9

INFINITE SERIES

9.1 SEQUENCES

DEFINITION

*An **infinite sequence** is a real function whose domain is the set of all positive integers.*

A sequence a can be displayed in the form

$$a(1), a(2), \ldots, a(n), \ldots.$$

The value $a(n)$ is called the nth *term* of the sequence and is usually written a_n. The whole sequence is denoted by

$$\langle a_n \rangle = a_1, a_2, \ldots, a_n, \ldots.$$

Hyperintegers, which were introduced in Section 3.8, are a basic tool in this chapter. Since a_n is defined for every positive integer n, a_H is defined for every positive infinite hyperinteger H.

EXAMPLE 1 If the sequence is simple enough one can look at the first few terms and guess the general rule for computing the nth term. For instance:

$1, 1, 1, 1, 1, \ldots$	$a_n = 1$
$-1, 0, 1, 2, 3, \ldots$	$a_n = n - 2$
$-2, -4, -6, -8, -10, \ldots$	$a_n = -2n$
$1, -1, 1, -1, 1, \ldots$	$a_n = (-1)^{n-1}$
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$	$a_n = \frac{1}{n}$

The graph of a sequence will look like a collection of dots whose x-coordinates are spaced one apart. Some examples of graphs of sequences are shown in Figure 9.1.1.

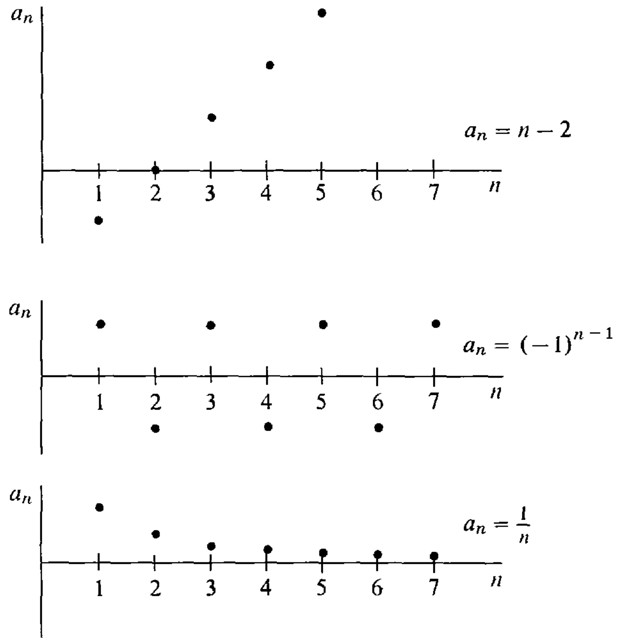

Figure 9.1.1

EXAMPLE 2 The sequence

$$3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots$$

is defined by the rule

$$a_n = \pi \text{ to } n \text{ decimal places},$$

that is, $a_n = \dfrac{m}{10^n}$ where m is the integer such that $\dfrac{m}{10^n} \leq \pi < \dfrac{m+1}{10^n}$.

EXAMPLE 3 The number $n!$, read n factorial, is defined as the product of the first n positive integers;

$$n! = 1 \cdot 2 \cdot \cdots \cdot n$$

$\langle n! \rangle$ is an important sequence. Its first few terms are

$$1, 2, 6, 24, 120, 720, \ldots.$$

By convention, $0!$ is defined by $0! = 1$.

DEFINITION

An infinite sequence $\langle a_n \rangle$ is said to ***converge*** *to a real number L if a_H is infinitely close to L for all positive infinite hyperintegers H (Figure 9.1.2). L is called the* ***limit*** *of the sequence and is written*

$$L = \lim_{n \to \infty} a_n.$$

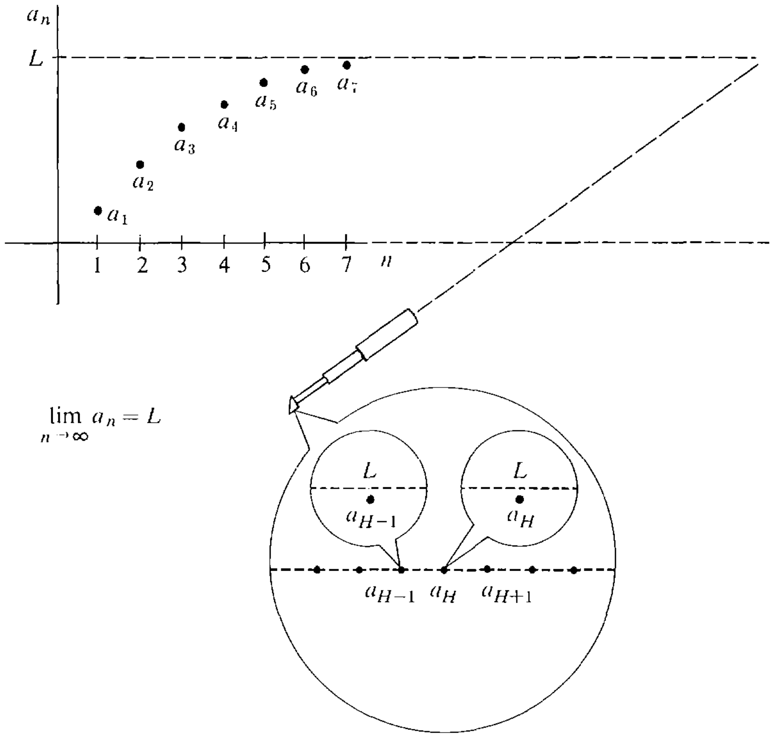

Figure 9.1.2

A sequence which does not converge to any real number is said to ***diverge****. If a_H is positive infinite for all positive infinite hyperintegers H, the sequence is said to* ***diverge to*** *∞, and we write*

$$\lim_{n \to \infty} a_n = \infty.$$

Sequences can diverge to $-\infty$, and also diverge without diverging to ∞ or to $-\infty$.

Throughout this chapter, H and K will always be used for positive infinite hyperintegers. One can often determine whether or not a sequence converges by examining the values of a_H for infinite H. The definition gives us some convenient working rules.

(1) If a_H is infinitely close to L for all H, the sequence converges to L.

(2) If we can find a_H and a_K which are not infinitely close to each other, the sequence diverges.

(3) If at least one a_H is infinite, the sequence diverges.

(4) If all the a_H are positive infinite, the sequence diverges to ∞.

EXAMPLE 1 (Continued)

$\lim_{n \to \infty} 1 = 1$, converges, because $a_H = 1$ for all H.

$\lim_{n \to \infty} n - 2 = \infty$, diverges, because $H - 2$ is positive infinite for all H.

$\lim_{n\to\infty}(-2n) = -\infty$, diverges, because $-2H$ is negative infinite for all H.

$\lim_{n\to\infty}(-1)^n$ is undefined, diverges, because $(-1)^{2H} = 1$ but $(-1)^{2H+1} = -1$.

$\lim_{n\to\infty}\frac{1}{n} = 0$, converges, because $\frac{1}{H}$ has standard part zero.

EXAMPLE 2 (Continued) The sequence

$$3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots, a_n, \ldots$$

where $a_n = (\pi$ to n decimal places), converges to π. That is,

$$\lim_{n\to\infty} a_n = \pi.$$

PROOF Let H be positive infinite. For some K,

$$\frac{K}{10^H} \le \pi < \frac{K+1}{10^H}.$$

Then
$$a_H = \frac{K}{10^H}, \qquad a_H \le \pi \le a_H + \frac{1}{10^H}.$$

But $1/10^H$ is infinitesimal, so $a_H \approx \pi$.

EXAMPLE 3 (Continued) $\lim_{n\to\infty} n! = \infty$.

PROOF For any $n > 1$, we have

$$(n-1)! \ge 1, \qquad n! = n\cdot(n-1)! \ge n.$$

Therefore for positive infinite H, $H! \ge H$ is positive infinite.

Given a function $f(x)$ defined for all $x \ge 1$, we can form the sequence

$$f(1), f(2), \ldots, f(n), \ldots.$$

The graph of the sequence $\langle f(n)\rangle$ is the collection of dots on the curve $y = f(x)$ where the x-coordinate is a positive integer (Figure 9.1.3).

If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$ because $f(H) \approx L$ for any positive infinite H.

EXAMPLE 4 $\lim_{n\to\infty}\frac{4n^2+1}{n^2+3n} = \lim_{x\to\infty}\frac{4x^2+1}{x^2+3x} = 4.$

Similarly, if $\lim_{x\to\infty} f(x) = \infty$ then $\lim_{n\to\infty} f(n) = \infty$.

EXAMPLE 5 $\lim_{n\to\infty}\ln(n) = \lim_{x\to\infty}\ln(x) = \infty.$

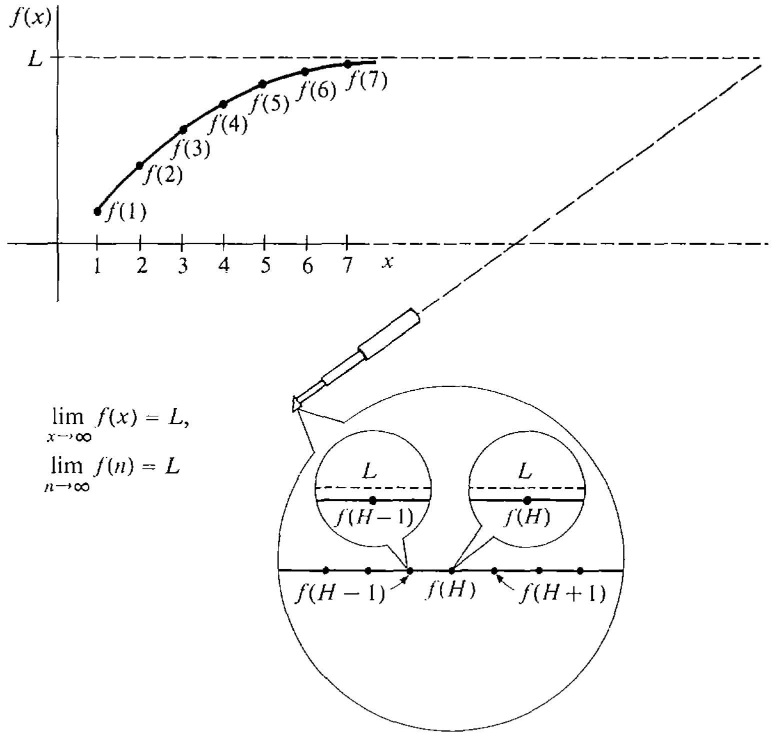

Figure 9.1.3

If $\lim_{x \to 0^+} f(x) = L$, then $\lim_{n \to \infty} f(1/n) = L$. If H is positive infinite, then $\varepsilon = 1/H$ is infinitesimal and

$$f(1/H) = f(\varepsilon) \approx L.$$

EXAMPLE 6 $\lim_{n \to \infty} c^{1/n} = \lim_{x \to 0^+} c^x = c^0 = 1$, if $c > 0$.

EXAMPLE 7 Evaluate the limits

(a) $\lim_{n \to \infty} \left(1 + \frac{1}{c}\right)^n$ where $c > 0$,

(b) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^c$ where $c > 0$,

(c) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

The answers are

(a) $\lim_{n \to \infty} \left(1 + \frac{1}{c}\right)^n = \lim_{x \to \infty} \left(1 + \frac{1}{c}\right)^x = \infty.$

(b) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^c = \lim_{x \to 0^+} (1 + x)^c = 1.$

(c) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$

The limit $\lim_{n\to\infty}(1 + 1/n)^n = e$ is closely related to compound interest. Suppose a bank pays interest on one dollar at the rate of 100% per year. If the interest is compounded n times per year the dollar will grow to $(1 + 1/n)$ after $1/n$ years, to $(1 + 1/n)^k$ after k/n years, and thus to $(1 + 1/n)^n$ after one year. Since $\lim_{n\to\infty}(1 + 1/n)^n = e$, one dollar will grow to e dollars if the interest is compounded continuously for one year, and to e^t dollars after t years.

More generally, suppose the account initially has a dollars and the bank pays interest at the rate of b% per year. If the interest is compounded n times per year, the account will grow as follows:

0 years	a
$\frac{1}{n}$ years	$a\left(1 + \frac{b}{100}\cdot\frac{1}{n}\right)$
$\frac{k}{n}$ years	$a\left(1 + \frac{b}{100}\cdot\frac{1}{n}\right)^k$
1 year	$a\left(1 + \frac{b}{100}\cdot\frac{1}{n}\right)^n$

If the interest is compounded continuously the account will grow in one year to

$$\lim_{n\to\infty} a\left(1 + \frac{b}{100}\cdot\frac{1}{n}\right)^n.$$

We can evaluate this limit by setting $x = \frac{100}{b}n$, $n = \frac{b}{100}x$.

$$\lim_{n\to\infty} a\left(1 + \frac{b}{100}\cdot\frac{1}{n}\right)^n = \lim_{x\to\infty} a\left(1 + \frac{1}{x}\right)^{bx/100} = ae^{b/100}.$$

Thus the account grows to $ae^{b/100}$ dollars after one year and to $ae^{bt/100}$ dollars after t years.

Sometimes we may wish to know how rapidly a sequence grows. If two sequences approach ∞ and their quotient also approaches ∞,

$$\lim_{n\to\infty} a_n = \infty, \qquad \lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty}\frac{a_n}{b_n} = \infty,$$

the sequence $\langle a_n\rangle$ is said to *grow faster* than the sequence $\langle b_n\rangle$. For each infinite H, both a_H and b_H are infinite. But a_H/b_H is still infinite, so a_H is infinite even compared to b_H.

THEOREM 1

Each of the following sequences approaches ∞.

$$\lim_{n\to\infty} n! = \infty,$$
$$\lim_{n\to\infty} b^n = \infty \qquad \text{if } b > 1,$$
$$\lim_{n\to\infty} n^c = \infty \qquad \text{if } c > 0,$$
$$\lim_{n\to\infty} \ln(n) = \infty.$$

Moreover, each sequence in the list grows faster than the next one,

(i) $\lim_{n\to\infty} \dfrac{n!}{b^n} = \infty \qquad (b > 1),$

(ii) $\lim_{n\to\infty} \dfrac{b^n}{n^c} = \infty \qquad (b > 1, \quad c > 0),$

(iii) $\lim_{n\to\infty} \dfrac{n^c}{\ln(n)} = \infty \qquad (c > 0).$

PROOF Let H be positive infinite. We already know that $\ln H$ is positive infinite. We must show that each of the following are also positive infinite.

$$\frac{H!}{b^H}, \qquad \frac{b^H}{H^c}, \qquad \frac{H^c}{\ln H}.$$

It is easier to show that their logarithms are positive infinite. We need the fact that, by l'Hospital's rule for ∞/∞,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0,$$

so $\dfrac{\ln K}{K} \approx 0$ for all infinite K.

(i) $\dfrac{H!}{b^H}$. Let $m > b$. Then

$$\ln\left(\frac{H!}{b^H}\right) = \ln 1 + \cdots + \ln(m-1) + \ln m + \cdots + \ln H - H\ln b$$
$$> (H - m)\ln m - H\ln b = H(\ln m - \ln b) - m\ln m.$$

Since $m > b$, $\ln m > \ln b$, and $\ln(H!/b^H)$ is positive infinite.

(ii) $\dfrac{b^H}{H^c}$.

$$\ln\left(\frac{b^H}{H^c}\right) = H\ln b - c\ln H = H\left(\ln b - c\frac{\ln H}{H}\right).$$

Since $b > 1$, $\ln b > 0$. $\dfrac{\ln H}{H}$ is infinitesimal. Therefore $\ln(b^H/H^c)$ is positive infinite.

(iii) $\dfrac{H^c}{\ln H}$. Let $K = \ln H$.

$$\ln\left(\frac{H^c}{\ln H}\right) = c\ln H - \ln(\ln H) = K\left(c - \frac{\ln K}{K}\right).$$

Since $c > 0$, K is infinite, and $(\ln K)/K$ is infinitesimal, $\ln(H^c/\ln H))$ is positive infinite.

Note: For $n = 1$, the term $n^c/(\ln n)$ is undefined, so we should start the sequence with $n = 2$.

EXAMPLE 8 From Theorem 1, the following sequences all approach ∞.

$$\frac{\sqrt{2}}{\ln 2}, \frac{\sqrt{3}}{\ln 3}, \frac{\sqrt{4}}{\ln 4}, \ldots, \frac{\sqrt{n}}{\ln(n)}, \ldots$$

$$\frac{2^1}{1^{10}}, \frac{2^2}{2^{10}}, \frac{2^3}{3^{10}}, \frac{2^4}{4^{10}}, \ldots, \frac{2^n}{n^{10}}, \ldots$$

$$\frac{1!}{100^1}, \frac{2!}{100^2}, \frac{3!}{100^3}, \frac{4!}{100^4}, \ldots, \frac{n!}{100^n}, \ldots$$

If $\lim_{n \to \infty} a_n = \infty$, then $\lim_{n \to \infty} 1/a_n = 0$ because $1/a_H$ will be infinitesimal.

COROLLARY

(i) $\lim_{n \to \infty} b^{-n} = 0$ if $b > 1$.

(ii) $\lim_{n \to \infty} n^{-c} = 0$ if $c > 0$.

Like other types of limits, limits of sequences have an ε, N condition. It will be used later to prove theorems on series.

THEOREM 2 (ε, N **Condition for Limits of Sequences**)

$$\lim_{n \to \infty} a_n = L$$

if and only if for every real number $\varepsilon > 0$ there is a positive integer N such that the numbers

$$a_N, a_{N+1}, a_{N+2}, \ldots, a_{N+m}, \ldots$$

are all within ε of L.

The proof is similar to that of the ε, δ condition for limits of functions. The ε, N condition says intuitively that a_n gets close to L as the integer n gets large.

A similar condition can be formulated for $\lim_{n \to \infty} a_n = \infty$.

THEOREM 3 (ε, N **Condition for Infinite Limits**)

$$\lim_{n \to \infty} a_n = \infty$$

if and only if for every real number B, there is a positive integer N such that the numbers

$$a_N, a_{N+1}, a_{N+2}, \ldots, a_{N+m}, \ldots$$

are all greater than B.

We conclude this section with another useful criterion for convergence.

CAUCHY CONVERGENCE TEST FOR SEQUENCES

A sequence $\langle a_n \rangle$ converges if and only if

(1) $$a_H \approx a_K \text{ for all infinite } H \text{ and } K.$$

PROOF First suppose $\langle a_n \rangle$ converges, say $\lim_{n\to\infty} a_n = L$. Then for all infinite H and K,

$$a_H \approx L \approx a_K.$$

Now assume Equation 1 and let H be infinite. There are three cases to consider.

Case 1 a_H is finite. Then for all infinite K,

$$st(a_K) = st(a_H),$$

so the sequence converges to $st(a_H)$.

Case 2 a_H is positive infinite. For each finite m, $a_H \geq a_m + 1$. Among the hyperintegers $\{1, 2, \ldots, H-1\}$, there must be a largest element M such that $a_H \geq a_M + 1$. But this largest M cannot be finite, and since $a_M \not\approx a_H$, M cannot be infinite. Therefore Case 2 cannot arise.

Case 3 a_H is negative infinite. By a similar argument this case cannot arise.

Therefore, only Case 1 is possible, whence $\langle a_n \rangle$ converges.

PROBLEMS FOR SECTION 9.1

In Problems 1–8, find the nth term of the sequence.

