7

TRIGONOMETRIC FUNCTIONS

7.1 TRIGONOMETRY

In this chapter we shall study the *trigonometric functions*, i.e., the sine and cosine function and other functions that are built up from them. Let us start from the beginning and introduce the basic concepts of trigonometry.

The *unit circle* $x^2 + y^2 = 1$ has radius 1 and center at the origin.

Two points P and Q on the unit circle determine an *arc* $\widehat{PQ}$, an *angle* $\angle POQ$, and a *sector* POQ. The arc starts at P and goes counterclockwise to Q along the circle. The sector POQ is the region bounded by the arc $\widehat{PQ}$ and the lines OP and OQ. As Figure 7.1.1 shows, the arcs $\widehat{PQ}$ and $\widehat{QP}$ are different.

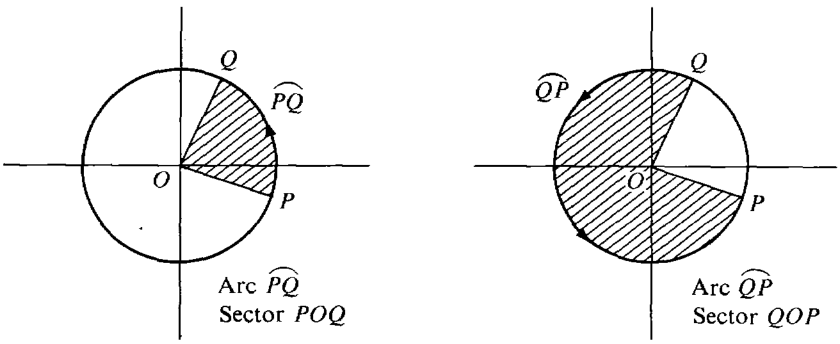


Figure 7.1.1

Trigonometry is based on the notion of the *length* of an arc. Lengths of curves were introduced in Section 6.3. Although that section provides a useful background, this chapter can also be studied independently of Chapter 6. As a starting point we shall give a formula for the length of an arc in terms of the area of a sector. (This formula was proved as a theorem in Section 6.3 but can also be taken as the definition of arc length.)

DEFINITION

*The **length** of an arc $\widehat{PQ}$ on the unit circle is equal to twice the area of the sector POQ, $s = 2A$.*

This formula can be seen intuitively as follows. Consider a small arc $\widehat{PQ}$ of length Δs (Figure 7.1.2). The sector POQ is a thin wedge which is almost a right triangle of altitude one and base Δs. Thus $\Delta A \sim \frac{1}{2}\Delta s$. Making Δs infinitesimal and adding up, we get $A = \frac{1}{2}s$.

The number $\pi \sim 3.14159$ is defined as the area of the unit circle. Thus the unit circle has circumference 2π.

The area of a sector POQ is a definite integral. For example, if P is the point $P(1, 0)$ and the point $Q(x, y)$ is in the first quadrant, then we see from Figure 7.1.3 that the area is

$$A(x) = \tfrac{1}{2}x\sqrt{1 - x^2} + \int_x^1 \sqrt{1 - t^2}\, dt.$$

Notice that $A(x)$ is a continuous function of x. The length of an arc has the following basic property.

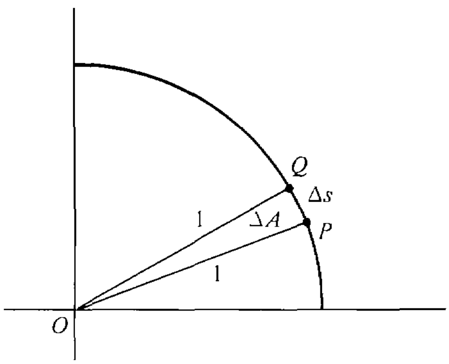


Figure 7.1.2

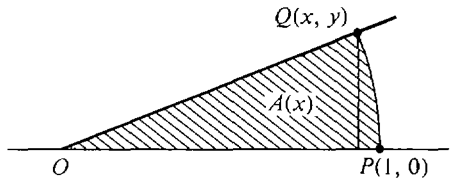


Figure 7.1.3

THEOREM 1

Let P be the point $P(1, 0)$. For every number s between 0 and 2π there is a point Q on the unit circle such that the arc $\widehat{PQ}$ has length s.

PROOF We give the proof for s between 0 and $\pi/2$, whence

$$0 \le \tfrac{1}{2}s \le \pi/4.$$

Let $A(x)$ be the area of the sector POQ where $Q = Q(x, y)$ (Figure 7.1.4). Then $A(0) = \pi/4$, $A(1) = 0$ and the function $A(x)$ is continuous for $0 \le x \le 1$. By the Intermediate Value Theorem there is a point x_0 between 0 and 1 where the sector has area $\frac{1}{2}s$,

$$A(x_0) = \tfrac{1}{2}s.$$

Therefore the arc $\widehat{PQ}$ has length

$$2A(x_0) = s.$$

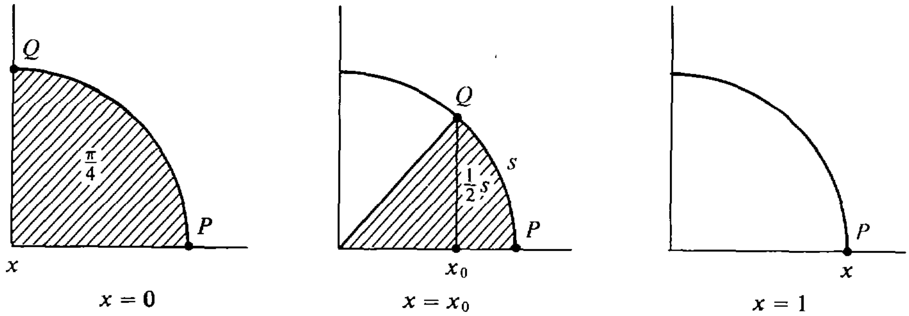


Figure 7.1.4

Arc lengths are used to measure angles. Two units of measurement for angles are radians (best for mathematics) and degrees (used in everyday life).

DEFINITION

*Let P and Q be two points on the unit circle. The measure of the angle $\angle POQ$ in **radians** is the length of the arc $\widehat{PQ}$. A **degree** is defined as*

$$1^\circ = \pi/180 \text{ radians},$$

whence the measure of $\angle POQ$ in degrees is $180/\pi$ times the length of $\widehat{PQ}$.

Approximately, $1^\circ \sim 0.01745$ radians,
$1 \text{ radian} \sim 57^\circ 18' = (57\tfrac{18}{60})^\circ$.

A complete revolution is 360° or 2π radians. A straight angle is 180° or π radians. A right angle is 90° or $\pi/2$ radians.

It is convenient to take the point $(1, 0)$ as a starting point and measure arc length around the unit circle in a counterclockwise direction. Imagine a particle which moves with speed one counterclockwise around the circle and is at the point $(1, 0)$ at time $t = 0$. It will complete a revolution once every 2π units of time. Thus if the particle is at the point P at time t, it will also be at P at all the times $t + 2k\pi$, k an integer. Another way to think of the process is to take a copy of the real line, place the origin at the point $(1, 0)$, and wrap the line around the circle infinitely many times with the positive direction going counterclockwise. Then each point on the circle will correspond to an infinite family of real numbers spaced 2π apart (Figure 7.1.5).

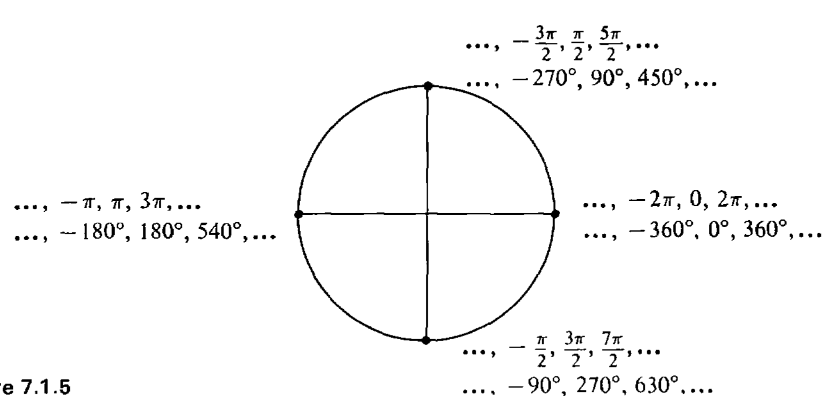


Figure 7.1.5

The Greek letters θ (theta) and ϕ (phi) are often used as variables for angles or circular arc lengths.

DEFINITION

Let $P(x, y)$ be the point at counterclockwise distance θ around the unit circle starting from $(1, 0)$. x is called the ***cosine*** *of θ and y the* ***sine*** *of θ,*

$$x = \cos\theta, \qquad y = \sin\theta.$$

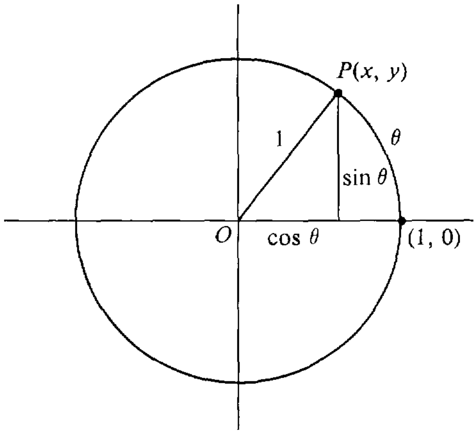


Figure 7.1.6

Cos θ and sin θ are shown in Figure 7.1.6. Geometrically, if θ is between 0 and $\pi/2$ so that the point $P(x, y)$ is in the first quadrant, then the radius OP is the hypotenuse of a right triangle with a vertical side $\sin\theta$ and horizontal side $\cos\theta$. By Theorem 1, $\sin\theta$ and $\cos\theta$ are real functions defined on the whole real line. We write $\sin^n\theta$ for $(\sin\theta)^n$, and $\cos^n\theta$ for $(\cos\theta)^n$. By definition $(\cos\theta, \sin\theta) = (x, y)$ is a point on the unit circle $x^2 + y^2 = 1$, so we always have

$$\sin^2\theta + \cos^2\theta = 1.$$

Also, $$-1 \le \sin\theta \le 1, \qquad -1 \le \cos\theta \le 1.$$

Sin θ and cos θ are *periodic functions* with period 2π. That is,

$$\sin(\theta + 2\pi n) = \sin\theta,$$
$$\cos(\theta + 2\pi n) = \cos\theta$$

for all integers n. The graphs of $\sin\theta$ and $\cos\theta$ are infinitely repeating waves which oscillate between -1 and $+1$ (Figure 7.1.7).

For infinite values of θ, the values of $\sin\theta$ and $\cos\theta$ continue to oscillate between -1 and 1. Thus the limits

$$\lim_{\theta\to\infty}\sin\theta, \qquad \lim_{\theta\to-\infty}\sin\theta,$$
$$\lim_{\theta\to\infty}\cos\theta, \qquad \lim_{\theta\to-\infty}\cos\theta,$$

do not exist. Figure 7.1.8 shows parts of the hyperreal graph of $\sin\theta$, for positive and negative infinite values of θ, through infinite telescopes.

The motion of our particle traveling around the unit circle with speed one

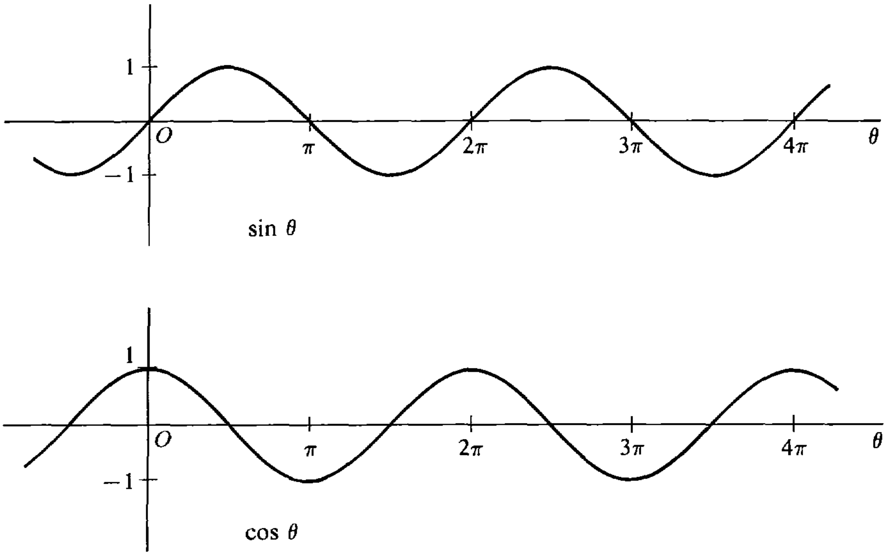


Figure 7.1.7

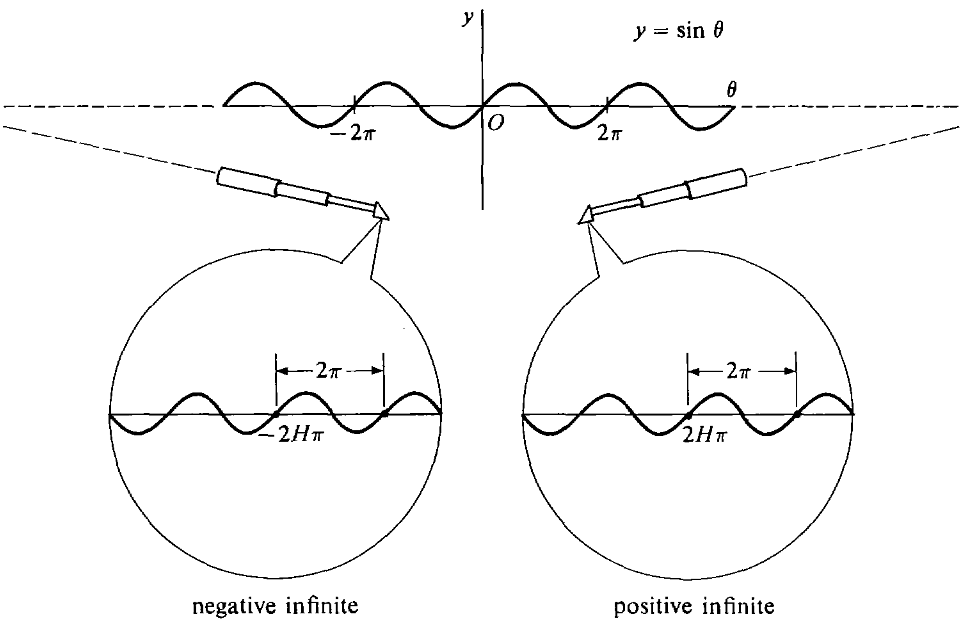


Figure 7.1.8

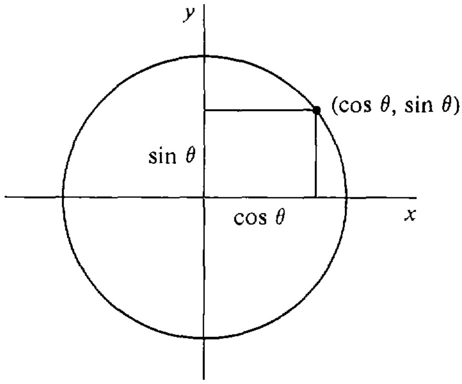


Figure 7.1.9

starting at (1, 0) (Figure 7.1.9) has the parametric equations

$$x = \cos\theta, \qquad y = \sin\theta.$$

The following table shows a few values of $\sin\theta$ and $\cos\theta$, for θ in either radians or degrees.

Table 7.1.1

θ in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
θ in degrees	0°	30°	45°	60°	90°	135°	180°	270°	360°
$\sin\theta$	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{2}/2$	0	-1	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	$-\sqrt{2}/2$	-1	0	1

DEFINITION

The other trigonometric functions are defined as follows.

tangent: $$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

cotangent: $$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

secant: $$\sec\theta = \frac{1}{\cos\theta}$$

cosecant: $$\csc\theta = \frac{1}{\sin\theta}$$

These functions are defined everywhere except where there is a division by zero. They are periodic with period 2π. Their graphs are shown in Figure 7.1.10.

When θ is strictly between 0 and $\pi/2$, trigonometric functions can be described as the ratio of two sides of a right triangle with an angle θ. Let a be the side opposite θ, b the side adjacent to θ, c the hypotenuse as in Figure 7.1.11. Comparing this triangle

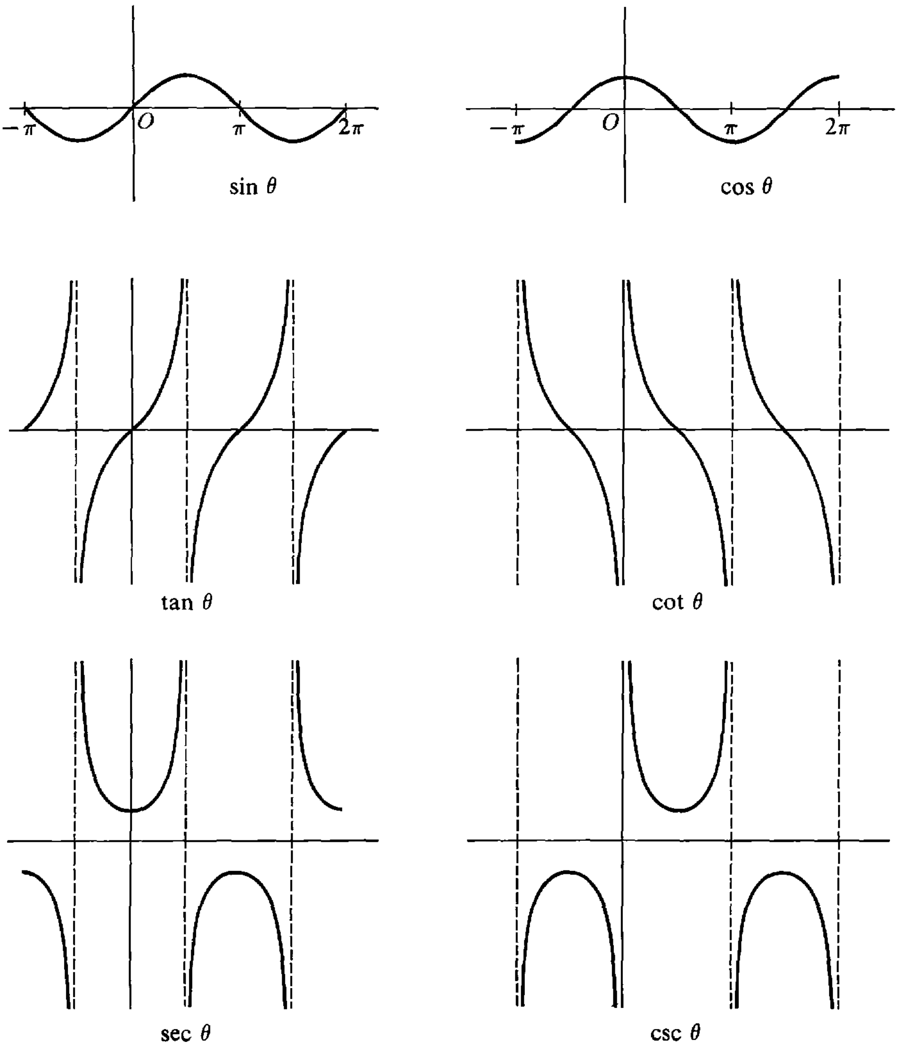


Figure 7.1.10

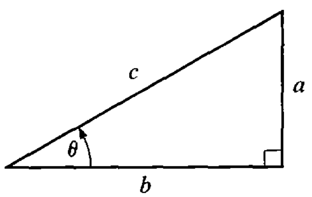


Figure 7.1.11

with a similar triangle whose hypotenuse is a radius of the unit circle, we see that

$$\sin\theta = \frac{a}{c}, \qquad \sec\theta = \frac{c}{b}, \qquad \tan\theta = \frac{a}{b},$$

$$\cos\theta = \frac{b}{c}, \qquad \csc\theta = \frac{c}{a}, \qquad \cot\theta = \frac{b}{a}.$$

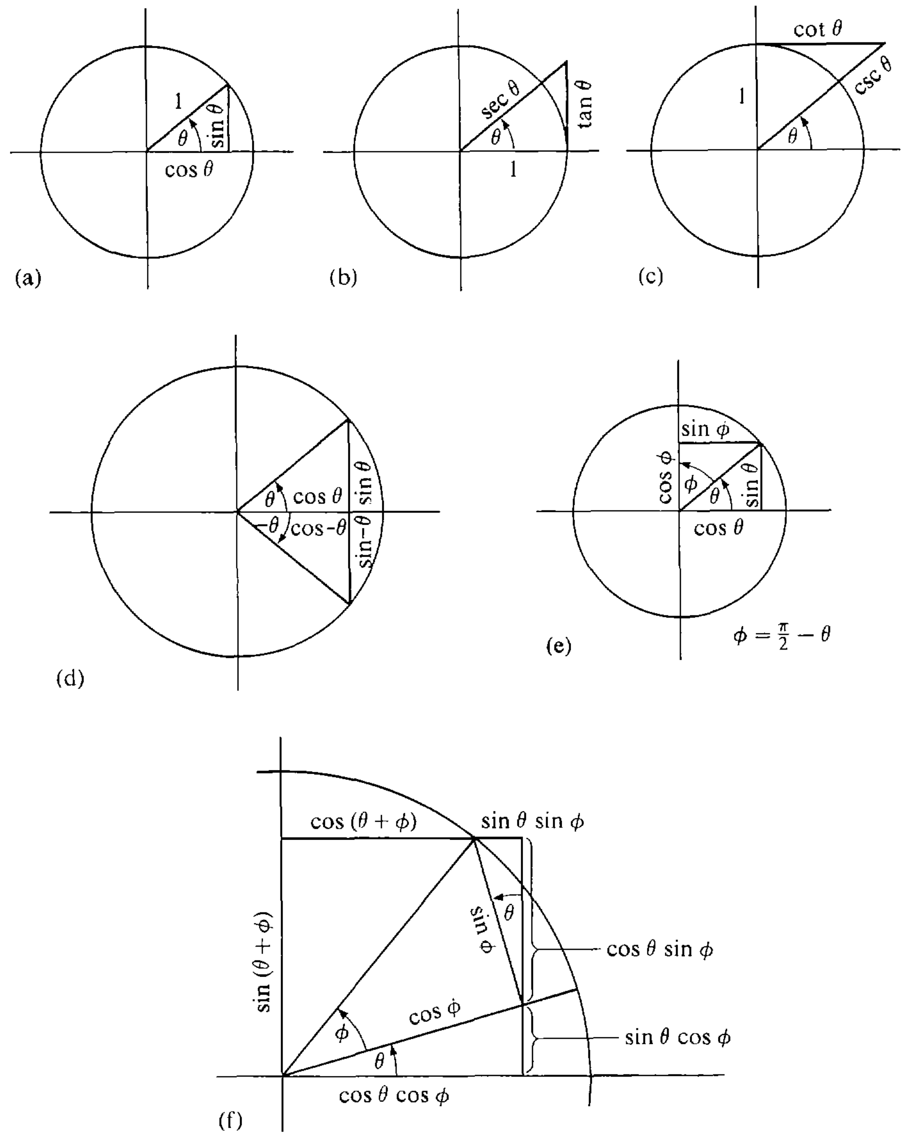


Figure 7.1.12 (Continued)

Here is a table of trigonometric identities. The diagrams in Figure 7.1.12 suggest possible proofs. ((6) and (7) are called the *addition formulas*.)

