3

CONTINUOUS FUNCTIONS

3.1 HOW TO SET UP A PROBLEM

In applications, a calculus problem is often presented verbally, and it is up to you to set up the problem in mathematical terms. The problem can usually be described mathematically by a list of equations and inequalities. The next two sections contain several examples that illustrate the process of setting up a problem. The examples in this section are from algebra and geometry, and those in the next section are from calculus.

It is sometimes hard to see how to begin on a story problem. It is helpful to break the process up into three steps:

Step 1 Draw a diagram if possible, and label all quantities involved.

Step 2 Write the given information as a set of equations and/or inequalities.

Step 3 Solve the mathematical problem, and interpret the mathematical solution to answer the original story problem.

EXAMPLE 1 According to a treasure map, a buried treasure is located due east of a cave and is 200 paces from a tree. The tree is 30 paces east and 40 paces north of the cave. How far is the treasure from the cave?

The solution of this problem uses the quadratic formula, which will be needed throughout the calculus course. We review it here.

QUADRATIC FORMULA If $a \neq 0$, then

$$ax^2 + bx + c = 0$$

if and only if

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We solve Example 1 in three steps.

Step 1 Draw a diagram and label all quantities involved. In Figure 3.1.1, we put the cave at the origin and let x be the distance from the cave to the target along the x-axis. The tree is at the point (30, 40), and the treasure is at the point $(x, 0)$.

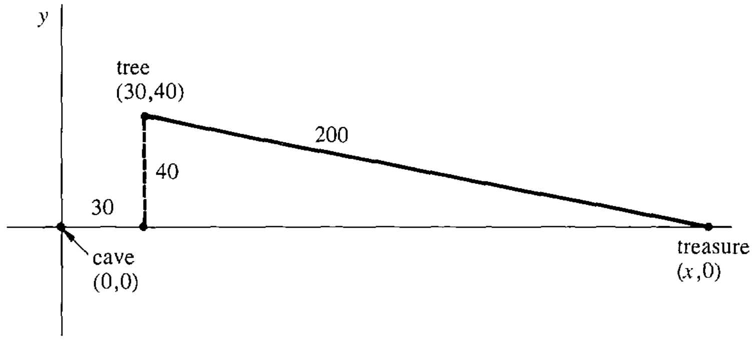


Figure 3.1.1

Step 2 Write the known information as a system of formulas. By the distance formula, we have

$$200 = \sqrt{(x-30)^2 + (0-40)^2}, \quad x \geq 0.$$

The inequality $x \geq 0$ arises because the treasure is east of the cave.

Step 3 Solve for x. We square the Distance Formula.

$$40{,}000 = (x-30)^2 + (0-40)^2 = x^2 - 60x + 900 + 1600$$
$$= x^2 - 60x + 2500$$
$$x^2 - 60x - 37{,}500 = 0$$

To find x we use the Quadratic Formula.

$$x = \frac{60 \pm \sqrt{(60)^2 - 4(-37{,}500)}}{2} = \frac{60 \pm \sqrt{153{,}600}}{2}$$
$$= 30 \pm \sqrt{38{,}400}$$

INTERPRET THE SOLUTION Since $x \geq 0$, we reject the negative solution. Thus $x = 30 + \sqrt{38{,}400} \sim 226$ paces. The treasure is approximately 226 paces from the cave.

Most calculus problems involve two or more variables.

EXAMPLE 2 A six-foot man stands near a ten-foot lamppost. Find the length of his shadow as a function of his distance from the lamppost.

Step 1 Draw a diagram and label all the quantities involved. In Figure 3.1.2, we let

$$x = \text{man's distance from lamppost},$$
$$s = \text{length of his shadow}.$$

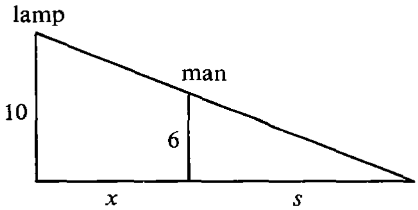


Figure 3.1.2

Step 2 Write the known information as a system of formulas. By similar triangles we have

$$\frac{s}{6} = \frac{s + x}{10}, \qquad x \geq 0.$$

The inequality $x \geq 0$ arises because the distance cannot be negative.

Step 3 Solve for s as a function of x.

$$10s = 6s + 6x,$$
$$4s = 6x,$$
$$s = \tfrac{3}{2}x.$$

INTERPRET THE SOLUTION $s = \frac{3}{2}x, \qquad x \geq 0.$

The domain of the function is $[0, \infty)$. The length of the shadow is $\frac{3}{2}$ times the distance from the lamppost. In this problem, x is the independent variable and s depends on x.

EXAMPLE 3 Two ships start at the same point at time $t = 0$. One ship moves north at 30 miles per hour, while the other ship moves east at 40 miles per hour. Find the distance between the two ships as a function of time.

Step 1 The ships start at the origin; the y-axis points north; and the x-axis points east. The diagram is shown in Figure 3.1.3. x and y are the distances of the east- and north-moving ships from the origin, and z is the distance between the ships, all in miles. t is the time in hours.

Step 2 $t \geq 0, \qquad y = 30t, \qquad x = 40t, \qquad z = \sqrt{x^2 + y^2}.$

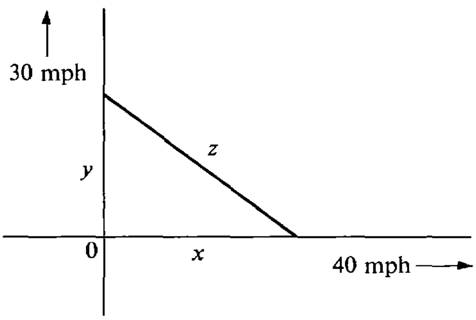


Figure 3.1.3

Step 3 $z = \sqrt{(30t)^2 + (40t)^2} = \sqrt{2500t^2} = 50t.$

INTERPRET THE SOLUTION $z = 50t, \qquad t \geq 0.$

t is the independent variable, and x, y, z all depend on t. The distance between the ships is $50t$ miles, where t is the time in hours.

EXAMPLE 4 A brush fire starts along a straight line segment of length 20 ft and expands in all directions at the rate of 2 ft per second. Find the burned out area as a function of time.

Step 1 A = total burned out area
A_1 = area of left semicircle
A_2 = area of central rectangle
A_3 = area of right semicircle
s = distance of spread of fire
t = time

The diagram is shown in Figure 3.1.4.

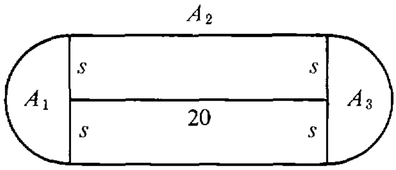


Figure 3.1.4

Step 2 $s = 2t, \qquad t \geq 0.$
$A_1 = \frac{1}{2}\pi s^2, \qquad A_2 = 20(2s), \qquad A_3 = \frac{1}{2}\pi s^2.$
$A = A_1 + A_2 + A_3.$

Step 3 $A_1 = \frac{1}{2}\pi(2t)^2 = 2\pi t^2.$
$A_2 = 20 \cdot 2 \cdot 2t = 80t.$
$A_3 = \frac{1}{2}\pi(2t)^2 = 2\pi t^2.$
$A = 2\pi t^2 + 80t + 2\pi t^2 = 4\pi t^2 + 80t.$

INTERPRET THE SOLUTION The burned out area is $A = 4\pi t^2 + 80t$ sq ft, $t \geq 0$, where t is time in seconds.

An algebraic identity that comes up frequently in calculus problems is

$$(a - b)(a + b) = a^2 - b^2.$$

Sometimes it occurs in the form

$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b.$$

EXAMPLE 5 The area of square A is twelve square units greater than the area of square B, and the side of A is three units greater than the side of B. Find the areas of A and B.

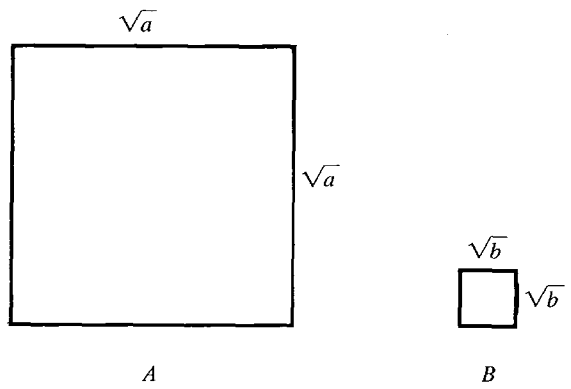


Figure 3.1.5

Step 1 Let a be the area of A and b the area of B. See Figure 3.1.5.

Step 2 The sides of the squares have length $\sqrt{a}$ and $\sqrt{b}$ respectively. Thus

$$a - b = 12, \qquad \sqrt{a} - \sqrt{b} = 3.$$

Step 3 We find $\sqrt{a} + \sqrt{b}$.

$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b,$$
$$3(\sqrt{a} + \sqrt{b}) = 12,$$
$$\sqrt{a} + \sqrt{b} = 4.$$

Adding the equations $\sqrt{a} + \sqrt{b} = 4$ and $\sqrt{a} - \sqrt{b} = 3$, we obtain $2\sqrt{a} = 7$, $\sqrt{a} = \frac{7}{2}$, $a = \frac{49}{4}$. Subtracting the equations gives $2\sqrt{b} = 1$, $\sqrt{b} = \frac{1}{2}$, $b = \frac{1}{4}$.

INTERPRET THE SOLUTION The area of square A is $\frac{49}{4}$ square units, and the area of square B is $\frac{1}{4}$ square units.

PROBLEMS FOR SECTION 3.1

1 Find the perimeter p of a square as a function of its area A.

2 A piece of clay in the shape of a cube of side s is rolled into a sphere of radius r. Find r as a function of s.

3 Find the volume V of a sphere as a function of its surface area S.

4 Find the area A of a rectangle of perimeter 4 as a function of the length x.

5 Find the distance z between the origin and a point on the parabola $y = 1 - x^2$ as a function of x.

6 Express the perimeter p of a right triangle as a function of the base x and height y.

7 Four small squares of side x are cut from the corners of a large cardboard square of side s. The sides are then folded up to form an open top box. Find the volume of the box as a function of s and x.

8 A ladder of length L is propped up against a wall with its bottom at distance x from the wall. Find the height y of the top of the ladder as a function of x.

9 A man of height y stands 3 ft from a ten foot high lamp. Find the length s of his shadow as a function of y.

10 One ship traveling north at 30 mph passes the origin at time $t = 0$ hours. A second ship moving east at 30 mph passes the origin at $t = 1$. Find the distance z between them as a function of t.

11 A ball is thrown from ground level, and its path follows the equations

$$x = bt, \qquad y = t - 16t^2.$$

How far does it travel in the x direction before it hits the ground?

12 A circular weedpatch is initially 2 ft in radius. It grows so that its radius increases by 1 ft/day. Find its area after five days.

13 A rectangle originally has length l and width w. Its shape changes so that its length increases by one unit per second while its width decreases by 2 units per second. Find its area as a function of l, w and time t.

14 At p units of pollution per item, a product can be made at a cost of $2 + 1/p$ dollars per item. x items are to be produced with a total pollution of one unit. Find the cost.

15 In economics, the profit in producing and selling x items is equal to the revenue minus the cost,

$$P(x) = R(x) - C(x).$$

If a product can be manufactured at a cost of \$10 per item and x items can be sold at a price of $100 - \sqrt{x}$ per item, find the profit as a function of x.

16 Suppose the demand for a commodity at price p is $x = 1000/\sqrt{p}$, that is, $x = 1000/\sqrt{p}$ items can be sold at a price of p dollars per item. If it costs $100 + 10x$ dollars to produce x items, find the profit as a function of the selling price p.

3.2 RELATED RATES

In a related rates problem, we are given the rate of change of one quantity and wish to find the rate of change of another. Such problems can often be solved by implicit differentiation.

EXAMPLE 1 The point of a fountain pen is placed on an ink blotter, forming a circle of ink whose area increases at the constant rate of 0.03 in.^{2}/sec. Find the rate at which the radius of the circle is changing when the circle has a radius of $\frac{1}{2}$ inch. We solve the problem in four steps.

Step 1 Label all quantities involved and draw a diagram.

$$t = \text{time} \qquad A = \text{area} \qquad r = \text{radius of circle}$$

Both A and r are functions of t. The diagram is shown in Figure 3.2.1.

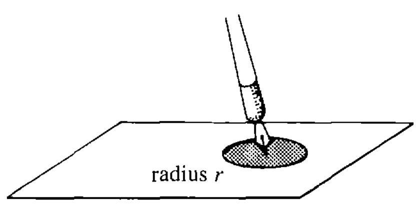


Figure 3.2.1

Step 2 Write the given information in the form of equations.

$$dA/dt = 0.03, \qquad A = \pi r^2.$$

The problem is to find dr/dt when $r = 1/2$.

Step 3 Differentiate both sides of the equation $A = \pi r^2$ with respect to t.

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}, \quad \text{whence} \quad 0.03 = 2\pi r\frac{dr}{dt}.$$

Step 4 Set $r = \frac{1}{2}$ and solve for dr/dt.

$$0.03 = 2\pi\frac{1}{2}\cdot\frac{dr}{dt}, \quad \text{so} \quad \frac{dr}{dt} = \frac{0.03}{\pi}\text{in./sec.}$$

EXAMPLE 2 A 10 foot ladder is propped against a wall. The bottom end is being pulled along the floor away from the wall at the constant rate of 2 ft/sec. Find the rate at which the top of the ladder is sliding down the wall when the bottom end is 5 ft from the wall. *Warning:* although the bottom end of the ladder is being moved at a constant rate, the rate at which the top end moves will vary with time.

Step 1 t = time,
x = distance of the bottom end from the wall,
y = height of the top end above the floor.

The diagram is shown in Figure 3.2.2.

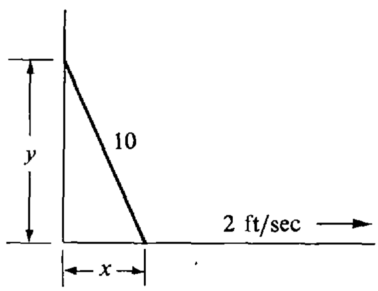


Figure 3.2.2

Step 2 $dx/dt = 2, \qquad x^2 + y^2 = 10^2 = 100.$

Step 3 We differentiate both sides of $x^2 + y^2 = 100$ with respect to t.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0, \quad \text{whence} \quad 4x + 2y\frac{dy}{dt} = 0.$$

Step 4 Set $x = 5$ ft and solve for dy/dt. We first find the value of y when $x = 5$.

$$x^2 + y^2 = 100, \qquad y = \sqrt{100 - x^2} = \sqrt{100 - 5^2} = \sqrt{75}.$$

Then we can solve for dy/dt,

$$4x + 2y\frac{dy}{dt} = 0,$$

$$4 \cdot 5 + 2\sqrt{75}\frac{dy}{dt} = 0,$$

$$\frac{dy}{dt} = -\frac{4 \cdot 5}{2\sqrt{75}} = -\frac{2}{\sqrt{3}}\text{ft/sec.}$$

The sign of dy/dt is negative because y is decreasing.

Related rates problems have the following form.

Given:

(a) Two quantities which depend on time, say x and y.
(b) The rate of change of one of them, say dx/dt.
(c) An equation showing the relationship between x and y.

(Usually this information is given in the form of a verbal description of a physical situation and part of the problem is to express it in the form of an equation.)

The problem: Find the rate of change of y, dy/dt, at a certain time t_0. (The time t_0 is sometimes specified by giving the value which x, or y, has at that time.)

Related rates problems can frequently be solved in four steps as we did in the examples.

Step 1 Label all quantities in the problem and draw a picture. If the labels are x, y, and t (time). the remaining steps are as follows:

Step 2 Write an equation for the given rate of change dx/dt. Write another equation for the given relation between x and y.

Step 3 Differentiate both sides of the equation relating x and y with respect to t. We choose the time t as the independent variable. The result is a new equation involving x, y, dx/dt, and dy/dt.

Step 4 Set $t = t_0$ and solve for dy/dt. It may be necessary to find the values of x, y, and dx/dt at $t = t_0$ first.

The hardest step is usually Step 2, because one has to start with the given verbal description of the problem and set it up as a system of formulas. Sometimes more than two quantities that depend on time are given. Here is an example with three.

EXAMPLE 3 One car moves north at 40 mph (miles per hour) and passes a point P at time 1:00. Another car moves east at 60 mph and passes the same point P at time 2:30. How fast is the distance between the two cars changing at the time 2:00?

It is not even easy to tell whether the two cars are getting closer or farther away at time 2:00. This is part of the problem.

Step 1 t = time,
y = position of the first car travelling north,
x = position of the second car travelling east,
z = distance between the two cars.

In the diagram in Figure 3.2.3, the point P is placed at the origin.

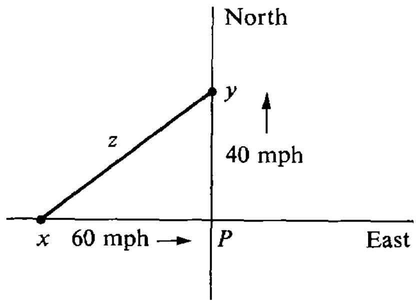


Figure 3.2.3

Step 2
$$\frac{dy}{dt} = 40, \quad \frac{dx}{dt} = 60, \qquad x^2 + y^2 = z^2.$$

Step 3
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}, \quad \text{whence} \quad 60x + 40y = z\frac{dz}{dt}.$$

Step 4 We first find the values of x, y, and z at the time $t = 2$ hrs. We are given that when $t = 1$, $y = 0$. In the next hour the car goes 40 miles, so at $t = 2$, $y = 40$. We are given that at time $t = 2\frac{1}{2}$, $x = 0$. One-half hour before that the car was 30 miles to the left of P, so at $t = 2$, $x = -30$. To sum up,

$$\text{at } t = 2, \qquad y = 40 \quad \text{and} \quad x = -30.$$

We can now find the value of z at $t = 2$,

$$z = \sqrt{x^2 + y^2} = \sqrt{(-30)^2 + 40^2} = 50.$$

Finally, we solve for dz/dt at $t = 2$,

$$60 \cdot (-30) + 40 \cdot 40 = 50\frac{dz}{dt}, \qquad \frac{dz}{dt} = \frac{-1800 + 1600}{50} = -4.$$

The negative sign shows that z is decreasing. Therefore at 2:00 the cars are getting closer to each other at the rate of 4 mph.

EXAMPLE 4 The population of a country is growing at the rate of one million people per year, while gasoline consumption is decreasing by one billion gallons per year. Find the rate of change of the per capita gasoline consumption when the population is 30 million and total gasoline consumption is 15 billion gallons per year.

By the per capita gasoline consumption we mean the total consumption divided by the population.

Step 1 t = time
x = population
y = gasoline consumption
z = per capita gasoline consumption.

Step 2 At $t = t_0$,

$$dx/dt = 1 \text{ million} = 10^6$$
$$dy/dt = -1 \text{ billion} = -10^9$$
$$z = y/x.$$

Step 3
$$\frac{dz}{dt} = \frac{x(dy/dt) - y(dx/dt)}{x^2},$$
$$\frac{dz}{dt} = \frac{-10^9x - 10^6y}{x^2}.$$

Step 4 At $t = t_0$, we are given
$$x = 30 \text{ million} = 30 \times 10^6,$$
$$y = 15 \text{ billion} = 15 \times 10^9.$$
Thus
$$\frac{dz}{dt} = \frac{-10^9 \cdot 30 \cdot 10^6 - 10^6 \cdot 15 \cdot 10^9}{(30 \cdot 10^6)^2}$$
$$= -\frac{45 \cdot 10^{15}}{900 \cdot 10^{12}} = -50.$$

The per capita gasoline consumption is decreasing at the annual rate of 50 gallons per person.

We conclude with another example from economics. In this example the independent variable is the quantity x of a commodity. The quantity x which can be sold at price p is called the *demand function* $D(p)$,
$$x = D(p).$$
When a quantity x is sold at price p, the *revenue* is the product
$$R = px.$$
The additional revenue from the sale of an additional unit of the commodity is called the *marginal revenue* and is given by the derivative
$$\text{marginal revenue} = dR/dx.$$

EXAMPLE 5 Suppose the demand for a product is equal to the inverse of the square of the price. Find the marginal revenue when the price is \$10 per unit.

Step 1 p = price, x = demand, R = revenue.

Step 2 $x = 1/p^2$, $R = px$.

Step 3
$$\frac{dR}{dx} = p\frac{dx}{dx} + x\frac{dp}{dx} = p + x\frac{dp}{dx},$$
$$\frac{dx}{dp} = -2p^{-3},$$
so by the Inverse Function Rule,
$$\frac{dp}{dx} = \frac{1}{dx/dp} = -\frac{1}{2p^{-3}} = -\frac{1}{2}p^3.$$
Substituting,
$$\frac{dR}{dx} = p + \left(\frac{1}{p^2}\right)\left(-\frac{1}{2}p^3\right) = \frac{1}{2}p.$$

Step 4 We are given $p = \$10$. Therefore the marginal revenue is
$$dR/dx = \$5.$$

An additional unit sold would bring in an additional revenue of \$5.

Here is a list of formulas from plane and solid geometry which will be useful in related rates problems. We always let A = area and V = volume.

Rectangle with sides a and b: $A = ab$, perimeter $= 2a + 2b$

Triangle with base b and height h: $A = \frac{1}{2}bh$

Circle of radius r: $A = \pi r^2$, circumference $= 2\pi r$

Sector (pie slice) of a circle of radius r and central angle θ (measured in radians): $A = \frac{1}{2}r^2\theta$

Rectangular solid with sides a, b, c: $V = abc$

Sphere of radius r: $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$

Right circular cylinder, base of radius r, height of h: $V = \pi r^2 h$, $A = 2\pi rh$

Prism with base of area B and height h: $V = Bh$

Right circular cone, base of radius r, height h: $V = \pi r^2 h/3$, $A = \pi r\sqrt{r^2 + h^2}$

PROBLEMS FOR SECTION 3.2

1 Each side of a square is expanding at the rate of 5 cm/sec. How fast is the area changing when the length of each side is 10 cm?

2 The area of a square is decreasing at the constant rate of 2 sq cm/sec. How fast is the length of each side decreasing when the area is 1 sq cm?

3 The vertical side of a rectangle is expanding at the rate of 1 in./sec, while the horizontal side is contracting at the rate of 1 in./sec. At time $t = 1$ sec the rectangle is a square whose sides are 2 in. long. How fast is the area of the rectangle changing at time $t = 2$ sec?

4 Each edge of a cube is expanding at the rate of 1 in./sec. How fast is the volume of the cube changing when the volume is 27 cu in.?

5 Two cars pass point P at approximately the same time, one travelling north at 50 mph, the other travelling west at 60 mph. Find the rate of change of the distance between the two cars one hour after they pass the point P.