1	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$	2	$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
3	$-1, 2, -3, 4, -5, 6, \ldots$	4	$2, 5, 10, 17, 26, 37, \ldots$
5	$1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \ldots$	6	$1, 3, 6, 10, 15, \ldots$
7	$2, 4, 16, 256, \ldots$	8	$0.6, 0.61, 0.616, 0.6161, \ldots$

Determine whether the following sequences converge, and find the limits when they exist.

9	$a_n = \sqrt{n}$	10	$a_n = \dfrac{n+2}{n}$
11	$a_n = n - \dfrac{n^2}{n+1}$	12	$a_n = n(-1)^n$
13	$a_n = \dfrac{(-1)^n}{\sqrt{n}}$	14	$a_n = \dfrac{n!}{n^3}$
15	$a_n = \dfrac{n}{(\ln(n))^2}$	16	$a_n = \sqrt[n]{n}$
17	$a_n = \ln(\ln(n))$	18	$a_n = \sqrt{n^2+n} - n$
19	$a_n = \left(\dfrac{n-1}{n}\right)^n$	20	$a_n = \dfrac{3n^2-2n+4}{2n^2-n+1}$
21	$a_n = \dfrac{n^2+1}{n^3+4}$	22	$a_n = \dfrac{n^3-2}{n^2+5}$

23 $a_n = \dfrac{2^n + 3^n}{2^n - 3^n}$

24 $a_n = 2^n - n^2$

25 $a_n = n! - 10^n$

26 $a_n = \dfrac{n! + 2}{(n + 1)! + 1}$

27 $a_n = \dfrac{\ln(n)}{\ln(\ln(n))}$

28 $a_n = (n!)^{1/n}$

29 $a_n = \dfrac{(n + 1)^n}{n^{n+1}}$

☐ 30 Formulate an ε, N condition for $\lim_{n\to\infty} a_n = -\infty$.

☐ 31 Show that if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ then $\lim_{n\to\infty} (a_n + b_n) = L + M$.

☐ 32 Show that if $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} ca_n = cL$.

9.2 SERIES

The sum of finitely many real numbers $a_1, a_2, \ldots, a_n$ is again a real number $a_1 + a_2 + \cdots + a_n$. Sometimes we wish to form the sum of an infinite sequence of real numbers,

$$a_1 + a_2 + \cdots + a_n + \cdots.$$

For example, if a man walks halfway across a room of unit width, then half of the remaining distance, then half the remaining distance again, and so forth, the total distance he will travel is an infinite sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots.$$

In n steps he will travel $1 - \dfrac{1}{2^n}$ units,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

Thus he will get closer and closer to the other side of the room, and we have the limit

$$\lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}\right) = 1.$$

It is natural to call this limit the infinite sum,

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots.$$

We can go from this example to the general notion of an infinite sum. When we wish to find the sum of an infinite sequence $\langle a_n \rangle$ we call it an *infinite series* and write it in the form

$$a_1 + a_2 + \cdots + a_n + \cdots.$$

Given an infinite sequence $\langle a_n \rangle$, each finite sum

$$a_1 + \cdots + a_n$$

is defined. This sum is called the nth *partial sum* of the series. Thus, with each infinite series

$$a_1 + a_2 + \cdots + a_n + \cdots,$$

there are associated two sequences, the *sequence of terms*,

$$a_1, a_2, \ldots, a_n, \ldots,$$

and the *sequence of partial sums*,

$$S_1, S_2, \ldots, S_n, \ldots \qquad \text{where } S_n = a_1 + \cdots + a_n.$$

For each positive hyperreal number H, the *infinite partial sum*

$$S_H = a_1 + \cdots + a_H$$

is also defined, by the Extension Principle.

The sum of an infinite series will be a real number which is close to the nth partial sum for large n, and infinitely close to the infinite partial sums. Before stating the definition precisely, let us examine some infinite series and their partial sum sequences, and guess at their sums.

Table 9.2.1

Series	Partial sums	Sum
$1 + 0.1 + 0.01 + 0.001 + \cdots$	$1, 1.1, 1.11, 1.111, \ldots$	$1\frac{1}{9}$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$	$1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{15}{16}, \ldots$	2
$1 - 1 + 1 - 1 + 1 - 1 + \cdots$	$1, 0, 1, 0, 1, 0, \ldots$	?
$1 + 1 + 1 + 1 + 1 + \cdots$	$1, 2, 3, 4, 5, \ldots$	∞
$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$	$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \ldots$	?
$3 + 0.1 + 0.04 + 0.001 + \cdots$	$3, 3.1, 3.14, 3.141, \ldots$	π

DEFINITION

*The **sum** of an infinite series is defined as the limit of the sequence of partial sums if the limit exists,*

$$a_1 + a_2 + \cdots + a_n + \cdots = \lim_{n\to\infty} (a_1 + \cdots + a_n).$$

*The series is said to **converge** to a real number S, **diverge**, or **diverge to** ∞, if the sequence of partial sums converges to S, diverges, or diverges to ∞, respectively.*

The sum of an infinite series can often be found by looking at the infinite partial sums $a_1 + \cdots + a_H$. Corresponding to our working rules for limits of sequences, we have the following rules for sums of series.

(1) If the value of every infinite partial sum is finite with standard part S, then the series converges to S,

$$a_1 + \cdots + a_n + \cdots = S.$$

(2) If there are two infinite partial sums which are not infinitely close to each other, the series diverges.

(3) If there is an infinite partial sum whose value is infinite, then the series diverges.

(4) If all infinite partial sums have positive infinite values, the series diverges to ∞,

$$a_1 + \cdots + a_n + \cdots = \infty.$$

Given an infinite series, we often wish to answer two questions. Does the series converge? What is the sum of the series? Our next theorem gives a formula for the sum of an important kind of series, the geometric series.

For each constant c, the series

$$1 + c + c^2 + \cdots + c^n + \cdots$$

is called the *geometric series* for c.

THEOREM 1

If $|c| < 1$, *the geometric series converges and*

$$1 + c + c^2 + \cdots + c^n + \cdots = \frac{1}{1 - c}.$$

PROOF For each n we have

$$\begin{aligned}(1 - c)(1 + c + c^2 + \cdots + c^n) \\ &= (1 + c + c^2 + \cdots + c^n) - (c + c^2 + \cdots + c^n + c^{n+1}) \\ &= 1 - c^{n+1}.\end{aligned}$$

The nth partial sum is therefore

$$1 + c + c^2 + \cdots + c^n = \frac{1 - c^{n+1}}{1 - c}.$$

The infinite partial sum up to H is

$$1 + c + \cdots + c^H = \frac{1 - c^{H+1}}{1 - c}.$$

Since $|c| < 1$, c^{H+1} is infinitesimal, so

$$1 + c + \cdots + c^H \approx \frac{1}{1 - c}.$$

EXAMPLE 1 $1 + 0.1 + 0.01 + 0.001 + \cdots = \dfrac{1}{1 - 1/10} = 1\dfrac{1}{9}.$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{1}{1 - (-1/2)} = \frac{2}{3}.$$

EXAMPLE 2 Every sequence $S_1, S_2, S_3, \ldots, S_n, \ldots$

is the partial sum sequence of an infinite series, namely

$$S_1 + (S_2 - S_1) + (S_3 - S_2) + \cdots + (S_{n+1} - S_n) + \cdots.$$

For example,

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$$

is the partial sum sequence of

$$1 + \left(\frac{1}{2} - 1\right) + \left(\frac{1}{3} - \frac{1}{2}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n}\right) + \cdots$$

or

$$1 - \frac{1}{2} - \frac{1}{6} - \cdots - \frac{1}{n(n+1)} - \cdots.$$

The Cauchy Convergence Test from the preceding section takes on the following form for series.

CAUCHY CONVERGENCE TEST FOR SERIES

$a_1 + a_2 + \cdots + a_n + \cdots$ converges if and only if

(1) *for all infinite $H < K$, $a_{H+1} + a_{H+2} + \cdots + a_K \approx 0$.*

DISCUSSION The sum in (1) is just the difference in partial sums,

$$a_{H+1} + a_{H+2} + \cdots + a_K = S_K - S_H.$$

A very important consequence of the Cauchy Convergence Criterion is that all the infinite terms of a convergent series must be infinitesimal. We state this consequence as a corollary, which is illustrated in Figure 9.2.1.

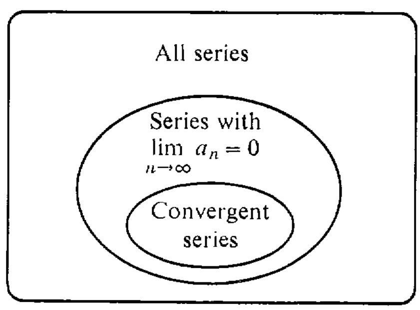

Figure 9.2.1

COROLLARY

If the series $a_1 + a_2 + \cdots + a_n + \cdots$ converges, then $\lim_{n\to\infty} a_n = 0$. That is, $a_K \approx 0$ for every infinite K.

PROOF This is true by the Cauchy Criterion, with $K = H + 1$.

Warning: The converse of this corollary is false. It is possible for a sequence to have $\lim_{n\to\infty} a_n = 0$ and yet diverge. We shall give an example later (Example 3).

The Cauchy Convergence Criterion and its corollary can often be used to show that a series diverges. Table 9.2.2 sums up the various possibilities. In this table it is understood that

$$a_1 + a_2 + \cdots + a_n + \cdots$$

is an infinite series and H, K are positive infinite hyperintegers with $H < K$.

Table 9.2.2 Cauchy Convergence and Divergence Tests

Hypothesis	Conclusion
all $a_{H+1} + \cdots + a_K \approx 0$	Converges
all $a_K \approx 0$	none
some $a_{H+1} + \cdots + a_K \not\approx 0$	Diverges
some $a_K \not\approx 0$	Diverges

We shall give many other convergence tests later on in this chapter. For convenience there is a summary of all these tests at the end of Section 9.6.

THEOREM 2

(i) *If* $|c| \geq 1$ *the geometric series* $1 + c + c^2 + \cdots + c^n + \cdots$ *diverges.*

(ii) ***The harmonic series*** $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} + \cdots$ *diverges.*

PROOF (i) For infinite H the term c^H is not infinitesimal, so the series diverges.

(ii) Intuitively this can be seen by writing

$$\begin{aligned} &1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + \cdots \\ &\geq 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) + \cdots \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots = \infty. \end{aligned}$$

Instead we can use the Cauchy Test. We see that for each n,

$$\frac{1}{2^n + 1} + \frac{1}{2^n + 2} + \cdots + \frac{1}{2^{n+1}} \geq 2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Therefore for infinite H,

$$\frac{1}{2^H + 1} + \frac{1}{2^H + 2} + \cdots + \frac{1}{2^{H+1}} \geq \frac{1}{2}.$$

Since the above sum is not infinitesimal the series diverges.

EXAMPLE 3 The harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is the example promised in our warning. It has the property that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

and yet the series diverges.

PROBLEMS FOR SECTION 9.2

In Problems 1–13 find the nth partial sum, determine whether the series converges, and find the sum when it exists.

1 $1 + \frac{1}{3} + \frac{1}{9} + \cdots + (\frac{1}{3})^n + \cdots$

2 $1 - \frac{1}{3} + \frac{1}{9} - \cdots + (-\frac{1}{3})^n + \cdots$

3 $1 + \frac{3}{4} + \frac{9}{16} + \cdots + (\frac{3}{4})^n + \cdots$

4 $1 - 2 + 4 - 8 + \cdots + (-2)^n + \cdots$

5 $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{24}\right) + \cdots + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) + \cdots$

6 $(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_{n+1}) + \cdots$ where $\lim_{n\to\infty} a_n = 0$. This is called a *telescoping series*.

7 $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots$. *Hint:* $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

8 $\ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \cdots + \ln\frac{n}{n+1} + \cdots$

9 $1 - 2 + 3 - 4 + \cdots + n(-1)^{n-1} + \cdots$

10 $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{n(n+2)} + \cdots$

11 $\frac{3}{1^2\cdot 2^2} + \frac{5}{2^2\cdot 3^2} + \cdots + \frac{2n+1}{n^2(n+1)^2} + \cdots$. *Hint:* $\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$

12 $\ln\frac{4}{3} + \ln\frac{9}{8} + \ln\frac{16}{15} + \cdots + \ln\frac{n^2}{n^2-1} + \cdots$

13 $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} + \cdots$

In Problems 14–19, show that the series diverges.

14 $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots$

15 $\frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \cdots + \frac{(-1)^{n-1}n}{2n+1} + \cdots$

16 $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} + \cdots$

17 $\frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \cdots + \frac{1}{3n+1} + \cdots$

18 $1 + \sqrt{2} + \sqrt[3]{3} + \cdots + \sqrt[n]{n} + \cdots$

19 $\ln 1 + \ln 2 + \ln 3 + \cdots + \ln n + \cdots$

20 A ball bounces along a street. On each bounce it goes $\frac{4}{5}$ as far as it did on the previous bounce. If the first bounce is one foot long, how far will the ball go before it stops bouncing?

21 Two students are sharing a loaf of bread. Student A eats half of the loaf, then student B eats half of what's left, then A eats half of what's left, and so on. How much of the loaf will each student eat?

22 In the Problem 21, how much will each student eat if only $\frac{1}{5}$ of the remaining loaf is eaten at each turn?

23 Three students A, B, C take turns eating a loaf of bread, taking $\frac{1}{3}$ of the remaining loaf at each turn. How much will each student eat?

9.3 PROPERTIES OF INFINITE SERIES

It is convenient to use capital sigmas, $\sum$, for partial sums and infinite series, as we did for finite and infinite Riemann sums. We write

$$S_m = \sum_{n=1}^{m} a_n = a_1 + a_2 + \cdots + a_m$$

for the mth partial sum,

$$S_H = \sum_{n=1}^{H} a_n = a_1 + a_2 + \cdots + a_H$$

for an infinite partial sum, and

$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

for the infinite series. Thus S is the standard part of S_H,

$$\sum_{n=1}^{\infty} a_n = st\left(\sum_{n=1}^{H} a_n\right).$$

Sometimes we start counting from zero instead of one. For example, the formula for the sum of a geometric series can be written

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}, \qquad \text{where } |c| < 1.$$

Infinite series are similar to definite integrals. Table 9.3.1 compares and contrasts the two notions.

Table 9.3.1

Infinite series	Definite integral
$\sum_{n=1}^{\infty} a_n$	$\int_a^b f(x)\,dx$
Finite partial sum $\sum_{n=1}^{m} a_n = a_1 + \cdots + a_m$	Finite Riemann sum $\sum_a^b f(x)\,\Delta x = f(x_1)\,\Delta x + \cdots + f(x_m)\,\Delta x$
Infinite partial sum $\sum_{n=1}^{H} a_n = a_1 + \cdots + a_H$	Infinite Riemann sum $\sum_a^b f(x)\,dx = f(x_1)\,dx + \cdots + f(x_H)\,dx$
$\sum_{n=1}^{\infty} a_n = st\left(\sum_{n=1}^{H} a_n\right)$	$\int_a^b f(x)\,dx = st\left(\sum_a^b f(x)\,dx\right)$
$\sum_{n=1}^{\infty} a_n = \lim_{m\to\infty}\left(\sum_{n=1}^{m} a_n\right)$	$\int_a^b f(x)\,dx = \lim_{\Delta x\to 0^+}\left(\sum_a^b f(x)\,\Delta x\right)$

The difference between them is that the infinite series is formed by adding up the terms of an infinite sequence, while the definite integral is formed by adding up the values of $f(x)\,dx$ for x between a and b. The definite integral of a continuous function always exists. But the problem of whether an improper integral converges is similar to the problem of whether an infinite series converges.

Here are some basic theorems about infinite series which are like theorems about integrals.

THEOREM 1

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ *are convergent.*

(i) Constant Rule *For any constant* c, $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.

(ii) Sum Rule $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

(iii) Inequality Rule *If* $a_n \le b_n$ *for all* n *then* $\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$.

PROOF To illustrate we prove (ii). For any H,

$$(a_1 + b_1) + \cdots + (a_H + b_H) = (a_1 + \cdots + a_H) + (b_1 + \cdots + b_H).$$

Taking standard parts we get the Sum Rule.

EXAMPLE 1 For any constant b, and any $|c| < 1$,

$$b + bc + bc^2 + \cdots + bc^n + \cdots = b(1 + c + c^2 + \cdots + c^n + \cdots) = \frac{b}{1-c}.$$

The next theorem corresponds to the Addition Property for integrals,

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

DEFINITION

The series $$\sum_{n=m+1}^{\infty} a_n = a_{m+1} + a_{m+2} + \cdots + a_{m+n} + \cdots$$

is defined as $$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \cdots + b_n + \cdots$$

where $b_n = a_{m+n}$. *This series is called a* ***tail*** *of the original series* $\sum_{n=1}^{\infty} a_n$.

THEOREM 2

A series $\sum_{n=1}^{\infty} a_n$ *converges if and only if its tail* $\sum_{n=m+1}^{\infty} a_n$ *converges for any* m. *The sum of a convergent series is equal to the* m*th partial sum plus the remaining tail,*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{m} a_n + \sum_{n=m+1}^{\infty} a_n,$$

or

$$a_1 + \cdots + a_n + \cdots = (a_1 + \cdots + a_m) + (a_{m+1} + \cdots + a_{m+n} + \cdots).$$

PROOF First assume the tail converges. For any infinite H, we have

$$a_1 + \cdots + a_H = (a_1 + \cdots + a_m) + (a_{m+1} + \cdots + a_H),$$

or

$$\sum_{n=1}^{H} a_n = \sum_{n=1}^{m} a_n + \sum_{n=m+1}^{H} a_n.$$

Taking standard parts,

$$st\left(\sum_{n=1}^{H} a_n\right) = \sum_{n=1}^{m} a_n + \sum_{n=m+1}^{\infty} a_n.$$

Therefore the series converges and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{m} a_n + \sum_{n=m+1}^{\infty} a_n.$$

If we assume the series converges we can prove the tail converges in a similar way.

EXAMPLE 2 The series $\quad \dfrac{1}{5^3} + \dfrac{1}{5^4} + \dfrac{1}{5^5} + \cdots = \sum_{n=3}^{\infty} \left(\dfrac{1}{5}\right)^n$

is a tail of the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n.$$

Its sum can be found in two ways.

(a) $\displaystyle \sum_{n=3}^{\infty} \left(\frac{1}{5}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n - \sum_{n=0}^{2} \left(\frac{1}{5}\right)^n = \frac{1}{1 - \frac{1}{5}} - \left(1 + \frac{1}{5} + \frac{1}{25}\right) = \frac{1}{100}.$

(b) $\displaystyle \sum_{n=3}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{5^3} \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{125} \cdot \frac{1}{1 - \frac{1}{5}} = \frac{1}{125} \cdot \frac{5}{4} = \frac{1}{100}.$

COROLLARY 1

If $\sum_{n=1}^{\infty} a_n$ converges, then the tails $\sum_{n=m}^{\infty} a_n$ approach zero as m approaches ∞,

$$\lim_{m \to \infty} \left(\sum_{n=m}^{\infty} a_n\right) = 0.$$

PROOF If H is infinite, then

$$\sum_{n=1}^{H} a_n \approx \sum_{n=1}^{\infty} a_n,$$

so

$$\sum_{n=H+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{H} a_n \approx 0.$$

COROLLARY 2

If a series $\sum_{n=1}^{\infty} a_n$ converges, then it remains convergent if finitely many terms are added, deleted, or changed.

PROOF If a_m is the last term changed, then the tail

$$\sum_{n=m+1}^{\infty} a_n$$

is left unchanged, so it still converges.

