A Canonical Hidden-Variable Space*

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Abstract

The hidden-variable question is whether or not various properties — randomness or correlation, for example — that are observed in the outcomes of an experiment can be explained via introduction of extra (hidden) variables which are unobserved by the experimenter. The question can be asked in both the classical and quantum domains. In the latter, it is fundamental to the interpretation of the quantum formalism (Bell, 1964, Kochen and Specker, 1967, and others). In building a suitable mathematical model of an experiment, the physical set-up will guide us on how to model the observable variables — i.e., the measurement and outcome spaces. But, by definition, we cannot know what structure to put on the hidden-variable space. Nevertheless, we show that, under a measure-theoretic condition, the hidden-variable question can be put into a canonical form. The condition is that the σ-algebras on the measurement and outcome spaces are countably generated. An argument using a classical result on isomorphisms of measure algebras then shows that the hidden-variable space can always be taken to be the unit interval equipped with the Lebesgue measure on the Borel sets.

1 Introduction

Consider an experiment in which Alice can make one of several measurements on her part of a certain system and Bob can make one of several measurements on his part of the system.

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Each pair of measurements (one by Alice and one by Bob) leads to a pair of outcomes (one for Alice and one for Bob). We keep track of the frequency distribution of the different pairs of outcomes that arise. This situation can be abstracted to an **empirical model**, which, for each pair of measurements, specifies a probability measure on pairs of outcomes.

An associated **hidden-variable model** is obtained by starting with the empirical model and then appending to it extra variables that are assumed to be present in a more complete theory of how the data are generated. The uses of hidden-variable analysis include: (i) seeing if a deterministic account can be given of the observed data, and (ii) seeing if a common-cause account can be given of correlations in the observed data.

Arguably, the most famous context for hidden-variable analysis is quantum mechanics (QM). Starting with von Neumann (1932), and including, most famously, Einstein, Podolsky, and Rosen (1935), Bell (1964), and Kochen and Specker (1967), a vast literature has grown up around the question of whether or not a hidden-variable formulation of QM is possible. The watershed no-go theorems of Bell and Kochen-Specker give conditions under which the answer is no.

Hidden variables are variables above and beyond those which are part of the actual experiment, and are therefore unobserved. This poses a question: *What can one assume about the structure of the space on which a hidden variable lives?* Choosing a good empirical model includes choosing measurement and outcome spaces that incorporate appropriate physical features (say, discreteness, connectedness, or other features). But, since it is unobserved, there is no such guide to choosing a hidden-variable space.

This may or may not be a serious obstacle. The question often under study is whether or not a hidden-variable model exists that exhibits desired properties such as determinism or common-cause correlation. For a positive answer, we may be satisfied with showing that, at least for a certain choice of hidden-variable space, such a model exists. But, some of the most important results — including the Bell and Kochen-Specker theorems — are negative answers, asserting that no hidden-variable model with certain properties exists. For a non-existence result to be definitive, we need to search over all (not just some) hidden-variable spaces.

Our main result is that there is a **canonical hidden-variable model**. To be more specific, fix an empirical model. Suppose there is an associated hidden-variable model that yields, for each pair of measurements, the same probability measure over joint outcomes. We will say that the hidden-variable model **realizes** the empirical model. We want to know if we can put this hidden-variable model into a canonical form. More than this, to be useful, such a canonical hidden-variable model must preserve properties — determinism, common-cause correlation, etc. — satisfied by the original hidden-variable model.

We show that, under a measure-theoretic condition, such a canonical model exists: *If the σ-algebras on the measurement and outcome spaces are countably generated, then the hidden-variable space can always be taken to be the unit interval equipped with the Lebesgue measure on the Borel sets.* Note that if a probability space has a countably generated σ-algebra, then the associated probability algebra is separable. The key to our result is the classical theorem that any two separable atomless probability algebras are isomorphic. This theorem can be found in Carathéodory (1939) and Halmos and von Neumann (1942). It is also a special case of Maharam's Theorem (Maharam, 1942).

