma(\delta, t)$ as the part of Γ^{ap} that matters at the index t .

Lemma 4.9. The following are equivalent.

- (i) Γ is finitely satisfiable in \mathcal{M}' .
- (ii) Γ has a distribution in \mathcal{M}' .
- (iii) Γ has an accurate distribution in \mathcal{M}' .

Proof. (ii) \Rightarrow (i): By Remark 2.26, \mathcal{M}' is \aleph_1 -saturated, so by Remark 4.6, Γ is finitely satisfiable in \mathcal{M}' if and only if Γ^{ap} is finitely satisfiable in \mathcal{M}' . If δ is a distribution of Γ in \mathcal{M}' and Ψ is a finite subset of Γ^{ap} , then $\delta(\Psi) \in \mathcal{D}$, so by (c), the Loś Theorem, and the \aleph_1 -saturation of \mathcal{M}' , Ψ is satisfiable in \mathcal{M}' .

(i) \Rightarrow (iii): Suppose Γ^{ap} is finitely satisfiable in \mathcal{M}' . Since \mathcal{D} is regular, \mathcal{D} has a regularizing set \mathcal{X} . Thus $\mathcal{X} \subseteq \mathcal{D}$, $|\mathcal{X}| = \lambda$, and each $t \in I$ belongs to only finitely many $X \in \mathcal{X}$. So there is an injective function $h: \Gamma^{ap} \rightarrow \mathcal{X}$. For $\psi \in \Gamma^{ap}$ let $\delta(\{\psi\}) = h(\psi)$. Then the mapping δ defined by condition (c) of Definition 4.7 is an accurate distribution of Γ in \mathcal{M}' . \square

Since \mathcal{D} is regular and I is infinite, we may choose sets $J_1 \supseteq J_2 \supseteq \dots$ in \mathcal{D} such that $\bigcap_n J_n$ is empty. Put $J_0 = I$, and for each $t \in I$, let $n(t)$ be the greatest $n \in \mathbb{N}$ such that $t \in J_n$.

Lemma 4.10. The following are equivalent.

- (i) Γ is satisfiable in \mathcal{M}' .
- (ii) Γ has a multiplicative distribution in \mathcal{M}' .
- (iii) Γ is finitely satisfiable in \mathcal{M}' and every accurate distribution of Γ in \mathcal{M}' has a multiplicative refinement.

Proof. (i) \Rightarrow (ii): Suppose \vec{c} satisfies Γ in \mathcal{M}' . Let \mathcal{X} regularize \mathcal{D} and let $h: \Gamma^{ap} \rightarrow \mathcal{X}$ be injective. For each $\Psi \in \mathcal{P}_{\aleph_0}(\Gamma^{ap})$, let

$$\delta(\Psi) = \bigcap_{\psi \in \Psi} (h(\psi) \cap \{t \in I : \mathcal{M}_t \models \psi(\vec{c}[t], A[t]) = 0\}).$$

It is clear that δ is multiplicative, and each $t \in I$ belongs to $\delta(\Psi)$ for only finitely many Ψ . Consider a formula $\psi(\vec{x}, A) \in \Gamma^{ap}$. For some $\theta \in \Gamma$ and $n \in \mathbb{N}$, we have $\psi(\vec{x}, A) = (\theta(\vec{a}, A) \div 2^{-n})$, and $\mathcal{M}' \models \theta(\vec{c}, A)$. By the Loś Theorem,

$$\{t \in I : \mathcal{M}_t \models \psi(\vec{c}[t], A[t]) = 0\} \in \mathcal{D}.$$

Therefore δ maps $\mathcal{P}_{\aleph_0}(\Gamma^{ap})$ into \mathcal{D} , and hence δ is a multiplicative distribution of Γ in \mathcal{M}' .

(ii) \Rightarrow (i): Now suppose δ is a multiplicative distribution of Γ in \mathcal{M}' . Let $t \in I$. By (a), the set

$$\Psi = \{\psi \in \Gamma^{ap} : t \in \delta(\{\psi\})\}$$

belongs to $\mathcal{P}_{\aleph_0}(\Gamma^{ap})$. By multiplicity, $\delta(\Psi) = \bigcap_{\psi \in \Psi} \delta(\{\psi\})$, so $t \in \delta(\Psi) \in \mathcal{D}$. By 4.7 (b), there is a tuple $\vec{c}[t]$ in \mathcal{M}_t such that

$$(\forall \psi \in \Psi) \mathcal{M}_t \models \psi(\vec{c}[t], A[t]) \leq 2^{-n(t)}.$$

Then for each $\Psi \in \mathcal{P}_{\aleph_0}(\Gamma^{ab})$ and $n \in \mathbb{N}$, we have

$$X := \delta(\Psi) \cap J_n \in \mathcal{D}, \quad (\forall t \in X) (\forall \psi \in \Psi) \mathcal{M}_t \models \psi(\vec{c}[t], A[t]) \leq 2^{-n}.$$

By the Loś Theorem, for each $\psi \in \Gamma^{ab}$ and $n \in \mathbb{N}$ we have $\mathcal{M}' \models \psi(\vec{c}, A) \leq 2^{-n}$, so $\mathcal{M}' \models \psi(\vec{c}, A) = 0$.

(iii) \Rightarrow (ii): Assume (iii). By Lemma 4.9, Γ has an accurate distribution in \mathcal{M}' , so by (iii), Γ has a multiplicative distribution in \mathcal{M}' .

(i) \Rightarrow (iii): Suppose \vec{c} satisfies Γ in \mathcal{M}' and δ is an accurate distribution of Γ in \mathcal{M}' . For $\Psi \in \mathcal{P}_{\aleph_0}(\Gamma^{ap})$, let

$$\delta'(\Psi) = \left\{ t \in \delta(\Psi) : \mathcal{M}_t \models \max_{\psi \in \Psi} \psi(\vec{c}[t], A[t]) = 0 \right\}.$$

Then δ' is a distribution of Γ in \mathcal{M}' that refines δ . Suppose $t \in \delta'(\Psi) \cap \delta'(\Theta)$. Then

$$t \in \delta(\Psi) \cap \delta(\Theta) \text{ and } \mathcal{M}_t \models \max_{\psi \in \Psi \cup \Theta} \psi(\vec{c}[t], A[t]) = 0.$$

Since δ is accurate, $t \in \delta(\Psi \cup \Theta)$, so $t \in \delta'(\Psi \cup \Theta)$ and δ' is multiplicative. \square

Proof of Theorem 4.4. Let $\mathcal{M}' = \prod^{\mathcal{D}} \mathcal{M}_t$ and $\mathcal{N}' = \prod^{\mathcal{D}} \mathcal{N}_t$. We suppose that \mathcal{M}' is λ^+ -saturated, and prove that \mathcal{N}' is λ^+ -saturated. Let B be a λ -sequence of elements of \mathcal{N}' , and assume that $\Gamma(u, B)$ is a set of at most λ formulas with parameters from B and at most u free that is finitely satisfiable in \mathcal{N}' . By Lemma 2.11, to show that \mathcal{N}' is λ^+ -saturated it suffices to prove that $\Gamma(u, B)$ is satisfiable in \mathcal{N}' . By Lemma 4.9, $\Gamma(u, B)$ has a distribution δ_1 in \mathcal{N}' . For each $\psi \in \Gamma^{ap}(u, B)$, we have $\psi = (\theta(\psi) \div 2^{-m(\psi)})$ for some $\theta(\psi) \in \Gamma(u, A)$ and $m(\psi) \in \mathbb{N}$, and we define $\psi^+ = (\theta(\psi) \div 2^{-(m(\psi)+1)})$. Then $\psi \mapsto \psi^+$ is a bijection from $\Gamma^{ap}(u, B)$ onto itself. Define $\Psi^+ = \{\psi^+ : \psi \in \Psi\}$.

For each $t \in I$, let $m(t) = \max\{m(\psi) : \psi \in \Gamma(\delta_1, t)\}$. By Lemma 4.3, there is a λ -sequence A of elements of \mathcal{M}' such that for every $t \in I$,

- (1) For every $e[t] \in \mathcal{N}_t$ there exists $c[t] \in \mathcal{M}_t$ such that for every ψ in $\Gamma(\delta_1, t)$, $\theta(\psi)^{\mathcal{N}_t}(e[t], B[t])$ is within $2^{-(m(t)+1)}$ of $\theta(\psi)^{\mathcal{M}_t}(c[t], A[t])$.
- (2) For every $c[t] \in \mathcal{M}_t$ there exists $e[t] \in \mathcal{N}_t$ such that for every ψ in $\Gamma(\delta_1, t)$, $\theta(\psi)^{\mathcal{N}_t}(e[t], B[t])$ is within $2^{-(m(t)+1)}$ of $\theta(\psi)^{\mathcal{M}_t}(c[t], A[t])$.

It follows from (1) that the mapping $\delta_2 : \mathcal{P}_{\aleph_0}(\Gamma^{ap}(u, A)) \rightarrow \mathcal{D}$ such that

$$\delta_2(\Psi(u, A)) = \delta_1(\Psi^+(u, B))$$

is a distribution of $\Gamma(u, A)$ in \mathcal{M}' , so by Lemma 4.9, $\Gamma(u, A)$ is finitely satisfiable in \mathcal{M}' . Since \mathcal{M}' is λ^+ -saturated, $\Gamma(u, A)$ is satisfiable in \mathcal{M}' . Then by Lemma 4.10, there is a multiplicative distribution δ_3 of $\Gamma(u, A)$ in \mathcal{M}' . It follows from (2) that the mapping $\delta_4 : \mathcal{P}_{\aleph_0}(\Gamma^{ap}(u, B)) \rightarrow \mathcal{D}$ such that

$$\delta_4(\Psi(u, B)) = \delta_3(\Psi^+(u, A))$$

is a multiplicative distribution of $\Gamma(u, B)$ in \mathcal{N}' , so by Lemma 4.10, $\Gamma(u, B)$ is satisfiable in \mathcal{N}' . \square

5. EVERY CONTINUOUS THEORY IS \trianglelefteq -EQUIVALENT TO A METRIC THEORY

In this short section we will prove the following.

Theorem 5.1. *For every continuous theory T there is a metric theory U (which may have a different vocabulary) such that $T \trianglelefteq U$.*

We will see that Theorem 5.1 is an easy consequence of Theorem 3.5 above and results about premetric expansions from the paper [Ke23] (Fact 5.3–5.7 below).

We say that a sequence $\langle \varphi_m(\vec{x}, \vec{y}) \rangle_{m \in \mathbb{N}}$ of formulas is *Cauchy* in a V -structure \mathcal{M} if for each $\varepsilon > 0$ there exists m such that for all $k \geq m$,

$$\mathcal{M} \models (\sup_{\vec{x}})(\sup_{\vec{y}}) |\varphi_m(\vec{x}, \vec{y}) - \varphi_k(\vec{x}, \vec{y})| \leq \varepsilon.$$

If $\langle \varphi_m(\vec{x}, \vec{y}) \rangle_{m \in \mathbb{N}}$ is Cauchy in \mathcal{M} , then there is a unique mapping from $M^{|\vec{x}|} \times M^{|\vec{y}|}$ into $[0, 1]$, denoted by $[\lim \varphi_m]^{\mathcal{M}}$, such that

$$(\forall \vec{b} \in M^{|\vec{x}|})(\forall \vec{c} \in M^{|\vec{y}|}) [\lim \varphi_m]^{\mathcal{M}}(\vec{b}, \vec{c}) = \lim_{m \rightarrow \infty} \varphi_m^{\mathcal{M}}(\vec{b}, \vec{c}).$$

Definition 5.2. Let T be a V -theory, let D be a predicate symbol that may or may not belong to V , and let $V_D = V \cup \{D\}$. We say that a V_D -theory T_e is a *premetric expansion* of T if:

- (i) There is a metric signature L_e over V_D with distance predicate D such that T_e is an L_e -metric theory.
- (ii) There is a Cauchy sequence $\langle d \rangle = \langle d_m \rangle$ of V -formulas such that the models of T_e are exactly the V_D -structures of the form $\mathcal{M}_e = (\mathcal{M}, [\lim d_m]^{\mathcal{M}})$, where \mathcal{M} is a model of T .

It follows that \mathcal{M}_e is an L_e -premetric structure, which we call the *premetric expansion* of \mathcal{M} to T_e .

Fact 5.3. (*Expansion Theorem, Theorem 3.3.4 of [Ke23]*) *Every continuous theory T has a premetric expansion.*

Fact 5.4. (*By Propositions 3.4.5 and 4.3.4 of [Ke23]*) *Every reduced V -structure \mathcal{M} has a unique topology that is metrizable by a metric D (not unique) such that (\mathcal{M}, D) is a premetric expansion of \mathcal{M} .*

The following fact can be used to show that when T_e is a premetric expansion of T , various properties hold for T if and only if they hold for T_e .

Fact 5.5. (*Lemma 4.2.1 in [Ke23]*) *Suppose T_e is a premetric expansion of T . Then for every continuous formula $\varphi(\vec{x})$ in the vocabulary of T_e , and every real $\varepsilon > 0$, there is a formula $\theta(\vec{x})$ in the vocabulary of T such that*

$$T_e \models (\sup_{\vec{x}}) |\varphi(\vec{x}) - \theta(\vec{x})| \leq \varepsilon.$$

Fact 5.6. (*Proposition 3.1.6 of [Ke23]*). *Suppose T_e is a premetric expansion of T . Then for every pre-ultraproduct $\prod^{\mathcal{D}} \mathcal{M}_t$ of models of T , $(\prod^{\mathcal{D}} \mathcal{M}_t)_e = \prod^{\mathcal{D}} ((\mathcal{M}_t)_e)$.*

Fact 5.7. (*Proposition 4.1.6 of [Ke23]*) *If \mathcal{M}_e is a premetric expansion of \mathcal{M} , then for each infinite cardinal κ , \mathcal{M}_e is κ -saturated if and only if \mathcal{M} is κ -saturated.*

Theorem 5.8. *If T is a continuous theory and T_e is a premetric expansion of T , then $T_e \trianglelefteq T$.*

Proof of Theorem 5.8. Let \mathcal{M} be a model of T . By Fact 5.6, $\prod^{\mathcal{D}}(\mathcal{M}_e) = (\prod^{\mathcal{D}}\mathcal{M})_e$, so $\prod^{\mathcal{D}}(\mathcal{M}_e)$ is a premetric expansion of $\prod^{\mathcal{D}}\mathcal{M}$. By Fact 5.7, $\prod^{\mathcal{D}}(\mathcal{M}_e)$ is λ^+ -saturated if and only if $\prod^{\mathcal{D}}\mathcal{M}$ is λ^+ -saturated. Therefore by Theorem 3.5, we have $T_e \trianglelefteq T$. \square

Proof of Theorem 5.1. By Fact 5.3, T has a pre-metric expansion T_e . By definition, T_e is a metric theory. By Theorem 5.8, $T_e \trianglelefteq T$. \square

6. STABLE THEORIES

In view of the Theorem 5.8, from here on we may confine our attention to metric theories.

Convention 6.1. *For the rest of this paper, L will be a metric signature over V with distance predicate $d \in V$, and T will be an L -metric theory.*

Thus by Corollary 2.5, every model of T is an L -premetric structure.

In this section we study the \trianglelefteq ordering on stable metric theories. Theorem 6.22 (iii) below will show that if T, U are metric theories, T is stable, and U is unstable, then $T \triangleleft U$.

The papers [BBHU08] and [BU09] gave equivalent definitions of a stable metric theory. Here we will use the definition in [BU09] (Definition 6.2 below).

For every result proved in this section, the corresponding result for first order theories was proved by Shelah in Section VI of [Sh78]. The arguments in this section are similar to the arguments in [Sh78], but generalized to the continuous case. Many first order results about Morley sequences and indiscernible sets have been extended to continuous model logic (e.g. in [BU09], [BY03], [BY09], [BY13], and [EG12]), but we will need to extend some additional results about indiscernible sets from first order logic to continuous logic here. It will be convenient to use the properties of a stable independence relation (Definition 6.5) rather than the notion of a non-forking extension of a type, and sometimes to use ultraproducts of real-valued structures rather than just metric structures.

Definition 6.2. (Local stability. See Definition 7.1 of [BU09]).

(i) A continuous formula $\varphi(\vec{x}, \vec{y})$ is *unstable with bounds* (s, r) in T if $0 \leq s < r \leq 1$ and in some model $\mathcal{M} \models T$ there is an infinite sequence $\langle \vec{a}_h, \vec{b}_h \rangle_{h \in \mathbb{N}}$ such that whenever $h < k$, $\varphi^{\mathcal{M}}(\vec{a}_h, \vec{b}_k) \leq s$ and $\varphi^{\mathcal{M}}(\vec{a}_k, \vec{b}_h) \geq r$. φ is *unstable* in T if it is unstable in T for some (s, r) , and is *stable* in T otherwise.