Warning: Although the convergence properties of a series are not affected by changing finitely many terms, the value of the sum, if finite, is affected.

EXAMPLE 3 Here is a convergent geometric series.

$$\frac{1}{5^0} + \frac{1}{5^1} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} + \cdots = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4} = 1.25.$$

The following series still converges by Corollary 2. Find its sum.

$$3 - 8 + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} + \cdots.$$

We have

$$3 - 8 + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} + \cdots = 3 - 8 + \frac{1}{5^3}\left(\frac{1}{5^0} + \frac{1}{5^1} + \frac{1}{5^2} + \cdots\right)$$

$$= (3 - 8) + \frac{1}{5^3} \cdot \frac{5}{4} = -5 + \frac{1}{100}$$

$$= -4.99.$$

PROBLEMS FOR SECTION 9.3

Find the sum of the following series.

1 $\frac{1}{7^2} + \frac{1}{7^3} + \cdots + \frac{1}{7^{n+2}} + \cdots$

2 $\frac{2}{1} + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots + \frac{2^{n+1}}{3^n} + \cdots$

3 $(1 + 1) + (\frac{1}{3} + \frac{1}{5}) + (\frac{1}{9} + \frac{1}{25}) + \cdots + (3^{-n} + 5^{-n}) + \cdots$

4 $\sum_{n=0}^{\infty} \left(-\frac{2}{7}\right)^n$

5 $\sum_{n=3}^{\infty} 5 \cdot 4^{-n}$

6 $1 + \frac{1}{5} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^9} + \cdots + \frac{1}{7^n} + \cdots$

7 $6^2 + 6 + 1 + 6^{-1} + 6^{-2} + \cdots + 6^{-n} + \cdots$

8 $\sum_{n=0}^{\infty} \frac{3^n + 4^n}{5^n}$

9 $8.88888\ldots = 8 + 8 \cdot 10^{-1} + 8 \cdot 10^{-2} + \cdots + 8 \cdot 10^{-n} + \cdots$

10 $2.36666\ldots = 2.3 + 6 \cdot 10^{-2} + 6 \cdot 10^{-3} + 6 \cdot 10^{-4} + \ldots$

11 $5.434343\ldots = 5 + 43 \cdot 100^{-1} + 43 \cdot 100^{-2} + 43 \cdot 100^{-3} + \cdots$

12 $0.286286286\ldots$

13 $492.315041041041041\ldots$

14 Prove the Constant Rule $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.

15 Prove that the repeating decimal 0.142857142857142857... is a rational number.

9.4 SERIES WITH POSITIVE TERMS

By a *positive term series*, we mean a series in which every term is greater than zero. For example, the geometric series

$$1 + c + c^2 + \cdots + c^n + \cdots$$

is a positive term series if $c > 0$ but not if $c \leq 0$. We call a sequence $S_1, S_2, \ldots, S_n, \ldots$ *increasing* if $S_m < S_n$ whenever $m < n$. It is easy to see that

$$a_1 + a_2 + \cdots + a_n + \cdots$$

is a positive term series if and only if its partial sum sequence is increasing. We are going to give several tests for the convergence of a positive term series. The starting point is the following theorem.

THEOREM 1

An increasing sequence $\langle S_n \rangle$ either converges or diverges to ∞.

Geometrically, this says that, as n gets large, the graph of the sequence either levels out at a limit L or the value of S_n gets large (Figure 9.4.1). We omit the proof. (The proof is given in the Epilogue at the end of the book.)

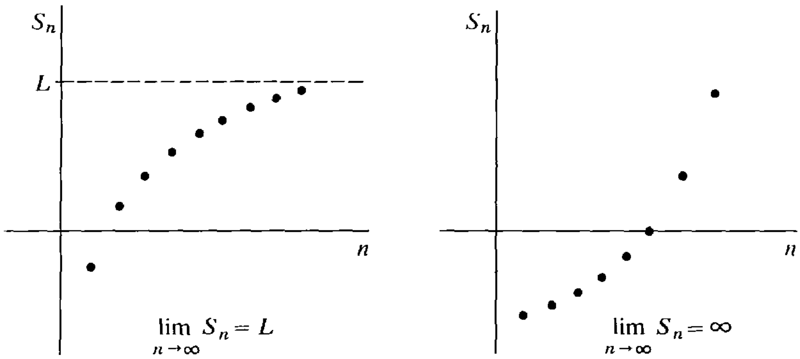

Figure 9.4.1

Theorem 1 has an equivalent form for positive term series because the partial sum sequence of a positive term series is increasing.

THEOREM 1 (Second Form)

A positive term series either converges or diverges to ∞.

EXAMPLE 1 The harmonic series diverges to ∞,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \infty.$$

This is because it is a positive term series and we have shown that it diverges.

EXAMPLE 2 If $0 < a$ the geometric series

$$1 + a + a^2 + \cdots + a^n + \cdots$$

is a positive term series. It converges when $a < 1$ and diverges to ∞ when $a \geq 1$.

Remark Theorem 1 shows that to determine whether a positive term series converges, we need only look at one infinite partial sum. If it is finite the series converges and if it is infinite the series diverges to ∞.

COMPARISON TEST

Let c be a positive constant. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive term series and $a_n \leq cb_n$ for all n.

(i) *If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.*

(ii) *If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.*

PROOF (i) Suppose $\sum_{n=1}^{\infty} b_n$ converges to S. The Constant Rule gives $cS = \sum_{n=1}^{\infty} cb_n$. Each finite partial sum of $\sum_{n=1}^{\infty} a_n$ is less than cS,

$$\sum_{n=1}^{m} a_n \leq \sum_{n=1}^{m} c\, b_n < cS.$$

Therefore, an infinite partial sum $\sum_{n=1}^{H} a_n$ is less than cS and hence finite. It follows that $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ cannot converge by part (i).

To use the Comparison Test we compare a series whose convergence or divergence is unknown with one which is known.

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} 6^n/(7^n - 5^n)$ for convergence. Intuitively, the 7^n should overcome the -5^n, so we shall compare with $6^n/7^n$. The simplest approach is to factor out 7^n. We have

$$\frac{6^n}{7^n - 5^n} = \frac{6^n}{7^n(1 - (5/7)^n)} \leq \frac{6^n}{7^n(2/7)} = \frac{7}{2}\left(\frac{6}{7}\right)^n.$$

The geometric series $\sum_{n=1}^{\infty} (6/7)^n$ is convergent, so the given series converges.

EXAMPLE 4 Test for convergence: $\sum_{n=1}^{\infty} n^2/(n^3 + 1)$. We have $n^3 + 1 \leq 2n^3$, so

$$\frac{n^2}{n^3 + 1} \geq \frac{n^2}{2n^3} = \frac{1}{2} \cdot \frac{1}{n}.$$

The harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges, whence the given series diverges.

Sometimes the following comparison test is easier to use.

LIMIT COMPARISON TEST

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive term series and c a positive real number. Suppose that

$$a_K \le cb_K \qquad \textit{for all infinite } K.$$

Then:

(i) *If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.*
(ii) *If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.*

PROOF Assume $\sum_{n=1}^{\infty} b_n$ converges. Let H and K be infinite. By the Cauchy Convergence Test (Section 9.2),

$$b_{H+1} + b_{H+2} + \cdots + b_K \approx 0.$$

Hence
$$0 \le a_{H+1} + \cdots + a_K \le cb_{H+1} + \cdots + cb_K = c(b_{H+1} + \cdots + b_K) \approx 0.$$

It follows that
$$a_{H+1} + \cdots + a_K \approx 0$$

and $\sum_{n=1}^{\infty} a_n$ converges.

EXAMPLE 5 Test $\sum_{n=2}^{\infty} \dfrac{1}{(\ln n)^p}$ where p is a positive constant.

We compare this series with the divergent series

$$\sum_{n=2}^{\infty} \frac{1}{n}.$$

Let H be positive infinite. Then by Theorem 1 in Section 9.1,

$$\ln H < H^{1/p},$$
$$(\ln H)^p < H,$$
$$\frac{1}{(\ln H)^p} > \frac{1}{H}.$$

By the Limit Comparison Test, the given series $\sum_{n=2}^{\infty} 1/(\ln n)^p$ diverges.

For our last test we need another theorem which is similar to Theorem 1.

THEOREM 2

If the function $F(x)$ increases for $x \ge 1$, then $\lim_{x\to\infty} F(x)$ either exists or is infinite.

This says that the curve $y = F(x)$ is either asymptotic to some horizontal line $y = L$ or increases indefinitely, as illustrated in Figure 9.4.2.

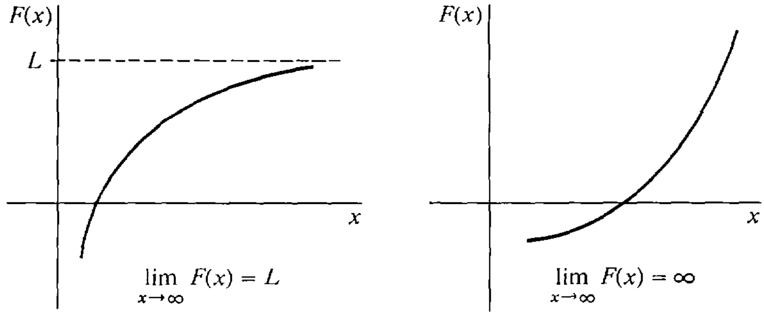

Figure 9.4.2

INTEGRAL TEST

Suppose f is a continuous decreasing function and $f(x) > 0$ for all $x \geq 1$. Then the improper integral

$$\int_1^\infty f(x)\,dx$$

and the infinite series

$$\sum_{n=1}^{\infty} f(n)$$

either both converge or both diverge to ∞.

Discussion Figure 9.4.3 suggests that

$$\sum_{n=2}^{\infty} f(n) < \int_1^\infty f(x)\,dx < \sum_{n=1}^{\infty} f(n)$$

so the series and the integral should both converge or both diverge to ∞. The Integral Test shows that the integral $\int_1^\infty f(x)\,dx$ and the series $\sum_{n=1}^\infty f(n)$ have the same convergence properties. However, their values, when finite, are different. In fact, we can see from Figure 9.4.3(c) that the integral is less than the series sum,

$$\int_1^\infty f(x)\,dx < \sum_{n=1}^{\infty} f(n).$$

PROOF As we can see from Figure 9.4.3, for each m we have

$$\sum_{n=2}^{m} f(n) \leq \int_1^m f(x)\,dx \leq \sum_{n=1}^{m-1} f(n).$$

The improper integral is defined by

$$\int_1^\infty f(x)\,dx = \lim_{u \to \infty} \int_1^u f(x)\,dx.$$

Since $f(x)$ is always positive, the function $F(u) = \int_1^u f(x)\,dx$ is increasing, so by Theorem 2, the limit either exists or is infinite. Hence the improper integral either converges or diverges to ∞.

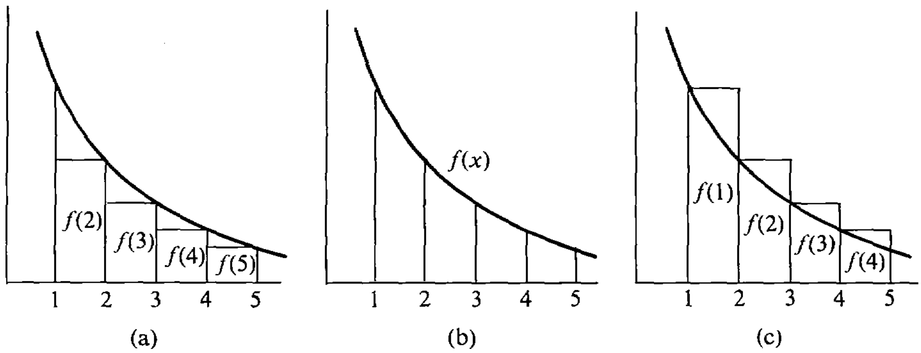

Figure 9.4.3 The Integral Test

Case 1 $\int_1^\infty f(x)\,dx = S$ converges. For infinite H we have

$$\sum_{n=2}^{H} f(n) \leq \int_1^H f(x)\,dx \approx S;$$

thus the infinite partial sum is finite. Hence the tail $\sum_{n=2}^{\infty} f(n)$ and the series $\sum_{n=1}^{\infty} f(n)$ converge.

Case 2 $\int_1^\infty f(x)\,dx$ diverges to ∞. Since $\int_1^H f(x)\,dx \leq \sum_{n=1}^{H-1} f(n)$, the infinite partial sum has infinite value, whence the series $\sum_{n=1}^{\infty} f(n)$ diverges to ∞.

The series $\sum_{n=1}^{\infty} 1/n^p$, where p is constant, is called the *p series.*

COROLLARY

The p series $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.

PROOF

Case 1 $p = 1$. The p series is just $\sum_{n=1}^{\infty} 1/n = \infty$.

Case 2 $p > 1$. The improper integral converges,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{u\to\infty} \int_1^u x^{-p}\,dx$$
$$= \lim_{u\to\infty} \frac{u^{1-p} - 1}{1 - p} = -\frac{1}{1-p}.$$

Therefore the p series converges.

Case 3 $p < 1$. The improper integral diverges to ∞, $\int_1^\infty (1/x^p)\,dx = \lim_{u\to\infty} \int_1^u x^{-p}\,dx = \lim_{u\to\infty} (u^{1-p} - 1)/(1 - p) = \infty$.
Therefore the p series diverges to ∞.

EXAMPLE 6 The p series

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$$

converges because $4/3 > 1$. The p series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges to ∞ because $1/2 < 1$.

The p series is often used in the Comparison Tests.

EXAMPLE 7 Test the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

for convergence.

If H is positive infinite then by Theorem 1 in Section 9.1,

$$\ln H < H^c,$$

$$\frac{\ln H}{H^2} < \frac{H^c}{H^2} = \frac{1}{H^{2-c}}, \qquad \text{for real } c > 0.$$

Now take c so that $0 < c < 1$. Then $2 - c > 1$ so the p series $\sum_{n=1}^{\infty} 1/n^{2-c}$ converges. By the Limit Comparison Test, the given series $\sum_{n=1}^{\infty} (\ln n)/n^2$ converges.

EXAMPLE 8 Use the Integral Test to test the improper integral $\int_3^{\infty} ((\ln x)/x^2)\,dx$ for convergence.

By Example 7 the series $\sum_{n=3}^{\infty} (\ln n)/n^2$ converges. For $x > 1$ the function $f(x) = (\ln x)/x^2$ is continuous, positive, and has derivative

$$f'(x) = x^{-3}(1 - 2\ln x).$$

Thus for $x > \sqrt{e}$, $f'(x) < 0$ and $f(x)$ is decreasing. Therefore the Integral Test applies and the improper integral converges.

PROBLEMS FOR SECTION 9.4

Test the following series for convergence.

1 $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+4}$

2 $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4n-3}$

3 $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3}$

4 $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2+2}$

5 $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$

6 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$

7 $\displaystyle\sum_{n=3}^{\infty} \frac{n+3}{n(n+1)(n-2)}$

8 $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n+1)(n^2+1)}$

9 $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n+1)(n+2)}$

10 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

11 $\displaystyle\sum_{n=0}^{\infty} \sqrt{n+1} - \sqrt{n}$

12 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+2} - \sqrt{n}}{n}$

13 $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+1}{2n^4-1}$

14 $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^3+4}}$

15 $\sum_{n=0}^{\infty} \frac{n}{\sqrt{n^3+1}}$

16 $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{3n+2}$

17 $\sum_{n=1}^{\infty} n^{-n}$

18 $\sum_{n=1}^{\infty} \frac{1}{2^n - n}$

19 $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

20 $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$

21 $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n}$

22 $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$

23 $\sum_{n=0}^{\infty} \frac{1}{n!}$

24 $\sum_{n=0}^{\infty} \frac{5^n}{3^n + 4^n}$

25 $\sum_{n=0}^{\infty} \frac{5^n + 6^n}{2^n + 7^n}$

26 $\sum_{n=1}^{\infty} \frac{1}{2 + \ln n}$

27 $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

28 $\sum_{n=1}^{\infty} \frac{\ln n}{n\sqrt{n}}$

29 $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

30 $\sum_{n=1}^{\infty} \frac{1}{\ln(n^2+1)}$

31 $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

32 $\sum_{n=0}^{\infty} \left(\frac{\pi}{2} - \arctan n\right)$

33 $\sum_{n=2}^{\infty} \frac{1}{\ln(n!)}$

Use the Integral Test to determine whether the following improper integrals converge or diverge.

34 $\int_2^{\infty} \frac{dx}{\ln x}$

35 $\int_2^{\infty} \frac{dx}{x^2 + \ln x}$

36 $\int_2^{\infty} \frac{1}{x + \ln x}\,dx$

37 $\int_1^{\infty} \frac{x+1}{x^3 + x^2 + 1}\,dx$

38 $\int_3^{\infty} \frac{\ln x\,dx}{x\sqrt{x}}$

39 $\int_0^{\infty} e^{-x^2}\,dx$

40 $\int_1^{\infty} x^{-x}\,dx$

□ 41 Prove that if each a_n is positive and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

□ 42 Using Theorem 1 (page 539), prove that a negative term series either converges or diverges to $-\infty$.

9.5 ALTERNATING SERIES

An *alternating series* is a series in which the odd numbered terms are positive and the even numbered terms are negative, or vice versa. An example is the geometric series

$$\sum_{n=1}^{\infty} a^n, \qquad a < 0.$$

Given any positive term series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots,$$

the series $$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

and $$\sum_{n=1}^{\infty} (-1)^{n} a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$

are alternating series. Here is a test for convergence of alternating series.

ALTERNATING SERIES TEST

Assume that

(i) $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *is an alternating series.*

(ii) *The terms* a_n *are decreasing,* $a_1 > a_2 > \cdots > a_n > \cdots$.

(iii) *The terms approach zero,* $\lim_{n\to\infty} a_n = 0$.

Then the series converges to a sum $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = S$. *Moreover, the sum* S *is between any two consecutive partial sums,*

$$S_{2n} < S < S_{2n+1}.$$

Discussion We see from the graph in Figure 9.5.1 that the partial sums S_n alternately increase and decrease, but the change is less each time. The value of S_n "vibrates" back and forth and the vibration damps down around the limit S.

PROOF The sequence of even partial sums is increasing.

$$S_2 < S_4 < \cdots < S_{2n} < \cdots,$$

because $S_4 = S_2 + (a_3 - a_4)$, $S_6 = S_4 + (a_5 - a_6)$, etc.

The sequence of odd partial sums is decreasing,

$$S_1 > S_3 > S_5 > \cdots,$$

for $S_3 = S_1 - (a_2 - a_3)$, $S_5 = S_3 - (a_4 - a_5)$, etc.

