1

REAL AND HYPERREAL NUMBERS

Chapter 1 takes the student on a direct route to the point where it is possible to study derivatives. Sections 1.1 through 1.3 are reviews of precalculus material and can be skipped in many calculus courses. Section 1.4 gives an intuitive explanation of the hyperreal numbers and how they can be used to find slopes of curves. This section has no problem set and is intended as the basis for an introductory lecture. The main content of Chapter 1 is in the last two sections, 1.5 and 1.6. In these sections, the student will learn how to work with the hyperreal numbers and in particular how to compute standard parts. Standard parts are used at the beginning of the next chapter to find derivatives of functions. Sections 1.5 and 1.6 take the place of the beginning chapter on limits found in traditional calculus texts.

For the benefit of the interested student, we have included an Epilogue at the end of the book that presents the theory underlying this chapter.

1.1 THE REAL LINE

Familiarity with the real number system is a prerequisite for this course. A review of the rules of algebra for the real numbers is given in the appendix. For convenience, these rules are also listed in a table inside the front cover. The letter R is used for the set of all real numbers. We think of the real numbers as arranged along a straight line with the integers (whole numbers) marked off at equal intervals, as shown in Figure 1.1.1. This line is called the *real line.*

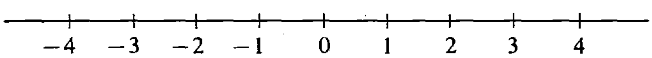

Figure 1.1.1 The real line.

In grade school and high school mathematics, the real number system is constructed gradually in several stages. Beginning with the positive integers, the systems of integers, rational numbers, and finally real numbers are built up. One way to construct the set of real numbers is as the set of all nonterminating decimals.

After constructing the real numbers, it is possible to prove the familiar rules for sums, differences, products, quotients, exponents, roots, and order. In this course, we take it for granted that these rules are familiar to the student, so that we can proceed as quickly as possible to the calculus.

Before going on, we pause to recall two special points that are important in the calculus. First, *division by zero is never allowed.* Expressions such as

$$\frac{2}{0}, \quad \frac{0}{0}, \quad \frac{x}{0}, \quad \frac{5}{1+3-4}$$

are always considered to be *undefined.*

Second, a positive real number c always has two square roots, $\sqrt{c}$ and $-\sqrt{c}$, and $\sqrt{c}$ always stands for the positive square root. Negative real numbers do not have real square roots. *For each positive real number c, $\sqrt{c}$ is positive and $\sqrt{-c}$ is undefined.*

On the other hand, every real number has one real cube root. If $c > 0$, c has the positive cube root $\sqrt[3]{c}$, and $-c$ has the negative cube root $\sqrt[3]{-c} = -\sqrt[3]{c}$.

In calculus, we often deal with sets of real numbers. By a *set S* of real numbers, we mean any collection of real numbers, called *members* of S, *elements* of S, or *points* in S.

A simple but important kind of set is an *interval.* Given two real numbers a and b with $a < b$, the *closed interval* $[a, b]$ is defined as the set of all real numbers x such that $a \leq x$ and $x \leq b$, or more concisely, $a \leq x \leq b$.

The *open interval* (a, b) is defined as the set of all real numbers x such that $a < x < b$. Closed and open intervals are illustrated in Figure 1.1.2.

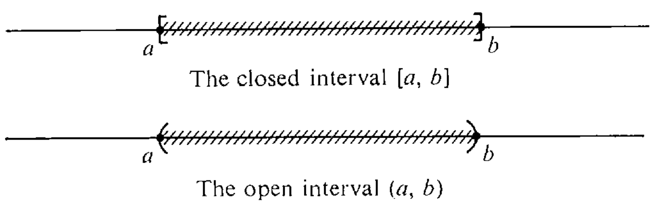

Figure 1.1.2

For both open and closed intervals, the number a is called the *lower endpoint*, and b the *upper endpoint*. The difference between the closed interval $[a, b]$ and the open interval (a, b) is that the endpoints a and b are elements of $[a, b]$ but are not elements of (a, b). When $a \leq x \leq b$, we say that x is *between* a and b; when $a < x < b$, we say that x is *strictly between* a and b.

Three other types of sets are also counted as open intervals: the set (a, ∞) of all real numbers x greater than a; the set $(-\infty, b)$ of all real numbers x less than b, and the whole real line R. The real line R is sometimes denoted by $(-\infty, \infty)$. The symbols ∞ and $-\infty$, read "infinity" and "minus infinity," do not stand for numbers; they are only used to indicate an interval with no upper endpoint, or no lower endpoint.

Besides the open and closed intervals, there is one other kind of interval, called a *half-open interval.* The set of all real numbers x such that $a \leq x < b$ is a half-open interval denoted by $[a, b)$. The set of all real numbers x such that $a \leq x$ is also a half-open interval and is written $[a, \infty)$. Here is a table showing the various kinds of intervals.

Table 1.1.1 Kinds of Intervals

Type	Symbol	Defining Formula
Closed	$[a, b]$	$a \leq x \leq b$
Open	(a, b)	$a < x < b$
Open	(a, ∞)	$a < x$
Open	$(-\infty, b)$	$x < b$
Open	$(-\infty, \infty)$	
Half-open	$[a, b)$	$a \leq x < b$
Half-open	$[a, \infty)$	$a \leq x$
Half-open	$(a, b]$	$a < x \leq b$
Half-open	$(-\infty, b]$	$x \leq b$

We list some other important examples of sets of real numbers.

(1) The empty set $\varnothing$, which has no elements.
(2) The finite set $\{a_1, \ldots, a_n\}$, whose only elements are the numbers $a_1, a_2, \ldots, a_n$.
(3) The set of all x such that $x \neq 0$.
(4) The set $N = \{1, 2, 3, 4, \ldots\}$ of all positive integers.
(5) The set $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ of all integers.
(6) The set Q of all rational numbers. A rational number is a quotient m/n where m and n are integers and $n \neq 0$.

While real numbers correspond to points on a line, ordered pairs of real numbers correspond to points on a plane. This correspondence gives us a way to draw pictures of calculus problems and to translate physical problems into the language of calculus. It is the starting point of the subject called *analytic geometry.*

An *ordered pair* of real numbers, (a, b), is given by the first number a and the second number b. For example, (1, 3), (3, 1), and (1, 1) are three different ordered pairs. Following tradition, we use the same symbol for the open interval (a, b) and the ordered pair (a, b). However the open interval and ordered pair are completely different things. It will always be quite obvious from the context whether (a, b) stands for the open interval or the ordered pair.

We now explain how ordered pairs of real numbers correspond to points in a plane. A system of *rectangular coordinates* in a plane is given by a horizontal and a vertical copy of the real line crossing at zero. The horizontal line is called the *horizontal axis,* or *x-axis,* while the vertical line is called the *vertical axis,* or *y-axis.* The point where the two axes meet is called the *origin* and corresponds to the ordered pair (0, 0). Now consider any point P in the plane. A vertical line through P will cross the x-axis at a real number x_0, and a horizontal line through P will cross the y-axis at a real number y_0. The ordered pair (x_0, y_0) obtained in this way corresponds to the point P. (See Figure 1.1.3.) We sometimes call P the point (x_0, y_0) and sometimes write $P(x_0, y_0)$. x_0 is called the *x-coordinate* of P and y_0 the *y-coordinate* of P.

Conversely, given an ordered pair (x_0, y_0) of real numbers there is a corresponding point $P(x_0, y_0)$ in the plane. $P(x_0, y_0)$ is the point of intersection of the vertical line crossing the x-axis at x_0 and the horizontal line crossing the y-axis at y_0. We have described a one-to-one correspondence between all points in the plane and all ordered pairs of real numbers.

From now on, we shall simplify things by identifying points in the plane with ordered pairs of real numbers, as shown in Figure 1.1.4.

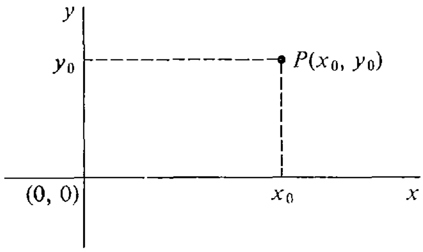

Figure 1.1.3

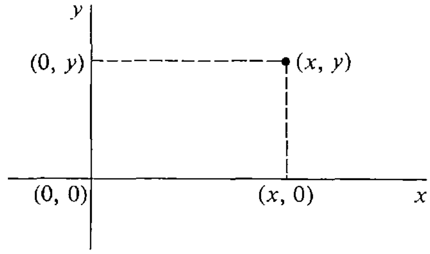

Figure 1.1.4

DEFINITION

*The (x, y) **plane** is the set of all ordered pairs (x, y) of real numbers. The **origin** is the point $(0, 0)$. The x-**axis** is the set of all points of the form $(x, 0)$, and the y-**axis** is the set of all points of the form $(0, y)$.*

The x- and y-axes divide the rest of the plane into four parts called *quadrants*. The quadrants are numbered I through IV, as shown in Figure 1.1.5.