2 Empirical and Hidden-Variable Models

Alice has a space of possible measurements, which is a measurable space $(Y_a, \mathcal{Y}_a)$, and a space of possible outcomes, which is a measurable space $(X_a, \mathcal{X}_a)$. Likewise, Bob has a space of possible measurements, which is a measurable space $(Y_b, \mathcal{Y}_b)$, and a space of possible outcomes, which is a measurable space $(X_b, \mathcal{X}_b)$. We will restrict attention to bipartite systems. (We comment later on the extension to more than two parts.) There is also a hidden-variable space, which is an unspecified measurable space $(\Lambda, \mathcal{L})$. Write

$$
\begin{aligned}
(X, \mathcal{X}) &= (X_a, \mathcal{X}_a) \otimes (X_b, \mathcal{X}_b), \\
(Y, \mathcal{Y}) &= (Y_a, \mathcal{Y}_a) \otimes (Y_b, \mathcal{Y}_b), \\
\Psi &= (X, \mathcal{X}) \otimes (Y, \mathcal{Y}), \\
\Omega &= (X, \mathcal{X}) \otimes (Y, \mathcal{Y}) \otimes (\Lambda, \mathcal{L}).
\end{aligned}
$$

Definition 1. An **empirical model** is a probability measure e on Ψ.

We see that an empirical model describes an experiment in which the pair of measurements $y = (y_a, y_b) \in Y$ is randomly chosen according to the probability measure $\text{marg}_Y e$, and y and the joint outcome $x = (x_a, x_b) \in X$ are distributed according to e.

Definition 2. A **hidden-variable model** is a probability measure p on Ω.

Definition 3. We say that a hidden-variable model p **realizes** an empirical model e if $e = \text{marg}_\Psi p$. We say that two hidden-variable models are (**realization-**)**equivalent** if they realize the same empirical model.

3 Preliminaries

Throughout, we use the following two conventions. First, when p is a probability measure on a product space $(X, \mathcal{X}) \otimes (Y, \mathcal{Y})$ and $q = \text{marg}_X p$, then for each $J \in \mathcal{X}$ we write

$$p(J) = p(J \times Y) = q(J),$$

and for each q-integrable $f : X \to \mathbb{R}$ we write

$$\int_J f(x)\, dp = \int_{J \times Y} f(x)\, dp = \int_J f(x)\, dq.$$

Thus, in particular, a statement holds for p-almost all $x \in X$ if and only if it holds for q-almost all $x \in X$.

Second, when p is a probability measure on a product space $(X, \mathcal{X}) \otimes (Y, \mathcal{Y}) \otimes (Z, \mathcal{Z})$, and $J \in \mathcal{X}$, we let $p[J||\mathcal{Z}]$ be the function from Z into $[0, 1]$ such that

$$p[J||\mathcal{Z}]_z = p[J \times Y \times Z|\{X \times Y, \emptyset\} \otimes \mathcal{Z}]_{(x,y,z)} = \text{E}[1_{J \times Y \times Z}|\{X \times Y, \emptyset\} \otimes \mathcal{Z}].$$

Note that $\{X \times Y, \emptyset\}$ is the trivial σ-algebra over $X \times Y$.

We use similar notation for (finite) products with factors to the left of $(X, \mathcal{X})$ or to the right of $(Z, \mathcal{Z})$. Note that if $q = \text{marg}_{X \times Z} p$, then $q[J||\mathcal{Z}] = p[J||\mathcal{Z}]$. We also use the analogous notation for expected values of random variables: Given an integrable function $f : X \to \mathbb{R}$, we write $\text{E}[f||\mathcal{Z}]$ for the conditional expectation $\text{E}[f \circ \pi|\{X \times Y, \emptyset\} \otimes \mathcal{Z}]$where π is the projection from $X \times Y \times Z$ to X.

Lemma 1. *The mapping $z \mapsto p[J||\mathcal{Z}]_z$ is the p-almost surely unique $\mathcal{Z}$-measurable function $f : Z \to [0, 1]$ such that for each set $L \in \mathcal{Z}$,*

$$\int_L f(z)\, dp = p(J \times L).$$

Proof. Existence: Let $f(z) = p[J||\mathcal{Z}]_z$. Using the definition of $p[J||\mathcal{Z}]$, we see that

$$\int_L f(z)\, dp = \int_{X \times Y \times L} \text{E}[1_{J \times Y \times Z}|\{X \times Y, \emptyset\} \otimes \mathcal{Z}]\, dp =$$
$$\int_{X \times Y \times L} 1_{J \times Y \times Z}\, dp = p((X \times Y \times L) \cap (J \times Y \times Z)) = p(J \times L),$$

Uniqueness: Let f and g be two such functions and let $L = \{z : f(z) < g(z)\}$. Then $L \in \mathcal{Z}$. If $p(J) = 0$, then $f(z) = g(z) = 0$, p-almost surely. Next suppose $p(J) > 0$. If