(ii) T is stable if every continuous formula $\varphi(\vec{x}, \vec{y})$ is stable in T .

Note that if $\varphi(\vec{x}, \vec{y})$ is unstable in T , then there is a model \mathcal{M} as in (i) that has a countable dense set.

Corollary 6.3. *T is unstable if and only if there is a continuous formula $\theta(\vec{x}, \vec{y})$ that is unstable for $(0, 1)$ in T . That is, in some model $\mathcal{M} \models T$ there is an infinite sequence $\langle \vec{a}_h, \vec{b}_h \rangle_{h \in \mathbb{N}}$ such that whenever $h < k$, $\varphi^{\mathcal{M}}(\vec{a}_h, \vec{b}_k) = 0$ and $\varphi^{\mathcal{M}}(\vec{a}_k, \vec{b}_h) = 1$.*

Proof. Suppose $\varphi(\vec{x}, \vec{y})$ is unstable for (s, r) in T . There is a continuous function $f: [0, 1] \rightarrow [0, 1]$ that maps $[0, s]$ to 0 and maps $[r, 1]$ to 1. Then the formula $f(\varphi(\vec{x}, \vec{y}))$ is unstable for $(0, 1)$ in T . \square

As is often done in the literature, we assume for convenience that there is an uncountable inaccessible cardinal Υ . That assumption simplifies things but can be avoided. We call a set *small* if it has cardinality less than Υ . By a *monster structure* we mean a Υ -saturated metric structure of cardinality $\leq \Upsilon$.

Fact 6.4. *Up to isomorphism, T has a unique monster model \mathfrak{M} . Every small model of T is elementarily embeddable in \mathfrak{M} .*

Proof. The proof that T has a unique reduced Υ -saturated model \mathfrak{M} of cardinality $\leq \Upsilon$ is similar to the proof of the analogous result in first order model theory. Since \mathfrak{M} is reduced and \aleph_1 -saturated, it is a metric structure by Remark 2.15. \square

We recall some notation from the literature. We sometimes write AB for $A \cup B$. We write $A \equiv_B A'$ if $\mathfrak{M}_{BA} \equiv \mathfrak{M}_{BA'}$. Given an n -tuple \vec{a} in the monster model \mathfrak{M} of T and a subset B of \mathfrak{M} , the *type* of \vec{a} over B is the set $\text{tp}(\vec{a}/B)$ of all formulas $\varphi(\vec{x}, B)$ with parameters in B such that $\mathfrak{M} \models \varphi(\vec{a}, B) = 0$. The set of all n -types over B is $S_n(B) = \{\text{tp}(\vec{a}/B) : \vec{a} \in \mathfrak{M}^n\}$. For each $p \in S_n(B)$ and formula $\psi(\vec{x}, B)$ with $|\vec{x}| = n$, p^ψ is the unique $r \in [0, 1]$ such that $|\psi - r| \in p$.

The following definition is equivalent to Definition 14.13 of [BBHU08], in view of Theorem 8.10 of [BU09].

Definition 6.5. A *stable independence relation* for T is a ternary relation \downarrow on the small subsets of the monster model \mathfrak{M} of T that has the following properties:

- *Invariance under automorphisms of \mathfrak{M} .*
- *Symmetry:* $A \downarrow_C B$ if and only if $B \downarrow_C A$.
- *Transitivity:* $A \downarrow_C BD$ if and only if $A \downarrow_C B$ and $A \downarrow_{BC} D$.
- *Finite Character:* $A \downarrow_C B$ if and only if $\vec{a} \downarrow_C B$ for all finite $\vec{a} \subseteq A$.
- *Existence:* For all A, B, C there exists A' such that $A' \equiv_C A$ and $A' \downarrow_C B$.
- *Strong Local Character:* For each finite \vec{a} , there exists $C \subseteq B$ of cardinality $\leq \aleph_0$ with $\vec{a} \downarrow_C B$.
- *Stationarity:* For all small B , algebraically closed C , and tuples \vec{a}, \vec{a}' ,

$$\text{If } \vec{a} \equiv_C \vec{a}', \quad \vec{a} \downarrow_C B, \quad \vec{a}' \downarrow_C B, \quad \text{then } \vec{a} \equiv_{BC} \vec{a}'.$$

Fact 6.6. (See Theorems 14.6 and 14.14 of [BBHU08].) *T is stable if and only if there is a stable independence relation for T , and also if and only if there is a unique stable independence relation for T .*

Convention 6.7. *Hereafter, when T is a stable theory, \downarrow will denote the unique stable independence relation for T .*

By a proper tuple we mean a tuple of distinct elements. Let $n \in \mathbb{N}$. A set $J \subseteq \mathfrak{M}^n$ is called *B -indiscernible* if every two proper tuples of elements of J of the same length have the same type over B . A sequence $J = \langle \vec{a}_h \rangle_{h \in \mathbb{N}}$ of elements of \mathfrak{M}^n is *B -indiscernible* if every two strictly increasing finite subsequences of J of the same length have the same type over B . Indiscernible means \emptyset -indiscernible.

Lemma 6.8. *Suppose T is stable. Then every B -indiscernible sequence is a B -indiscernible set.*

Proof. The proof is the same as the proof of the corresponding result in first order logic (see, for example, Proposition 7.1 in [Pi83]), but using the continuous notion of a stable formula in Definition 6.2 instead of the first order notion of a stable formula. \square

The next lemma is a continuous analogue of Definition III.1.5 and Lemma III.1.7 in [Sh78].

Lemma 6.9. *Suppose T is stable, $n \in \mathbb{N}$, $\mathcal{J} \subseteq \mathfrak{M}^n$ is an infinite indiscernible set, and $B \subseteq \mathfrak{M}$ is small. There is a unique type $\text{Av}(\mathcal{J}, B) \in S_k(B)$ such that for each formula $\psi(\vec{x}, B)$ with $|\vec{x}| = n$ and $\varepsilon > 0$, for all but finitely many $\vec{a} \in \mathcal{J}$ we have*

$$|\psi^{\mathfrak{M}}(\vec{a}, B) - \text{Av}(\mathcal{J}, B)^\psi| < \varepsilon.$$

Proof. Consider a formula $\psi(\vec{x}, B)$. We have $\psi^{\mathfrak{M}}(\vec{x}, B) = \psi^{\mathfrak{M}}(\vec{x}, \vec{b})$ for some finite $\vec{b} \subseteq B$. It suffices to show that there is a unique value $t(\psi) \in [0, 1]$ such that for every $\varepsilon > 0$, for all but finitely many $\vec{a} \in \mathcal{J}$ we have

$$|\psi^{\mathfrak{M}}(\vec{a}, \vec{b}) - t(\psi)| < \varepsilon,$$

because then

$$\text{Av}(\mathcal{J}, B) = \{\psi(\vec{x}, \vec{b}) : t(\psi) = 0\}$$

has the required property. It is clear that there is at most one such value $t(\psi)$.

Let us call a set $X \subseteq [0, 1]$ *broad* if X is a closed interval, and $\psi^{\mathfrak{M}}(\vec{a}, \vec{b}) \in X$ for all but finitely many $\vec{a} \in \mathcal{J}$. Note that any intersection of finitely many broad sets is broad.

Claim 6.9.1: Suppose $0 \leq s < r \leq 1$. Then at least one of the intervals $[0, r], [s, 1]$ is broad.

Proof of Claim 6.9.1. Assume not. Then there are infinitely many $\vec{a} \in \mathcal{J}$ such that $\psi^{\mathfrak{M}}(\vec{a}, \vec{b}) < s$, and infinitely many $\vec{c} \in \mathcal{J}$ such that $\psi^{\mathfrak{M}}(\vec{c}, \vec{b}) > r$. Hence there is an infinite subsequence $\langle \vec{c}_h \rangle_{h \in \mathbb{N}}$ of \mathcal{J} such that for all even $h \in \mathbb{N}$ we have $\psi^{\mathfrak{M}}(\vec{c}_h, \vec{b}) < s$, and for all odd $j \in \mathbb{N}$ we have $\psi^{\mathfrak{M}}(\vec{c}_j, \vec{b}) > r$. Taking $\vec{b}_h = \vec{b}$ for all h , we see that $\psi(\vec{x}, \vec{y})$ is unstable in T . But since T is stable by hypothesis, $\psi(\vec{x}, \vec{y})$ is stable in T by Fact 6.6. This proves Claim 6.9.1.

Claim 6.9.2: For each positive $k \in \mathbb{N}$ there exists a broad set X_k of length $\leq 2/k$.

Proof of Claim 6.9.2. There is a least integer $m_0 \leq k$ such that $[0, m_0/k]$ is broad, and a greatest integer $m_1 \leq k$ such that $[m_1/k, 1]$ is broad. Then the set

$$X_k = [0, m_0/k] \cap [m_1/k, 1] = [m_1/k, m_0/k]$$

is broad, but neither of the sets $[0, (m_0 - 1)/k], [(m_1 + 1)/k, 1]$ is broad. By Claim 6.9.1, we cannot have $m_0 - 1 > m_1 + 1$. Hence $m_0 \leq m_1 + 2$, so the interval X_k has length $\leq 2/k$. This proves Claim 6.9.2.

Since $[0, 1]$ is compact, $\bigcap_{n=1}^{\infty} X_k$ is non-empty, and contains exactly one point $t \in [0, 1]$. The result now follows with $t(\psi) = t$. \square

Corollary 6.10. *Suppose T is stable, $n \in \mathbb{N}$, $\mathcal{J} \subseteq \mathfrak{M}^n$ is an infinite indiscernible set, and $B \subseteq \mathfrak{M}$ is small. Then:*

- (i) *If \mathcal{J} is B -indiscernible, then each $\vec{a} \in \mathcal{J}$ realizes $\text{Av}(\mathcal{J}, B)$.*
- (ii) *For any infinite $\mathcal{H} \subseteq \mathcal{J}$, $\text{Av}(\mathcal{H}, B) = \text{Av}(\mathcal{J}, B)$.*

Proof. (i) is clear.

(ii) By Lemma 6.9 we have $\text{Av}(\mathcal{J}, B) \subseteq \text{Av}(\mathcal{H}, B)$. Since $\text{Av}(\mathcal{J}, B) \in S_n(B)$ and $\text{Av}(\mathcal{H}, B) \in S_n(B)$, $\text{Av}(\mathcal{H}, B) = \text{Av}(\mathcal{J}, B)$. \square

We will only need the following lemma for subsets of \mathfrak{M} , but a slightly more complicated argument will prove the corresponding result for subsets of \mathfrak{M}^n .

Lemma 6.11. *Suppose T is stable, and C is small and algebraically closed. For each element $a \in \mathfrak{M} \setminus C$ there is an infinite C -indiscernible set $\mathcal{J} \subseteq \mathfrak{M}$ such that $a \in \mathcal{J}$, and $X \downarrow_C Y$ for any disjoint $X, Y \subseteq \mathcal{J}$.*

Proof. The proof is the same as the corresponding argument in first order logic. By Existence and Symmetry, for any X and Y we have $X \downarrow_C \emptyset$ and $\emptyset \downarrow_C Y$. By Existence, one can inductively build a sequence $\mathcal{J} = \langle a_h \rangle_{h \in \mathbb{N}}$ in \mathfrak{M} such that $a_0 = a$, and for each $n \in \mathbb{N}$ we have

$$a_n \equiv_C a \text{ and } a_n \downarrow_C \{a_h : h < n\}.$$

By Anti-reflexivity and Finite Character, $a_n \neq a_h$ whenever $h < n$, so \mathcal{J} is infinite. Using Symmetry and Transitivity for \downarrow , one can show by induction on n that $X \downarrow_C Y$ for each pair X, Y of disjoint subsets of $\{a_0, \dots, a_n\}$. By Symmetry and Finite Character, it follows that $X \downarrow_C Y$ for any disjoint $X, Y \subseteq \mathcal{J}$.

By Lemma 6.8, to show that \mathcal{J} is a C -indiscernible set, it suffices to show that \mathcal{J} is a C -indiscernible sequence. This means that for all $n \in \mathbb{N}$, $X \equiv_C Y$ for all strictly increasing subsequences

$$X = \{x_1, \dots, x_n\}, Y = \{y_1, \dots, y_n\}$$

of \mathcal{J} of length n . We argue by induction on n . The result holds trivially for $n = 1$. Suppose $(X, u), (Y, v)$ are strictly increasing subsequences of \mathcal{J} of length $n + 1$, and $X \equiv_C Y$. Without loss of generality we may assume that u is equal to or before v in \mathcal{J} . Then $u \equiv_C v$, $X \downarrow_C v$, and $Y \downarrow_C v$, so by Stationarity we have $X \equiv_{Cv} Y$ and hence $(X, v) \equiv_C (Y, v)$. By a similar argument but using Symmetry, we have $u \equiv_{CX} v$, and hence

$$(X, u) \equiv_C (X, v) \equiv_C (Y, v).$$

This completes the induction. \square

The sequence \mathcal{J} in the above proof is called a *Morley sequence* in type $\text{tp}(a_0/C)$. A proof of the following fact for first order logic using Morley sequences can be found, for example, in [Ad09]. The same proof works for continuous logic (see also [EG12]).

Fact 6.12. *Every stable independence relation \downarrow has the following additional properties:*

- *Base Monotonicity:* If $C \subseteq D \subseteq B$ and $A \downarrow_C B$, then $A \downarrow_D B$.
- *Normality:* If $A \downarrow_C B$ then $AC \downarrow_C B$.
- *Anti-reflexivity:* If $a \downarrow_C a$ then a belongs to the algebraic closure of C .

Lemma 6.13. *Suppose T is stable and \mathcal{J} is an indiscernible set. Then for any set $B \subseteq \mathfrak{M}$ there is a set $X \subseteq \mathcal{J}$ such that $|X| \leq |B| + \aleph_1$, and $\mathcal{J} \setminus X$ is $(B \cup \bigcup X)$ -indiscernible.*

Proof. The argument is exactly the same as the proof of Corollary III.3.5 of [Sh78], but applied to continuous rather than first order logic. \square

Lemma 6.14. *Suppose T is stable, $\mathcal{M} \prec \mathfrak{M}$ is small and \aleph_1 -saturated, and every indiscernible set $\mathcal{J} \subseteq \mathcal{M}$ of cardinality \aleph_0 can be extended to an indiscernible set $\mathcal{H} \subseteq \mathcal{M}$ of cardinality \aleph_1 . Then \mathcal{M} is \aleph_1 -saturated.*

Proof. The proof is similar to the proof of Lemma III.3.10 in [Sh78]. Suppose \mathcal{M} is not κ -saturated. Then $\kappa > \aleph_1$, and there is a set $B \subseteq M$ of cardinality $|B| < \kappa$ and an element $a \in \mathfrak{M}$ such that no element of M realizes $\text{tp}(a/B)$. By Strong Local Character and Base Monotonicity, there is a countable algebraically closed set $C \subseteq M$ such that $a \perp_C M$.

By Lemma 6.11, there is a Morley sequence $\mathcal{J} = \langle a_h \rangle_{h \in \mathbb{N}}$ in $\text{tp}(a/M)$, with $a_0 = a$. By Corollary 6.10 (i), each $a_h \in \mathcal{J}$ realizes $\text{Av}(\mathcal{J}, M)$. Let $\mathcal{J}_e = \{a_h : h \text{ is even}\}$ and $\mathcal{J}_o = \{a_h : h \text{ is odd}\}$. Since \mathcal{M} is \aleph_1 -saturated and C is countable, there is a sequence \mathcal{J}' of elements of M such that $(\mathcal{J}_e, \mathcal{J}_o) \equiv_C (\mathcal{J}_e, \mathcal{J}')$. Then $\mathcal{J}_e \cup \mathcal{J}'$ is C -indiscernible. By Corollary 6.10 (ii),

$$\text{Av}(\mathcal{J}, M) = \text{Av}(\mathcal{J}_e, M) = \text{Av}(\mathcal{J}', M).$$

By hypothesis, \mathcal{J}' can be extended to an indiscernible set $\mathcal{H} \subseteq M$ of cardinality κ . By Lemma 6.13, there is a set $X \subseteq \mathcal{H}$ such that $|X| \leq |BC| + \aleph_1 < \kappa$ and $\mathcal{H}' = \mathcal{H} \setminus X$ is BC -indiscernible. Then \mathcal{H}' is infinite. By Corollary 6.10 (ii),

$$\text{Av}(\mathcal{J}', M) = \text{Av}(\mathcal{H}, M) = \text{Av}(\mathcal{H}', M),$$

so

$$\text{Av}(\mathcal{J}, BC) = \text{Av}(\mathcal{H}', BC).$$

Pick an element $a' \in \mathcal{H}'$. Then $a' \in M$. By Corollary 6.10 (i), a realizes $\text{Av}(\mathcal{J}, BC)$ and a' realizes $\text{Av}(\mathcal{H}', BC)$. But then $a \equiv_{BC} a'$, which contradicts the fact that no element of M realizes $\text{tp}(a/B)$. So \mathcal{M} is κ -saturated after all. \square

The following definition is from Shelah [Sh78].