In Figure 1.1.6, $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two different points in the (x, y) plane. As we move from P to Q, the coordinates x and y will change by amounts that we denote by Δx and Δy. Thus

$$\text{change in } x = \Delta x = x_2 - x_1,$$
$$\text{change in } y = \Delta y = y_2 - y_1.$$

The quantities Δx and Δy may be positive, negative, or zero. For example, when $x_2 > x_1$, Δx is positive, and when $x_2 < x_1$, Δx is negative. Using Δx and Δy we define the basic notion of distance.

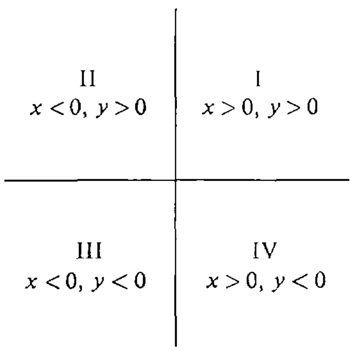

Figure 1.1.5 Quadrants

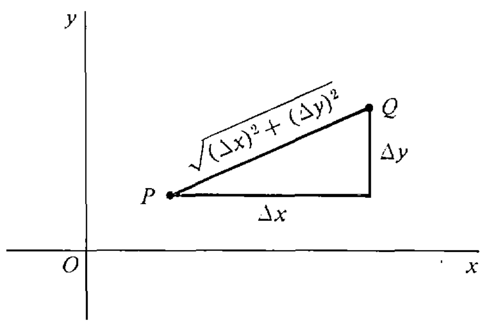

Figure 1.1.6

DEFINITION

*The **distance** between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is the quantity*

$$\text{distance}(P, Q) = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

When we square both sides of the distance formula, we obtain

$$[\text{distance}(P, Q)]^2 = (\Delta x)^2 + (\Delta y)^2.$$

One can also get this formula from the Theorem of Pythagoras in geometry: *The square of the hypotenuse of a right triangle is the sum of the squares of the sides.*

EXAMPLE 1 Find the distance between $P(7, 2)$ and $Q(4, 6)$ (see Figure 1.1.7).

$$\Delta x = 4 - 7 = -3, \qquad \Delta y = 6 - 2 = 4.$$
$$\text{distance } (P, Q) = \sqrt{(-3)^2 + 4^2} = 5.$$

We often deal with sets of points in the plane as well as on the line. One way to describe a set of points in the plane is by an equation or inequality in two variables, say x and y. A solution of an equation in x and y is a point (x_0, y_0) in the plane for which the equation is true. The set of all solutions is called the *locus*, or *graph*, of the equation. The circle is an important example of a set of points in the plane.

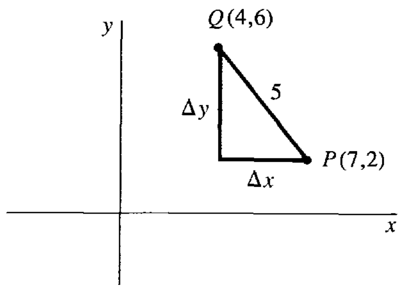

Figure 1.1.7

DEFINITION OF CIRCLE

The set of all points in the plane at distance r from a point P is called the ***circle of radius*** *r and* ***center*** *P.*

Using the distance formula, we see that the circle of radius r and center at the origin (Figure 1.1.8) is the locus of the equation

$$x^2 + y^2 = r^2.$$

The circle of radius r and center at $P(h, k)$ (Figure 1.1.8) is the locus of the equation

$$(x - h)^2 + (y - k)^2 = r^2.$$

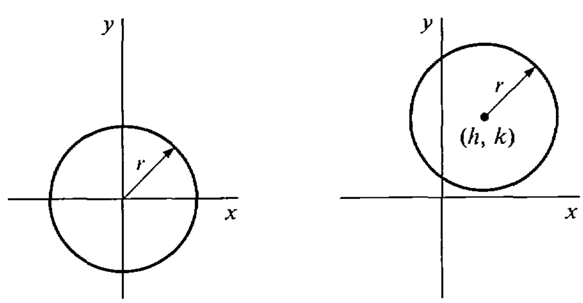

Figure 1.1.8 $x^2 + y^2 = r^2$ $(x - h)^2 + (y - k)^2 = r^2$

For example, the circle with radius 3 and center at $P(2, -4)$ has the equation

$$(x - 2)^2 + (y + 4)^2 = 9.$$

PROBLEMS FOR SECTION 1.1

In Problems 1–6, find the distance between the points P and Q.

1	$P(2, 9), Q(-1, 13)$	**2**	$P(1, -2), Q(2, 10)$
3	$P(0, 0), Q(-2, -3)$	**4**	$P(-1, -1), Q(4, 4)$
5	$P(6, 1), Q(-7, 1)$	**6**	$P(5, 10), Q(9, 10)$

Sketch the circles given in Problems 7–12.

7	$x^2 + y^2 = 4$	**8**	$x^2 + y^2 = \frac{1}{4}$
9	$(x - 1)^2 + (y + 2)^2 = 1$	**10**	$(x + 2)^2 + (y + 3)^2 = 9$
11	$(x - 1)^2 + (y - 1)^2 = 2$	**12**	$(x + 3)^2 + (y - 4)^2 = 25$

13 Find the equation of the circle of radius 2 with center at (3, 0).

14 Find the equation of the circle of radius $\sqrt{3}$ with center at $(-1, -2)$.

☐ **15** There are two circles of radius 2 that have centers on the line $x = 1$ and pass through the origin. Find their equations.

☐ **16** Find the equation of the circle that passes through the three points (0, 0), (0, 1), (2, 0).

☐ **17** Find the equation of the circle one of whose diameters is the line segment from $(-1, 0)$ to (5, 8).

1.2 FUNCTIONS OF REAL NUMBERS

The next two sections are about real numbers only. The calculus deals with problems in which one quantity depends on one or more others. For example, the area of a circle depends on its radius. The length of a day depends on both the latitude and the date. The price of an object depends on the supply and the demand. The way in which one quantity depends on one or more others can be described mathematically by a function of one or more variables.

DEFINITION

*A **real function of one variable** is a set f of ordered pairs of real numbers such that for every real number a one of the following two things happens:*

(i) *There is exactly one real number b for which the ordered pair (a, b) is a member of f. In this case we say that $f(a)$ is defined and we write $f(a) = b$. The number b is called the value of f at a.*

(ii) *There is no real number b for which the ordered pair (a, b) is a member of f. In this case we say that $f(a)$ is undefined.*

Thus $f(a) = b$ means that the ordered pair (a, b) is an element of f.

Here is one way to visualize a function. Imagine a black box labeled f as in Figure 1.2.1. Inside the box there is some apparatus, which we can't see. On both the left and right sides of the box there is a copy of the real line, called the input line and

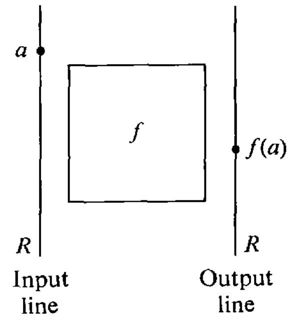

Figure 1.2.1

output line, respectively. Whenever we point to a number a on the input line, either one point b will light up on the output line to tell us that $f(a) = b$, or else nothing will happen, in which case $f(a)$ is undefined.

A second way to visualize a function is by drawing its graph. The *graph* of a real function f of one variable is the set of all points $P(x, y)$ in the plane such that $y = f(x)$. To draw the graph, we plot the value of x on the horizontal, or x-axis and the value of $f(x)$ on the vertical, or y-axis. How can we tell whether a set of points in the plane is the graph of some function? By reading the definition of a function again, we have an answer.

A set of points in the plane is the graph of some function f if and only if for each vertical line one of the following happens:

(1) Exactly one point on the line belongs to the set.
(2) No point on the line belongs to the set.