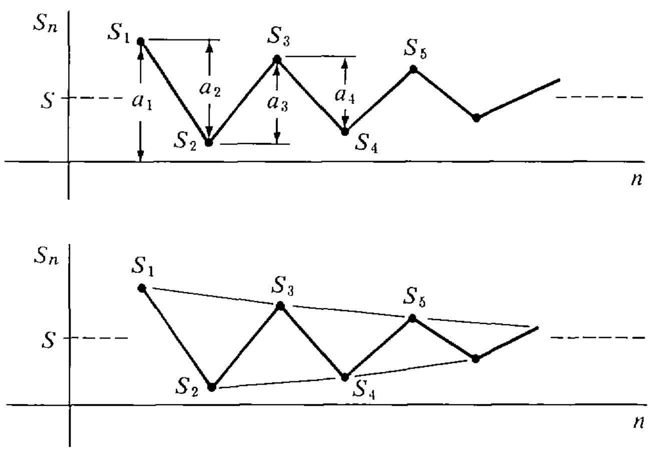

Figure 9.5.1

It follows that each even partial sum is less than S_1,

$$S_1 > S_1 - a_2 = S_2, \qquad S_1 > S_3 - a_4 = S_4, \qquad S_1 > S_5 - a_6 = S_6, \quad \text{etc.}$$

Theorem 1 (Section 9.4) shows that the increasing sequence of even partial sums converges,

$$\lim_{n \to \infty} S_{2n} = S.$$

Given any infinite H, $a_{2H+1} \approx 0$ and $S_{2H} \approx S$, so

$$S_{2H+1} = S_{2H} + a_{2H+1} \approx S.$$

Therefore the sequence of all partial sums converges to S, and

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \lim_{n \to \infty} S_n = S.$$

Finally, since the even partial sums are increasing and the odd partial sums are decreasing, we have the estimate

$$S_{2n} < S < S_{2n+1}.$$

Figure 9.5.2 shows a graph of the partial sums.

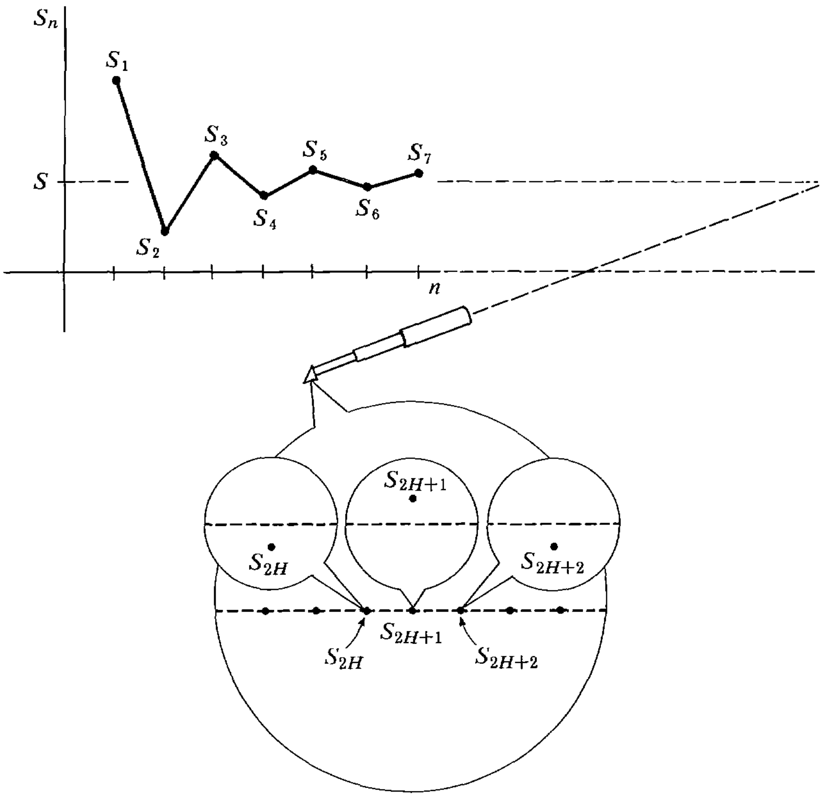

Figure 9.5.2

EXAMPLE 1 The *alternating harmonic series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

converges by the Alternating Series Test, because $\frac{1}{n}$ is decreasing and approaches zero as $n \to \infty$. The partial sums are

$$1, \tfrac{1}{2}, \tfrac{5}{6}, \tfrac{7}{12}, \tfrac{47}{60}, \tfrac{37}{60}, \ldots$$

or

$$\tfrac{60}{60}, \tfrac{30}{60}, \tfrac{50}{60}, \tfrac{35}{60}, \tfrac{47}{60}, \tfrac{37}{60}, \ldots.$$

The sum S is between any two consecutive partial sums, for example

$$\tfrac{37}{60} < S < \tfrac{47}{60}.$$

EXAMPLE 2 The alternating series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \cdots + (-1)^{n+1}\frac{n+1}{n} + \cdots$$

diverges. The terms $(n + 1)/n$ are decreasing, but their limit is one instead of zero,

$$\lim_{n \to \infty} \frac{n+1}{n} = 1.$$

The Cauchy Test for Divergence in Section 9.2 shows that if the terms a_n do not converge to zero the series diverges.

We have now built up quite a long list of convergence tests. The next section contains one more important test, the Ratio Test. At the end of that section is a summary of all the convergence tests with hints on when to use them.

PROBLEMS FOR SECTION 9.5

Test the following alternating series for convergence.

1 $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$

2 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

3 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{10n+5}$

4 $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

5 $\sum_{n=1}^{\infty} (-1)^n n^{-2}$

6 $\sum_{n=1}^{\infty} (-1)^n n^{-1/3}$

7 $\sum_{n=1}^{\infty} (-1)^n \sqrt[n]{\tfrac{1}{2}}$

8 $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

9 $\sum_{n=2}^{\infty} \frac{n(-1)^{n+1}}{\ln n}$

10 $\sum_{n=1}^{\infty} (-1)^n$

11 $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{2^n}$

12 $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^n}{n!}$

13 $\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(\ln n)}$

14 $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$

15 $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2}$

16 $\sum_{n=1}^{\infty} (-1)^{n+1}\left(1 - \frac{1}{n}\right)$

17 $\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1} - \sqrt{n})$

18 $\sum_{n=0}^{\infty} (-1)^n \frac{2^n + 1}{3^n - 2}$

19 $\sum_{n=0}^{\infty} (-1)^n \frac{2^{n-2} + 1}{2^{n+3} + 5}$

20 $\sum_{n=1}^{\infty} (-1)^{n+1}\left(1 + \frac{1}{n}\right)^{-n}$

21 Approximate the series $\sum_{n=1}^{\infty} (-1)^{n+1}n^{-3}$ to two decimal places.

22 Approximate the series $1 - \frac{2}{10} + \frac{3}{100} - \frac{4}{1000} + \cdots$ to four decimal places.

23 Approximate $\sum_{n=0}^{\infty} (-1)^n/n!$ to two decimal places.

24 Approximate $\sum_{n=1}^{\infty} (-n)^{-n}$ to three decimal places.

9.6 ABSOLUTE AND CONDITIONAL CONVERGENCE

Consider a series $\sum_{n=1}^{\infty} a_n$ which has both positive and negative terms. We may form a new series $\sum_{n=1}^{\infty} |a_n|$ whose terms are the absolute values of the terms of the given series. If all the terms a_n are nonzero, then $|a_n| > 0$ so $\sum_{n=1}^{\infty} |a_n|$ is a positive term series.

If $\sum_{n=1}^{\infty} a_n$ is already a positive term series, then $|a_n| = a_n$ and the series is identical to its absolute value series $\sum_{n=1}^{\infty} |a_n|$.

Sometimes it is simpler to study the convergence of the absolute value series $\sum_{n=1}^{\infty} |a_n|$ than of the given series $\sum_{n=1}^{\infty} a_n$. This is because we have at our disposal all the convergence tests for positive term series from the preceding sections.

DEFINITION

*A series $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent** if its absolute value series $\sum_{n=1}^{\infty} |a_n|$ is convergent. A series which is convergent but not absolutely convergent is called **conditionally convergent**.*

THEOREM 1

Every absolutely convergent series is convergent. That is, if the absolute value series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Discussion This theorem shows that if a positive term series $\sum_{n=1}^{\infty} b_n$ is convergent, then it remains convergent if we make some or all of the terms b_n negative, because the new series will still be absolutely convergent.

Given an arbitrary series $\sum_{n=1}^{\infty} a_n$, the theorem shows that exactly one of the following three things can happen:

The series is absolutely convergent.
The series is conditionally convergent.
The series is divergent.

PROOF OF THEOREM 1 We use the Sum Rule. Assume $\sum_{n=1}^{\infty} |a_n|$ converges and let

$$b_n = a_n + |a_n|.$$

Then $a_n = b_n - |a_n|$ and

$$b_n = \begin{cases} 2|a_n| & \text{if } a_n > 0, \\ 0 & \text{if } a_n < 0. \end{cases}$$

(See Figure 9.6.1). Both $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} b_n$ have nonnegative terms. Moreover, $\sum_{n=1}^{\infty} |a_n|$ converges and $b_n \leq 2|a_n|$. By the Comparison Test, $\sum_{n=1}^{\infty} b_n$ converges. Then using the Sum and Constant Rules,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} |a_n|$$

converges.

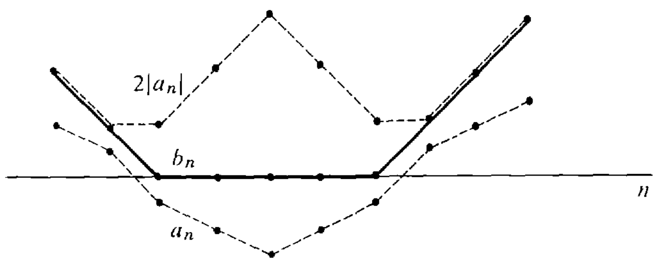

Figure 9.6.1

EXAMPLE 1 The alternating series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots,$$

is absolutely convergent, because its absolute value series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is convergent.

EXAMPLE 2 The alternating harmonic series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots$$

is conditionally convergent. It converges by the Alternating Series Test. But its absolute value series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots$$

diverges.

Given a series

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_k + \cdots,$$

one can form a new series by listing the terms in a different order, for example

$$a_1 + a_3 + a_2 + a_5 + a_4 + \cdots.$$

Such a series is called a *rearrangement* of $\sum_{n=1}^{\infty} a_n$. The difference between absolute convergence and conditional convergence is shown emphatically by the following pair of theorems.

THEOREM 2

A. *Every rearrangement of an absolutely convergent series is also convergent and has the same sum.*

B. *Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series.*
   (i) *The series has a rearrangement which diverges to ∞.*
   (ii) *The series has another rearrangement which diverges to $-\infty$.*
   (iii) *For each real number r, the series has a rearrangement which converges to r.*

We shall not prove these theorems. Instead we give a pair of rearrangements of the conditionally convergent series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots,$$

one diverging to ∞ and the other converging to -1.

The alternating series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

conditionally converges to a number between $\tfrac{1}{2}$ and 1.

To get a rearrangement which diverges to ∞, we write down terms in the following order:

1st positive term,	1st negative term,
next 2 positive terms,	2nd negative term,
next 4 positive terms,	3rd negative term,
⋮	
next 2^m positive terms,	mth negative term,
⋮	

We thus obtain the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{4} + \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} + \tfrac{1}{13} - \tfrac{1}{6} + \cdots.$$

Each block of 2^m positive terms adds up to at least $\tfrac{1}{4}$,

$$1 \geq \tfrac{1}{4},$$
$$\tfrac{1}{3} + \tfrac{1}{5} \geq 2 \times \tfrac{1}{8} = \tfrac{1}{4},$$
$$\tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} + \tfrac{1}{13} \geq 4 \times \tfrac{1}{16} = \tfrac{1}{4},$$
$$\tfrac{1}{15} + \cdots + \tfrac{1}{29} \geq 8 \times \tfrac{1}{32} = \tfrac{1}{4}.$$

However, all the negative terms except $-\tfrac{1}{2}$ and $-\tfrac{1}{4}$ have absolute value $\leq \tfrac{1}{6}$. Hence after the mth negative term the partial sum is more than

$$\frac{m}{4} - \frac{m}{6} - \frac{1}{2} - \frac{1}{4} = \frac{m}{12} - \frac{3}{4}.$$

Therefore the partial sums, and hence the series, diverge to ∞.

To get a rearrangement which converges conditionally to -1 we proceed as follows:

Write down negative terms until the partial sum is below -1, then positive terms until the partial sum is above -1, then negative terms until the partial sum is below -1, and so on.

The mth time the partial sum goes above -1, it must be between -1 and $-1 + (1/m)$. The mth time it goes below -1 it must be between -1 and $-1-(1/m)$. Therefore the series converges to -1.

The comparison tests for positive term series give us tests for absolute convergence.

COMPARISON TEST

If $|a_n| \le c|b_n|$ and $\sum_{n=1}^{\infty} b_n$ is absolutely convergent then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

LIMIT COMPARISON TEST

Let c be a positive real number. If

$$|a_K| \le c|b_K| \qquad \textit{for all infinite } K$$

and $\sum_{n=1}^{\infty} b_n$ is absolutely convergent then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

The above tests do not help to distinguish between conditional convergence and divergence. Theorem 2 in Section 9.2 is often useful as a test for divergence.

There is another test which can be used either to show that a series is absolutely convergent or that a series is divergent.

RATIO TEST

Suppose the limit of the ratio $|a_{n+1}|/|a_n|$ exists or is ∞,

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L.$$

(i) *If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

(ii) *If $L > 1$, or $L = \infty$, the series diverges.*

(iii) *If $L = 1$, the test gives no information and the series may converge absolutely, converge conditionally, or diverge.*

PROOF (i) Choose b with $L < b < 1$. By the ε, N condition, there is an N such that all the ratios

$$\frac{|a_{N+1}|}{|a_N|}, \frac{|a_{N+2}|}{|a_{N+1}|}, \dots, \frac{|a_{N+k+1}|}{|a_{N+k}|}, \dots$$

are less than b. Therefore with $c = |a_N|$,

$$|a_{N+1}| < cb, \qquad |a_{N+2}| < cb^2, \ldots, |a_{N+n}| < cb^n, \ldots.$$

The geometric series $\sum_{n=1}^{\infty} b^n$ converges, so by the Comparison Test, the tail $\sum_{n=N}^{\infty} |a_n|$ converges. Therefore the absolute value series $\sum_{n=1}^{\infty} |a_n|$ converges.

(ii) By the ε, N condition there is an N such that the ratios

$$\frac{|a_{N+1}|}{|a_N|}, \ldots, \frac{|a_{N+n+1}|}{|a_{N+n}|}, \ldots$$

are all greater than one. Therefore

$$|a_N| < |a_{N+1}| < \cdots < |a_{N+n}| < \cdots.$$

It follows that the terms a_n do not converge to zero, so the series $\sum_{n=1}^{\infty} a_n$ diverges.

The Ratio Test is useful even for positive term series, and is often effective for series involving $n!$ and a^n.

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} \frac{1}{n!}$.

$$\lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

so by the Ratio Test the series converges.

EXAMPLE 4 Test $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$.

$$\lim_{n\to\infty} \left(\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}\right) = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

e is greater than one, so by the Ratio Test the series diverges.

EXAMPLE 5 The Ratio Test does not apply to either of the series

$$\sum_{n=1}^{\infty} \frac{1}{n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2},$$

since $$\lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1, \qquad \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = 1.$$

SUMMARY OF SERIES CONVERGENCE TESTS

A. *Particular Series*

(1) *Geometric Series*

$\sum_{n=0}^{\infty} c^n$ converges to $\frac{1}{1-c}$ if $|c| < 1$,

diverges if $|c| \geq 1$.

(2) *Harmonic Series*

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(3) *p Series*

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$,

diverges if $p \leq 1$.

B. *Tests for Positive and Alternating Series*

In the tests below, assume $a_n \geq 0$ for all n.

(1) *Convergence versus Divergence to* ∞

Let H be infinite.

$\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{H} a_n$ is finite,

diverges to ∞ if $\sum_{n=1}^{H} a_n$ is infinite.

(2) *Comparison Test*

Suppose $a_n \leq cb_n$ for all n.

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

Hint: Often a series can be compared with one of the particular series above: a geometric, harmonic, or p series.

(3) *Limit Comparison Test*

Suppose $a_K \leq cb_K$ for all infinite K.

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

Hint: Try this test if the Comparison Test almost works.

(4) *Integral Test*

Suppose f is continuous, decreasing, and positive for $x \geq 1$.

If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} f(n)$ converges.

If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} f(n)$ diverges.

Hint: This test may be useful if a_n comes from a continuous function $f(x)$.

(5) *Alternating Series Test*

$\sum_{n=1}^{\infty} (-1)^n a_n$ converges if the a_n are decreasing and approach 0.

Hint: This is usually the simplest test if you see a $(-1)^n$ in the expression.

C. *Tests for General Series*

(1) *Definition of Convergence*

$\sum_{n=1}^{\infty} a_n$ converges if and only if the partial sum series $\sum_{n=1}^{k} a_n = S_k$ converges.

(2) *Cauchy Convergence Test*

$\sum_{n=1}^{\infty} a_n$ converges if for all infinite H and $K > H$,

$$a_{H+1} + \cdots + a_K \approx 0,$$

diverges if for some infinite H and $K > H$,

$$a_{H+1} + \cdots + a_K \not\approx 0,$$

diverges if $\lim_{n\to\infty} a_n \neq 0$.

Hint: This test is useful for showing a series diverges.

(3) *Constant and Sum Rules*

Sums and constant multiples of convergent series converge.

(4) *Tail Rule*

$\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=m}^{\infty} a_n$ converges.

(5) *Absolute Convergence*

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Hint: Remember that $\sum_{n=1}^{\infty} |a_n|$ is a positive term series. Thus tests in group B may be applied to $\sum_{n=1}^{\infty} |a_n|$.

(6) *Ratio Test*

Suppose $\lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = L$.

$\sum_{n=1}^{\infty} a_n$ converges absolutely if $L < 1$,

diverges if $L > 1$.

Hint: This is useful if a_n involves a factorial. Watch for $\left(\dfrac{n+1}{n}\right)^n$ in the ratio because $\lim_{n\to\infty} \left(\dfrac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$.

If the limit L is one, try another test because the Ratio Test gives no information.

PROBLEMS FOR SECTION 9.6

Problems 1–20: For each of the first 20 problems in Section 9.5, determine whether the alternating series is absolutely convergent, conditionally convergent, or divergent.

Problems 21–44: Apply the Ratio Test to the given series. Possible answers are "convergent," "divergent," or "Ratio Test gives no information."

21 $\sum_{n=1}^{\infty} 3^n$

22 $\sum_{n=1}^{\infty} \dfrac{1}{2^n}$

23 $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$

24 $\sum_{n=2}^{\infty} n^2$

25 $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$

26 $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$

27 $\sum_{n=1}^{\infty} \dfrac{5^n}{3^n + 4^n}$

28 $\sum_{n=1}^{\infty} \dfrac{5^n}{6^n - 5^n}$

29 $\sum_{n=1}^{\infty} \dfrac{n!}{n^n}$

30 $\sum_{n=1}^{\infty} \dfrac{n^n}{(2n)!}$

31 $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

32 $\sum_{n=1}^{\infty} \frac{3^n(n!)}{n^n}$

33 $\sum_{n=1}^{\infty} \frac{e^n(n!)}{n^n}$

34 $\sum_{n=1}^{\infty} \frac{4^n(n!)^2}{(2n)!}$

35 $\sum_{n=1}^{\infty} \frac{3^n(n!)^2}{(2n)!}$

36 $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

37 $\sum_{n=2}^{\infty} \frac{10^n}{(\ln n)^n}$

38 $\sum_{n=3}^{\infty} \frac{1}{(\ln(\ln n))^n}$

39 $\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

40 $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(n!)^2}$

41 $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$

42 $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)}$

43 $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

44 $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$

9.7 POWER SERIES

So far we have studied series of constants,

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_n + \cdots.$$

One can also form a *series of functions*

$$\sum_{n=0}^{\infty} f_n(x) = f_0(x) + f_1(x) + \cdots + f_n(x) + \cdots.$$

Such a series will converge for some values of x and diverge for others. The *sum* of the series is a new function

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

which is defined at each point x_0 where the series converges. We shall concentrate on a particular kind of series of functions called a power series. Its importance will be evident in the next section where we show that many familiar functions are sums of power series.