$p(J \times L) > 0$, then $p(L) > 0$, and

$$0 < \int_L g(z)\,dp - \int_L f(z)\,dp = \int_L (g(z) - f(z))\,dp = 0,$$

a contradiction. Therefore $p(J \times L) = 0$, so $p(L) = 0$ and hence $f(z) \geq g(z)$, p-almost surely. Similarly, $g(z) \geq f(z)$, p-almost surely, so $f(z) = g(z)$, p-almost surely. □

Corollary 1. *Let q be the marginal of p on $X \times Z$. Then, for each $J \in \mathcal{X}$, we have $p[J||\mathcal{Z}] = q[J||\mathcal{Z}]$, q-almost surely.*

Given probability measures p on $(X, \mathcal{X}) \otimes (Y, \mathcal{Y})$ and r on $(Y, \mathcal{Y})$, we say that p is an **extension** of r if $r = \mathrm{marg}_Y p$. We say that two probability measures p and q on $(X, \mathcal{X}) \otimes (Y, \mathcal{Y})$ **agree on** Y if $\mathrm{marg}_Y p = \mathrm{marg}_Y q$.

4 Properties of Hidden-Variable Models

Whenever we write an equation involving conditional probabilities, it will be understood to mean that the equation holds p-almost surely. All expressions below which are given for Alice have counterparts for Bob, with a and b interchanged.

The properties of hidden-variable models we now list are all stated for infinite measurement and outcome spaces. In the sources we give, these properties were stated for the finite case only.

We begin with Bell (1964) locality:

Definition 4. The hidden-variable model p satisfies **locality** if for every $J_a \in \mathcal{X}_a$, $J_b \in \mathcal{X}_b$, we have

$$p[J_a \times J_b||\mathcal{Y} \otimes \mathcal{L}] = p[J_a||\mathcal{Y}_a \otimes \mathcal{L}] \times p[J_b||\mathcal{Y}_b \otimes \mathcal{L}].$$

The next two properties come from Jarrett (1984) (and were given these names by Shimony, 1986).

Definition 5. The hidden-variable model p satisfies **parameter independence** if for every $J_a \in \mathcal{X}_a$ we have

$$p[J_a||\mathcal{Y} \otimes \mathcal{L}] = p[J_a||\mathcal{Y}_a \otimes \mathcal{L}].$$

Definition 6. The hidden-variable model p satisfies **outcome independence** if for every $J_a \in \mathcal{X}_a$, $J_b \in \mathcal{X}_b$, we have

$$p[J_a \times J_b||\mathcal{Y} \otimes \mathcal{L}] = p[J_a||\mathcal{Y} \otimes \mathcal{L}] \times p[J_b||\mathcal{Y} \otimes \mathcal{L}].$$

The name of the next property is due to Dickson (2005).

Definition 7. The hidden-variable model p satisfies **λ-independence** if for every event $L \in \mathcal{L}$,

$$p[L||\mathcal{Y}]_y = p(L).$$

Remark 1. We observe:

1. The λ-independence property for p depends only on $\text{marg}_{Y\times\Lambda}p$.

2. Any hidden-variable model p such that Λ is a singleton satisfies λ-independence.

By Remark 1, we have:

Lemma 2. *The following are equivalent:*

1. p satisfies λ-independence.

2. $\text{marg}_{Y\times\Lambda}p = \text{marg}_Y p \otimes \text{marg}_\Lambda p$.

3. The σ-algebras $\mathcal{Y}$ and $\mathcal{L}$ are independent with respect to p, i.e.,

$$p(K \times L) = p(K) \times p(L)$$

   for every $K \in \mathcal{Y}, L \in \mathcal{L}$.

The distinction between strong and weak determinism in the next two definitions is from Brandenburger and Yanofsky (2008).