A vertical line crossing the x-axis at a point a will meet the set in exactly one point (a, b) if $f(a)$ is defined and $f(a) = b$, and the line will not meet the set at all if $f(a)$ is undefined. Try this rule out on the sets of points shown in Figure 1.2.2.

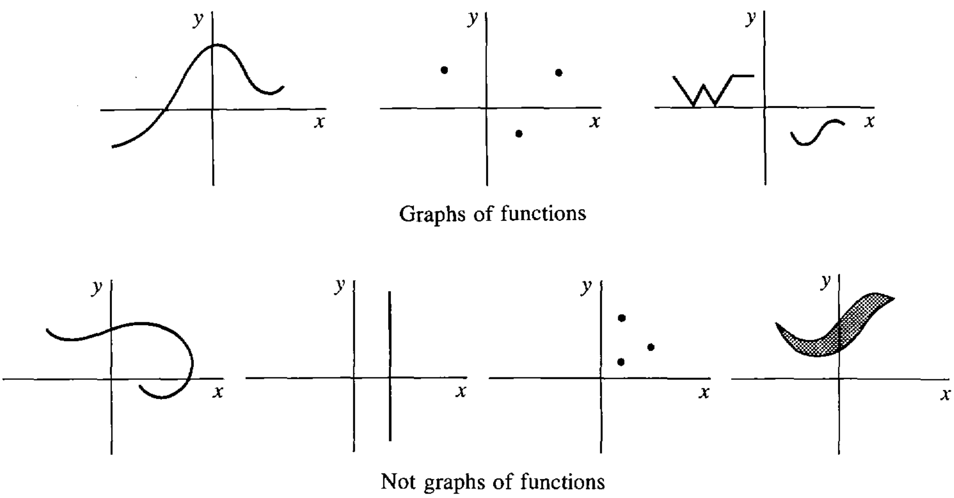

Figure 1.2.2

Here are two examples of real functions of one variable. Each function will be described in two ways: the black box approach, where a rule is given for finding the value of the function at each real number, and the graph method, where an equation is given for the graph of the function.

EXAMPLE 1 The *square function.*

The square function is defined by the rule

$$f(x) = x^2$$

for each number x. The value of $f(a)$ is found by squaring a. For instance, the values of $f(0)$, $f(2)$, $f(-3)$, $f(r)$, $f(r+1)$ are

$$f(0) = 0, \quad f(2) = 4, \quad f(-3) = 9,$$
$$f(r) = r^2, \quad f(r+1) = r^2 + 2r + 1.$$

The graph of the square function is the parabola with the equation $y = x^2$. The graph of $y = x^2$, with several points marked in, is shown in Figure 1.2.3.

EXAMPLE 2 The *reciprocal function.*

The reciprocal function g is given by the rule

$$g(x) = \frac{1}{x}.$$

$g(x)$ is defined for all nonzero x, but is undefined at $x = 0$. Find the following values if they are defined: $g(0)$, $g(2)$, $g(-\frac{1}{3})$, $g(\frac{7}{4})$, $g(r+1)$.

$$g(0) \text{ is undefined}, \quad g(2) = \tfrac{1}{2}, \quad g(-\tfrac{1}{3}) = -3,$$
$$g(\tfrac{7}{4}) = \tfrac{4}{7}, \quad g(r+1) = \frac{1}{r+1}.$$

The graph of the reciprocal function has the equation $y = 1/x$. This equation can also be written in the form $xy = 1$. The graph is shown in Figure 1.2.4.

In Examples 1 and 2 we have used the variables x and y in order to describe a function. A *variable* is a letter which stands for an arbitrary real number; that is, it "varies" over the real line. In the equation $y = x^2$, the value of y depends on the value of x; for this reason we say that x is the *independent variable* and y the *dependent variable* of the equation.

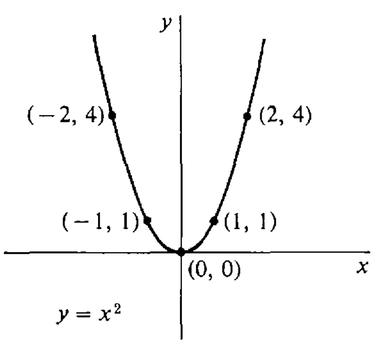

Figure 1.2.3

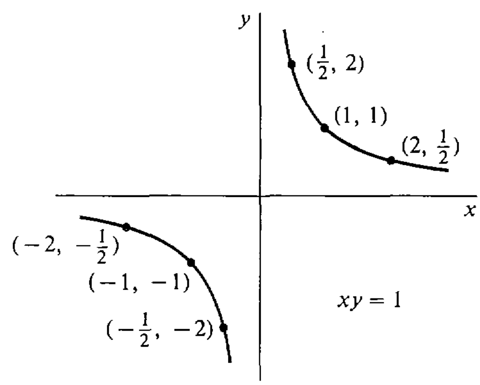

Figure 1.2.4

In describing a function, we do not always use x and y; sometimes other variables are more convenient, especially in problems involving several functions. The variable t is often used to denote time.

It is important to distinguish between the symbol f and the expression $f(x)$. f by itself stands for a *function*. $f(x)$ is called a *term* and stands for the value of the function at x. The need for this distinction is illustrated in the next example.

EXAMPLE 3 Let h be the function given by the rule

$$h(t) = t^3 + 1.$$

t is a variable, h is a function, and $h(t)$ is a term. The following expressions are also terms: $h(\frac{1}{2})$, $h(x)$, $h(t^3)$, $h(t^3) + 1$, $h(t^3 + 1)$, $h(x) - h(t)$, $h(t + \Delta t)$, $h(t + \Delta t) - h(t)$. Find the values of each of these terms.

The values are computed by careful substitution.

$$\begin{aligned}
h(\tfrac{1}{2}) &= (\tfrac{1}{2})^3 + 1 = 1\tfrac{1}{8}. \\
h(x) &= x^3 + 1. \\
h(t^3) &= (t^3)^3 + 1 = t^9 + 1. \\
h(t^3) + 1 &= [(t^3)^3 + 1] + 1 = t^9 + 2. \\
h(t^3 + 1) &= (t^3 + 1)^3 + 1 = t^9 + 3t^6 + 3t^3 + 2. \\
h(x) - h(t) &= [x^3 + 1] - [t^3 + 1] = x^3 - t^3. \\
h(t + \Delta t) &= (t + \Delta t)^3 + 1 = t^3 + 3t^2\,\Delta t + 3t\,\Delta t^2 + \Delta t^3 + 1. \\
h(t + \Delta t) - h(t) &= [(t + \Delta t)^3 + 1] - [t^3 + 1] \\
&= [t^3 + 3t^2\,\Delta t + 3t\,\Delta t^2 + \Delta t^3 + 1] - [t^3 + 1] \\
&= 3t^2\,\Delta t + 3t\,\Delta t^2 + \Delta t^3.
\end{aligned}$$

The graph of h is given by the equation $x = t^3 + 1$. In this equation, t is the independent variable and x is the dependent variable. In Figure 1.2.5, the five points

$$h(-1) = 0, \qquad h(-\tfrac{1}{2}) = \tfrac{7}{8}, \qquad h(0) = 1, \qquad h(\tfrac{1}{2}) = 1\tfrac{1}{8}, \qquad h(1) = 2$$

are plotted and the graph is drawn.

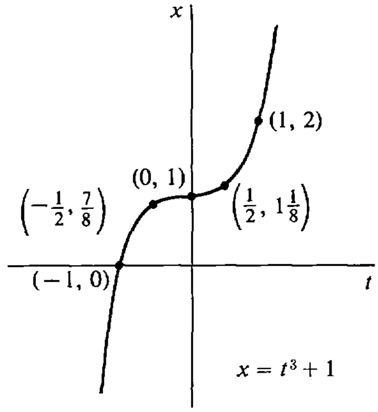

Figure 1.2.5

DEFINITION

*The **domain** of a real function f of one variable is the set of all real numbers x such that $f(x)$ is defined.*

*The **range** of f is the set of all values $f(x)$ where x is in the domain of f.*

EXAMPLE 1 (Continued) The domain of the square function is the set R of all real numbers. The range is the interval $[0, \infty)$ of all nonnegative reals.

EXAMPLE 2 (Continued) Both the domain and the range of the reciprocal function are equal to the set of all real x such that $x \neq 0$.

When a function is described by a rule, it is understood that the domain is the set of all real numbers for which the rule is meaningful.