6 A cup in the form of a right circular cone with radius r and height h is being filled with water at the rate of 5 cu in./sec. How fast is the level of the water rising when the volume of the water is equal to one half the volume of the cup?

7 A spherical balloon is being inflated at the rate of 10 cu in./sec. Find the rate of change of the area when the balloon has radius 6 in.

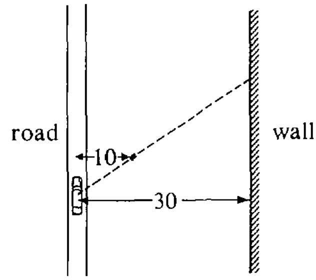
8 A snowball melts at the rate equal to twice its surface area, with area in square inches and melting measured in cubic inches per hour. How fast is the radius shrinking?

9 A ball is dropped from a height of 100 ft, at which time its shadow is 500 ft from the ball. How fast is the shadow moving when the ball hits the ground? The ball falls with velocity gt ft/sec, and the shadow is cast by the sun. Here $g = 32$ and t = time after drop.

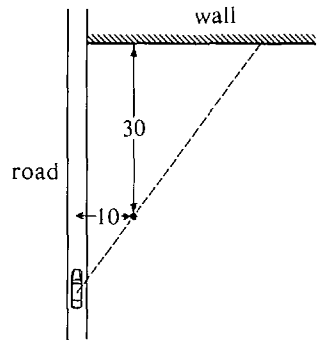
10 A 6 foot man walks away from a 10 foot high lamp at the rate of 3 ft/sec. How fast is the tip of his shadow moving?

11 A car is moving along a road at 60 mph. To the right of the road is a bush 10 ft away

and a parallel wall 30 ft away. Find the rate of motion of the shadow of the bush on the wall cast by the car headlights.



12 A car moves along a road at 60 mph. There is a bush 10 ft to the right of the road, and a wall 30 ft behind the bush is perpendicular to the road. Find the rate of motion of the shadow of the bush on the wall when the car is 26 ft from the bush.



13 An airplane passes directly above a train at an altitude of 6 miles. If the airplane moves north at 500 mph and the train moves north at 100 mph, find the rate at which the distance between them is increasing two hours after the airplane passes over the train.

14 A rectangle has constant area, but its length is growing at the rate of 10 ft/sec. Find the rate at which the width is decreasing when the rectangle is 3 ft long and 1 ft wide.

15 A cylinder has constant volume, but its radius is growing at the rate of 1 ft/sec. Find the rate of change of its height when the radius and height are both 1 ft.

16 A country has constant national income, but its population is growing at the rate of one million people per year. Find the rate of change of the per capita income (national income divided by population) when the population is 20 million and the national income is 20 billion dollars.

17 If at time t a country has a birth rate of $1{,}000{,}000t$ births per year and a death rate of $300{,}000\sqrt{t}$ deaths per year, how fast is the population growing?

18 The population of a country is 10 million and is increasing at the rate of 500,000 people per year. The national income is \$10 billion and is increasing at the rate of \$100 million per year. Find the rate of change of the per capita income.

19 Work Problem 18 assuming that the population is decreasing by 500,000 per year.

20 Sand is poured at the rate of 4 cu in./sec and forms a conical pile whose height is equal to the radius of its base. Find the rate of increase of the height when the pile is 12 in. high.

21 A circular clock has radius 5 in. At time t minutes past noon, how fast is the area of the sector of the circle between the hour and minute hand increasing? ($t \leq 60$).

22 The demand x for a commodity at price p is $x = 1/(1 + \sqrt{p})$. Find the *marginal revenue*, that is, the change in revenue per unit change in x, when the price is \$100 per unit.

23 x units of a commodity can be produced at a total cost of $y = 100 + 5x$. The *average cost* is defined as the total cost divided by x. Find the change in average cost per unit change in x (the marginal average cost) when $x = 100$.

24 The demand for a commodity at price p is $x = 1/(p + p^3)$. Find the change of the price per unit change in x, dp/dx, when the price is 3 dollars per unit.

25 In one day a company can produce x items at a total cost of $200 + 3x$ dollars and can sell x items at a price of $5 - x/1000$ dollars per item. *Profit* is defined as revenue minus cost. Find the change in profit per unit change in the number of items x (marginal profit).

26 In one day a company can produce x items at a total cost of $200 + 3x$ dollars and can sell $x = 1000/\sqrt{y}$ items at a price of y dollars per item.

(a) Find the change in profit per dollar change in the price y (the marginal profit with respect to price).

(b) Find the change in profit per unit change in x (the marginal profit).

27 An airplane P flies at 400 mph one mile above a line L on the surface. An observer is at the point O on L. Find the rate of change (in radians per hour) of the angle θ between the line L and the line OP from the observer to the airplane when $\theta = \pi/6$.

28 A train 20 ft wide is approaching an observer standing in the middle of the track at 100 ft/sec. Find the rate of increase of the angle subtended by the train (in radians per second) when the train is 20 ft from the observer.

29 Find the rate of increase of e^{2x+3y} when $x = 0$, $y = 0$, $dx/dt = 5$, and $dy/dt = 4$.

30 Find the rate of change of $\ln A$ where A is the area of a rectangle of sides x and y when $x = 1, y = 2, dx/dt = 3, dy/dt = -2$.

3.3 LIMITS

The notion of a limit is closely related to that of a derivative, but it is more general. In this chapter f will always be a real function of one variable. Let us recall the definition of the slope of f at a point a:

S is the slope of f at a if whenever Δx is infinitely close to but not equal to zero, the quotient

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

is infinitely close to S.

We now define the limit. c and L are real numbers.

DEFINITION

*L is the **limit** of $f(x)$ as x approaches c if whenever x is infinitely close to but not equal to c, $f(x)$ is infinitely close to L.*

In symbols,

$$\lim_{x \to c} f(x) = L$$

if whenever $x \approx c$ but $x \neq c$, $f(x) \approx L$. When there is no number L satisfying the above definition, we say that the limit of $f(x)$ as x approaches c *does not exist.*

Notice that the limit

$$\lim_{x \to c} f(x)$$

depends only on the values of $f(x)$ for x infinitely close but not equal to c. The value $f(c)$ itself has no influence at all on the limit. In fact, it very often happens that

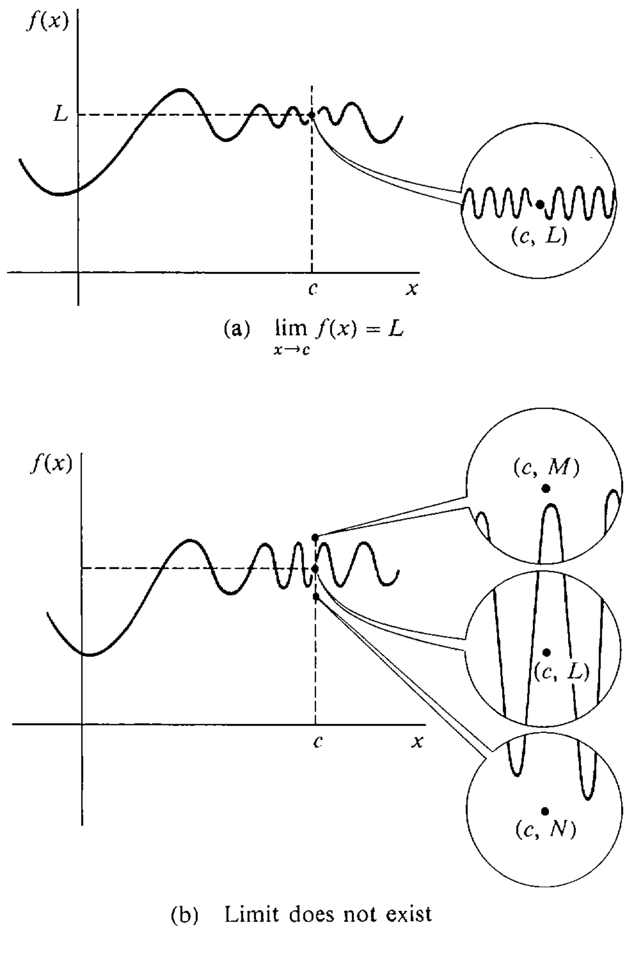
$$\lim_{x \to c} f(x)$$

exists but $f(c)$ is undefined.

Figure 3.3.1(a) shows a typical limit. Looking at the point (c, L) through an infinitesimal microscope, we can see the entire portion of the curve with $x \approx c$ because $f(x)$ will be infinitely close to L and hence within the field of vision of the microscope.

In Figure 3.3.1(b), part of the curve with $x \approx c$ is outside the field of vision of the microscope, and the limit does not exist.

Our first example of a limit is the slope of a function.



(a) $\lim_{x \to c} f(x) = L$

Figure 3.3.1 (b) Limit does not exist

THEOREM 1

The slope of f at a is given by the limit

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

Verbally, the slope of f at a is the limit of the ratio of the change in $f(x)$ to the change in x as the change in x approaches zero. The theorem is seen by simply comparing the definitions of limit and slope. The slope exists exactly when the limit exists; and when they do exist they are equal. Notice that the ratio

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

is undefined when $\Delta x = 0$.

The slope of f at a is also equal to the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This is seen by setting

$$\Delta x = x - a,$$
$$x = a + \Delta x.$$

Then when $x \approx a$ but $x \neq a$, we have $\Delta x \approx 0$ but $\Delta x \neq 0$; and

$$\frac{f(x) - f(a)}{x - a} = \frac{f(a + \Delta x) - f(a)}{\Delta x} \approx f'(a).$$

Sometimes a limit can be evaluated by recognizing it as a derivative and using Theorem 1 above.

EXAMPLE 1 Evaluate $\lim_{\Delta x \to 0} \frac{(3 + \Delta x)^2 - 9}{\Delta x}$.

Let $F(x) = x^2$. The given limit is just $F'(3)$,

$$F'(3) = \lim_{\Delta x \to 0} \frac{F(3 + \Delta x) - F(3)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(3 + \Delta x)^2 - 9}{\Delta x},$$

$$F'(3) = 2 \cdot 3 = 6.$$

Therefore $$\lim_{\Delta x \to 0} \frac{(3 + \Delta x)^2 - 9}{\Delta x} = 6.$$

The symbol x in

$$\lim_{x \to c} f(x)$$

is an example of a "dummy variable." The value of the limit does not depend on x at all. However, it does depend on c. If we replace c by a variable u, we obtain a new function

$$L(u) = \lim_{x \to u} f(x).$$

A limit $\lim_{x \to c} f(x)$ is usually computed as follows.

Step 1 Let x be infinitely close but not equal to c, and simplify $f(x)$.

Step 2 Compute the standard part $st(f(x))$.

CONCLUSION If the limit $\lim_{x \to c} f(x)$ exists, it must equal $st(f(x))$.

EXAMPLE 1 (Continued) Instead of using the derivative, we can directly compute

$$\lim_{\Delta x \to 0} \frac{(3 + \Delta x)^2 - 9}{\Delta x}.$$

Step 1 Let $\Delta x \approx 0$, but $\Delta x \neq 0$. Then

$$\frac{(3 + \Delta x)^2 - 9}{\Delta x} = \frac{9 + 6\,\Delta x + \Delta x^2 - 9}{\Delta x} = \frac{6\,\Delta x + \Delta x^2}{\Delta x} = 6 + \Delta x.$$

Step 2 Taking standard parts,

$$st\frac{(3 + \Delta x)^2 - 9}{\Delta x} = st(6 + \Delta x) = 6.$$

Therefore the limit is equal to 6. (See Figure 3.3.2.)

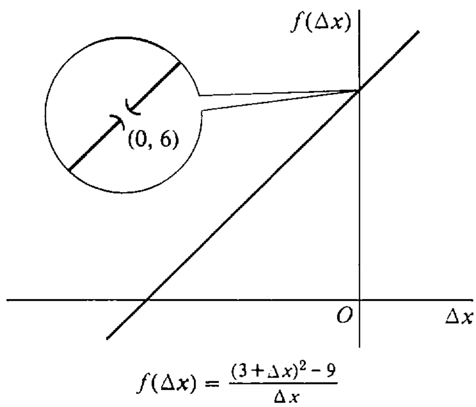


Figure 3.3.2

EXAMPLE 2 Find $\lim_{t \to 4} (t^2 + 3t - 5)$.

Step 1 Let t be infinitely close to but not equal to 4.

Step 2 We take the standard part.

$$st(t^2 + 3t - 5) = 4^2 + 3 \cdot 4 - 5 = 23,$$

so the limit is 23.

EXAMPLE 3 Find $\lim_{x \to 2} \frac{x^2 + 3x - 10}{x^2 - 4}$.

Step 1 This time the term inside the limit is undefined at $x = 2$. Taking $x \approx 2$ but $x \neq 2$, we have

$$\frac{x^2 + 3x - 10}{x^2 - 4} = \frac{(x-2)(x+5)}{(x-2)(x+2)} = \frac{x+5}{x+2}.$$

Step 2 $$st\left(\frac{x^2 + 3x - 10}{x^2 - 4}\right) = st\left(\frac{x+5}{x+2}\right) = \frac{2+5}{2+2} = \frac{7}{4}.$$

Thus $$\lim_{x \to 2}\left(\frac{x^2 + 3x - 10}{x^2 - 4}\right) = \frac{7}{4}.$$

EXAMPLE 4 Find $\lim_{x \to 0}\left(\frac{(2/x) + 3}{(3/x) - 1}\right)$

Step 1 Taking $x \approx 0$ but $x \neq 0$,

$$\frac{(2/x) + 3}{(3/x) - 1} = \frac{2 + 3x}{3 - x}.$$

Step 2 $$st\left(\frac{(2/x) + 3}{(3/x) - 1}\right) = st\left(\frac{2 + 3x}{3 - x}\right) = \frac{2}{3}.$$

Thus the limit exists and equals $\frac{2}{3}$.

EXAMPLE 5 Find $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$.

Step 1 Taking $x \approx 9$ and $x \neq 9$,

$$\frac{\sqrt{x} - 3}{x - 9} = \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)} = \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} = \frac{1}{\sqrt{x} + 3}.$$

Step 2 $$st\left(\frac{\sqrt{x} - 3}{x - 9}\right) = st\left(\frac{1}{\sqrt{x} + 3}\right) = \frac{1}{\sqrt{9} + 3} = \frac{1}{6},$$

so the limit exists and is $\frac{1}{6}$.

Our rules for standard parts in Chapter 1 lead at once to rules for limits. We list these rules in Table 3.3.1. The limit rules can be applied whenever the two limits $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist.

Table 3.3.1

Standard Part Rule	Limit Rule
$st(kb) = k\, st(b)$, k real	$\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$
$st(a + b) = st(a) + st(b)$	$\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
$st(ab) = st(a) \cdot st(b)$	$\lim_{x \to c} (f(x)g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
$st(a/b) = st(a)/st(b)$, if $\lvert st(b) \rvert > 0$	$\lim_{x \to c} (f(x)/g(x)) = \lim_{x \to c} f(x)/\lim_{x \to c} g(x)$, if $\lim_{x \to c} g(x) \neq 0$
$st(\sqrt[n]{a}) = \sqrt[n]{st(a)}$, if $a > 0$	$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$, if $\lim_{x \to c} f(x) > 0$

EXAMPLE 6 Find $\lim_{x \to 1} (x^2 - 2x)\sqrt{(x^2 - 1)/(x - 1)}$.

All the limits involved exist, so we can use the limit rules to compute the limit as follows. First we find the limit of the expression inside the radical.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2.$$

Now we find the answer to the original problem.

$$\begin{aligned}\lim_{x \to 1} (x^2 - 2x)\sqrt{(x^2 - 1)/(x - 1)} &= \lim_{x \to 1} (x^2 - 2x)\sqrt{\lim_{x \to 1} (x^2 - 1)/(x - 1)} \\ &= (1 - 2)\sqrt{2} = -\sqrt{2}.\end{aligned}$$

There are three ways in which a limit $\lim_{x \to c} f(x)$ can fail to exist:

(1) $f(x)$ is undefined for some x which is infinitely close but not equal to c.

(2) $f(x)$ is infinite for some x which is infinitely close but not equal to c.

(3) The standard part of $f(x)$ is different for different numbers x which are infinitely close but not equal to c.

EXAMPLE 7 $\lim_{x \to 0} \sqrt{x}$ does not exist because $\sqrt{x}$ is undefined for negative infinitesimal x. (See Figure 3.3.3(a).)

EXAMPLE 8 $\lim_{x \to 0} 1/x^2$ does not exist because $1/x^2$ is infinite for infinitesimal $x \neq 0$. (See Figure 3.3.3(b).)

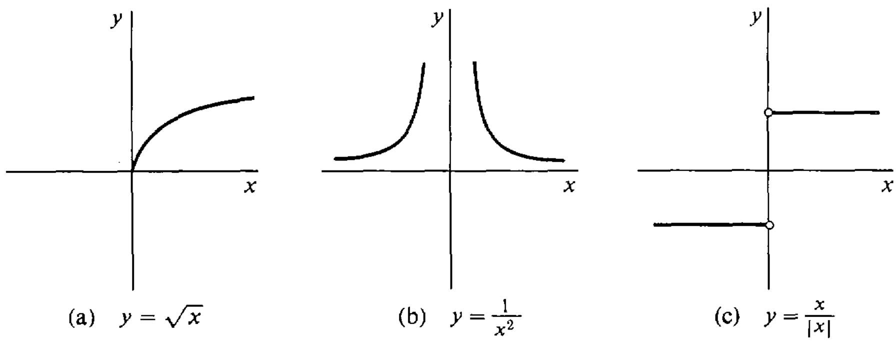


Figure 3.3.3

EXAMPLE 9 $\lim_{x \to 0} x/|x|$ does not exist because

$$st\left(\frac{x}{|x|}\right) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

(See Figure 3.1.3(c).)

In the above examples the function behaves differently on one side of the point 0 than it does on the other side. For such functions, one-sided limits are useful.

We say that

$$\lim_{x \to c^+} f(x) = L$$

if whenever $x > c$ and $x \approx c$, $f(x) \approx L$.

$$\lim_{x \to c^-} f(x) = L$$

means that whenever $x < c$ and $x \approx c$, $f(x) \approx L$. These two kinds of limits, shown in Figure 3.3.4, are called the *limit from the right* and the *limit from the left.*

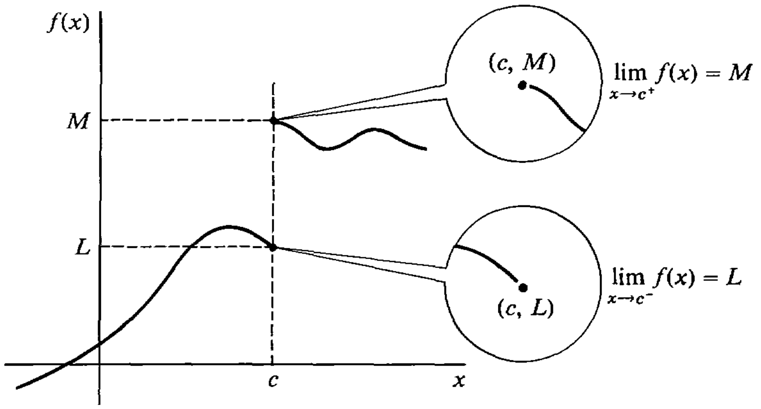


Figure 3.3.4 One-sided limits.

THEOREM 2

A limit has value L,

$$\lim_{x \to c} f(c) = L,$$

if and only if both one-sided limits exist and are equal to L,

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

PROOF If $\lim_{x \to c} f(x) = L$, it follows at once from the definition that both one-sided limits are L.

Assume that both one-sided limits are equal to L. Let $x \approx c$, but $x \neq c$. Then either $x < c$ or $x > c$. If $x < c$, then because $\lim_{x \to c^-} f(x) = L$, we have $f(x) \approx L$. On the other hand if $x > c$, then $\lim_{x \to c^+} f(x) = L$ gives $f(x) \approx L$. Thus in either case $f(x) \approx L$. This shows that $\lim_{x \to c} f(x) = L$.

When a limit does not exist, it is possible that neither one-sided limit exists, that just one of them exists, or that both one-sided limits exist but have different values.

EXAMPLE 7 (Continued) $\lim_{x \to 0^+} \sqrt{x} = 0$, and $\lim_{x \to 0^-} \sqrt{x}$ does not exist.

EXAMPLE 8 (Continued) Neither $\lim_{x \to 0^+} 1/x^2$ nor $\lim_{x \to 0^-} 1/x^2$ exists.

EXAMPLE 9 (Continued) $\lim_{x \to 0^+} x/|x| = 1$, and $\lim_{x \to 0^-} x/|x| = -1$.

PROBLEMS FOR SECTION 3.3

In each problem below, determine whether or not the limit exists. When the limit exists, find its value. With a calculator, compute some values as x approaches its limit, and see what happens.

1 $\lim_{t \to 4} 3t^2 + t + 1$

2 $\lim_{\Delta x \to -1} \frac{\Delta x^2 + 2\Delta x + 1}{\Delta x + 1}$

3 $\lim_{x \to c} \sqrt{c - x}$

4 $\lim_{y \to 0} \frac{1}{y^5}$

5 $\lim_{x \to 2} \frac{x}{x^2 - 4}$

6 $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

7 $\lim_{v \to 8} \frac{\sqrt{8} - \sqrt{v}}{v - 8}$

8 $\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$

9 $\lim_{u \to 1} \frac{\sqrt[3]{u} - 1}{u - 1}$

10 $\lim_{t \to 0} \frac{t^3 - 2t^2 + 4}{3t^2 - 5t + 7}$

11 $\lim_{y \to 0} (\sqrt{1 + 1/y} - \sqrt{1/y}$

12 $\lim_{x \to 0} \frac{(a + x)^2 - a^2}{x}$

13 $\lim_{y \to -1} \frac{y^2 + 1}{y + 1}$

14 $\lim_{x \to 1} \frac{|x - 1|}{x - 1}$

15 $\lim_{x \to 1^+} \frac{|x - 1|}{x - 1}$

16 $\lim_{x \to c^-} \sqrt{c - x}$

17 $\lim_{z \to 1} \sqrt{z + \sqrt{z + \sqrt{z}}}$

18 $\lim_{x \to a} \sqrt{|a - x|}$

19 $\lim_{x \to 0^+} x\sqrt{1 + x^{-2}}$

20 $\lim_{x \to 0^-} x\sqrt{1 + x^{-2}}$

21 $\lim_{t \to 0} \frac{1 + 2t^{-1}}{3 - 4t^{-1}}$

22 $\lim_{x \to 0} \frac{3 + 4x^{-1} - 5x^{-2}}{6 - x^{-1} + 3x^{-2}}$

23 $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$

24 $\lim_{\Delta x \to 0} \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x} \quad (x \neq 0)$

25 $\lim_{\Delta t \to 0} \frac{\sqrt{t + \Delta t} - \sqrt{t}}{\Delta t} \quad (t > 0)$

26 $\lim_{\Delta t \to 0} \frac{(t + \Delta t)^{1/5} - t^{1/5}}{\Delta t} \quad (t > 0)$

27 $\lim_{\Delta x \to 0} \frac{(x - \Delta x)^3 - x^3}{\Delta x}$

28 $\lim_{\Delta x \to 0} \frac{\frac{x + \Delta x}{x + \Delta x + 1} - \frac{x}{x + 1}}{\Delta x} \quad (x \neq -1)$

29 $\lim_{\Delta x \to 0^-} \frac{|(1 + \Delta x)^3 - (1 + \Delta x)|}{\Delta x}$

30 $\lim_{\Delta x \to 0^+} \frac{|(1 + \Delta x)^3 - (1 + \Delta x)|}{\Delta x}$

31 $\lim_{\Delta x \to 0^-} \frac{\sqrt{1 - (1 + \Delta x)^2}}{\Delta x}$

3.4 CONTINUITY

Intuitively, a curve $y = f(x)$ is continuous if it forms an unbroken line, that is, whenever x_1 is close to x_2, $f(x_1)$ is close to $f(x_2)$. To make this intuitive idea into a mathematical definition, we substitute "infinitely close" for "close."