Definition 8. The hidden-variable model p satisfies **strong determinism** if for each $J_a \in \mathcal{X}_a$, we have

$$p[J_a||\mathcal{Y}_a \otimes \mathcal{L}]_{(y_a,\lambda)} \in \{0,1\}.$$

Definition 9. The hidden-variable model p satisfies **weak determinism** if for every $J_a \in \mathcal{X}_a$, $J_b \in \mathcal{X}_b$, we have

$$p[J_a \times J_b||\mathcal{Y} \otimes \mathcal{L}]_{(y,\lambda)} \in \{0,1\}.$$

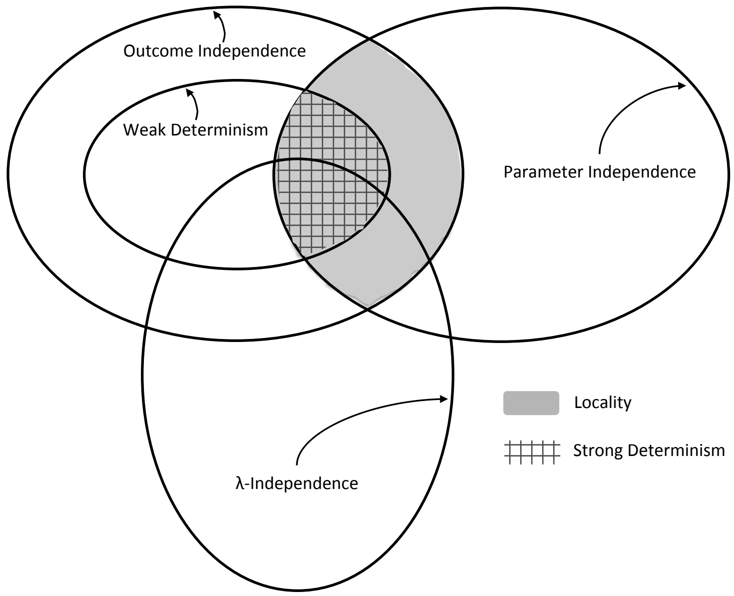

Figure 1

In Brandenburger and Keisler (2016), we establish how the various properties we have listed above are related, in the case that the outcome spaces X_b and X_b are finite. The Venn diagram above summarizes these relationships. The same relationships hold for general (infinite) X_a and X_b.

To see this, we first show that the relationships automatically carry over to the case that the σ-algebras $\mathcal{X}_a$ and $\mathcal{X}_b$ are finite. In this case, let X'_a be the finite set of atoms of $\mathcal{X}_a$ and let $\mathcal{X}'_a$ be the power set of X'_a. Define X'_b and $\mathcal{X}'_b$ similarly. Then the space

$$\Omega' = (X', \mathcal{X}') \otimes (Y, \mathcal{Y}) \otimes (\Lambda, \mathcal{L})$$

behaves exactly like the space Ω. For each hidden-variable model p, let p' be the hidden-variable model on Ω' such that $p'(\{x'\} \times K \times L) = p(x' \times K \times L)$, and, for each empirical model e, let e' be the empirical model such that $e'(\{x'\} \times K) = e(x' \times K)$. It is clear that p realizes e if and only if p' realizes e', that p' has the same hidden-variable space as p, and that each of the properties we have listed holds for p' if and only if it holds for p. It follows that the same Venn diagram of relationships holds for the case that the σ-algebras $\mathcal{X}_a$ and $\mathcal{X}_b$ are finite.

Next, we extend this observation to the case of general $\mathcal{X}_a$ and $\mathcal{X}_b$.

Definition 10. Let $\mathcal{F}_a$ and $\mathcal{F}_b$ be finite subalgebras of $\mathcal{X}_a$ and $\mathcal{X}_b$, respectively, and let $\mathcal{F} = \mathcal{F}_a \otimes \mathcal{F}_b$. For each empirical model e, the **finite restriction** $e \upharpoonright \mathcal{F}$ of e to $\mathcal{F}$ is the restriction of e to the σ-algebra $\mathcal{F} \otimes \mathcal{Y}$. For each hidden-variable model p, the **finite restriction** $p \upharpoonright \mathcal{F}$ of p to $\mathcal{F}$ is the restriction of p to the σ-algebra $\mathcal{F} \otimes \mathcal{Y} \otimes \mathcal{L}$.

The following lemma is an easy consequence of the definitions.

Lemma 3. *We have:*

1. p realizes e if and only if every finite restriction of p realizes the corresponding finite restriction of e.

2. p satisfies locality, parameter independence, outcome independence, λ-independence, strong determinism, or weak determinism if and only if every finite restriction of p satisfies the corresponding property.

From this it follows that the same Venn diagram of relationships holds for the case of general $\mathcal{X}_a$ and $\mathcal{X}_b$.

Finally in this section, we note that the definitions and relationships we have given extend immediately to systems with more than two parts, except that parameter independence must now be stated in terms of sets of parts instead of individual parts.