(1) $\sin^2\theta + \cos^2\theta = 1$ (Figure 7.1.12(a))

(2) $\tan^2\theta + 1 = \sec^2\theta$ (Figure 7.1.12(b))

(3) $\cot^2\theta + 1 = \csc^2\theta$ (Figure 7.1.12(c))

(4) $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$ (Figure 7.1.12(d))

(5) $\sin(\pi/2 - \theta) = \cos\theta$, $\cos(\pi/2 - \theta) = \sin\theta$ (Figure 7.1.12(e))

(6) $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ (Figure 7.1.12(f))

(7) $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ (Figure 7.1.12(f))

PROBLEMS FOR SECTION 7.1

In Problems 1–6, derive the given identity using the formula $\sin^2\theta + \cos^2\theta = 1$ and the addition formulas for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$.

1 $\tan^2\theta + 1 = \sec^2\theta$

2 $\cos^2\theta + \cos^2\theta\cot^2\theta = \cot^2\theta$

3 $\sin 2\theta = 2\sin\theta\cos\theta$

4 $\cos 2\theta = \cos^2\theta - \sin^2\theta$

5 $\sin^2(\frac{1}{2}\theta) = \dfrac{1 - \cos\theta}{2}$

6 $\tan(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

In Problems 7–10, find all values of θ for which the given equation is true.

7 $\sin\theta = \cos\theta$

8 $\sin\theta\cos\theta = 0$

9 $\sec\theta = 0$

10 $5\sin 3\theta = 0$

11 Find a value of θ where $\sin 2\theta$ is not equal to $2\sin\theta$.

Determine whether the limits exist in Problems 12–17.

12 $\lim_{x\to\infty} \sin x$

13 $\lim_{x\to\infty} \dfrac{\sin x}{x}$

14 $\lim_{x\to\infty} x\sin x$

15 $\lim_{x\to 0} x\cos(1/x)$

16 $\lim_{x\to 0} \cot x$

17 $\lim_{x\to 0} \tan x$

18 Find all values of θ where $\tan\theta$ is undefined.

19 Find all values of θ where $\csc\theta$ is undefined.

7.2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

THEOREM 1

The functions $x = \cos\theta$ and $y = \sin\theta$ are continuous for all θ.

PROOF We give the proof for θ in the first quadrant, $0 < \theta < \pi/2$. Let $\Delta\theta$ be infinitesimal and consider Figure 7.2.1.

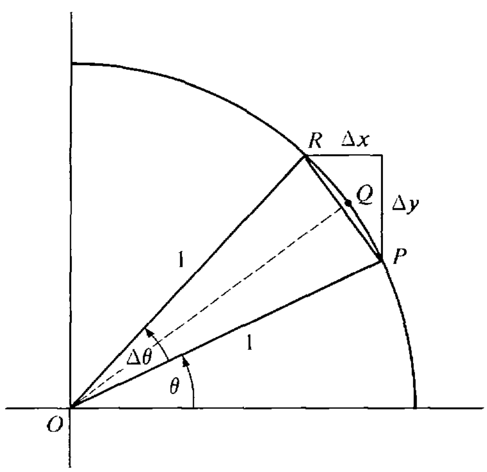


Figure 7.2.1

Let $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$ be the length of the line PR. Then

$$0 < \text{Area of quadrilateral } QPOR \le \text{Area of sector } POR,$$
$$0 < \tfrac{1}{2}\Delta s \le \tfrac{1}{2}\Delta\theta.$$

Thus Δs is infinitesimal. It follows that Δx and Δy are infinitesimal, whence the functions $x = \cos\theta$, $y = \sin\theta$ are continuous.

THEOREM 2

The functions $x = \cos\theta$ *and* $y = \sin\theta$ *are differentiable for all* θ, *and*

$$d(\sin\theta) = \cos\theta\, d\theta,$$
$$d(\cos\theta) = -\sin\theta\, d\theta.$$

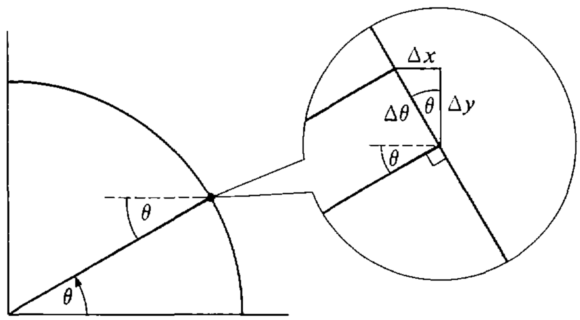


Figure 7.2.2

Discussion Intuitively, the small triangle in Figure 7.2.2 is infinitely close to a right triangle with angle θ and hypotenuse $\Delta\theta$, whence

$$\frac{\Delta y}{\Delta\theta} \approx \cos\theta, \qquad \frac{\Delta x}{\Delta\theta} \approx -\sin\theta.$$

Notice that Δx is negative while Δy is positive when θ is in the first quadrant. The proof of Theorem 2 uses a lemma.

LEMMA

(i) $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$ (ii) $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0.$

PROOF (i) We show that for any nonzero infinitesimal $\Delta\theta$,

$$\frac{\sin\Delta\theta}{\Delta\theta} \approx 1.$$

When $\Delta\theta$ is positive we draw the figure shown in Figure 7.2.3. We have

$$\text{Area of triangle } QOR < \text{area of sector } QOR < \text{area of triangle } QOS,$$
$$\tfrac{1}{2}\sin\Delta\theta < \tfrac{1}{2}\Delta\theta < \tfrac{1}{2}\tan\Delta\theta.$$

Then $$\frac{\sin\Delta\theta}{\tan\Delta\theta} < \frac{\sin\Delta\theta}{\Delta\theta} < \frac{\sin\Delta\theta}{\sin\Delta\theta}, \qquad \cos\Delta\theta < \frac{\sin\Delta\theta}{\Delta\theta} < 1.$$

Since $\cos\theta$ is continuous, $\cos\Delta\theta \approx 1$, whence $\dfrac{\sin\Delta\theta}{\Delta\theta} \approx 1$. The case $\Delta\theta < 0$ is similar.

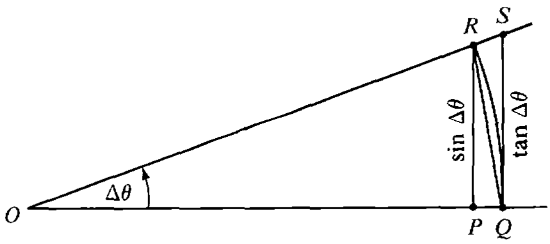


Figure 7.2.3

(ii) We compute the standard part of $(\cos \Delta\theta - 1)/\Delta\theta$.

$$st\left(\frac{\cos \Delta\theta - 1}{\Delta\theta}\right) = st\left(\frac{\cos^2 \Delta\theta - 1}{\Delta\theta(\cos \Delta\theta + 1)}\right) = st\left(\frac{-\sin^2 \Delta\theta}{\Delta\theta(\cos \Delta\theta + 1)}\right)$$

$$= -st\left(\frac{\sin \Delta\theta}{\Delta\theta}\right)\frac{st(\sin \Delta\theta)}{st(\cos \Delta\theta + 1)} = -1 \cdot \frac{0}{2} = 0.$$

PROOF OF THEOREM 2 Let $\Delta\theta$ be a nonzero infinitesimal. Then

$$\frac{d(\sin \theta)}{d\theta} = st\left(\frac{\sin(\theta + \Delta\theta) - \sin \theta}{\Delta\theta}\right)$$

$$= st\left(\frac{\sin \theta \cos \Delta\theta + \cos \theta \sin \Delta\theta - \sin \theta}{\Delta\theta}\right)$$

$$= st\left(\frac{\sin \theta(\cos \Delta\theta - 1) + \cos \theta \sin \Delta\theta}{\Delta\theta}\right)$$

$$= \sin \theta \, st\left(\frac{\cos \Delta\theta - 1}{\Delta\theta}\right) + \cos \theta \, st\left(\frac{\sin \Delta\theta}{\Delta\theta}\right)$$

$$= \sin \theta \cdot 0 + \cos \theta \cdot 1 = \cos \theta.$$

Here is a second proof that the derivative of the sine is the cosine. It uses the formula for the length of a curve in Section 6.3.

ALTERNATE PROOF OF THEOREM 2 (Optional) Let $0 \le \theta \le \pi/2$ and

$$x = \cos \theta, \qquad y = \sin \theta.$$

Then (x, y) is a point on the unit circle as shown in Figure 7.2.4.

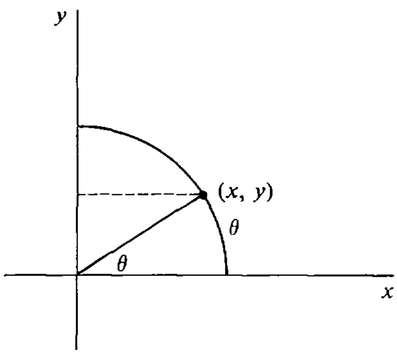


Figure 7.2.4

Take y as the independent variable. Then

$$x = \sqrt{1 - y^2}, \qquad \frac{dx}{dy} = \frac{-y}{\sqrt{1 - y^2}} = -\frac{y}{x}.$$

θ is the length of the arc from 0 to y, so

$$\theta = \int_0^y \sqrt{1 + (dx/dy)^2}\, dy.$$

By the Second Fundamental Theorem of Calculus,

$$\frac{d\theta}{dy} = \sqrt{1 + (dx/dy)^2} = \sqrt{1 + (y^2/x^2)} = \frac{\sqrt{x^2 + y^2}}{x} = \frac{1}{x}.$$

Then by the Chain Rule,

$$\frac{dy}{d\theta} = \frac{1}{d\theta/dy} = x,$$

and

$$\frac{dx}{d\theta} = \frac{dx}{dy}\frac{dy}{d\theta} = -\frac{y}{x} \cdot x = -y.$$

Substituting $\cos\theta$ for x and $\sin\theta$ for y,

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta, \qquad \frac{d(\cos\theta)}{d\theta} = -\sin\theta.$$

We can now find the derivatives of all the trigonometric functions by using the Quotient Rule

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

THEOREM 3

(i) $d(\sin\theta) = \cos\theta\, d\theta, \qquad d(\cos\theta) = -\sin\theta\, d\theta,$

(ii) $d(\tan\theta) = \sec^2\theta\, d\theta, \qquad d(\cot\theta) = -\csc^2\theta\, d\theta,$

(iii) $d(\sec\theta) = \sec\theta\tan\theta\, d\theta, \qquad d(\csc\theta) = -\csc\theta\cot\theta\, d\theta.$

PROOF We prove the formula for $d(\tan\theta)$ and leave the rest as problems.

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

$$d(\tan\theta) = d\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\cos\theta\, d(\sin\theta) - \sin\theta\, d(\cos\theta)}{\cos^2\theta}$$

$$= \frac{\cos\theta\cos\theta - \sin\theta(-\sin\theta)}{\cos^2\theta}d\theta = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}d\theta$$

$$= \frac{1}{\cos^2\theta}d\theta = \sec^2\theta\, d\theta.$$

These formulas lead at once to new integration formulas.

THEOREM 4

(i) $\int \cos\theta\, d\theta = \sin\theta + C, \qquad \int \sin\theta\, d\theta = -\cos\theta + C.$

(ii) $\int \sec^2\theta\, d\theta = \tan\theta + C, \qquad \int \csc^2\theta\, d\theta = -\cot\theta + C.$

(iii) $\int \sec\theta\tan\theta\, d\theta = \sec\theta + C, \qquad \int \csc\theta\cot\theta\, d\theta = -\csc\theta + C.$

We are not yet able to evaluate the integrals $\int \tan\theta\, d\theta$, $\int \cot\theta\, d\theta$, $\int \sec\theta\, d\theta$, $\int \csc\theta\, d\theta$. These integrals will be found in the next chapter.

EXAMPLE 1 Find the derivative of $y = \tan^2(3x)$.

$$dy = 2\tan 3x\, d(\tan 3x) = 2\tan 3x \sec^2 3x\, d(3x)$$
$$= 6\tan 3x \sec^2 3x\, dx.$$

EXAMPLE 2 Evaluate $\lim_{t\to\pi/2} \dfrac{\cos t}{t - \pi/2}$.

This is a limit of the form 0/0 because

$$\lim_{t\to\pi/2} \cos t = 0, \qquad \lim_{t\to\pi/2}\left(t - \frac{\pi}{2}\right) = 0.$$

By l'Hospital's Rule (Section 5.2),

$$\lim_{t\to\pi/2} \frac{\cos t}{t - \pi/2} = \lim_{t\to\pi/2} \frac{-\sin t}{1} = -\sin\left(\frac{\pi}{2}\right) = -1.$$

EXAMPLE 3 A particle travels around a vertical circle of radius r_0 with constant angular velocity $\omega = d\theta/dt$, beginning with $\theta = 0$ at time $t = 0$. If the sun is directly overhead, find the position, velocity, and acceleration of the shadow.

Let us center the circle at the origin in the (x, y) plane (Figure 7.2.5). Then

$$x = r_0\cos\theta, \qquad y = r_0\sin\theta.$$

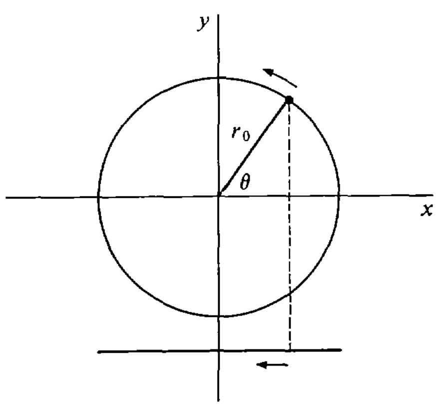


Figure 7.2.5

At time t, θ has the value $\theta = \omega t$. So the motion of the particle is given by the parametric equations

$$x = r_0 \cos(\omega t), \qquad y = r_0 \sin(\omega t).$$

The shadow is directly below the particle, and its position is given by the x-component

$$x = r_0 \cos(\omega t).$$

The velocity and acceleration of the shadow are

$$v = \frac{dx}{dt} = -r_0 \omega \sin(\omega t),$$

$$a = \frac{dv}{dt} = -r_0 \omega^2 \cos(\omega t).$$

EXAMPLE 4 A light beam on a 100 ft tower rotates in a vertical circle at the rate of one revolution per second. Find the speed of the spot of light moving along the ground at a point 1000 ft from the base of the tower.

We start by drawing the picture in Figure 7.2.6.

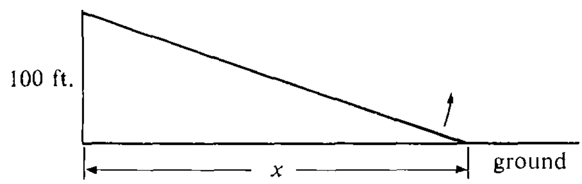


Figure 7.2.6

Assume the rotation is counterclockwise. Let t be time and let x and θ be as in the figure. Then

$$\frac{d\theta}{dt} = 2\pi \text{ radians/sec}, \qquad x = 100 \tan \theta \text{ ft}.$$

We wish to find dx/dt when $x = 1000$.

$$\frac{dx}{dt} = 100 \sec^2 \theta \frac{d\theta}{dt} = 200\pi \sec^2 \theta.$$

When $x = 1000$,

$$\sec^2 \theta = 1 + \tan^2 \theta = 1 + (x/100)^2 = 1 + 10^2 = 101.$$

Therefore $$\frac{dx}{dt} = 20200\pi \sim 63{,}000 \text{ ft/sec}.$$

EXAMPLE 5 Find $\int \sin^3 t \cos t\, dt$. Let $u = \sin t$, $du = \cos t\, dt$.

Then $$\int \sin^3 t \cos t\, dt = \int u^3\, du = \frac{u^4}{4} + C = \frac{\sin^4 t}{4} + C.$$

EXAMPLE 6 Find the area under one arch of the curve $y = \cos x$.

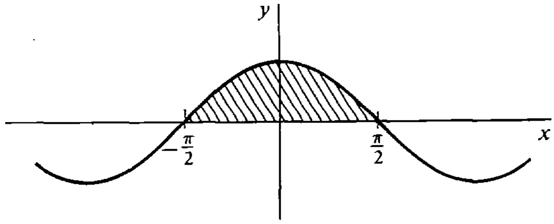


Figure 7.2.7

From Figure 7.2.7 we see that one arch lies between the limits $x = -\pi/2$ and $x = \pi/2$, therefore the area is

$$\int_{-\pi/2}^{\pi/2} \cos t\, dt = \sin t \Big]_{-\pi/2}^{\pi/2} = 1 - (-1) = 2.$$

Trigonometric identities can often be used to get an integral into a form which is easy to evaluate.

EXAMPLE 7 Evaluate $\int \sec^4 x\, dx$. Using the identity $\sec^2 x = 1 + \tan^2 x$, we have

$$\int \sec^4 x\, dx = \int (1 + \tan^2 x)\sec^2 x\, dx$$
$$= \int (1 + \tan^2 x)\, d(\tan x) = \tan x + \frac{\tan^3 x}{3} + C.$$

EXAMPLE 8 Find $\int \sqrt{1 - \cos x}\, dx$. Using the identity $\sin^2 x + \cos^2 x = 1$, we have

$$\sqrt{1 - \cos x} = \frac{\sqrt{1 - \cos x}\sqrt{1 + \cos x}}{\sqrt{1 + \cos x}} = \frac{\sqrt{1 - \cos^2 x}}{\sqrt{1 + \cos x}}$$
$$= \frac{\sqrt{\sin^2 x}}{\sqrt{1 + \cos x}} = \frac{|\sin x|}{\sqrt{1 + \cos x}}.$$

Case 1 In an interval where $\sin x \geq 0$,

$$\int \sqrt{1 - \cos x}\, dx = \int \frac{\sin x}{\sqrt{1 + \cos x}}\, dx = \int -\frac{1}{\sqrt{1 + \cos x}}\, d(1 + \cos x)$$
$$= -2\sqrt{1 + \cos x} + C.$$

Case 2 In an interval where $\sin x \leq 0$,

$$\int \sqrt{1 - \cos x}\, dx = 2\sqrt{1 + \cos x} + C.$$

PROBLEMS FOR SECTION 7.2

In Problems 1–14, find the derivative.

1 $y = \sin 5x$

2 $y = 3\cos^2 x$

3 $x = \sin(3\theta^2)$

4 $y = \sec^3 x$

5 $x = \tan(4\theta - 3)$

6 $y = x \sin x$

7 $u = a \sin \theta + b \cos \theta$

8 $u = \sin(a\theta) + \cos(b\theta)$

9 $y = \cos \sqrt{x}$

10 $y = \sqrt{\cos x}$

11 $y = \tan(\sin \theta)$

12 $y = \sin \theta \tan \theta$

13 $u = \dfrac{1}{2 + \csc(3t)}$

14 $y = \cot(t^2 + 3t - 2)$

15 Find dy/dx where $x = \sin^2 y$

16 Find dy/dx where $y = \tan(xy)$

In Problems 17–24, evaluate the limit if it exists.

17 $\lim_{\theta \to \pi/3} 2 \sin^2 \theta$

18 $\lim_{x \to 0} \csc x$

19 $\lim_{x \to 0^+} \csc x$

20 $\lim_{t \to 0} \dfrac{\sin^2 t}{t}$

21 $\lim_{t \to 0} \dfrac{\sin(2t)}{t}$

22 $\lim_{\theta \to \pi} \dfrac{\sin \theta}{\pi - \theta}$

23 $\lim_{t \to 0} \dfrac{\sin(t^2)}{t \sin t}$

24 $\lim_{\theta \to \pi/2} (\sec \theta - \tan \theta)$

In Problems 25–34, find the maxima, minima, inflection points, and limits when necessary, and sketch the curve for $0 \le x \le 2\pi$.

25 $y = 3 \sin x$

26 $y = \sin x \cos x$

27 $y = \sin^2 x$

28 $y = \cos(2x)$

29 $y = \sin\left(x - \dfrac{\pi}{4}\right)$

30 $y = \sec x$

31 $y = \tan x$

32 $y = 1 - \cos x$

33 $y = \csc^2 x$

34 $y = x + \sin x$

☐ 35 Show that at $\lim_{x \to 0} \sin(1/x)$ does not exist.