EXAMPLE 3 (Continued) The function h given by the rule

$$h(t) = t^3 + 1$$

has the whole real line as its domain and as its range.

EXAMPLE 4 Let f be the function given by the rule

$$f(x) = \sqrt{1 - x^2}.$$

Thus $f(x)$ is the positive square root of $1 - x^2$. The domain of f is the closed interval $[-1, 1]$. The range of f is $[0, 1]$.

For instance,

$$f(-2) \text{ is undefined}, \qquad f(-1) = 0, \qquad f(0) = 1,$$
$$f(\tfrac{1}{2}) = \sqrt{\tfrac{3}{4}}, \qquad f(1) = 0, \qquad f(2) \text{ is undefined}.$$

The graph of f is given by the equation $y = \sqrt{1 - x^2}$.

The equation can also be written in the form

$$x^2 + y^2 = 1, \quad y \geq 0.$$

The graph is just the upper half of the unit semicircle, shown in Figure 1.2.6.

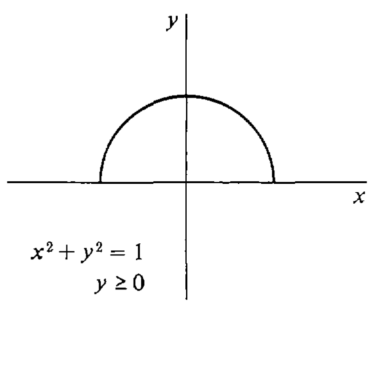

Figure 1.2.6

Sometimes a function is described by explicitly giving its domain in addition to a rule.

EXAMPLE 5 Let g be the function whose domain is the closed interval $[1, 2]$ with the rule

$$g(x) = x^2.$$

The domain and rule can be written in concise form with an equation and extra inequalities,

$$g(x) = x^2, \qquad 1 \leq x \leq 2.$$

Note that

$$g(0) \text{ is undefined} \qquad g(1) = 1$$
$$g(2) = 4 \qquad g(3) \text{ is undefined.}$$

The graph is described by the formulas

$$y = x^2, \qquad 1 \leq x \leq 2$$

and is drawn in Figure 1.2.7.

Some especially important functions are the *constant functions*, the *identity function*, and the *absolute value function.*

A real number is sometimes called a *constant*. This name is used to emphasize the difference between a fixed real number and a variable.

For a given real number c, the function f with the rule

$$f(x) = c$$

is called the *constant function* with value c. It has domain R and range $\{c\}$.

EXAMPLE 6 The constant function with value 5 is described by the rule

$$f(x) = 5.$$

Thus $\qquad f(0) = 5, \qquad f(-3) = 5, \qquad f(1{,}000{,}000) = 5.$

The graph (Figure 1.2.8) of the constant function with value 5 is given by the equation $y = 5$.

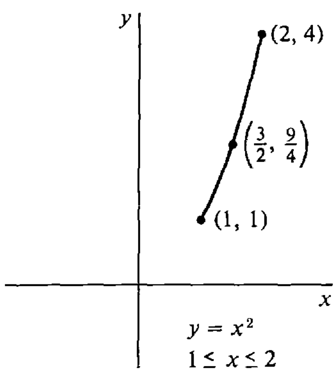

Figure 1.2.7

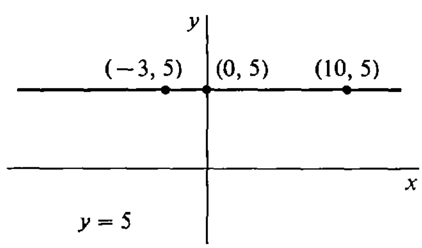

Figure 1.2.8

EXAMPLE 7 The function f given by the rule

$$f(x) = x$$

is called the *identity function.*

The graph (Figure 1.2.9) of the identity function is the straight line with the equation $y = x$.

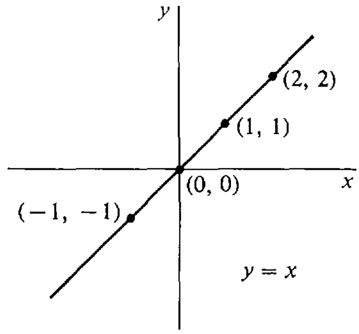

Figure 1.2.9

The *absolute value function* is defined by a rule which is divided into two cases.

DEFINITION

The ***absolute value function*** | | *is defined by*

$$|x| = \begin{cases} x & \text{if } x \geq 0. \\ -x & \text{if } x < 0. \end{cases}$$

The absolute value of x gives the distance between x and 0. It is always positive or zero. For example,

$$|3| = 3, \qquad |-3| = 3, \qquad |0| = 0.$$

The domain of the absolute value function is the whole real line R while its range is the interval $[0, \infty)$.

The absolute value function can also be described by the rule

$$|x| = \sqrt{x^2}.$$

Its graph is given by the equation $y = \sqrt{x^2}$. The graph is the V shown in Figure 1.2.10.

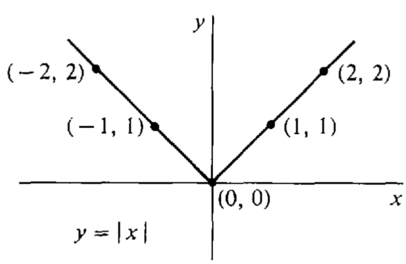

Figure 1.2.10

If a and b are two points on the real line, then from the definition of $|x|$ we see that

$$|a - b| = \begin{cases} a - b & \text{if } a \geq b, \\ b - a & \text{if } b \geq a. \end{cases}$$

Thus $|a - b|$ is the difference between the larger and the smaller of the two numbers. In other words, $|a - b|$ is the *distance* between the points a and b, as illustrated in Figure 1.2.11.

Figure 1.2.11

For example, $|2 - 5| = 3$, $|4 - (-4)| = 8$. Here are some useful facts about absolute values.

THEOREM 1

Let a and b be real numbers.

(i) $|-a| = |a|$.
(ii) $|ab| = |a| \cdot |b|$.
(iii) *If* $b \neq 0$, $|a/b| = |a|/|b|$.

PROOF We use the equation $|x| = \sqrt{x^2}$.

(i) $|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$.
(ii) $|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a| \cdot |b|$.
(iii) The proof is similar to (ii).

Warning The equation $|a + b| = |a| + |b|$ is *false* in general. For example, $|2 + (-3)| = 1$, while $|2| + |(-3)| = 5$.

Functions arise in a great variety of situations. Here are some examples.

Geometry:

πr^2 = area of a circle of radius r

$4\pi r^2$ = surface area of a sphere of radius r

$\frac{4}{3}\pi r^3$ = volume of a sphere of radius r

$\sin\theta$ = the sine of the angle θ

Physics:

$s(t)$ = distance a particle travels from time 0 to t

$v(t)$ = velocity of a particle at time t

$a(t)$ = acceleration of a particle at time t

$p(y)$ = water pressure at depth y below the surface

$C = \frac{5}{9}(F - 32)$ = Celsius temperature as a function of Fahrenheit temperature

Economics:

$f(t)$ = population at time t
$p(t)$ = price of a commodity at time t
$c(x)$ = cost of x items of a commodity
$D(p)$ = demand for a commodity at price p, i.e., the amount which can be sold at price p

Functions of two or more variables can be dealt with in a similar way. Here is the precise definition of a function of two variables.

DEFINITION

*A **real function of two variables** is a set f of ordered triples of real numbers such that for every ordered pair of real numbers (a, b) one of the following two things occurs:*

(i) *There is exactly one real number c for which the ordered triple (a, b, c) is a member of f. In this case, $f(a, b)$ is defined and we write:*

$$f(a, b) = c.$$

(ii) *There is no real number c for which the ordered triple (a, b, c) is a member of f. In this case $f(a, b)$ is called undefined.*

If f is a real function of two variables, then the value of $f(x, y)$ depends on both the value of x and the value of y when $f(x, y)$ is defined.

A real function f of two variables can be visualized as a black box with *two* input lines and one output line, as in Figure 1.2.12.

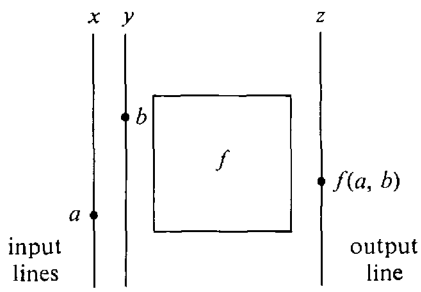

Figure 1.2.12

The *domain* of a real function f of two variables is the set of all pairs of real numbers (x, y) such that $f(x, y)$ is defined.