5 A Canonical Hidden-Variable Space

Given a hidden-variable model p, we let $\ell = \mathrm{marg}_\Lambda p$ and call the probability space $(\Lambda, \mathcal{L}, \ell)$ the **hidden-variable space of** p. We now state and prove our result on the existence of a canonical hidden-variable space.

Theorem 1. *Assume that the σ-algebras $\mathcal{X}$ and $\mathcal{Y}$ are countably generated. Then every hidden-variable model p is realization-equivalent to a hidden-variable model $\overline{p}$ with hidden-variable space $([0,1], \mathcal{U}, u)$, where $\mathcal{U}$ is the set of Borel subsets of $[0,1]$ and u is Lebesgue measure on $\mathcal{U}$. Moreover, for each of the properties of parameter independence and outcome independence (and, therefore, locality), λ-independence, strong determinism, and weak determinism, if p has the property then so does $\overline{p}$.*

To begin the proof of Theorem 1, we may assume without loss of generality that p has an atomless hidden-variable model, because the product $p \otimes u$ of p with the Lebesgue unit interval is realization-equivalent to p and has the atomless hidden-variable model $(\Lambda, \mathcal{L}, \ell) \otimes ([0,1], \mathcal{U}, u)$, and if p has any of the five properties above, then so does $p \otimes u$.

Since $\mathcal{X}$ and $\mathcal{Y}$ are countably generated, $\mathcal{X}_a, \mathcal{X}_b, \mathcal{Y}_a$, and $\mathcal{Y}_b$ are countably generated. Let $\mathcal{X}_a^0$ be a countable subset of $\mathcal{X}_a$ that generates the σ-algebra $\mathcal{X}_a$, and let $\mathcal{Y}_a^0$ be a countable subset of $\mathcal{Y}_a$ that generates the σ-algebra $\mathcal{Y}_a$. Define $\mathcal{X}_b^0$ and $\mathcal{Y}_b^0$ similarly. Let $\mathcal{U}_0$ be the family of all open subintervals of $[0,1]$ with rational endpoints. Let

$$(A_{an}, A_{bn}, B_{an}, B_{bn}, C_n)$$

be an enumeration of the countable set $\mathcal{X}_a^0 \times \mathcal{X}_b^0 \times \mathcal{Y}_a^0 \times \mathcal{Y}_b^0 \times \mathcal{U}_0$. The Cartesian products $A_{an} \times A_{bn} \times B_{an} \times B_{bn} \times C_n$ generate the σ-algebra $\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{U}$. For each n, let $f_n \colon \Lambda \to [0,1]$ be a representative of the family of p-almost surely unique functions

$$\lambda \mapsto p[A_{an} \times A_{bn} \times B_{an} \times B_{bn} || \mathcal{L}]_\lambda,$$

and let $D_n = f_n^{-1}(C_n)$.

To continue the proof, we need the following lemma.

Lemma 4. *There is a countably generated σ-algebra $\mathcal{D} \subseteq \mathcal{L}$ such that each of the sets D_n belongs to $\mathcal{D}$, and the restriction of ℓ to $\mathcal{D}$ is atomless.*

Proof. Since ℓ is atomless, it follows from a result of Sierpinski (1922) that for each set $L \in \mathcal{L}$ there is a set $L' \in \mathcal{L}$ such that $L' \subseteq L$ and $\ell(L') = \ell(L)/2$. Then, by the Axiom of Choice, there is a function $F : \mathcal{L} \to \mathcal{L}$ such that for each $L \in \mathcal{L}$, $F(L) \subseteq L$ and $\ell(F(L)) = \ell(L)/2$. Let $\mathcal{E}_0$ be the algebra of subsets of Λ generated by $\{D_n : n \in \mathbb{N}\}$. For each $m \in \mathbb{N}$, let $\mathcal{E}_{m+1}$ be the algebra of subsets of Λ generated by $\mathcal{E}_m \cup \{F(L) : L \in \mathcal{E}_m\}$. Let $\mathcal{E} = \bigcup_m \mathcal{E}_m$ and let $\mathcal{D}$ be the σ-algebra generated by $\mathcal{E}$. Clearly, each D_n belongs to $\mathcal{D}$, and $\mathcal{E}$ is countable, so $\mathcal{D}$ is countably generated.