DEFINITION

*f is said to be **continuous** at a point c if:*

(i) *f is defined at c;*

(ii) *whenever x is infinitely close to c, $f(x)$ is infinitely close to $f(c)$.*

If f is not continuous at c it is said to be *discontinuous* at c.

When f is continuous at c, the entire part of the curve where $x \approx c$ will be visible in an infinitesimal microscope aimed at the point $(c, f(c))$, as shown in Figure 3.4.1(a). But if f is discontinuous at c, some values of $f(x)$ where $x \approx c$ will either be undefined or outside the range of vision of the microscope, as in Figure 3.4.1(b).

Continuity, like the derivative, can be expressed in terms of limits. Again the proof is immediate from the definitions.

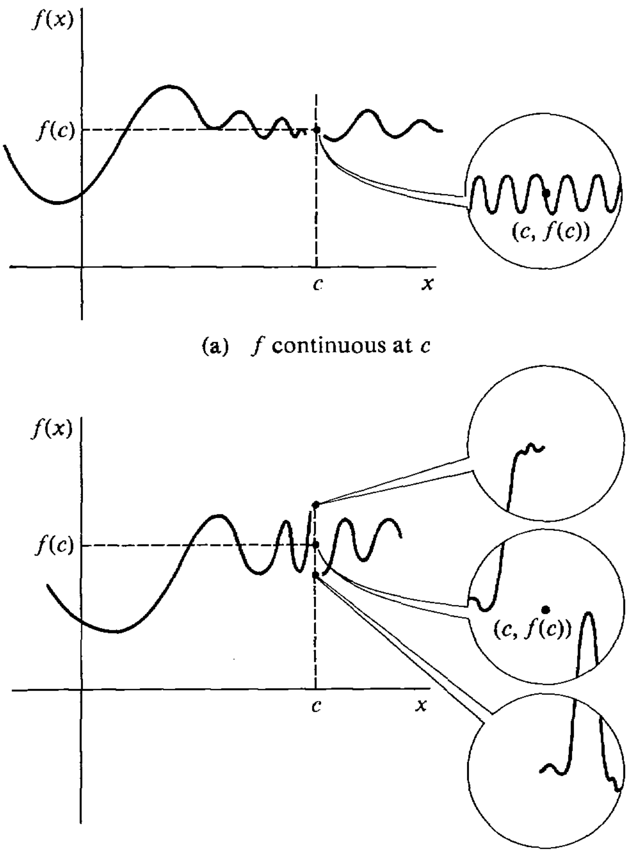


Figure 3.4.1 (b) f discontinuous at c

THEOREM 1

f is continuous at c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

As an application, we have a set of rules for combining continuous functions. They can be proved either by the corresponding rules for limits (Table 3.3.1 in Section 3.3) or by computing standard parts.

THEOREM 2

Suppose f and g are continuous at c.

(i) *For any constant k, the function $k \cdot f(x)$ is continuous at c.*
(ii) *$f(x) + g(x)$ is continuous at c.*
(iii) *$f(x) \cdot g(x)$ is continuous at c.*
(iv) *If $g(c) \neq 0$, then $f(x)/g(x)$ is continuous at c.*
(v) *If $f(c)$ is positive and n is an integer, then $\sqrt[n]{f(x)}$ is continuous at c.*

By repeated use of Theorem 2, we see that all of the following functions are continuous at c.

Every polynomial function.
Every rational function $f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials and $g(c) \neq 0$.
The functions $f(x) = x^r$, r rational and x positive.

Sometimes a function $f(x)$ will be undefined at a point $x = c$ while the limit

$$L = \lim_{x \to c} f(x)$$

exists. When this happens, we can make the function continuous at c by defining $f(c) = L$.

EXAMPLE 1 Let $f(x) = \dfrac{x^2 + x - 2}{x - 1}$.

At any point $c \neq 1$, f is continuous. But $f(1)$ is undefined so f is discontinuous at 1. However,

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 2)}{x - 1} = 3.$$

We can make f continuous at 1 by defining

$$f(x) = \begin{cases} \dfrac{x^2 + x - 2}{x - 1} & \text{if } x \neq 1, \\ 3 & \text{if } x = 1. \end{cases}$$

(See Figure 3.4.2.)

In terms of a dependent variable $y = f(x)$, the definition of continuity takes the following form, where $\Delta y = f(c + \Delta x) - f(c)$.

y is continuous at $x = c$ if:

(i) *y is defined at $x = c$.*
(ii) *Whenever Δx is infinitesimal, Δy is infinitesimal.*

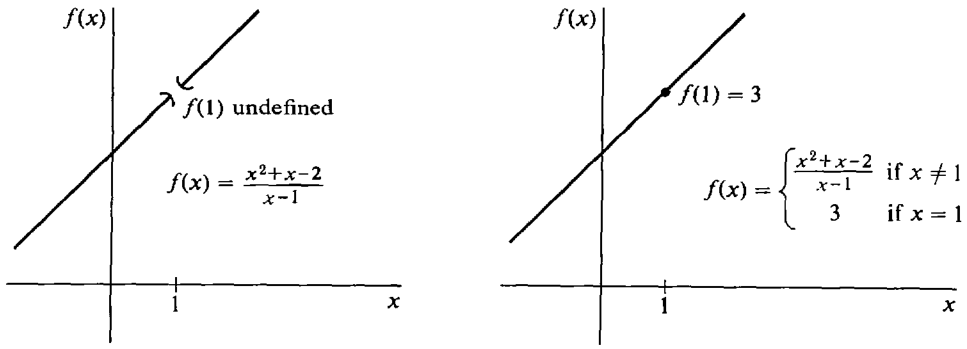


Figure 3.4.2

To summarize, given a function $y = f(x)$ defined at $x = c$, all the statements below are equivalent.

(1) f is continuous at c.
(2) Whenever $x \approx c$, $f(x) \approx f(c)$.
(3) Whenever $st(x) = c$, $st(f(x)) = f(c)$.
(4) $\lim_{x \to c} f(x) = f(c)$.
(5) y is continuous at $x = c$.
(6) Whenever Δx is infinitesimal, Δy is infinitesimal.

Our next theorem is that differentiability implies continuity. That is, the set of differentiable functions at c is a subset of the set of continuous functions at c. (See Figure 3.4.3.)

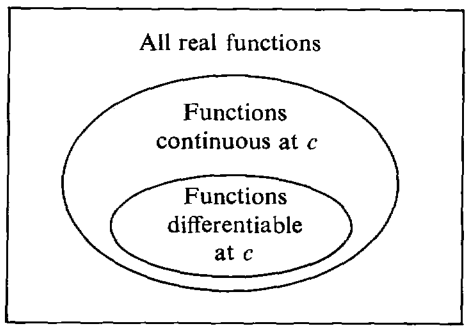


Figure 3.4.3

THEOREM 3

If f is differentiable at c then f is continuous at c.

PROOF Let $y = f(x)$, and let Δx be a nonzero infinitesimal. Then $\Delta y/\Delta x$ is infinitely close to $f'(c)$ and is therefore finite. Thus $\Delta y = \Delta x(\Delta y/\Delta x)$ is the product of an infinitesimal and a finite number, so Δy is infinitesimal.

For example, the transcendental functions $\sin x$, $\cos x$, e^x are continuous for all x, and $\ln x$ is continuous for $x > 0$. Theorem 3 can be used to show that combinations of these functions are continuous.

EXAMPLE 2 Find as large a set as you can on which the function

$$f(x) = \frac{\sin x \ln(x + 1)}{x^2 - 4}$$

is continuous.

$\sin x$ is continuous for all x. $\ln(x + 1)$ is continuous whenever $x + 1 > 0$, that is, $x > -1$. The numerator $\sin x \ln(x + 1)$ is thus continuous whenever $x > -1$. The denominator $x^2 - 4$ is continuous for all x but is zero when $x = \pm 2$. Therefore $f(x)$ is continuous whenever $x > -1$ and $x \neq 2$.

The next two examples give functions which are continuous but *not* differentiable at a point c.

EXAMPLE 3 The function $y = x^{1/3}$ is continuous but not differentiable at $x = 0$. (See Figure 3.4.4(a).) We have seen before that it is not differentiable at $x = 0$. It is continuous because if Δx is infinitesimal then so is

$$\Delta y = (0 + \Delta x)^{1/3} - 0^{1/3} = (\Delta x)^{1/3}.$$

EXAMPLE 4 The absolute value function $y = |x|$ is continuous but not differentiable at the point $x = 0$. (See Figure 3.4.4(b).)

We have already shown that the derivative does not exist at $x = 0$. To see that the function is continuous, we note that for any infinitesimal Δx,

$$\Delta y = |0 + \Delta x| - |0| = |\Delta x|$$

and thus Δy is infinitesimal.

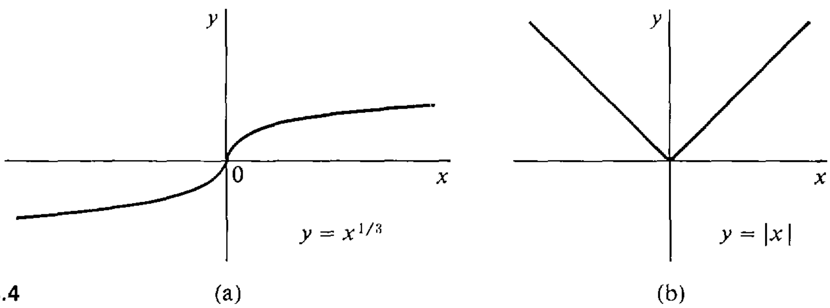


Figure 3.4.4

The path of a bouncing ball is a series of parabolas shown in Figure 3.4.5. The curve is continuous everywhere. At the points $a_1, a_2, a_3, \ldots$ where the ball bounces, the curve is continuous but not differentiable. At other points, the curve is both continuous and differentiable.

In the classical kinetic theory of gases, a gas molecule is assumed to be moving at a constant velocity in a straight line except at the instant of time when it collides with another molecule or the wall of the container. Its path is then a broken line in space, as in Figure 3.4.6.

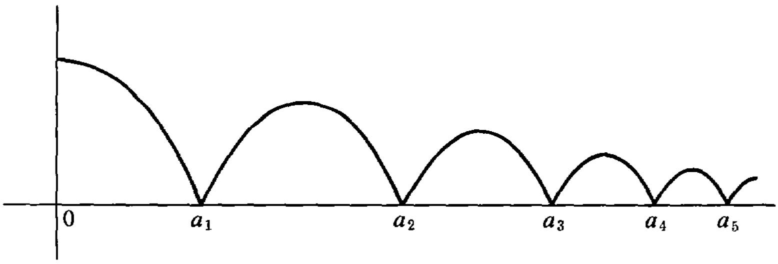


Figure 3.4.5 Path of Bouncing Ball

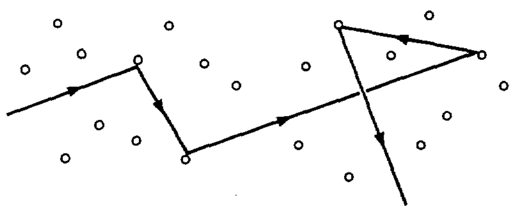


Figure 3.4.6

The position in three dimensional space at time t can be represented by three functions

$$x = f(t), \qquad y = g(t), \qquad z = h(t).$$

All three functions, f, g, and h are continuous for all values of t. At the time t of a collision, at least one and usually all three derivatives dx/dt, dy/dt, dz/dt will be undefined because the speed or direction of the molecule changes abruptly. At any other time t, when no collision is taking place, all three derivatives dx/dt, dy/dt, dz/dt will exist.

The functions we shall ordinarily encounter in this book will be defined and have a derivative at all but perhaps a finite number of points of an interval. The graph of such a function will be a smooth curve where the derivative exists. At points where the curve has a sharp corner (like 0 in $|x|$) or a vertical tangent line (like 0 in $x^{1/3}$), the function is continuous but not differentiable (see Figure 3.4.7). At points where the function is undefined or there is a jump, or the value approaches infinity or oscillates wildly, the function is discontinuous (see Figure 3.4.8).

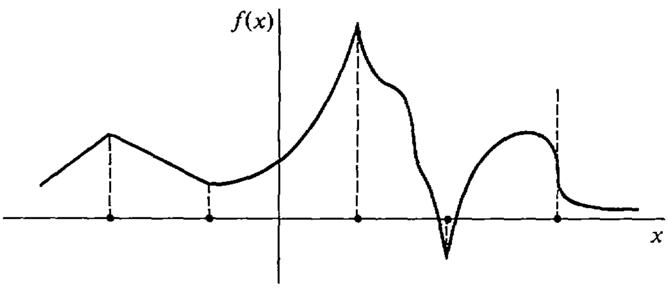


Figure 3.4.7 Points where f is continuous but nondifferentiable

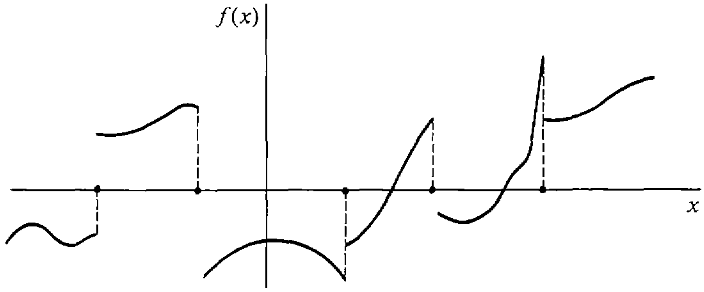


Figure 3.4.8 Points where f is discontinuous

The next theorem is similar to the Chain Rule for derivatives.

THEOREM 4

If f is continuous at c and G is continuous at $f(c)$, then the function

$$g(x) = G(f(x))$$

is also continuous at c. That is, a continuous function of a continuous function is continuous.

PROOF Let x be infinitely close to but not equal to c. Then

$$st(g(x)) = st(G(f(x))) = G(st(f(x))) = G(f(c)) = g(c).$$

For example, the following functions are continuous:

$$\begin{aligned} f(x) &= \sqrt{x^2 + 1}, && \text{all } x \\ g(x) &= |x^3 - x|, && \text{all } x \\ h(x) &= (1 + \sqrt{x})^{1/3}, && x > 0 \\ j(x) &= e^{\sin x}, && \text{all } x \\ k(x) &= \ln|x|, && \text{all } x \neq 0 \end{aligned}$$

Here are two examples illustrating two types of discontinuities.

EXAMPLE 5 The function $g(x) = \dfrac{x^2 - 3x + 4}{4(x - 1)(x - 2)}$ is continuous at every real point except $x = 1$ and $x = 2$. At these two points $g(x)$ is undefined (Figure 3.4.9).

EXAMPLE 6 The *greatest integer function* $[x]$, shown in Figure 3.4.10, is defined by

$$[x] = \text{the greatest integer } n \text{ such that } n \leq x.$$

Thus $[x] = 0$ if $0 \leq x < 1$, $[x] = 1$ if $1 \leq x < 2$, $[x] = 2$ if $2 \leq x < 3$, and so on. For negative x, we have $[x] = -1$ if $-1 \leq x < 0$, $[x] = -2$

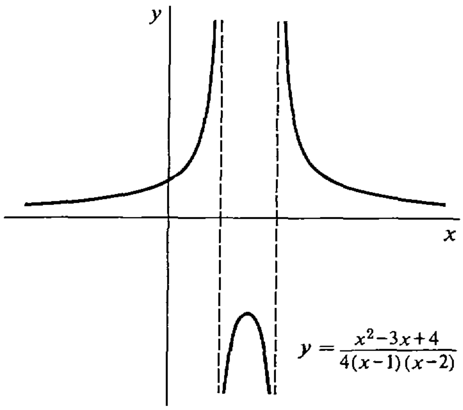


Figure 3.4.9

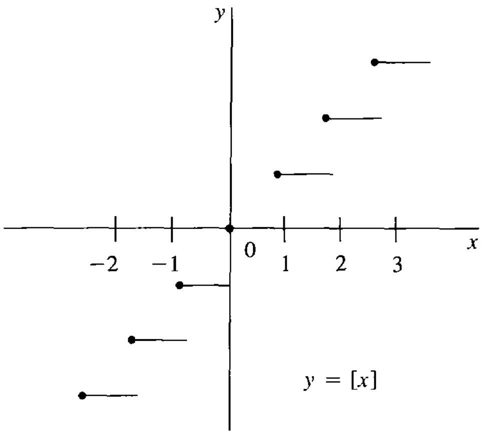


Figure 3.4.10

if $-2 \le x < -1$, and so on. For example,

$$[7.362] = 7, \qquad [\pi] = 3, \qquad [-2.43] = -3.$$

For each integer n, $[n]$ is equal to n. The function $[x]$ is continuous when x is not an integer but is discontinuous when x is an integer n. At an integer n, both one-sided limits exist but are different,

$$\lim_{x \to n^-} f(x) = n - 1, \qquad \lim_{x \to n^+} f(x) = n.$$

The graph of $[x]$ looks like a staircase. It has a step, or jump discontinuity, at each integer n. The function $[x]$ will be useful in the last section of this chapter. Some hand calculators have a key for either the greatest integer function or for the similar function that gives $[x]$ for positive x and $[x] + 1$ for negative x.

Functions which are "continuous on an interval" will play an important role in this chapter. Intervals were discussed in Section 1.1. Recall that closed intervals have the form

$$[a, b],$$

open intervals have one of the forms

$$(a, b), \qquad (a, \infty), \qquad (-\infty, b), \qquad (-\infty, \infty),$$

and half-open intervals have one of the forms

$$[a, b), \qquad (a, b], \qquad [a, \infty), \qquad (-\infty, b].$$

In these intervals, a is called the *lower endpoint* and b, the *upper endpoint*. The symbol $-\infty$ indicates that there is no lower endpoint, while ∞ indicates that there is no upper endpoint.

DEFINITION

We say that f is ***continuous on an open interval I*** *iff f is continuous at every point c in I. If in addition f has a derivative at every point of I, we say that f is* ***differentiable on I.***

To define what is meant by a function continuous on a closed interval, we introduce the notions of continuous from the right and continuous from the left, using one-sided limits.

DEFINITION

f is ***continuous from the right*** *at c if $\lim_{x \to c^+} f(x) = f(c)$.*

f is ***continuous from the left*** *at c if $\lim_{x \to c^-} f(x) = f(c)$.*

EXAMPLE 6 (Continued) The greatest integer function $f(x) = [x]$ is continuous from the right but not from the left at each integer n because

$$[n] = n, \qquad \lim_{x \to n^+}[x] = n, \qquad \lim_{y \to n^-}[x] = n - 1.$$

It is easy to check that f is continuous at c if and only if f is continuous from both the right and left at c.

DEFINITION

f is said to be ***continuous on the closed interval*** *$[a, b]$ if f is continuous at each point c where $a < c < b$, continuous from the right at a, and continuous from the left at b.*

Figure 3.4.11 shows a function f continuous on $[a, b]$.

EXAMPLE 7 The semicircle

$$y = \sqrt{1 - x^2},$$

shown in Figure 3.4.12, is continuous on the closed interval $[-1, 1]$. It is

Figure 3.4.11 f is continuous on the interval $[a, b]$

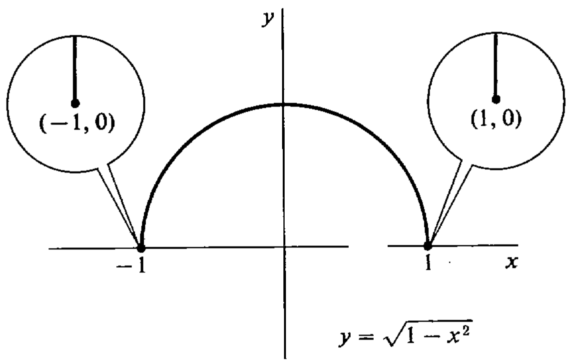


Figure 3.4.12

differentiable on the open interval $(-1, 1)$. To see that it is continuous from the right at $x = -1$, let Δx be positive infinitesimal. Then

$$y = \sqrt{1 - (-1)^2} = 0$$

$$y + \Delta y = \sqrt{1 - (-1 + \Delta x)^2} = \sqrt{1 - (1 - 2\,\Delta x + \Delta x^2)}$$

$$= \sqrt{2\,\Delta x - \Delta x^2} = \sqrt{(2 - \Delta x)\,\Delta x}.$$

Thus

$$\Delta y = \sqrt{(2 - \Delta x)\,\Delta x}.$$

The number inside the radical is positive infinitesimal, so Δy is infinitesimal. This shows that the function is continuous from the right at $x = -1$. Similar reasoning shows it is continuous from the left at $x = 1$.

PROBLEMS FOR SECTION 3.4

In Problems 1–17, find the set of all points at which the function is continuous.