Definition 6.15. Let \mathcal{D} be a regular ultrafilter. The *lower cofinality* $\text{lcf}(\mathcal{D})$ is the least cardinal κ such that there is a co-initial set C of κ infinite elements of the ultrapower $(\mathbb{N}, \leq)_{\mathcal{D}}$. That is, C is a set of infinite elements of $(\mathbb{N}, \leq)_{\mathcal{D}}$, $|C| = \kappa$, and for each infinite element $e \in (\mathbb{N}, \leq)_{\mathcal{D}}$, we have $(\mathbb{N}, \leq)_{\mathcal{D}} \models c \leq e$ for some $c \in C$.

Definition 6.16. Let \mathcal{D} be a regular ultrafilter. The *lower cardinality* $\text{lca}(\mathcal{D})$ is the least cardinality of an infinite initial segment of the ultrapower $(\mathbb{N}, \leq)_{\mathcal{D}}$. That is, the least infinite cardinal λ such that for some $a \in (\mathbb{N}, \leq)_{\mathcal{D}}$, $\lambda = |\{b : (\mathbb{N}, \leq)_{\mathcal{D}} \models b \leq a\}|$.

Note that for every regular ultrafilter \mathcal{D} over I , we have

$$\aleph_0 < \text{lcf}(\mathcal{D}) \leq \text{lca}(\mathcal{D}) \leq |2^I|.$$

Fact 6.17. (Theorem VI.3.12 in [Sh78]). If $\aleph_0 < \kappa \leq \mu = \mu^{\aleph_0} \leq |2^I|$, then there is a regular ultrafilter \mathcal{D} over I such that $\text{lcf}(\mathcal{D}) = \kappa$ and $\text{lca}(\mathcal{D}) = \mu$.

Definition 6.18. Let $\varepsilon \in [0, 1]$, and Δ be a finite set of formulas with free variables included in \vec{y} that contains at least the formula $d(y_0, y_1)$. A sequence $\mathcal{J} \subseteq \mathcal{M}$ is (Δ, ε) -indiscernible in \mathcal{M} if for all proper $|\vec{y}|$ -tuples \vec{b}, \vec{c} of elements of \mathcal{J} and every $\varphi(\vec{y}) \in \Delta$, we have

$$|\varphi(\vec{b})^{\mathcal{M}} - \varphi(\vec{c})^{\mathcal{M}}| \leq \varepsilon.$$

\mathcal{J} is r -separated if $d^{\mathcal{M}}(u, v) \geq r$ for some $u, v \in \mathcal{J}$.

We make some easy observations about (Δ, ε) -indiscernibility. Let $1 \geq r > \varepsilon$. If \mathcal{J} is r -separated, then any superset of \mathcal{J} in \mathcal{M} is r -separated. Since $d(y_0, y_1)$ is in Δ , if \mathcal{J} is (Δ, ε) -indiscernible in \mathcal{M} and r -separated, then two elements $u, v \in \mathcal{J}$ are distinct if and only if $d^{\mathcal{M}}(u, v) \geq r - \varepsilon$.

For each $k \in \mathbb{N}$, there is a single formula $\varphi(\vec{z})$ with $|\vec{z}| = k$ such that for each k -tuple \mathcal{H} in \mathcal{M} , $\varphi^{\mathfrak{M}}(\mathcal{H})$ holds if and only if \mathcal{H} is (Δ, ε) -indiscernible in T and r -separated. \mathcal{J} is indiscernible in T if and only if it is $(\Delta, 1/n)$ -indiscernible in T for all non-empty finite Δ and all $0 < n \in \mathbb{N}$. The property of (Δ, ε) -indiscernibility gets stronger as Δ gets larger and ε gets smaller.

The proof of the next lemma is similar to the argument for Theorem VI.5.1 (2) in [Sh78], but is simpler because the argument in [Sh78] also dealt with three other more difficult cases at the same time.

Lemma 6.19. *If T is stable, \mathcal{D} is a regular ultrafilter, and $\mathcal{M} \models T$, then $\mathcal{M}_{\mathcal{D}}$ is $\text{lca}(\mathcal{D})$ -saturated.*

Proof. By Remark 2.26, $\mathcal{M}_{\mathcal{D}}$ is \aleph_1 -saturated. By Lemma 6.14, it suffices to prove that every countably infinite indiscernible set $\mathcal{J} = \{a_h : h \in \mathbb{N}\}$ in $\mathcal{M}_{\mathcal{D}}$ can be extended to an indiscernible set \mathcal{H} in $\mathcal{M}_{\mathcal{D}}$ of cardinality $\geq \text{lca}(\mathcal{D})$.

\mathcal{J} is $2r$ -separated for some $1 \geq r > 0$. Let $\mathcal{M}' = (\mathcal{M}, S, \in)$ be the two-sorted real-valued structure where one sort is M , the other sort is the set S of finite subsets of M , and \in is the \in relation on $M \times S$. (\mathcal{M}' is not a metric structure.) Let V and V' be the vocabularies of \mathcal{M} and \mathcal{M}' . Let \mathbf{z} be a variable of sort S . For every finite set $\Delta(\vec{y})$ of strict V -formulas containing $d(y_0, y_1)$ and every $1/r < n \in \mathbb{N}$, there is a V' -formula $\varphi_{\Delta, n, r}(\mathbf{z})$ saying that the set \mathbf{z} is $(\Delta, 1/n)$ -indiscernible and r -separated. By Remark 2.26, the ultrapower $\mathcal{M}'_{\mathcal{D}}$ is \aleph_1 -saturated. The set of V' -formulas

$$\Gamma(\mathbf{z}) = \{a_h \in \mathbf{z} : h \in \mathbb{N}\} \cup \{\varphi_{\Delta, n, r}(\mathbf{z}) : \Delta \text{ finite and strict}, 0 < n \in \mathbb{N}\}$$

is finitely satisfiable in $\mathcal{M}'_{\mathcal{D}}$, because each finite subset $\Gamma_0 \subseteq \Gamma$ is satisfied by the element $b_{0\mathcal{D}}$ of $S_{\mathcal{D}}$ where for each $t \in I$, $b_0[t]$ is the set of $a_h[t]$ such that a_h occurs in Γ_0 . $\Gamma(\mathbf{z})$ is countable because there are only countably many strict V -formulas. Therefore $\Gamma(\mathbf{z})$ is satisfied in $\mathcal{M}'_{\mathcal{D}}$ by some element $b_{\mathcal{D}}$ of sort S in $\mathcal{M}'_{\mathcal{D}}$. It follows that the set

$$\mathcal{H} = \{a \in M_{\mathcal{D}} : \mathcal{M}'_{\mathcal{D}} \models a \in b\}$$

contains \mathcal{J} and is indiscernible in $\mathcal{M}_{\mathcal{D}}$. Since \mathcal{H} is infinite, we have

$$|\mathcal{H}| = \left| \prod_{\mathcal{D}} b[t] \right| \geq \text{lca}(\mathcal{D}),$$

as required. \square

Lemma 6.20. *Suppose \mathcal{D} is a regular ultrafilter, and the ultrapower $\mathcal{M}_{\mathcal{D}}$ is $\text{lcf}(\mathcal{D})^+$ -saturated. Then $\text{Th}(\mathcal{M})$ is stable.*

Proof. Suppose $\varphi(\vec{x}, \vec{y})$ is an unstable formula in $\text{Th}(\mathcal{M})$. By Theorem 3.5, we may assume that $\mathcal{M} \prec \mathfrak{M}$. There are dyadic rationals $r > s$ in $[0, 1]$, and a sequence $\langle (\vec{a}_k, \vec{b}_k) \rangle_{k \in \mathbb{N}}$ of tuples in \mathcal{M} such that whenever $h < j < k$ in \mathbb{N} we have $\varphi^{\mathcal{M}}(\vec{a}_j, \vec{b}_h) \geq r$ and $\varphi^{\mathcal{M}}(\vec{a}_j, \vec{b}_k) \leq s$. For simplicity we give the proof in the case that \vec{x}, \vec{y} are 1-tuples, and that $\varphi^{\mathcal{M}}(a_j, b_h) \geq r$ and $\varphi^{\mathcal{M}}(a_j, b_k) \leq s$ whenever $h \leq j < k$. (The proof in general is essentially the same). We may take the b_k to be distinct. Consider the countable first order structure (K, \leq^K) where $K = \{b_m : m \in \mathbb{N}\}$ and $\leq^K = \{(b_m, b_n) : m \leq n\}$. Let f be the unique isomorphism from (\mathbb{N}, \leq) onto (K, \leq^K) , and $f_{\mathcal{D}}$ be the corresponding isomorphism from $(\mathbb{N}, \leq)_{\mathcal{D}}$ onto $(K, \leq^K)_{\mathcal{D}}$.

Note that $K \subseteq M$, so $K_{\mathcal{D}} \subseteq M_{\mathcal{D}}$. Let $\kappa = \text{lcf}(\mathcal{D})$. There is a co-initial set C of infinite elements of $(\mathbb{N}, \leq)_{\mathcal{D}}$ such that $|C| = \kappa$. Let $\Sigma(x)$ be the set of formulas

$$\Sigma(x) = \{r \leq \varphi(x, f_{\mathcal{D}}(n)) : n \in \mathbb{N}\} \cup \{\varphi(x, f_{\mathcal{D}}(c)) \leq s : c \in C\}.$$

(By Remark 2.8, we are using \leq as an alternate notation for \vdash). Then $\Sigma(x)$ is finitely satisfiable in $\mathcal{M}_{\mathcal{D}}$ and has cardinality κ .

We claim that $\Sigma(x)$ is not satisfiable in $\mathcal{M}_{\mathcal{D}}$. To see this, suppose a satisfies $\Sigma(x)$ in $\mathcal{M}_{\mathcal{D}}$. Then for some $X \in \mathcal{D}$, for every $t \in X$ there exists $e[t] \in \mathbb{N}$ such that $\varphi^{\mathcal{M}}(a[t], b_{e[t]}) \geq r$ and $\varphi^{\mathcal{M}}(a[t], b_{e[t]+1}) \leq s$. Then $\varphi^{\mathcal{M}_{\mathcal{D}}}(a, f_{\mathcal{D}}(e)) \geq r$ and $\varphi^{\mathcal{M}_{\mathcal{D}}}(a, f_{\mathcal{D}}(e+1)) \leq s$. Moreover, e is an infinite element of $(\mathbb{N}, \leq)_{\mathcal{D}}$, so $c \leq_{\mathcal{D}} e$ for some $c \in C$. Then $\varphi^{\mathcal{M}_{\mathcal{D}}}(a, f_{\mathcal{D}}(c)) \geq r$, which contradicts the assumption that a satisfies $\Sigma(x)$ in $\mathcal{M}_{\mathcal{D}}$. This shows that $\Sigma(x)$ is not satisfiable in $\mathcal{M}_{\mathcal{D}}$, so $\mathcal{M}_{\mathcal{D}}$ is not κ^+ -saturated. \square

Definition 6.21. *LCA* is the class of regular ultrafilters \mathcal{D} over sets I such that $|I| < \text{lca}(\mathcal{D})$.

LCF is the class of regular ultrafilters \mathcal{D} over sets I such that $|I| < \text{lcf}(\mathcal{D})$.

By Fact 6.17, *LCF* is a proper subclass of *LCA*. In fact, whenever $\aleph_1 < 2^{|I|}$, there is a regular ultrafilter \mathcal{D} over I such that $\mathcal{D} \in LCA \setminus LCF$.

Theorem 6.22.

- (i) T is stable if and only if $\text{Sat}(T) \supseteq LCA$.
- (ii) T is unstable if and only if $\text{Sat}(T) \subseteq LCF$.
- (iii) If T is stable and U is unstable, then $T \triangleleft U$.

Proof. Let \mathcal{M} be a model of T .

(i) Forward: Suppose T is stable. By Lemma 6.19, $\mathcal{M}_{\mathcal{D}}$ is $\text{lca}(\mathcal{D})$ -saturated. If $\mathcal{D} \in LCA$, then $|I| < \text{lca}(\mathcal{D})$, so $\mathcal{M}_{\mathcal{D}}$ is $|I|^+$ -saturated and $\mathcal{D} \in \text{Sat}(T)$. Therefore $\text{Sat}(T) \supseteq LCA$.

(ii) Suppose T is unstable. By Lemma 6.20, $\mathcal{M}_{\mathcal{D}}$ is not $\text{lcf}(\mathcal{D})^+$ -saturated. If $\mathcal{D} \in \text{Sat}(T)$, then $\mathcal{M}_{\mathcal{D}}$ is $|I|^+$ -saturated, so $|I| < \text{lcf}(\mathcal{D})$ and $\mathcal{D} \in LCF$. Thus $\text{Sat}(T) \subseteq LCF$. Now suppose $\text{Sat}(T) \subseteq LCF$. Then $\text{Sat}(T) \supseteq LCA$ fails, so T is unstable by (i) forward,

(i) Reverse: Suppose T is unstable. Take $\mathcal{D} \in LCA \setminus LCF$. By (ii), $\mathcal{D} \notin \text{Sat}(T)$, so $\text{Sat}(T) \supseteq LCA$ fails.

(iii) follows by (i), (ii), and Fact 6.17. \square

Corollary 6.23. *There are regular ultrafilters \mathcal{D} that saturate all stable metric theories but no unstable metric theories. The class of all such \mathcal{D} is $LCA \setminus LCF$.*

Proof. Let $\text{stb}_{\mathbb{M}}$ be the set of \trianglelefteq -equivalence classes of stable metric theories in \mathbb{M} , and $\text{stb}_{\mathbb{F}}$ be the set of \trianglelefteq -equivalence classes of stable first order theories in \mathbb{F} . By Fact 6.17,

$$\{\mathcal{D} : \text{Sat}_{\mathbb{F}}(\mathcal{D}) = \text{stb}_{\mathbb{F}}\} = LCA \setminus LCF \neq \emptyset.$$

By Theorem 6.22, we have

$$\{\mathcal{D} : \text{Sat}_{\mathbb{F}}(\mathcal{D}) = \text{stb}_{\mathbb{F}}\} = \{\mathcal{D} : \text{Sat}_{\mathbb{M}}(\mathcal{D}) = \text{stb}_{\mathbb{M}}\}.$$

\square

7. \trianglelefteq -MINIMAL THEORIES

Theorem 7.10 below gives a characterization of \trianglelefteq -minimal metric theories. Theorem 7.12 shows that there are exactly two \trianglelefteq -classes of stable metric theories, and that they are the lowest two \trianglelefteq -equivalence classes. We are indebted to James Hanson for pointing out an error in the definition of NFCP and the proof of Theorem 7.10 in an earlier version of this section. To correct that error we modified the definition of NFCP from the earlier version.

For first order logic, in [Sh78], Section VI, Shelah identified the first two classes in the partial ordering $(\mathbb{F}, \trianglelefteq)$.

Definition 7.1. We denote the set of first order theories that are stable but not \trianglelefteq -minimal by $\text{stb}\neg\text{min}_{\mathbb{F}}$. We denote the set of metric theories that are stable but not \trianglelefteq -minimal by $\text{stb}\neg\text{min}_{\mathbb{M}}$.

Fact 7.2. (Theorem VI.5.9 in [Sh78]),

- (i) $\text{stb}\neg\text{min}_{\mathbb{F}} \in \mathbb{F}$.
- (ii) For each $H \in \mathbb{F}$, either $H = \text{min}_{\mathbb{F}}$, $H = \text{stb}\neg\text{min}_{\mathbb{F}}$, or $\text{stb}\neg\text{min}_{\mathbb{F}} \triangleleft H$.

In Theorem 7.12 and Corollary 7.13 below, we will prove the analogous results for metric theories. In this section, T will be a complete metric theory and \mathcal{M} will denote an \aleph_1 -saturated model of T .

Definition 7.3. Let $\varepsilon \in [0, 1]$ and $\Sigma(\vec{y})$ be a set of formulas with parameters in \mathcal{M} (where \vec{y} is a possibly infinite sequence of variables). We say that $\Sigma(\vec{y})$ is ε -satisfiable in \mathcal{M} if there is a $|\vec{y}|$ -sequence of elements \vec{a} of \mathcal{M} such that $\psi^{\mathcal{M}}(\vec{a}) \leq \varepsilon$ for all $\psi \in \Sigma(\vec{y})$.