☐ 36 Let $f(x) = x \sin(1/x)$, with $f(0) = 0$. Show that f is continuous but not differentiable at $x = 0$.

In Problems 37–53, evaluate the integral.

37 $\int \sin(2t)\,dt$

38 $\int \sin x \cos x\,dx$

39 $\int \tan x \sec^3 x\,dx$

40 $\int \tan^2 \theta\,d\theta$

41 $\int \dfrac{1}{\sqrt{x}} \cos \sqrt{x}\,dx$

42 $\int t \sin(t^2 + 1)\,dt$

43 $\int \cot(5\theta) \csc(5\theta)\,d\theta$

44 $\int \sqrt{1 + \sin \theta}\,d\theta$

45 $\int \sec x \sqrt{\sec x - 1}\,dx$

46 $\int \dfrac{\sin \theta - \cos \theta}{(\sin \theta + \cos \theta)^2}\,d\theta$

47 $\int \dfrac{1}{1 + \sin \theta}\,d\theta$

48 $\int_0^{\pi} 3 \sin t\,dt$

49 $\int_{-\pi/4}^{\pi/4} \sec^2 \theta\,d\theta$

50 $\int_0^1 \sin(\pi x)\,dx$

51 $\int_{\pi/3}^{\pi/2} \sin\theta + \cos\theta \, d\theta$

52 $\int_0^{\pi/2} \sec^2 x \, dx$

53 $\int_0^{\pi/2} \cot x \csc x \, dx$

54 A revolving light one mile from shore sweeps out eight revolutions per minute. Find the velocity of the beam of light along the shore at the instant when it makes an angle of 45° with the shoreline.

55 A ball is thrown vertically upward from a point P so that its height at time t is $y = 100t - 16t^2$ feet. Q is another point on the surface 100 ft from P. At time $t = 5$ find the rate of change of the angle between the horizontal line QP and the line from Q to the ball.

56 Two hallways of width a and b meet at right angles. Find the length of the longest rod which can be slid on the floor around the corner.

57 Find the area under one arch of the curve $y = 3 \sin x$.

58 Find the area under one arch of the curve $y = \sin(3x)$.

59 Find the area of the region between the curves $y = \sin x \cos x$ and $y = \sin x, 0 \le x \le \pi/2$.

60 The region between the x-axis and the curve $y = \tan x, 0 \le x \le \pi/4$, is rotated about the x-axis. Find the volume of the solid of revolution.

61 The region between the x-axis and the curve $y = (\sin x)/x, \pi/2 \le x \le \pi$, is rotated about the y-axis. Find the volume of the solid of revolution.

62 Find the length of the parametric curve $x = 2\cos(3t), y = 2\sin(3t), 0 \le t \le 1$.

63 Find the length of the parametric curve $x = \cos^2 t, y = \sin^2 t, 0 \le t \le \pi/2$.

64 Find the length of the parametric curve $x = \cos^3 t, y = \sin^3 t, 0 \le t \le \pi/2$.

65 Find the area of the surface generated by rotating the curve in Problem 63 about the x-axis.

66 Find the area of the surface generated by rotating the curve in Problem 64 about the y-axis.

7.3 INVERSE TRIGONOMETRIC FUNCTIONS

Inverse functions were studied in Section 2.4. We now take up the topic again and apply it to trigonometric functions. A *binary relation* on the real numbers is any set of ordered pairs of real numbers. Thus a real function f of one variable is a binary relation such that for each x, either there is exactly one y with (x, y) in f or there is no y with (x, y) in f. (Other important relations are $x < y, x \le y, x \ne y, x = y$.)

DEFINITION

Let S be a binary relation on the real numbers. The ***inverse relation*** *of S is the set T of all ordered pairs (y, x) such that (x, y) is in S. If S and T are both functions they are called* ***inverse functions*** *of each other.*

The inverse of a function f may or may not be a function. For example, the inverse of $y = x^2$ is the relation $x = \pm\sqrt{y}$, which is not a function (Figure 7.3.1). But the inverse of $y = x^2, x \ge 0$, is the function $x = \sqrt{y}$ (Figure 7.3.2).

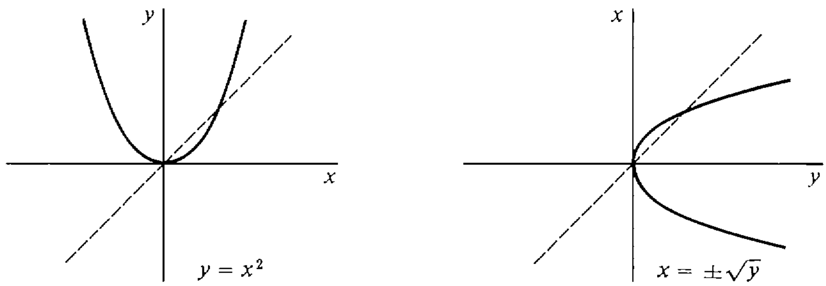


Figure 7.3.1

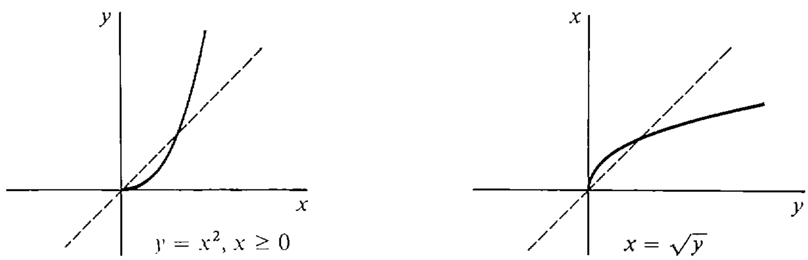


Figure 7.3.2

Geometrically, the graph of the inverse relation of $y = f(x)$ can be obtained by flipping the graph of $y = f(x)$ about the diagonal line $y = x$ (the dotted line in Figures 7.3.1 and 7.3.2). This flipping interchanges the x- and y-axes. This is because $f(x) = y$ means (x, y) is in f, and $g(y) = x$ means (y, x) is in g. It follows that:

If f and g are inverse functions then the range of f is the domain of g and vice versa.

Which functions have inverse functions? We can answer this question with a definition and a simple theorem.

DEFINITION

A real function f with domain X is said to be one-to-one if f never takes the same value twice, that is, for all $x_1 \neq x_2$ in X we have $f(x_1) \neq f(x_2)$.

THEOREM 1

f has an inverse function if and only if f is one-to-one.

PROOF The following statements are equivalent.

(1) f is a one-to-one function.
(2) For every y, either there is exactly one x with $f(x) = y$ or there is no x with $f(x) = y$.
(3) The equation $y = f(x)$ determines x as a function of y.
(4) f has an inverse function.

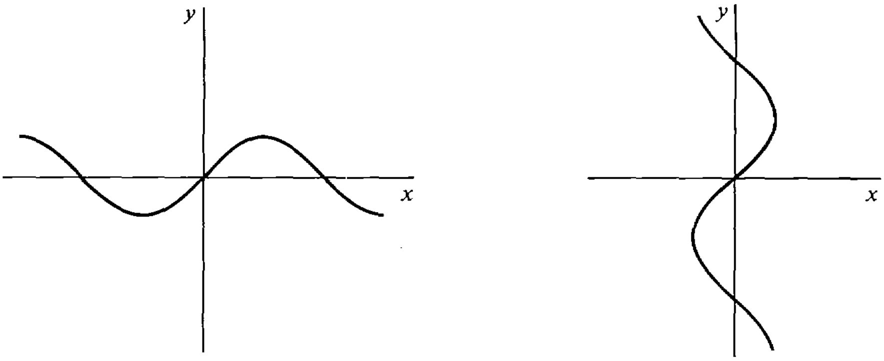
COROLLARY

Every function which is increasing on its domain I has an inverse function. So does every function decreasing on its domain I.

PROOF Let f be increasing on I. For any two points $x_1 \neq x_2$ in I, the value of f at the smaller of x_1, x_2 is less than the value of f at the greater, so $f(x_1) \neq f(x_2)$.

For example, the function $y = x^2$ is not one-to-one because $(-1)^2 = 1^2$, whence it has no inverse function. The function $y = x^2, x \geq 0$, is increasing on its domain $[0, \infty)$ and thus has an inverse.

Now let us examine the trigonometric functions. The function $y = \sin x$ is not one-to-one. For example, $\sin 0 = 0, \sin \pi = 0, \sin 2\pi = 0$, etc. We can see in Figure 7.3.3 that the inverse relation of $y = \sin x$ is not a function.



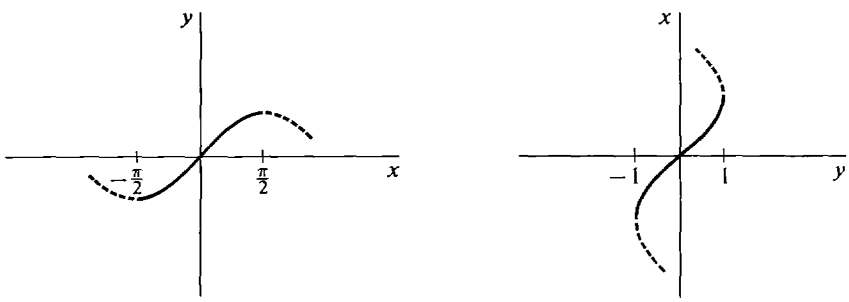
$y = \sin x$ not a function

Figure 7.3.3

However, the function $y = \sin x$ is increasing on the interval $[-\pi/2, \pi/2]$, because its derivative $\cos x$ is ≥ 0. So the sine function restricted to the interval $[-\pi/2, \pi/2]$,

$$y = \sin x, \qquad -\pi/2 \leq x \leq \pi/2,$$

has an inverse function shown in Figure 7.3.4. This inverse is called the *arcsine*



$y = \sin x, \ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ $x = \arcsin y$

Figure 7.3.4

function. It is written $x = \arcsin y$. Verbally, $\arcsin y$ is the angle x between $-\pi/2$ and $\pi/2$ whose sine is y.

The other trigonometric functions also are not one-to-one and thus do not have inverse functions. However, in each case we obtain a one-to-one function by restricting the domain to a suitable interval, either $[-\pi/2, \pi/2]$ or $[0, \pi]$. The resulting inverse functions are called the arccosine, arctangent, etc.

DEFINITION

The inverse trigonometric functions are defined as follows.

$x = \arcsin y$ *is the inverse of* $y = \sin x$, $\quad -\pi/2 \le x \le \pi/2$

$x = \arccos y$ *is the inverse of* $y = \cos x$, $\quad 0 \le x \le \pi$

$x = \arctan y$ *is the inverse of* $y = \tan x$, $\quad -\pi/2 \le x \le \pi/2$

$x = \operatorname{arccot} y$ *is the inverse of* $y = \cot x$, $\quad 0 \le x \le \pi$

$x = \operatorname{arcsec} y$ *is the inverse of* $y = \sec x$, $\quad 0 \le x \le \pi$

$x = \operatorname{arccsc} y$ *is the inverse of* $y = \csc x$, $\quad -\pi/2 \le x \le \pi/2$

The graphs of these functions are shown in Figure 7.3.5. The domains of the inverse trigonometric functions can be read off from the graphs, and are shown in the table below.

Table 7.3.1

Function	Domain
$\arcsin y$	$-1 \le y \le 1$
$\arccos y$	$-1 \le y \le 1$
$\arctan y$	whole real line
$\operatorname{arccot} y$	whole real line
$\operatorname{arcsec} y$	$y \le -1, \quad y \ge 1$
$\operatorname{arccsc} y$	$y \le -1, \quad y \ge 1$

We can prove the inverse trigonometric functions have these domains (i.e., the figures are correct) using the Intermediate Value Theorem. As an illustration we prove that $\arcsin y$ has domain $[-1, 1]$.

$\arcsin y$ is undefined outside $[-1, 1]$ because $-1 \le \sin x \le 1$ for all x. Suppose y_0 is in $[-1, 1]$. Then

$$\sin(-\pi/2) = -1 \le y_0 \le 1 = \sin(\pi/2).$$

$\sin x$ is continuous, so by the Intermediate Value Theorem there exists x_0 between $-\pi/2$ and $\pi/2$ such that $\sin x_0 = y_0$. Thus

$$\arcsin y_0 = x_0$$

and y_0 is in the domain of $\arcsin y$.

EXAMPLE 1 Find $\arccos(\sqrt{2}/2)$. From Table 7.1.1, $\cos(\pi/4) = \sqrt{2}/2$. Since $0 \le \pi/4 \le \pi$,

$$\arccos(\sqrt{2}/2) = \pi/4.$$

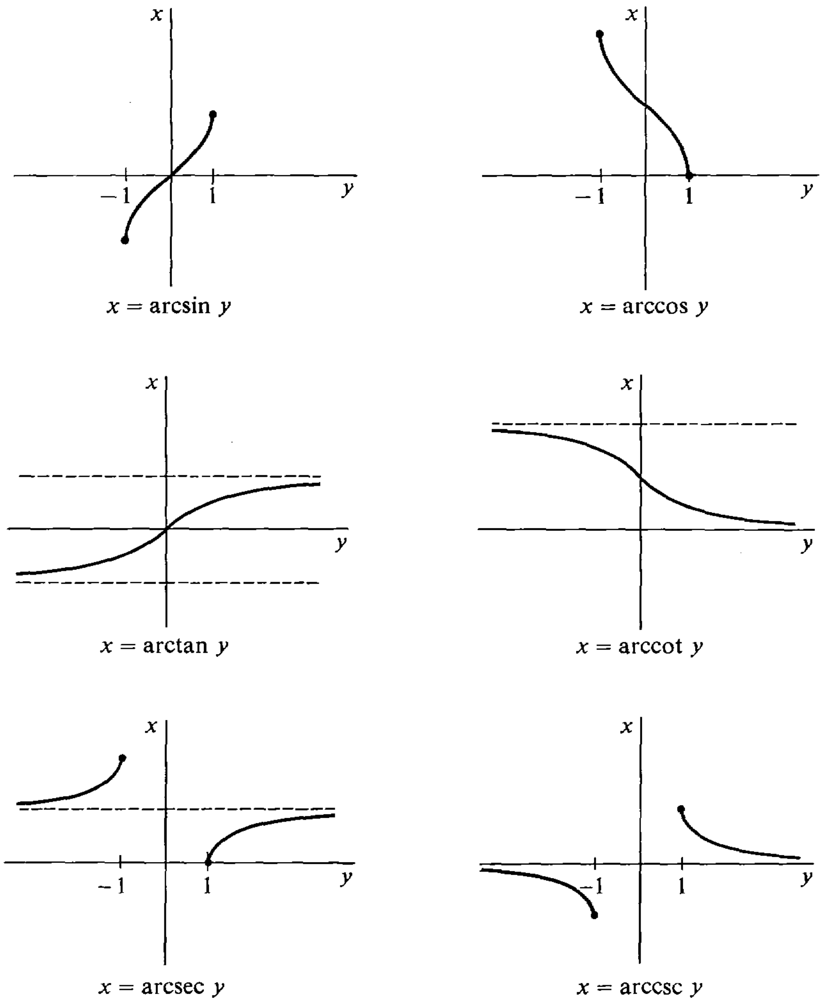


Figure 7.3.5

EXAMPLE 2 Find $\arcsin(-1)$. From Table 7.1.1, $\sin(3\pi/2) = -1$. But $3\pi/2$ is not in the interval $[-\pi/2, \pi/2]$. Using $\sin(\theta + 2n\pi) = \sin\theta$, we have

$$\sin(-\pi/2) = \sin(3\pi/2) = -1,$$

so

$$\arcsin(-1) = -\pi/2.$$

EXAMPLE 3 Find $\arctan(-\sqrt{3})$. We must find a θ in the interval $[-\pi/2, \pi/2]$ such that $\tan\theta = -\sqrt{3}$. From Table 7.1.1, $\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/3) = 1/2$. Then $\sin(-\pi/3) = -\sqrt{3}/2$, $\cos(-\pi/3) = 1/2$. So

$$\tan(-\pi/3) = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3},$$

$$\arctan(-\sqrt{3}) = -\pi/3.$$

EXAMPLE 4 Find cos (arctan y). Let $\theta = \arctan y$. Thus $\tan\theta = y$. Using

$$\sin^2\theta + \cos^2\theta = 1$$
$$\frac{\sin\theta}{\cos\theta} = y,$$

we solve for $\cos\theta$.

$$\sin\theta = y\cos\theta, \qquad (y\cos\theta)^2 + \cos^2\theta = 1,$$
$$\cos^2\theta\,(y^2+1) = 1, \qquad \cos^2\theta = \frac{1}{y^2+1}.$$

Thus $$\cos\theta = \pm\frac{1}{\sqrt{y^2+1}}.$$

By definition of arctan y, we know that $-\pi/2 \le \theta \le \pi/2$. In this interval, $\cos\theta \ge 0$. Therefore

$$\cos\theta = \frac{1}{\sqrt{y^2+1}}.$$

EXAMPLE 5 Show that $\arcsin y + \arccos y = \pi/2$ (Figure 7.3.6). Let $\theta = \arcsin y$. We have $y = \sin\theta = \cos(\pi/2 - \theta)$. Also, when $-\pi/2 \le \theta \le \pi/2$, we have

$$\pi/2 \ge -\theta \ge -\pi/2, \qquad \pi \ge \pi/2 - \theta \ge 0.$$

Thus $$\pi/2 - \theta = \arccos y,$$
$$\arcsin y + \arccos y = \theta + (\pi/2 - \theta) = \pi/2.$$

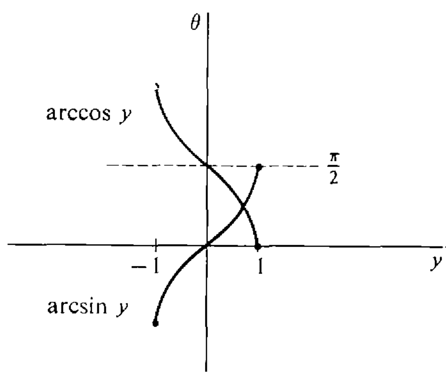


Figure 7.3.6

We shall next study the derivatives of the inverse trigonometric functions. Here is a general theorem which tells us when the derivative of the inverse function exists and gives a rule for computing its value.

INVERSE FUNCTION THEOREM

Suppose a real function f is differentiable on an open interval I and f has an inverse function g. Let x be a point in I where $f'(x) \ne 0$ and let $y = f(x)$. Then

(i) *$g'(y)$ exists,*

(ii) $g'(y) = \dfrac{1}{f'(x)}$.

We omit the proof that $g'(y)$ exists. Intuitively, the curve $y = f(x)$ has a nonhorizontal tangent line, so the curve $x = g(y)$ should have a nonvertical tangent line and thus $g'(y)$ should exist.

The Inverse Function Rule from Chapter 2 says that (ii) is true if we assume (i). The proof of (ii) from (i) is an application of the Chain Rule:

$$g(f(x)) = x, \qquad g'(f(x))f'(x) = 1, \qquad g'(y)f'(x) = 1, \qquad g'(y) = \frac{1}{f'(x)}.$$

The Inverse Function Theorem shows that all the inverse trigonometric functions have derivatives. We now evaluate these derivatives.

THEOREM 2

(i) $d(\arcsin x) = \dfrac{dx}{\sqrt{1-x^2}}$ (*where* $-1 < x < 1$).

$d(\arccos x) = -\dfrac{dx}{\sqrt{1-x^2}}$ (*where* $-1 < x < 1$).

(ii) $d(\arctan x) = \dfrac{dx}{1+x^2}$.

$d(\operatorname{arccot} x) = -\dfrac{dx}{1+x^2}$.

(iii) $d(\operatorname{arcsec} x) = \dfrac{dx}{|x|\sqrt{x^2-1}}$ (*where* $|x| > 1$).

$d(\operatorname{arccsc} x) = -\dfrac{dx}{|x|\sqrt{x^2-1}}$ (*where* $|x| > 1$).

PROOF We prove the first part of (i) and (iii). Since the derivatives exist we may use implicit differentiation.

(i) Let $y = \arcsin x$. Then

$$x = \sin y, \qquad -\pi/2 \le y \le \pi/2,$$
$$dx = \cos y\, dy.$$

From $\sin^2 y + \cos^2 y = 1$ we get

$$\cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}.$$

Since $-\pi/2 \le y \le \pi/2$, $\cos y \ge 0$. Then

$$\cos y = \sqrt{1-x^2}.$$

Substituting, $\quad dx = \sqrt{1-x^2}\, dy, \qquad dy = \dfrac{dx}{\sqrt{1-x^2}}.$

(iii) Let $y = \operatorname{arcsec} x$.

Then
$$x = \sec y, \qquad 0 \le y \le \pi,$$
$$dx = \sec y \tan y \, dy.$$

From $\tan^2 y + 1 = \sec^2 y$ we get $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$.

Since $0 \le y \le \pi$, $\tan y$ and $\sec y = \dfrac{1}{\cos y}$ have the same sign.

Therefore
$$\sec y \tan y \ge 0$$
and
$$dx = |\sec y||\tan y| \, dy = |x|\sqrt{x^2 - 1}\, dy,$$
$$dy = \frac{dx}{|x|\sqrt{x^2 - 1}}.$$

When we turn these formulas for derivatives around we get some surprising new integration formulas.

THEOREM 3

(i) $\displaystyle\int \frac{1}{\sqrt{1 - x^2}} dx = \arcsin x + C = -\arccos x + C.$ (*Provided that* $|x| < 1$).

(ii) $\displaystyle\int \frac{dx}{1 + x^2} = \arctan x + C = -\operatorname{arccot} x + C.$

(iii) $\displaystyle\int \frac{dx}{|x|\sqrt{x^2 - 1}} = \operatorname{arcsec} x + C = -\operatorname{arccsc} x + C.$ (*Provided that* $|x| > 1$).