DEFINITION

*A **power series** in x is a series of functions of the form*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots.$$

The nth finite partial sum of a power series is just a polynomial of degree n,

$$\sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \cdots + a_n x^n.$$

The infinite partial sums are polynomials of infinite degree,

$$\sum_{n=0}^{H} a_n x^n = a_0 + a_1 x + \cdots + a_H x^H.$$

At $x = 0$ every power series converges absolutely,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 0 + a_2 0^2 + \cdots = a_0.$$

(In a power series we use the convention $a_0 x^0 = a_0$.) If a power series converges absolutely at $x = u$, it also converges absolutely at $x = -u$, because the absolute value series $\sum_{n=0}^{\infty} |a_n u^n|$ and $\sum_{n=0}^{\infty} |a_n(-u)^n|$ are the same.

Intuitively, the smaller the absolute value $|x|$, the more likely the power series is to converge at x. This intuition is borne out in the following theorem.

THEOREM 1

(i) *If a power series*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

converges when $x = u$, then it converges absolutely whenever $|x| < |u|$.

(ii) *If a power series diverges when $x = v$, then it diverges whenever $|x| > |v|$.*

PROOF (i) Suppose the series $\sum_{n=0}^{\infty} a_n u^n$ converges. Then for any positive infinite H, $a_H u^H$ is infinitesimal. Let $|v| < |u|$. The ratio $b = |v|/|u|$ is then less than one. It follows that:

(1) The positive term geometric series $\sum_{n=0}^{\infty} b^n$ converges,

(2) $|a_H v^H| = \left| a_H u^H \left(\frac{v}{u}\right)^H \right| = |a_H u^H| b^H \le b^H.$

Now by the Limit Comparison Test, $\sum_{n=0}^{\infty} a_n v^n$ converges absolutely.

(ii) This follows trivially from (i). Let $\sum_{n=0}^{\infty} a_n v^n$ diverge and $|u| > |v|$. $\sum_{n=0}^{\infty} a_n u^n$ cannot converge because if it did $\sum_{n=0}^{\infty} a_n v^n$ would converge absolutely. Therefore $\sum_{n=0}^{\infty} a_n u^n$ diverges.

Theorem 1 shows that if a power series converges at $x = u$ and at $x = v$, then it converges absolutely at every point strictly between u and v. We conclude that the set of points where the power series converges is an interval, called the *interval of convergence.* (A rigorous proof that the set is an interval is given in the Epilogue.) The next corollary summarizes what we know about the interval of convergence.

COROLLARY

For each power series $\sum_{n=0}^{\infty} a_n x^n$, one of the following happens.

(i) *The series converges absolutely at $x = 0$ and diverges everywhere else.*

(ii) *The series converges absolutely on the whole real line $(-\infty, \infty)$.*

(iii) *The series converges absolutely at every point in an open interval* $(-r, r)$ *and diverges at every point outside the closed interval* $[-r, r]$. *At the endpoints* $-r$ *and* r *the series may converge or diverge, so the interval of convergence is one of the sets*

$$(-r, r), \quad [-r, r), \quad (-r, r], \quad [-r, r].$$

Figure 9.7.1 illustrates part (iii) of the Corollary. The number r is called the *radius of convergence* of the power series. In case (i) the radius of convergence is zero, and in case (ii) it is ∞. Once the radius of convergence is determined, we need only test the series at $x = r$ and $x = -r$ to find the interval of convergence.

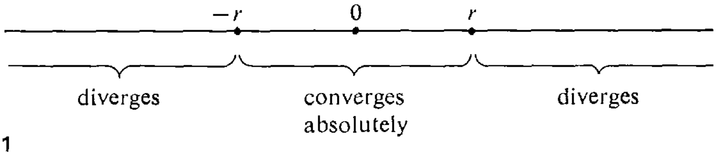

Figure 9.7.1

EXAMPLE 1 Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} b^n x^n, \qquad \text{where } b > 0.$$

This is just the geometric series

$$1 + bx + (bx)^2 + \cdots + (bx)^n + \cdots.$$

It converges absolutely when $|bx| < 1$, $|x| < 1/b$, and diverges when $|bx| > 1$, $|x| > 1/b$. So the radius of convergence is $r = 1/b$. At $x = r$ and at $x = -r$ the series diverges, because $b^n r^n = 1$. Thus the interval of convergence is $(-1/b, 1/b)$.

The Ratio Test can often be used to find the radius of convergence of a power series.

EXAMPLE 2 Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots.$$

We compute the limit

$$\lim_{n\to\infty} \frac{|x^{n+1}/(n+1)|}{|x^n/n|} = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|.$$

By the Ratio Test the series converges for $|x| < 1$ and diverges for $|x| > 1$, so the radius of convergence is $r = 1$.

At $x = 1$ the series is

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

which is divergent. At $x = -1$ the series is

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots + \frac{(-1)^n}{n} + \cdots$$

which converges by the Alternating Series Test. The interval of convergence is $[-1, 1)$.

EXAMPLE 3 Find the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots.$$

For all x we have

$$\lim_{n\to\infty} \frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = \lim_{n\to\infty} \frac{|x|}{n} = 0.$$

Therefore by the Ratio Test the series converges for all x. It has radius of convergence ∞, and interval of convergence $(-\infty, \infty)$.

EXAMPLE 4 Find the radius of convergence of

$$\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2x^2 + 6x^3 + \cdots.$$

For $x \neq 0$,
$$\lim_{n\to\infty} \frac{|(n+1)!\,x^{n+1}|}{|n!\,x^n|} = \lim_{n\to\infty} n|x| = \infty.$$

By the Ratio Test the series diverges for $x \neq 0$ and the radius of convergence is $r = 0$.

If we replace x by $x - c$ we obtain a *power series in* $x - c$,

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots.$$

The power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has the same radius of convergence as $\sum_{n=0}^{\infty} a_n x^n$, and the interval of convergence is simply moved over so that its center is c instead of 0. For example, if $\sum_{n=0}^{\infty} a_n x^n$ has interval of convergence $(-r, r]$, then

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

has interval of convergence $(c - r, c + r]$, illustrated in Figure 9.7.2.

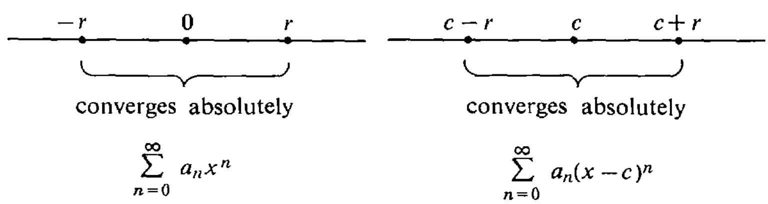

Figure 9.7.2

EXAMPLE 5 Find the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(x+5)^n = 1 + \frac{1}{2}(x+5) + \frac{(2!)^2}{4!}(x+5)^2 + \cdots.$$

We have

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,(n+1)!\,(x+5)^{n+1}/(2n+2)!}{(n!)(n!)(x+5)^n/(2n)!}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(n+1)^2(x+5)}{(2n+1)(2n+2)}\right| = \frac{|x+5|}{4}.$$

By the Ratio Test the series converges for $|x+5| < 4$ and diverges for $|x+5| > 4$. The radius of convergence is $r = 4$, and the interval of convergence is centered at -5. We note that

$$\frac{(k!)^2}{(2k)!} = \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdot\cdots\cdot\frac{n}{(2n-1)}\cdot\frac{n}{2n} > \left(\frac{1}{2}\cdot\frac{1}{2}\right)^n = \left(\frac{1}{4}\right)^n.$$

Therefore at $|x+5| = 4$,

$$\left|\frac{(n!)^2}{(2n)!}(x+5)^n\right| > \left(\frac{1}{4}\right)^n 4^n = 1.$$

Thus at $x + 5 = 4$ and $x + 5 = -4$ the terms do not approach zero and the series diverges. The interval of convergence is $(-9, -1)$.

PROBLEMS FOR SECTION 9.7

In Problems 1–25, find the radius of convergence.

1 $\sum_{n=0}^{\infty} 5x^n$

2 $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$

3 $\sum_{n=1}^{\infty} n^n x^n$

4 $\sum_{n=1}^{\infty} \sqrt[n]{n}\, x^n$

5 $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

6 $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$

7 $\sum_{n=1}^{\infty} \frac{n^{2n}}{(2n)!} x^n$

8 $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3} x^n$

9 $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x^n$

10 $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n} x^n$

11 $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

12 $\sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}$

13 $\sum_{n=2}^{\infty} \frac{n^n x^n}{(\ln n)^n}$

14 $\sum_{n=3}^{\infty} \frac{x^n}{(\ln(\ln n))^n}$

15 $\sum_{n=0}^{\infty} \frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n+1)}{n!} x^n$

16 $\sum_{n=0}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)} x^n$

17 $\sum_{n=0}^{\infty} \frac{x^n}{3^{(n^2)}}$

18 $\sum_{n=0}^{\infty} \frac{x^n}{5^{\sqrt{n}}}$

19 $\sum_{n=0}^{\infty} \frac{x^n}{5^{n\sqrt{n}}}$

20 $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^n}}$

21 $\sum_{n=1}^{\infty} \frac{n!x^n}{\sqrt{n^n}}$

22 $\sum_{n=1}^{\infty} 3^n x^{2n}$

23 $\sum_{n=1}^{\infty} \frac{x^{3n}}{5^n}$

24 $\sum_{n=1}^{\infty} \frac{x^{6n}}{n!}$

25 $\sum_{n=1}^{\infty} \frac{1}{n!} x^{(n^2)}$

In Problems 26–45, find the interval of convergence.

26 $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

27 $\sum_{n=0}^{\infty} 2x^n$

28 $\sum_{n=0}^{\infty} n3^n x^n$

29 $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

30 $\sum_{n=1}^{\infty} \frac{x^n}{6\sqrt{n}}$

31 $\sum_{n=2}^{\infty} \frac{2^n x^n}{\ln n}$

32 $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{3^n n^2}$

33 $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

34 $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\sqrt{n}}$

35 $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$

36 $\sum_{n=0}^{\infty} n!(x-3)^n$

37 $\sum_{n=0}^{\infty} \frac{(x-5)^n}{n!}$

38 $\sum_{n=0}^{\infty} \frac{(x+8)^n}{2^n}$

39 $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x^n$

40 $\sum_{n=0}^{\infty} (3^n + 4^n) x^n$

41 $\sum_{n=0}^{\infty} \frac{4^n}{3^n + 5^n} x^n$

42 $\sum_{n=0}^{\infty} 3^n x^{2n}$

43 $\sum_{n=1}^{\infty} \frac{x^{2n}}{n \cdot 5^n}$

44 $\sum_{n=1}^{\infty} \frac{e^n (x-4)^{2n}}{n^2}$

45 $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

9.8 DERIVATIVES AND INTEGRALS OF POWER SERIES

In the last section we concentrated on the problem of finding the interval of convergence of a power series. We shall now find the sums of some important power series. Our general plan will be as follows.

First, find the sums of two basic power series:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots,$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Then, starting with these basic power series, find the sums of other power series by differentiation and integration. (Based on Theorem 1.)

An especially useful property of power series is that they can be differentiated and integrated like polynomials. If we have a power series for a function $f(x)$, we can use Theorem 1 to immediately write down the power series for the derivative $f'(x)$ and integral $\int_0^x f(t)\,dt$.

THEOREM 1

Suppose $f(x)$ is the sum of a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with radius of convergence $r > 0$, and let $-r < x < r$. Then:

(i) *f has the derivative*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

(ii) *f has the integral*

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

(iii) *The power series in* (i) *and* (ii) *both have radius of convergence r.*

Discussion This theorem says that a power series can be differentiated and integrated term by term. Also, the radius of convergence remains the same. To differentiate or integrate each term of a power series we simply use the Power Rule.

$$n\text{th term of } f(x) = a_n x^n$$

$$\text{derivative} = \begin{cases} n a_n x^{n-1} & n \neq 0 \\ 0 & n = 0 \end{cases}$$

$$\text{integral} = \frac{a_n}{n+1} x^{n+1}$$

We postpone the proof of Theorem 1 until later.

EXAMPLE 1 Differentiate and integrate the power series $\sum_{n=0}^{\infty} n^2 x^n$, and find the radii of convergence.

By the Ratio Test this power series has radius of convergence $r = 1$, for

$$\lim_{n\to\infty} \frac{|(n+1)^2 x^{n+1}|}{|n^2 x^n|} = |x| \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = |x|.$$

$$\textit{Derivative:} \quad \frac{d}{dx}\left(\sum_{n=0}^{\infty} n^2 x^n\right) = \sum_{n=1}^{\infty} n^3 x^{n-1} = \sum_{m=0}^{\infty} (m+1)^3 x^m.$$

$$\textit{Integral:} \quad \int_0^x \left(\sum_{n=0}^{\infty} n^2 t^n\right) dt = \sum_{n=0}^{\infty} \frac{n^2}{n+1} x^{n+1} = \sum_{m=1}^{\infty} \frac{(m-1)^2}{m} x^m.$$

For convenience we rewrote the derivative as a power series in x^m where $m = n - 1$, and the integral as a power series in x^m where $m = n + 1$. Both the derivative and integral also have radius of convergence $r = 1$.

We are now ready to prove the power series formulas for $1/(1 - x)$ and e^x.

THEOREM 2

(i) $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad r = 1.$

(ii) $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots, \qquad r = \infty.$

PROOF (i) is just the geometric series for x. We proved in Section 9.2 that it converges to $1/(1 - x)$ for $|x| < 1$ and diverges for $|x| \geq 1$.

(ii) Let

$$y = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots.$$

At $x = 0$ we have $y = 1$. We can find dy/dx by Theorem 1.

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} = y.$$

The radius of convergence is ∞, so for all x,

$$\frac{dy}{dx} = y.$$

The general solution of this differential equation (see Section 8.6) is

$$y = Ce^x.$$

At $x = 0$, $1 = Ce^0 = C$. Therefore $y = e^x$.

We shall now get several new power series formulas starting from the power series for $1/(1 - x)$. We shall use the following methods:

A. Differentiate a power series.
B. Integrate a power series.
C. Substitute bu for x.
D. Substitute u^p for x.
E. Multiply a power series by a constant.
F. Multiply a power series by x^p.
G. Add two power series.

Methods C, D, and G may change the radius of convergence.

We start with

(1) $$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad r = 1.$$

Substitute $-u$ for x in Equation 1.

(2) $$\frac{1}{1+u} = 1 - u + u^2 - \cdots + (-1)^n u^n + \cdots, \qquad r = 1.$$

The radius of convergence is still $r = 1$ because when $|-u| < 1$, $|u| < 1$. Let us instead substitute $2u$ for x in Equation 1 and see what happens to the radius of convergence.

(3) $$\frac{1}{1-2u} = 1 + 2u + 2^2u^2 + \cdots + 2^n u^n + \cdots, \qquad r = \frac{1}{2}.$$

The radius of convergence in Equation 3 is $r = \frac{1}{2}$ because when $|2u| < 1$, $|u| < \frac{1}{2}$. For convenience we rewrite Equations 2 and 3 with x's instead of u's. Thus

(2) $$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots, \qquad r = 1.$$

(3) $$\frac{1}{1-2x} = 1 + 2x + 2^2x^2 + \cdots + 2^n x^n + \cdots, \qquad r = \frac{1}{2}.$$

By integrating $1/(1-x)$ and multiplying by -1 we get a power series for $\ln(1-x)$.

$$\int_0^x \frac{1}{1-t}\,dt = -\ln(1-x).$$

(4) $$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots, \qquad r = 1.$$

We next use the power series Equation 2 for $1/(1+x)$. Substitute x^2 for x in Equation 2.

(5) $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots, \qquad r = 1.$$

r is still 1 because if $|x^2| < 1$, $|x| < 1$. We obtain a power series for $\arctan x$ by integrating (5).

$$\int_0^x \frac{1}{1+t^2}\,dt = \arctan x.$$

(6) $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots, \qquad r = 1.$$

Finally let us differentiate the series (1) for $1/(1-x)$.

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

(7) $$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots, \qquad r = 1.$$

Let us begin again, this time with

(8) $$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots, \qquad r = \infty.$$

Substitute $-x$ for x in Equation 8.

(9) $$e^{-x} = 1 - x + \frac{x^2}{2!} - \cdots + \frac{(-1)^n x^n}{n!} + \cdots, \qquad r = \infty.$$

Using the formulas

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

we can obtain power series for $\cosh x$ and $\sinh x$. This is our first chance to use the method of adding power series.

(10) $$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots, \qquad r = \infty.$$

(11) $$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots, \qquad r = \infty.$$

Notice that the odd terms cancel out for $\cosh x$ and the even terms cancel out for $\sinh x$.

In Section 9.11 we shall obtain power series for $\sin x$ and $\cos x$ by another method.

We can easily get new power series by multiplying by x^p. For example, starting with the power series for $\ln(1-x)$, we obtain

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots, \qquad r = 1,$$

$$x\ln(1-x) = -x^2 - \frac{x^3}{2} - \frac{x^4}{3} - \cdots, \qquad r = 1,$$

$$x^2\ln(1-x) = -x^3 - \frac{x^4}{2} - \frac{x^5}{3} - \cdots, \qquad r = 1,$$

and so on. Since the series for $\ln(1-x)$ has no constant term, we may also divide by x to get a new power series. To cover the case $x = 0$, we let

$$f(x) = \begin{cases} \dfrac{\ln(1-x)}{x} & \text{if } x \neq 0, \\ -1 & \text{if } x = 0. \end{cases}$$

Then $$f(x) = -1 - \frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4} - \cdots, \qquad r = 1.$$

We can often get a power series formula for an indefinite integral which cannot be evaluated in other ways. For example, the integral

$$\int_0^x e^{-t^2}\,dt$$

is of central importance in probability theory. It is the area under the normal (bell-shaped) curve $y = e^{-x^2}$. This integral is not an elementary function at all, so the methods of integration in Chapter 9 will fail. However, we can easily find a power series for this integral. First substitute x^2 for x in Equation 9.

(12) $$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots, \qquad r = \infty.$$

Then integrate.

(13) $$\int_0^x e^{-t^2}\,dt = x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + \frac{(-1)^{n+1}x^{2n+1}}{(2n+1)n!} + \cdots, \qquad r = \infty.$$

PROOF OF THEOREM 1 It is easiest to prove (iii), then (ii), and finally (i).

(iii) The series $\sum_{n=1}^{\infty} na_n x^{n-1}$ and $\sum_{n=0}^{\infty} (a_n/(n+1))x^{n+1}$ have radius of convergence r.

Let $|x| < r$. We may choose c with $|x| < c < r$. Then $\sum_{n=0}^{\infty} a_n c^n$ converges absolutely. For positive infinite H, Theorem 1 in Section 9.1 (page 526) shows that $|c/x|^H/H$ is positive infinite, so $H|x/c|^H \approx 0$. Therefore

$$\left|Ha_Hx^{H-1}\right| = H\left|\frac{x}{c}\right|^H \cdot \left|\frac{1}{x}\cdot a_Hc^H\right| < \left|a_Hc^H\right|.$$

Then by the Limit Comparison Test, $\sum_{n=1}^{\infty} na_nx^{n-1}$ converges absolutely. Similarly $\sum_{n=0}^{\infty}(a_n/(n+1))x^{n+1}$ converges absolutely.