Corollary 7.4. Let $\Sigma(\vec{y})$ be a set of formulas with parameters in \mathcal{M} .

- (i) $\Sigma(\vec{y})$ is ε -satisfiable in \mathcal{M} if and only if the set of formulas

$$\{\psi(\vec{y}) \div \varepsilon : \psi \in \Sigma\}$$

is satisfiable in \mathcal{M} .

- (ii) If $\Sigma(\vec{y})$ is finite, $\Sigma(\vec{y})$ is ε -satisfiable in \mathcal{M} if and only if the sentence $(\inf_{\vec{y}}) \max(\Sigma(\vec{y})) \div \varepsilon$ holds in \mathcal{M} .

- (iii) If $\Sigma(\vec{y})$ is countable and every finite subset of $\Sigma(\vec{y})$ is ε -satisfiable in \mathcal{M} , then $\Sigma(\vec{y})$ is ε -satisfiable in \mathcal{M} .

Proof. (i) is a restatement of the definition, and (ii) follows easily from the fact that \mathcal{M} is \aleph_1 -saturated.

- (iii): By (i), the countable set of formulas

$$\{\psi(\vec{y}) \div \varepsilon : \psi(\vec{y}) \in \Sigma(\vec{y})\}$$

is finitely satisfiable in \mathcal{M} . Since \mathcal{M} is \aleph_1 -saturated, that set of formulas is satisfiable in \mathcal{M} . By (i), $\Sigma(\vec{y})$ is ε -satisfiable in \mathcal{M} . \square

Definition 7.5. Let \vec{x} be an n -tuple of distinct variables, $B \subseteq \mathcal{M}$, and $\Delta(\vec{x})$ be a finite set of formulas with free variables among \vec{x} . By an n -tuple over (\vec{y}, B) we mean an n -tuple (u_0, \dots, u_{n-1}) where each u_i is in $\vec{y} \cup B$, and no variable or parameter appears more than once in \vec{u} . By a Δ -formula over (\vec{y}, B) we mean a formula $\psi(\vec{u})$ obtained by a formula $\psi(\vec{x}) \in \Delta$ by replacing \vec{x} by an n -tuple \vec{u} over (\vec{y}, B) .

In the case that $B = M$, we sometimes say “over (\vec{y}, \mathcal{M}) ” instead of “over (\vec{y}, B) ”.

Definition 7.6. T has the *non-finite cover property* (NFCP) in \mathcal{M} (a better name would be “finite satisfaction property”) if T is stable, and for each non-empty finite set $\Delta(\vec{x})$ of formulas and $\varepsilon \in (0, 1]$, there is an $m(\Delta, \varepsilon) \in \mathbb{N}$ such that for every finite set $\Sigma(\vec{y})$ of Δ -formulas over (\vec{y}, \mathcal{M}) , if every subset of $\Sigma(\vec{y})$ of cardinality $\leq m(\Delta, \varepsilon)$ is satisfiable in \mathcal{M} , then $\Sigma(\vec{y})$ is ε -satisfiable in \mathcal{M} .

T has the NFCP if T has the NFCP in every \aleph_1 -saturated model of T .

We will see in Corollary 7.11 below that a first order theory has the NFCP if and only if it does not have the finite cover property as defined in [Ke67].

Lemma 7.7. *If T has the NFCP in some \aleph_1 -saturated model of T , then T has the NFCP.*

Proof. Suppose T has the NFCP in \mathcal{M} , and \mathcal{N} is another \aleph_1 -saturated model of T . For every countable sequence $\vec{b} \subseteq \mathcal{M}$ there is a countable sequence $\vec{c} \subseteq \mathcal{N}$ such that $(\mathcal{M}, \vec{b}) \equiv (\mathcal{N}, \vec{c})$. By Corollary 7.4, a countable set of formulas $\Sigma(\vec{y}, \vec{b})$ is ε -satisfiable in \mathcal{M} if and only if $\Sigma(\vec{y}, \vec{c})$ is ε -satisfiable in \mathcal{N} . It follows that T has the NFCP in \mathcal{N} as well as in \mathcal{M} . \square

Corollary 7.8. *Suppose T has the NFCP and $\Delta(\vec{x})$, ε , and $m(\Delta, \varepsilon)$ are as in Definition 7.6. Then for every countable set $\Sigma(\vec{y})$ of Δ -formulas over (\vec{y}, \mathcal{M}) , if every subset of $\Sigma(\vec{y})$ of cardinality $\leq m(\Delta, \varepsilon)$ is satisfiable in \mathcal{M} , then $\Sigma(\vec{y})$ is ε -satisfiable in \mathcal{M} .*

Proof. By Corollary 7.4. \square

Lemma 7.9. *Suppose T has the NFCP. Then for each $r/2 > \varepsilon' > \varepsilon \geq 0$ and finite set of formulas $\Delta(\vec{x})$ containing $d(x_0, x_1)$, there exists $k \in \mathbb{N}$ such that every r -separated finite (Δ, ε) -indiscernible set in \mathcal{M} of cardinality $\geq k$ can be extended to an (Δ, ε') -indiscernible set in \mathcal{M} of cardinality \aleph_0 .*

Proof. Let $n = |\vec{x}|$ and \vec{x}' be an n -tuple of new distinct variables, and let Δ' be the finite set of formulas

$$\Delta' = \Delta \cup \{|\varphi(\vec{x}) - \varphi(\vec{x}')| \div \varepsilon : \varphi(\vec{x}) \in \Delta\}.$$

Suppose that \mathcal{J} is (Δ, ε) -indiscernible in \mathcal{M} , r -separated, and at most countable. Let \vec{y} be a countable sequence of variables and let

$$\Sigma(\vec{y}, \mathcal{J}) = \{|\varphi(\vec{u}) - \varphi(\vec{v})| \div \varepsilon : \vec{u}, \vec{v} \text{ are } n\text{-tuples over } (\vec{y}, \mathcal{J}) \text{ and } \varphi \in \Delta\}.$$

Then $\Sigma(\vec{y}, \mathcal{J})$ is a countable set of Δ' -formulas over (\vec{y}, \mathcal{J}) .

Claim 7.9.1. *Assume that \vec{a} is a countable sequence in \mathcal{M} that $(\varepsilon' - \varepsilon)$ -satisfies $\Sigma(\vec{y}, \mathcal{J})$. Then \vec{a} is disjoint from \mathcal{J} and $\mathcal{J} \cup \vec{a}$ is (Δ, ε') -indiscernible in \mathcal{M} .*

Proof of Claim 7.9.1. Work in \mathcal{M} . Since \mathcal{J} is r -separated, for each distinct b, c in \mathcal{J} we have $d(b, c) \geq r - \varepsilon$. Then for each $a_i \in \vec{a}$ and $c \in \mathcal{J}$ we have $|d(a_i, c) - d(b, c)| \leq \varepsilon'$, so $d(a_i, c) \geq r - 2\varepsilon' > 0$ and hence $a_i \notin \mathcal{J}$. For each pair of n -tuples \vec{u}, \vec{v} over (\vec{y}, \mathcal{J}) and $\varphi(\vec{x}) \in \Delta$, the formula $|\varphi(\vec{u}) - \varphi(\vec{v})| \div \varepsilon$ belongs to $\Sigma(\vec{y}, \mathcal{J})$. Since \vec{a} $(\varepsilon' - \varepsilon)$ -satisfies $\Sigma(\vec{y}, \mathcal{J})$ and \mathcal{J} is (Δ, ε) -indiscernible, \vec{a} satisfies $|\varphi(\vec{u}) - \varphi(\vec{v})| \leq \varepsilon'$. Therefore $\mathcal{J} \cup \vec{a}$ is (Δ, ε') -indiscernible in \mathcal{M} . This proves Claim 7.9.1. \square

Now let $m := m(\Delta', \varepsilon' - \varepsilon)$ and $k := 4mn$.

Claim 7.9.2. *Assume \mathcal{J} has cardinality $\geq k$. Then every subset of $\Sigma(\vec{y}, \mathcal{J})$ of cardinality $\leq m$ is satisfiable in \mathcal{M} .*

Proof of Claim 7.9.2. Let $\Sigma_0(\vec{z})$ be a subset of $\Sigma(\vec{y}, \mathcal{J})$ of cardinality $\leq m$ where \vec{z} is the finite subsequence of \vec{y} whose terms actually occur in $\Sigma_0(\vec{z})$. $\Sigma_0(\vec{z})$ is contained in $\Sigma(\vec{y}, \mathcal{H})$ for some $\mathcal{H} \subseteq \mathcal{J}$ of cardinality at most $2mn$. Also, the length of \vec{z} is at most $2mn$. Since \mathcal{J} has cardinality $\geq k = 4mn$, there is a sequence \vec{c} of distinct elements of $\mathcal{J} \setminus \mathcal{H}$ of length $|\vec{z}|$. Since \mathcal{J} is (Δ, ε) -indiscernible in \mathcal{M} , $\Sigma_0(\vec{z})$ is satisfied by \vec{c} in \mathcal{M} . This proves Claim 7.9.2. \square

By Corollary 7.8 and Claim 7.9.2, $\Sigma(\vec{y}, \mathcal{J})$ is $(\varepsilon' - \varepsilon)$ -satisfiable by some infinite sequence \vec{a} in \mathcal{M} . Then by Claim 7.9.1, $\mathcal{J} \cup \vec{a}$ is (Δ, ε') -indiscernible in \mathcal{M} and has cardinality \aleph_0 . \square

The following theorem gives a characterization of \triangleleft -minimal theories.

Theorem 7.10. *Let $T = \text{Th}(\mathcal{M})$. The following are equivalent:*

- (i) T is \triangleleft -minimal.
- (ii) $\text{Sat}(T) \not\supseteq LCA$.
- (iii) T has the NFCEP.
- (iv) T is stable and satisfies the conclusion of Lemma 7.9.

Proof. (i) \Rightarrow (ii): Assume (i). By Lemma 3.10, $\text{Sat}(T)$ is the class of all regular ultrafilters, so $\text{Sat}(T) \supseteq LCA$. By Fact 6.17, if $|I| \geq 2^{\aleph_0}$ then there is a regular ultrafilter \mathcal{D} over I such that $\text{lca}(\mathcal{D}) = 2^{\aleph_0} \leq |I|$, so $\mathcal{D} \in \text{Sat}(T) \setminus LCA$ and (ii) holds.

(ii) \Rightarrow (iii): Assume (ii) holds but (iii) fails. By (ii) and Theorem 6.22 (i), T is stable. By (ii), there is an ultrafilter \mathcal{D} over I such that $\mathcal{D} \in \text{Sat}(T) \setminus LCA$. Since $\mathcal{D} \notin LCA$, $\text{lca}(\mathcal{D}) \leq |I|$. Since (iii) fails, there is non-empty finite set of formulas $\Delta(\vec{x})$, and an $\varepsilon > 0$ such that for each $m \in \mathbb{N}$ there is a finite tuple \vec{b}_m of parameters in \mathcal{M} and a finite set $\Sigma_m(\vec{y}, \vec{b}_m)$ of Δ -formulas over (\vec{y}, \vec{b}_m) with the following property:

Each subset of Σ_m of cardinality $\leq m$ is satisfiable in \mathcal{M} , but Σ_m is not ε -satisfiable in \mathcal{M} .

Let $f(m)$ be the smallest cardinality of a set Σ_m with the above property. Then $m < f(m) \in \mathbb{N}$, and in \mathcal{M} , every subset of Σ_m of cardinality $\leq m$ is satisfiable, and every proper subset of Σ_m is ε -satisfiable, but Σ_m is not ε -satisfiable. Whenever $f(0) < \ell \in \mathbb{N}$, let $g(\ell)$ be the greatest $m \in \mathbb{N}$ such that $f(m) \leq \ell$. For each $\ell \leq f(0)$, let $g(\ell) = 0$. Then $g(\ell) < \max(\ell, f(0))$ for all $\ell \in \mathbb{N}$, and $\lim_{\ell \rightarrow \infty} g(\ell) = \infty$.

By the definition of $\text{lca}(\mathcal{D})$, there is an $\eta \in (\mathbb{N}, \leq)_{\mathcal{D}}$ such that

$$|\{\beta: (\mathbb{N}, \leq)_{\mathcal{D}} \models \beta \leq \eta\}| = \text{lca}(\mathcal{D}).$$

Then in $(\mathbb{N}, \leq, g)_{\mathcal{D}}$, $g(\eta)$ is infinite and $g(\eta) \leq \eta$, so

$$|\{\beta: (\mathbb{N}, \leq, g)_{\mathcal{D}} \models \beta \leq g(\eta)\}| = \text{lca}(\mathcal{D}).$$

For each $t \in I$, let $\Sigma[t](\vec{y})$ be the set of formulas $\Sigma_{g(\eta[t])}(\vec{y})$, which is finite because $\eta[t] \in \mathbb{N}$. Since Δ is finite, if $\psi[t](\vec{x}) \in \Delta$ for each $t \in I$ and

$$\{t \in I: \psi[t](\vec{y}, \vec{b}[t]) \in \Sigma[t](\vec{y})\} \in \mathcal{D},$$

then there is a unique Δ -formula $\theta(\vec{y}, \vec{b}_{\mathcal{D}})$ over $(\vec{y}, \mathcal{M}_{\mathcal{D}})$ such that

$$\{t \in I: \psi[t](\vec{y}, \vec{b}[t]) = \theta(\vec{y}, \vec{b}[t])\} \in \mathcal{D}.$$

Let $\Sigma_{\mathcal{D}}(\vec{y})$ be the set of all Δ -formulas $\theta(\vec{y}, \vec{b}_{\mathcal{D}})$ over $(\vec{y}, \mathcal{M}_{\mathcal{D}})$ such that

$$\{t: \theta(\vec{y}, \vec{b}[t]) \in \Sigma[t]\} \in \mathcal{D}.$$

Then $|\Sigma_{\mathcal{D}}(\vec{y})| = \text{lca}(\mathcal{D})$, and in $\mathcal{M}_{\mathcal{D}}$, every finite subset of $\Sigma_{\mathcal{D}}(\vec{y})$ is ε -satisfiable, but $\Sigma_{\mathcal{D}}(\vec{y})$ is not ε -satisfiable. By Remark 2.26, the ultrapower $\mathcal{M}_{\mathcal{D}}$ is an \aleph_1 -saturated model of T . Let

$$\Gamma_{\mathcal{D}}(\vec{y}) = \{\theta(\vec{y}, \vec{b}_{\mathcal{D}}) \div \varepsilon: \theta(\vec{y}, \vec{b}_{\mathcal{D}}) \in \Sigma_{\mathcal{D}}(\vec{y})\}.$$

Then $\Gamma_{\mathcal{D}}(\vec{y})$ is a set of formulas with parameters in $\mathcal{M}_{\mathcal{D}}$ of cardinality $\text{lca}(\mathcal{D})$ that is finitely satisfiable but not satisfiable in $\mathcal{M}_{\mathcal{D}}$, so $\mathcal{M}_{\mathcal{D}}$ is not $\text{lca}(\mathcal{D})^+$ -saturated. Since $\text{lca}(\mathcal{D}) \leq I$, $\mathcal{M}_{\mathcal{D}}$ is not I^+ -saturated, contradicting the hypothesis that $\mathcal{D} \in \text{Sat}(T)$. Therefore (ii) implies (iii).

By Lemma 7.9, (iii) implies (iv).

(iv) \Rightarrow (i): Assume (iv). Our argument will be similar to the proof of Theorem VI.5.1 (1) of [Sh78] and of Lemma 6.19 above. Let \mathcal{D} be a regular ultrafilter over I , and let \mathcal{M} be an \aleph_1 -saturated model of T . We prove that $\mathcal{M}_{\mathcal{D}}$ is $2^{|I|}$ -saturated, and hence $|I|^+$ -saturated and $\mathcal{D} \in \text{Sat}(T)$. Since \mathcal{D} is arbitrary, this will prove (i).

By Remark 2.26, $\mathcal{M}_{\mathcal{D}}$ is \aleph_1 -saturated. Suppose that $\mathcal{J} = \langle a_h \rangle_{h \in \mathbb{N}}$ is an indiscernible ω -sequence in $\mathcal{M}_{\mathcal{D}}$.