From part (i), $\arcsin x$ and $-\arccos x$ must differ only by a constant. We already knew this from Example 5,

$$\arcsin x = -\arccos x + \pi/2.$$

Before now we were not able to find the area of the regions under the curves

$$y = \frac{1}{\sqrt{1 - x^2}}, \qquad y = \frac{1}{1 + x^2}, \qquad y = \frac{1}{x\sqrt{x^2 - 1}}.$$

It is a remarkable and quite unexpected fact that these areas are given by inverse trigonometric functions.

EXAMPLE 6 (a) Find the area of the region under the curve

$$y = \frac{1}{1 + x^2}$$

for $-1 \le x \le 1$.

(b) Find the area of the region under the same curve for $-\infty < x < \infty$. The regions are shown in Figure 7.3.7.

(a) $\displaystyle A = \int_{-1}^{1} \frac{1}{1 + x^2} dx = \arctan x \Big]_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$

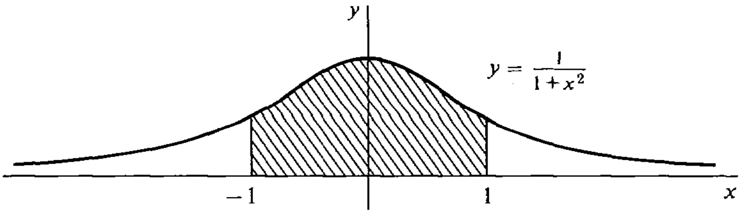


Figure 7.3.7

(b) $$A = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$
$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx$$
$$= \lim_{a\to-\infty} (\arctan 0 - \arctan a) + \lim_{b\to\infty} (\arctan b - \arctan 0)$$
$$= -\lim_{a\to-\infty} \arctan a + \lim_{b\to\infty} \arctan b.$$

From the graph of arctan x we see that the first limit is $-\pi/2$ and the second limit is $\pi/2$, so

$$A = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi.$$

Thus the region under $y = 1/(1+x^2)$ has exactly the same area as the unit circle, and half of this area is between $x = -1$ and $x = 1$.

EXAMPLE 7 Find $\displaystyle\int_{-2}^{-\sqrt{2}} \frac{1}{x\sqrt{x^2-1}}\,dx.$

The region is shown in Figure 7.3.8. Since x is negative, $x = -|x|$. Thus

$$\int_{-2}^{-\sqrt{2}} \frac{1}{x\sqrt{x^2-1}}\,dx = \int_{-2}^{-\sqrt{2}} -\frac{1}{|x|\sqrt{x^2-1}}\,dx$$
$$= -\operatorname{arcsec} x\Big]_{-2}^{-\sqrt{2}} = -(\operatorname{arcsec}(-\sqrt{2}) - \operatorname{arcsec}(-2))$$
$$= -\left(\frac{3\pi}{4} - \frac{2\pi}{3}\right) = -\frac{\pi}{12}.$$

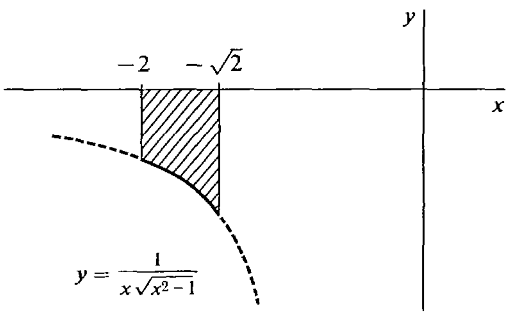


Figure 7.3.8

PROBLEMS FOR SECTION 7.3

In Problems 1–9, evaluate the given expression.

1 $\arcsin(\sqrt{3}/2)$

2 $\arcsin(-1/2)$

3 $\arctan(-1)$

4 $\sec(\arctan(-1))$

5 $\operatorname{arcsec} 2$

6 $\arcsin(\cos\pi)$

7 $\sin(\arccos x)$

8 $\cot(\operatorname{arcsec} x)$

9 $\arcsin(\cos x), \quad 0 \le x \le \pi$

10 Prove the identity $\arctan(-x) = -\arctan x$.

11 Prove $\arctan(1/x) = \operatorname{arccot} x$, for $0 < x$.

12 Prove $\arccos(-x) = \pi - \arccos x$.

13 Prove $\arctan x + \operatorname{arccot} x = \pi/2$.

Find the derivatives in Problems 14–25.

14 $y = \arcsin(x/2)$

15 $y = \operatorname{arcsec}(5x - 2)$

16 $y = (\arcsin x)^2$

17 $y = \arcsin(x^2)$

18 $y = \arctan\sqrt{x}$

19 $s = t\arcsin t$

20 $y = x\operatorname{arcsec} x$

21 $y = \arcsin x + \sqrt{1 - x^2}$

22 $y = \arccos x + (x/\sqrt{1 - x^2})$

23 $y = x\arcsin x + \sqrt{1 - x^2}$

24 $u = \operatorname{arcsec} t + \sqrt{t^2 - 1}$

25 $y = \arctan(1/\sqrt{x})$

26 Evaluate $\lim_{x\to\infty} \operatorname{arccsc} x$.

27 Evaluate $\lim_{x\to-\infty} \arctan x$.

28 Evaluate $\lim_{x\to 0} \dfrac{\arcsin x}{x}$.

29 Evaluate $\lim_{x\to\infty} \dfrac{\operatorname{arccot} x}{\operatorname{arccsc} x}$.

In Problems 30–47 evaluate the integrals.

30 $\displaystyle\int \frac{dx}{1 + 4x^2}$

31 $\displaystyle\int \frac{dx}{9 + x^2}$

32 $\displaystyle\int \frac{dx}{\sqrt{4 - x^2}}$

33 $\displaystyle\int \frac{dx}{\sqrt{x - x^2}}$

34 $\displaystyle\int \frac{\cos x}{1 + \sin^2 x}\,dx$

35 $\displaystyle\int \frac{dx}{x\sqrt{4x^2 - 1}}, \quad x > 1$

36 $\displaystyle\int \frac{x\,dx}{x^4 + 1}$

37 $\displaystyle\int \frac{x\,dx}{\sqrt{1 - x^4}}$

38 $\displaystyle\int \frac{dx}{(1 + x)\sqrt{x}}$

39 $\displaystyle\int \frac{dx}{x\sqrt{x - 1}}$

40 $\displaystyle\int \frac{\arctan x}{1 + x^2}\,dx$

41 $\displaystyle\int \frac{\arcsin x}{\sqrt{1 - x^2}}\,dx$

42 $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1 + x^2}$

43 $\displaystyle\int_1^2 \frac{1}{x\sqrt{x^2 - 1}}\,dx$

44 $\displaystyle\int_{-\sqrt{2}}^{-1} \frac{1}{x\sqrt{x^2 - 1}}\,dx$

45 $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1 - x^2}}\,dx$

46 $\displaystyle\int_0^{\infty} \frac{dx}{25x^2 + 1}$

47 $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2}$

48 Find the area of the region bounded by the x-axis and the curve $y = 1/\sqrt{1 - x^2}$, $-1 < x < 1$.

49 Find the area of the region under the curve $y = 1/(x\sqrt{x^2 - 1})$, $1 \le x < \infty$.

50 Find the area of the region bounded below by the line $y = \frac{1}{2}$ and above by the curve $y = 1/(x^2 + 1)$.

7.4 INTEGRATION BY PARTS

One reason it is harder to integrate than differentiate is that for derivatives there is both a Sum Rule and a Product Rule,

$$d(u + v) = du + dv, \qquad d(uv) = u\,dv + v\,du$$

while for integrals there is only a Sum Rule,

$$\int du + dv = \int du + \int dv.$$

The Sum Rule for integrals is obtained in a simple way by reversing the sum rule for derivatives.

There is a way to turn the Product Rule for derivatives into a rule for integrals. It no longer looks like a product rule, and is called integration by parts. Integration by parts is a basic method which is needed for many integrals involving trigonometric functions (and later exponential functions).

INDEFINITE INTEGRATION BY PARTS

Suppose, for x in an open interval I, that u and v depend on x and that du and dv exist. Then

$$\int u\,dv = uv - \int v\,du.$$

PROOF We use the Product Rule

$$u\,dv + v\,du = d(uv), \qquad u\,dv = d(uv) - v\,du.$$

Integrating both sides with x as the independent variable,

$$\int u\,dv = \int (d(uv) - v\,du) = \int d(uv) - \int v\,du = uv - \int v\,du.$$

No constant of integration is needed because there are indefinite integrals on both sides of the equation.

Integration by parts is useful whenever $\int v\,du$ is easier to evaluate than a given integral $\int u\,dv$.

EXAMPLE 1 Evaluate $\int x \sin x\, dx$. Our plan is to break $x \sin x\, dx$ into a product of the form $u\, dv$, evaluate the integrals $\int dv$ and $\int v\, du$, and then use integration by parts to get $\int u\, dv$. There are several choices we might make for u and dv, and not all of them lead to a solution of the problem. Some guesswork is required.

First try: $u = \sin x, dv = x\, dx$. $\int dv = \int x\, dx = \frac{1}{2}x^2 + C$. Take $v = \frac{1}{2}x^2$. Next we find du and try to evaluate $\int v\, du$.

$$du = \cos x\, dx, \qquad \int v\, du = \int \tfrac{1}{2}x^2 \cos x\, dx.$$

This integral looks harder than the one we started with, so we shall start over with another choice of u and dv.

Second try: $u = x, dv = \sin x\, dx$.

$$\int dv = \int \sin x\, dx = -\cos x + C.$$

We take $v = -\cos x$. This time we find du and easily evaluate $\int v\, du$.

$$du = dx, \qquad \int v\, du = \int -\cos x\, dx = -\sin x + C_1.$$

Finally we use the rule

$$\int u\, dv = uv - \int v\, du,$$

$$\int x \sin x\, dx = x(-\cos x) - (-\sin x + C_1),$$

or

$$\int x \sin x\, dx = -x \cos x + \sin x + C.$$

EXAMPLE 2 Evaluate $\int \arcsin x\, dx$. A choice of u and dv which works is

$$u = \arcsin x, \qquad dv = dx.$$

We may take $v = x$. Then

$$du = \frac{dx}{\sqrt{1 - x^2}},$$

$$\int v\, du = \int \frac{x\, dx}{\sqrt{1 - x^2}} = -\sqrt{1 - x^2} + C_1.$$

Finally,

$$\int \arcsin x\, dx = x \arcsin x - (-\sqrt{1 - x^2} + C_1),$$

$$\int \arcsin x\, dx = x \arcsin x + \sqrt{1 - x^2} + C.$$

This integral and the similar formula for $\int \arccos x\, dx$ are included in our table at the end of the book. We shall see how to integrate the other inverse trigonometric functions in the next chapter.

EXAMPLE 3 Evaluate $\int x^2 \sin x\, dx$. This requires two integrations by parts.

Step 1

$$u = x^2, \qquad dv = \sin x\, dx,$$

$$du = 2x\, dx, \qquad \int dv = \int \sin x\, dx = -\cos x + C.$$

We take $v = -\cos x$.

$$\int x^2 \sin x\, dx = uv - \int v\, du = -x^2 \cos x + \int 2x \cos x\, dx.$$

Step 2 Evaluate $\int 2x \cos x\, dx$.

$$u_1 = 2x, \qquad dv_1 = \cos x\, dx,$$

$$du_1 = 2\, dx, \qquad \int dv_1 = \int \cos x\, dx = \sin x + C.$$

We take $v_1 = \sin x$.

$$\int 2x \cos x\, dx = u_1 v_1 - \int v_1\, du_1$$

$$= 2x \sin x - \int 2 \sin x\, dx$$

$$= 2x \sin x + 2 \cos x + C.$$

Combining the two steps,

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

Sometimes integration by parts will yield an equation in which the given integral occurs on both sides. One can often solve for the answer.

EXAMPLE 4 Evaluate $\int \sin^2 \theta\, d\theta$. Let

$$u = \sin \theta, \qquad dv = \sin \theta\, d\theta.$$

Then

$$du = \cos \theta\, d\theta, \qquad v = -\cos \theta.$$

$$\int \sin^2 \theta\, d\theta = -\sin \theta \cos \theta - \int -\cos^2 \theta\, d\theta$$

$$= -\sin \theta \cos \theta + \int \cos^2 \theta\, d\theta$$

$$= -\sin \theta \cos \theta + \int (1 - \sin^2 \theta)\, d\theta$$

$$= -\sin \theta \cos \theta + \theta - \int \sin^2 \theta\, d\theta.$$

We solve this equation for $\int \sin^2 \theta\, d\theta$,

$$\int \sin^2 \theta\, d\theta = -\tfrac{1}{2} \sin \theta \cos \theta + \tfrac{1}{2}\theta + C.$$

Here is another way to evaluate $\int \sin^2\theta\,d\theta$. Instead of using integration by parts, we can use the half-angle formula

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}.$$

This is derived from the addition formula,

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta,$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}.$$

Then $$\int \sin^2\theta\,d\theta = \int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{1}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta\,d\theta$$
$$= \frac{1}{2}\int d\theta - \frac{1}{4}\int \cos 2\theta\,d(2\theta) = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C.$$

This answer agrees with Example 4 because

$$\sin 2\theta = \sin(\theta+\theta) = 2\sin\theta\cos\theta,$$

so $$\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta = \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta.$$

Integration by parts requires a great deal of guesswork. Given a problem $\int h(x)\,dx$ we try to find a way to split $h(x)\,dx$ into a product $f(x)g'(x)\,dx$ where we can evaluate both of the integrals $\int g'(x)\,dx$ and $\int g(x)f'(x)\,dx$.

Definite integrals take the following form when integration by parts is applied.

DEFINITE INTEGRATION BY PARTS

If $u = f(x)$ and $v = g(x)$ have continuous derivatives on an open interval I, then for a, b in I,

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx.$$

PROOF The Product Rule gives

$$f(x)g'(x)\,dx + g(x)f'(x)\,dx = d(f(x)g(x)).$$

Then by the Fundamental Theorem of Calculus,

$$\int_a^b (f(x)g'(x) + g(x)f'(x))\,dx = f(x)g(x)\Big]_a^b,$$

and the desired result follows by the Sum Rule.

If we plot $u = f(x)$ on one axis and $v = g(x)$ on the other, we get a picture of definite integration by parts (Figure 7.4.1). The picture is easier to interpret if we change variables in the definite integrals and write the formula for integration by

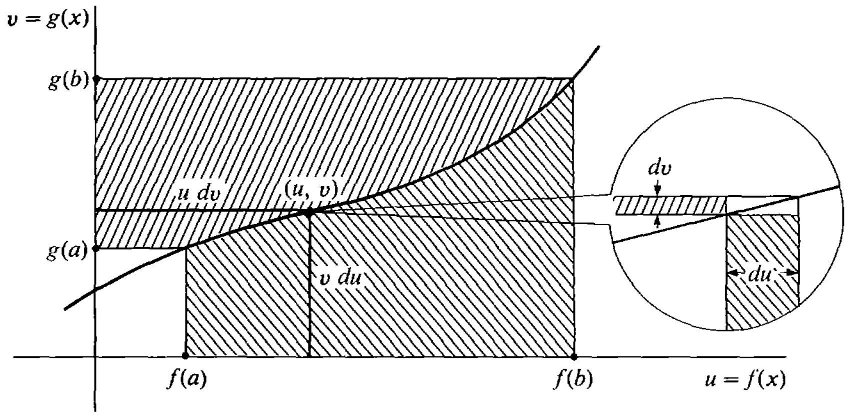


Figure 7.4.1 Definite Integration by Parts

parts in the form

$$\int_{g(a)}^{g(b)} u\,dv + \int_{f(a)}^{f(b)} v\,du = f(b)g(b) - f(a)g(a).$$

EXAMPLE 5 Evaluate $\int_0^\pi x \sin x\,dx$ (Figure 7.4.2). Take $u = x, dv = \sin x\,dx$ as in Example 1. Then $v = -\cos x$ and

$$\begin{aligned}\int_0^\pi x \sin x\,dx &= -x\cos x\Big]_0^\pi - \int_0^\pi -\cos x\,dx\\ &= -x\cos x\Big]_0^\pi + \sin x\Big]_0^\pi\\ &= (-\pi(-1) + 0\cdot 1) + (0 - 0) = \pi.\end{aligned}$$

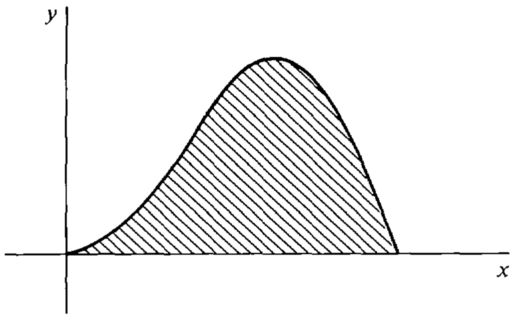


Figure 7.4.2

PROBLEMS FOR SECTION 7.4

Evaluate the integrals in Problems 1–35.

1 $\int x \cos x\,dx$

2 $\int \arccos x\,dx$

3 $\int t^2 \cos t\, dt$

4 $\int x \arctan x\, dx$

5 $\int t \sin(2t - 1)\, dt$

6 $\int \arcsin(3t)\, dt$

7 $\int x^2 \sin(4x)\, dx$

8 $\int x \operatorname{arcsec} x\, dx$

9 $\int x^3 \operatorname{arcsec} x\, dx$

10 $\int x^3 \sin x\, dx$

11 $\int \sin \sqrt{x}\, dx$

12 $\int \sin \theta \tan^2 \theta\, d\theta$

13 $\int \arctan \sqrt{x}\, dx$

14 $\int x \tan x \sec^2 x\, dx$

15 $\int \frac{x^3}{\sqrt{x^2 - 1}}\, dx$

16 $\int \cos^2 \theta\, d\theta$

17 $\int x \sin x \cos x\, dx$

18 $\int t \sin^2 t\, dt$

19 $\int \sin \theta \sin(2\theta)\, d\theta$

20 $\int \cos x \cos(3x)\, dx$

21 $\int \sin x \cos(5x)\, dx$

22 $\int \cos x \cot^4 x\, dx$

23 $\int t^3 \sin(t^2)\, dt$

24 $\int x^3 \cos(2x^2 - 1)\, dx$

25 $\int \frac{1}{x^3} \sin\left(\frac{1}{x}\right) dx$

26 $\int \sin \theta \cos \theta \cos(\sin \theta)\, d\theta$

27 $\int t^3 \sqrt{t^2 + 4}\, dt$

28 $\int \frac{1}{x^3} \sqrt{\frac{1}{x} - 1}\, dx$

29 $\int_0^{\pi/2} \theta \cos \theta\, d\theta$

30 $\int_0^{1/2} \arcsin x\, dx$

31 $\int_0^{\pi} \sin^2 \theta\, d\theta$

32 $\int_0^1 \arcsin x\, dx$

33 $\int_0^1 x \operatorname{arccot} x\, dx$

34 $\int_0^{\infty} x \operatorname{arccot} x\, dx$

35 $\int_1^2 t \operatorname{arcsec} t\, dt$

36 Find the volume of the solid of revolution generated by rotating the region under the curve $y = \sin x, 0 \le x \le \pi$, about (a) the x-axis, (b) the y-axis.

☐ 37 Prove that if f is a differentiable function of x, then

$$\int f(x)\, dx = xf(x) - \int xf'(x)\, dx.$$

☐ 38 If u and v are differentiable functions of x, show that

$$\int u^2\, dv = u^2 v - 2 \int uv\, du.$$

☐ 39 Show that if f' and g are differentiable for all x, then

$$\int g(x)g'(x)f''(g(x))\, dx = f'(g(x))g(x) - f(g(x)) + C.$$

7.5 INTEGRALS OF POWERS OF TRIGONOMETRIC FUNCTIONS

It is often possible to transform an integral into one of the forms

$$\int \sin^n u\, du, \qquad \int \cos^n u\, du, \qquad \int \tan^n u\, du, \quad \text{etc.}$$

These integrals can be evaluated by means of *reduction formulas*, which express the integral of the nth power of a trigonometric function in terms of the $(n-2)$nd power. The easiest reduction formulas to prove are those for the tangent and cotangent, so we shall give them first.

THEOREM 1

Let $n \neq 1$. Then

(i) $$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.$$

(ii) $$\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx.$$

PROOF We recall that

$$\tan^2 x = \sec^2 x - 1, \qquad d(\tan x) = \sec^2 x\, dx.$$

Then
$$\begin{aligned}\int \tan^n x\, dx &= \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2} x\,(\sec^2 x - 1)\, dx \\ &= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx \\ &= \int \tan^{n-2} x\, d(\tan x) - \int \tan^{n-2} x\, dx \\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.\end{aligned}$$

These reduction formulas are true for any rational number $n \neq 1$. They are most useful, however, when n is a positive integer.

EXAMPLE 1 $$\int \tan^2 x\, dx = \frac{\tan x}{1} - \int \tan^0 x\, dx = \tan x - x + C.$$

EXAMPLE 2 $$\int \tan^4 x\, dx = \frac{\tan^3 x}{3} - \int \tan^2 x\, dx = \frac{\tan^3 x}{3} - \tan x + x + C.$$

EXAMPLE 3 $$\int \tan^3 x\, dx = \frac{\tan^2 x}{2} - \int \tan x\, dx.$$

We will evaluate $\int \tan x\, dx$ in the next chapter.

Each time we use the reduction formula the exponent in the integral goes down by two. By repeated use of the reduction formulas we can integrate any even power of $\tan x$ or $\cot x$. We can also work the integral of any odd power of $\tan x$ or $\cot x$ down to an expression involving $\int \tan x$ or $\int \cot x$.

The reduction formulas for the other trigonometric functions are obtained by using integration by parts.