Now let $|x| > r$. Using the same test we can show that $\sum_{n=1}^{\infty} na_nx^{n-1}$ and $\sum_{n=0}^{\infty}(a_n/(n+1))x^{n+1}$ diverge. Therefore both series have radius of convergence r.

(ii) $\displaystyle\int_0^x f(t)\,dt = \sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}.$

Let $0 < c < r$. Our proof has three main steps. First, get an error estimate for the difference between $f(t)$ and the mth partial sum. Second, show that $f(t)$ is continuous for $-c \le t \le c$. Third, show that $f(t)$ has the required integral.

The series $\sum_{n=0}^{\infty} a_nc^n$ converges absolutely. Let E_m be the tail

$$E_m = \sum_{n=m+1}^{\infty} |a_n|c^n.$$

Then
$$\lim_{m\to\infty} E_m = 0.$$

Moreover, for $-c \le t \le c$,

$$\left|\sum_{n=m+1}^{\infty} a_nt^n\right| \le \sum_{n=m+1}^{\infty} |a_nt^n| \le E_m.$$

Therefore E_m is an error estimate for $f(t)$ minus the partial sum,

(14)
$$-E_m \le f(t) - \sum_{n=0}^{m} a_nt^n \le E_m.$$

We now prove f is continuous on $[-c, c]$. Since c was chosen arbitrarily between 0 and r, it will follow that f is continuous on $(-r, r)$. Let $t \approx u$ in $[-c, c]$. For each finite m,

$$\left|f(t) - \sum_{n=0}^{m} a_nt^n\right| \le E_m,$$

$$\left|\sum_{n=0}^{m} a_nt^n - \sum_{n=0}^{m} a_nu^n\right| \approx 0,$$

$$\left|\sum_{n=0}^{m} a_nu^n - f(u)\right| \le E_m.$$

Therefore
$$st|f(t) - f(u)| \le E_m + 0 + E_m.$$

Since the E_m's approach zero, it follows that $f(t) \approx f(u)$. Hence f is continuous on $[-c, c]$.

To prove the integral formula we integrate both sides of Equation 14 from 0 to x. Let $0 < x$.

$$-E_mx \le \int_0^x f(t)\,dt - \sum_{n=0}^{m}\frac{a_n}{n+1}x^{n+1} \le E_mx.$$

Again since E_m approaches zero, we conclude that

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

The case $x < 0$ is similar.

(i) $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

Let
$$g(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}.$$

Integrating term by term,

$$\int_0^x g(t)\,dt = \sum_{n=1}^{\infty} a_n x^n.$$

Thus
$$\int_0^x g(t)\,dt = f(x) - a_0 = f(x) - f(0).$$

By the Fundamental Theorem of Calculus, $g(x) = f'(x)$.

In part (i) of the proof we needed part (iii) to be sure that the series for $g(t)$ converges for $-r < t < r$, and part (ii) to justify the term by term integration.

PROBLEMS FOR SECTION 9.8

In Problems 1–10 find power series for $f'(x)$ and for $\int_0^x f(t)\,dt$.

1 $f(x) = \sum_{n=0}^{\infty} 10^n x^n$

2 $f(x) = \sum_{n=1}^{\infty} n^{-n} x^n$

3 $f(x) = \sum_{n=1}^{\infty} n^{-3} x^n$

4 $f(x) = \sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

5 $f(x) = \sum_{n=1}^{\infty} \frac{n+1}{n} x^n$

6 $f(x) = \sum_{n=1}^{\infty} \sqrt{n}\sqrt{n+1}\, x^n$

7 $f(x) = \sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

8 $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{3^n} x^n$

9 $f(x) = \sum_{n=0}^{\infty} x^{2n}$

10 $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} x^{2n}$

In Problems 11–34 find a power series for the given function and determine its radius of convergence.

11 $f(x) = \dfrac{1}{1+3x}$

12 $f(x) = \dfrac{1}{1-x^2}$

13 $f(x) = \arctan(4x^2)$

14 $f(x) = \ln(1 - 3x^2)$

15 $f(x) = x\ln(1 + 2x)$

16 $f(x) = \dfrac{\arctan x}{x}$ if $x \neq 0$, $f(0) = 1$

17 $f(x) = e^{-4x}$

18 $f(x) = x^2 e^x$

19 $f(x) = \sinh(3x)$

20 $f(x) = \cosh(x^2)$

21 $f(x) = \int_0^x \ln(1 + 2t^2)\,dt$

22 $f(x) = \int_0^x \arctan(t^3)\,dt$

23 $f(x) = \int_0^x e^{t^3}\,dt$

24 $f(x) = \int_0^x \sinh(t^2)\,dt$

25 $f(x) = \int_0^x t \ln(1 - t)\, dt$

26 $f(x) = \int_0^x \frac{t^2}{1 + t^2}\, dt$

27 $f(x) = \int_0^x \frac{\ln(1 + t)}{t}\, dt$

28 $f(x) = \int_0^x \frac{e^t - 1}{t}\, dt$

29 $f(x) = \frac{2x}{(1 + x^2)^2}$ *Hint*: $f(x) = \frac{d}{dx}\frac{-1}{1 + x^2}$.

30 $f(x) = \frac{1}{(1 + x^2)^2}$

31 $f(x) = \frac{2x}{1 + x^4}$ *Hint*: $f(x) = \frac{d}{dx}\arctan(x^2)$.

32 $f(x) = \frac{1}{(1 - x)^3}$

33 $f(x) = \arctan x + \arctan(2x)$

34 $f(x) = \sinh x + x \cosh x$

35 Check the formulas $d(\sinh x)/dx = \cosh x$, $d(\cosh x)/dx = \sinh x$ by differentiating the power series.

☐ **36** Prove that if the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has finite radius of convergence r, then the power series

$$f(bx) = \sum_{n=0}^{\infty} a_n (bx)^n$$

has radius of convergence r/b $(b > 0)$.

☐ **37** Prove that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has finite radius of convergence r, then

$$f(x^2) = \sum_{n=0}^{\infty} a_n x^{2n}$$

has radius of convergence $\sqrt{r}$.

☐ **38** Prove that if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

have radii of convergence r and s respectively and $r \leq s$, then $f(x) + g(x)$ has a radius of convergence of at least r.

☐ **39** Show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence r, then for any positive integer p,

$$x^p f(x) = \sum_{n=0}^{\infty} a_n x^{n+p}$$

has radius of convergence r.

☐ **40** Evaluate $\sum_{n=1}^{\infty} n x^n$, $|x| < 1$, using the derivative of the power series $\sum_{n=0}^{\infty} x^n$.

☐ **41** Evaluate $\sum_{n=1}^{\infty} n^2 x^n$, $|x| < 1$, using the first and second derivatives of $\sum_{n=0}^{\infty} x^n$.

9.9 APPROXIMATIONS BY POWER SERIES

Power series are one of the most important methods of approximation in mathematics. Consider a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots.$$

The partial sums give approximate values for the function,

$$f(x) \sim a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

and the tails E_n give the error in the approximation,

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + E_n.$$

If we can estimate the error E_n we can compute approximate values for $f(x)$ to any desired degree of accuracy.

In this section we shall give two simple methods of estimating the error. A more general method will be given in the next section. Our first method is to use the Alternating Series Test. It can be applied whenever a power series is alternating.

EXAMPLE 1 Approximate $\ln(1\frac{1}{2})$ within 0.01.

We use the power series for $\ln(1 - x)$,

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots, \qquad r = 1.$$

Setting $1 - x = 1\frac{1}{2}$, $x = -\frac{1}{2}$,

$$\ln\left(1\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 8} - \frac{1}{4 \cdot 16} + \frac{1}{5 \cdot 32} - \cdots.$$

This is an alternating series. The last term shown is less than 0.01,

$$\frac{1}{5 \cdot 32} = \frac{1}{160} \sim 0.006.$$

By the Alternating Series Test, the error in each partial sum is less than the next term. So

$$\ln\left(1\frac{1}{2}\right) \sim \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 8} - \frac{1}{4 \cdot 16}, \qquad \text{error} \le \frac{1}{5 \cdot 32},$$

or $\qquad \ln(1\frac{1}{2}) \sim 0.401, \qquad \text{error} \le 0.006.$

The actual value is $\ln(1\frac{1}{2}) \sim 0.405$.

EXAMPLE 2 Approximate $\arctan\frac{1}{2}$ within 0.001.

The power series for $\arctan x$ is

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots, \qquad r = 1.$$

Setting $x = \frac{1}{2}$,

$$\arctan\frac{1}{2} = \frac{1}{2} - \frac{1}{3 \cdot 8} + \frac{1}{5 \cdot 32} - \frac{1}{7 \cdot 128} + \frac{1}{9 \cdot 512} - \cdots.$$

This an an alternating series. The last term is less than 0.001,

$$\frac{1}{9 \cdot 512} \sim 0.0002.$$

Therefore

$$\arctan\frac{1}{2} \sim \frac{1}{2} - \frac{1}{3 \cdot 8} + \frac{1}{5 \cdot 32} - \frac{1}{7 \cdot 128}, \qquad \text{error} \le 0.0002.$$

Adding up, $\qquad \arctan\frac{1}{2} \sim 0.4635, \qquad \text{error} \le 0.0002.$

The series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad r = 1$$

can be used to approximate π. We start with

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, \qquad \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

Setting $x = 1/\sqrt{3}$ in the series,

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 - \frac{1}{7}\left(\frac{1}{\sqrt{3}}\right)^7 + \cdots,$$

or

$$\frac{\sqrt{3}}{6}\pi = 1 - \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{5}\left(\frac{1}{3}\right)^2 - \frac{1}{7}\left(\frac{1}{3}\right)^3 + \frac{1}{9}\left(\frac{1}{3}\right)^4 - \cdots.$$

This is an alternating series, so

$$\frac{\sqrt{3}}{6}\pi \sim 1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \frac{1}{729}, \qquad \text{error} \le \frac{1}{11}\left(\frac{1}{3}\right)^5,$$

$$\frac{\sqrt{3}}{6}\pi \sim 0.9072, \qquad \text{error} \le 0.0004.$$

Dividing everything by $\sqrt{3}/6$ we get

$$\pi \sim 3.1426, \qquad \text{error} \le 0.0013.$$

EXAMPLE 3 Approximate e^{-1} within 0.001.

The power series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots, \qquad r = \infty.$$

Setting $x = -1$,

$$e^{-1} = 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \cdots.$$

The series alternates and the last term is less than 0.001, so

$$e^{-1} \sim 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720}, \qquad \text{error} \le \tfrac{1}{5040} \sim 0.0002.$$

Adding up, $\qquad e^{-1} \sim 0.36806, \qquad \text{error} \le 0.0002.$

The actual value is $e^{-1} \sim 0.36788$.

Our second method of approximation is to start with a known error estimate for the geometric series and carefully keep track of the error each time we make a new series.

We recall the formula for the partial sum of a geometric series.

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x} - \frac{x^{n+1}}{1 - x}.$$

Thus $$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + E_n, \qquad E_n = \frac{x^{n+1}}{1-x}.$$

This formula is valid for all x, but the error E_n approaches zero only when x is within the interval of convergence $(-1, 1)$.

EXAMPLE 4 Approximate $1/(1 - 0.02)$ to six decimal places. Take $x = 0.02$.

$$\begin{aligned}\frac{1}{1-0.02} &= 1 + 0.02 + (0.02)^2 + (0.02)^3 + E_4 \\ &= 1 + 0.02 + 0.0004 + 0.000008 + E_4 \\ &= 1.020408 + E_4.\end{aligned}$$

The error E_4 after four terms is

$$E_4 = \frac{(0.02)^4}{1-0.02} = \frac{0.00000016}{0.98} < \frac{0.00000016}{0.8} = 0.00000020.$$

So $1/(1 - 0.02) \sim 1.020408$ to six places.

Suppose we wish to approximate $\ln \frac{1}{2}$ within 0.01. If in the series

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \qquad r = 1$$

we set $1 - x = \frac{1}{2}$, $x = \frac{1}{2}$, we get

$$\ln\frac{1}{2} = -\frac{1}{2} - \frac{1}{2 \cdot 4} - \frac{1}{3 \cdot 8} - \frac{1}{4 \cdot 16} - \cdots.$$

We know this series converges, but to be sure of an approximation within 0.01 we need an error estimate. The next example shows how to get such an error estimate.

EXAMPLE 5 Given a constant c where $-1 < c < 1$, find a simple error estimate for the power series

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots$$

valid for $-1 < x \le c$.

We start with the equation

(1) $$\frac{1}{1-t} = (1 + t + t^2 + \cdots + t^n) + E_n, \qquad E_n = \frac{t^{n+1}}{1-t}.$$

For $-1 < t \le c$ we have

$$1 - t \ge 1 - c, \qquad |E_n| \le \frac{|t|^{n+1}}{1-c}.$$

Integrating Equation 1 from 0 to x we have

(2) $$-\ln(1-x) = \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{n+1}}{n+1}\right) + \int_0^x E_n\,dt$$

and $$\left|\int_0^x E_n\,dt\right| \le \int_0^x \frac{|t|^{n+1}}{1-c}\,dt = \frac{|x|^{n+2}}{(1-c)(n+2)}.$$

Multiplying Equation 2 by -1 and setting $m = n + 1$ we have the following error estimate for $\ln(1-x)$, valid for $-1 < x \le c$.

(3) $$\ln(1-x) \sim \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^m}{m}\right),$$
$$\text{error} \le \frac{|x|^{m+1}}{(1-c)(m+1)}.$$

EXAMPLE 6 Use Example 5 to approximate $\ln\frac{1}{2}$ within 0.01. We set $c = x = \frac{1}{2}$ in Equation 3.

$$\ln\frac{1}{2} \sim -\frac{1}{2} - \frac{1}{2\cdot 4} - \frac{1}{3\cdot 8} - \frac{1}{4\cdot 16} - \cdots - \frac{1}{m\cdot 2^m},$$
$$|\text{error}| \le \frac{(1/2)^{m+1}}{\frac{1}{2}(m+1)} = \frac{1}{(m+1)2^m}.$$

Table 9.9.1 shows approximate values and error estimates.

Table 9.9.1

m	$\frac{1}{m\cdot 2^m}$	Approximate value for $\ln\frac{1}{2}$: $-\frac{1}{2} - \frac{1}{2\cdot 4} - \cdots - \frac{1}{m\cdot 2^m}$	Error estimate $\frac{1}{(m+1)2^m}$
1	0.5000	−0.5000	0.2500
2	0.1250	−0.6250	0.0833
3	0.04167	−0.6667	0.0313
4	0.01563	−0.6823	0.0125
5	0.00625	−0.6886	0.0052

We see that the error estimate drops below 0.01 when $m = 5$.

So $$\ln\tfrac{1}{2} \sim -0.689, \qquad \text{error} \le 0.01.$$

Since $\ln\frac{1}{2} = -\ln 2$, we also have

$$\ln 2 \sim 0.689, \qquad \text{error} \le 0.01.$$

A more rapidly converging series for $\ln 2$ can be obtained in the following way. Any number $a > 1$ can be put in the form

$$a = \frac{1+x}{1-x}, \qquad 0 < x < 1.$$

We simply take $$x = \frac{a-1}{a+1}.$$

By the rules of logarithms,

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x).$$

We can subtract two series by the Sum Rule, whence

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots, \qquad r = 1,$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots, \qquad r = 1,$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots, \qquad r = 1.$$

This power series is convenient because half of the terms are zero.

EXAMPLE 7 Find an error estimate for the power series for $\ln((1+x)/(1-x))$ valid for $-c \le x \le c$. Use it to approximate ln 2 within 0.00001.

From Example 5 we have the following error estimates for $\ln(1+x)$ and $-\ln(1-x)$ valid for $-c \le x \le c$.

$$\ln(1+x) \sim x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{m+1}\frac{x^m}{m},$$

$$\text{error} \le \frac{|x|^{m+1}}{(1-c)(m+1)}.$$

$$-\ln(1-x) \sim x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^m}{m},$$

$$\text{error} \le \frac{|x|^{m+1}}{(1-c)(m+1)}.$$

We add the two sums and error estimates,

$$\ln\left(\frac{1+x}{1-x}\right) \sim 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots + \frac{2x^{2m-1}}{2m-1},$$

$$\text{error} \le \frac{2|x|^{2m+1}}{(1-c)(2m+1)}.$$

We wish to choose x so that $(1+x)/(1-x) = 2$. Solving for x we get $x = \frac{1}{3}$. Now set $c = \frac{1}{3}$ and $x = \frac{1}{3}$. The error estimate for $x = \frac{1}{3}$ is

$$\frac{2|x|^{2m+1}}{(1-c)(2m+1)} = \frac{1}{(2m+1)3^{2m}}.$$

Table 9.9.2

m	$\dfrac{2}{(2m-1)3^{2m-1}}$	Approximate value for ln 2: $\dfrac{2}{1\cdot 3} + \dfrac{2}{3\cdot 27} + \cdots + \dfrac{2}{(2m-1)3^{2m-1}}$	Error estimate: $\dfrac{1}{(2m+1)3^{2m}}$
1	0.666667	0.666667	0.037037
2	0.024691	0.691358	0.002469
3	0.001646	0.693004	0.000196
4	0.000131	0.693134	0.000017
5	0.000011	0.693146	0.000002

The error estimate drops below 0.00001 when $m = 5$. Thus

$$\ln 2 \sim 0.693146, \qquad \text{error} \le 0.00001.$$

EXAMPLE 8 Find the sum of the alternating harmonic series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots.$$

Our first guess is to set $x = -1$ in the power series

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \qquad r = 1.$$

This suggests to us the sum

$$\ln 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots.$$

We know the series converges to something by the Alternating Series Test. For $-1 < x < 1$ the series converges to $\ln(1 - x)$. But $x = -1$ is an endpoint of the interval of convergence and the general theorem on integrating a power series does not apply. So we must go back to the beginning and use the equation

$$\frac{1}{1 - t} = (1 + t + \cdots + t^n) + \frac{t^{n+1}}{1 - t}.$$

For $t \le 0$, $|t^{n+1}/(1 - t)| \le |t^{n+1}|$, whence

$$\frac{1}{1 - t} = (1 + t + \cdots + t^n) + E_n, \qquad |E_n| \le |t^{n+1}|.$$

Integrating from 0 to x,

$$-\ln(1 - x) = \left(x + \frac{x^2}{2} + \cdots + \frac{x^{n+1}}{n + 1}\right) + F_n, \qquad |F_n| \le \left|\frac{x^{n+2}}{n + 2}\right|.$$

This holds for all $x \le 0$.

Now we set $x = -1$ and see that the error term $|F_n| \le 1/(n + 2)$ approaches zero. This proves that $\ln 2$ really is the sum of the alternating harmonic series,

$$\ln 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots.$$

The alternating harmonic series converges very slowly, because after n terms the error estimate is only $1/(n + 1)$.

PROBLEMS FOR SECTION 9.9

Problems 1–12 below are to be done using a power series with an error estimate. If a hand calculator is available they can be worked with the errors reduced by an additional factor of 1000.