1 $f(x) = 3x^2 + 5x + 4$

2 $f(x) = \dfrac{5x + 2}{x^2 + 1}$

3 $f(x) = \sqrt{x + 2}$

4 $f(x) = \dfrac{x}{x + 2}$

5 $f(x) = \sqrt{|x - 2| + 1}$

6 $f(x) = \dfrac{x + 3}{|x + 3|}$

7 $f(x) = \dfrac{x}{x^2 + x}$

8 $f(x) = \dfrac{x + 2}{(x - 1)(x - 3)^{1/3}}$

9 $f(x) = \sqrt{4 - x^2}$

10 $f(x) = \sqrt{x^2 - 4}$

11 $f(x) = \dfrac{1}{x - (1/(x + 1))}$

12 $g(x) = \dfrac{1}{x} + \dfrac{1}{x - 1}$

13 $g(x) = \dfrac{x - 2}{x - 3} + \dfrac{x - 3}{x - 2}$

14 $g(x) = \sqrt{x^3 - x}$

15 $g(x) = \sqrt[4]{x^2 - x^3}$

16 $f(t) = \sqrt{t^{-2} - 1}$

17 $f(t) = \sqrt{t^{-1} - 1}$

18 Show that $f(x) = \sqrt{x}$ is continuous from the right at $x = 0$.

19 Show that $f(x) = \sqrt{1 - x}$ is continuous from the left at $x = 1$.

20 Show that $f(x) = \sqrt{1 - |x|}$ is continuous on the closed interval $[-1, 1]$.

21 Show that $f(x) = \sqrt{x} + \sqrt{2 - x}$ is continuous on the closed interval $[0, 2]$.

22 Show that $f(x) = \sqrt{9 - x^2}$ is continuous on the closed interval $[-3, 3]$.

23 Show that $f(x) = \sqrt{x^2 - 9}$ is continuous on the half-open intervals $(-\infty, -3]$ and $[3, \infty)$.

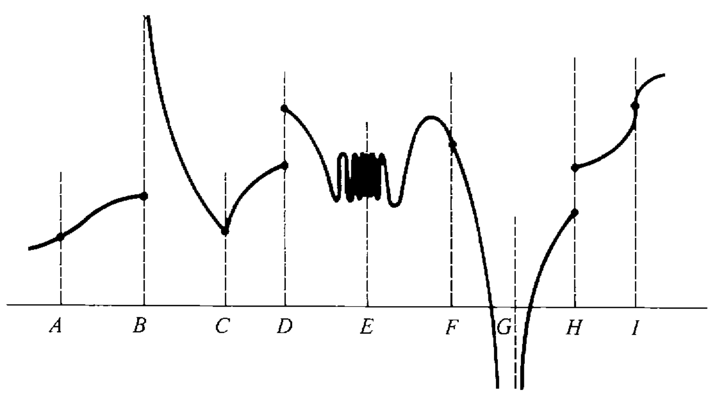
□ 24 Suppose the function $f(x)$ is continuous on the closed interval $[a, b]$. Show that there is a function $g(x)$ which is continuous on the whole real line and has the value $g(x) = f(x)$ for x in $[a, b]$.

□ 25 Suppose $\lim_{x \to c} f(x) = L$. Prove that the function $g(x)$, defined by $g(x) = f(x)$ for $x \neq c$ and $g(x) = L$ for $x = c$, is continuous at c.

26 In the curve $y = f(x)$ illustrated below, identify the points $x = c$ where each of the following happens:

(a) f is discontinuous at $x = c$

(b) f is continuous but not differentiable at $x = c$.



3.5 MAXIMA AND MINIMA

Let us assume throughout this section that f is a real function whose domain is an interval I, and furthermore that f is continuous on I. A problem that often arises is that of finding the point c where $f(c)$ has its largest value, and also the point c where $f(c)$ has its smallest value. The derivative turns out to be very useful in this problem. We begin by defining the concepts of maximum and minimum.

DEFINITION

Let c be a real number in the domain I of f.

(i) *f has a **maximum** at c if $f(c) \geq f(x)$ for all real numbers x in I. In this case f(c) is called the **maximum value** of f.*

(ii) *f has a **minimum** at c if $f(c) \leq f(x)$ for all real numbers x in I. f(c) is then called the **minimum value** of f.*

When we look at the graph of a continuous function f on I, the maximum will appear as the highest peak and the minimum as the lowest valley (Figure 3.5.1).

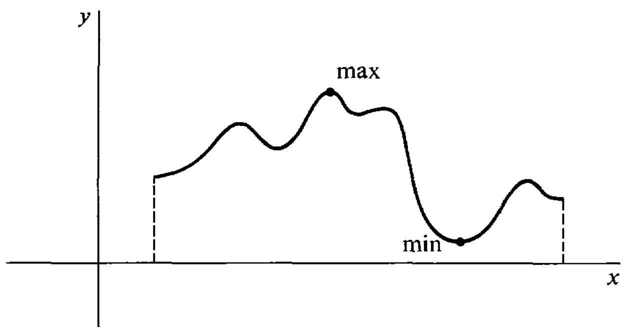


Figure 3.5.1 Maximum and Minimum

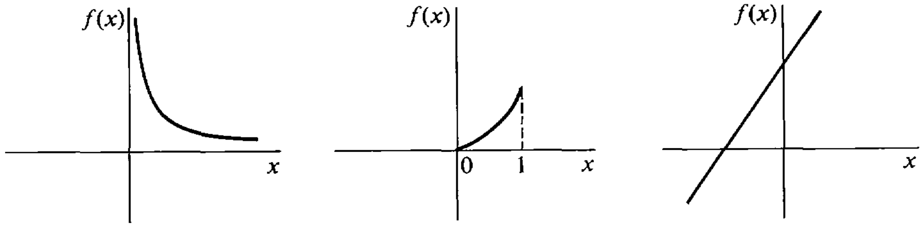
In general, all of the following possibilities can arise:

f has no maximum in its domain I.
f has a maximum at exactly one point in I.
f has a maximum at several different points in I.

However even if f has a maximum at several different points, f can have only one maximum value. Because if f has a maximum at c_1 and also at c_2, then $f(c_1) \geq f(c_2)$ and $f(c_2) \geq f(c_1)$, and therefore $f(c_1)$ and $f(c_2)$ are equal.

EXAMPLE 1 Each of the following functions, graphed in Figure 3.5.2, have no maximum and no minimum:

(a) $f(x) = 1/x, \quad 0 < x.$
(b) $f(x) = x^2, \quad 0 < x < 1.$
(c) $f(x) = 2x + 3.$



(a) $f(x) = \frac{1}{x}, 0 < x$ (b) $f(x) = x^2, 0 < x < 1$ (c) $f(x) = 2x + 3$

Figure 3.5.2 No Maximum or Minimum

EXAMPLE 2 The function $f(x) = x^2 + 1$ has no maximum. But f has a minimum at $x = 0$ with value 1, because for $x \neq 0$, we always have $x^2 > 0, x^2 + 1 > 1$. The graph is shown in Figure 3.5.3.

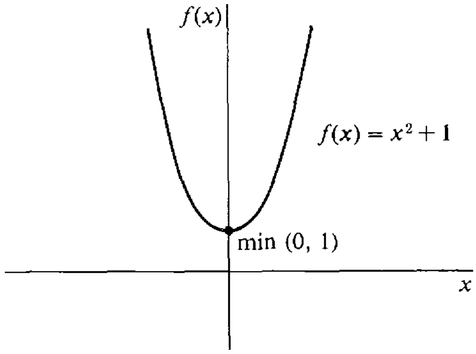


Figure 3.5.3

The use of the derivative in finding maxima and minima is based on the Critical Point Theorem. It shows that the maxima and minima of a function can only occur at certain points, called *critical points*. The theorem will be stated now, and its proof is given at the end of this section.

CRITICAL POINT THEOREM

Let f have domain I. Suppose that c is a point in I and f has either a maximum or a minimum at c. Then one of the following three things must happen:

(i) *c is an endpoint of I,*
(ii) *$f'(c)$ is undefined,*
(iii) *$f'(c) = 0$.*

We shall say that c is a *critical point* of f if either (i), (ii), or (iii) happens. The three types of critical points are shown in Figure 3.5.4. When I is an open interval, (i) cannot arise since the endpoints are not elements of I. But when I is a closed

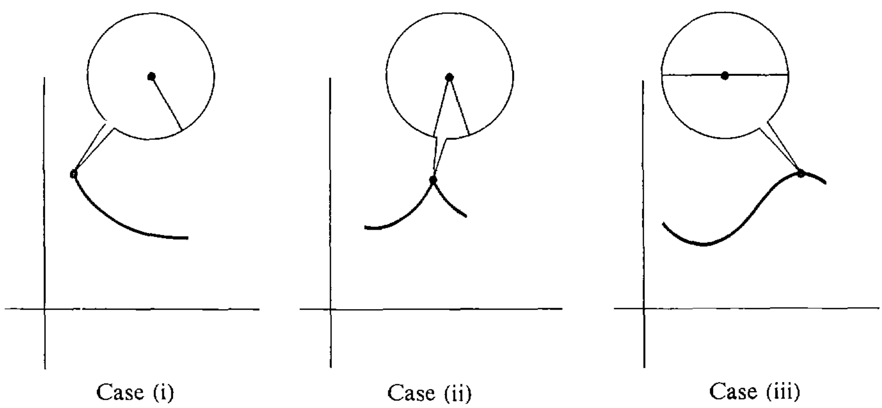


Figure 3.5.4 Critical Point Theorem

interval, the two endpoints of I will always be among the critical points. Geometrically the theorem says that if f has a maximum or minimum at c, then either c is an endpoint of the curve, or there is a sharp corner at c, or the curve has a horizontal slope at c. Thus at a maximum there is either an endpoint, a sharp peak, or a horizontal summit.

The Critical Point Theorem has some important applications to economics. Here is one example. Some other examples are described in the problem set.

EXAMPLE 3 Suppose a quantity x of a commodity can be produced at a total cost $C(x)$ and sold for a total revenue of $R(x)$, $0 < x < \infty$. The *profit* is defined as the difference between the revenue and the cost,

$$P(x) = R(x) - C(x).$$

Show that if the profit has a maximum at x_0, then the marginal cost is equal to the marginal revenue at x_0,

$$R'(x_0) = C'(x_0).$$

In this problem it is understood that $R(x)$ and $C(x)$ are differentiable functions, so that the marginal cost and marginal revenue always exist. Therefore $P'(x)$ exists and

$$P'(x) = R'(x) - C'(x).$$

Assume $P(x)$ has a maximum at x_0. Since $(0, \infty)$ has no endpoints and $P'(x_0)$ exists, the Critical Point Theorem shows that $P'(x_0) = 0$. Thus

$$P'(x_0) = R'(x_0) - C'(x_0) = 0$$

and

$$R'(x_0) = C'(x_0).$$

DEFINITION

*An **interior point** of an interval I is an element of I which is not an endpoint of I.*

For example, if I is an open interval, then every point of I is an interior point of I. But if I is a closed interval $[a, b]$, then the set of all interior points of I is the open interval (a, b) (Figure 3.5.5).

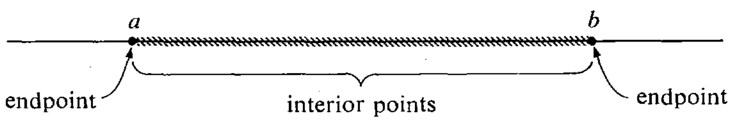


Figure 3.5.5

An interior point of I which is a critical point of f is called an *interior critical point*. There are a number of tests to determine whether or not f has a maximum at a given interior critical point. Here are two such tests. In both tests we assume that f is continuous on its domain I.

DIRECT TEST

Suppose c is the only interior critical point of f, and u, v are points in I with $u < c < v$.

(i) *If* $f(c) > f(u)$ *and* $f(c) > f(v)$, *then* f *has a maximum at* c *and nowhere else.*

(ii) *If* $f(c) < f(u)$ *and* $f(c) < f(v)$, *then* f *has a minimum at* c *and nowhere else.*

(iii) *Otherwise,* f *has neither a maximum nor a minimum at* c.

The three cases in the Direct Test are shown in Figure 3.5.6. The advantage of the Direct Test is that one can determine whether f has a maximum or minimum at c by computing only the three values $f(u), f(v)$, and $f(c)$ instead of computing all values of $f(x)$.

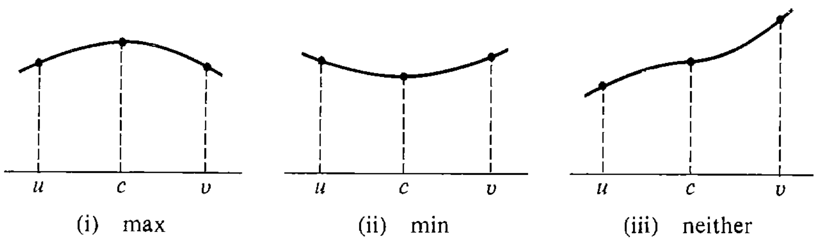


Figure 3.5.6

PROOF OF THE DIRECT TEST We must prove that if two points of I are on the same side of c, their values are on the same side of $f(c)$. Suppose, for instance, that $u_1 < u_2 < c$ (Figure 3.5.7). On the closed interval $[u_1, c]$ the only

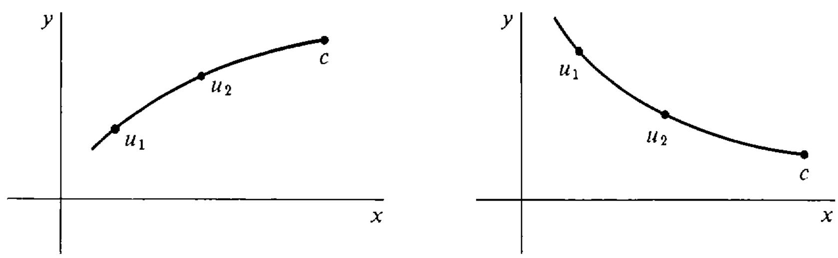


Figure 3.5.7

critical points are the endpoints. Thus when we restrict f to this interval, it has a maximum at one endpoint and a minimum at the other. If the maximum is at c, then $f(u_1)$ and $f(u_2)$ are both less than $f(c)$; if the minimum is at c, then $f(u_1)$ and $f(u_2)$ are both greater than $f(c)$. A similar proof works when $c < v_1 < v_2$. Note: our proof used the fact that f has a maximum and minimum on each closed interval. That fact, called the Extreme Value Theorem, will be proved on page 164.

SECOND DERIVATIVE TEST

Suppose c *is the only interior critical point of* f *and that* $f'(c) = 0$.

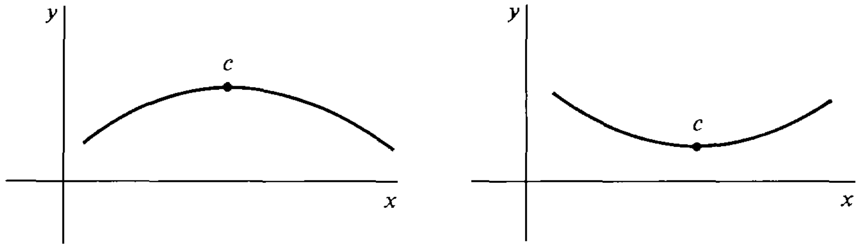
(i) *If* $f''(c) < 0$, f *has a maximum at* c *and nowhere else.*

(ii) *If* $f''(c) > 0$, f *has a minimum at* c *and nowhere else.*

We omit the proof and give a simple intuitive argument instead. (See Figure 3.5.8.) Since $f'(c) = 0$, the curve is horizontal at c. If $f''(c)$ is negative the slope is decreasing. This means that the curve climbs up until it levels off at c and then falls down, so it has a maximum at c. On the other hand, if $f''(c)$ is positive, the slope is increasing, so the curve falls down until it reaches a minimum at c and then climbs up. This argument makes it easy to remember which way the inequalities go in the test.

The Second Derivative Test fails when $f''(c) = 0$ and when $f''(c)$ does not exist. When the Second Derivative Test fails any of the following things can still happen:

(1) f has a maximum at $x = c$.
(2) f has a minimum at $x = c$.
(3) f has neither a maximum nor a minimum at $x = c$.



(i) $f''(c) < 0$, max (ii) $f''(c) > 0$, min

Figure 3.5.8

In most maximum and minimum problems, there is only one critical point except for the endpoints of the interval. We develop a method for finding the maximum and minimum in that case.

METHOD FOR FINDING MAXIMA AND MINIMA

When to use: f is continuous on its domain I, and f has exactly one interior critical point.

Step 1 Differentiate f.

Step 2 Find the unique interior critical point c of f.

Step 3 Test to see whether f has a maximum or minimum at c. The Direct Test or the Second Derivative Test may be used.

This method can be applied to an open or half-open interval as well as a closed interval. The Second Derivative Test is more convenient because it requires only the single computation $f''(c)$, while the Direct Test requires the three computations $f(u)$, $f(v)$, and $f(c)$. However, the Direct Test always works while the Second Derivative Test sometimes fails.

We illustrate the use of both tests in the examples.

EXAMPLE 4 Find the point on the line $y = 2x + 3$ which is at minimum distance from the origin.

The distance is given by

$$z = \sqrt{x^2 + y^2},$$

and substituting $2x + 3$ for y,

$$z = \sqrt{x^2 + (2x + 3)^2} = \sqrt{5x^2 + 12x + 9}.$$

This is defined on the whole real line.

Step 1 $\dfrac{dz}{dx} = \dfrac{10x + 12}{2\sqrt{5x^2 + 12x + 9}} = \dfrac{5x + 6}{z}$.

Step 2 $\dfrac{dz}{dx} = 0$ only when $5x + 6 = 0$, or $x = -\frac{6}{5}$.

Step 3 $\dfrac{d^2z}{dx^2} = \dfrac{5z - (5x + 6)(dz/dx)}{z^2}$.

At $x = -\frac{6}{5}$, $5x + 6 = 0$ and $z > 0$ so $d^2z/dx^2 = 5/z > 0$. By the Second Derivative Test, z has a minimum at $x = -\frac{6}{5}$.

CONCLUSION The distance is a minimum at $x = -\frac{6}{5}$, $y = 2x + 3 = \frac{3}{5}$. The minimum distance is $z = \sqrt{x^2 + y^2} = \sqrt{\frac{9}{5}}$. This is shown in Figure 3.5.9.

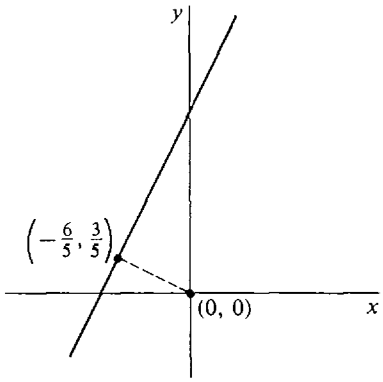


Figure 3.5.9

EXAMPLE 5 Find the minimum of $f(x) = x^6 + 10x^4 + 2$.

Step 1 $f'(x) = 6x^5 + 40x^3 = x^3(6x^2 + 40)$.

Step 2 $f'(x) = 0$ only when $x = 0$.

Step 3 The Second Derivative Test fails, because

$$f''(x) = 30x^4 + 120x^2, \qquad f''(0) = 0.$$

We use the Direct Test. Let $u = -1$, $v = 1$. Then

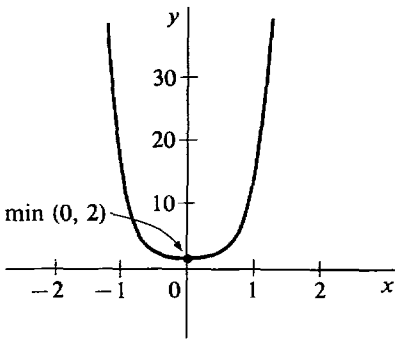
$$f(0) = 2, \qquad f(-1) = 13, \qquad f(1) = 13.$$

Hence f has a minimum at 0, as shown in Figure 3.5.10.

EXAMPLE 6 Find the maximum of $f(x) = 1 - x^{2/3}$.

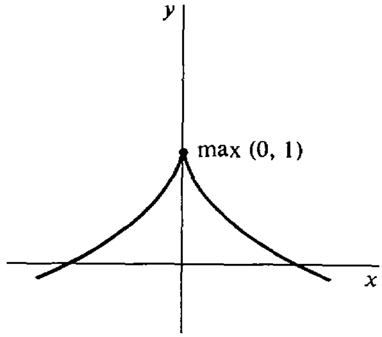
Step 1 $f'(x) = -(\frac{2}{3})x^{-1/3}$.

Step 2 $f'(x)$ is undefined at $x = 0$, and this the only critical point.



$f(x) = x^6 + 10x^4 + 2$

Figure 3.5.10



$f(x) = 1 - x^{2/3}$

Figure 3.5.11

Step 3 We use the Direct Test. Let $u = -1, v = 1$.

$$f(0) = 1, \qquad f(-1) = 0, \qquad f(1) = 0.$$

Thus f has a maximum at $x = 0$, as shown in Figure 3.5.11.

If f has more than one interior critical point, the maxima and minima can sometimes be found by dividing the interval into two or more parts.

EXAMPLE 7 Find the maximum and minimum of $f(x) = x/(x^2 + 1)$.

Step 1 $f'(x) = \dfrac{(x^2+1) - 2x^2}{(x^2+1)^2} = \dfrac{1 - x^2}{(x^2+1)^2}.$

Step 2 $f'(x) = 0$, when $x = -1$ and $x = 1$. There are two interior critical points. We divide the interval $(-\infty, \infty)$ on which f is defined into the two subintervals $(-\infty, 0]$ and $[0, \infty)$. On each of these subintervals, f has just one interior critical point.

Step 3 We shall use the direct test for the subinterval $(-\infty, 0]$. At the critical point -1, we have $f(-1) = -\frac{1}{2}$. By direct computation, we see that $f(-2) = -\frac{2}{5}$ and $f(0) = 0$. Both of these values are greater than $-\frac{1}{2}$. This shows that the restriction of f to the subinterval $(-\infty, 0]$ has a minimum at $x = -1$. Moreover, $f(x)$ is always ≥ 0 for x in the other subinterval $[0, \infty)$. Therefore f has a minimum at -1 for the whole interval $(-\infty, \infty)$. In a similar way, we can show that f has a maximum at $x = 1$.

CONCLUSION f has a minimum at $x = -1$ with value $f(-1) = -\frac{1}{2}$, and a maximum at $x = 1$ with value $f(1) = \frac{1}{2}$. (See Figure 3.5.12.)

The Critical Point Theorem can often be used to show that a curve has no maximum or minimum on an open interval $I = (a, b)$. The theorem shows that:

If $y = f(x)$ has no critical points in (a, b), the curve has no maximum or minimum on (a, b).

If $y = f(x)$ has just one critical point $x = c$ in (a, b) and two points x_1 and

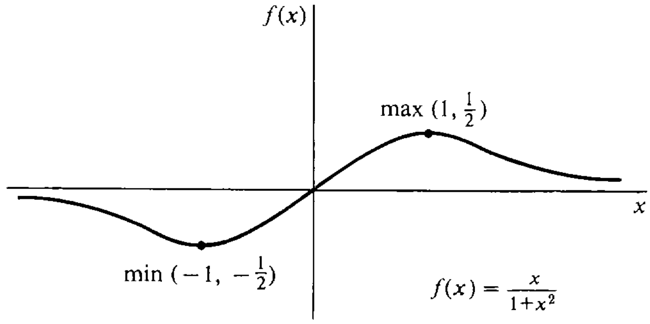


Figure 3.5.12

x_2 are found where $f(x_1) < f(c) < f(x_2)$, then the curve has no maximum or minimum on (a, b).

EXAMPLE 8 $f(x) = x^3 - 1$. Test for maxima and minima.

Step 1 $f'(x) = 3x^2$.

Step 2 $f'(x) = 0$ only when $x = 0$.

Step 3 The Second Derivative Test fails, because $f''(x) = 6x$, $f''(0) = 0$.
By direct computation, $f(0) = -1$, $f(-1) = -2$, $f(1) = 0$.
Therefore f has neither a minimum nor a maximum at $x = 0$.