THEOREM 2

Let $n \neq 0$. Then

(i) $$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

(ii) $$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

PROOF (i) Break the term $\sin^n x\,dx$ into two parts,

$$\sin^n x\,dx = \sin^{n-1} x(\sin x\,dx).$$

We shall let
$$u = \sin^{n-1} x, \qquad v = -\cos x,$$
$$du = (n-1)\sin^{n-2} x \cos x\,dx, \qquad dv = \sin x\,dx,$$

and use integration by parts. Then

$$\begin{aligned}\int \sin^n x\,dx &= \int u\,dv = uv - \int v\,du\\ &= -\sin^{n-1} x \cos x - \int (n-1)(-\cos x)\sin^{n-2} x \cos x\,dx\\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1 - \sin^2 x)\,dx\\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.\end{aligned}$$

We find that $\int \sin^n x\,dx$ appears on both sides of the equation, and we solve for it,

$$n\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx,$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

We already know the integrals

$$\int \sin x\,dx = -\cos x + C, \qquad \int \cos x\,dx = \sin x + C.$$

We can use the reduction formulas to integrate any positive power of $\sin x$ or $\cos x$. Again, the formulas are true where n is any rational number, $n \neq 0$.

EXAMPLE 4 $\int \sin^2 x\, dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}\int dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x + C.$

$$\int \cos^2 x\, dx = \tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}\int dx = \tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}x + C.$$

EXAMPLE 5 $\int \cos^3 x\, dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x\, dx$

$$= \tfrac{1}{3}\cos^2 x \sin x + \tfrac{2}{3}\sin x + C.$$

THEOREM 3

Let $m \neq 1$. Then

(i) $\displaystyle\int \sec^m x\, dx = \frac{1}{m-1}\sec^{m-1} x \sin x + \frac{m-2}{m-1}\int \sec^{m-2} x\, dx.$

(ii) $\displaystyle\int \csc^m x\, dx = -\frac{1}{m-1}\csc^{m-1} x \cos x + \frac{m-2}{m-1}\int \csc^{m-2} x\, dx.$

PROOF (ii) This can be done by integration by parts, but it is easier to use Theorem 2. Let $n = 2 - m$. For $m \neq 2$, $n \neq 0$ and Theorem 2 gives

$$\int \sin^{2-m} x\, dx = -\frac{1}{2-m}\sin^{1-m} x \cos x + \frac{1-m}{2-m}\int \sin^{-m} x\, dx,$$

$$\int \csc^{m-2} x\, dx = \frac{1}{m-2}\csc^{m-1} x \cos x + \frac{m-1}{m-2}\int \csc^m x\, dx,$$

whence $\displaystyle\int \csc^m x\, dx = -\frac{1}{m-1}\csc^{m-1} x \cos x + \frac{m-2}{m-1}\int \csc^{m-2} x\, dx.$

For $m = 2$ the formula is already known,

$$\int \csc^2 x\, dx = -\cot x + C = -\csc x \cos x + C.$$

These reduction formulas can be used to integrate any even power of $\sec x$ or $\csc x$, and to get the integral of any odd power of $\sec x$ or $\csc x$ in terms of $\int \sec x$ or $\int \csc x$. We shall find $\int \sec x$ and $\int \csc x$ in the next chapter.

EXAMPLE 6 $\int \sec^3 x\, dx = \frac{1}{2}\sec^2 x \sin x + \frac{1}{2}\int \sec x\, dx.$

EXAMPLE 7 $\int \sec^4 x\, dx = \frac{1}{3}\sec^3 x \sin x + \frac{2}{3}\int \sec^2 x\, dx$

$$= \tfrac{1}{3}\sec^3 x \sin x + \tfrac{2}{3}\tan x + C.$$

By using the identity $\sin^2 x + \cos^2 x = 1$ we can evaluate any integral of the form $\int \sin^m x \cos^n x\, dx$ where m and n are positive integers. If either m or n is odd we let $u = \sin x$ or $u = \cos x$ and transform the integral into a polynomial in u.

EXAMPLE 8 $\int \sin^4 x \cos^3 x\, dx$. Let $u = \sin x$, $du = \cos x\, dx$.

$$\begin{aligned}\int \sin^4 x \cos^3 x\, dx &= \int u^4(1 - u^2)\, du \\ &= \tfrac{1}{5}u^5 - \tfrac{1}{7}u^7 + C \\ &= \tfrac{1}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + C.\end{aligned}$$

This method also works for an odd power of $\sin x$ times any power of $\cos x$, and vice versa.

EXAMPLE 9 $\int \sqrt{\cos x}\, \sin^3 x\, dx$. Let $u = \cos x$, $du = -\sin x\, dx$.

$$\begin{aligned}\int \sqrt{\cos x}\, \sin^3 x\, dx &= \int \sqrt{u}(1 - u^2)(-1)\, du \\ &= \int -u^{1/2} + u^{5/2}\, du = -\tfrac{2}{3}u^{3/2} + \tfrac{2}{7}u^{7/2} + C \\ &= -\tfrac{2}{3}(\cos x)^{3/2} + \tfrac{2}{7}(\cos x)^{7/2} + C.\end{aligned}$$

EXAMPLE 10 $\int \sin^5 x\, dx$. Let $u = \cos x$, $du = -\sin x\, dx$.

$$\begin{aligned}\int \sin^5 x\, dx &= \int (1 - u^2)^2(-1)\, du \\ &= -\int (1 - 2u^2 + u^4)\, du = -u + \tfrac{2}{3}u^3 - \tfrac{1}{5}u^5 + C \\ &= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C.\end{aligned}$$

If m and n are both even, the integral $\int \sin^m x \cos^n x\, dx$ can be transformed into the integral of a sum of even powers of $\sin x$. Then the reduction formula can be used.

EXAMPLE 11

$$\begin{aligned}\int \sin^4 x \cos^4 x\, dx &= \int \sin^4 x(1 - \sin^2 x)^2\, dx \\ &= \int \sin^4 x - 2\sin^6 x + \sin^8 x\, dx.\end{aligned}$$

We can also evaluate integrals of the form

$$\int \tan^m x \sec^n x\, dx,$$

$$\int \cot^m x \csc^n x\, dx.$$

EXAMPLE 12 When m is even use $\tan^2 x = \sec^2 x - 1$.

$$\int \tan^4 x \sec x \, dx = \int (\sec^2 x - 1)^2 \sec x \, dx$$

$$= \int \sec^5 x - 2\sec^3 x + \sec x \, dx.$$

Now the reduction formula for $\sec x$ can be used.

EXAMPLE 13 When m is odd use the new variable $u = \sec x$ or $u = -\csc x$.

$$\int \cot^3 x \csc^3 x \, dx = \int \cot^2 x \csc^2 x (\cot x \csc x \, dx)$$

$$= \int (u^2 - 1)u^2 \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= -\frac{\csc^5 x}{5} + \frac{\csc^3 x}{3} + C.$$

PROBLEMS FOR SECTION 7.5

Evaluate the integrals in Problems 1–32.

1 $\int \frac{\sin^3 t}{\cos^2 t} dt$

2 $\int \sin^2(2t) \, dt$

3 $\int \cot^2 x \, dx$

4 $\int \sin^3(5u) \, du$

5 $\int \cos^4 x \, dx$

6 $\int \frac{1}{\sin^4 x} dx$

7 $\int \tan^3 x \sec^4 x \, dx$

8 $\int \tan^6 \theta \, d\theta$

9 $\int \sin^2 x \cos^3 x \, dx$

10 $\int \cot \theta \csc^2 \theta \, d\theta$

11 $\int \cot^2 \theta \csc^2 \theta \, d\theta$

12 $\int \sin x (\cos x)^{3/2} \, dx$

13 $\int (\tan x)^{3/2} \sec^4 x \, dx$

14 $\int \sec^4(3u - 1) \, du$

15 $\int \sec^2 \theta \csc^2 \theta \, d\theta$

16 $\int \frac{\sin^2 \theta}{1 - \cos \theta} d\theta$

17 $\int \frac{1 - \cos \theta}{\sin^2 \theta} d\theta$

18 $\int_0^{\pi/2} \sin^3 x \cos x \, dx$

19 $\int_0^{\pi/3} \tan^3 \theta \sec \theta \, d\theta$

20 $\int_0^{\pi/2} \sqrt{\cos x} \sin x \, dx$

21 $\int_0^{\pi/4} \tan^4 x \, dx$

22 $\int_0^{\pi/2} \tan^2 x \, dx$

23 $\int_0^{\pi} \sin^4\theta\, d\theta$

24 $\int \sin^3(2u)\cos^3(2u)\, du$

25 $\int \frac{\cos^2\sqrt{x}}{\sqrt{x}}\, dx$

26 $\int x\tan(x^2)\sec^2(x^2)\, dx$

27 $\int x\sin^3 x\, dx$

28 $\int x\tan^3 x\sec^2 x\, dx$

29 $\int x\sin^2 x\cos x\, dx$

30 $\int \sin^6\theta\cos^5\theta\, d\theta$

31 $\int \tan^4\theta\sec^6\theta\, d\theta$

32 $\int \sin^2 x\cos^2 x\, dx$

In Problems 33–39, express the given integral in terms of

$$\int \tan x\, dx, \quad \int \cot x\, dx, \quad \int \sec x\, dx, \quad \int \csc x\, dx.$$

33 $\int \sec^3 x\, dx$

34 $\int \cot^3 x\, dx$

35 $\int \tan^2 x\sec x\, dx$

36 $\int \csc^5 x\, dx$

37 $\int \cot^2 x\csc^3 x\, dx$

38 $\int \tan^4 x\sec x\, dx$

39 $\int \frac{\sin x + \cos x}{\sin x\cos x}\, dx$

40 Check the reduction formula for $\int \sin^n x\, dx$ by differentiating both sides of the equation. Do the same for $\int \tan^n x\, dx$ and $\int \sec^n x\, dx$.

□ 41 Find a reduction formula for $\int x^n \sin x\, dx$ using integration by parts.

42 Find the volume of the solid generated by rotating the region under the curve $y = \sin^2 x$, $0 \le x \le \pi$, about (a) the x-axis, (b) the y-axis.

43 Find the volume of the solid generated by rotating the region under the curve $y = \sin x\cos x$, $0 \le x \le \pi/2$, about (a) the x-axis, (b) the y-axis.

7.6 TRIGONOMETRIC SUBSTITUTIONS

Integrals containing one of the terms

$$\sqrt{a^2 + x^2}, \quad \sqrt{a^2 - x^2}, \quad \text{or} \quad \sqrt{x^2 - a^2}$$

can often be integrated by a *trigonometric substitution*. The idea is to take x, a, and the square root as the three sides of a right triangle and use one of its acute angles as a new variable θ. The three kinds of trigonometric substitutions are shown in Figure 7.6.1. These figures do not have to be memorized. Just remember that the sides must be labeled so that

$$(\text{opposite})^2 + (\text{adjacent})^2 = (\text{hypotenuse})^2.$$

These substitutions frequently give an integral of powers of trigonometric functions discussed in the preceding section.

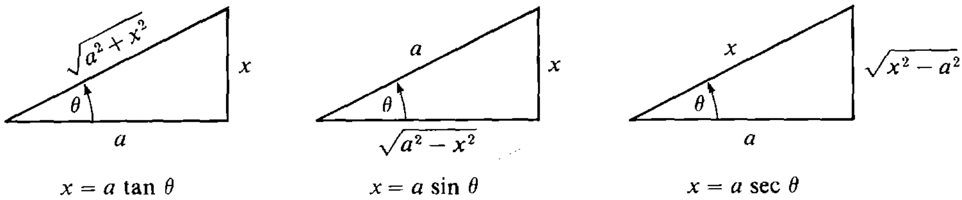


Figure 7.6.1

EXAMPLE 1 Find $\int (a^2 + x^2)^{-3/2}\, dx$.

Let $\theta = \arctan(x/a)$. Then from Figure 7.6.2,

$$x = a\tan\theta, \qquad dx = a\sec^2\theta\, d\theta, \qquad \sqrt{a^2+x^2} = a\sec\theta.$$

So

$$\begin{aligned}\int (a^2+x^2)^{-3/2}\,dx &= \int (a\sec\theta)^{-3}\, a\sec^2\theta\, d\theta \\ &= \frac{1}{a^2}\int (\sec\theta)^{-1}\,d\theta = \frac{1}{a^2}\int \cos\theta\, d\theta \\ &= \frac{1}{a^2}\sin\theta + C = \frac{1}{a^2}\frac{\tan\theta}{\sec\theta} + C = \frac{x}{a^2\sqrt{a^2+x^2}} + C.\end{aligned}$$

EXAMPLE 2 Find $\int\sqrt{x^2-a^2}\,dx$.

Let $\theta = \operatorname{arcsec}(x/a)$ (Figure 7.6.3), so

$$x = a\sec\theta, \qquad dx = a\tan\theta\sec\theta\, d\theta, \qquad \sqrt{x^2-a^2} = a\tan\theta.$$

So

$$\begin{aligned}\int\sqrt{x^2-a^2}\,dx &= \int a\tan\theta\, a\tan\theta\sec\theta\,d\theta = a^2\int \tan^2\theta\sec\theta\,d\theta \\ &= a^2\int (\sec^2\theta - 1)\sec\theta\,d\theta \\ &= a^2\int \sec^3\theta\,d\theta - a^2\int\sec\theta\,d\theta \\ &= \left(\tfrac{1}{2}a^2\sec^2\theta\sin\theta + \tfrac{1}{2}a^2\int\sec\theta\,d\theta\right) - a^2\int\sec\theta\,d\theta \\ &= \tfrac{1}{2}a^2\sec^2\theta\sin\theta - \tfrac{1}{2}a^2\int\sec\theta\,d\theta \\ &= \tfrac{1}{2}x\sqrt{x^2-a^2} - \tfrac{1}{2}a^2\int\sec\theta\,d\theta.\end{aligned}$$

This is as far as we can go on this problem until we find out how to integrate $\int\sec\theta\,d\theta$ in the next chapter.

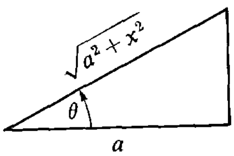


Figure 7.6.2

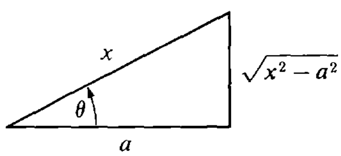


Figure 7.6.3

EXAMPLE 3 $\int \frac{1}{x^2\sqrt{a^2-x^2}}\,dx$. Let $\theta = \arcsin(x/a)$ (Figure 7.6.4). Then

$$x = a\sin\theta, \qquad dx = a\cos\theta\,d\theta, \qquad \sqrt{a^2-x^2} = a\cos\theta.$$

$$\int \frac{1}{x^2\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a^2\sin^2\theta\, a\cos\theta}\,a\cos\theta\,d\theta = \int \frac{1}{a^2\sin^2\theta}\,d\theta$$

$$= \frac{1}{a^2}\int \csc^2\theta\,d\theta = -\frac{1}{a^2}\cot\theta + C$$

$$= -\frac{1}{a^2}\frac{\sqrt{a^2-x^2}}{x} + C.$$

EXAMPLE 4 $\int \frac{\sqrt{x^2-a^2}}{x}\,dx$. Put $\theta = \operatorname{arcsec}(x/a)$ (Figure 7.6.5). Then

$$x = a\sec\theta, \qquad dx = a\tan\theta\sec\theta\,d\theta, \qquad \sqrt{x^2-a^2} = a\tan\theta.$$

$$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \int \frac{a\tan\theta}{a\sec\theta}\,a\tan\theta\sec\theta\,d\theta = a\int \tan^2\theta\,d\theta$$

$$= a\int \sec^2\theta\,d\theta - a\int d\theta = a\tan\theta - a\theta + C$$

$$= \sqrt{x^2-a^2} - a\operatorname{arcsec}(x/a) + C.$$

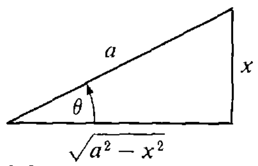


Figure 7.6.4

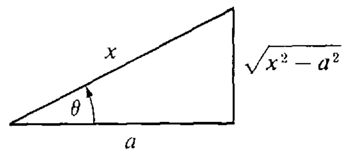


Figure 7.6.5

To keep track of a trigonometric substitution, it is a good idea to actually draw the triangle and label the sides.

EXAMPLE 5 The basic integrals:

(a) $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C,$$

(b) $$\int \frac{dx}{1+x^2}\,dx = \arctan x + C,$$

(c) $$\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec} x + C, \qquad x > 1$$

can be evaluated very easily by a trigonometric substitution.

(a) $$\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

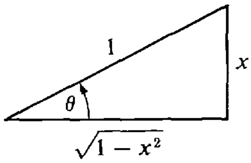


Figure 7.6.6

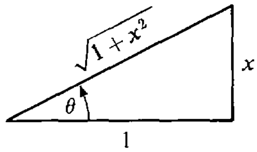


Figure 7.6.7

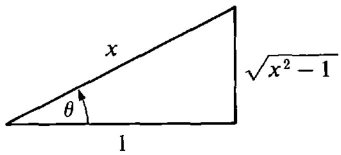


Figure 7.6.8

Let $\theta = \arcsin x$ (Figure 7.6.6). Then $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\sqrt{1-x^2} = \cos\theta$.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta = \theta + C,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C.$$

(b) $$\int \frac{dx}{1+x^2}$$

Let $\theta = \arctan x$ (Figure 7.6.7). Then $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $\sqrt{1+x^2} = \sec\theta$.

$$\int \frac{dx}{1+x^2} = \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \int d\theta = \theta + C,$$

$$\int \frac{dx}{1+x^2} = \arctan x + C.$$

(c) $$\int \frac{dx}{x\sqrt{x^2-1}}, \qquad x > 1.$$

Let $\theta = \operatorname{arcsec} x$ (Figure 7.6.8). Then $x = \sec\theta$, $dx = \tan\theta\sec\theta\, d\theta$, $\sqrt{x^2-1} = \tan\theta$.

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\tan\theta\sec\theta}{\sec\theta\tan\theta}\,d\theta = \int d\theta = \theta + C,$$

$$\int \frac{dx}{x\sqrt{x^2+1}} = \operatorname{arcsec} x + C, \qquad x > 1.$$

It is therefore more important to remember the method of trigonometric substitution than to remember the integration formulas (a), (b), (c).

PROBLEMS FOR SECTION 7.6

Draw the appropriate triangle and evaluate using trigonometric substitutions.

1 $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$

2 $\displaystyle\int \sqrt{a^2-x^2}\,dx$

3 $\displaystyle\int \frac{x^3\,dx}{\sqrt{9+x^2}}$

4 $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$

5 $\int (4 - x^2)^{-3/2}\,dx$

6 $\int (1 - 3x^2)^{3/2}\,dx$

7 $\int \frac{\sin\theta\,d\theta}{\sqrt{2 - \cos^2\theta}}$

8 $\int \frac{dx}{\sqrt{x}\sqrt{1 - x}}$

9 $\int \frac{dx}{x^2(1 + x^2)}$

10 $\int \frac{x^4\,dx}{9 + x^2}$

11 $\int \frac{x^2\,dx}{\sqrt{4 - x^2}}$

12 $\int \frac{\sqrt{9 - 4x^2}}{x^4}\,dx$

13 $\int x^2\sqrt{1 - x^2}\,dx$

14 $\int \sqrt{x}\sqrt{1 - x}\,dx$

15 $\int \sqrt{4x - x^2}\,dx$

16 $\int \frac{\sqrt{x}}{\sqrt{1 - x}}\,dx$

17 $\int \frac{x^3}{\sqrt{4x^2 - 1}}\,dx$

18 $\int \frac{dx}{x^3\sqrt{x^2 - 3}}$

19 $\int \frac{x^3}{\sqrt{a^2 - x^2}}\,dx$

20 $\int x^3\sqrt{1 + a^2x^2}\,dx$

21 $\int \frac{\sqrt{x^4 - 1}}{x}\,dx$

22 $\int x\sqrt{1 - x^4}\,dx$

23 $\int \frac{x^3}{(a^2 + x^2)^{3/2}}\,dx$

24 $\int_0^2 \sqrt{4 - x^2}\,dx$

25 $\int_{-1}^1 \frac{x^2}{\sqrt{1 - x^2}}\,dx$

26 $\int_0^4 \frac{dx}{(9 + x^2)^{3/2}}$

27 $\int_0^\infty \frac{dx}{(9 + x^2)^{3/2}}$

28 $\int_2^4 \frac{\sqrt{x^2 - 2}}{x}\,dx$

29 $\int_2^\infty \frac{\sqrt{x^2 - 2}}{x}\,dx$

30 $\int_0^\infty \frac{x^3}{\sqrt{1 + x^2}}\,dx$

31 $\int x \arcsin x\,dx$

32 $\int x \arccos x\,dx$

33 $\int x^2 \arcsin x\,dx$

34 $\int x^3 \arctan x\,dx$

35 $\int x^{-3} \arcsin x\,dx$

36 $\int x^{-3} \arctan x\,dx$

37 Find the surface area generated by rotating the ellipse $x^2 + 4y^2 = 1$ about the x-axis.

7.7 POLAR COORDINATES

The position of a point in the plane can be described by its distance and direction from the origin. In measuring direction we take the x-axis as the starting point. Let X be the point $(1, 0)$ on the x-axis and let P be a point in the plane as in Figure 7.7.1.

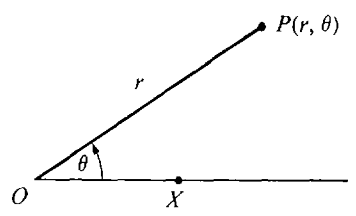


Figure 7.7.1

A pair of *polar coordinates* of P is given by (r, θ) where r is the distance from the origin to P and θ is the angle XOP.

Each pair of real numbers (r, θ) determines a point P in polar coordinates. To find P we first rotate the line OX through an angle θ, forming a new line OX', and then go out a distance r along the line OX'. If θ is negative then the rotation is in the negative, or clockwise direction. If r is negative the distance is measured along the line OX' in the direction away from X' (see Figure 7.7.2).