Let $r = d^{\mathcal{M}_{\mathcal{D}}}(a_0, a_1)/2$. Then $r > 0$ and \mathcal{J} is $2r$ -separated. By Lemma 6.14, to prove that $\mathcal{M}_{\mathcal{D}}$ is $2^{|I|}$ -saturated it is enough to show that \mathcal{J} can be extended to an indiscernible sequence of cardinality $2^{|I|}$ in $\mathcal{M}_{\mathcal{D}}$. Let $\mathcal{M}' = (\mathcal{M}, S, \in)$ be the two-sorted real-valued structure where one sort is M , the other sort is the set S of ω -sequences \mathbf{b} of elements of M such that $d^{\mathcal{M}}(\mathbf{b}_i, \mathbf{b}_j) \geq r$ whenever $i < j$, \mathbf{z} is a variable of sort S , and \in is the predicate on $M \times S$ such that the formula $a \in \mathbf{z}$ has value 0 if a occurs in \mathbf{z} , and has value 1 otherwise. (Again, \mathcal{M}' is not a metric structure.) By Remark 2.26, $\mathcal{M}'_{\mathcal{D}}$ is \aleph_1 -saturated. For each $t \in I$, the sequence $\mathcal{J}[t] = \langle a_h[t] \rangle_{h \in \mathbb{N}}$ is an element of sort S in \mathcal{M}' . Let $\mathcal{J}_{\mathcal{D}}$ be the corresponding element of sort S in $\mathcal{M}'_{\mathcal{D}}$.

Let $\varphi_{\Delta, n, r}(\mathbf{z})$ be the V' -formula saying that the sequence \mathbf{z} is $(\Delta, 1/n)$ -indiscernible and r -separated. Let

$$\Gamma(\mathbf{z}) = \{a_h \in \mathbf{z}: h \in \mathbb{N}\} \cup \{\varphi_{\Delta, n, r}(\mathbf{z}): \Delta \text{ finite and strict, } n \in \mathbb{N} \cap (2/r, \infty)\},$$

which is a countable set of V' -formulas.

Claim 7.10.1. Suppose \mathbf{b} satisfies $\Gamma(\mathbf{z})$ in $\mathcal{M}'_{\mathcal{D}}$ and

$$\mathcal{H} = \{a \in \mathcal{M}_{\mathcal{D}}: \mathcal{M}'_{\mathcal{D}} \models a \in \mathbf{b}\}.$$

Then $\mathcal{H} \supseteq \mathcal{J}$, \mathcal{H} is indiscernible and r -separated in $\mathcal{M}_{\mathcal{D}}$, and $|\mathcal{H}| = 2^{|I|}$.

Proof of Claim 7.10.1. Since \mathbf{z} has sort S , $\mathbf{b}[t]$ has sort S in \mathcal{M}' for each $t \in I$, and \mathbf{b} is the element of the ultrapower corresponding to $\langle \mathbf{b}[t] \rangle_{t \in I}$. The set of formulas $\Gamma(\mathbf{z})$ guarantees that $a_h \in \mathcal{H}$ for each $h \in \mathbb{N}$ so $\mathcal{H} \supseteq \mathcal{J}$. Also, \mathcal{H} is $(\Delta, 1/n)$ -indiscernible and r -separated in $\mathcal{M}_{\mathcal{D}}$ for each $n \in \mathbb{N}$, and hence \mathcal{H} is indiscernible and r -separated in $\mathcal{M}_{\mathcal{D}}$. For each $t \in I$, the set

$$\mathcal{H}[t] = \{c \in \mathcal{M}: \mathcal{M}' \models c \in \mathbf{b}[t]\}$$

has cardinality \aleph_0 . Then by Fact 2.27, we have

$$|\mathcal{H}| = |\mathbb{N}_{\mathcal{D}}| = \aleph_0^{|I|} = 2^{|I|}.$$

□

Claim 7.10.2. $\Gamma(\mathbf{z})$ is finitely satisfiable in $\mathcal{M}'_{\mathcal{D}}$.

Proof of Claim 7.10.2. For each $n \in \mathbb{N} \cap (2/r, \infty)$ and finite Δ , let

$$\Gamma_{n,\Delta}(\mathbf{z}) = \{a_h \in \mathbf{z} : h \leq n\} \cup \{\varphi_{\Delta,n,r}(\mathbf{z})\}.$$

Each $\Gamma_{n,\Delta}(\mathbf{z})$ is finite. Every finite subset of $\Gamma(\mathbf{z})$ is implied by some $\Gamma_{n,\Delta}(\mathbf{z})$, because the formulas $\varphi_{\Delta,n,r}(\mathbf{z})$ become stronger as Δ and n increase. Hence it suffices to prove that each $\Gamma_{n,\Delta}(\mathbf{z})$ is satisfiable in $\mathcal{M}'_{\mathcal{D}}$. Fix a finite Δ and $n \in \mathbb{N} \cap (2/r, \infty)$. By (iv), the conclusion of Lemma 7.9 holds. Whenever $n \in \mathbb{N} \cap (2/r, \infty)$ we have $r/2 > 1/n$. So there exists $2 \leq k \in \mathbb{N}$ such that whenever $\Delta' \subseteq \Delta$, every finite r -separated $(\Delta', 1/(n+1))$ -indiscernible set in \mathcal{M} of cardinality $> k$ can be extended to a $(\Delta', 1/n)$ -indiscernible set in \mathcal{M} of cardinality \aleph_0 .

There is a V' -formula $\theta(u_0, \dots, u_k, \mathbf{z})$ saying that $u_i \in \mathbf{z}$ for each $i \leq k$, and \vec{u} is $(\Delta, 1/(n+1))$ -indiscernible and r -separated. Let X be the set of $t \in I$ such that $\theta(a_0[t], \dots, a_k[t], \mathcal{J}[t]) = 0$ in \mathcal{M}' .

Let $\psi(u_0, \dots, u_k, \mathbf{z})$ be the V' -formula saying that either $u_i \notin \mathbf{z}$ for some $i \leq k$, or $d(u_i, u_j) \leq r$ for some $i < j \leq k$, or $|\delta(\vec{v}) - \delta(\vec{w})| \geq 1/(n+1)$ for some $\delta(\vec{x}) \in \Delta$ and some pair of $|\vec{x}|$ -subtuples \vec{v}, \vec{w} of \vec{u} . Let Y be the set of $t \in I$ such that $\psi(a_0[t], \dots, a_k[t], \mathcal{J}[t]) = 0$ in \mathcal{M}' .

One can check that for every (\vec{u}, \mathbf{z}) in \mathcal{M}' , either $\theta^{\mathcal{M}'}(\vec{u}, \mathbf{z}) = 0$ or $\psi^{\mathcal{M}'}(\vec{u}, \mathbf{z}) = 0$. Therefore $X \cup Y = I$. Since \mathcal{J} is indiscernible and $2r$ -separated in $\mathcal{M}_{\mathcal{D}}$, $\psi(\vec{u}, \mathcal{J})$ has value > 0 in $\mathcal{M}_{\mathcal{D}}$. Therefore $Y \notin \mathcal{D}$, and hence $X \in \mathcal{D}$. By the Łoś Theorem, $\theta(a_0, \dots, a_k, \mathcal{J}_{\mathcal{D}}) = 0$ in $\mathcal{M}'_{\mathcal{D}}$. So (a_0, \dots, a_k) is $(\Delta, 1/(n+1))$ -indiscernible and r -separated in $\mathcal{M}_{\mathcal{D}}$.

For each $t \in X$, $(a_0[t], \dots, a_k[t])$ has cardinality $> k$ and is $(\Delta, 1/(n+1))$ -indiscernible and r -separated in \mathcal{M} , so $(a_0[t], \dots, a_k[t])$ can be extended to a $(\Delta, 1/n)$ -indiscernible ω -sequence $\mathbf{c}[t]$ in \mathcal{M} . Therefore $\Gamma(\mathbf{c}[t])$ holds in \mathcal{M}' for each $t \in X$, and by the Łoś Theorem, $\mathbf{c}_{\mathcal{D}}$ satisfies $\Gamma(\mathbf{z})$ in $\mathcal{M}'_{\mathcal{D}}$. This proves Claim 7.10.2. \square

Since $\Gamma(\mathbf{z})$ is countable and finitely satisfiable in the \aleph_1 -saturated structure $\mathcal{M}'_{\mathcal{D}}$, there is an element \mathbf{b} that satisfies $\Gamma(\mathbf{z})$ in $\mathcal{M}'_{\mathcal{D}}$. By Claim 7.10.1, \mathcal{H} contains \mathcal{J} and is indiscernible and r -separated in $\mathcal{M}_{\mathcal{D}}$, and $|\mathcal{H}| = 2^{|I|}$. Then by Lemma 6.14, $\mathcal{M}_{\mathcal{D}}$ is $2^{|I|}$ -saturated. As mentioned above, this prove (i). \square

Corollary 7.11. *A first order theory has the NFPCP if and only if it does not have the finite cover property as defined in [Ke67].*

Proof. Let T be a complete first order theory. By Theorem VI.5.8 (i) of [Sh78], T is $\trianglelefteq_{\mathbb{F}}$ -minimal if and only if T does not have the FCP in the sense of [Ke67]. By Theorem 7.10, T is \trianglelefteq -minimal if and only if T has the NFPCP. By Lemma 3.10 (i), $\text{Sat}(T)$ is the class of all regular ultrafilters iff T is $\trianglelefteq_{\mathbb{F}}$ -minimal, and also iff T is \trianglelefteq -minimal. \square

Theorem 7.12. (i) *A metric theory T is stable but not \trianglelefteq -minimal if and only if $\text{Sat}(T) = LCA$.*

(ii) *There are exactly two \trianglelefteq -equivalence classes of stable metric theories, the set of \trianglelefteq -minimal theories and the set of stable non- \trianglelefteq -minimal theories.*

Proof. (i) By Theorem 6.22, T is stable iff $\text{Sat}(T) = LCA$ or $\text{Sat}(T) \supsetneq LCA$. By Theorem 7.10, T is \trianglelefteq -minimal iff $\text{Sat}(T) \supsetneq LCA$. So (i) holds.

(ii) If T is \leq -minimal, then T has the NFCP by Theorem 7.10, so T is stable. There are first order theories that are stable and not \leq -minimal, so the set of stable non- \leq -minimal theories is non-empty. If T and U are both stable but not minimal, then by (i), $\text{Sat}(T) = LCA = \text{Sat}(U)$, so the set of stable non- \leq -minimal theories belongs to \mathbb{M} . \square

As corollaries, we identify the first two elements in the partial ordering (\mathbb{M}, \leq) , and show that there is a \mathcal{D} such that $\text{Sat}_{\mathbb{M}}(\mathcal{D}) = \{\min_{\mathbb{M}}\}$.

Corollary 7.13. *For each $H \in \mathbb{M}$, one of the following holds:*

$$H = \min_{\mathbb{M}}, \quad H = \text{stb}\neg\min_{\mathbb{M}}, \quad \text{stb}\neg\min_{\mathbb{M}} \triangleleft H.$$

Proof. We have $\min_{\mathbb{M}} \in \mathbb{M}$ by Lemma 3.10, and $\text{stb}\neg\min_{\mathbb{M}} \in \mathbb{M}$ by Theorem 7.12. Therefore, $H \in \mathbb{M} \setminus \{\min_{\mathbb{M}}, \text{stb}\neg\min_{\mathbb{M}}\}$ if and only if the theories in H are unstable. By Fact 6.17, LCF is a proper subclass of LCA , so by Theorem 6.22, if the theories in H are unstable then $\text{stb}\neg\min_{\mathbb{M}} \triangleleft H$. \square

Corollary 7.14.

$$\{\mathcal{D} : \text{Sat}_{\mathbb{M}}(\mathcal{D}) = \{\min_{\mathbb{M}}\}\} = \{\mathcal{D} : \text{Sat}_{\mathbb{F}}(\mathcal{D}) = \{\min_{\mathbb{F}}\}\} = \{\mathcal{D} : \mathcal{D} \notin LCA\} \neq \emptyset.$$

Proof. It follows from Section VI of [Sh78] that

$$\{\mathcal{D} : \text{Sat}_{\mathbb{F}}(\mathcal{D}) = \{\min_{\mathbb{F}}\}\} = \{\mathcal{D} : \mathcal{D} \notin LCA\} \neq \emptyset.$$

We prove

$$\{\mathcal{D} : \text{Sat}_{\mathbb{M}}(\mathcal{D}) = \{\min_{\mathbb{M}}\}\} = \{\mathcal{D} : \text{Sat}_{\mathbb{F}}(\mathcal{D}) = \{\min_{\mathbb{F}}\}\}.$$

Assume that $\text{Sat}_{\mathbb{F}}(\mathcal{D}) \neq \{\min_{\mathbb{F}}\}$. Then there exists $H \in \text{Sat}_{\mathbb{F}}(\mathcal{D})$ such that $H \neq \min_{\mathbb{F}}$. By Corollaries 3.13 and 3.16, $\tau(H) \neq \min_{\mathbb{M}}$ and $\tau(H) \in \text{Sat}_{\mathbb{M}}(\mathcal{D})$, so $\text{Sat}_{\mathbb{M}}(\mathcal{D}) \neq \{\min_{\mathbb{M}}\}$.

Assume that $\text{Sat}_{\mathbb{F}}(\mathcal{D}) = \{\min_{\mathbb{F}}\}$. Then $\mathcal{D} \notin LCA$. Suppose $T \in \text{Sat}_{\mathbb{M}}(\mathcal{D})$. Then $\text{Sat}(T) \neq LCA$, so by Theorem 7.12 (i), T is either unstable or \leq -minimal. But if T is unstable, then $\mathcal{D} \in LCF$ by Theorem 6.22 (ii), contradicting $\mathcal{D} \notin LCA$, so T is stable. Therefore T is \leq -minimal, so $\text{Sat}_{\mathbb{M}}(\mathcal{D}) = \{\min_{\mathbb{M}}\}$. \square

8. \leq -MAXIMAL THEORIES AND SOP_2

Shelah [Sh78] introduced a family of “strict order properties” of first order theories, including the property SOP_2 . Malliaris and Shelah [MS16a] proved that every first order theory with SOP_2 is maximal in (\mathbb{F}, \leq) , and conjectured that the converse also holds. In this section we introduce a metric analogue SOP_2 , and show that metric theories that have SOP_2 are maximal in (\mathbb{M}, \leq) .

We refer to [Sh78], [MS15], and the expository article [Co12] for background about SOP_2 and related notions in first order logic.

We now introduce some notation for working with binary trees. We identify each $n \in \mathbb{N}$ with the set $n = \{0, \dots, n-1\}$. Let $\{0, 1\}^{<\mathbb{N}}$ be the set of finite sequences of elements of $\{0, 1\}$, let $\{0, 1\}^{\mathbb{N}}$ be the set of infinite sequences of elements of $\{0, 1\}$, and let $\{0, 1\}^n$ be the set of n -tuples of elements of $\{0, 1\}$. Thus $\{0, 1\}^{<\mathbb{N}} = \bigcup_n \{0, 1\}^n$. We regard the elements $\{0, 1\}^{<\mathbb{N}}$ as nodes of the full binary tree, and regard the elements of $\{0, 1\}^{\mathbb{N}}$ as branches of the full binary tree. For $\sigma \in \{0, 1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $\sigma|n = \langle \sigma(k) \rangle_{k < n} \in \{0, 1\}^n$. For $\mu, \nu \in \{0, 1\}^{<\mathbb{N}}$, note that $\mu \subseteq \nu$ iff μ is an initial segment of ν . We call μ, ν *comparable* if $\mu \subseteq \nu$ or $\nu \subseteq \mu$, and *incomparable* otherwise.

Definition 8.1. A metric theory T has SOP_2 if there is a formula $\theta(\vec{x}, \vec{z})$, called an SOP_2 formula for T , such that in some model $\mathcal{M} \models T$ there are $|\vec{z}|$ -tuples $\langle \vec{b}_\mu \rangle_{\mu \in \{0,1\}^{<\mathbb{N}}}$ such that for each $\sigma \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$,

$$(1) \quad \mathcal{M} \models (\inf_{\vec{x}}) \max\{\theta(\vec{x}, \vec{b}_{\sigma|k}) : k < n\} = 0,$$

but for each incomparable pair $\mu, \nu \in \{0,1\}^{<\mathbb{N}}$ we have

$$(2) \quad \mathcal{M} \models (\inf_{\vec{x}}) \max(\theta(\vec{x}, \vec{b}_\mu), \theta(\vec{x}, \vec{b}_\nu)) = 1.$$

Remark 8.2. A first order theory has SOP_2 in the above sense if and only if it has SOP_2 in the sense of [Sh78].

Corollary 8.3. T has SOP_2 if and only if for some $0 \leq s < r \leq 1$, Definition 8.1 holds with (1) and (2) replaced by

$$(3) \quad \mathcal{M} \models (\inf_{\vec{x}}) \max\{\theta(\vec{x}, \vec{b}_{\sigma|k}) : k < n\} \leq s,$$

and

$$(4) \quad \mathcal{M} \models (\inf_{\vec{x}}) \max(\theta(\vec{x}, \vec{b}_\mu), \theta(\vec{x}, \vec{b}_\nu)) \geq r.$$

We call θ an SOP_2 -formula with bounds (s, r) in T .