CONCLUSION Since $x = 0$ is the only critical point of f and f doesn't have a maximum or minimum there, we conclude that f has no maximum and no minimum as shown in Figure 3.5.13.

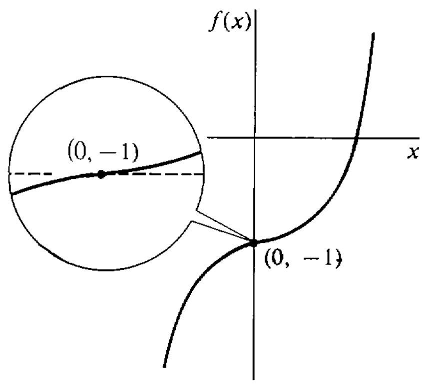


Figure 3.5.13

PROOF OF THE CRITICAL POINT THEOREM Assume that neither (i) nor (ii) holds; that is, assume that c is not an endpoint of I and $f'(c)$ exists. We must show that (iii) is true; i.e., $f'(c) = 0$. We give the proof for the case that f has a maximum at c. Let $x = c$, and let $\Delta x > 0$ be infinitesimal. Then

$$f(c + \Delta x) \le f(c), \qquad f(c - \Delta x) \le f(c).$$

(See Figure 3.5.14.) Therefore

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} \le 0 \le \frac{f(c - \Delta x) - f(c)}{-\Delta x}.$$

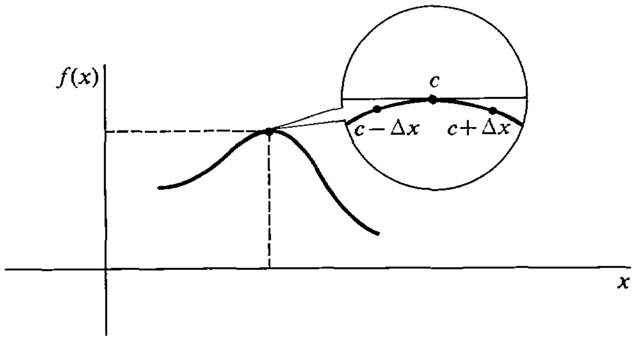


Figure 3.5.14 Proof of the Critical Point Theorem

Taking standard parts,

$$f'(c) = st\left(\frac{f(c+\Delta x) - f(c)}{\Delta x}\right) \leq 0,$$

and also,

$$0 \leq st\left(\frac{f(c-\Delta x) - f(c)}{-\Delta x}\right) = f'(c).$$

Therefore $f'(c) = 0$.

PROBLEMS FOR SECTION 3.5

In Problems 1–36, find the unique interior critical point and determine whether it is a maximum, a minimum, or neither.

1 $f(x) = x^2$

2 $f(x) = 1 - x^2$

3 $f(x) = x^4 + 2$

4 $f(x) = x^4 + 3x^2 + 5$

5 $f(x) = x^3 + 2$

6 $f(x) = x^3 - 3x^2 + 3x$

7 $f(x) = 3x^2 + 2x - 5$

8 $f(x) = 2(x-1)^4 + (x-1)^2 + 6$

9 $f(x) = x^{4/5}$

10 $f(x) = 2 - (x+1)^{2/3}$

11 $f(x) = \dfrac{1}{x^2 - 1}, \quad -1 < x < 1$

12 $f(x) = \dfrac{1}{x^2 + 1}$

13 $f(x) = x^{1/3} + 1$

14 $f(x) = 4 - x^{1/5}$

15 $f(x) = x^2 - x^{-1}, \quad x < 0$

16 $f(x) = x^2 - x^{-1}, \quad x > 0$

17 $f(x) = x^{-1} - (x-3)^{-1}, \quad 0 < x < 3$

18 $f(x) = x + x^{-1}, \quad 0 < x$

19 $f(x) = \sqrt{4 - x^2}, \quad -2 \leq x \leq 2$

20 $f(x) = (4 - x^2)^{-1/2}, \quad -2 < x < 2$

21 $y = \sin x + x, \quad 0 \leq x \leq 2\pi$

22 $y = \sin^2 x, \quad 0 < x < \pi$

23 $y = e^{-x^2}$

24 $y = e^{x^2 - 1}$

25 $y = \dfrac{1}{\cos x}, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

26 $y = \ln(\sin x), \quad 0 < x < \pi$

27 $y = xe^x$

28 $y = x \ln x, \quad 0 < x < \infty$

29 $y = x - \ln x, \quad 0 < x < \infty$

30 $y = e^x - x$

31 $f(x) = |x - 3|$

32 $f(x) = 3 + |1 - x|$

33 $f(x) = 2 - |x|$

34 $f(x) = 2|x| - x$

35 $f(x) = \sqrt{x} + \sqrt{1 - x}$, $0 \le x \le 1$

36 $f(x) = \sqrt{x} + \sqrt{9 - 3x}$, $0 \le x \le 3$

37 Find the shortest distance between the line $y = 1 - 4x$ and the origin.

38 Find the shortest distance between the curve $y = 2/x$ and the origin.

39 Find the minimum of the curve $f(x) = x^m - mx$, $x > 0$, where m is an integer ≥ 2.

40 Find the maximum of $f(x) = x^m - mx$, $x < 0$, where m is an odd integer ≥ 2.

In Problems 41–44, find the maximum and minimum of the given curve.

41 $f(x) = \dfrac{x}{x^2 + 4}$

42 $f(x) = \dfrac{3x + 4}{x^2 + 1}$

43 $f(x) = \dfrac{x}{x^4 + 1}$

44 $f(x) = \dfrac{x^3}{x^4 + 1}$

3.6 MAXIMA AND MINIMA—APPLICATIONS

Maximum and minimum problems arise in both the physical and social sciences. We give three examples.

EXAMPLE 1 A woman wishes to rent a house. If she lives x miles from her work, her transportation cost will be cx dollars per year, while her rent will be $25c/(x + 1)$ dollars per year. How far should she live from work to minimize her rent and transportation expenses?

Let y be her expenses in dollars per year. Then

$$y = cx + \frac{25c}{x + 1}.$$

The problem is to find the minimum value of y in the interval $0 \le x < \infty$.

Step 1 $\dfrac{dy}{dx} = c - \dfrac{25c}{(x + 1)^2}$.

Step 2 To find x such that $dy/dx = 0$ we set $dy/dx = 0$ and solve for x.

$$c - \frac{25c}{(x + 1)^2} = 0, \quad c = \frac{25c}{(x + 1)^2}, \quad (x + 1)^2 = 25, \quad x + 1 = \pm 5.$$

Then $x = 4$ or $x = -6$. We reject $x = -6$ because $0 \le x$. The only interior critical point is $x = 4$.

Step 3 We use the Direct Test.

At $x = 0$, $y = c \cdot 0 + 25c/(0 + 1) = 25c$.

At $x = 4$, $y = 4c + 25c/(4 + 1) = 9c$.

At $x = 9$, $y = 9c + 25c/(9 + 1) = 11.5c$.

CONCLUSION y has its minimum at $x = 4$ miles. So the woman should live four miles from work. (See Figure 3.6.1.)

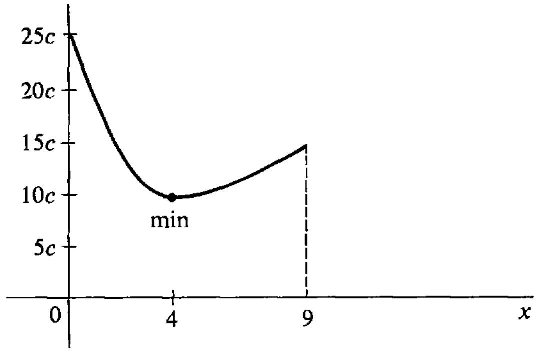


Figure 3.6.1

EXAMPLE 2 A farmer plans to use 1000 feet of fence to enclose a rectangular plot along the bank of a straight river. Find the dimensions which enclose the maximum area.

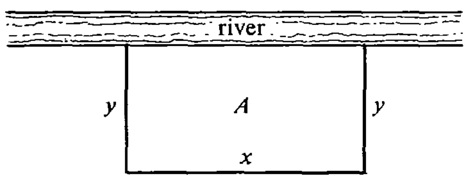


Figure 3.6.2

Let x be the dimension of the side along the river, and y be the other dimension, as in Figure 3.6.2. Call the area A.

No fencing is needed on the side of the plot bordering the river. The given information is expressed by the following system of formulas.

$$A = xy, \quad x + 2y = 1000, \quad 0 \le x \le 1000.$$

The problem is to find the values of x and y at which A is maximum. In this problem A is expressed in terms of two variables instead of one. However, we can select x as the independent variable, and then both y and A are functions of x. We find an equation for A as a function of x alone by eliminating y.

$$x + 2y = 1000, \qquad y = \frac{1000 - x}{2}.$$

$$A = xy = \frac{x(1000 - x)}{2} = 500x - \frac{1}{2}x^2.$$

We then find the maximum of A in the closed interval $0 \le x \le 1000$.

Step 1 $dA/dx = 500 - x$.

Step 2 $dA/dx = 0$ when $x = 500$. This is the unique interior critical point.

Step 3 We use the Second Derivative Test: $d^2A/dx^2 = -1$. Therefore A has a maximum at the critical point $x = 500$.

CONCLUSION The maximum area occurs when the plot has dimensions $x = 500$ ft and $y = (1000 - x)/2 = 250$ft (Figure 3.6.3).

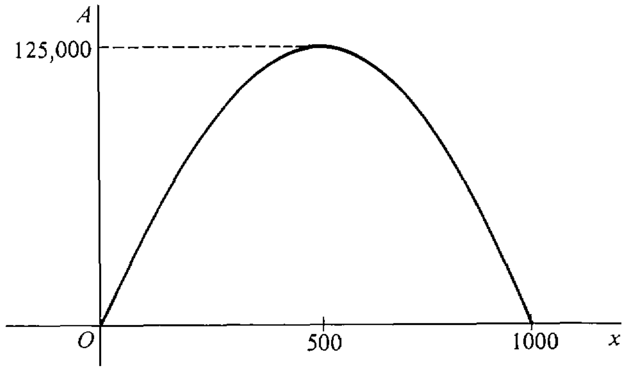


Figure 3.6.3

EXAMPLE 3 Find the shape of the cylinder of maximum volume which can be inscribed in a given sphere.

The *shape* of a right circular cylinder can be described by the ratio of the radius of its base to its height. This ratio for the inscribed cylinder of maximum volume should be a number which does not depend on the radius of the sphere. For example, we should get the same shape whether the radius of the sphere is given in inches or centimeters.

Let r be the radius of the given sphere, x the radius of the base of the cylinder, h its height, and V its volume. First, we draw a sketch of the problem in Figure 3.6.4.

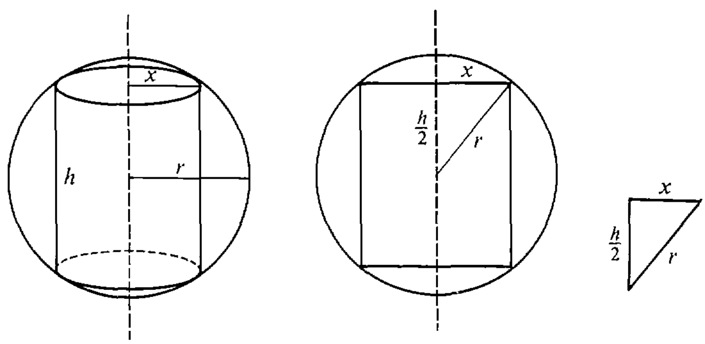


Figure 3.6.4

From the sketch we can read off the formulas

$$V = \pi x^2 h, \quad x^2 + (\tfrac{1}{2}h)^2 = r^2, \quad 0 \le x \le r.$$

r is a constant. We select x as the independent variable, while h and V are functions of x. To solve the problem we shall find the value of x where V is a maximum and then compute the ratio x/h at this point to describe the shape of the cylinder. The answer x/h should not depend on the constant r. We give two methods of solution.

FIRST SOLUTION Express V as a function of x by eliminating h.

$$x^2 + (\tfrac{1}{2}h)^2 = r^2,$$

$$h = 2\sqrt{r^2 - x^2}.$$

$$V = \pi x^2 h = 2\pi x^2\sqrt{r^2 - x^2}.$$

The problem is to find the maximum of V in the interval $0 \le x \le r$.

Step 1 $\dfrac{dV}{dx} = 4\pi x\sqrt{r^2 - x^2} - \dfrac{2\pi x^3}{\sqrt{r^2 - x^2}}, \quad (x < r).$

Step 2 There is one critical point at $x = r$, where dV/dx does not exist. We set $dV/dx = 0$ and solve for x to find the other critical points.

$$4\pi x\sqrt{r^2 - x^2} - \frac{2\pi x^3}{\sqrt{r^2 - x^2}} = 0, \quad 4\pi x(r^2 - x^2) - 2\pi x^3 = 0,$$

$$2\pi x(2r^2 - 3x^2) = 0, \qquad x = 0 \quad \text{or} \quad x = \pm r\sqrt{\tfrac{2}{3}}.$$

We reject $x = -r\sqrt{\frac{2}{3}}$ because $0 \le x \le r$. The only interior critical point is $x = r\sqrt{\frac{2}{3}}$.

Step 3 We use the Direct Test.

At $x = 0$, $\quad V = 0$.

At $x = r\sqrt{\frac{2}{3}}$, $\quad V = \dfrac{4\pi r^3}{3\sqrt{3}}$.

At $x = r$, $\quad V = 0$.

CONCLUSION The maximum of V is at $x = r\sqrt{\frac{2}{3}}$ (see Figure 3.6.5). At that point, $h = 2\sqrt{r^2 - x^2} = 2r/\sqrt{3}$. Then the ratio of x to h is

$$x/h = 1/\sqrt{2}.$$

Notice that, as we expected, this number does not depend on r.

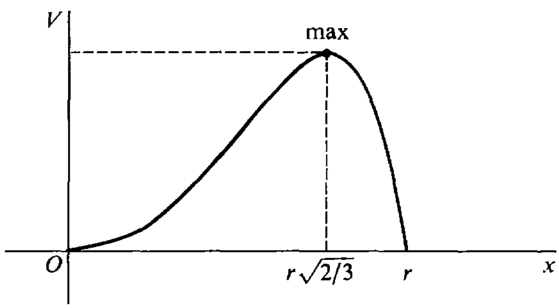


Figure 3.6.5

SECOND SOLUTION Instead of eliminating h and expressing V as a function of x, we shall use the equations in their original form and find the critical points by implicit differentiation.

Step 1 $V = \pi x^2 h, \quad dV/dx = 2\pi x h + \pi x^2\, dh/dx.$

We find dh/dx by implicit differentiation.

$$x^2 + (\tfrac{1}{2}h)^2 = r^2, \quad 2x + \tfrac{1}{2}h\frac{dh}{dx} = 0, \quad \frac{dh}{dx} = -\frac{4x}{h}, \qquad (h \ne 0).$$

Then $$\frac{dV}{dx} = 2\pi x h + \pi x^2\left(-\frac{4x}{h}\right) = 2\pi x h - \frac{4\pi x^3}{h}, \qquad (h \ne 0).$$

Step 2 When $h = 0$ we have $x = r$, which is an endpoint. When $h \ne 0$ we set $dV/dx = 0$ and solve for x.

$$2\pi xh - 4\pi x^3/h = 0, \qquad xh - 2x^3/h = 0,$$
$$xh^2 = 2x^3, \qquad x = 0 \quad \text{or} \quad x = \pm h/\sqrt{2}.$$

We reject $x = -h/\sqrt{2}$ because x and $h \geq 0$. $x = 0$ is an endpoint. Thus $x = h/\sqrt{2}$ is the unique interior critical point.

Step 3 We use the Direct Test. At $x = 0$, $V = 0$. At $x = h/\sqrt{2}$, $V = \pi h^3/2$. At $x = r$, $V = 0$.

CONCLUSION The maximum of V is at $x = h/\sqrt{2}$. At that point the ratio of x to h is $x/h = 1/\sqrt{2}$.

The second method of solution may be better in a problem where it is hard or impossible to find explicit equations for the dependent variables (like h and V) as functions of the independent variable.

PROBLEMS FOR SECTION 3.6

1 Split 20 into the sum of two numbers $x \geq 0$ and $y \geq 0$ such that the product of x and y^2 is a maximum.

2 Find two numbers $x \geq 0$ and $y \geq 0$ such that $x + y = 8$ and $x^2 + y^2$ is a minimum.

3 Find two numbers $x \geq 1$ and $y \geq 1$ such that $xy = 50$ and $2x + y$ is a maximum.

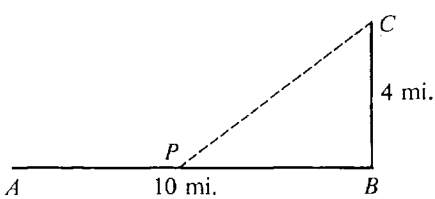
4 Find the rectangle with perimeter 8 which has maximum area.

5 Find the maximum value of x^3y if x and y belong to $[0, 1]$ and $x + y = 1$.

6 A rectangular box which is open at the top can be made from a 10 by 12 inch piece of metal by cutting a square from each corner and bending up the sides. Find the dimensions of the box with greatest volume.

7 A poster of total area 400 sq in. is to have a margin of 4 in. at the top and bottom and 3 in. at each side. Find the dimensions which give the largest printed area.

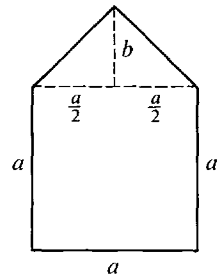
8 A man can travel 5 mph along the path AB and 3 mph off the path as shown in the figure. Find the quickest route APC from the point A to the point C.



9 Find the dimensions of the right triangle of maximum area whose hypotenuse has length one.

10 Find the dimensions of the isosceles triangle of maximum area which has perimeter 3.

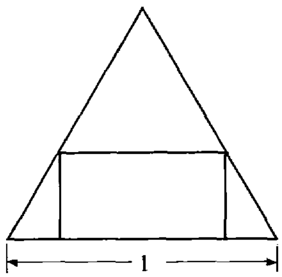
11 Find the five-sided figure of maximum area which has the shape of a square topped by an isosceles triangle, and such that the sum of the height of the figure and the perimeter of the square is 20 ft.



12 A wire of length L is to be divided into two parts; one part will be bent into a square and the other into a circle. How should the wire be divided to make the sum of the areas of the square and circle as large as possible? As small as possible?

13 Find the area of the largest rectangle which can be inscribed in a semicircle of radius r.

14 Find the dimensions of the rectangle of maximum area which can be inscribed in an equilateral triangle as shown in the figure.

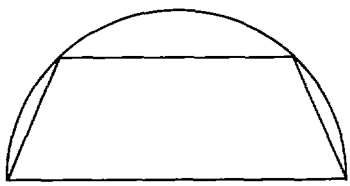


15 Find the shape of the right circular cylinder of maximum volume which can be inscribed in a right circular cone of height 3 and base of radius 1.

16 Find the shape of the right circular cone of maximum volume which can be inscribed in a given sphere.

17 Find the shape of the cylinder of maximum volume such that the sum of the height and the circumference of the base is equal to 4.

18 Find the shape of the largest trapezoid which can be inscribed in a semicircle as shown in the figure.



19 If a farmer plants x units of wheat in his field, $0 \le x \le 100$, the yield will be $10x - x^2/10$ units. How much wheat should he plant for the maximum yield?

20 In Problem 19 above, it costs the farmer \$100 for each unit of wheat he plants, and he is able to sell each unit he harvests for \$50. How much should he plant to maximize his profit?

21 A professional football team has a stadium which seats 60,000. It is found that x tickets can be sold at a price of $p = 10 - x/10{,}000$ dollars per ticket. Find the values of x and p at which the total money received will be a maximum.

22 In Problem 21 a tax of \$1 per ticket is added onto the price. Find x and p so that the total revenue after taxes is a maximum.

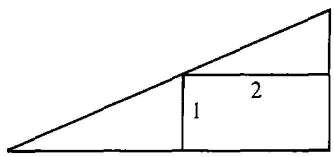
23 A store can buy up to 300 seconds of advertising time daily on the radio at the rate of \$2/sec for the first 100 sec, and \$1/sec thereafter. x seconds on the radio increases daily sales by $32\sqrt{x}$ dollars. How many seconds on the radio will yield the maximum profit?

24 Work Problem 23 if the cost of advertising time is \$1/sec for the first 100 sec and \$2/sec thereafter.

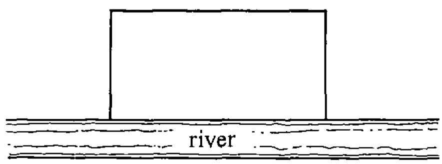
25 Find the real number which most exceeds its square.

26 Find the rectangle of area 9 which has the smallest perimeter.

27 Find the right triangle of smallest area in which a 1 by 2 rectangle can be inscribed as shown in the figure.



28 A farmer wishes to enclose 10,000 sq ft of land along a river by three sides of fence as shown in the figure. Find the dimensions which require the minimum length of fence.



29 Find the shortest distance between the line $y = 1 - 4x$ and the origin.

30 Find the shortest distance between the curve $y = 2/x$ and the origin.

31 A warehouse is to be built in the shape of a rectangular solid with a square base. The cost of the roof per unit area is three times the cost of the walls. Find the shape which will enclose the maximum volume for a given cost.

32 A rectangular box with volume 1 cu ft is to be made with a square base and no top. Find the dimensions which require the smallest amount of material.

33 Find the dimensions of the right circular cylinder of volume 1 cu ft which has the smallest surface area (top plus bottom plus sides).

34 Find the dimensions of the right circular cone of smallest volume which can be circumscribed about a sphere of radius r.

35 Given two real numbers a and b, find x such that $(x - a)^2 + (x - b)^2$ is a minimum.

36 The area of a sector of a circle with radius r and central angle θ is $A = \frac{1}{2}r^2\theta$, and its arc has length $s = r\theta$. Find r and θ so that $0 < \theta < 2\pi$, the sector has area 1, and the perimeter is a minimum.

37 Show that among all right circular cylinders of volume 1 cu ft which are open at both ends, there is no maximum or minimum surface area.

38 The population of a country at time $t = 0$ is 50 million and is increasing at the rate of one million people per year. The national income at time t is $(20{,}000 + t^2)$ million dollars per year. At what time $t \geq 0$ is the per capita income (= national income $\div$ population) a minimum?

39 A man estimates that he can paint his house in x hours of his spare time if he buys equipment costing $200 + 2000/x^2$ dollars, and that his spare time is worth \$2/hr. How many hours should he take?

40 An artisan can produce x items at a total cost of $100 + 5x$ dollars and sell x items at a price of $10 - x/100$ dollars per item. Find the value of x which gives the maximum profit.