We show that the restriction of ℓ to $\mathcal{D}$ is atomless. Let $\mathcal{D}'$ be the set of all $D \in \mathcal{D}$ that can be approximated by sets in $\mathcal{E}$ with respect to ℓ, that is,

$$\mathcal{D}' = \{D \in \mathcal{D} : (\forall r > 0)(\exists E \in \mathcal{E})\ell(E \triangle D) < r\}.$$

It is clear that $\mathcal{E} \subseteq \mathcal{D}'$, and that $\mathcal{D}'$ is closed under finite unions and intersections. The set $\mathcal{D}'$ is also closed under unions of countable chains, because if $L_n \in \mathcal{D}'$ and $L_n \subseteq L_{n+1}$ for each n, and $L = \bigcup_n L_n$, then for each $r > 0$ there exists $n \in \mathbb{N}$ and $E \in \mathcal{E}$ such that $\ell(L \triangle L_n) < r/2$ and $\ell(E \triangle L_n) < r/2$. Therefore $\ell(L \triangle E) < r$, so $L \in \mathcal{D}'$. It follows that $\mathcal{D}' = \mathcal{D}$. Now suppose $D \in \mathcal{D}$, $\ell(D) > 0$, and $r > 0$. Then $D \in \mathcal{D}'$, so there is a set

$G \in \mathcal{E}$ such that $\ell(D \triangle G) < r$. We have $F(G) \in \mathcal{E}$, $\ell(F(G)) = \ell(G)/2$, and $F(G) \subseteq G$. Then $D \cap F(G) \in \mathcal{D}$, and, by taking r small enough, we can guarantee that $\ell(D) > \ell(D \cap F(G)) > 0$. This shows that the restriction of ℓ to $\mathcal{D}$ is atomless, and proves the lemma. □

Continuing the proof of Theorem 1, we know by Carathéodory (1939) and Halmos and von Neumann (1942) (and a special case in Maharam, 1942) that any two separable atomless probability algebras are isomorphic. Therefore, the measure algebras of $(\Lambda, \mathcal{D}, \ell)$ and $([0,1], \mathcal{U}, u)$ are isomorphic. This isomorphism maps the equivalence class (modulo null sets) of each $D \in \mathcal{D}$ to the equivalence class of a set $h(D) \in \mathcal{U}$ such that $u(h(D)) = \ell(D)$.

Let $\overline{p}$ be the probability measure on $\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{U}$ such that for each $R \in \mathcal{X}$, $S \in \mathcal{Y}$, and $D \in \mathcal{D}$ we have $\overline{p}(R \times S \times h(D)) = p(R \times S \times D)$. It follows that the hidden-variable space of $\overline{p}$ is $([0,1], \mathcal{U}, \ell)$. Moreover, $\overline{p}$ and p agree on $X \times Y$, so $\overline{p}$ is realization-equivalent to p. For each $S \in \mathcal{Y}$ and $D \in \mathcal{D}$ we have

$$\overline{p}(S) = p(S), \ \ \overline{p}(h(D)) = p(D), \ \ \overline{p}(S \times h(D)) = p(S \times D).$$

Therefore, if p has λ-independence, then $\overline{p}$ has λ-independence by Lemma 2.

The σ-algebra $\mathcal{D}$ is large enough so that for each $K \in \mathcal{Y}$, the function $p[J_a \times J_b \times K || \mathcal{L}]$ is $\mathcal{D}$-measurable. It follows that

$$p[J_a \times J_b || \mathcal{Y} \otimes \mathcal{L}] = p[J_a \times J_b || \mathcal{Y} \otimes \mathcal{D}],$$

$$p[J_a || \mathcal{Y}_a \otimes \mathcal{L}] = p[J_a || \mathcal{Y}_a \otimes \mathcal{D}],$$

and

$$p[J_b || \mathcal{Y}_b \otimes \mathcal{L}] = p[J_b || \mathcal{Y}_b \otimes \mathcal{D}].$$

From the definition of $\overline{p}$, one can see that joint distributions of the functions

$$p[J_a \times J_b || \mathcal{Y} \otimes \mathcal{D}], \ \ p[J_a || \mathcal{Y}_a \otimes \mathcal{D}], \ \ p[J_b || \mathcal{Y}_b \otimes \mathcal{D}]$$

and

$$\overline{p}[J_a \times J_b || \mathcal{Y} \otimes \mathcal{U}], \ \ \overline{p}[J_a || \mathcal{Y}_a \otimes \mathcal{U}], \ \ \overline{p}[J_b || \mathcal{Y}_b \otimes \mathcal{U}]$$

are the same. It follows that for each of the properties of parameter independence, outcome independence, strong determinism, and weak determinism, if p has the property then so does $\overline{p}$.

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