The most important examples of real functions of two variables are the sum, difference, product, and quotient functions:

$$f(x,y) = x + y, \qquad f(x, y) = xy,$$
$$f(x, y) = x - y, \qquad f(x, y) = x/y.$$

The sum, difference, and product functions have the whole plane as domain. The domain of the quotient function is the set of all ordered pairs (x, y) such that $y \neq 0$.

Here are some examples of functions of two or more variables arising in applications.

Geometry:

ab = area of a rectangle of sides a and b

abc = volume of a rectangular solid

$\frac{1}{2}bh$ = area of a triangle with base b and height h

$\pi r^2 h$ = volume of a cylinder with circular base of radius r and height h

$\frac{1}{3}\pi r^2 h$ = volume of a cone with circular base of radius r and height h

$\sqrt{x^2 + y^2}$ = distance from the origin to (x, y)

Physics:

$F = ma$ = force required to give a mass m an acceleration a

$\rho(x, y, z)$ = density of a three-dimensional object at the point (x, y, z)

$F = Gm_1m_2/s^2$ = gravitational force between objects of mass m_1 and m_2 at distance s

$m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ = relativistic mass of an object with rest mass m_0 and velocity v

Economics:

$c(x, y)$ = cost of x items of one commodity and y items of another commodity

$D_1(p_1, p_2)$ = demand for commodity one when commodity one has price p_1 and commodity two has price p_2

PROBLEMS FOR SECTION 1.2

For each of the following functions (Problems 1–8), make a table showing the value of $f(x)$ when $x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. Put a * where $f(x)$ is undefined. Example:

$$f(x) = \frac{1}{x}$$

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x)$	-1	-2	*	2	1

1 $f(x) = x/3$

2 $f(x) = 3$

3 $f(x) = 3x^3 - 5x^2 + 2$

4 $f(x) = 1/(x - 1)$

5 $f(x) = \sqrt{-x}$

6 $f(x) = |x|$

7 $f(x) = |x - \frac{1}{2}| + |x + \frac{1}{2}|$

8 $f(x) = \sqrt{x^2 - 1}$

9 Is the set of ordered pairs $\{(3, 2), (0, 1), (4, 2)\}$ a function?

10 Is the set of ordered pairs $\{(0, 2), (3, 6), (3, 4)\}$ a function?

11 If f is the function $f(x) = 1 + x + x^2$, find $f(2), f(t), f(t + \Delta t), f(1 + t + t^2), f(g(t))$.

12 If $f(x) = 1/x$, find $f(t), f(t + \Delta t), f(t^2), f(1/t), f(g(t))$.

13 If $f(x) = x\sqrt{x}$, find $f(t), f(t + \Delta t), f(t^2), f(\sqrt{t}), f(g(t))$.

14 If $f(x) = ax + b$, find $f(ct + d), f(t^2), f(1/t), f(t/a), f(g(t))$.

For each of the following functions (Problems 15–20), find $f(x + \Delta x) - f(x)$.

15 $f(x) = 4x + 1$

16 $f(x) = x^2 - x$

17 $f(x) = x^{-2}$

18 $f(x) = x^4$

19 $f(x) = \sqrt{x}$

20 $f(x) = 4$

21 Find the domain of the function $f(x) = 1/(x^2 - 1)$.

22 Find the domain of the function $f(z) = \sqrt{z^2 - 1}$.

23 What is the domain of the function $f(x) = \sqrt{x}$?

24 What is the domain of the function $f(t) = \sqrt{|t|}$?

25 What is the domain of the function $f(x) = 1/\sqrt{1 - x^2}$?

□ 26 Show that if a and b have the same sign then $|a + b| = |a| + |b|$, and if a and b have opposite signs then $|a + b| < |a| + |b|$.

1.3 STRAIGHT LINES

DEFINITION

*Let $P(x_0, y_0)$ be a point and let m be a real number. The **line** through P with slope m is the set of all points $Q(x, y)$ with*

$$y - y_0 = m(x - x_0).$$

*This equation is called the **point-slope equation** of the line (See Figure 1.3.1.) The **vertical line** through P is the set of all points $Q(x, y)$ with $x = x_0$. Vertical lines do not have slopes.*

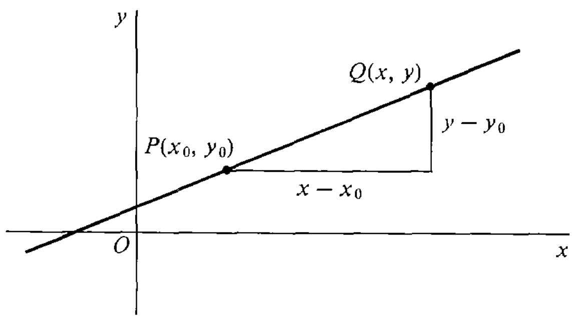

Figure 1.3.1

The slope is a measure of the direction of the line. Figure 1.3.2 shows lines with zero, positive, and negative slopes.

The line that crosses the y-axis at the point $(0, b)$ and has slope m has the simple equation.

$$y = mx + b.$$

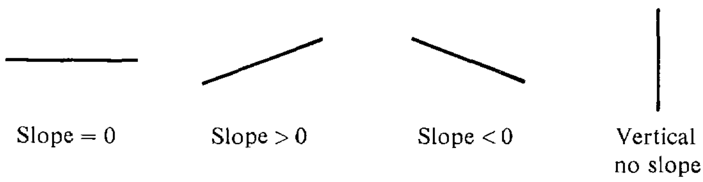

Figure 1.3.2

This is called the *slope-intercept* equation for the line. We can get it from the point-slope equation by setting $x_0 = 0$ and $y_0 = b$.

EXAMPLE 1 The line through the point $P(-1, 2)$ with slope $m = -\frac{1}{2}$ (Figure 1.3.3) has the point-slope equation

$$y - 2 = (x - (-1))\cdot(-\tfrac{1}{2}), \quad \text{or} \quad y - 2 = -\tfrac{1}{2}(x + 1).$$

The slope-intercept equation is

$$y = -\tfrac{1}{2}x + 1\tfrac{1}{2}.$$

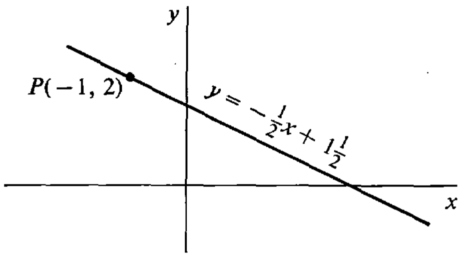

Figure 1.3.3

We now describe the functions whose graphs are nonvertical lines.

DEFINITION

*A **linear function** is a function f of the form*

$$f(x) = mx + b,$$

where m and b are constants.

The graph of a linear function is just the line with slope-intercept equation

$$y = mx + b.$$

This is the line through $(0, b)$ with slope m.

If two points on a line are known, the slope can be found as follows.

THEOREM 1

Suppose a line L passes through two distinct points $P(x_1, y_1)$ and $Q(x_2, y_2)$. If $x_1 = x_2$, then the line L is vertical. If $x_1 \neq x_2$, then the slope of the line L is equal to the change in y divided by the change in x,

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

PROOF Suppose $x_1 \neq x_2$, so L is not vertical. Let m be the slope of L. L has the point-slope formula

$$y - y_1 = m(x - x_1).$$

Substituting y_2 for y and x_2 for x, we see that $m = (y_2 - y_1)/(x_2 - x_1)$.

Theorem 1 shows why the slope of a line is a measure of its direction. Sometimes Δx is called the *run* and Δy the *rise*. Thus the slope is equal to the rise divided by the run. A large positive slope means that the line is rising steeply to the right, and a small positive slope means the line rises slowly to the right. A negative slope means that the line goes downward to the right. These cases are illustrated in Figure 1.3.4.