41 A manufacturer can produce any number of buttons at a cost of two cents per button and can sell x buttons at a price of $1000/\sqrt{x}$ cents per button. How many buttons should be produced for maximum profit?

3.7 DERIVATIVES AND CURVE SKETCHING

If we compute n values of $f(x)$,

$$f(x_1), f(x_2), \ldots, f(x_n),$$

we obtain n points through which the curve $y = f(x)$ passes. The first and second derivatives tell us something about the shape of the curve in the intervals between these points and permit a much more accurate plot of the curve. It is especially helpful to know the signs of the first two derivatives.

When the first derivative is positive the curve is increasing from left to right, and when the first derivative is negative the curve is decreasing from left to right. When the first derivative is zero the curve is horizontal. These facts can be proved as a theorem if we define exactly what is meant by increasing and decreasing (see Figures 3.7.1 and 3.7.2).

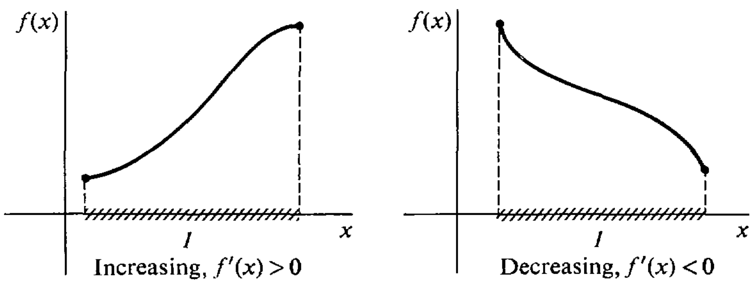


Figure 3.7.1

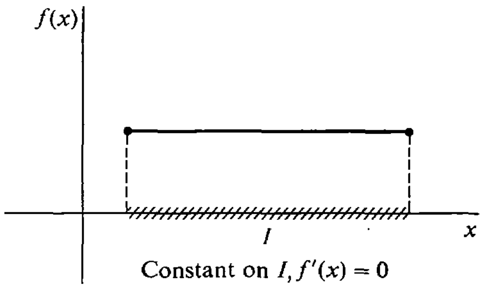


Figure 3.7.2

DEFINITION

*A function f is said to be **constant** on an interval I if:*

$$f(x_1) = f(x_2) \qquad \text{for all } x_1, x_2 \text{ in } I.$$

*f is **increasing** on I if:*

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I.$$

*f is **decreasing** on I if:*

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I.$$

THEOREM 1

Suppose f is continuous on I and has a derivative at every interior point of I.

(i) *If $f'(x) = 0$ for all interior points x of I, then f is constant on I.*

(ii) *If $f'(x) > 0$ for all interior points x of I, then f is increasing on I.*

(iii) *If $f'(x) < 0$ for all interior points x of I, then f is decreasing on I.*

A proof will be given in the next section.

EXAMPLE 1 The curve $y = x^3 + x - 1$ has derivative $dy/dx = 3x^2 + 1$. The derivative is always positive, so the curve is always increasing (Figure 3.7.3).

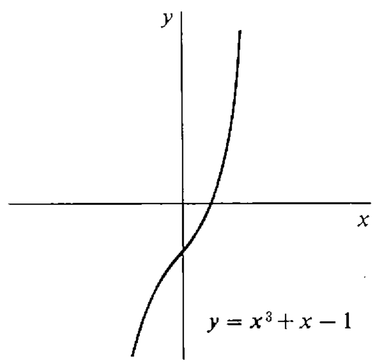


Figure 3.7.3

Let us now turn to the second derivative. It is the rate of change of the slope of the curve, so it has something to do with the way in which the curve is changing direction. When the second derivative is positive, the slope is increasing, and we would expect the curve to be concave upward, i.e., shaped like a ∪. When the second derivative is negative the slope is decreasing, so the curve should be shaped like ∩ (see Figure 3.7.4).

A precise definition of concave upward or downward can be given by comparing the curve with the chord (straight line segment) connecting two points on the curve.

DEFINITION

*Let f be defined on I. The curve $y = f(x)$ is **concave upward** on I if for any two points $x_1 < x_2$ in I and any value of x between x_1 and x_2, the curve at x is below the chord which meets the curve at x_1 and x_2.*

*The curve $y = f(x)$ is **concave downward** on I if for any two points $x_1 < x_2$ in I and any value of x between x_1 and x_2, the curve at x is above the chord which meets the curve at x_1 and x_2* (see Figure 3.7.5).

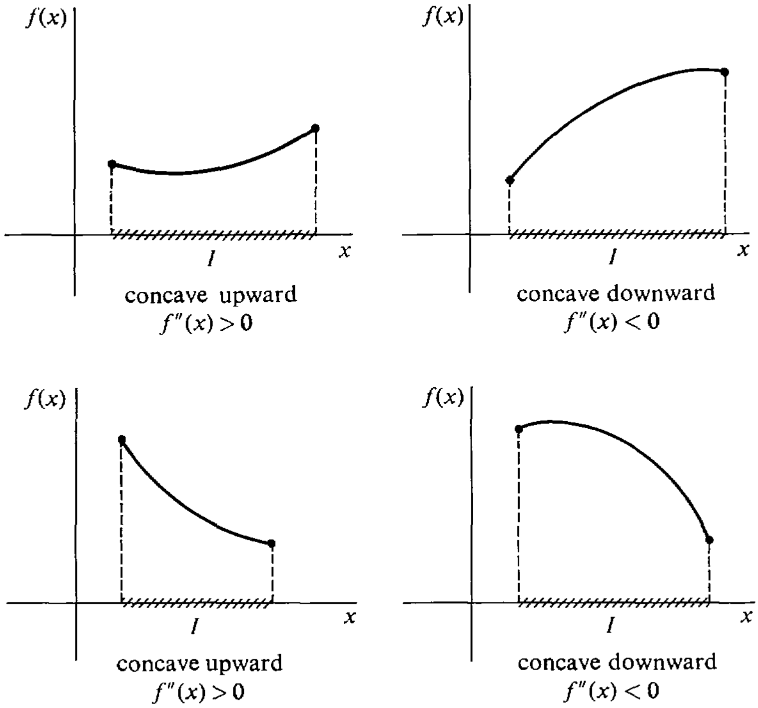


Figure 3.7.4

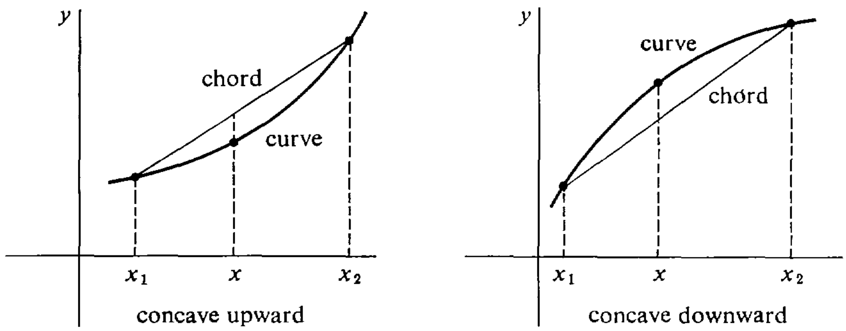


Figure 3.7.5

The next theorem gives the geometric meaning of the sign of the second derivative.

THEOREM 2

Suppose f is continuous on I and f has a second derivative at every interior point of I.

(i) *If $f''(x) > 0$ for all interior points x of I, then f is concave upward on I.*

(ii) *If $f''(x) < 0$ for all interior points x of I, then f is concave downward on I.*

We have already explained the intuitive reason for Theorem 2. The proof is omitted. Theorem 1 tells what happens when f' always has the same sign on an open interval I, while Theorem 2 does the same thing for f''. To use these results we need another theorem that tells us that certain functions always have the same sign on I.

THEOREM 3

Suppose g is continuous on I, and $g(x) \neq 0$ for all x in I.

(i) *If $g(c) > 0$ for at least one c in I, then $g(x) > 0$ for all x in I.*
(ii) *If $g(c) < 0$ for at least one c in I, then $g(x) < 0$ for all x in I.*

The two cases are shown in Figure 3.7.6. We give the proof in the next section.

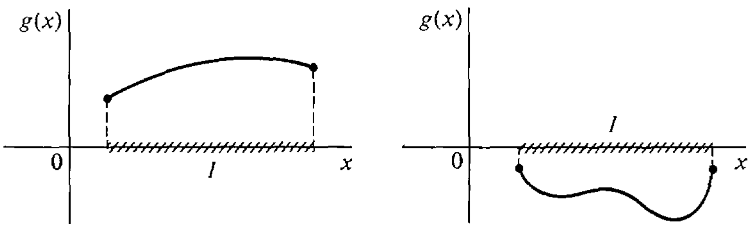


Figure 3.7.6 $g(x) > 0$ for all x in I $g(x) < 0$ for all x in I

Let us show with some simple examples how we can use the first and second derivatives in sketching curves. The three theorems above and the tests for minima and maxima are all helpful.

EXAMPLE 1 (Continued) $y = x^3 + x - 1$. We have

$$\frac{dy}{dx} = 3x^2 + 1,$$

$$\frac{d^2y}{dx^2} = 6x.$$

dy/dx is always positive, while $d^2y/dx^2 = 0$ at $x = 0$. We make a table of values for y and its first two derivatives at $x = 0$ and at a point to the right and left side of 0.

x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
−1	−3	4	−6
0	−1	1	0
1	1	4	6

With the aid of Theorems 1–3, we can draw the following conclusions:

(a) $dy/dx > 0$ and the curve is increasing for all x.
(b) $d^2y/dx^2 < 0$ for $x < 0$; concave downward.
(c) $d^2y/dx^2 > 0$ for $x > 0$; concave upward.

At the point $x = 0$, the curve changes from concave downward to concave upward. This is called a *point of inflection.*

To sketch the curve we first plot the three values of y shown in the table, then sketch the slope at these points as shown in Figure 3.7.7, then fill in a smooth curve, which is concave downward or upward as required.

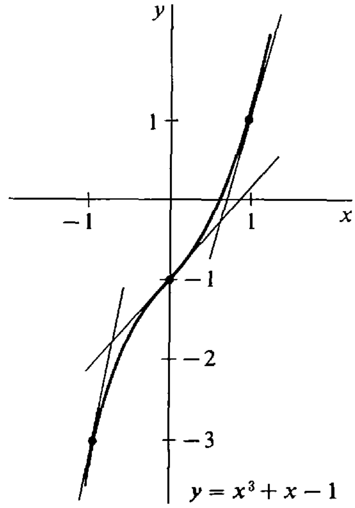


Figure 3.7.7

EXAMPLE 2 Sketch the curve $y = 2x - x^2$.

$$\frac{dy}{dx} = 2 - 2x, \qquad \frac{d^2y}{dx^2} = -2.$$

We see that $dy/dx = 0$ when $x = 1$, a critical point. d^2y/dx^2 is never zero because it is constant. We make a table of values including the critical point $x = 1$ and points to the right and left of it.

x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
−1	−3	4	−2
0	0	2	−2
1	1	0	−2
2	0	−2	−2
3	−3	−4	−2

CONCLUSIONS

(a) $dy/dx > 0$ for $x < 1$; increasing.
(b) $dy/dx < 0$ for $x > 1$; decreasing.
(c) $d^2y/dx^2 < 0$ for all x; concave downward.
(d) $dy/dx = 0$, $d^2y/dx^2 < 0$ at $x = 1$; maximum.

The curve is shown in Figure 3.7.8.

In general a curve $y = f(x)$ may go up and down several times. To sketch it we need to determine the intervals on which it is increasing or decreasing, and concave upward or downward. Here are some things which may happen at the endpoints of these intervals.

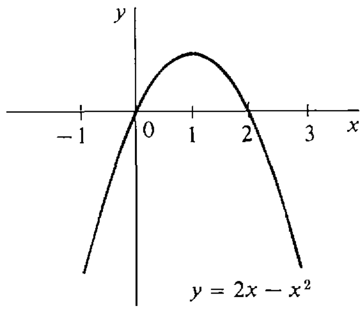


Figure 3.7.8

DEFINITION

Let c be an interior point of I.

f has a ***local maximum*** *at c if* $f(c) \geq f(x)$ *for all x in some open interval* (a_0, b_0) *containing c.*

f has a ***local minimum*** *at c if* $f(c) \leq f(x)$ *for all x in some open interval* (a_0, b_0) *containing c. (The interval* (a_0, b_0) *may be only a small subinterval of I.)*

f has a ***point of inflection*** *at c if f changes from one direction of concavity to the other at c.*

These definitions are illustrated in Figure 3.7.9.

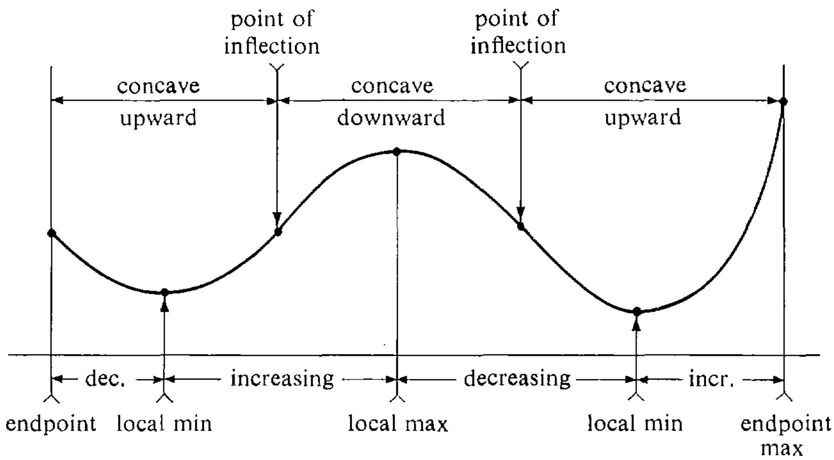


Figure 3.7.9

We may now describe the steps in sketching a curve. We shall stick to the simple case where f and its first two derivatives are continuous on a closed interval $[a, b]$, and either are never zero or are zero only finitely many times. (Curve plotting in a more general situation is discussed in Chapter 5 on limits.)

Step 1 Compute dy/dx and d^2y/dx^2.

Step 2 Find all points where $dy/dx = 0$ and all points where $d^2y/dx^2 = 0$.

Step 3 Pick a few points

$$a = x_0, x_1, x_2, \ldots, x_n = b$$

in the interval $[a, b]$. They should include both endpoints, all points where the first or second derivative is zero, and at least one point between any two consecutive zeros of dy/dx or d^2y/dx^2.

Step 4 At each of the points $x_0, \ldots, x_n$, compute the values of y and dy/dx and determine the sign of d^2y/dx^2. Make a table.

Step 5 From the table draw conclusions about where y is increasing or decreasing, where y has a local maximum or minimum, where the curve is concave upward or downward, and where it has a point of inflection. Use Theorems 1–3 of this section and the tests for maxima and minima.

Step 6 Plot the values of y and indicate slopes from the table. Then connect them with a smooth curve which agrees with the conclusions of Step 5.

EXAMPLE 3 $y = x^4/2 - x^2, \quad -2 \le x \le 2.$

Step 1 $dy/dx = 2x^3 - 2x. \quad d^2y/dx^2 = 6x^2 - 2.$

Step 2 $dy/dx = 0$ at $x = -1, 0, 1.$

Step 3 $d^2y/dx^2 = 0$ at $x \pm \sqrt{\frac{1}{3}}$. $\quad -2, -1, -\sqrt{\frac{1}{3}}, 0, \sqrt{\frac{1}{3}}, 1, 2.$

Step 4

x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
-2	4	-12	$+$
-1	$-\frac{1}{2}$	0	$+$
$-\sqrt{\frac{1}{3}}$	$-\frac{5}{18}$	$4/(3\sqrt{3})$	0
0	0	0	$-$
$\sqrt{\frac{1}{3}}$	$-\frac{5}{18}$	$-4/(3\sqrt{3})$	0
1	$-\frac{1}{2}$	0	$+$
2	4	12	$+$

Step 5 We indicate the conclusions schematically in Figure 3.7.10.

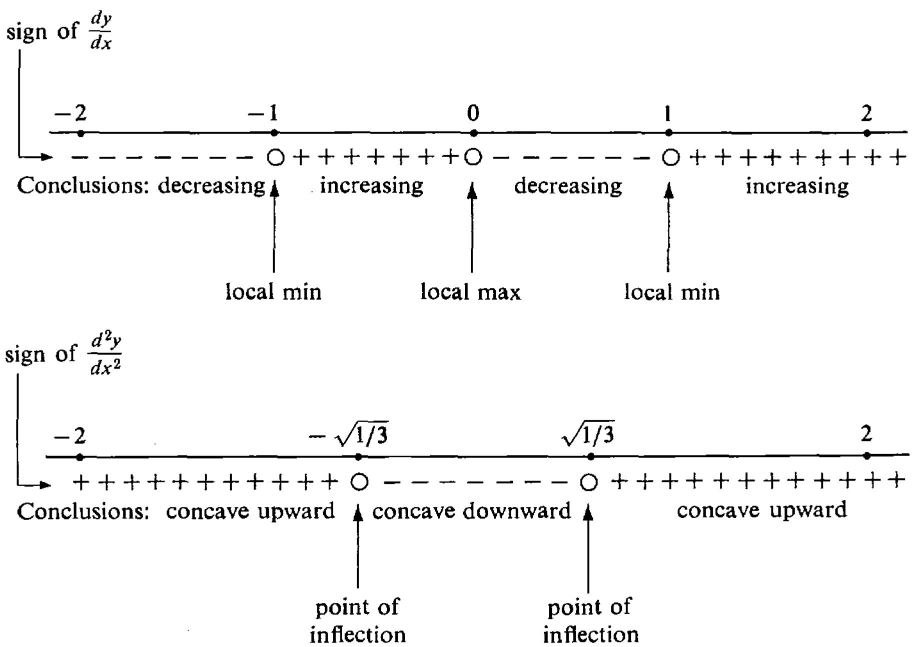


Figure 3.7.10

Step 6 The curve is W-shaped, as shown in Figure 3.7.11.

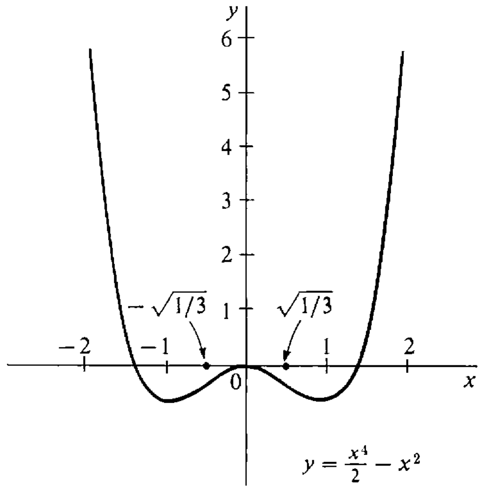


Figure 3.7.11

PROBLEMS FOR SECTION 3.7

Sketch each of the curves given below by the six-step process explained in the text. For each curve, give a table showing all the critical points, local maxima and minima, intervals on which the curve is increasing or decreasing, points of inflection, and intervals on which the curve is concave upward or downward.

1 $y = x^2 + 2, \quad -2 \le x \le 2$

2 $y = 1 - x^2, \quad -2 \le x \le 2$

3 $y = x^2 - 2x, \quad -2 \le x \le 2$

4 $y = \frac{1}{2}x^2 + x, \quad -2 \le x \le 2$

5 $y = 2x^2 - 4x + 3, \quad 0 \le x \le 2$

6 $y = -x^2 - 2x + 6, \quad -4 \le x \le 0$

7 $y = x^4, \quad -2 \le x \le 2$

8 $y = x^5, \quad -2 \le x \le 2$

9 $y = x^3 + x^2 + x, \quad -2 \le x \le 2$

10 $y = x^3 + x^2 - x, \quad -2 \le x \le 2$

11 $y = \frac{1}{3}x^3 + x^2 + x, \quad -2 \le x \le 2$

12 $y = -x^3 + 12x - 12, \quad -3 \le x \le 3$

13 $y = x^4 + 4x^3 + 2, \quad -4 \le x \le 2$

14 $y = \frac{1}{4}x^4 - x, \quad -2 \le x \le 2$

15 $y = x^2 - \frac{1}{2}x^4, \quad -2 \le x \le 2$

16 $y = x^2(x - 2)^2, \quad -1 \le x \le 3$

17 $y = 1/x, \quad -4 \le x \le -\frac{1}{4}$ and $\frac{1}{4} \le x \le 4$

18 $y = 1/x + x, \quad -4 \le x \le -\frac{1}{4}$ and $\frac{1}{4} \le x \le 4$

19 $y = x^{-2}, \quad -2 \le x \le -\frac{1}{2}$ and $\frac{1}{2} \le x \le 2$

20 $y = x + x^{-2}, \quad -2 \le x \le -\frac{1}{2}$ and $\frac{1}{2} \le x \le 2$

21 $y = \dfrac{x - 1}{x + 1}, \quad 0 \le x \le 10$

22 $y = \dfrac{2x}{x + 1}, \quad 0 \le x \le 10$

23 $y = \dfrac{1}{x^2 + 1}, \quad -4 \le x \le 4$

24 $y = \dfrac{x}{x^2 + 1}, \quad -4 \le x \le 4$

25 $y = \dfrac{x^2}{x^2 + 1}, \quad -2 \le x \le 2$

26 $y = \dfrac{1}{x^2 - 1}, \quad -\dfrac{9}{10} \le x \le \dfrac{9}{10}$

27	$y = \sqrt{x}, \quad \frac{1}{4} \le x \le 4$	**28**	$y = 2\sqrt{x} - x, \quad \frac{1}{4} \le x \le 4$
29	$y = 1/\sqrt{x}, \quad \frac{1}{4} \le x \le 4$	**30**	$y = x^{1/2} + x^{-1/2}, \quad \frac{1}{4} \le x \le 4$
31	$y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$	**32**	$y = \sqrt{9 + x^2}, \quad -4 \le x \le 4$
33	$y = \sin x \cos x, \quad 0 \le x \le 2\pi$	**34**	$y = \sin x + \cos x, \quad 0 \le x \le 2\pi$
35	$y = 3\sin(\frac{1}{2}x), \quad 0 \le x \le 2\pi$	**36**	$y = \sin^2 x, \quad 0 \le x \le 2\pi$
37	$y = \tan x, \quad -\pi/3 \le x \le \pi/3$	**38**	$y = 1/\cos x, \quad -\pi/3 \le x \le \pi/3$
39	$y = e^{-x}, \quad -2 \le x \le 2$	**40**	$y = e^{(1/2)x}, \quad -2 \le x \le 2$
41	$y = \ln x, \quad 1/e \le x \le e$	**42**	$y = (\ln x)^2, \quad 1/e \le x \le e$
43	$y = xe^{-x}, \quad -1 \le x \le 3$	**44**	$y = x - e^x, \quad -2 \le x \le 2$
45	$y = x\ln x, \quad e^{-2} \le x \le e$	**46**	$y = x - \ln x, \quad e^{-2} \le x \le e$
47	$y = xe^x, \quad -3 \le x \le 1$	**48**	$y = e^{-x^2}, \quad -2 \le x \le 2$
49	$y = e^x/x, \quad \frac{1}{4} \le x \le 4$	**50**	$y = \ln(1 + x^2), \quad -3 \le x \le 3$

3.8 PROPERTIES OF CONTINUOUS FUNCTIONS

This section develops some theory that will be needed for integration in Chapter 4. We begin with a new concept, that of a *hyperinteger*. The hyperintegers are to the integers as the hyperreal numbers are to the real numbers. The hyperintegers consist of the ordinary finite integers, the positive infinite hyperintegers, and the negative infinite hyperintegers. The hyperintegers have the same algebraic properties as the integers and are spaced one apart all along the hyperreal line as in Figure 3.8.1.