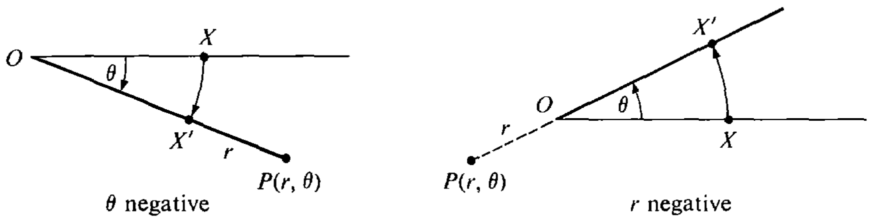


Figure 7.7.2

EXAMPLE 1 Plot the following points in polar coordinates.

$$(2, \pi/4), \quad (-1, \pi/4), \quad (3, 3\pi/4), \quad (2, -\pi/4), \quad (-4, -\pi/4).$$

The solution is shown in Figure 7.7.3.

Each point P has infinitely many different polar coordinate pairs. We see in Figure 7.7.4 that the point $P(3, \pi/2)$ has all the coordinates

$$\left.\begin{matrix}(3, \pi/2 + 2n\pi), \\ (-3, 3\pi/2 + 2n\pi),\end{matrix}\right\} n \text{ an integer.}$$

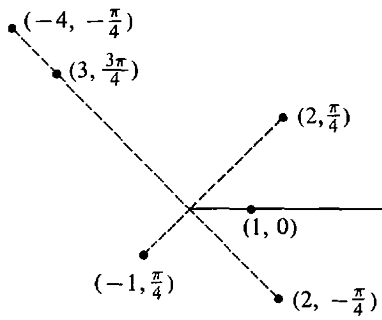


Figure 7.7.3

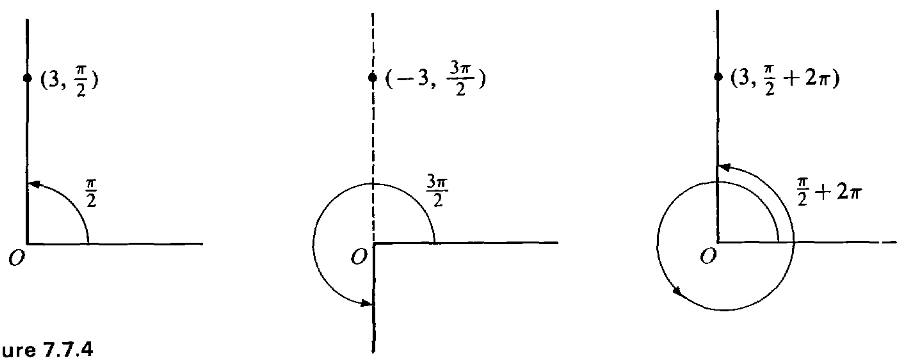


Figure 7.7.4

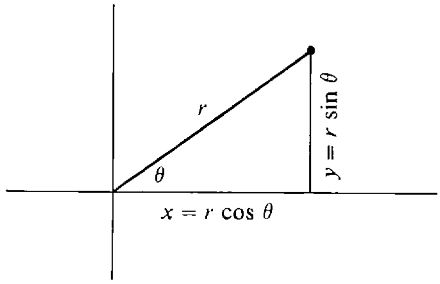


Figure 7.7.5

Any coordinate pair $(0, \theta)$ with $r = 0$ determines the origin. As we see in Figure 7.7.5, the coordinates of a point P in rectangular and in polar coordinates are related by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The *graph*, or *locus in polar coordinates* of a system of formulas in the variables r, θ is the set of all points $P(r, \theta)$ for which the formulas are true.

EXAMPLE 2 The graph of the equation $r = a$ is the circle of radius a centered at the origin (Figure 7.7.6(a)). The graph of the equation $\theta = b$ is a straight line through the origin (Figure 7.7.6(b)).

EXAMPLE 3 The graph of the system of formulas

$$r = \theta, \qquad 0 \le \theta$$

is the spiral of Archimedes formed by moving a pencil along the line OX while the line is rotating, with the pencil moving at the same speed as the point X. The graph is shown in Figure 7.7.6(c).

An equation in rectangular coordinates can readily be transformed into an equation in polar coordinates with the same graph by using $x = r\cos\theta$, $y = r\sin\theta$.

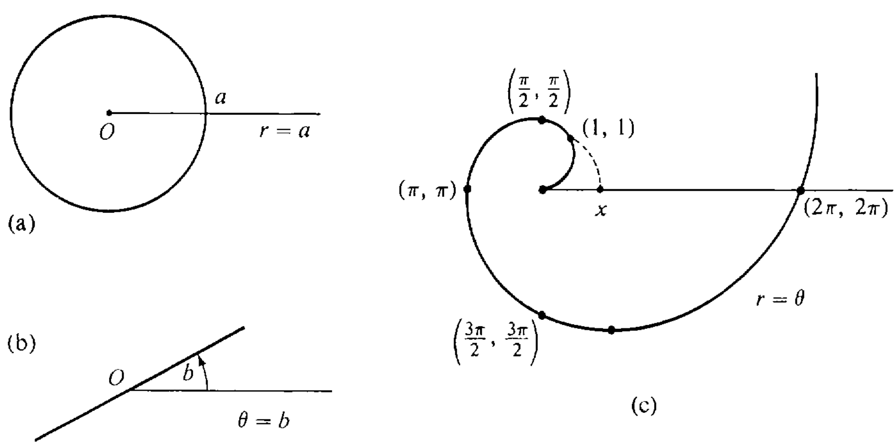


Figure 7.7.6

Here are the polar equations for various types of straight lines. Examples of their graphs are shown in Figure 7.7.7.

(1) Line through the origin (not vertical).
Rectangular equation: $y = mx$.
Polar equation: $r \sin\theta = mr\cos\theta$,
or: $\tan\theta = m$.

(2) Horizontal line (not through origin).
Rectangular equation: $y = b$.
Polar equation: $r\sin\theta = b$,
or: $r = b\csc\theta$.

(3) Vertical line (not through origin).
Rectangular equation: $x = a$.
Polar equation: $r\cos\theta = a$,
or: $r = a\sec\theta$.

(4) Vertical line through origin.
Rectangular equation: $x = 0$.
Polar equation: $r\cos\theta = 0$,
or: $\theta = \pi/2$.

(5) Other lines.
Rectangular equation: $y = mx + b$.
Polar equation: $r\sin\theta = mr\cos\theta + b$,
or:

$$r = \frac{b}{\sin\theta - m\cos\theta}.$$

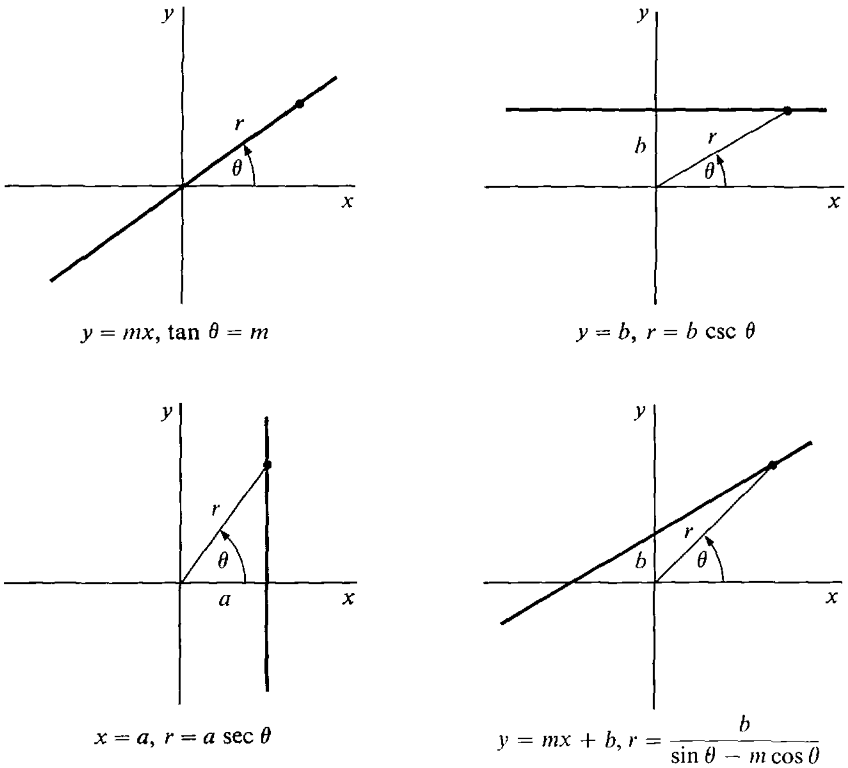


Figure 7.7.7

EXAMPLE 4 The parabola $y = x^2$ has the polar equation

$$r \sin\theta = (r\cos\theta)^2, \quad \text{or} \quad r = \frac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta.$$

EXAMPLE 5 The curve $y = 1/x$ has the polar equation

$$r\sin\theta = \frac{1}{r\cos\theta}, \quad \text{or} \quad r^2 = \sec\theta\csc\theta.$$

The graph is shown in Figure 7.7.8.

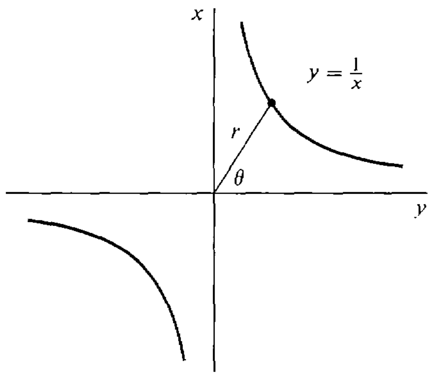


Figure 7.7.8

Some curves have much simpler equations in polar coordinates than in rectangular coordinates.

EXAMPLE 6 The graph of the equation

$$r = a\sin\theta$$

is the circle one of whose diameters is the line from the origin to a point a above the origin.

This can be seen from Figure 7.7.9, if we remember that a diameter and a point on the circle form a right triangle.

As θ increases, the point $(a\sin\theta, \theta)$ goes around this circle once for every π radians.

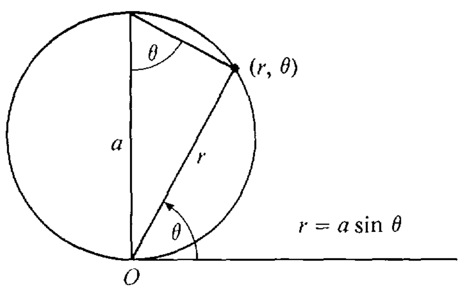


Figure 7.7.9

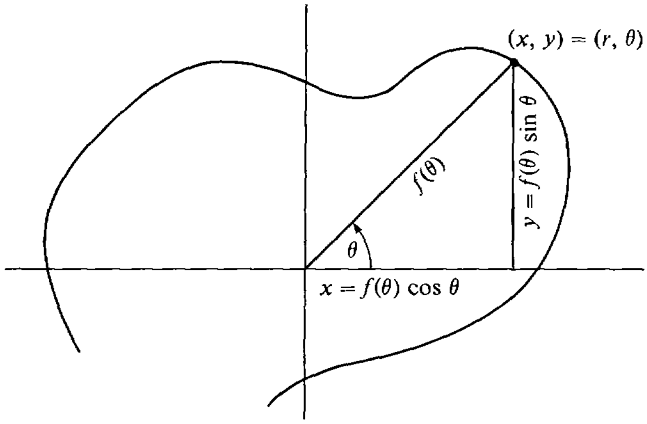


Figure 7.7.10

An equation $r = f(\theta)$ in polar coordinates has the same graph as the pair of parametric equations

$$x = f(\theta)\cos\theta, \qquad y = f(\theta)\sin\theta$$

in rectangular coordinates. This can be seen from Figure 7.7.10.

EXAMPLE 7

(a) The spiral $r = \theta$ has the parametric equations

$$x = \theta\cos\theta, \qquad y = \theta\sin\theta.$$

(b) The circle $r = a\sin\theta$ has the parametric equations

$$x = a\sin\theta\cos\theta, \qquad y = a\sin^2\theta.$$

PROBLEMS FOR SECTION 7.7

1 Plot the following points in polar coordinates:

(a) $(2, \pi/3)$ (b) $(-3, \pi/2)$ (c) $(1, 4\pi/3)$

(d) $(-2, -\pi/4)$ (e) $(\frac{1}{2}, \pi)$ (f) $(0, 3\pi/2)$

In Problems 2–12, find an equation in polar coordinates which has the same graph as the given equation in rectangular coordinates.

2 $y = 3x$ **3** $y = 5x + 2$

4 $y = -4$ **5** $x = 2$

6 $xy^2 = 1$ **7** $y = x^2 + 1$

8 $x^2 + y^2 = 5$ **9** $y = 3x^2 - 2x$

10 $y = x^3$ **11** $y = x^2 + y^2$

12 $y = \sin x$

In Problems 13–20, sketch the given curve in polar coordinates.

13 $r = \cos\theta$ **14** $r = -\sec\theta$

15 $r = \sin(\theta + \pi/4)$ **16** $r = \theta, \quad \theta \le 0$

17 $r = 1 + \theta^2/\pi^2$

18 $r = \dfrac{1}{\sin\theta + \cos\theta}$

19 $r = \cot\theta \csc\theta$

20 $r^2 = -2\sec\theta\csc\theta$

In Problems 21–24, find rectangular parametric equations for the given curves.

21 $r = \sin(3\theta)$

22 $r = \sec\theta\csc\theta$

23 $r = \theta^2$

24 $r = \tan\theta$

25 Prove that if $f(\theta) = f(-\theta)$ then the curve $r = f(\theta)$ is symmetric about the x-axis. That is, if (x, y) is on the curve then so is $(x, -y)$.

26 Prove that if $f(\theta) = f(\pi + \theta)$ then the curve $r = f(\theta)$ is symmetric about the origin. That is, if (x, y) is on the curve so is $(-x, -y)$.

27 Prove that if $f(\theta) = f(\pi - \theta)$ then the curve $r = f(\theta)$ is symmetric about the y-axis.

7.8 SLOPES AND CURVE SKETCHING IN POLAR COORDINATES

Derivatives can be used to measure direction in polar as well as in rectangular coordinates. We begin with two theorems, one about the direction of a curve at the origin (an unusual point in polar coordinates) and the other about the direction of a curve elsewhere. Then we shall use these theorems for sketching curves.

THEOREM 1

At any value θ_0 where the curve $r = f(\theta)$ passes through the origin, the curve is tangent to the line $\theta = \theta_0$.

More precisely, if $r = 0$ at $\theta = \theta_0$ but $r \neq 0$ for all $\theta \neq \theta_0$ in some neighborhood of θ_0, then

$$\lim_{\theta\to\theta_0} \frac{\Delta y}{\Delta x} = \tan\theta_0, \qquad \lim_{\theta\to\theta_0} \frac{\Delta x}{\Delta y} = \cot\theta_0.$$

PROOF Suppose $\cos\theta_0 \neq 0$, so $\tan\theta_0$ exists. Let $\Delta\theta$ be a nonzero infinitesimal. Then $\Delta r \neq 0$ and r changes from 0 to Δr. We compute $\Delta y/\Delta x$.

$$\begin{aligned}
\Delta y &= (0 + \Delta r)\sin(\theta_0 + \Delta\theta) - 0\sin\theta_0 \\
&= \Delta r \sin(\theta_0 + \Delta\theta), \\
\Delta x &= \Delta r \cos(\theta_0 + \Delta\theta), \\
\frac{\Delta y}{\Delta x} &= \frac{\Delta r \sin(\theta_0 + \Delta\theta)}{\Delta r \cos(\theta_0 + \Delta\theta)} = \tan(\theta_0 + \Delta\theta).
\end{aligned}$$

Taking standard parts,

$$\lim_{\theta\to\theta_0} \frac{\Delta y}{\Delta x} = \tan\theta_0.$$

Similarly, when $\sin\theta_0 \neq 0$,

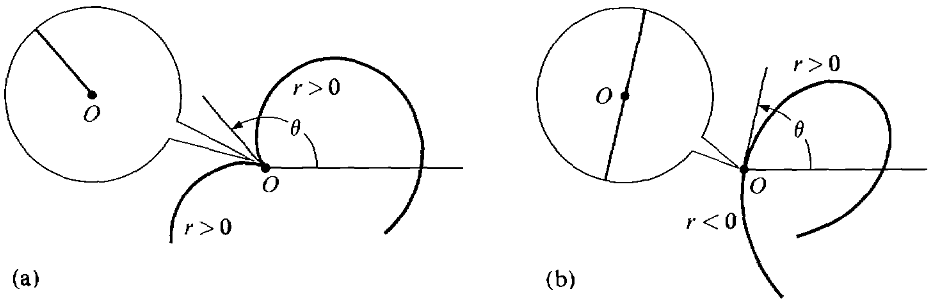
$$\lim_{\theta\to\theta_0} \frac{\Delta x}{\Delta y} = \cot\theta_0.$$

Both limits were given in the theorem to cover the case where the curve is vertical and $\tan \theta_0$ is undefined.

The theorem tells us that if $r = 0$ at θ_0, the curve must approach the origin from the θ_0 direction. Figure 7.8.1 shows two cases.

(a) If r has a local maximum or minimum at θ_0, then r has the same sign on both sides of θ_0. In this case the curve has a cusp at θ_0.

(b) If r has no local maximum or minimum at θ_0, then r is positive on one side of θ_0 and negative on the other side. In this case the curve crosses the origin at θ_0.



(a) (b)

Figure 7.8.1

We now consider points other than the origin. In rectangular coordinates, the slope of a curve $y = f(x)$ at a point P is $dy/dx = \tan \phi$ where ϕ is the angle between the x-axis and the line tangent to the curve at P as shown in Figure 7.8.2.

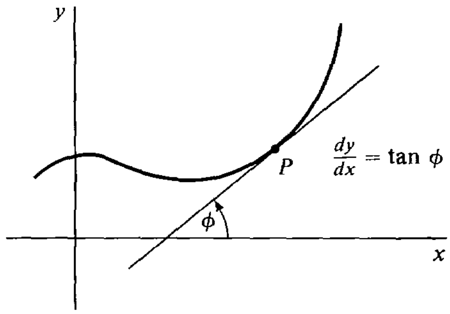


Figure 7.8.2

When $r \neq 0$ in polar coordinates, a useful measure of the direction of the curve at a point P is $\tan \psi$, where ψ is the angle between the radius OP and the tangent line at P (see Figure 7.8.3).

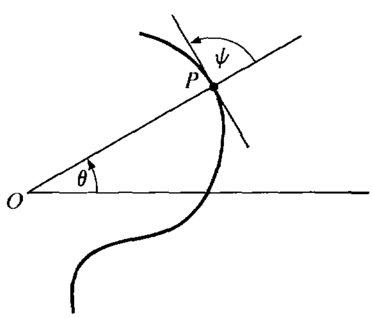


Figure 7.8.3

The following theorem gives a simple formula for $\tan\psi$ when $r \neq 0$.

THEOREM 2

Suppose $r = f(\theta)$ is a curve in polar coordinates and $dr/d\theta$ exists at a point P where $r \neq 0$. Let L be the line tangent to the curve at P and let ψ be the angle between OP and L. Then

$$\cot\psi = \frac{1}{r}\frac{dr}{d\theta}.$$

If $dr/d\theta \neq 0$,

$$\tan\psi = \frac{r}{dr/d\theta}$$

DISCUSSION When $r = 0$, P is the origin so the line OP and angle ψ are undefined.

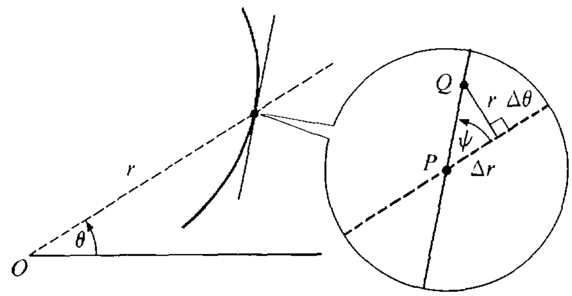


Figure 7.8.4

The formula can be seen intuitively in Figure 7.8.4.

$\Delta\theta$ is infinitesimal. As we move from the point $P(r, \theta)$ to the point $Q(r + \Delta r, \theta + \Delta\theta)$ on the curve, the change in the direction perpendicular to OP will be very close to $r\,\Delta\theta$, so we have

$$\frac{\Delta r}{r\Delta\theta} \approx \cot\psi, \qquad \frac{1}{r}\frac{dr}{d\theta} = \cot\psi.$$

We shall postpone the proof to the end of this section.

We can use Theorem 2 in curve sketching as follows.

(a) In an interval where $\tan\psi > 0$, the curve is going away from the origin as θ increases because $dr/d\theta$ has the same sign as r.

(b) Where $\tan\psi < 0$, the curve is going toward the origin as θ increases because $dr/d\theta$ has the opposite sign as r.

(c) Where r has either a local maximum or minimum and $dr/d\theta$ exists, the curve is going in a direction perpendicular to the radius. This is because $dr/d\theta = 0$ so $\cot\psi = 0$.

Each of these cases is shown in Figure 7.8.5.

Polar coordinates are best suited for trigonometric functions, which have the property that $f(\theta) = f(\theta + 2\pi)$. We shall therefore concentrate on the interval $0 \leq \theta < 2\pi$.

Suppose that the function $r = f(\theta)$ is differentiable for $0 \leq \theta \leq 2\pi$. The following steps may be used in sketching the curve.