1 Approximate $\ln(1.2)$ within 0.01.

2 Approximate $\arctan(\frac{1}{10})$ within 10^{-7}.

3 Approximate $e^{-1/4}$ within 0.00001.

4 Approximate $\int_0^1 e^{-t^2}\,dt$ within 0.01.

5 Approximate $\int_0^{1/2} \frac{1}{1 + t^3}\,dt$ within 0.0001.

6 Approximate $\int_0^{1/2} \ln(1 + t^2)\,dt$ within 0.001.

7 Approximate $\int_0^{1/3} \frac{1}{t} \arctan(t)\,dt$ within 0.0001.

8 Approximate $\int_0^{1/2} \arctan(t^2)\,dt$ within 0.00001.

9 Approximate $1/(1 - 0.003)$ within 0.0001.

10 Approximate $\ln 3$ within 0.1 by the method of Example 6. *Hint:* $\ln 3 = -\ln(1 - x)$ where $x = \frac{2}{3}$.

11 Approximate $\ln 3$ within 0.001 by the method of Example 7.

12 (a) Approximate $\ln(1\frac{1}{2})$ within 0.00001 by the method of Example 7.
(b) Approximate $\ln 3$ within 0.00002 using the formula $\ln 3 = \ln 2 + \ln(1\frac{1}{2})$.

In Problems 13–18 find a power series approximation with an error estimate for $f(x)$ valid for $-\frac{1}{2} \le x \le \frac{1}{2}$. Then approximate $f(\frac{1}{2})$ within 0.01.

13 $f(x) = \dfrac{x}{1 - x}$

14 $f(x) = \int_0^x \ln(1 - t)\,dt$

15 $f(x) = \dfrac{1}{1 - x^2}$.
Hint: $x^2 = \frac{1}{4}$ when $x = \frac{1}{2}$.

16 $f(x) = \int_0^x \dfrac{1}{1 - t^3}\,dt$

17 $f(x) = \int_0^x \dfrac{\ln(1 - t)}{t}\,dt$

18 $f(x) = \int_0^x \ln(1 - t^2)\,dt$

□ **19** Using the power series for $\arctan x$ at $x = 1$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.$$

□ **20** Using the power series for $\int_0^x \ln(1 + t)\,dt$ at $x = 1$, show that

$$2 \ln 2 - 1 = \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} - \cdots.$$

9.10 TAYLOR'S FORMULA

If we wish to express $f(x)$ as a power series in $x - c$, we need two things:

(1) A sequence of polynomials which approximate $f(x)$ near $x = c$,

(1) $$a_0, a_0 + a_1(x - c), \ldots, a_0 + a_1(x - c) + \cdots + a_n(x - c)^n, \ldots.$$

(2) An estimate for the error E_n between $f(x)$ and the nth polynomial,

(2) $$f(x) = a_0 + a_1(x - c) + \cdots + a_n(x - c)^n + E_n.$$

In the last section the formula

$$\frac{1}{1 - x} = 1 + x + \cdots + x^n + E_n, \qquad E_n = \frac{x^{n+1}}{1 - x}$$

was used to obtain power series approximations. A much more general formula of this type is Taylor's Formula. In Taylor's Formula the nth polynomial $P_n(x)$ is chosen so that its value and first n derivatives agree with $f(x)$ at $x = c$.

The tangent line at $x = c$,

$$P_1(x) = f(c) + f'(c)(x - c),$$

has the same value and first derivative as $f(x)$ at $x = c$. A polynomial of degree two with the same value and first two derivatives as $f(x)$ at c is

$$P_2(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2.$$

$P_1(x)$ and $P_2(x)$ are the first and second *Taylor polynomials* of $f(x)$ (see Figure 9.10.1).

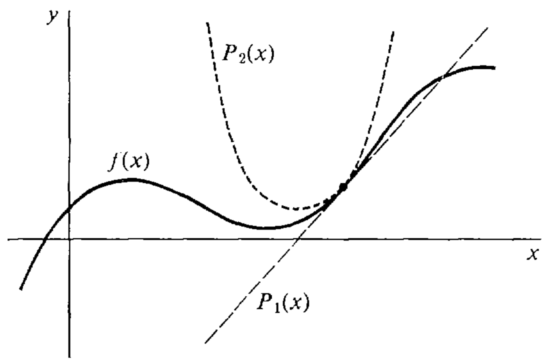

Figure 9.10.1 First and Second Taylor Polynomials

To continue the procedure we need a formula for the nth derivative of a polynomial.

LEMMA 1

Let $P(x)$ be a polynomial in $x - c$ of degree n.

$$P(x) = a_0 + a_1(x - c) + a_2(x - c) + \cdots + a_n(x - c)^n.$$

For each $m \leq n$, the mth derivative of $P(x)$ at $x = c$ divided by $m!$ is equal to the coefficient a_m,

$$\frac{P^{(m)}(c)}{m!} = a_m.$$

PROOF Consider one term $a_k(x - c)^k$. Its mth derivative is

$$\begin{aligned} k(k-1)\cdots(k-m+1)a_k(x-c)^{k-m} &\quad \text{if } m < k, \\ m!\,a_m &\quad \text{if } m = k, \\ 0 &\quad \text{if } m > k. \end{aligned}$$

At $x = c$, the mth derivative of $a_k(x - c)^k$ is:

$$0 \text{ if } m < k, \qquad m!\,a_m \text{ if } m = k, \qquad 0 \text{ if } m > k.$$

It follows that $P^{(m)}(c) = m!\,a_m$.

This lemma shows us how to find a polynomial $P(x)$ whose value and first n derivatives agree with $f(x)$ at $x = c$. The mth coefficient of $P(x)$ must be

$$a_m = \frac{f^{(m)}(c)}{m!}.$$

DEFINITION

Let $f(x)$ have derivatives of all orders at $x = c$. The nth ***Taylor polynomial*** *of $f(x)$ at $x = c$ is the polynomial*

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n.$$

By Lemma 1, $P_n(x)$ is the unique polynomial of degree n whose value and first n derivatives at $x = c$ agree with $f(x)$,

$$P_n(c) = f(c),\ P_n'(c) = f'(c), \ldots, P_n^{(n)}(c) = f^{(n)}(c).$$

The difference between $f(x)$ and the nth *Taylor polynomial is called the* nth ***Taylor remainder,***

$$R_n(x) = f(x) - P_n(x).$$

Thus

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x).$$

EXAMPLE 1 Find the first five Taylor polynomials of $\sin x$ at $x = 0$. We work them out in Table 9.10.1.

Table 9.10.1

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$P_k(x)$
0	$\sin x$	0	0
1	$\cos x$	1	x
2	$-\sin x$	0	x
3	$-\cos x$	-1	$x - x^3/3!$
4	$\sin x$	0	$x - x^3/3!$
5	$\cos x$	1	$x - x^3/3! + x^5/5!$

Since the even degree terms are zero, the $2n$th Taylor polynomial is the same as the $(2n - 1)$st. Figure 9.10.2 compares the first and third Taylor polynomials with $\sin x$.

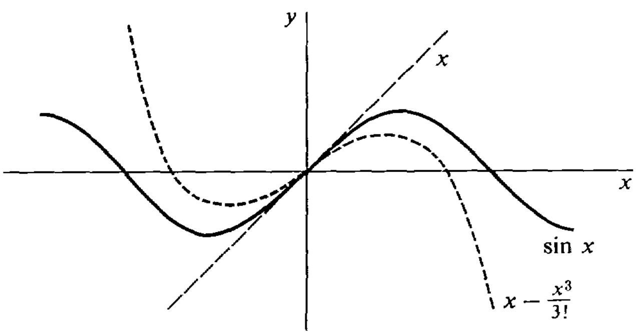

Figure 9.10.2

We can easily find the Taylor polynomials of $f(x)$ by differentiating. Let us now try to find a formula for the Taylor remainders. The Mean Value Theorem gives a formula for the Taylor remainder, $R_0(x)$.

MEAN VALUE THEOREM (Repeated)

Suppose $f(t)$ is differentiable at all t between c and d. Then

$$f'(t_0) = \frac{f(d) - f(c)}{d - c}$$

for some point t_0 strictly between c and d.

When we replace d by x, this gives the formula

$$f(x) = f(c) + R_0(x), \qquad R_0(x) = f'(t_0)(x - c).$$

Taylor's Formula is a generalization of the Mean Value Theorem which gives the nth Taylor remainder.

TAYLOR'S FORMULA

Suppose the $(n + 1)$st derivative $f^{(n+1)}(t)$ exists for all t between c and x. Then

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where
$$R_n(x) = \frac{f^{(n+1)}(t_n)}{(n + 1)!}(x - c)^{n+1}$$

for some point t_n strictly between c and x.

Notice that the remainder term looks just like the $(n + 1)$st term of a Taylor polynomial except that $f^{(n+1)}(c)$ is replaced by $f^{(n+1)}(t_n)$.

When $c = 0$ Taylor's Formula is sometimes called *MacLaurin's Formula.*

Taylor's Formula can be used to get an estimate of the error $R_n(x)$ between $f(x)$ and the Taylor polynomial $P_n(x)$. For example if

$$|f^{(n+1)}(t)| \le M_{n+1}$$

for all t between c and x, then we obtain the error estimate

$$|R_n(x)| \le \frac{M_{n+1}}{(n + 1)!}|x - c|^{n+1}.$$

Taylor polynomials with the error estimate are of great practical value in obtaining approximations. In the next example we use Taylor's Formula to approximate the value of e.

EXAMPLE 2 Find MacLaurin's Formula for $f(x) = e^x$.

The nth derivative is $f^{(n)}(x) = e^x$, $f^{(n)}(0) = 1$. MacLaurin's Formula is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x), \qquad R_n(x) = e^{t_n}\frac{x^{n+1}}{(n + 1)!}$$

for some t_n between 0 and x. For t between 0 and x the value of e^t is always less than or equal to $3^{|x|}$, for

$$e^t \le e^{|x|} \le 3^{|x|}.$$

We therefore have the formula

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x), \qquad |R_n(x)| \le 3^{|x|} \cdot \frac{|x|^{n+1}}{(n+1)!}. \tag{3}$$

The formula (3) can be used to approximate e^x. Let us set $x = 1$ and approximate e. The error estimate is now

$$3^{|x|} \cdot \frac{|x|^{n+1}}{(n+1)!} = \frac{3}{(n+1)!}.$$

n	$1/n!$	Approximate value for e $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$	Error estimate $\frac{3}{(n+1)!}$
2	0.500000	2.500000	0.500000
3	0.166667	2.666667	0.125000
4	0.041667	2.708333	0.025000
5	0.008333	2.716667	0.004167
6	0.001389	2.718056	0.000594
7	0.000198	2.718254	0.000075
8	0.000025		

This compares with $e = 2.718282$.

EXAMPLE 3 Find MacLaurin's Formula for $f(x) = \sin x$. The derivatives are

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0\\
f'(x) &= \cos x & f'(0) &= 1\\
f''(x) &= -\sin x & f''(0) &= 0\\
f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1\\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0\\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1\\
&\vdots & &\vdots
\end{aligned}$$

MacLaurin's Formula for $2n$ terms is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + R_{2n}(x),$$

$$R_{2n}(x) = (-1)^n \cos t \frac{x^{2n+1}}{(2n+1)!}.$$

For all t, $|\cos t| \le 1$, so we have the error estimate

$$|R_{2n}(x)| \le \frac{|x|^{2n+1}}{(2n+1)!}.$$

MacLaurin's Formula can be used to approximate $\sin x$ (with x in radians) when x is close to zero. We approximate $\sin(18°)$ as follows.

$$x = 18° = \frac{\pi}{10} \sim 0.31415927 \text{ radians.}$$

n	$(-1)^{n-1}\dfrac{x^{2n-1}}{(2n-1)!}$	Approximate value of $P_{2n}(x)$	Error estimate $\lvert x\rvert^{2n+1}/(2n+1)!$
1	0.31415927	0.31415927	0.00516771
2	−0.00516771	0.30899156	0.00002550
3	0.00002550	0.30901706	0.00000006

Thus $\sin(18°) \sim 0.3090171$ to seven places.

The proof of Taylor's Formula uses the following generalized form of the Mean Value Theorem.

GENERALIZED MEAN VALUE THEOREM

Suppose f and g are differentiable at all t between c and d, and that $g'(t) \neq 0$ for t strictly between c and d. Then

$$\frac{f'(t_0)}{g'(t_0)} = \frac{f(d) - f(c)}{g(d) - g(c)}$$

for some point t_0 strictly between c and d.

This theorem can be illustrated graphically by plotting the parametric equations $x = g(t)$, $y = f(t)$ in the (x, y) plane, as in Figure 9.10.3.

If $f(c) = 0$ and $g(c) = 0$, the formula in the theorem takes on the simpler form

$$\frac{f'(t_0)}{g'(t_0)} = \frac{f(d)}{g(d)}.$$

This is the form which will be used in the proof of Taylor's Formula.

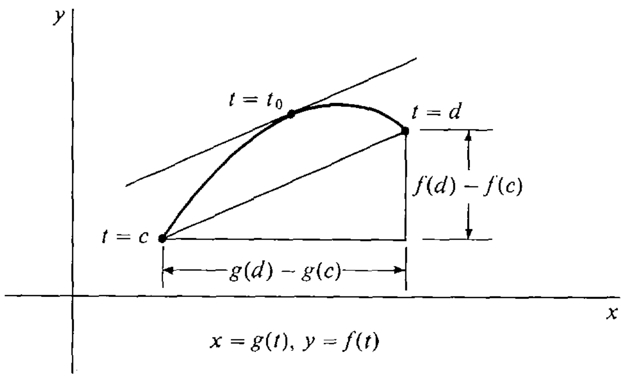

Figure 9.10.3

PROOF OF THE GENERALIZED MEAN VALUE THEOREM Introduce the new function

$$h(t) = f(t)(g(d) - g(c)) - g(t)(f(d) - f(c)).$$

Then $h(t)$ is also differentiable at all points between c and d. Furthermore, at the endpoints c and d we have

$$h(c) = f(c)g(d) - f(d)g(c) = h(d).$$

We may therefore apply Rolle's Theorem, whence there is a point t_0 strictly between c and d such that $h'(t_0) = 0$. Differentiating $h(t)$, we get

$$h'(t) = f'(t)(g(d) - g(c)) - g'(t)(f(d) - f(c)).$$

Therefore at $t = t_0$,

$$0 = f'(t_0)(g(d) - g(c)) - g'(t_0)(f(d) - f(c)).$$

$g'(t)$ is never zero. Also, $g(c) \neq g(d)$ because otherwise Rolle's Theorem would give a t with $g'(t) = 0$. We may therefore divide out and obtain the desired formula

$$\frac{f'(t_0)}{g'(t_0)} = \frac{f(d) - f(c)}{g(d) - g(c)}.$$

PROOF OF TAYLOR'S FORMULA Let $F(x) = R_n(x)$, $G(x) = (x - c)^{n+1}$. Then $F(x) = f(x) - P_n(x)$. $f(x)$ and the nth Taylor polynomial $P_n(x)$ have the same value and first n derivatives at $x = c$. Therefore

$$F(c) = F'(c) = F''(c) = \cdots = F^{(n)}(c) = 0.$$

We also see that

$$G(c) = G'(c) = G''(c) = \cdots = G^{(n)}(c) = 0.$$

Using the Generalized Mean Value Theorem $n + 1$ times, we have

$$\frac{F'(t_0)}{G'(t_0)} = \frac{F(x)}{G(x)} \quad \text{for some } t_0 \text{ strictly between } c \text{ and } x;$$

$$\frac{F''(t_1)}{G''(t_1)} = \frac{F'(t_0)}{G'(t_0)} \quad \text{for some } t_1 \text{ strictly between } c \text{ and } t_0;$$

$$\vdots$$

$$\frac{F^{(n+1)}(t_n)}{G^{(n+1)}(t_n)} = \frac{F^{(n)}(t_{n-1})}{G^{(n)}(t_{n-1})} \quad \text{for some } t_n \text{ strictly between } c \text{ and } t_{n-1}.$$

It follows that

$$\frac{F^{(n+1)}(t_n)}{G^{(n+1)}(t_n)} = \frac{F(x)}{G(x)}.$$

Either

$$x < t_0 < t_1 < \cdots < t_n < c \quad \text{or} \quad x > t_0 > t_1 > \cdots > t_n > c,$$

so t_n is strictly between c and x. The $(n + 1)$st derivatives of $F(t)$ and $G(t)$ are

$$F^{(n+1)}(t) = f^{(n+1)}(t) - 0, \qquad G^{(n+1)}(t) = (n + 1)!$$

Substituting, we have

$$\frac{f^{(n+1)}(t_n)}{(n+1)!} = \frac{R_n(x)}{(x-c)^{n+1}},$$

and Taylor's Formula follows at once.

PROBLEMS FOR SECTION 9.10

In Problems 1–8, find MacLaurin's Formula for $f(x)$, and use it to approximate $f(\frac{1}{2})$ within 0.01. (If a hand calculator is available, the approximations should be found within 0.0001.)

1	$f(x) = \cos x$	2	$f(x) = \sinh x$
3	$f(x) = \sin(2x)$	4	$f(x) = 100e^x$
5	$f(x) = \sin x \cos x$	6	$f(x) = \sqrt{1+x}$
7	$f(x) = (4+x)^{-3/2}$	8	$f(x) = (1-x)^{1/3}$

In Problems 9–18 find the first two nonzero terms in MacLaurin's Formula and use it to approximate $f(\frac{1}{3})$.

9	$f(x) = \tan x$	10	$f(x) = \sec x$
11	$f(x) = \arcsin x$	12	$f(x) = \sin(e^x)$
13	$f(x) = \ln(1 + \sin x)$	14	$f(x) = \sqrt{x^2+1}$
15	$\int_0^x e^{t^2}\,dt$	16	$\int_0^x \sin(t^2)\,dt$
17	$\int_0^x \sin(\ln(1+t))\,dt$	18	$\int_0^x \arcsin(t^2)\,dt$

19 Find Taylor's Formula for $f(x) = e^x$ in powers of $x - 2$.

20 Find Taylor's Formula for $f(x) = \ln x$ in powers of $x - 10$.

21 Find Taylor's Formula for $f(x) = x^p$ in powers of $x - 1$, where p is a constant real number.

22 Find Taylor's Formula for $f(x) = \sin x$ in powers of $x - \pi$.

9.11 TAYLOR SERIES

DEFINITION

If we continue the Taylor polynomial (by adding three dots at the end) we obtain a power series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

This series is called the ***Taylor series*** *for the function $f(x)$ about the point $x = c$.*

The Taylor series about the point $x = 0$ is called the ***MacLaurin series,***

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

At $x = c$ the Taylor series about the point c converges to $f(c)$. But we have no assurance that the Taylor series converges to $f(x)$ at any other point x. There are three possibilities and all of them arise:

(1) *The Taylor series diverges at x.*

(2) *The Taylor series converges but to a value different than $f(x)$.* (For an example, see Problem 28 at the end of this section.)

(3) *The Taylor series converges to $f(x)$; i.e., $f(x)$ is equal to the sum of its Taylor series.*

Theorem 1 shows that if we already know a function $f(x)$ is the sum of a power series, then that power series must be the Taylor series of $f(x)$.

THEOREM 1

Suppose $f(x)$ is equal to the sum of a power series with radius of convergence $r > 0$,

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n.$$

Then the power series is the same as the Taylor series for f about c. In other words, a_n is just $f^{(n)}(c)/n!$ for $n = 0, 1, 2, \ldots$.

Discussion A function which is equal to the sum of a power series in $x - c$ (with nonzero radius of convergence) is called *analytic* at c. The theorem shows that every analytic function is equal to the sum of its Taylor series.