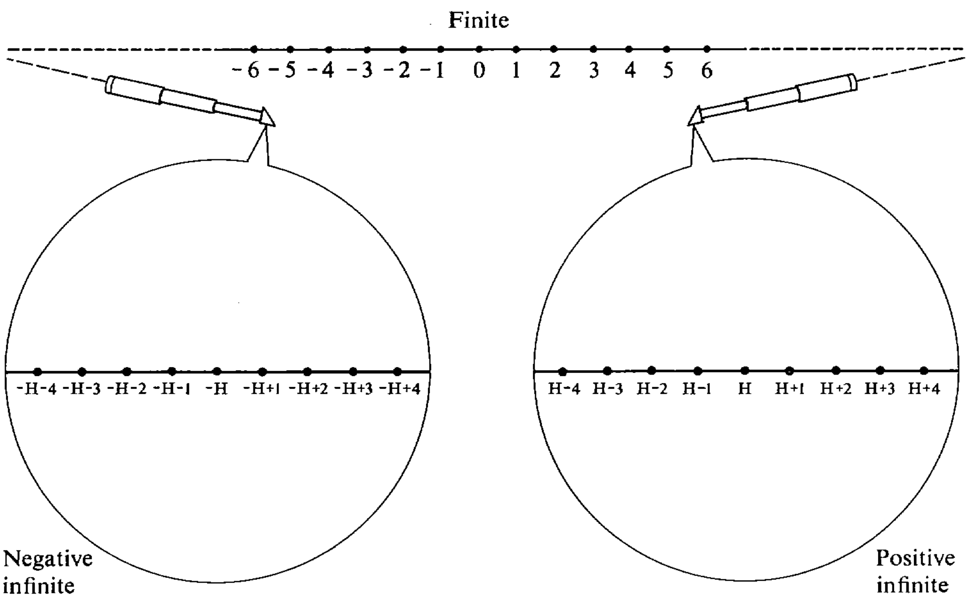


Figure 3.8.1 The Set of Hyperintegers

The rigorous definition of the hyperintegers uses the greatest integer function $[x]$ introduced in Section 3.4, Example 6. Remember that for a real number x, $[x]$ is the greatest integer n such that $n \le x$. A real number y is itself an integer if and only if $y = [x]$ for some real x. To get the hyperintegers, we apply the function $[x]$ to hyperreal numbers x (see Figure 3.8.2).

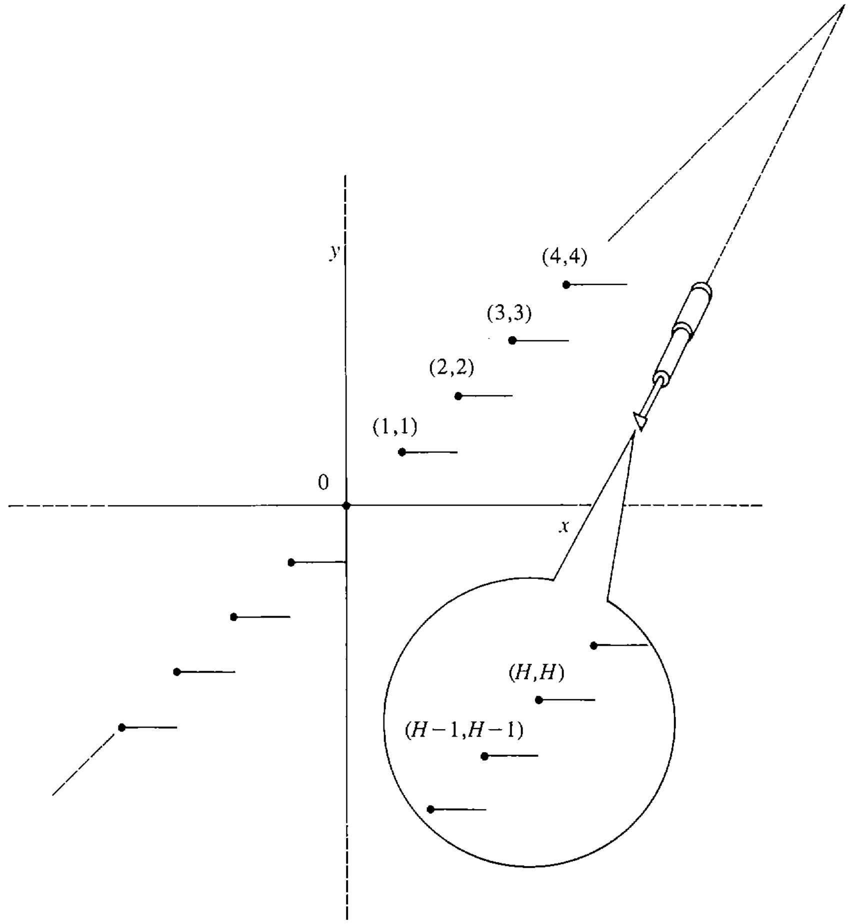


Figure 3.8.2

DEFINITION

*A **hyperinteger** is a hyperreal number y such that $y = [x]$ for some hyperreal x.*

When x varies over the hyperreal numbers, $[x]$ is the greatest hyperinteger y such that $y \leq x$. Because of the Transfer Principle, every hyperreal number x is between two hyperintegers $[x]$ and $[x] + 1$,

$$[x] \leq x < [x] + 1.$$

Also, sums, differences, and products of hyperintegers are again hyperintegers.

We are now going to use the hyperintegers. In sketching curves we divided a closed interval $[a, b]$ into finitely many subintervals. For theoretical purposes in the calculus we often divide a closed interval into a finite or infinite number of equal subintervals. This is done as follows.

Given a closed real interval $[a, b]$, a *finite partition* is formed by choosing a positive integer n and dividing $[a, b]$ into n equal parts, as in Figure 3.8.3. Each part will be a subinterval of length $t = (b - a)/n$. The n subintervals are

$$[a, a + t], [a + t, a + 2t], \ldots, [a + (n - 1)t, b].$$

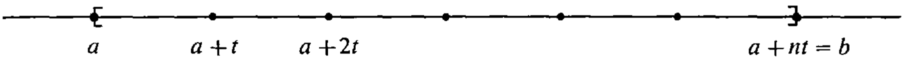


Figure 3.8.3

The endpoints

$$a, a + t, a + 2t, \ldots, a + (n - 1)t, a + nt = b$$

are called *partition points.*

The real interval $[a, b]$ is contained in the *hyperreal interval* $[a, b]^*$, which is the set of all hyperreal numbers x such that $a \le x \le b$. An infinite partition is applied to the hyperreal interval $[a, b]^*$ rather than the real interval. To form an infinite partition of $[a, b]^*$, choose a positive infinite hyperinteger H and divide $[a, b]^*$ into H equal parts as shown in Figure 3.8.4. Each subinterval will have the same infinitesimal length $\delta = (b - a)/H$. The H subintervals are

$$[a, a + \delta], [a + \delta, a + 2\delta], \ldots, [a + (K - 1)\delta, a + K\delta], \ldots, [a + (H - 1)\delta, b],$$

and the partition points are

$$a, a + \delta, a + 2\delta, \ldots, a + K\delta, \ldots, a + H\delta = b,$$

where K runs over the hyperintegers from 1 to H. Every hyperreal number x between a and b belongs to one of the infinitesimal subintervals,

$$a + (K - 1)\delta \le x < a + K\delta.$$

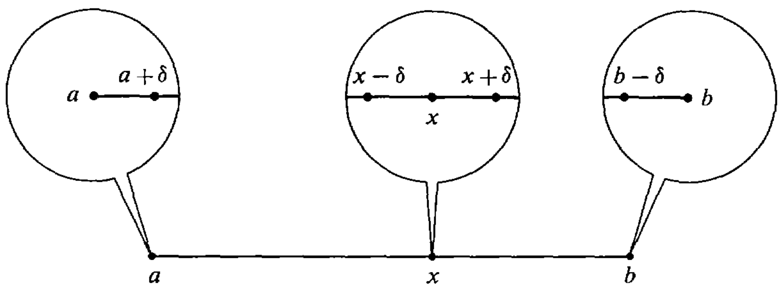
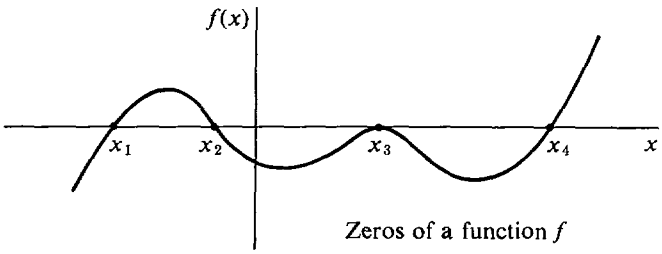


Figure 3.8.4 An infinite partition

We shall now use infinite partitions to sketch the proofs of three basic results, called the Intermediate Value Theorem, the Extreme Value Theorem, and Rolle's Theorem. The use of these results will be illustrated by studying zeros of continuous functions. By a *zero* of a function f we mean a point c where $f(c) = 0$. As we can see in Figure 3.8.5, the zeros of f are the points where the curve $y = f(x)$ intersects the x-axis.



Zeros of a function f

Figure 3.8.5

INTERMEDIATE VALUE THEOREM

Suppose the real function f is continuous on the closed interval $[a, b]$ and $f(x)$ is positive at one endpoint and negative at the other endpoint. Then f has a zero in the interval (a, b); that is, $f(c) = 0$ for some real c in (a, b).

Discussion There are two cases illustrated in Figure 3.8.6:

$$f(a) < 0 < f(b) \quad \text{and} \quad f(a) > 0 > f(b).$$

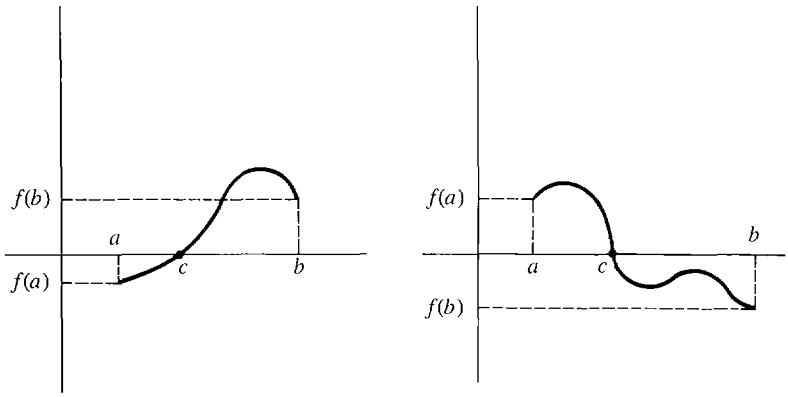


Figure 3.8.6

In the first case, the theorem says that if a continuous curve is below the x-axis at a and above it at b, then the curve must intersect the x-axis at some point c between a and b. Theorem 3 in the preceding Section 3.7 on curve sketching is simply a reformulation of the Intermediate Value Theorem.

SKETCH OF PROOF We assume $f(a) < 0 < f(b)$. Let H be a positive infinite hyperinteger and partition the interval $[a, b]^*$ into H equal parts

$$a, a + \delta, a + 2\delta, \ldots, a + H\delta = b.$$

Let $a + K\delta$ be the last partition point at which $f(a + K\delta) < 0$. Thus

$$f(a + K\delta) < 0 \leq f(a + (K + 1)\delta).$$

Since f is continuous, $f(a + K\delta)$ is infinitely close to $f(a + (K + 1)\delta)$. We conclude that $f(a + K\delta) \approx 0$ (Figure 3.8.7). We take c to be the standard part of $a + K\delta$, so that

$$f(c) = st(f(a + K\delta)) = 0.$$

EXAMPLE 1 The function

$$f(x) = \frac{1}{1 + x} - x - \sqrt{x} - \sqrt[3]{x},$$

which is shown in Figure 3.8.8, is continuous for $0 \leq x \leq 1$. Moreover,

$$f(0) = 1, \qquad f(1) = \tfrac{1}{2} - 3 = -2\tfrac{1}{2}.$$

The Intermediate Value Theorem shows that $f(x)$ has a zero $f(c) = 0$ for some c between 0 and 1.

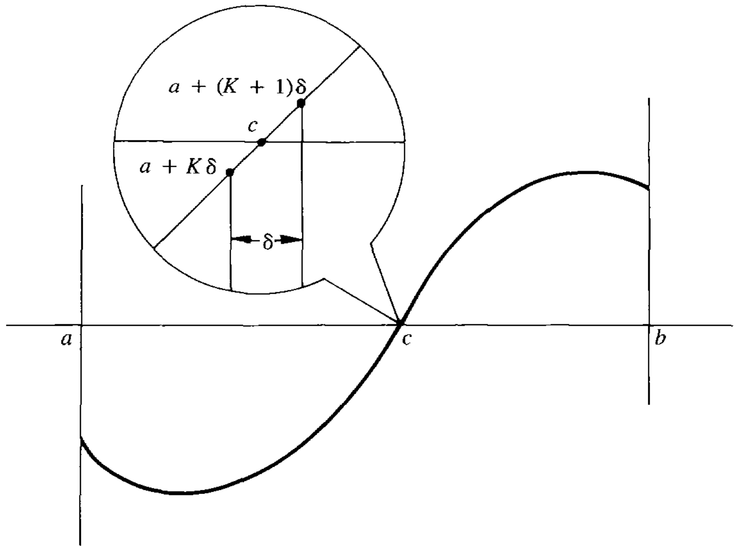


Figure 3.8.7

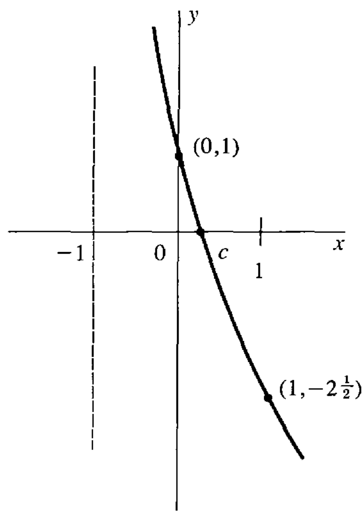


Figure 3.8.8

The Intermediate Value Theorem can be used to prove Theorem 3 of Section 3.7 on curve sketching:

Suppose g is a continuous function on an interval I, and $g(x) \neq 0$ *for all x in I.*

(i) *If* $g(c) > 0$ *for at least one c in I, then* $g(x) > 0$ *for all x in I.*
(ii) *If* $g(c) < 0$ *for at least one c in I, then* $g(x) < 0$ *for all x in I.*

PROOF (i) Let $g(c) > 0$ for some c in I. If $g(x_1) < 0$ for some other point x_1 in I, then by the Intermediate Value Theorem there is a point x_2 between c and x_1 such that $g(x_2) = 0$, contrary to hypothesis (Figure 3.8.9). Therefore we conclude that $g(x) > 0$ for all x in I.

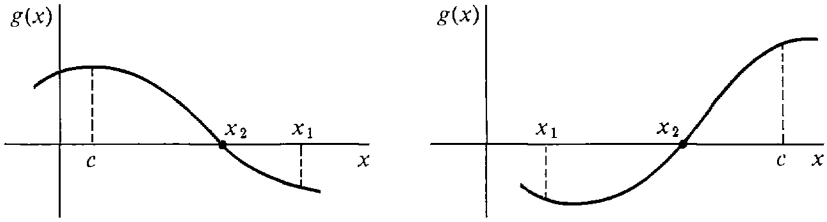


Figure 3.8.9

EXTREME VALUE THEOREM

Let f be continuous on its domain, which is a closed interval $[a, b]$. Then f has a maximum at some point in $[a, b]$, and a minimum at some point in $[a, b]$.

Discussion We have seen several examples of functions that do not have maxima on an *open* interval, such as $f(x) = 1/x$ on $(0, \infty)$, or $g(x) = 2x$ on $(0, 1)$. The Extreme Value Theorem says that on a closed interval a continuous function always has a maximum.

SKETCH OF PROOF Form an infinite partition of $[a, b]^*$,

$$a, a + \delta, a + 2\delta, \ldots, a + H\delta = b.$$

By the Transfer Principle, there is a partition point $a + K\delta$ at which $f(a + K\delta)$ has the largest value. Let c be the standard part of $a + K\delta$ (see Figure 3.8.10). Any point u of $[a, b]^*$ lies in a subinterval, say

$$a + L\delta \leq u < a + (L + 1)\delta.$$

We have
$$f(a + K\delta) \geq f(a + L\delta),$$

and taking standard parts,

$$f(c) \geq f(u).$$

This shows that f has a maximum at c.

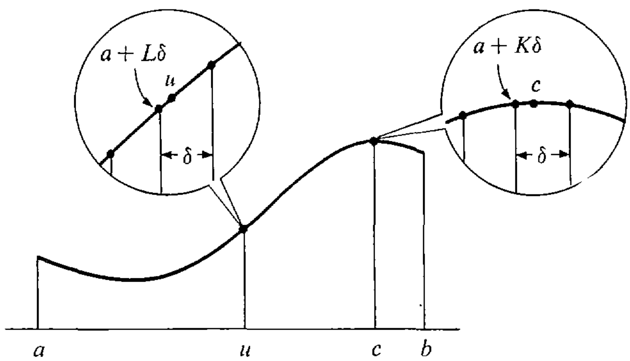


Figure 3.8.10 Proof of the Extreme Value Theorem

ROLLE'S THEOREM

Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

$$f(a) = f(b) = 0,$$

then there is at least one point c strictly between a and b where f has derivative zero; i.e.,

$$f'(c) = 0 \qquad \text{for some } c \text{ in } (a, b).$$

Geometrically, the theorem says that a differentiable curve touching the x-axis at a and b must be horizontal for at least one point strictly between a and b.

PROOF We may assume that $[a, b]$ is the domain of f. By the Extreme Value Theorem, f has a maximum value M and a minimum value m in $[a, b]$. Since $f(a) = 0$, $m \leq 0$ and $M \geq 0$ (see Figure 3.8.11).

Case 1 $M = 0$ and $m = 0$. Then f is the constant function $f(x) = 0$, and therefore $f'(c) = 0$ for all points c in (a, b).

Case 2 $M > 0$. Let f have a maximum at c, $f(c) = M$. By the Critical Point Theorem, f has a critical point at c. c cannot be an endpoint because the value of $f(x)$ is zero at the endpoints and positive at $x = c$. By hypothesis, $f'(x)$ exists at $x = c$. It follows that c must be a critical point of the type $f'(c) = 0$.

Case 3 $m < 0$. We let f have a minimum at c. Then as in Case 2, c is in (a, b) and $f'(c) = 0$.

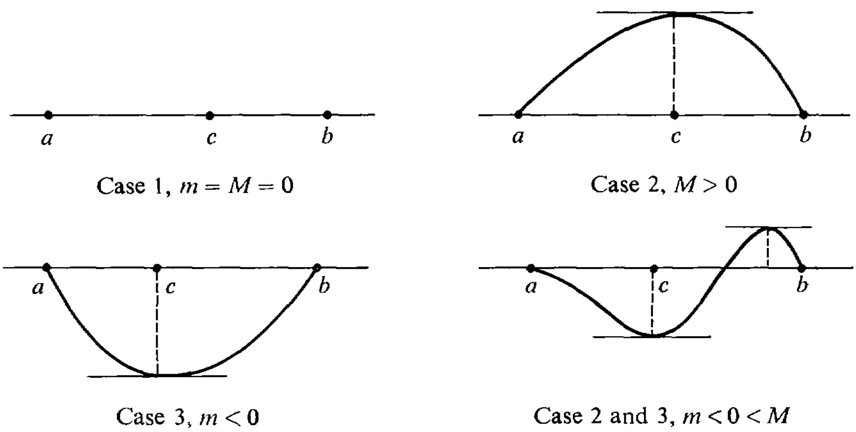


Figure 3.8.11 Rolle's Theorem

EXAMPLE 2 $f(x) = (x - 1)^2(x - 2)^3$, $a = 1$, $b = 2$. The function f is continuous and differentiable everywhere (Figure 3.8.12). Moreover, $f(1) = f(2) = 0$. Therefore by Rolle's Theorem there is a point c in $(1, 2)$ with $f'(c) = 0$.

Let us find such a point c. We have

$$f'(x) = 3(x - 1)^2(x - 2)^2 + 2(x - 1)(x - 2)^3 = (x - 1)(x - 2)^2(5x - 7).$$

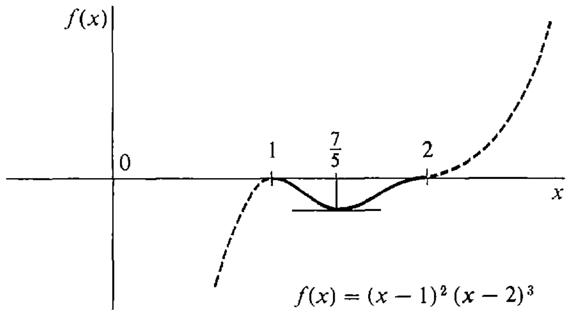


Figure 3.8.12

Notice that $f'(1) = 0$ and $f'(2) = 0$. But Rolle's Theorem says that there is another point c which is in the *open* interval $(1, 2)$ where $f'(c) = 0$. The required value for c is $c = \frac{7}{5}$ because $f'(\frac{7}{5}) = 0$ and $1 < \frac{7}{5} < 2$.

EXAMPLE 3 Let $f(x) = \dfrac{x^4}{2} - x^2, \quad a = -\sqrt{2}, \quad b = \sqrt{2}.$

Then $f(a) = f(b) = 0$.

Rolle's Theorem says that there is at least one point c in $(-\sqrt{2}, \sqrt{2})$ at which $f'(c) = 0$. As a matter of fact there are three such points,

$$c = -1, \quad c = 0, \quad c = 1.$$

We can find these points as follows:

$$f'(x) = 2x^3 - 2x = 2x(x^2 - 1),$$
$$f'(x) = 0 \quad \text{when} \quad x = 0 \quad \text{or} \quad x = \pm 1.$$

The function is drawn in Figure 3.8.13.