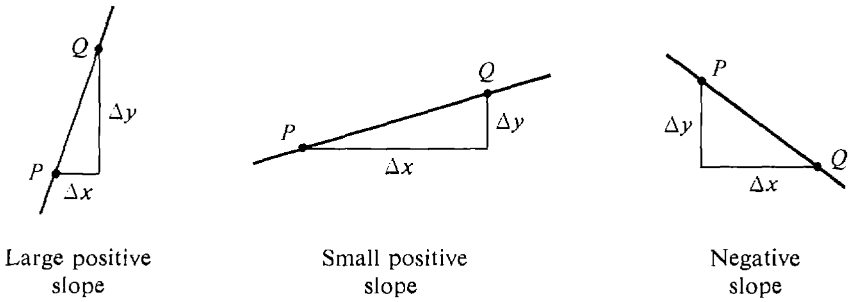

Figure 1.3.4

There is exactly one line L passing through two distinct points $P(x_1, y_2)$ and $Q(x_2, y_2)$. If $x_1 \neq x_2$, we see from Theorem 1 that L has the equation

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

This is called the *two-point* equation for the line.

EXAMPLE 2 Given $P(3, 1)$ and $Q(1, 4)$, find the changes in x and y, the slope, and the equation of the line through P and Q. (See Figure 1.3.5.)

$$\Delta x = 1 - 3 = -2, \qquad \Delta y = 4 - 1 = 3.$$

The line through P and Q has slope $\Delta y/\Delta x = -\frac{3}{2}$, and its equation is

$$y - 1 = -\tfrac{3}{2}(x - 3).$$

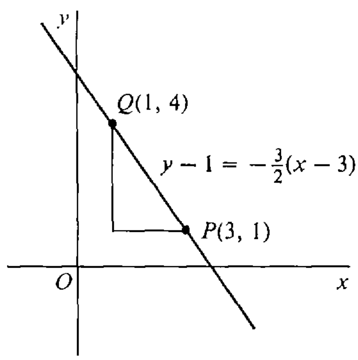

Figure 1.3.5

EXAMPLE 3 Given $P(1, -1)$ and $Q(1, 2)$, as in Figure 1.3.6,

$$\Delta x = 1 - 1 = 0, \qquad \Delta y = 2 - (-1) = 3.$$

The line through P and Q is the vertical line $x = 1$.

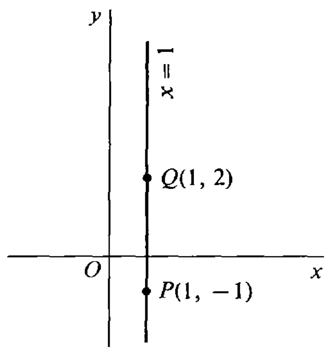

Figure 1.3.6

EXAMPLE 4 A particle moves along the y-axis with constant velocity. At time $t = 0$ sec, it is at the point $y = 3$ ft. At time $t = 2$ sec, it is at the point $y = 11$ ft. Find the velocity and the equation for the motion.

The velocity is defined as the distance moved divided by the time elapsed, so the velocity is

$$v = \frac{\Delta y}{\Delta t} = \frac{11 - 3}{2 - 0} = 4 \text{ ft/sec.}$$

If the motion of the particle is plotted in the (t, y) plane as in Figure 1.3.7,

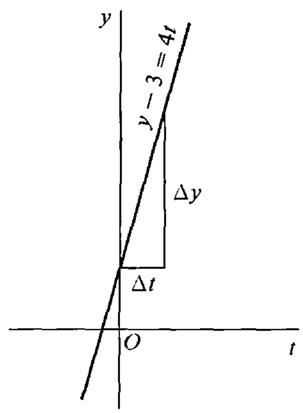

Figure 1.3.7

the result is a line through the points $P(0, 3)$ and $Q(2, 11)$. The velocity, being the ratio of Δy to Δt, is just the slope of this line. The line has the equation

$$y - 3 = 4t.$$

Suppose a particle moving with constant velocity is at the point $y = y_1$ at time $t = t_1$, and at the point $y = y_2$ at time $t = t_2$. Then the velocity is $v = \Delta y/\Delta t$. The motion of the particle plotted on the (t, y) plane is the line passing through the two points (t_1, y_1) and (t_2, y_2), and the velocity is the slope of this line.

An equation of the form

$$Ax + By + C = 0$$

where A and B are not both zero is called a *linear equation*. The reason for this name is explained by the next theorem.

THEOREM 2

Every linear equation determines a line.

PROOF

Case 1 $B = 0$. The equation $Ax + C = 0$ can be solved for x, $x = -C/A$. This is a vertical line.

Case 2 $B \neq 0$. In this case, we can solve the given equation for y, and the result is

$$y = \frac{-Ax - C}{B}, \qquad y = -\frac{A}{B}x - \frac{C}{B}.$$

This is a line with slope $-A/B$ crossing the y-axis at $-C/B$.

EXAMPLE 5 Find the slope of the line $6x - 2y + 7 = 0$.

The answer is $m = -A/B = -6/(-2) = 3$.

To draw the graph of a linear equation, find two points on the line and draw the line through them with a ruler.

EXAMPLE 6 Draw the graph of the line $4x + 2y + 3 = 0$.

First solve for y as a function of x:

$$y = -2x - \tfrac{3}{2}.$$

Next select any two values for x, say $x = 0$ and $x = 1$, and compute the corresponding values of y.

When $x = 0, \quad y = -\tfrac{3}{2}$.

When $x = 1, \quad y = -\tfrac{7}{2}$.

Finally, plot the two points $(0, -\tfrac{3}{2})$ and $(1, -\tfrac{7}{2})$, and draw the line through them. (See Figure 1.3.8.)

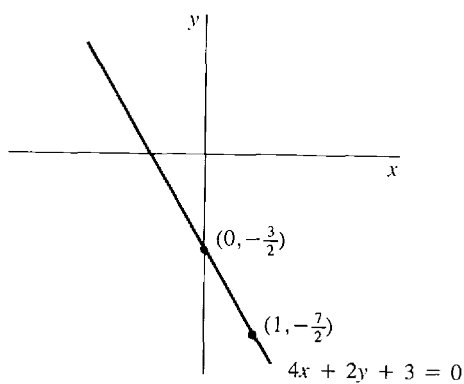

Figure 1.3.8

PROBLEMS FOR SECTION 1.3

In Problems 1–8, find the slope and equation of the line through P and Q.

1	$P(1, 2)$, $Q(3, 4)$	2	$P(1, -3)$, $Q(0, 2)$
3	$P(-4, 1)$, $Q(-4, 2)$	4	$P(2, 5)$, $Q(2, 7)$
5	$P(3, 0)$, $Q(0, 1)$	6	$P(0, 0)$, $Q(10, 4)$
7	$P(1, 3)$, $Q(3, 3)$	8	$P(6, -2)$, $Q(1, -2)$

In Problems 9–16, find the equation of the line with slope m through the point P.

9	$m = 2$, $P(3, 3)$	10	$m = 3$, $P(-2, 1)$
11	$m = -\frac{1}{2}$, $P(1, -4)$	12	$m = -1$, $P(2, 4)$
13	$m = 5$, $P(0, 0)$	14	$m = -2$, $P(0, 0)$
15	$m = 0$, $P(7, 4)$	16	vertical line, $P(4, 5)$

In Problems 17–22, a particle moves with constant velocity and has the given positions y at the given times t. Find the velocity and the equation of motion.

17 $y = 0$ at $t = 0$, $y = 2$ at $t = 1$

18 $y = 3$ at $t = 0$, $y = 1$ at $t = 2$

19 $y = 4$ at $t = 1$, $y = 2$ at $t = 5$

20 $y = 1$ at $t = 2$, $y = 3$ at $t = 3$

21 $y = 4$ at $t = 0$, $y = 4$ at $t = 1$

22 $y = 1$ at $t = 3$, $y = -2$ at $t = 6$

23 A particle moves with constant velocity 3, and at time $t = 2$ is at the point $y = 8$. Find the equation for its motion.

24 A particle moves with constant velocity $\frac{1}{4}$, and at time $t = 0$ is at $y = 1$. Find the equation for its motion.

In Problems 25–30, find the slope of the line with the given equation, and draw the line.

25	$3x - 2y + 5 = 0$	26	$x + y - 1 = 0$
27	$2x - y = 0$	28	$6x + 2y = 0$
29	$3x + 4y = 6$	30	$-2x + 4y = -1$

31 Show that the line that crosses the x-axis at $a \neq 0$ and the y-axis at $b \neq 0$ has the equation $(x/a) + (y/b) - 1 = 0$.

32 What is the equation of the line through the origin with slope m?

33 Find the points at which the line $ax + by + c = 0$ crosses the x- and y-axes. (Assume that $a \neq 0$ and $b \neq 0$.)