Proof. Similar to the proof of Corollary 6.3. \square

Fact 8.4. (Malliaris and Shelah [MS16a], Theorem 11.11). Every first order theory with SOP_2 is maximal in $(\mathbb{F}, \trianglelefteq)$.

Let \mathcal{K}_{bt} be the first order structure $(\{0,1\}^{<\mathbb{N}}, \subseteq)$ (a full binary tree). The first order theory $\text{Th}(\mathcal{K}_{bt})$ has SOP_2 . We will need the following characterization of good regular ultrafilters, which was shown and used in [MS16a] to prove Fact 8.4.

Fact 8.5. (By Theorem 10.26 and Lemma 11.6 in [MS16a]) A regular ultrafilter \mathcal{D} is good if and only if in the pre-ultrapower $\mathcal{K}_{bt}^{\mathcal{D}}$, every pairwise comparable set of cardinality $\leq |I|$ has an upper bound.

Theorem 8.6. Every metric theory T with SOP_2 is maximal in $(\mathbb{M}, \trianglelefteq)$.

Proof. Let $\mathcal{D} \in \text{Sat}(T)$. Then \mathcal{D} is a regular ultrafilter over a set I , and for every model \mathcal{M} of T , the pre-ultrapower $\mathcal{M}^{\mathcal{D}}$ is $|I|^+$ saturated. By Lemma 3.10 (ii), it suffices to prove that \mathcal{D} is good. Suppose A is a pairwise comparable subset of $\mathcal{K}_{bt}^{\mathcal{D}}$ of cardinality $\leq |I|$. By Fact 8.5 above, it is enough to show that the set of formulas $\Gamma = \{a \subseteq u : a \in A\}$ is satisfiable in $\mathcal{K}_{bt}^{\mathcal{D}}$. Since $\mathcal{K}_{bt}^{\mathcal{D}} \equiv \mathcal{K}_{bt}$, every finite pairwise comparable subset of $\mathcal{K}_{bt}^{\mathcal{D}}$ has a greatest element, so Γ is finitely satisfiable in $\mathcal{K}_{bt}^{\mathcal{D}}$.

There is an SOP_2 -formula $\theta(\vec{x}, \vec{z})$ for T , a model \mathcal{M} of T , and $|\vec{z}|$ -tuples $\langle \vec{b}_\mu \rangle_{\mu \in \{0,1\}^{<\mathbb{N}}}$ in \mathcal{M} that satisfy Conditions (1) and (2) of Definition 8.1. For each $a \in A$, let \vec{b}_a be the $|\vec{z}|$ -tuple in $\mathcal{M}^{\mathcal{D}}$ such that for each $t \in I$, $\vec{b}_a[t] = \vec{b}_{a[t]}$. Consider a finite $A_0 \subseteq A$. By the Łoś Theorem, there is a set $X_0 \in \mathcal{D}$ such that for each $t \in X_0$ the set $\{a[t] : a \in A_0\}$ is pairwise comparable in \mathcal{K}_{bt} . By Condition (1), for each $t \in X_0$ we have

$$\mathcal{M} \models (\inf_{\vec{x}}) \max\{\theta(\vec{x}, \vec{b}_a[t]) : a \in A_0\} = 0,$$

so by the Łoś Theorem,

$$\mathcal{M}^{\mathcal{D}} \models (\inf_{\vec{x}}) \max\{\theta(\vec{x}, \vec{b}_a) : a \in A_0\} = 0.$$

By Remark 2.26, $\mathcal{M}^{\mathcal{D}}$ is \aleph_1 -saturated, so there exists \vec{x} in $\mathcal{M}^{\mathcal{D}}$ such that

$$\mathcal{M}^{\mathcal{D}} \models \max\{\theta(\vec{x}, \vec{b}_a) : a \in A_0\} = 0.$$

Thus the set of formulas

$$\Gamma' = \{\theta(\vec{x}, \vec{b}_a) : a \in A\}$$

is finitely satisfiable in $\mathcal{M}^{\mathcal{D}}$. Since $\mathcal{M}^{\mathcal{D}}$ is $|I|^+$ -saturated, there is a tuple \vec{x} that satisfies Γ' in $\mathcal{M}^{\mathcal{D}}$.

Let \mathcal{X} regularize \mathcal{D} , so $\mathcal{X} \subseteq \mathcal{D}$, $|\mathcal{X}| = |I|$, and each $t \in I$ belongs to only finitely many $X \in \mathcal{X}$. Pick an injective function $h: A \rightarrow \mathcal{X}$. Consider an element $t \in I$. Let $A_t = \{a \in A : t \in h(a)\}$ be the finite set of $a \in A$ that “matter” for t . Let

$$B_t = \{a \in A_t : \mathcal{M} \models \theta(\vec{x}[t], \vec{b}_a[t]) < 1\}.$$

By Condition (2), any two elements of B_t are comparable in $\{0, 1\}^{<\aleph}$. Let $u[t] = \bigcup B_t$, the maximum element of B_t in \mathcal{K}_{bt} (with the convention that $u[t]$ is the empty sequence if B_t is empty). Then $u \in \mathcal{K}_{bt}^{\mathcal{D}}$.

Fix an element $a \in A$. Let

$$Y = h(a) \cap \{t \in I : a \in B_t\} = h(a) \cap \{t \in I : \mathcal{M} \models \theta(\vec{x}[t], \vec{b}_a[t]) < 1\}.$$

By the Loś Theorem and the fact that $\mathcal{M}^{\mathcal{D}} \models \theta(\vec{x}, \vec{b}_a) = 0$, we have $Y \in \mathcal{D}$. For each $t \in Y$ we have $\mathcal{K}_{bt} \models a[t] \subseteq u[t]$, so by the Loś Theorem again, $\mathcal{K}_{bt}^{\mathcal{D}} \models a \subseteq u$. Therefore u satisfies Γ in $\mathcal{K}_{bt}^{\mathcal{D}}$, and the proof is complete. \square

9. \leq -MINIMAL UNSTABLE THEORIES AND THE INDEPENDENCE PROPERTY

The theory T_{rg} of the random (or Rado) graph is the first order theory with a single binary relation R whose axioms say that R is symmetric and reflexive, and that for any two disjoint finite sets of elements X, Y there is an element z such that $R(z, x) \wedge \neg R(z, y)$ for all $x \in X$ and $y \in Y$. T_{rg} is \aleph_0 -categorical, simple, unstable, and has quantifier elimination.

In Theorem 9.7 below, we show that T_{rg} is \leq -minimal among all unstable metric theories. It will follow that the equivalence class of T_{rg} is the third element of (\mathbb{M}, \leq) and is not maximal in (\mathbb{M}, \leq) . This is a continuous analogue of the result of Malliaris that T_{rg} is \leq -minimal among all unstable first order theories (Fact 9.6 below).

Shelah, in Theorem II.4.7 of [Sh78], proved that a first order theory is unstable if and only if it has the strict order property or the independence property. Dzamonja and Shelah [DS04] proved that a first order theory with the strict order property has SOP_2 , so a first order theory is unstable if and only if it has either SOP_2 or the independence property. In [BY09], Ben Yaacov introduced the continuous analogue of the independence property, which is equivalent to Definition 9.1 below. We will show in Theorem 9.5 below that a metric theory is unstable if and only if it has either the SOP_2 or the independence property.

Definition 9.1. T has the *independence property* if there is a formula $\varphi(\vec{x}, \vec{y})$ and a model \mathcal{M} of T such that for each $m \in \mathbb{N}$ there are $|\vec{x}|$ -tuples $\langle \vec{a}_Z \rangle_{Z \subseteq m}$ and $|\vec{y}|$ -tuples $\langle \vec{b}_n \rangle_{n < m}$ such that for all $n < m$ and $Z \subseteq m$,

$$n \in Z \Rightarrow \varphi^{\mathcal{M}}(\vec{a}_Z, \vec{b}_n) = 0, \quad n \notin Z \Rightarrow \varphi^{\mathcal{M}}(\vec{a}_Z, \vec{b}_n) = 1.$$

We also say that $\varphi(\vec{x}, \vec{y})$ has the independence property for T (in \mathcal{M}).

Remark 9.2. If $\varphi(\vec{x}, \vec{y})$ has the independence property for T , then for every \aleph_1 -saturated model \mathcal{M} of T , $\varphi(\vec{x}, \vec{y})$ has the independence property for T in \mathcal{M} .

Lemma 9.3. T has the independence property if and only if there exist $0 \leq s < r \leq 1$ and a formula $\varphi(\vec{x}, \vec{y})$ such that Definition 9.1 holds with the displayed formulas replaced by

$$n \in Z \Rightarrow \varphi^{\mathcal{M}}(\vec{a}_Z, \vec{b}_n) \leq s, \quad n \notin Z \Rightarrow \varphi^{\mathcal{M}}(\vec{a}_Z, \vec{b}_n) \geq r.$$

Proof. Similar to the proof of Corollary 6.3. \square

Lemma 9.4. Let \mathcal{M} be an \aleph_1 -saturated model of T . The following are equivalent:

- (i) T has the independence property.
- (ii) There are real numbers $0 \leq s < r \leq 1$, a formula $\varphi(\vec{x}, \vec{y})$, a tuple \vec{a} in \mathcal{M} , and an indiscernible sequence $\langle \vec{b}_n \rangle_{n \in \mathbb{N}}$ in \mathcal{M} such that $\varphi^{\mathcal{M}}(\vec{a}, \vec{b}_n) \leq s$ for all even n , and $\varphi^{\mathcal{M}}(\vec{a}, \vec{b}_n) \geq r$ for all odd n .
- (iii) There is a formula $\varphi(\vec{x}, \vec{y})$, a tuple \vec{a} in \mathcal{M} , and an indiscernible sequence $\langle \vec{b}_n \rangle_{n \in \mathbb{N}}$ in \mathcal{M} such that $\lim_{n \rightarrow \infty} \varphi^{\mathcal{M}}(\vec{a}, \vec{b}_n)$ does not exist.

Proof. It is trivial that (ii) implies (iii). If (iii) holds, then it is easily seen that (ii) holds with $\langle \vec{b}_n \rangle_{n \in \mathbb{N}}$ replaced by some infinite subsequence of $\langle \vec{b}_n \rangle_{n \in \mathbb{N}}$. Assume (i). Then there exist $0 \leq s < r \leq 1$ and a formula $\varphi(\vec{x}, \vec{y})$ satisfying the conditions of Lemma 9.3. By \aleph_1 -saturation and Ramsey's theorem, (ii) holds.

Assume (ii), and let $r, s, \varphi(\vec{x}, \vec{y}), \vec{a}$, and $\langle \vec{b}_n \rangle_{n \in \mathbb{N}}$ be as in (ii). Then

$$n \in Z \Rightarrow \varphi^{\mathcal{M}}(\vec{a}, \vec{b}_{f(n)}) \leq s, \quad n \notin Z \Rightarrow \varphi^{\mathcal{M}}(\vec{a}, \vec{b}_{f(n)}) \geq r.$$

Since $\langle \vec{b}_n \rangle_{n \in \mathbb{N}}$ is indiscernible in \mathcal{M} , it follows from Lemma 9.3 that T has the independence property, so (i) holds. \square

In [BY09], the independence property was defined for metric theories in a different and more complicated way. Lemma 5.4 in [BY09] is the same as Lemma 9.4 above except that T is a metric theory, and (i) refers to the independence property as defined in [BY09]. Thus a metric theory T has the independence property in the sense of Definition 9.1 if and only if it has the independence property in the sense of [BY09]. Here is a continuous analogue of Shelah's result that a first order theory is unstable iff it has the strict order property or the independence property.⁵

Theorem 9.5. A metric theory is unstable if and only if it has either SOP_2 or the independence property.

Proof. It is easily seen that if T has the independence property, then T is unstable. If T has SOP_2 , then T is \leq -maximal by Theorem 8.6, so T is unstable by Theorem 6.22.

Suppose T is unstable and does not have the independence property. By Corollary 6.3, there is a formula $\varphi(\vec{x}, \vec{y})$ that is unstable for $(0, 1)$ in T . Thus in some \aleph_1 -saturated model $\mathcal{M} \models T$ there is an infinite sequence $\langle \vec{a}_n, \vec{b}_n \rangle_{n \in \mathbb{N}}$ of (\vec{x}, \vec{y}) -tuples such that whenever $h < k$, $\varphi^{\mathcal{M}}(\vec{a}_h, \vec{b}_k) = 0$ and $\varphi^{\mathcal{M}}(\vec{a}_k, \vec{b}_h) = 1$. We may also take $\langle \vec{a}_n, \vec{b}_n \rangle_{n \in \mathbb{N}}$ so that $\varphi^{\mathcal{M}}(\vec{a}_h, \vec{b}_h) = 0$ for all h .

⁵In [Kh19], Khanaki proved another continuous analogue of Shelah's result. He introduced a property of metric theories called $wSOP$, and showed that every unstable metric theory has either $wSOP$ or the independence property.

Let $\vec{u} = (\vec{x}, \vec{y})$, let $\vec{v} = (\vec{x}_*, \vec{y}_*)$ be a tuple of new variables with the same length as \vec{u} , and let $\vec{c}_h = (\vec{a}_h, \vec{b}_h)$. Then $\theta(\vec{u}, \vec{c}_h)$ is the formula $\varphi(\vec{x}, \vec{b}_h)$, and $\theta(\vec{c}_h, \vec{c}_k)$ is the formula $\varphi(\vec{a}_h, \vec{b}_k)$. $\theta(\vec{u}, \vec{v})$ is unstable for $(0, 1)$ in T with $|\vec{u}| = |\vec{v}|$, and whenever $h < k$ we have $\theta^{\mathcal{M}}(\vec{c}_h, \vec{c}_k) = 0$ and $\theta^{\mathcal{M}}(\vec{c}_k, \vec{c}_h) = 1$. Using Ramsey's theorem and \aleph_1 -saturation, there is an indiscernible family $\mathcal{J} = \langle \vec{c}_m \rangle_{m \in \mathbb{Q}}$ in \mathcal{M} indexed by the set \mathbb{Q} of dyadic rationals in $(0, 1)$ such that whenever $h < k$ we have $\theta^{\mathcal{M}}(\vec{c}_h, \vec{c}_k) = 0$ and $\theta^{\mathcal{M}}(\vec{c}_k, \vec{c}_h) = 1$, and $\theta^{\mathcal{M}}(\vec{c}_h, \vec{c}_h) = 0$.

Let $\vec{u} \sqsubset \vec{v}$ be the formula

$$\vec{u} \sqsubset \vec{v} : \max(\theta(\vec{u}, \vec{v}), 1 - \theta(\vec{v}, \vec{u})).$$

Thus $\mathcal{M} \models \vec{u} \sqsubset \vec{v} = 0$ if and only if $\theta^{\mathcal{M}}(\vec{u}, \vec{v}) = 0$ and $\theta^{\mathcal{M}}(\vec{v}, \vec{u}) = 1$. Let $\beta(\vec{u}, \vec{t}, \vec{v})$ be the formula saying that \vec{t} is "between" \vec{u} and \vec{v} :

$$\beta(\vec{u}, \vec{t}, \vec{v}) : \max(\vec{u} \sqsubset \vec{v}, \vec{u} \sqsubset \vec{t}, \vec{t} \sqsubset \vec{v}).$$

For each $n \in \mathbb{N}$, let $\psi_n(\vec{t}, \langle \vec{u}_m \rangle_{m < 2^{n+1}})$ be the formula saying that \vec{t} is between \vec{u}_m and \vec{u}_{m+1} for each even $m < 2^{n+1}$. Formally,

$$\psi_n(\vec{t}, \langle \vec{u}_m \rangle_{m < 2^{n+1}}) : \max_{h < 2^n} \beta(\vec{u}_{2h}, \vec{t}, \vec{u}_{2h+1}).$$

By indiscernibility, for each $n \in \mathbb{N}$ and increasing sequence $\langle h(0), \dots, h(2^n - 1) \rangle$ in \mathbb{Q} , the sentence $(\inf_{\vec{t}}) \psi_n(\vec{t}, \langle \vec{c}_{h(m)} \rangle_{m < 2^{n+1}})$ has the same value r_n in \mathcal{M} . It is clear that $r_0 = 0$, because $\beta^{\mathcal{M}}(\vec{c}_h, \vec{c}_k, \vec{c}_\ell) = 0$ when $h < k < \ell$.