PROOF Since power series can be differentiated term by term within its interval of convergence, all the nth derivatives $f^{(n)}(c)$ exist. Let us compute $f^{(n)}(x)$ and set $x = c$.

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n, \qquad f(c) = a_0$$

$$f'(x) = \sum_{n=1}^{\infty} na_n(x - c)^{n-1}, \qquad f'(c) = a_1$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x - c)^{n-2}, \qquad f''(c) = 2!a_2$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x - c)^{n-3}, \qquad f'''(c) = 3!a_3$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x - c)^{n-k}, \qquad f^{(k)}(c) = k!a_k.$$

Thus for each n,

$$a_n = f^{(n)}(c)/n!,$$

and the original power series is the same as the Taylor series of $f(x)$,

$$\sum_{n=0}^{\infty} a_n(x-c)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

EXAMPLE 1 Let $f(x)$ be a polynomial in $x - c$,

$$f(x) = a_0 + a_1(x-c) + \cdots + a_n(x-c)^n.$$

This is just a power series with all but the first $n + 1$ coefficients equal to zero. So by Theorem 1, the Taylor series of the polynomial is just the polynomial itself followed by infinitely many zeros,

$$a_0 + a_1(x-c) + \cdots + a_n(x-c)^n + 0 + 0 + \cdots.$$

We can also see this directly from Lemma 1 of the last section, namely

$$\frac{f^{(m)}(c)}{m!} = a_m \qquad \text{for } m \le n.$$

Here is a review of the power series obtained earlier in this chapter. By Theorem 1, they are all MacLaurin series.

(1) $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \qquad |x| < 1$$

(2) $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots, \qquad |x| < 1$$

(3) $$\frac{1}{1-2x} = 1 + 2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \cdots, \qquad |x| < \tfrac{1}{2}$$

(4) $$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \qquad |x| < 1$$

(5) $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots, \qquad |x| < 1$$

(6) $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots, \qquad |x| < 1$$

(7) $$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots, \qquad |x| < 1$$

(8) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

(9) $$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

(10) $$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

(11) $$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

(12) $$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

(13) $$\int_0^x e^{-t^2}\,dt = x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \cdots$$

(14) $$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \cdots, \qquad |x| < 1$$

At the end of this section we shall add three important power series to our list:

(15) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

(16) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

(17) $$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots, \qquad |x| < 1,$$

where p is constant.

The last series is called the *binomial series.*

It is interesting to observe that the derivatives of an analytic function at zero can be read directly from the MacLaurin series. Sometimes it is quite hard to compute the derivative directly but easy to take it from the MacLaurin series.

EXAMPLE 2 Find the sixth derivative of $f(x) = 1/(1+x^2)$ at $x = 0$.

If we try to differentiate directly we will be hopelessly bogged down at about the third derivative. But from the MacLaurin series we see that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots,$$

$$\frac{f^{(6)}(0)}{6!}x^6 = -x^6,$$

$$\frac{f^{(6)}(0)}{6!} = -1,$$

$$f^{(6)}(0) = -6! = -720.$$

Suppose we are given a function $f(x)$ and a point c, and we wish to represent $f(x)$ as the sum of a power series in $x - c$. This will be possible for some functions (the analytic functions), but not for all. Theorem 1 shows that if there is such a power series it is the Taylor series for $f(x)$. Thus we use the following steps to represent $f(x)$ as a power series.

Step 1 Compute all the derivatives $f^{(n)}(c), n = 0, 1, 2, \ldots$. If these derivatives do not all exist, $f(x)$ is not the sum of a power series in powers of $x - c$.

Step 2 Write down the Taylor series of $f(x)$ at $x = c$ and find its radius of convergence r.

Step 3 If possible, show that $f(x)$ is equal to the sum of its Taylor series for $c - r < x < c + r$.

We shall now use Steps 1–3 to obtain the power series for $\sin x$, $\cos x$, and $(1 + x)^p$.

THE POWER SERIES FOR sin x

Step 1 This step was carried out in the preceding section. The values of $f^{(n)}(0)$ for $n = 0, 1, 2, \ldots$ are

$$0, 1, 0, -1, 0, 1, 0, -1, \ldots.$$

Step 2 The MacLaurin series for sin x is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots.$$

Let $b_n = (-1)^{n-1}\dfrac{x^{2n-1}}{(2n-1)!}$. We use the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\frac{x^2}{2n(2n+1)} = 0.$$

Therefore the series converges for all x and has radius of convergence ∞.

Step 3 We use MacLaurin's Formula,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + R_{2n}(x),$$

$$|R_{2n}(x)| \le \frac{|x|^{2n+1}}{(2n+1)!}.$$

Let us show that the remainders approach zero. We have

$$\lim_{n\to\infty}\frac{|x|^{2n+1}}{(2n+1)!} = 0, \qquad \lim_{n\to\infty} R_{2n}(x) = 0.$$

Since the even terms are zero, $R_{2n-1}(x) = R_{2n}(x)$. Therefore

$$\lim_{n\to\infty} R_n(x) = 0.$$

Conclusion: Since the remainders approach zero, the MacLaurin polynomials approach sin x. So for all x,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots.$$

THE POWER SERIES FOR cos x

This power series can be found by the same method as was used for sin x. However, it is simpler to differentiate the power series for sin x.

$$\frac{d(\sin x)}{dx} = \cos x.$$

$$\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots.$$

THE BINOMIAL SERIES FOR $(1 + x)^p$

Let us first consider the case where p is a nonnegative integer m, whence $(1 + x)^m$ is a polynomial. The *Binomial Theorem* states that for nonnegative integers m,

$$(a + b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \cdots + \frac{m(m-1)\cdots(m-k+1)}{k!}a^{m-k}b^k + \cdots + b^m.$$

Setting $a = 1$ and $b = x$ we obtain a finite power series for $(1 + x)^m$,

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-k+1)}{k!}x^k + \cdots + x^m.$$

When $p < 0$, and when $p > 0$ but p is not an integer, we shall see that $(1 + x)^p$ is the sum of a similar power series but with infinitely many terms. Let $g(x) = (1 + x)^p$.

Step 1 By differentiation we see that

$$g'(x) = p(1 + x)^{p-1},$$
$$g''(x) = p(p-1)(1 + x)^{p-2},$$
$$g^{(n)}(x) = p(p-1)\cdots(p-n+1)(1 + x)^{p-n}.$$

Thus at $x = 0$,

$$g(0) = 1,$$
$$g'(0) = p,$$
$$g''(0) = p(p-1),$$
$$g^{(n)}(0) = p(p-1)\cdots(p-n+1).$$

Step 2 The MacLaurin series is

$$f(x) = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots.$$

We use the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{p(p-1)\cdots(p-n)/(n+1)!}{p(p-1)\cdots(p-n+1)/n!}\right| |x| = \frac{|p-n|}{n+1}|x|.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|p-n|}{n+1}|x| = |x|.$$

Therefore the series converges for $|x| < 1$, diverges for $|x| > 1$, and has radius of convergence $r = 1$. We denote the sum by $f(x)$.

Step 3 We wish to show that the sum $f(x)$ is equal to $(1 + x)^p$ for $|x| < 1$. In this case, the MacLaurin Formula does not give the needed information (see Problem 27 at the end of this section). Instead we show that the quotient $f(x)/(1 + x)^p$ has derivative zero for $|x| < 1$. We have

$$\frac{d}{dx}[f(x)(1 + x)^{-p}] = \frac{f'(x)(1 + x) - pf(x)}{(1 + x)^{p+1}}.$$

It suffices to show that

$$f'(x)(1 + x) = pf(x) \quad \text{or} \quad f'(x) + xf'(x) = pf(x).$$

Let us compute $f'(x)$ and $xf'(x)$.

$$f'(x) = p + p(p-1)x + \frac{p(p-1)(p-2)}{2!}x^2 + \frac{p(p-1)(p-2)(p-3)}{3!}x^3 + \cdots,$$

$$xf'(x) = px + p(p-1)x^2 + \frac{p(p-1)(p-2)}{2!}x^3 + \cdots.$$

Adding the power series, we have

$$f'(x) + xf'(x) = p + p[(p-1)+1]x + \frac{p(p-1)}{2!}[(p-2)+2]x^2 + \frac{p(p-1)(p-2)}{3!}[(p-3)+3]x^3 + \cdots$$

$$= p\left[1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots\right]$$

$$= pf(x).$$

Thus $\quad f'(x) + xf'(x) = pf(x), \quad \dfrac{d}{dx}[f(x)(1+x)^{-p}] = 0.$

We conclude that for some constant C,

$$f(x)(1+x)^{-p} = C.$$

At $x = 0$, $f(x) = 1 = (1+x)^{-p}$. Hence $C = 1$. This shows that $(1+x)^p = f(x)$ for $|x| < 1$.

Thus we have the binomial series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots, \qquad |x| < 1.$$

EXAMPLE 3 Find the power series for arcsin x.

Recall that for $|x| < 1$,

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x (1-t^2)^{-1/2}\,dt.$$

We start with the binomial series with $p = -\frac{1}{2}$ and obtain the following power series by substitution and integration. They are valid for $|x| < 1$.

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2!}x^2 - \cdots + (-1)^n\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n n!}x^n + \cdots$$

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots + \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n n!}x^n + \cdots$$

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots + \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n n!}x^{2n} + \cdots$$

$$\arcsin x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots + \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n n!(2n+1)}x^{2n+1} + \cdots.$$

PROBLEMS FOR SECTION 9.11

1 Find $f^{(4)}(0)$ where $f(x) = 1/(1 - 2x^2)$.

2 Find $f^{(5)}(0)$ where $f(x) = x/(1 + x^2)$.

3 Find $f^{(6)}(0)$ where $f(x) = xe^x$.

4 Find $f^{(8)}(0)$ where $f(x) = \cos(x^2)$.

5 Find $f^{(7)}(0)$ where $f(x) = x^2 \ln(1 + x)$.

6 Find $f^{(6)}(0)$ where $f(x) = (\arctan x)/x$ if $x \neq 0$, and $f(0) = 1$.

In Problems 7–24, find a power series converging to $f(x)$ and determine the radius of convergence.

7 $f(x) = e^{x/2}$

8 $f(x) = x^2e^x$

9 $f(x) = \sqrt{1 + 2x}$

10 $f(x) = (1 - 4x)^{-1/3}$

11 $f(x) = \cos\sqrt{x}$

12 $f(x) = \arcsin(x^3)$

13 $f(x) = \dfrac{\sin x}{x}$ if $x \neq 0$, $f(0) = 1$

14 $f(x) = \dfrac{1 - \cos x}{x^2}$ if $x \neq 0$, $f(0) = \frac{1}{2}$.

15 $f(x) = \sqrt{1 - x^2}$

16 $f(x) = \dfrac{x}{(1 + x)^4}$

17 $f(x) = \int_0^x \sin(t^3)\,dt$

18 $f(x) = \int_0^x t^{-1} \sin t\,dt$

19 $f(x) = \int_0^x t^{-2} \sinh(t^2)\,dt$

20 $f(x) = \int_0^x \ln(1 + t^2)\,dt$

21 $f(x) = \int_0^x (1 + t^2)^{1/3}\,dt$

22 $f(x) = \int_0^x \sqrt{1 - t^3}\,dt$

23 $f(x) = \int_0^x \dfrac{\arcsin t}{t}\,dt$

24 $f(x) = \int_0^x \arcsin(t^2)\,dt$

25 Find the Taylor series for $\ln x$ in powers of $x - 1$.

26 Find the Taylor series for $\sin x$ in powers of $x - \pi/4$.

□ 27 Use Taylor's Formula to prove that the binomial series converges to $(1 + x)^p$ when $-\frac{1}{2} \leq x < 1$. (The proof in the text shows that it actually converges to $(1 + x)^p$ for $-1 < x < 1$.)

□ 28 Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ e^{-1/x^2} & \text{if } x \neq 0, \end{cases}$$

Show that $f^{(n)}(0) = 0$ for all integers n; so for $x \neq 0$ the MacLaurin series converges but to zero instead of to $f(x)$.

EXTRA PROBLEMS FOR CHAPTER 9

Determine whether the sequences 1–5 converge and find the limits when they exist.

1 $a_n = \left(1 + \dfrac{1}{n^2}\right)^n$

2 $a_n = \left(1 + \dfrac{1}{\sqrt{n}}\right)^n$

3 $a_n = (1 + n)^{1/n}$

4 $a_n = n! - 10^n$

□ 5 $a_n = n^n/n!$ (*Hint:* Show that $a_{n+1} \geq 2a_n$.)

Determine whether the series 6–12 converge and find the sums when they exist.

6 $1 + \frac{3}{7} + \frac{9}{49} + \cdots + (\frac{3}{7})^n + \cdots$

7 $1 - 1.1 + 1.11 - 1.111 + 1.1111 - \cdots$

8 $$\left(1 - \frac{1}{8}\right) + \left(\frac{1}{8} - \frac{1}{27}\right) + \left(\frac{1}{27} - \frac{1}{64}\right) + \cdots + \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \cdots$$

9 $6 + 19 + 3 + \frac{4}{25} + \frac{8}{125} + \frac{16}{625} + \cdots + (\frac{2}{5})^n + \cdots$

10 $\sum_{n=0}^{\infty} \frac{7^n - 6^n}{5^n}$

11 $\sum_{n=0}^{\infty} \frac{7 \cdot 5^n}{6^n}$

12 $\sum_{n=0}^{\infty} \frac{2n - 3}{5n + 6}$

Test the series 13–23 for convergence.

13 $\sum_{n=0}^{\infty} \frac{3n - 7}{10n + 9}$

14 $\sum_{n=1}^{\infty} \frac{5}{6n^2 + n - 1}$

15 $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1 + 2\sqrt{n} + 3n}$

16 $\sum_{n=1}^{\infty} ne^{-n}$

17 $\sum_{n=2}^{\infty} (\ln(n))^{-n}$

18 $\sum_{n=2}^{\infty} n^{-\ln n}$

19 $\sum_{n=2}^{\infty} \ln n^{-\ln n}$

20 $\sum_{n=3}^{\infty} (-1)^n/\sqrt{\ln n}$

21 $\sum_{n=1}^{\infty} (-1)^n\left(1 - \frac{1}{n^2}\right)$

22 $\sum_{n=1}^{\infty} (-1)^n n^{-1/n}$

23 $\sum_{n=1}^{\infty} (-1)^n \frac{n^{1/n}}{n}$

24 Test the integral $\int_0^{\infty} e^{-\sqrt{x}}\, dx$ for convergence.

25 Test the integral $\int_2^{\infty} (\ln x)^{-x}\, dx$ for convergence.

26 Approximate the series $\sum_{n=1}^{\infty} (-1)^n(1/n^3)$ to three decimal places.

Test the series 27–30 by the Ratio Test.

27 $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

28 $\sum_{n=1}^{\infty} \frac{2^n(n!)}{n^n}$

29 $\sum_{n=2}^{\infty} \frac{n^n}{(\ln n)^n}$

30 $\sum_{n=1}^{\infty} \frac{100^n(n!)^3}{(3n)!}$

Find the radius of convergence of the power series in Problems 31–35.

31 $\sum_{n=0}^{\infty} 2^n n^3 x^n$

32 $\sum_{n=1}^{\infty} n^{-n} x^n$

33 $\sum_{n=1}^{\infty} \frac{x^n}{(n!)^{1/n}}$

34 $\sum_{n=1}^{\infty} x^n \sqrt{n!/n^n}$

35 $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^{2n}$

36 Find the interval of convergence of $\sum_{n=2}^{\infty} (x + 10)^n/(\ln n)$.

37 Find the power series and radius of convergence for $f'(x)$ and $\int_0^x f(t)\, dt$ where

$$f(x) = \sum_{n=1}^{\infty} n^a(n + 1)^b 2^n x^n.$$

38 Find a power series for $f(x) = 1/(1 + 2x^3)$ and determine its radius of convergence.

39 Find a power series for

$$f(x) = \int_0^x \frac{\arctan(t^2)}{t^2}\,dt$$

and determine its radius of convergence.

40 Approximate $\int_0^{1/2} \frac{\arctan(t^2)}{t^2}\,dt$ within 0.0001.

41 Approximate $\int_0^{1/4} t \ln(1-t)\,dt$ within 0.001.

42 Approximate $e^{1/5}$ within 10^{-7}.

43 Approximate $\int_0^{1/2} e^{\sin t}\,dt$ within 0.01.

44 Find a power series for $(1 + x^3)^{-3/2}$ and give its radius of convergence.

45 Find a power series for $\int_0^x (1 + 2t^2)^{-2/3}\,dt$ and determine its radius of convergence.

☐ 46 Prove that any repeating decimal

$$0.b_1b_2\ldots b_nb_1b_2\ldots b_nb_1b_2\ldots b_n\ldots$$

(where each of $b_1, \ldots, b_n$ is a digit from the set $\{0, 1, \ldots, 9\}$) is equal to a rational number.

☐ 47 Approximately how many terms of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/n + \cdots$ are needed to reach a partial sum of at least 50? *Hint:* Compare with $\int_1^n (1/t)\,dt$.

☐ 48 Suppose $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} b_n$ is either finite or ∞. Prove that $\sum_{n=1}^{\infty} (a_n + b_n) = \infty$.

☐ 49 Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent positive term series and $\sum_{n=1}^{\infty} b_n$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$. Prove that $\sum_{n=1}^{\infty} b_n$ converges and has the same sum. *Hint:* Show that each finite partial sum of $\sum_{n=1}^{\infty} a_n$ is less than or equal to each infinite partial sum of $\sum_{n=1}^{\infty} b_n$, and vice versa.

☐ 50 Give a rearrangement of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which diverges to $-\infty$.

☐ 51 Suppose $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = S$, and $a_n \le c_n \le b_n$ for all n. Prove that $\sum_{n=1}^{\infty} c_n = S$.

☐ 52 Prove the following result using the Limit Comparison Test.
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive term series and suppose $\lim_{n\to\infty} (a_n/b_n)$ exists. If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

☐ 53 Multiplication of Power Series.
Prove that if $f(x) = \sum_{n=0}^{\infty} a_nx^n$ and $g(x) = \sum_{n=0}^{\infty} b_nx^n$ then $f(x)g(x) = \sum_{n=0}^{\infty} c_nx^n$ where

$$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0.$$

Hint: First prove the corresponding formula for partial sums, then take the standard part of an infinite partial sum.

☐ 54 Suppose $f(x)$ is the sum of a power series for $|x| < r$ and let $g(x) = f(x^2)$. Prove that for each n,

$$g^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \dfrac{n!}{(n/2)!} f^{(n/2)}(0) & \text{if } n \text{ is even.} \end{cases}$$

☐ 55 Show that if $p \le -1$ then the binomial series

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

diverges at $x = 1$ and $x = -1$. *Hint:* Cauchy Test.

If $p \ge 1$, the series converges at $x = 1$ and $x = -1$. *Hint:* Compare with $\sum_{n=1}^{\infty} 1/n^2$. *Note:* The cases $-1 < p < 1$ are more difficult. It turns out that if $-1 < p < 0$ the series converges at $x = 1$ and diverges at $x = -1$. If $p \ge 0$ the series converges at $x = 1$ and $x = -1$.

☐ 56 Prove that e is irrational, that is, $e \ne a/b$ for all integers a, b. *Hint:* Suppose $e = a/b$, $e^{-1} = b/a$. Let $c = e^{-1} - \sum_{n=0}^{a} (-1)^n/n!$. Then $|c| \ge 1/a!$ but $|c| \le 1/(a+1)!$.