34 Let C denote Celsius temperature and F Fahrenheit temperature. Thus, $C = 0$ and $F = 32$ at the freezing point of water, while $C = 100$ and $F = 212$ at the boiling point of water. Use the two-point formula to find the linear equation relating C and F.

1.4 SLOPE AND VELOCITY; THE HYPERREAL LINE

In Section 1.3 the slope of the line through the points (x_1, y_1) and (x_2, y_2) is shown to be the ratio of the change in y to the change in x,

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

If the line has the equation

$$y = mx + b,$$

then the constant m is the slope.

What is meant by the slope of a *curve*? The differential calculus is needed to answer this question, as well as to provide a method of computing the value of the slope. We shall do this in the next chapter. However, to provide motivation, we now describe intuitively the method of finding the slope.

Consider the parabola

$$y = x^2.$$

The slope will measure the direction of a curve just as it measures the direction of a line. The slope of this curve will be different at different points on the x-axis, because the direction of the curve changes.

If (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$ are two points on the curve, then the "average slope" of the curve between these two points is defined as the ratio of the change in y to the change in x,

$$\text{average slope} = \frac{\Delta y}{\Delta x}.$$

This is exactly the same as the slope of the straight line through the points (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$, as shown in Figure 1.4.1.

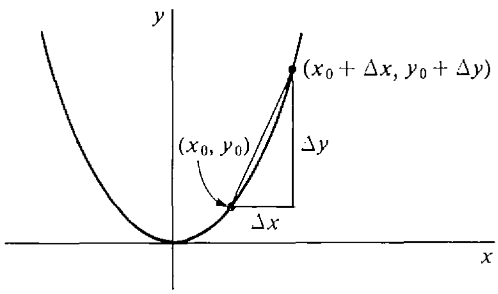

Figure 1.4.1

Let us compute the average slope. The two points (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$ are on the curve, so

$$y_0 = x_0^2,$$

$$y_0 + \Delta y = (x_0 + \Delta x)^2.$$

Subtracting,

$$\Delta y = (x_0 + \Delta x)^2 - x_0^2.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x}.$$

This can be simplified,

$$\frac{\Delta y}{\Delta x} = \frac{x_0^2 + 2x_0\,\Delta x + (\Delta x)^2 - x_0^2}{\Delta x}$$

$$= \frac{2x_0\,\Delta x + (\Delta x)^2}{\Delta x} = 2x_0 + \Delta x.$$

Thus the average slope is

$$\frac{\Delta y}{\Delta x} = 2x_0 + \Delta x.$$

Notice that this computation can only be carried out when $\Delta x \neq 0$, because at $\Delta x = 0$ the quotient $\Delta y/\Delta x$ is undefined.

Reasoning in a nonrigorous way, the actual slope of the curve at the point (x_0, y_0) can be found thus. Let Δx be very small (but not zero). Then the point $(x_0 + \Delta x, y_0 + \Delta y)$ is close to (x_0, y_0), so the average slope between these two points is close to the slope of the curve at (x_0, y_0);

$$[\text{slope at } (x_0, y_0)] \text{ is close to } 2x_0 + \Delta x.$$

We neglect the term Δx because it is very small, and we are left with

$$[\text{slope at } (x_0, y_0)] = 2x_0.$$

For example, at the point (0, 0) the slope is zero, at the point (1, 1) the slope is 2, and at the point $(-3, 9)$ the slope is -6. (See Figure 1.4.2.)

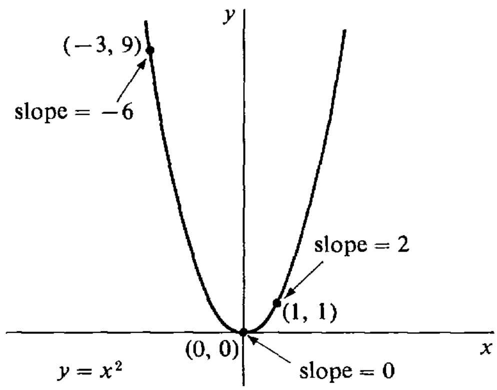

Figure 1.4.2

The whole process can also be visualized in another way. Let t represent time, and suppose a particle is moving along the y-axis according to the equation $y = t^2$. That is, at each time t the particle is at the point t^2 on the y-axis. We then ask: what is meant by the *velocity* of the particle at time t_0? Again we have the difficulty that the velocity is different at different times, and the calculus is needed to answer the question in a satisfactory way. Let us consider what happens to the particle between a time t_0 and a later time $t_0 + \Delta t$. The time elapsed is Δt, and the distance moved is $\Delta y = 2t_0 \Delta t + (\Delta t)^2$. If the velocity were constant during the entire interval of time, then it would just be the ratio $\Delta y/\Delta t$. However, the velocity is changing during the time interval. We shall call the ratio $\Delta y/\Delta t$ of the distance moved to the time elapsed the "average velocity" for the interval;

$$v_{\text{ave}} = \frac{\Delta y}{\Delta t} = 2t_0 + \Delta t.$$

The average velocity is not the same as the velocity at time t_0 which we are after. As a matter of fact, for $t_0 > 0$, the particle is speeding up; the velocity at time t_0 will be somewhat less than the average velocity for the interval of time between t_0 and $t_0 + \Delta t$, and the velocity at time $t_0 + \Delta t$ will be somewhat greater than the average.

But for a very small increment of time Δt, the velocity will change very little, and the average velocity $\Delta y/\Delta t$ will be close to the velocity at time t_0. To get the velocity v_0 at time t_0, we neglect the small term Δt in the formula

$$v_{\text{ave}} = 2t_0 + \Delta t,$$

and we are left with the value

$$v_0 = 2t_0.$$

When we plot y against t, the velocity is the same as the slope of the curve $y = t^2$, and the average velocity is the same as the average slope.

The trouble with the above intuitive argument, whether stated in terms of slope or velocity, is that it is not clear when something is to be "neglected." Nevertheless, the basic idea can be made into a useful and mathematically sound method of finding the slope of a curve or the velocity. What is needed is a sharp distinction between numbers which are small enough to be neglected and numbers which aren't. Actually, no real number except zero is small enough to be neglected. To get around this difficulty, we take the bold step of introducing a new kind of number, which is infinitely small and yet not equal to zero.

A number ε is said to be *infinitely small*, or *infinitesimal*, if

$$-a < \varepsilon < a$$

for every positive real number a. Then the only *real* number that is infinitesimal is zero. We shall use a new number system called the *hyperreal numbers*, which contains all the real numbers and also has infinitesimals that are not zero. Just as the real numbers can be constructed from the rational numbers, the hyperreal numbers can be constructed from the real numbers. This construction is sketched in the Epilogue at the end of the book. In this chapter, we shall simply list the properties of the hyperreal numbers needed for the calculus.

First we shall give an intuitive picture of the hyperreal numbers and show how they can be used to find the slope of a curve. The set of all hyperreal numbers is denoted by R^*. Every real number is a member of R^*, but R^* has other elements too. The infinitesimals in R^* are of three kinds: positive, negative, and the real number 0. The symbols $\Delta x, \Delta y, \ldots$ and the Greek letters ε (epsilon) and δ (delta) will be used for infinitesimals. If a and b are hyperreal numbers whose difference $a - b$ is infinitesimal, we say that a is *infinitely close to* b. For example, if Δx is infinitesimal then $x_0 + \Delta x$ is infinitely close to x_0. If ε is positive infinitesimal, then $-\varepsilon$ will be a negative infinitesimal. $1/\varepsilon$ will be an *infinite positive number*, that is, it will be greater than any real number. On the other hand, $-1/\varepsilon$ will be an *infinite negative number*, i.e., a number less than every real number. Hyperreal numbers which are not infinite numbers are called *finite numbers*. Figure 1.4.3 shows a drawing of the hyperreal line. The circles represent "infinitesimal microscopes" which are powerful enough to show an infinitely small portion of the hyperreal line. The set R of real numbers is scattered among the finite numbers. About each real number c is a portion of the hyperreal line composed of the numbers infinitely close to c (shown under an infinitesimal microscope for $c = 0$ and $c = 100$). The numbers infinitely close to 0 are the infinitesimals.

In Figure 1.4.3 the finite and infinite parts of the hyperreal line were separated from each other by a dotted line. Another way to represent the infinite parts of the hyperreal line is with an "infinite telescope" as in Figure 1.4.4. The field of view of an infinite telescope has the same scale as the finite portion of the hyperreal line, while the field of view of an infinitesimal microscope contains an infinitely small portion of the hyperreal line blown up.