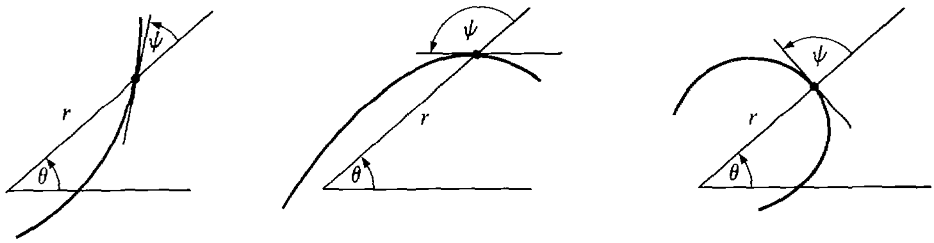


Figure 7.8.5

Step 1 Compute $dr/d\theta$.

Step 2 Find all points where $r = 0$ or $dr/d\theta = 0$.

Step 3 Sketch $y = f(x)$ in rectangular coordinates. (A method for doing this is given in Section 3.9.)

Step 4 Compute r, $dr/d\theta$, and $\tan\psi = r(dr/d\theta)$ at the points where $r = 0$ or $dr/d\theta = 0$ and at least one point between. Make a table, and test for local maxima or minima.

Step 5 Draw a smooth curve using the rectangular graph of step three and the table of step four.

EXAMPLE 1 Sketch the curve $r = 1 + \cos\theta$.

Step 1 $dr/d\theta = -\sin\theta$.

Step 2 $r = 0$ when $\theta = \pi$. $dr/d\theta = 0$ when $\theta = 0, \pi$.

Step 3 See Figure 7.8.6.

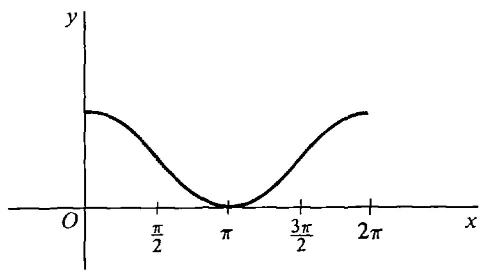


Figure 7.8.6

Step 4

θ	$r = 1 + \cos\theta$	$dr/d\theta$	$\tan\psi$	Comments
0	2	0	—	max
$\pi/2$	1	-1	-1	$\|r\|$ decreasing
π	0	0	—	min, cusp at 0
$3\pi/2$	1	1	1	$\|r\|$ increasing

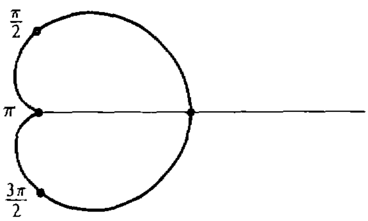


Figure 7.8.7

Step 5 We draw the curve in Figure 7.8.7. The curve is called a *cardioid* because of its heart shape.

EXAMPLE 2 Sketch the curve $r = \sin 2\theta$.

Step 1 $dr/d\theta = 2\cos 2\theta$.

Step 2 $r = 0$ at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. $dr/d\theta = 0$ at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Step 3 See Figure 7.8.8.

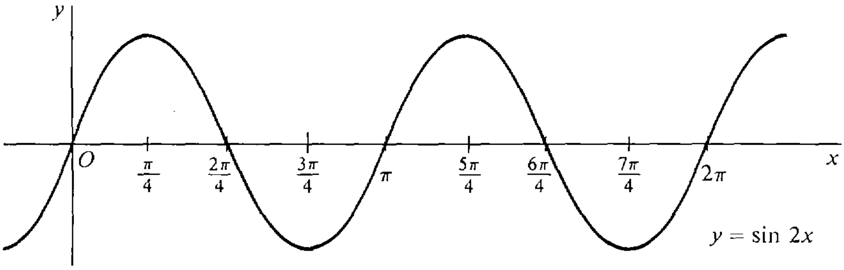


Figure 7.8.8

Step 4 We take values at intervals of $\frac{\pi}{8}$ beginning at $\theta = 0$. We can save some time by observing that the values from π to 2π are the same as those from 0 to π.

θ	$r = \sin 2\theta$	$dr/d\theta$	$\tan\psi$	Comments
0 and π	0	2	0	crosses origin
$\pi/8$ and $9\pi/8$	$\sqrt{2}/2$	$\sqrt{2}$	1/2	$\lvert r\rvert$ increasing
$2\pi/8$ and $10\pi/8$	1	0	—	max
$3\pi/8$ and $11\pi/8$	$\sqrt{2}/2$	$-\sqrt{2}$	$-1/2$	$\lvert r\rvert$ decreasing
$4\pi/8$ and $12\pi/8$	0	-2	0	crosses origin
$5\pi/8$ and $13\pi/8$	$-\sqrt{2}/2$	$-\sqrt{2}$	1/2	$\lvert r\rvert$ increasing
$6\pi/8$ and $14\pi/8$	-1	0	—	min
$7\pi/8$ and $15\pi/8$	$-\sqrt{2}/2$	$\sqrt{2}$	$-1/2$	$\lvert r\rvert$ decreasing

Step 5 We plot the points and trace out the curve as θ increases from 0 to 2π. Figure 7.8.9 shows the curve at various stages of development. The graph looks like a four-leaf clover.

If r approaches ∞ as θ approaches 0 or π, the curve may have a horizontal

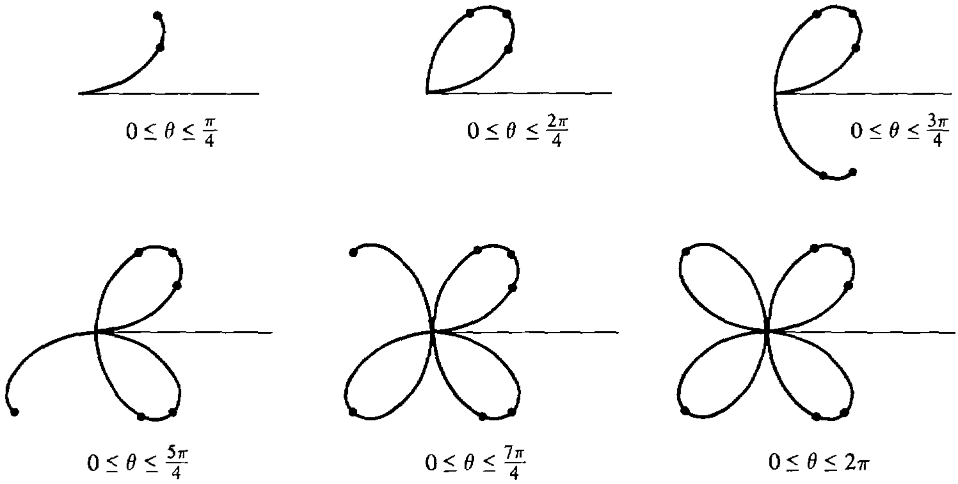


Figure 7.8.9

asymptote which can be found by computing the limit of y. At $\theta = \pi/2$ or $3\pi/2$ there may be vertical asymptotes. The method is illustrated in the following example.

EXAMPLE 3 Sketch $r = \tan(\frac{1}{2}\theta)$.

Step 1 $dr/d\theta = \frac{1}{2}\sec^2(\frac{1}{2}\theta)$.

$$y = r\sin\theta = \sin\tfrac{1}{2}\theta \sin\theta/\cos\tfrac{1}{2}\theta$$
$$= \sin\tfrac{1}{2}\theta(2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta)/\cos\tfrac{1}{2}\theta = 2\sin^2(\tfrac{1}{2}\theta).$$

Step 2 $r = 0$ at $\theta = 0$.
r is undefined at $\theta = \pi$.
$dr/d\theta$ is never 0.

Step 3 See Figure 7.8.10.

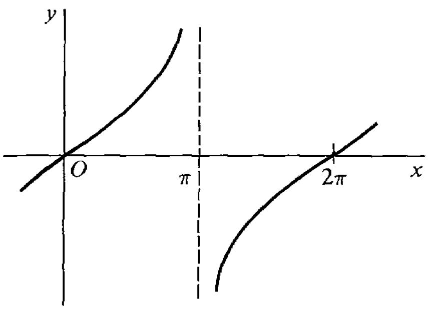


Figure 7.8.10

Step 4

θ	r or $\lim r$	$\lim y$	$dr/d\theta$	$\tan\psi$	Comments
0	0		1/2		crosses origin
$\pi/2$	1		1	1	$\lvert r\rvert$ increasing
$\theta \to \pi^-$	∞	2			asymptote $y = 2$
$\theta \to \pi^+$	$-\infty$	2			asymptote $y = 2$
$3\pi/2$	-1		1	-1	$\lvert r\rvert$ decreasing

Step 5 The curve crosses itself at the point $x = 0, y = 1$, because this point has both polar coordinates

$$(r = 1, \theta = \pi/2), (r = -1, \theta = 3\pi/2).$$

Figure 7.8.11 shows the graph for various stages of development.

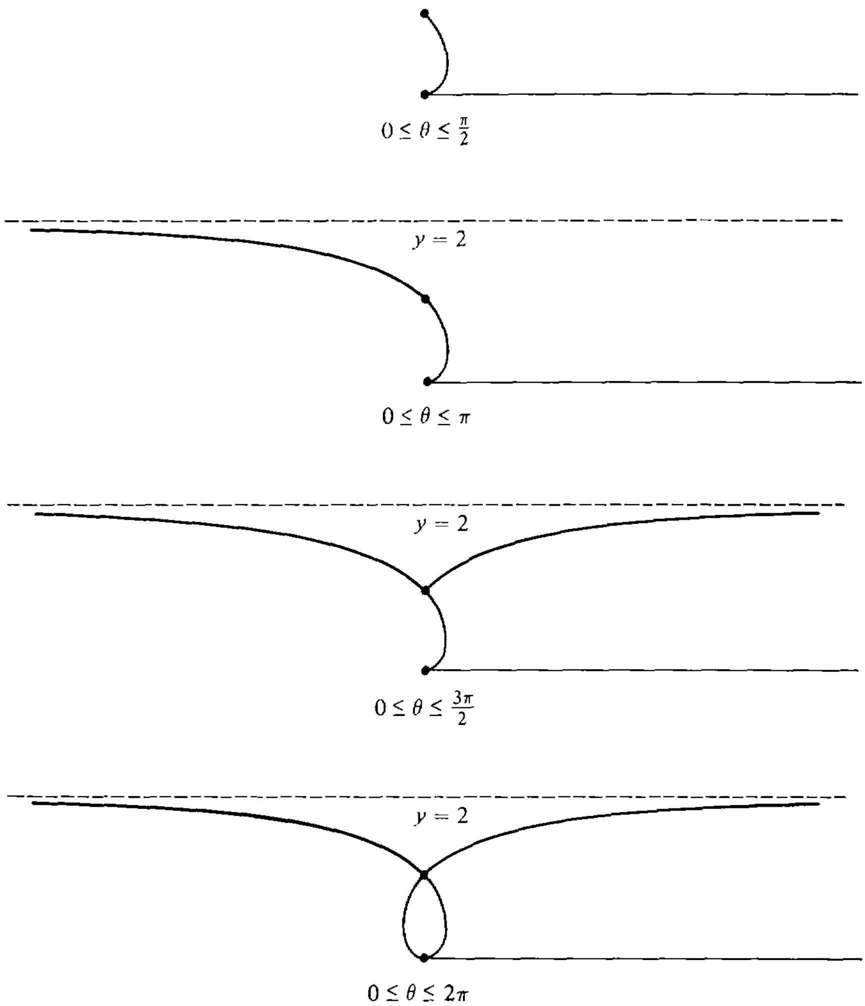


Figure 7.8.11

PROOF OF THEOREM 2 Assume the curve is not vertical at the point P, that is, $dx \neq 0$. Since

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we have
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + (dr/d\theta)\sin\theta}{-r\sin\theta + (dr/d\theta)\cos\theta}.$$

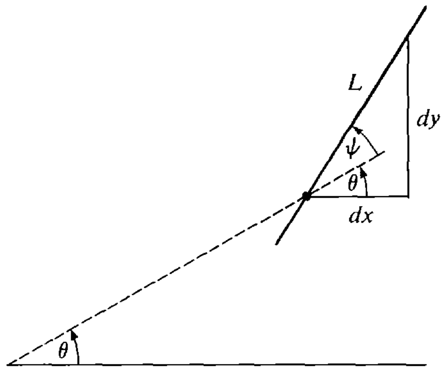


Figure 7.8.12

By the definition of the tangent line L (see Figure 7.8.12),

$$\frac{dy}{dx} = \frac{\text{change in } y \text{ along } L}{\text{change in } x \text{ along } L} = \frac{\sin(\theta + \psi)}{\cos(\theta + \psi)}.$$

Using the addition formulas,

$$\frac{dy}{dx} = \frac{\sin\theta\cos\psi + \cos\theta\sin\psi}{\cos\theta\cos\psi - \sin\theta\sin\psi}.$$

Thus $$\frac{r\cos\theta + (dr/d\theta)\sin\theta}{-r\sin\theta + (dr/d\theta)\cos\theta} = \frac{\sin\psi\cos\theta + \cos\psi\sin\theta}{-\sin\psi\sin\theta + \cos\psi\cos\theta}.$$

Multiplying out and canceling, we get

$$r\cos\psi(\sin^2\theta + \cos^2\theta) = \frac{dr}{d\theta}\sin\psi(\sin^2\theta + \cos^2\theta),$$

whence $$r\cos\psi = \frac{dr}{d\theta}\sin\psi, \qquad \frac{1}{r}\frac{dr}{d\theta} = \cot\psi.$$

If the curve is vertical at P we may use the same proof but with dx/dy instead of dy/dx.

PROBLEMS FOR SECTION 7.8

In Problems 1–6, find $\tan\psi$, where ψ is the angle between a line through the origin and the curve.

1 $r = \theta$

2 $r = \sin\theta$

3 $r = \cos\theta$

4 $r = \sec\theta$

5 $r = 1 + \cos\theta$

6 $r = \sin(2\theta)$

In Problems 7–25, sketch the given curve in polar coordinates by the method described in the text; $0 \le \theta \le 2\pi$ unless stated otherwise.

7 $r = \sin\theta + \cos\theta$

8 $r = 2 + 2\sin\theta$

9 $r = 1\frac{1}{2} + \sin\theta$

10 $r = 2 + \cos\theta$

11 $r = \frac{1}{2} + \cos\theta$

12 $r = \cos(\frac{1}{2}\theta), \quad 0 \le \theta \le 4\pi$

13 $r = \sin(\frac{1}{3}\theta), \quad 0 \le \theta \le 6\pi$

14 $r = \sin^2\theta$

15 $r = 1 + 3\cos^2(2\theta)$

16 $r = \sin^2(3\theta)$

17 $r = \tan\theta$

18 $r = \sec(\frac{1}{2}\theta), \quad 0 < \theta < 4\pi$

19 $r = 1 + \sec\theta$

20 $r = \dfrac{1}{1 - \cos\theta}$

21 $r = \dfrac{1}{1 + \sin\theta}$

22 $r = \cot(2\theta)$

23 $r = \pi/\theta, \quad 0 < \theta < \infty$

24 $r = 1 + \pi/\theta, \quad 0 < \theta < \infty$

25 $r = \sqrt{\pi/\theta}, \quad 0 < \theta < \infty$

In Problems 26–29, find the points where x and y have maxima and minima.

26 $r = 1 + \cos\theta$

27 $r = 1 + \sin^2\theta$

28 $r = \sin(2\theta)$

29 $r = \frac{3}{2} + \cos\theta$

30 Find all points where the curves $r = 1 + \cos\theta$ and $r = 3\cos\theta$ intersect.

31 Find all points where the curves $r = \frac{1}{2}$ and $r = \sin(2\theta)$ intersect. *Warning:* The points (r, θ) and $(-r, \pi + \theta)$ are the same.

32 Find all points where the curves $r = \cos\theta$ and $r = \sin(2\theta)$ intersect.

7.9 AREA IN POLAR COORDINATES

In this section we derive a formula for the area of a region in polar coordinates. Section 6.3 on the length of a curve in rectangular coordinates should be studied before this and the following section.

Our starting point for areas in rectangular coordinates was the formula for the area of a rectangle. In polar coordinates our starting point is the formula for the area of a sector of a circle.

THEOREM 1

A sector of a circle with radius r and central angle θ has area

$$A = \tfrac{1}{2}r^2\theta.$$

An arc of a circle with radius r and central angle θ has length

$$s = r\theta.$$

PROOF Consider a sector POQ shown in Figure 7.9.1. To simplify notation let O be the origin, and put the sector POQ in the first quadrant with P on the

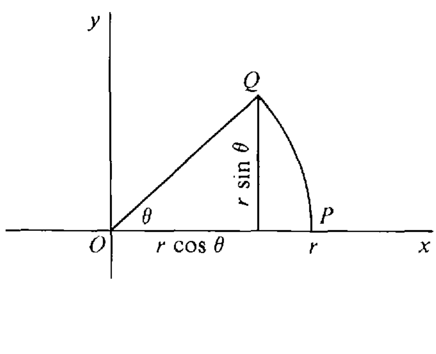


Figure 7.9.1

x-axis. Then

$$P = (r, 0), \qquad Q = (r\cos\theta, r\sin\theta).$$

The arc QP has the equation

$$y = \sqrt{r^2 - x^2}, \qquad r\cos\theta \le x \le r.$$

We see from the figure that

$$A = \tfrac{1}{2}r^2 \sin\theta\cos\theta + \int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\,dx.$$

Integrating by the trigonometric substitution $x = r\sin\phi$, we get

$$\int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\,dx = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\cos\theta.$$

Therefore $A = \frac{1}{2}r^2\theta$. By definition, $A = \frac{1}{2}rs$, so

$$s = \frac{2A}{r} = r\theta.$$

The next theorem gives the formula for area in polar coordinates.

THEOREM 2

Let $r = f(\theta)$ be continuous and $r \ge 0$ for $a \le \theta \le b$, where $b \le a + 2\pi$. Then the region R bounded by the curve $r = f(\theta)$ and the lines $\theta = a$ and $\theta = b$ has area

$$A = \tfrac{1}{2}\int_a^b f(\theta)^2\,d\theta.$$

Discussion Imagine a point P moving along the curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$. The line OP will sweep out the region R in Figure 7.9.2. Since $b \le a + 2\pi$, the line will complete at most one revolution, so no point of R will be counted more than once.

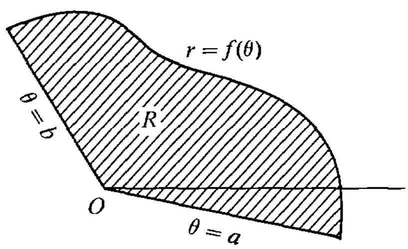


Figure 7.9.2

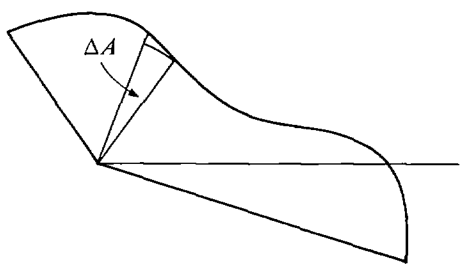


Figure 7.9.3

The formula for area can be seen intuitively by considering an infinitely small wedge ΔA of R between θ and $\theta + \Delta\theta$. (Figure 7.9.3). The wedge is almost a sector of a circle of radius $f(\theta)$ with central angle $\Delta\theta$, so

$$\Delta A \approx \tfrac{1}{2}f(\theta)^2\,\Delta\theta \qquad \text{(compared to } \Delta\theta\text{)}.$$

By the Infinite Sum Theorem,

$$A = \tfrac{1}{2}\int_a^b f(\theta)^2\,d\theta.$$

The actual proof follows this intuitive idea but the area of ΔA must be computed more carefully.

PROOF Let $\Delta\theta$ be positive infinitesimal and let θ be a hyperreal number between a and $b - \Delta\theta$. Consider the wedge of R with area ΔA between θ and $\theta + \Delta\theta$. Since $f(\theta)$ is continuous, it has a minimum value m and maximum value M between θ and $\theta + \Delta\theta$, and furthermore,

$$m \approx f(\theta), \qquad M \approx f(\theta).$$

The sector between θ and $\Delta\theta$ of radius m is inscribed in ΔA while the sector of radius M is circumscribed about ΔA.

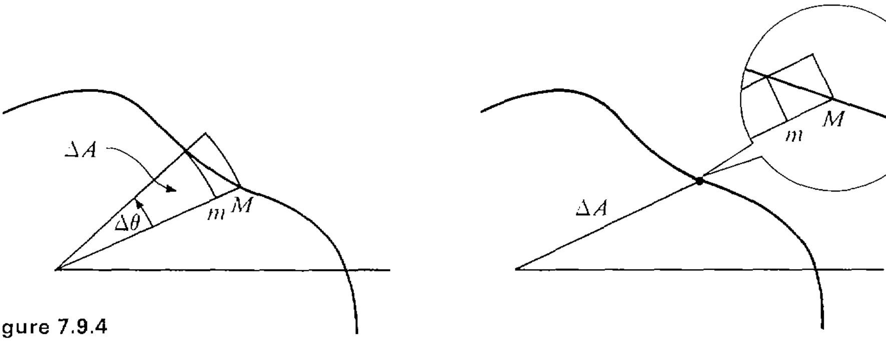


Figure 7.9.4

(Figure 7.9.4 shows the inscribed and circumscribed sectors for real $\Delta\theta$ and infinitesimal $\Delta\theta$.) By Theorem 1, the two sectors have areas $\frac{1}{2}m^2\,\Delta\theta$ and $\frac{1}{2}M^2\,\Delta\theta$. Moreover, ΔA is between those two areas,

$$\tfrac{1}{2}m^2\,\Delta\theta \le \Delta A \le \tfrac{1}{2}M^2\,\Delta\theta,$$
$$\tfrac{1}{2}m^2 \le \Delta A/\Delta\theta \le \tfrac{1}{2}M^2.$$

Taking standard parts,

$$\tfrac{1}{2}f(\theta)^2 \le st(\Delta A/\Delta\theta) \le \tfrac{1}{2}f(\theta)^2.$$

Therefore $$\Delta A/\Delta\theta \approx \tfrac{1}{2}f(\theta)^2,$$

and by the Infinite Sum Theorem,

$$A = \tfrac{1}{2}\int_a^b f(\theta)^2\,d\theta.$$

Theorem 1 is also true in the case that $r = f(\theta)$ is continuous and $r \le 0$.