If there were infinitely many n such that $r_n = 0$, there would be an indiscernible subsequence $\langle \vec{a}_n \rangle_{n \in \mathbb{N}}$ of \mathcal{J} and a tuple \vec{t} in \mathcal{M} such that $\theta^{\mathcal{M}}(\vec{t}, \vec{a}_n) = 1$ for each even n and $\theta^{\mathcal{M}}(\vec{t}, \vec{a}_n) = 0$ for each odd n . But by Lemma 9.4, this would contradict the hypothesis that T does not have the independence property. Therefore there is a greatest $N \in \mathbb{N}$ such that $r_N = 0$, and thus $r_{N+1} > 0$.

We will show that ψ_N is an SOP_2 formula with bounds $(0, r_{N+1})$ for T . Then by Corollary 8.3, it will follow that T has SOP_2 .

For each $h < k$ in \mathbb{Q} , let $\vec{d}_{h,k}$ be the sequence with 2^{N+1} elements evenly spaced between \vec{c}_h and \vec{c}_k , so $\vec{d}_{h,k} = \langle \vec{c}_{h+m\delta} \rangle_{m < 2^{N+1}}$ where $\delta = (k - h)/2^{N+1}$. It follows that whenever $h_1 < k_1 < h_2 < k_2$ in \mathbb{Q} , we have

$$(5) \quad \mathcal{M} \models (\inf_{\vec{t}}) \psi_N(\vec{t}, \vec{d}_{h_1, k_1}) = 0,$$

but

$$(6) \quad \mathcal{M} \models (\inf_{\vec{t}}) \max[\psi_N(\vec{t}, \vec{d}_{h_1, k_1}), \psi_N(\vec{t}, \vec{d}_{h_2, k_2})] \geq r_{N+1} > 0.$$

For each node μ in the binary tree $\{0, 1\}^{<\mathbb{N}}$, pick dyadic rationals $h_\mu, k_\mu \in \mathbb{Q}$ such that

$$h_\mu < h_{\mu 0} < k_{\mu 0} < h_{\mu 1} < k_{\mu 1} < k_\mu.$$

This gives us a nested binary tree of intervals in \mathbb{Q} . By (5) and (6), Ψ_N is an SOP_2 formula with bounds $(0, r_N)$ in T with the parameters $\vec{b}_\mu = \vec{d}_{h_\mu, k_\mu}$. \square

Fact 9.6. (Lemma 5.3 in Malliaris [Ma12]) T_{r_g} is \leq -minimal among all unstable first order theories, that is, $T_{r_g} \trianglelefteq S$ for every unstable first order theory S .

We now improve Fact 9.6 by showing that T_{r_g} is \leq -minimal among all unstable metric theories.

Theorem 9.7. $T_{rg} \leq T$ for every unstable metric theory T .

Proof. Suppose T is unstable. If T has SOP_2 , then T is \leq -maximal by Theorem 8.6, and hence $T_{rg} \leq T$.

Now suppose T does not have SOP_2 . By Theorem 9.5, T has the independence property. Then there is a continuous formula $\varphi(\vec{x}, \vec{y})$ that has the independence property for T . Let $\mathcal{D} \in \text{Sat}(T)$, and \mathcal{M} be an \aleph_1 -saturated model of T . Then $\mathcal{M}^{\mathcal{D}}$ is $|I|^+$ -saturated.

Let \mathcal{K}_{rg} be a first order structure which is a model of T_{rg} . We must show that $\mathcal{K}_{rg}^{\mathcal{D}}$ is $|I|^+$ -saturated. Let $A \subseteq \mathcal{K}_{rg}^{\mathcal{D}}$ have cardinality $|A| \leq |I|$, and let $p(z)$ be a set of formulas with parameters in A that is maximal consistent in $\mathcal{K}_{rg}^{\mathcal{D}}$. Since T_{rg} has elimination of quantifiers, and its vocabulary has only equality and the binary predicate symbol R which is symmetric in T_{rg} , two elements z_1, z_2 of $\mathcal{K}_{rg}^{\mathcal{D}}$ have the same type over A if and only if

$$\{a \in A : \mathcal{K}_{rg}^{\mathcal{D}} \models R(z_1, a)\} = \{a \in A : \mathcal{K}_{rg}^{\mathcal{D}} \models R(z_2, a)\}.$$

Therefore there are sets B, C such that $A = B \cup C$ and $p(z)$ is equivalent in $\mathcal{K}_{rg}^{\mathcal{D}}$ to the set of formulas

$$\Gamma = \Gamma(x, A) = \{R(z, b) : b \in B\} \cup \{\neg R(z, c) : c \in C\}.$$

Then Γ is finitely satisfiable in $\mathcal{K}_{rg}^{\mathcal{D}}$, and $\mathcal{K}_{rg}^{\mathcal{D}} \models b \neq c$ for all $b \in B$ and $c \in C$. To complete the proof it suffices to show that Γ is satisfiable in $\mathcal{K}_{rg}^{\mathcal{D}}$.

As in the proof of Theorem 8.6, let \mathcal{X} regularize \mathcal{D} , and pick an injective function $h : A \rightarrow \mathcal{X}$. For each $t \in I$, the sets

$$A_t = \{a \in A : t \in h(a)\} \subseteq \mathcal{K}_{rg}^{\mathcal{D}}, \quad A[t] = \{a[t] : a \in A, t \in h(a)\} \subseteq \mathcal{K}_{rg}$$

are finite. Since $\varphi(\vec{x}, \vec{y})$ has the independence property for T , for each $a \in A$ we may choose a $|\vec{y}|$ -tuple \hat{a} in $\mathcal{M}^{\mathcal{D}}$ such that for each $t \in I$ and set $Z \subseteq A[t]$ there is a tuple \vec{x}_Z in \mathcal{M} for which

$$a[t] \in Z \Rightarrow \varphi^{\mathcal{M}}(\vec{x}_Z, \hat{a}[t]) = 0, \quad a[t] \in A[t] \setminus Z \Rightarrow \varphi^{\mathcal{M}}(\vec{x}_Z, \hat{a}[t]) = 1.$$

Let Γ' be the set of continuous formulas

$$\Gamma' = \{\varphi(\vec{x}, \hat{b}) = 0, \varphi(\vec{x}, \hat{c}) = 1 : b \in B, c \in C\}.$$

Let B_0, C_0 be finite subsets of B, C . By the Łoś Theorem applied to $\mathcal{K}_{rg}^{\mathcal{D}}$, the set

$$X = \{t \in I : (\forall b \in B_0)(\forall c \in C_0)b[t] \neq c[t]\}$$

belongs to \mathcal{D} . For each $t \in X$ there is a tuple \vec{x} in \mathcal{M} such that

$$(\forall b \in B_0)\varphi^{\mathcal{M}}(\vec{x}, \hat{b}[t]) = 0, \quad (\forall c \in C_0)\varphi^{\mathcal{M}}(\vec{x}, \hat{c}[t]) = 1.$$

Hence by the Łoś Theorem, Γ' is finitely satisfiable in $\mathcal{M}^{\mathcal{D}}$. Since $\mathcal{M}^{\mathcal{D}}$ is $|I|^+$ -saturated, there is a tuple \vec{x} that satisfies Γ' in $\mathcal{M}^{\mathcal{D}}$.

We now show that Γ is satisfiable in $\mathcal{K}_{rg}^{\mathcal{D}}$ by finding an element $z \in \mathcal{K}_{rg}^{\mathcal{D}}$ that “matches” \vec{x} . Fix a real number $0 < r < 1/2$. Let $t \in I$. Note that if $b, c \in A$, $\varphi^{\mathcal{M}}(\vec{x}[t], \hat{b}[t]) \leq r$, and $\varphi^{\mathcal{M}}(\vec{x}[t], \hat{c}[t]) \geq 1 - r$, then $b[t] \neq c[t]$. So there is an element $z[t] \in \mathcal{K}_{rg}$ such that for each $a \in A_t$:

- $\varphi^{\mathcal{M}}(\vec{x}[t], \hat{a}[t]) \leq r \Rightarrow \mathcal{K}_{rg} \models R(z[t], a[t])$.
- $\varphi^{\mathcal{M}}(\vec{x}[t], \hat{a}[t]) \geq 1 - r \Rightarrow \mathcal{K}_{rg} \models \neg R(z[t], a[t])$.

Since \vec{x} satisfies Γ' in $\mathcal{M}^{\mathcal{D}}$, for each $b \in B$ and $c \in C$, the set

$$Y = \{t \in I : t \in h(b) \cap h(c), \varphi^{\mathcal{M}}(\vec{x}[t], \widehat{b}[t]) \leq r, \varphi^{\mathcal{M}}(\vec{x}[t], \widehat{c}[t]) \geq 1 - r\}$$

belongs to \mathcal{D} . The element $z[t]$ was chosen so that the set

$$\{t \in I : \mathcal{K}_{rg} \models R(z[t], b[t]) \wedge \neg R(z[t], c[t])\}$$

contains Y , and thus also belongs to \mathcal{D} . Therefore, by the Loś Theorem, $\mathcal{K}_{rg}^{\mathcal{D}} \models R(z, b) \wedge \neg R(z, c)$. This shows that z satisfies Γ in $\mathcal{K}_{rg}^{\mathcal{D}}$, so $\mathcal{D} \in \text{Sat}(T_{rg})$ and $T_{rg} \preceq T$. \square

10. \preceq -MINIMAL TP_2 THEORIES

In Theorem 10.5 and Corollary 10.9 below, we will prove that there are metric theories T that are \preceq -minimal among metric TP_2 theories and strictly above T_{rg} . These results are the continuous analogues of the first order Facts 10.1 and 10.2. Also, in Theorem 10.16, we will give a natural example of such a theory T . That example is T_{rg}^R , the randomization of T_{rg} . Our proofs will use the notion of a distribution of a set of continuous formulas from Section 3 above, as well as the earlier notion of a distribution of a set of first order formulas introduced by Malliaris in [Ma12].

Shelah [Sh80] defined simple theories in first order logic and proved that a first order theory is simple if and only if it has neither SOP_2 nor TP_2 . The theory T_{feq}^* is the model completion of the first order theory of infinitely many parameterized equivalence relations (see [DS04] for the precise definition). T_{feq}^* is TP_2 but not SOP_2 , and thus is not simple.

Fact 10.1. (Corollary 6.10 in Malliaris [Ma12]) T_{feq}^* is \preceq -minimal among first order theories with TP_2 .

Fact 10.2. (Malliaris and Shelah [MS13]) $T_{rg} \triangleleft S$ for any first order theory S that has TP_2 .

The following definition is equivalent to the first order notion in the case that T is a first order theory.

Definition 10.3. (Ben Yaacov [BY13]) A metric theory T has the *tree property of the second kind* (briefly, T is TP_2), if in some \aleph_1 -saturated model \mathcal{M} of T there is a continuous formula $\varphi(\vec{x}, \vec{y})$ and an array $\langle \vec{b}_{n,m} \rangle_{n,m \in \mathbb{N}}$ of $|\vec{y}|$ -tuples such that:

- (a) The sequences $\mathcal{J}_n = \langle \vec{b}_{n,m} \rangle_{m \in \mathbb{N}}$ are mutually indiscernible.
- (b) The sequence of sequences $\langle \mathcal{J}_n \rangle_{n \in \mathbb{N}}$ is indiscernible.
- (c) For each $n \in \mathbb{N}$, $\{\varphi(\vec{x}, \vec{b}_{n,m}) : m \in \mathbb{N}\}$ is not satisfiable in \mathcal{M} .
- (d) For each function $f : \mathbb{N} \rightarrow \mathbb{N}$, $\{\varphi(\vec{x}, \vec{b}_{n,f(n)}) : n \in \mathbb{N}\}$ is satisfiable in \mathcal{M} .

Ben Yaacov [BY03] defined simple theories in the setting of compact abstract theories. [BY13] points out that the definition can be translated into the setting of continuous logic, and that no simple metric theory is TP_2 . It was shown in [MS20] that there are continuum many \preceq -equivalence classes of simple first order theories in (\mathbb{F}, \preceq) . Every simple first order theory is simple as a metric theory, so there are continuum many \preceq -equivalence classes of simple metric theories in (\mathbb{M}, \preceq) . In this section we will avoid the translation from compact abstract theories to metric theories, and work directly with the continuous properties SOP_2 and TP_2 .

Lemma 10.4. *Let $0 \leq \varepsilon < 1$. T is TP_2 if and only if Definition 10.3 holds with φ replaced by $\varphi \dot{-} \varepsilon$ in conditions (c) and (d).*

Proof. Say that $\varphi(\vec{x}, \vec{y})$ is an ε - TP_2 formula if Definition 10.3 holds with φ replaced by $\varphi \dot{-} \varepsilon$ in conditions (c) and (d). If φ is an ε - TP_2 formula, then $\varphi \dot{-} \varepsilon$ is a 0- TP_2 formula. Suppose φ is a 0- TP_2 formula. By indiscernibility and Remark 2.8, there exists $0 < \delta \leq 1$ such that whenever $n, m, m' \in \mathbb{N}$ and $m < m'$,

$$\mathcal{M} \models (\inf_{\vec{x}}) \max[\varphi(\vec{x}, \vec{b}_{n,m}), \varphi(\vec{x}, \vec{b}_{n,m'})] = \delta.$$

If $\varepsilon < \delta$, then φ is an ε - TP_2 formula. On the other hand, if $\varepsilon \geq \delta$, then $g(\varphi)$ is an ε - TP_2 formula, where $g: [0, 1] \rightarrow [0, 1]$ is the unary connective whose graph is the line from $(0, 0)$ to $(\delta, (\varepsilon + 1)/2)$ followed by the horizontal line from $(\delta, (\varepsilon + 1)/2)$ to $(1, (\varepsilon + 1)/2)$. \square

Theorem 10.5. *T_{feq}^* is \leq -minimal among TP_2 metric theories.*

Theorem 10.5 is the continuous analogue of Fact 10.1. To prove it we will use Fact 10.7 below, which was proved in [Ma12] along the way to proving Fact 10.1. Fact 10.7 will allow us to get around the difficulty that continuous logic does not have negation.

Definition 10.6. Say that \mathcal{D} solves (ω, ω) if for every first order structure \mathcal{K} , first order formula $\varphi(\vec{x}, \vec{y})$, and array $C = \langle c_{n,m} \rangle_{n,m \in \mathbb{N}}$ of elements of \mathcal{K} such that:

- For all n, m, m' with $m \neq m'$, $\mathcal{K} \models \neg[(\exists x)\varphi(\vec{x}, \vec{c}_{n,m}) \wedge \varphi(\vec{x}, c_{n,m'})]$,
- For each $f: \mathbb{N} \rightarrow \mathbb{N}$, the type $\{\varphi(\vec{x}, \vec{c}_{n,f(n)}): n \in \mathbb{N}\}$ is realized in \mathcal{K} ,

and for every set $A \subseteq C^{\mathcal{D}}$ of cardinality $|A| \leq \lambda$, if $\{\varphi(\vec{x}, \vec{a}): \vec{a} \in A\}$ is finitely satisfiable in $\mathcal{K}^{\mathcal{D}}$ then it is satisfiable in $\mathcal{K}^{\mathcal{D}}$.

Fact 10.7. *(By Lemmas 6.7 and 6.8 in [Ma12]). If \mathcal{D} solves (ω, ω) then \mathcal{D} saturates T_{feq}^* .*

Lemma 10.8. *If T is a metric TP_2 theory and \mathcal{D} saturates T , then \mathcal{D} solves (ω, ω) .*

Proof. Let \mathcal{M} be an \aleph_1 -saturated model of T . There is a continuous formula $\theta(\vec{u}, \vec{v})$ and an array of $|\vec{u}|$ -tuples $\langle \vec{b}_{n,m} \rangle_{n,m \in \mathbb{N}}$ in \mathcal{M} such that:

- (a) For all n, m, m' with $m \neq m'$, $\mathcal{M} \models (\inf_{\vec{u}}) \max[\theta(\vec{u}, \vec{b}_{n,m}), \theta(\vec{u}, b_{n,m'})] = 1$,
- (b) For each $f: \mathbb{N} \rightarrow \mathbb{N}$, the type $\{\theta(\vec{u}, \vec{b}_{n,f(n)}) = 0: n \in \mathbb{N}\}$ is realized in \mathcal{M} .