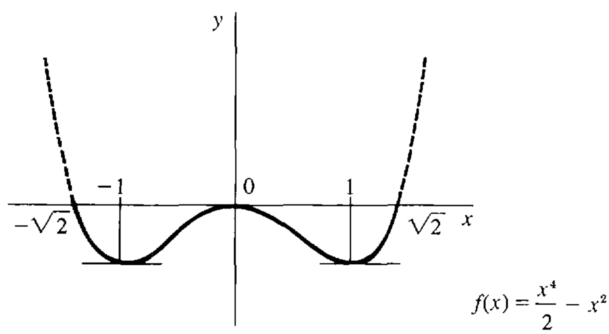


Figure 3.8.13

EXAMPLE 4 $f(x) = \sqrt{1 - x^2}$, $a = -1$, $b = 1$. Then $f(-1) = f(1) = 0$. The function f is continuous on $[-1, 1]$ and has a derivative at each point of $(-1, 1)$, as Rolle's Theorem requires (Figure 3.8.14). Note, however, that $f'(x)$ does not exist at either endpoint, $x = -1$ or $x = 1$. By Rolle's Theorem there is a point c in $(-1, 1)$ such that $f'(c) = 0$, $c = 0$ is such a point, because

$$f'(x) = -\frac{x}{\sqrt{1 - x^2}}, \qquad f'(0) = 0.$$

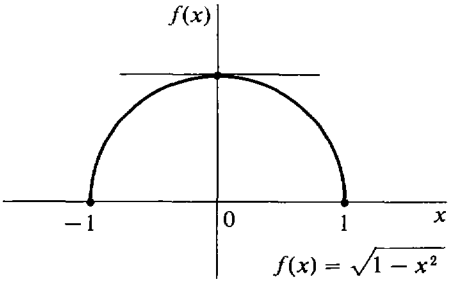


Figure 3.8.14

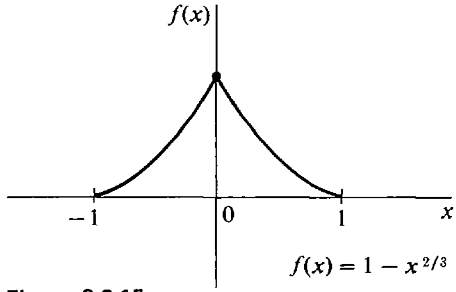


Figure 3.8.15

EXAMPLE 5 $f(x) = 1 - x^{2/3}$, $a = -1$, $b = 1$. Then $f(-1) = f(1) = 0$, and $f'(x) = -\frac{2}{3}x^{-1/3}$ for $x \neq 0$. $f'(0)$ is undefined. There is no point c in $(-1, 1)$ at which $f'(c) = 0$. Rolle's Theorem does not apply in this case because $f'(x)$ does not exist at one of the points of the interval $(-1, 1)$, namely at $x = 0$. In Figure 3.8.15, we see that instead of being horizontal at a point in the interval, the curve has a sharp peak.

Rolle's Theorem is useful in finding the number of zeros of a differentiable function f. It shows that between any two zeros of f there must be one or more zeros of f'. It follows that if f' has no zeros in an interval I, then f cannot have more than one zero in I.

EXAMPLE 6 How many zeros does the function $f(x) = x^3 + x + 1$ have? We use both Rolle's Theorem and the Intermediate Value Theorem.

Using Rolle's Theorem: $f'(x) = 3x^2 + 1$. For all x, $x^2 \geq 0$, and hence $f'(x) \geq 1$. Therefore $f(x)$ has at most one zero.

Using the Intermediate Value Theorem: We have $f(-1) = -1$, $f(0) = 1$. Therefore f has at least one zero between -1 and 0.

CONCLUSION f has exactly one zero, and it lies between -1 and 0 (see Figure 3.8.16).

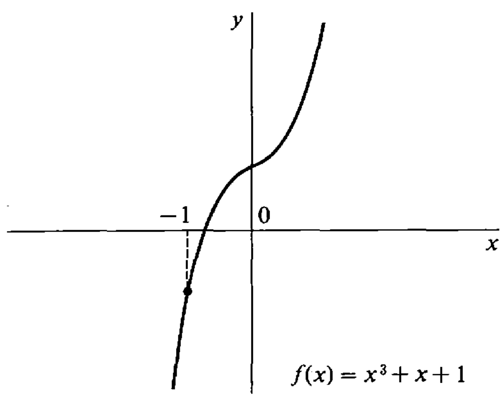


Figure 3.8.16

Our method of sketching curves in Section 3.7 depends on a consequence of Rolle's Theorem called the Mean Value Theorem. It deals with the average slope of a curve between two points.

DEFINITION

*Let f be defined on the closed interval $[a, b]$. The **average slope** of f between a and b is the quotient*

$$\text{average slope} = \frac{f(b) - f(a)}{b - a}.$$

We can see in Figure 3.8.17 that the average slope of f between a and b is equal to the slope of the line passing through the points $(a, f(a))$ and $(b, f(b))$. This is shown by the two-point equation for a line (Section 1.3). In particular, if f is already a linear function $f(x) = mx + c$, then the average slope of f between a and b is equal to the slope m of the line $y = f(x)$.

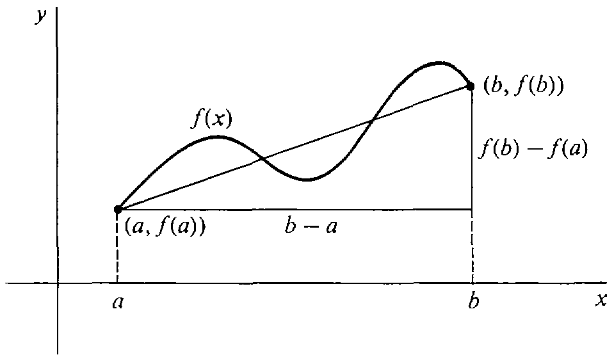


Figure 3.8.17 Average Slope

This is shown by the two-point equation for a straight line (Section 1.2). In particular, if f is already a linear function $f(x) = mx + c$, then the average slope of f between a and b is equal to the slope m of the straight line $y = f(x)$.

MEAN VALUE THEOREM

Assume that f is continuous on the closed interval $[a, b]$ and has a derivative at every point of the open interval (a, b). Then there is at least one point c in (a, b) where the slope $f'(c)$ is equal to the average slope of f between a and b,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark In the special case that $f(a) = f(b) = 0$, the Mean Value Theorem becomes Rolle's Theorem:

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0.$$

On the other hand, we shall use Rolle's Theorem in the proof of the Mean Value Theorem. The Mean Value Theorem is illustrated in Figure 3.8.18.

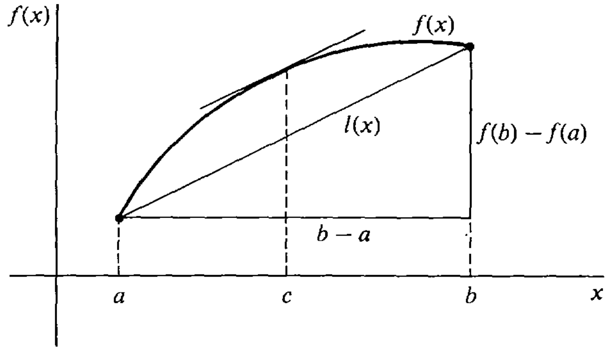


Figure 3.8.18 The Mean Value Theorem

PROOF OF THE MEAN VALUE THEOREM Let m be the average slope, $m = (f(b) - f(a))/(b - a)$. The line through the points $(a, f(a))$ and $(b, f(b))$ has the equation

$$l(x) = f(a) + m(x - a).$$

Let $h(x)$ be the distance of $f(x)$ above $l(x)$,

$$h(x) = f(x) - l(x).$$

Then h is continuous on $[a, b]$ and has the derivative

$$h'(x) = f'(x) - l'(x) = f'(x) - m$$

at each point in (a, b). Since $f(x) = l(x)$ at the endpoints a and b, we have

$$h(a) = 0, \qquad h(b) = 0.$$

Therefore Rolle's Theorem can be applied to the function h, and there is a point c in (a, b) such that $h'(c) = 0$. Thus

$$0 = h'(c) = f'(c) - l'(c) = f'(c) - m,$$

whence

$$f'(c) = m.$$

We can give a physical interpretation of the Mean Value Theorem in terms of velocity. Suppose a particle moves along the y-axis according to the equation $y = f(t)$. The *average velocity* of the particle between times a and b is the ratio

$$\frac{f(b) - f(a)}{b - a}$$

of the change in position to the time elapsed. The Mean Value Theorem states that there is a point of time c, $a < c < b$, when the velocity $f'(c)$ of the particle is equal to the average velocity between times a and b.

Theorems 1 and 2 in Section 3.7 on curve sketching are consequences of the Mean Value Theorem. As an illustration, we prove part (ii) of Theorem 1:

If $f'(x) > 0$ for all interior points x of I, then f is increasing on I.

PROOF Let $x_1 < x_2$ where x_1 and x_2 are points in I. By the Mean Value Theorem there is a point c strictly between x_1 and x_2 such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since c is an interior point of $I, f'(c) > 0$. Because $x_1 < x_2$, $x_2 - x_1 > 0$. Thus

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0, \qquad f(x_2) - f(x_1) > 0, \qquad f(x_2) > f(x_1).$$

This shows that f is increasing on I.

PROBLEMS FOR SECTION 3.8

In Problems 1–16, use the Intermediate Value Theorem to show that the function has at least one zero in the given interval.

1 $f(x) = x^4 - 2x^3 - x^2 + 1, \quad 0 \le x \le 1$

2 $f(x) = x^2 + x - 3/x, \quad 1 \le x \le 2$

3 $f(x) = \sqrt{x} + \sqrt{x+1} - x, \quad 4 \le x \le 9$

4 $f(x) = \sqrt{x} + 1/x^2 - x^2, \quad 1 \le x \le 2$

5 $f(x) = \dfrac{2}{1 + x\sqrt{x}} - \sqrt{x^2 + 2}, \quad 0 \le x \le 1$

6 $f(x) = x^5 + x - \sqrt{x+1}, \quad 0 \le x \le 1$

7 $f(x) = x^3 + x^2 - 1, \quad 0 \le x \le 1$

8 $f(x) = x^2 + 1 - \dfrac{3}{x+1}, \quad 0 \le x \le 1$

9 $f(x) = 1 - 3x + x^3, \quad 0 \le x \le 1$

10 $f(x) = 1 - 3x + x^3, \quad 1 \le x \le 2$

11 $f(x) = x^2 + \sqrt{x} - 1, \quad 0 \le x \le 1$

12 $f(x) = x^2 - (x+1)^{-1/2}, \quad 0 \le x \le 1$

13 $f(x) = \cos x - \frac{1}{10}, \quad 0 \le x \le \pi$

14 $f(x) = \sin x - 2\cos x, \quad 0 \le x \le \pi$

15 $f(x) = \ln x - \dfrac{1}{x}, \quad 1 \le x \le e$

16 $f(x) = e^x - 10x, \quad 1 \le x \le 10$

In Problems 17–30, determine whether or not f' has a zero in the interval (a, b). *Warning:* Rolle's Theorem may give a wrong answer unless all the hypotheses are met.

17 $f(x) = 5x^2 - 8x, \quad [a, b] = [0, \frac{8}{5}]$

18 $f(x) = 1 - x^{-2}, \quad [a, b] = [-1, 1]$

19 $f(x) = \sqrt{16 - x^4}, \quad [a, b] = [-2, 2]$

20 $f(x) = \sqrt{4 - x^{2/7}}, \quad [a, b] = [-128, 128]$

21 $f(x) = 1/x - x, \quad [a, b] = [-1, 1]$

22 $f(x) = (x-1)^2(x-2), \quad [a, b] = [1, 2]$

23 $f(x) = (x-4)^3 x^4, \quad [a, b] = [0, 4]$

24 $f(x) = \dfrac{(x-2)(x-4)}{x^3+x+2}$, $[a, b] = [2, 4]$

25 $f(x) = |x| - 1$, $[a, b] = [-1, 1]$

26 $f(x) = \dfrac{x(x-2)}{x-1}$, $[a, b] = [0, 2]$

27 $f(x) = x \sin x$, $[a, b] = [0, \pi]$

28 $f(x) = e^x \cos x$, $[a, b] = [-\pi/2, \pi/2]$

29 $f(x) = \tan x$, $[a, b] = [0, \pi]$

30 $f(x) = \ln(1 - \sin x)$, $[a, b] = [0, \pi]$

31 Find the number of zeros of $x^4 + 3x + 1$ in $[-2, -1]$.

32 Find the number of zeros of $x^4 + 2x^3 - 2$ in $[0, 1]$.

33 Find the number of zeros of $x^4 - 8x - 4$.

34 Find the number of zeros of $2x + \sqrt{x} - 4$.

In Problems 35–42, find a point c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.

35 $f(x) = x^2 + 2x - 1$, $[a, b] = [0, 1]$

36 $f(x) = x^3$, $[a, b] = [0, 3]$

37 $f(x) = x^{2/3}$, $[a, b] = [0, 1]$

38 $f(x) = \sqrt{x+1}$, $[a, b] = [0, 2]$

39 $f(x) = x + \sqrt{x}$, $[a, b] = [0, 4]$

40 $f(x) = 2 + (1/x)$, $[a, b] = [1, 2]$

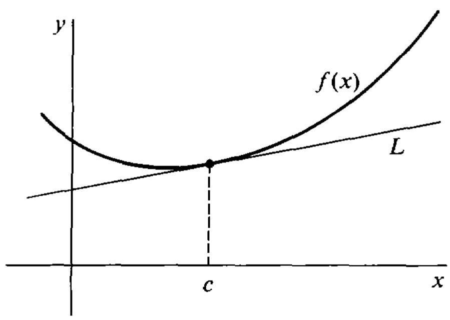
41 $f(x) = \dfrac{x-1}{x+1}$, $[a, b] = [0, 2]$

42 $f(x) = x\sqrt{x+1}$, $[a, b] = [0, 3]$

43 Use Rolle's Theorem to show that the function $f(x) = x^3 - 3x + b$ cannot have more than one zero in the interval $[-1, 1]$, regardless of the value of the constant b.

44 Suppose f, f', and f'' are all continuous on the interval $[a, b]$, and suppose f has at least three distinct zeros in $[a, b]$. Use Rolle's Theorem to show that f'' has at least one zero in $[a, b]$.

□ 45 Suppose that $f''(x) > 0$ for all real numbers x, so that the curve $y = f(x)$ is concave upward on the whole real line as illustrated in the figure. Let L be the tangent line to the curve at $x = c$. Prove that the line L lies below the curve at every point $x \neq c$.



EXTRA PROBLEMS FOR CHAPTER 3

1 Find the surface area A of a cube as a function of its volume V.

2 Find the length of the diagonal d of a rectangle as a function of its length x and width y.

3 An airplane travels for t hours at a speed of 300 mph. Find the distance x of travel as a function of t.

4 An airplane travels x miles at 500 mph. Find the travelling time t as a function of x.

5 A 5 foot tall woman stands at a distance x from a 9 foot high lamp. Find the length of her shadow as a function of x.

6 The sides and bottom of a rectangular box are made of material costing \$1/sq ft. and the top of material costing \$2/sq ft. Find the cost of the box as a function of the length x, width y, and height z feet.

7 A piece of dough with a constant volume of 10 cu in. is being rolled in the shape of a right circular cylinder. Find the rate of increase of its length when the radius is $\frac{1}{2}$ inch and is decreasing at $\frac{1}{10}$ inch per second.

8 Car A travels north at 60 mph and passes the point P at 1:00. Car B travels east at 40 mph and passes the point P at 3:00. Find the rate of change of the distance between the two cars at 2:00.

9 A cup of water has the shape of a cone with the apex at the bottom, height 4 in., and a circular top of radius 2 in. The loss of water volume due to evaporation is $0.01A$ cu in./sec where A is the water surface area. Find the rate at which the water level drops due to evaporation.

10 A country has a constant national income and its population is decreasing by one million people per year. Find the rate of change of the per capita income when the population is 50 million and the national income is 100 billion dollars.

11 Evaluate $\lim_{x \to 2} x^3 - 4x^2 + 3x - 1$

12 Evaluate $\lim_{x \to 3} \dfrac{(x^2 - 9)^2}{(x - 3)^2}$

13 Evaluate $\lim_{t \to 0^+} \dfrac{2 + t^{-1/2}}{3 - 4t^{-1/2}}$

14 Evaluate $\lim_{\Delta x \to 0^+} \dfrac{\sqrt{0 + \Delta x} - \sqrt{0}}{\Delta x}$.

15 Find the set of all points at which $f(x) = \sqrt{1 + x} + \sqrt{1 - x}$ is continuous.

16 Find the set of all points at which

$$g(x) = \frac{x - 2}{(x - 3)(x - 4)(x - 5)}$$

is continuous.

17 Find the set of all points at which $f(x) = \sqrt{(4 - x^2)(x^2 - 1)}$ is continuous.

18 Assume $a < b$. Show that $f(x) = \sqrt{(x - a)(b - x)}$ is continuous on the closed interval $[a, b]$.

19 Show that $g(x) = (x - 1)^{1/3}$ is continuous at every real number $x = c$.

20 Find the maximum and minimum of

$$f(x) = 4x^3 - 3x^2 + 2, \qquad -1 \leq x \leq 1.$$

21 Find the maximum and minimum of

$$f(x) = x + \frac{4}{x^2}, \qquad 1 \leq x \leq 4.$$

22 Find the maximum and minimum of

$$f(x) = |2x - 5| + 3, \qquad 0 \leq x \leq 10.$$

23 Find the maximum and minimum of

$$f(x) = 4 - 3x^{2/3}, \qquad -1 \leq x \leq 1.$$

24 Find the maximum and minimum of

$$f(x) = (x - 1)^{1/3} - 2, \qquad 0 \leq x \leq 2.$$

25 Find the rectangle of maximum area which can be inscribed in a circle of radius 1.

26 A box with a square base and no top is to be made with 10 sq ft of material. Find the dimensions which will have the largest volume.

27 In one day a factory can produce x items at a total cost of $c_0 + ax$ dollars and can sell x items at a price of $bx^{-1/3}$ dollars per item. How many items should be produced for a maximum daily profit?

28 Test the curve $f(x) = x^3 - 5x + 4$ for maxima and minima.

29 Test the curve $f(x) = 3x^4 + 4x^3 - 12x^2$ for maxima and minima.

30 The light intensity from a light source is equal to S/D^2 where S is the strength of the source and D the distance from the source. Two light sources A and B have strengths $S_A = 2$ and $S_B = 1$ and are located on the x-axis at $x_A = 0$ and $x_B = 10$. Find the point x, $0 < x < 10$, where the total light intensity is a minimum.

31 Find the right triangle of area $\frac{1}{2}$ with the smallest perimeter.

32 Find the points on the parabola $y = x^2$ which are closest to the point $(0, 2)$.

33 Find the number of zeros of $f(x) = x^3 - 8x^2 + 4x + 2$.

34 Find the number of zeros of $f(x) = x^3 - 2x^2 + 2x - 4$.

35 Sketch the curve $y = x^4 - x^3$, $-1 \le x \le 1$.

36 Sketch the curve $y = x^2 + x^{-2}$, $\frac{1}{2} \le x \le 2$.

37 Find all zeros of $f(x) = x^2 - 5x + 10$.

38 Show that the function $f(x) = x^6 - 5x^5 - 3x^2 + 4$ has at least one zero in the interval $[0, 1]$.

39 Show that the function $f(x) = \sqrt{x+1} + \sqrt[3]{x+8} - 2$ has at least one zero in the interval $[-1, 0]$.

40 Show that the equation $1 - x^2 = \sqrt{x}$ has at least one solution in the interval $[0, 1]$.

□ 41 Prove that $\lim_{x \to c} f(x)$ exists if and only if there is a function $g(x)$ such that
(a) $g(x)$ is continuous at $x = c$,
(b) $g(x) = f(x)$ whenever $x \neq c$.

□ 42 Let $S = \{a_1, \ldots, a_n\}$ be a finite set of real numbers. Show that the characteristic function of S,

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is in } S, \\ 0 & \text{otherwise,} \end{cases}$$

is discontinuous for x in S and continuous for x not in S.

□ 43 Show that the function $f(x) = \sqrt{|x|}$ is continuous but not differentiable at $x = 0$.

□ 44 Let

$$f(x) = \begin{cases} 1 & \text{if } 1 \le |x| \\ 1/n & \text{if } 1/n \le |x| < 1/(n-1), \quad n = 2, 3, 4, \ldots \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is continuous at $x = 0$ but discontinuous at $x = 1/n$ and $x = -1/n$, $n = 1, 2, 3, \ldots$.

□ 45 Let

$$f(x) = \begin{cases} 1 & \text{if } 1 \le |x| \\ 1/n^2 & \text{if } 1/n \le |x| < 1/(n-1), \quad n = 2, 3, \ldots \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is differentiable at $x = 0$ but discontinuous at $x = 1/n$ and $x = -1/n$, $n = 1, 2, 3, \ldots$.

□ 46 Suppose $f(x)$ is continuous on $[0, 1]$ and $f(0) = 1$, $f(1) = 0$. Prove that there is a point c in $(0, 1)$ such that $f(c) = c$.

☐ **47** Suppose $f(x)$ is continuous for all x, and $f(0) = 0$, $f(1) = 4$, $f(2) = 0$. Prove that there is a point c in $(0, 1)$ such that $f(c) = f(c + 1)$.

☐ **48** Prove that if $x = c$ is the only real solution of $f(x) = 0$, then $x = c$ is also the only hyperreal solution.

☐ **49** Prove that if n is odd, then the polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

has no maximum and no minimum.

☐ **50** Prove that if n is even then the polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

has no maximum.

☐ **51** Prove that if n is even then the polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

has a minimum. You may use the fact that there are only finitely many critical points.

☐ **52** Prove the First Derivative Test: Assume $f(x)$ is continuous on an interval I.

If $f'(a) > 0$ for all $a < c$ and $f'(b) < 0$ for all $b > c$, then f has a maximum at $x = c$.

If $f'(a) < 0$ for all $a < c$ and $f'(b) > 0$ for all $b > c$, then f has a minimum at $x = c$.

☐ **53** Suppose f is differentiable and $f'(x) > 1$ for all x. If $f(0) = 0$, show that $f(x) > x$ for all positive x.

☐ **54** Suppose $f''(x) > 0$ for all x. Show that for any two points P and Q above the curve $y = f(x)$, every point on the line segment PQ is above the curve $y = f(x)$.

☐ **55** Suppose $f(0) = A$ and $f'(x)$ has the constant value B for all x. Use the Mean Value Theorem to show that f is the linear function $f(x) = A + Bx$.

☐ **56** Suppose $f'(x)$ is continuous for all real x. Use the Mean Value Theorem to show that for all finite hyperreal b and nonzero infinitesimal Δx,

$$f'(b) \approx \frac{f(b + \Delta x) - f(b)}{\Delta x}.$$