Since $$A = \tfrac{1}{2}\int_a^b f(\theta)^2\,d\theta = \tfrac{1}{2}\int_a^b (-f(\theta))^2\,d\theta,$$

the region R bounded by the curve $r = f(\theta)$ has the same area as the region S bounded by the curve $r = -f(\theta)$. Both areas are positive. As we can see from Figure 7.9.5, S looks exactly like R but is on the opposite side of the origin.

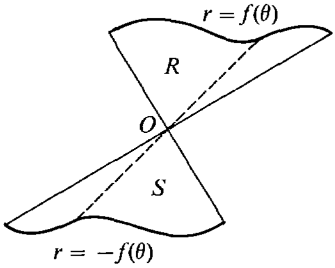


Figure 7.9.5

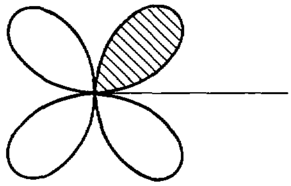


Figure 7.9.6

EXAMPLE 1 Find the area of one loop of the "four-leaf clover" $r = \sin 2\theta$. From Figure 7.9.6, we see that one loop is traced out when θ goes from 0 to $\pi/2$. Therefore the area is

$$A = \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\,d\theta = \frac{1}{2}\int_0^{\pi} \frac{1}{2}\sin^2\phi\,d\phi$$

$$= \frac{1}{4}\int_0^{\pi} \sin^2\phi\,d\phi = \frac{1}{4}\left(-\frac{1}{2}\sin\phi\cos\phi + \frac{1}{2}\phi\right)\Big]_0^{\pi} = \frac{1}{8}\pi.$$

As one would expect, all four loops have the same area.

On the loop from $\theta = \pi/2$ to $\theta = \pi$, the value of $r = \sin 2\theta$ is negative. However, the area is again

$$A = \frac{1}{2}\int_{\pi/2}^{\pi} \sin^2(2\theta)\,d\theta = \frac{1}{8}\pi.$$

Our next example shows why the hypothesis that r has the same sign for $a \le \theta \le b$ is needed in Theorem 2.

EXAMPLE 2 Find the area of the region inside the circle $r = \sin\theta$ (Figure 7.9.7).

The point (r, θ) goes around the circle once when $0 \le \theta \le \pi$ with r positive, and again when $\pi \le \theta \le 2\pi$ with r negative. The theorem says that we will get the correct area if we take either 0 and π, or π and 2π, as the limits of

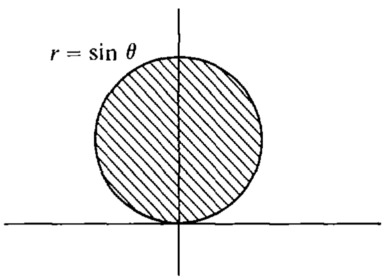


Figure 7.9.7

integration. Thus

$$A = \int_0^{\pi} \tfrac{1}{2}\sin^2\theta\, d\theta = \tfrac{1}{2}(-\tfrac{1}{2}\sin\theta\cos\theta + \tfrac{1}{2}\theta)\Big]_0^{\pi} = \tfrac{1}{4}(\pi - 0) = \pi/4.$$

Alternatively,

$$A = \int_{\pi}^{2\pi} \tfrac{1}{2}\sin^2\theta\, d\theta = \tfrac{1}{2}(-\tfrac{1}{2}\sin\theta\cos\theta + \tfrac{1}{2}\theta)\Big]_{\pi}^{2\pi} = \tfrac{1}{4}(2\pi - \pi) = \pi/4.$$

Since the curve is a circle of radius $\frac{1}{2}$, our answer $\pi/4$ agrees with the usual formula $A = \pi r^2$.

Integrating from 0 to 2π would count the area twice and give the wrong answer.

EXAMPLE 3 Find the area of the region inside both the circles $r = \sin\theta$ and $r = \cos\theta$.

The first thing to do is draw the graphs of both curves. The graphs are shown in Figure 7.9.8.

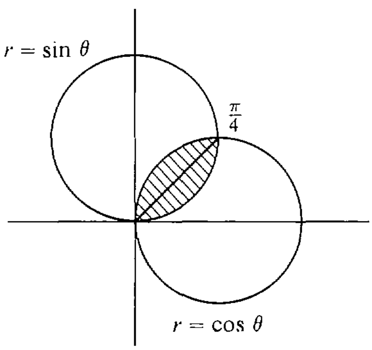


Figure 7.9.8

We see that the two circles intersect at the origin and at $\theta = \pi/4$. The region is divided into two parts, one bounded by $r = \sin\theta$ for $0 \le \theta \le \pi/4$ and the other bounded by $r = \cos\theta$ for $\pi/4 \le \theta \le \pi/2$. Thus

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2}\sin^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\cos^2\theta\, d\theta \\ &= \frac{1}{2}\left(-\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right)\Big]_0^{\pi/4} + \frac{1}{2}\left(\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right)\Big]_{\pi/4}^{\pi/2} \\ &= \frac{1}{2}\left[\left(-\frac{1}{4} - 0\right) + \left(\frac{\pi}{8} - 0\right) + \left(0 - \frac{1}{4}\right) + \left(\frac{\pi}{4} - \frac{\pi}{8}\right)\right] = \frac{\pi}{8} - \frac{1}{4}. \end{aligned}$$

PROBLEMS FOR SECTION 7.9

In Problems 1–13, find the area of the regions bounded by the following curves in polar coordinates.

1 $r = 2a\cos\theta$

2 $r = 1 + \cos\theta$

3 $r = \sqrt{\sin\theta}$

4 $r = 2 + \cos\theta$

5 The loop in $r = \tan(\frac{1}{2}\theta)$

6 One loop of $r = \cos(3\theta)$

7 One loop of $r = \sin^2\theta$

8 The large loop of $r = \frac{1}{2} + \cos\theta$

9 The small loop of $r = \frac{1}{2} + \cos\theta$

10 One loop of $r^2 = \cos(2\theta)$

11 $\theta = 0, \quad \theta = \pi/3, \quad r = \cos\theta$

12 $\theta = \pi/6, \quad \theta = \pi/3, \quad r = \sec\theta$

13 $r = \tan\theta, \quad r = \frac{1}{\sqrt{2}}\csc\theta$

14 Find the area of the region inside the curve $r = 2\cos\theta$ and outside the curve $r = 1$.

15 Find the area of the region inside the curve $r = 2\sin\theta$ and above the line $r = \frac{3}{2}\csc\theta$.

16 Find the area of the region inside the spiral $r = \theta, 0 \leq \theta \leq 2\pi$.

17 Find the area of the region inside the spiral $r = \sqrt{\theta}, 0 \leq \theta \leq 2\pi$.

18 Find the area of the region inside both of the curves $r = \sqrt{3}\cos\theta$, $r = \sin\theta$.

19 Find the area of the region inside both of the curves $r = 1 - \cos\theta$, $r = \cos\theta$.

20 The center of a circle of radius one is on the circumference of a circle of radius two. Find the area of the region inside both circles.

21 Find a formula for the area of the region between the curves $r = f(\theta)$ and $r = g(\theta)$, $a \leq \theta \leq b$, when $0 \leq f(\theta) \leq g(\theta)$.

7.10 LENGTH OF A CURVE IN POLAR COORDINATES

Consider a curve

$$r = f(\theta), \qquad a \leq \theta \leq b$$

in polar coordinates. The curve is called *smooth* if $f'(\theta)$ is continuous for θ between a and b. In Chapter 6 we obtained a formula for the length of a smooth parametric curve in rectangular coordinates. We may now apply this to get a formula for the length of a smooth curve in polar coordinates.

THEOREM

The length of a smooth curve

$$r = f(\theta), \qquad a \leq \theta \leq b$$

in polar coordinates which does not retrace itself is

$$s = \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta,$$

or equivalently

$$s = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta.$$

Discussion The formula can be seen intuitively as follows. We see from Figure 7.10.1 that

$$\Delta s \approx \sqrt{(r\,\Delta\theta)^2 + \Delta r^2} = \sqrt{r^2 + (\Delta r/\Delta\theta)^2}\,\Delta\theta$$
$$\approx \sqrt{r^2 + (dr/d\theta)^2}\,\Delta\theta \qquad \text{(compared to } \Delta\theta\text{)}.$$

By the Infinite Sum Theorem,

$$s = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta.$$

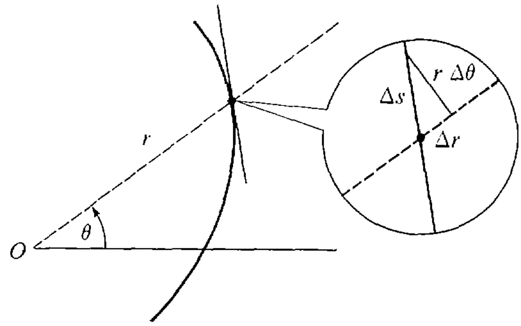


Figure 7.10.1

The length of a curve has already been defined using rectangular coordinates, and the theorem states that the new formula will give the same number for the length.

PROOF The curve is given in rectangular coordinates by the parametric equation

$$x = f(\theta)\cos\theta, \qquad y = f(\theta)\sin\theta.$$

The derivatives are

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta,$$

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta.$$

Since $f(\theta)$ and $f'(\theta)$ are continuous, $dx/d\theta$ and $dy/d\theta$ are continuous. Recall the length formula for parametric equations:

$$s = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

We compute

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= f(\theta)^2\sin^2\theta - 2f(\theta)f'(\theta)\sin\theta\cos\theta + f'(\theta)\cos^2\theta \\ &\quad + f(\theta)^2\cos^2\theta + 2f(\theta)f'(\theta)\sin\theta\cos\theta + f'(\theta)^2\sin^2\theta \\ &= f(\theta)^2(\sin^2\theta + \cos^2\theta) + f'(\theta)^2(\cos^2\theta + \sin^2\theta) \\ &= f(\theta)^2 + f'(\theta)^2.\end{aligned}$$

The desired formula now follows by substitution.

EXAMPLE 1 Find the length of the spiral $r = \theta^2$ from $\theta = \pi$ to $\theta = 4\pi$, shown in Figure 7.10.2.

$$\begin{aligned}s &= \int_\pi^{4\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \\ &= \int_\pi^{4\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_\pi^{4\pi} \sqrt{\theta^2 + 4}\,\theta\, d\theta.\end{aligned}$$

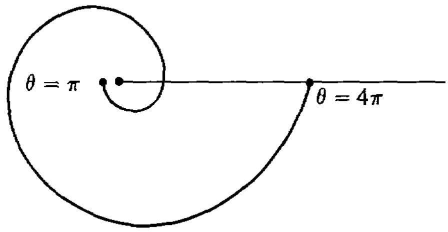


Figure 7.10.2

Let $u = \theta^2 + 4$, $du = 2\theta\, d\theta$. Then

$$s = \int_{\pi^2+4}^{16\pi^2+4} \tfrac{1}{2}\sqrt{u}\, du = \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2}\Big]_{\pi^2+4}^{16\pi^2+4}$$
$$= \tfrac{1}{3}((16\pi^2 + 4)^{3/2} - (\pi^2 + 4)^{3/2}).$$

EXAMPLE 2 Find the length of the curve $r = \sin\theta$ from $\theta = \alpha$ to $\theta = \beta$, shown in Figure 7.10.3. $dr/d\theta = \cos\theta$, so

$$s = \int_\alpha^\beta \sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = \int_\alpha^\beta d\theta = \beta - \alpha.$$

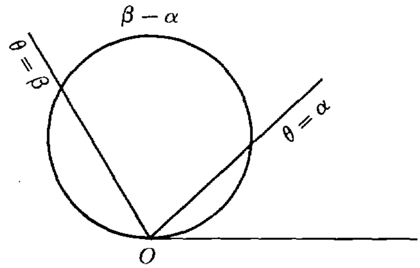


Figure 7.10.3

The graph of $r = \sin\theta$ is a circle of radius $\frac{1}{2}$ which passes through O. Example 2 proves that the length of an arc of the circle is equal to the angle formed by the ends of the arc and the origin. Note that if we take $\alpha = 0$ and $\beta = 2\pi$ we get an arc length of 2π, which is twice the circumference of the circle. This is because the point (r, θ) goes around the circle twice, once from $\theta = 0$ to $\theta = \pi$ and once from $\theta = \pi$ to $\theta = 2\pi$.

PROBLEMS FOR SECTION 7.10

In Problems 1–10, find the length in polar coordinates.

1 $r = 7, \quad 0 \le \theta \le 2\pi$

2 $r = \cos\theta, \quad \pi/4 \le \theta \le \pi/3$

3 $r = \sec\theta, \quad -\pi/4 \le \theta \le \pi/4$

4 $r = 6\theta^2, \quad 0 \le \theta \le \sqrt{5}$

5 $r = \theta^4, \quad 0 \le \theta \le 1$

6 $r = a\sin\theta + b\cos\theta, \quad 0 \le \theta \le \pi$

7 $r = 1 - \cos\theta, \quad 0 \le \theta \le \pi$

8 $r = 2 + 2\cos\theta, \quad 0 \le \theta \le 2\pi$

9 $r = \sin^2(\frac{1}{2}\theta), \quad 0 \le \theta \le 2\pi$

10 $r = \sin^3(\frac{1}{3}\theta), \quad 0 \le \theta \le 3\pi$

In Problems 11–14, set up an integral for the length of the curve.

11 $r = \sin(2\theta), \quad 0 \le \theta \le 2\pi$

12 $r = \tan\theta, \quad 0 \le \theta \le \pi/4$

13 $r = \theta, \quad 0 \le \theta \le b$

14 $r = \theta^n, \quad 0 \le \theta \le b$

□ 15 Show that the surface area generated by rotating the curve $r = f(\theta)$, $a \le \theta \le b$, about the y-axis is

$$A = \int_a^b 2\pi r \cos\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \qquad \text{(about } y\text{-axis)}.$$

(Assume $0 \le a < b \le \pi/2$.) Show that the corresponding formula for a rotation about the x-axis is

$$A = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \qquad \text{(about } x\text{-axis)}.$$

In Problems 16–21, find the surface area generated by rotating the curve about the given axis.

16 $r = \sin\theta, \quad 0 \le \theta \le \pi/3$, about y-axis

17 $r = a\sin\theta + b\cos\theta, \quad 0 \le \theta \le \pi/2$, about y-axis

18 $r = 1 + \cos\theta, \quad 0 \le \theta \le \pi/2$, about x-axis

19 $r = \sqrt{\cos(2\theta)}, \quad 0 \le \theta \le \pi/4$, about y-axis

20 $r = \sqrt{\cos(2\theta)}, \quad 0 \le \theta \le \pi/4$, about x-axis

21 $r = \cos^2(\frac{1}{2}\theta), \quad 0 \le \theta \le \pi/2$, about x-axis

EXTRA PROBLEMS FOR CHAPTER 7

1 Find dy/dx where $y = x + \sin x$.

2 Find dy/dx where $y = \sin(1/x)$.

3 Find $dy/d\theta$ where $y = \sqrt{\theta}\cos\theta$.

4 Find $dy/d\theta$ where $y = \sin(\tan\theta)$.

5 Evaluate $\lim_{\theta\to 0} \dfrac{\sin(4\theta)}{\sin(3\theta)}$.

6 Evaluate $\lim_{u\to 0} \dfrac{\cos(6u) - 1}{u^2}$.

7 Evaluate $\int \cos(\cos\theta)\sin\theta\, d\theta$.

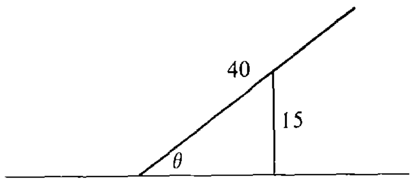
8 Evaluate $\int 3\sqrt{\sin x}\cos x\, dx$

9 Evaluate $\int_{-\pi/2}^{\pi/2} 4\cos\theta\, d\theta$.

10 Evaluate $\int_0^{\pi/2} \tan x \sec x\, dx$.

11 An airplane travels in a straight line at 600 mph at an altitude of 4 miles. Find the rate of change of the angle of elevation one minute after the airplane passes directly over an observer on the ground.

12 A 40 ft ladder is to be propped up against a 15 ft wall as shown in the figure. What angle should the ladder make with the ground if the horizontal distance the ladder extends beyond the wall is to be a maximum?



13 Find dy/dx where $y = \arccos\sqrt{x}$.

14 Find dy/dx where $y = \operatorname{arcsec}\sqrt{x}$.

15 Find du/dt where $u = \arctan t - t$.

16 Find du/dt where $u = \arcsin(1/t)$.

17 Evaluate $\lim_{x\to 0} \dfrac{\arctan x}{x}$.

18 Evaluate $\int \dfrac{dt}{a^2 + b^2t^2}$.

19 Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2-2x}}$, $x > 2$.

20 Evaluate $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$.

21 Evaluate $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$ in two ways, by change of variables and by parts.

22 Evaluate $\int x\sin(3x)\,dx$.

23 Evaluate $\int_0^1 \cos\sqrt{\theta}\,d\theta$.

24 Find the volume of the solid formed by rotating the region under the curve $y = x\sqrt{\sin x}$, $0 \le x \le \pi$, about the x-axis.

25 Find the volume of the solid generated by rotating the region under the curve $y = \tan x$, $0 \le x \le \pi/4$, about the x-axis.

26 Evaluate $\int \cot^4\theta\,d\theta$.

27 Evaluate $\int \tan^5\theta\sec^5\theta\,d\theta$.

28 Evaluate $\int (2x^2-1)^{-3/2}\,dx$.

29 Evaluate $\int \frac{\sqrt{2-x^2}}{x^2}\,dx$.

30 Evaluate $\int \frac{1}{(1+x^2)^2}\,dx$.

In Problems 31–34, sketch the given function in (a) rectangular coordinates, (b) polar coordinates. Let $0 \le \theta \le 2\pi$.

31 $r = 1 - \cos\theta$

32 $r = \cos(3\theta)$

33 $r = \frac{1}{2+\sin\theta}$

34 $r^2 = \cos(2\theta)$

35 Find the area of the polar region bounded by $r = 1 + \sin^2\theta$.

36 Find the area of the polar region bounded by $r = \sin\theta + \cos\theta$.

37 Find the area of the polar region inside both the curves $r = 1 - \cos\theta$ and $r = 1 + \cos\theta$.

38 Find the length in polar coordinates of the curve

$$r = \sin^4(\tfrac{1}{4}\theta), \qquad 0 \le \theta \le \pi.$$

39 Find the surface area generated by rotating the polar curve

$$r = 1 - \cos\theta, \qquad 0 \le \theta \le \pi/2,$$

about the x-axis.

□ 40 Use the Intermediate Value Theorem to prove that $\arctan y$ has domain $(-\infty, \infty)$.

□ 41 Use the Intermediate Value Theorem to prove that the domain of $\operatorname{arcsec} y$ is the set of all y such that $y \le -1$ or $y \ge 1$.

□ 42 Prove that if f is a differentiable function of x then

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx.$$

□ 43 If u and v are differentiable functions of x then

$$\int u^2\,dv = u^2v - 2\int uv\,du.$$

□ 44 Show that if f' and g are differentiable for all x then

$$\int g(x)g'(x)f''(g(x))\,dx = f'(g(x))g(x) - f(g(x)) + C.$$

☐ **45** Use integration by parts to prove the reduction formula

$$\int \frac{dx}{(1+x^2)^{m+1}} = \frac{1}{2m}\frac{x}{(1+x^2)^m} + \left(1 - \frac{1}{2m}\right)\int \frac{dx}{(1+x^2)^m}.$$

Hint:
$$\frac{1}{(1+x^2)^{m+1}} = \frac{1}{(1+x^2)^m} - \frac{x^2}{(1+x^2)^{m+1}}.$$

☐ **46** Suppose $y = f(x)$, $a < x < b$ and $x = g(y)$, $c < y < d$ are inverse functions and are strictly increasing. Let $y_0 = f(x_0)$. Prove that:

(a) If f is continuous at x_0, g is continuous at y_0.
(b) If $f'(x_0)$ exists and $f'(x_0) \neq 0$, then $g'(y_0)$ exists.

☐ **47** Justify the following formula for the area of the polar region bounded by the continuous curves

$$\theta = f(r), \qquad \theta = g(r), \qquad a \le r \le b,$$

where $0 \le f(r) \le g(r) \le 2\pi$.

$$A = \int_a^b r(g(r) - f(r))\,dr.$$

☐ **48** Justify the following formula for the mass of an object in the polar region $0 \le r \le f(\theta)$, $a \le \theta \le b$, with density $\rho(\theta)$ per unit area.

$$m = \int_a^b \tfrac{1}{2}\rho(\theta)(f(\theta))^2\,d\theta.$$

☐ **49** Justify the following formulas for the centroid of the polar region $0 \le r \le f(\theta)$, $a \le \theta \le b$.

$$\bar{x} = \frac{\int_a^b \frac{1}{3}\cos\theta(f(\theta))^3\,d\theta}{\int_a^b \frac{1}{2}(f(\theta))^2\,d\theta}, \qquad \bar{y} = \frac{\int_a^b \frac{1}{3}\sin\theta(f(\theta))^3\,d\theta}{\int_a^b \frac{1}{2}(f(\theta))^2\,d\theta}.$$

Hint: The centroid of a triangle is located on a median $\frac{2}{3}$ of the way from a vertex to the opposite side.

☐ **50** Find the centroid of the sector $0 \le r \le c$, $a \le \theta \le b$.

☐ **51** Find the centroid of the region bounded by the cardioid $r = 1 + \cos\theta$.