Let $\mathcal{K}, \varphi(\vec{x}, \vec{y})$, and $C = \langle c_{n,m} \rangle_{n,m \in \mathbb{N}}$ be as in Definition 10.6. Suppose

$$A = \{\vec{a}_\beta: \beta < \lambda\} \subseteq C^{\mathcal{D}},$$

and

$$\Gamma(\vec{x}, A) = \{\varphi(\vec{x}, \vec{a}_\beta): \beta < \lambda\}$$

is finitely satisfiable in $\mathcal{K}^{\mathcal{D}}$. To show that \mathcal{D} solves (ω, ω) , we must show that Γ is satisfiable in $\mathcal{K}^{\mathcal{D}}$. For each $\beta < \lambda$ and $t \in I$, we have $\vec{a}_\beta[t] = \vec{c}_{n_\beta[t], m_\beta[t]} \in C$ for some $n_\beta[t], m_\beta[t] \in \mathbb{N}$. For each $\Phi \in \mathcal{P}_{\aleph_0}(\Gamma)$ and $t \in I$, let

$$\Phi[t] = \{\varphi(\vec{x}, \vec{a}_\beta[t]): \varphi(\vec{x}, \vec{a}_\beta) \in \Phi\}.$$

For each $\alpha < \beta < \lambda$, let

$$X_{\alpha,\beta} = \{t \in I: n_\alpha[t] = n_\beta[t] \Rightarrow m_\alpha[t] = m_\beta[t]\}.$$

By 10.6, for each $\Phi \in \mathcal{P}_{\aleph_0}(\Gamma)$ and $t \in I$, $\Phi[t]$ is satisfiable in \mathcal{K} if and only if for all $\varphi(\vec{x}, \vec{a}_\alpha), \varphi(\vec{x}, \vec{a}_\beta) \in \Phi$ we have $t \in X_{\alpha, \beta}$. Since Γ is finitely satisfiable in $\mathcal{K}^{\mathcal{D}}$. By the Loś Theorem, for each $\alpha, \beta < \lambda$ we have $X_{\alpha, \beta} \in \mathcal{D}$.

Let $\vec{b}_\beta[t] = \vec{b}_{n_\beta[t], m_\beta[t]}$, $\vec{b}_\beta = \langle \vec{b}_{n_\beta[t], m_\beta[t]} \rangle_{t \in I} \in \mathcal{M}^{\mathcal{D}}$, and $B = \{\vec{B}_\beta : \beta < \lambda\}$. Let

$$\Sigma = \Sigma(\vec{u}, B) = \{\theta(\vec{u}, \vec{b}_\beta) : \beta < \lambda\}.$$

For $\Theta \in \mathcal{P}_{\aleph_0}(\Sigma)$, define $\Theta[t]$ similarly to $\Phi[t]$ above. By (a) and (b), $\Theta[t]$ is satisfiable in \mathcal{M} if and only if for all $\theta(\vec{u}, \vec{b}_\alpha), \theta(\vec{u}, \vec{b}_\beta) \in \Theta$ we have $t \in X_{\alpha, \beta}$. Then by the Loś Theorem, $\Sigma(\vec{u}, B)$ is finitely satisfiable in $\mathcal{M}^{\mathcal{D}}$. Since \mathcal{D} saturates T , $\Sigma(\vec{u}, B)$ is satisfied in $\mathcal{M}^{\mathcal{D}}$. By Lemma 4.10, Σ has a multiplicative distribution $\delta_1 : \mathcal{P}_{\aleph_0}(\Sigma^{ap}) \rightarrow \mathcal{D}$ in $\mathcal{M}^{\mathcal{D}}$. Let $0 < s < 1/2$. Then for each formula $\theta \in \Sigma(\vec{u}, B)$, the formula $\theta \dot{-} s$ belongs to Σ^{ap} . For each $\Theta \in \mathcal{P}_{\aleph_0}(\Sigma)$, let

$$\delta_2(\Theta) = \delta_1(\{\theta \dot{-} s : \theta \in \Theta\}).$$

δ_2 is not a continuous distribution because it is defined on finite subsets of Σ rather than Σ^{ap} . However,

- $\delta_2 : \mathcal{P}_{\aleph_0}(\Sigma) \rightarrow \mathcal{X}$ for some \mathcal{X} that regularizes \mathcal{D} .
- For each $\Theta \in \mathcal{P}_{\aleph_0}(\Sigma)$ and $t \in \delta_2(\Theta)$,

$$\mathcal{M} \models (\inf_{\vec{u}}) \max_{\theta \in \Theta} \theta(\vec{u}, B[t]) \leq s.$$

- δ_2 is multiplicative.

By (a) and (b), for each $\alpha < \beta < \lambda$ and $t \in \delta_2(\{\theta(\vec{u}, \vec{b}_\alpha), \theta(\vec{u}, \vec{b}_\beta)\})$, we still have $t \in X_{\alpha, \beta}$.

Finally, let $\delta_3 : \mathcal{P}_{\aleph_0}(\Gamma) \rightarrow \mathcal{D}$ be the unique multiplicative mapping such that $\delta_3(\{\varphi(\vec{x}, \vec{a}_\beta)\}) = \delta_2(\{\theta(\vec{u}, \vec{b}_\beta)\})$ for each $\beta < \lambda$. Then for each $\alpha < \beta < \lambda$ and $t \in \delta_3(\{\varphi(\vec{x}, \vec{a}_\alpha), \varphi(\vec{x}, \vec{a}_\beta)\})$, we have $t \in X_{\alpha, \beta}$. Therefore, for each $\Phi \in \mathcal{P}_{\aleph_0}(\Gamma)$, if $t \in \delta_3(\Phi)$ then $\Phi[t]$ is satisfiable in \mathcal{K} . This shows that δ_3 is a first order distribution of Γ in $\mathcal{K}^{\mathcal{D}}$, so by Lemma 4.10, Γ is satisfiable in $\mathcal{K}^{\mathcal{D}}$. \square

Proof of Theorem 10.5. Let T be a metric TP_2 theory. If \mathcal{D} saturates T , then \mathcal{D} solves (ω, ω) by Lemma 10.8, so \mathcal{D} saturates T_{feq}^* by Fact 10.7. Therefore $T_{feq}^* \trianglelefteq T$. \square

As a corollary, we get the continuous analogue of Fact 10.2.

Corollary 10.9. $T_{rg} \triangleleft T$ for every TP_2 metric theory T .

Proof. $T_{rg} \triangleleft T_{geq}^*$ by Fact 10.2, and $T_{geq}^* \trianglelefteq T$ by Theorem 10.5, so $T_{rg} \triangleleft T$. \square

Corollary 10.10. There is a regular ultrafilter \mathcal{D} that saturates T_{rg} , but saturates no TP_2 metric theory.

Proof. By Theorem 12.1 of [MS13], there is a \mathcal{D} that saturates T_{rg} but does not saturate T_{feq}^* . Then by Theorem 10.5, \mathcal{D} saturates no TP_2 metric theory. \square

Definition 10.11. \mathcal{D} is OK if every monotone function $g : \mathcal{P}_{\aleph_0}(I) \rightarrow \mathcal{D}$ such that $g(u) = g(v)$ whenever $|u| = |v|$ has a multiplicative refinement $f : \mathcal{P}_{\aleph_0}(I) \rightarrow \mathcal{D}$.

Clearly, every good ultrafilter is OK. OK ultrafilters were mentioned without a name in [Ke67]. In [Ku78], Kunen introduced the name OK ultrafilter and studied them from a topological viewpoint.

Fact 10.12. (By Lemma 8.8 of [Ma12] and Claim 6.1 in [MS15]) If \mathcal{D} saturates some TP_2 first order theory, then \mathcal{D} is OK.

Corollary 10.13. If \mathcal{D} saturates some TP_2 metric theory, then \mathcal{D} is OK.

Proof. If \mathcal{D} saturates T , then \mathcal{D} saturates T_{feq}^* by Theorem 10.5, so \mathcal{D} is OK by Fact 10.12. \square

We now review the *randomization* S^R of a first order theory S . S^R is the complete metric theory defined as follows (see [Ke99] and [BK09]).

Given a countable first order structure \mathcal{K} , $\mathcal{K}^{[0,1]}$ is the pre-metric structure with two sorts, the *random element* sort \mathbb{K} whose universe is the set of all Borel functions from $[0, 1]$ into \mathcal{K} , and the *event* sort \mathbb{E} whose universe is the set of all Borel subsets of $[0, 1]$. $\mathcal{K}^{[0,1]}$ has the Boolean operations in sort \mathbb{E} , a unary predicate μ of sort \mathbb{E} for measure, and for each first order formula $\varphi(\vec{v})$, a function $\llbracket \varphi(\vec{v}) \rrbracket$ of sort $\mathbb{K}^{|\vec{v}|} \rightarrow \mathbb{E}$. $\llbracket \varphi(\vec{a}) \rrbracket$ is the set $\{t \in [0, 1] : \mathcal{K} \models \varphi(\vec{a}(t))\}$, and for each Borel set X , $\mu(X)$ is the Lebesgue measure of X . If $\mathcal{K}_1 \equiv \mathcal{K}_2$, then $\mathcal{K}_1^{[0,1]} \equiv \mathcal{K}_2^{[0,1]}$, so there is a unique complete metric theory S^R , the *randomization* of S , such that for each countable model \mathcal{K} of S , $\mathcal{K}^{[0,1]}$ is a model of S^R . S^R is a metric theory with the distance predicate $d(x, y) = \mu(\llbracket x \neq y \rrbracket)$.

Fact 10.14. Let S be a first order theory.

(i) S^R is stable if and only if S is stable. (Ben Yaacov, Theorem 5.14 of [BK09]).

(ii) S^R has the independence property if and only if S has the independence property. (Theorem 4.10 of [BY13]).

(iii) If S has the independence property, then S^R is TP_2 . (Theorem 4.13 of [BY13]). Thus if S is simple but unstable then S^R is not simple.

Since the randomization S^R of a first order theory S shares many properties with S , one might expect that S^R is \trianglelefteq -equivalent to S . However, this is not the case when $S = T_{rg}$.

Corollary 10.15. $T_{rg} \triangleleft T_{rg}^R$.

Proof. T_{rg}^R is TP_2 by Fact 10.14 (iii). So $T_{rg} \triangleleft T_{rg}^R$ by Corollary 10.9. \square

Theorem 10.16. T_{rg}^R is \trianglelefteq -minimal among TP_2 metric theories.

Proof. Let \mathcal{N} be a model of T_{rg}^R . Let T be a TP_2 metric theory, \mathcal{M} be an \aleph_1 -saturated model of T , and \mathcal{D} be a regular ultrafilter over a set I of power λ such that $\mathcal{M}^{\mathcal{D}}$ is λ^+ -saturated. We must show that $\mathcal{N}^{\mathcal{D}}$ is λ^+ -saturated.

Let A be an infinite subset of $\mathcal{N}^{\mathcal{D}}$ of cardinality $|A| \leq \lambda$, and let $\Gamma = \Gamma(u, A)$ be a set of continuous formulas with one variable u and parameters from A that is complete in T_{rg}^R . It suffices to show that $\Gamma(u, A)$ is satisfiable in $\mathcal{N}^{\mathcal{D}}$.

For each finite subset $F \subseteq A$, the restriction of Γ to formulas with parameters from F is a complete type Γ_F in u over F . By Theorem 2.9 of [BK09], T_{rg}^R has quantifier elimination. Since T_{rg} also has quantifier elimination and its vocabulary has only the single symmetric binary predicate R , Γ_F is equivalent to the following continuous (but not strict) formula $\gamma_F(u)$ with parameters in F : Informally, for some probability measure ν_F over F , the formula $\gamma_F(u)$ says that for each $Z \subseteq F$, $\nu_F(Z)$ is the μ -measure of the set $Z = \{a \in F : R(u, a)\}$. Formally,

$$\gamma_F(u) := (\forall Z \subseteq F) \nu_F(Z) = \mu(\llbracket \bigwedge_{a \in Z} R(u, a) \wedge \bigwedge_{a \notin Z} \neg R(u, a) \rrbracket).$$

Therefore we may take $\Gamma(u, A)$ to be the set of formulas

$$\Gamma(u, A) = \{\gamma_F(u) : F \in \mathcal{P}_{\aleph_0}(A)\}.$$

Since $\Gamma(u, A)$ is complete in T_{rg}^R , it is finitely satisfiable in $\mathcal{N}^{\mathcal{D}}$. By Lemma 4.10, to show that $\Gamma(u, A)$ is satisfiable in $\mathcal{N}^{\mathcal{D}}$, it suffices to show that Γ has a multiplicative distribution in $\mathcal{N}^{\mathcal{D}}$.

Let $\kappa = |A|$. Then $\kappa \leq \lambda$. Since $\mathcal{M}^{\mathcal{D}}$ is λ^+ -saturated and T is TP_2 , in $\mathcal{M}^{\mathcal{D}}$ there is a formula $\varphi(\vec{x}, \vec{y})$ and family $\langle \vec{b}_{\alpha, \beta} \rangle_{\alpha, \beta \in \kappa \times \kappa}$ of $|\vec{y}|$ -tuples in $\mathcal{M}^{\mathcal{D}}$ such that:

- For each $\alpha < \kappa$ and $\beta < \beta' < \kappa$

$$\mathcal{M}^{\mathcal{D}} \models \max(\varphi(\vec{x}, \vec{b}_{\alpha, \beta}), \varphi(\vec{x}, \vec{b}_{\alpha, \beta'})) = 1.$$

- For each function $f: \kappa \rightarrow \kappa$ and $G \in \mathcal{P}_{\aleph_0}(\kappa)$,

$$\mathcal{M}^{\mathcal{D}} \models (\inf_{\vec{x}}) \max_{\alpha \in G} \varphi(\vec{x}, \vec{b}_{\alpha, f(\alpha)}) = 0.$$

Fix a function $f: \kappa \rightarrow \kappa$. Let $\vec{b}_\alpha = \vec{b}_{\alpha, f(\alpha)}$ for each $\alpha < \kappa$, and $B = \{\vec{b}_\alpha : \alpha < \kappa\}$. Let $\Sigma = \Sigma(\vec{x}, B) = \{\varphi(\vec{x}, \vec{b}_\alpha) : \alpha < \kappa\}$. Then Σ is finitely satisfiable in $\mathcal{M}^{\mathcal{D}}$. Γ^{ap} and Σ^{ap} both have cardinality κ , so there is a bijection g from Γ^{ap} onto Σ^{ap} . Since $\mathcal{M}^{\mathcal{D}}$ is λ^+ -saturated, Σ is satisfiable in $\mathcal{M}^{\mathcal{D}}$. By Lemma 4.9, $\Gamma(u, A)$ has an accurate distribution $\delta_1: \mathcal{P}_{\aleph_0}(\Gamma^{ap}) \rightarrow \mathcal{D}$ in $\mathcal{N}^{\mathcal{D}}$. By the proof of Lemma 4.9, Σ has a unique accurate distribution δ_2 in $\mathcal{M}^{\mathcal{D}}$ such that for each $\Psi \in \mathcal{P}_{\aleph_0}(\Gamma^{ap})$, $\delta_2(g(\Psi)) = \delta_1(\Psi)$.

By Lemma 4.10, Σ has a multiplicative distribution δ_3 in $\mathcal{M}^{\mathcal{D}}$ that is a refinement of δ_2 . Then

- $\delta_3: \mathcal{P}_{\aleph_0}(\Sigma^{ap}) \rightarrow \mathcal{X}$ where \mathcal{X} regularizes \mathcal{D} .
- For each $\Psi \in \mathcal{P}_{\aleph_0}(\Sigma^{ap})$ and $t \in \delta_3(\Psi)$,

$$\mathcal{M} \models (\inf_{\vec{x}}) \max\{\psi(\vec{x}, B[t]) : \psi \in \Psi\} = 0.$$

- $\delta_3(\Psi \cup \Theta) = \delta_3(\Psi) \cap \delta_3(\Theta)$ for all $\Psi, \Theta \in \mathcal{P}_{\aleph_0}(\Sigma^{ap})$.

It follows that the mapping δ_4 such that $\delta_4(\Psi) = \delta_3(g(\Psi))$ is a multiplicative refinement of δ_1 , and hence is a multiplicative distribution of Γ in $\mathcal{N}^{\mathcal{D}}$. \square

Corollary 10.17. $T_{rg}^R \trianglelefteq T_{feq}^*$.

Proof. By Theorems 10.5 and 10.16. \square

We conclude with an open question. In [MS16b], Theorem 11.4, Malliaris and Shelah proved that if there exists an uncountable compact cardinal then for all first order theories S, U , if S is simple and $U \trianglelefteq S$ then U is simple. Recall that a first order theory is simple if and only if it has neither SOP_2 nor TP_2 .

Question 10.18. *Suppose there exists an uncountable compact cardinal. If T, U are metric theories, $U \trianglelefteq T$, and T has neither SOP_2 nor TP_2 , must U have neither SOP_2 nor TP_2 ?*

A potential path to an affirmative answer would be to generalize the proof in [MS16b] to continuous logic.

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UNIVERSITY OF WISCONSIN-MADISON, DEPARTMENT OF MATHEMATICS, MADISON, WI 53706-1388

Email address: `keisler@math.wisc.edu`