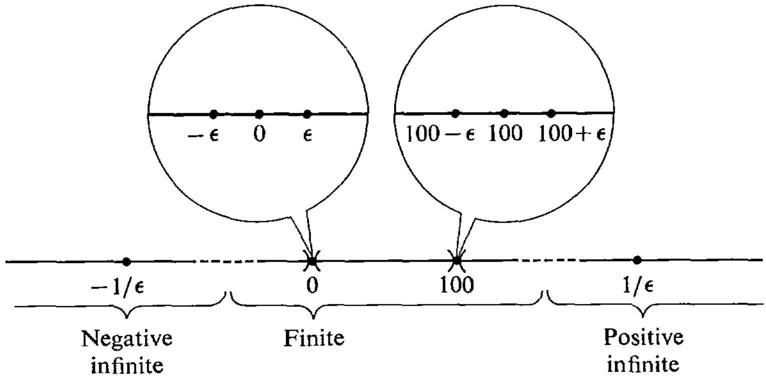

Figure 1.4.3

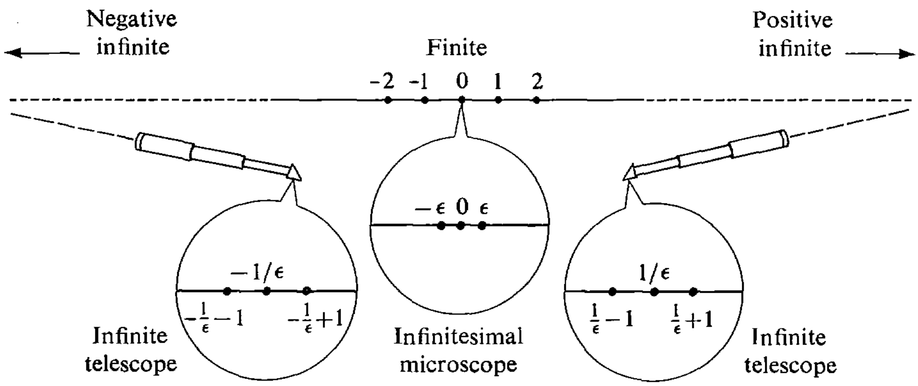

Figure 1.4.4

We have no way of knowing what a line in physical space is really like. It might be like the hyperreal line, the real line, or neither. However, in applications of the calculus it is helpful to imagine a line in physical space as a hyperreal line. The hyperreal line is, like the real line, a useful mathematical model for a line in physical space.

The hyperreal numbers can be algebraically manipulated just like the real numbers. Let us try to use them to find slopes of curves. We begin with the parabola $y = x^2$.

Consider a real point (x_0, y_0) on the curve $y = x^2$. Let Δx be either a positive or a negative infinitesimal (but not zero), and let Δy be the corresponding change in y. Then the slope at (x_0, y_0) is *defined* in the following way:

$$[\text{slope at } (x_0, y_0)] = \left[\text{the real number infinitely close to } \frac{\Delta y}{\Delta x}\right].$$

We compute $\frac{\Delta y}{\Delta x}$ as before: $\quad \frac{\Delta y}{\Delta x} = \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x} = 2x_0 + \Delta x.$

This is a hyperreal number, not a real number. Since Δx is infinitesimal, the hyperreal number $2x_0 + \Delta x$ is infinitely close to the real number $2x_0$. We conclude that

$$[\text{slope at } (x_0, y_0)] = 2x_0.$$

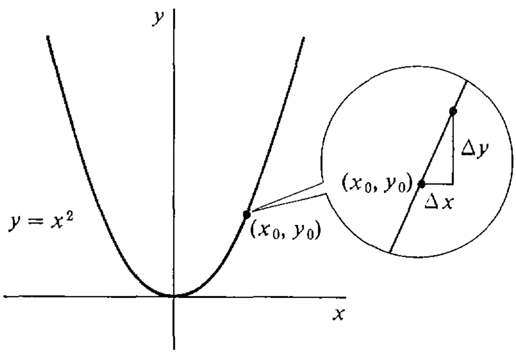

Figure 1.4.5

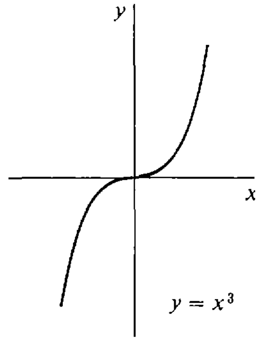

Figure 1.4.6

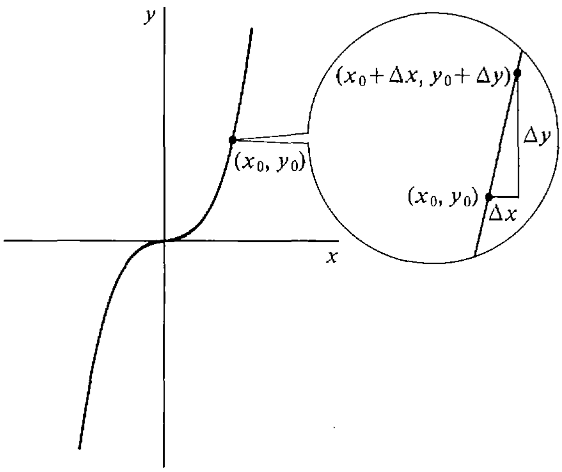

Figure 1.4.7

The process can be illustrated by the picture in Figure 1.4.5, with the infinitesimal changes Δx and Δy shown under a microscope.

The same method can be applied to other curves. The third degree curve $y = x^3$ is shown in Figure 1.4.6. Let (x_0, y_0) be any point on the curve $y = x^3$, and let Δx be a positive or a negative infinitesimal. Let Δy be the corresponding change in y along the curve. In Figure 1.4.7, Δx and Δy are shown under a microscope. We again define the slope at (x_0, y_0) by

$$[\text{slope at } (x_0, y_0)] = \left[\text{the real number infinitely close to } \frac{\Delta y}{\Delta x}\right].$$

We now compute the hyperreal number $\dfrac{\Delta y}{\Delta x}$.

$$\begin{aligned}
y_0 &= x_0^3, \\
y_0 + \Delta y &= (x_0 + \Delta x)^3, \\
\Delta y &= (x_0 + \Delta x)^3 - x_0^3, \\
\frac{\Delta y}{\Delta x} &= \frac{(x_0 + \Delta x)^3 - x_0^3}{\Delta x} \\
&= \frac{x_0^3 + 3x_0^2\,\Delta x + 3x_0(\Delta x)^2 + (\Delta x)^3 - x_0^3}{\Delta x} \\
&= \frac{3x_0^2\,\Delta x + 3x_0(\Delta x)^2 + (\Delta x)^3}{\Delta x},
\end{aligned}$$

and finally $$\frac{\Delta y}{\Delta x} = 3x_0^2 + 3x_0\,\Delta x + (\Delta x)^2.$$

In the next section we shall develop some rules about infinitesimals which will enable us to show that since Δx is infinitesimal,

$$3x_0\,\Delta x + (\Delta x)^2$$

is infinitesimal as well. Therefore the hyperreal number

$$3x_0^2 + 3x_0\,\Delta x + (\Delta x)^2$$

is infinitely close to the real number $3x_0^2$, whence

$$[\text{slope at } (x_0, y_0)] = 3x_0^2.$$

For example, at (0, 0) the slope is zero, at (1, 1) the slope is 3, and at (2, 8) the slope is 12.

We shall return to the study of the slope of a curve in Chapter 2 after we have learned more about hyperreal numbers. From the last example it is evident that we need to know how to show that two numbers are infinitely close to each other. This is our next topic.

1.5 INFINITESIMAL, FINITE, AND INFINITE NUMBERS

Let us summarize our intuitive description of the hyperreal numbers from Section 1.4. The real line is a subset of the hyperreal line; that is, each real number belongs to the set of hyperreal numbers. Surrounding each real number r, we introduce a collection of hyperreal numbers infinitely close to r. The hyperreal numbers infinitely close to zero are called infinitesimals. The reciprocals of nonzero infinitesimals are infinite hyperreal numbers. The collection of all hyperreal numbers satisfies the same algebraic laws as the real numbers. In this section we describe the hyperreal numbers more precisely and develop a facility for computation with them.

This entire calculus course is developed from three basic principles relating the real and hyperreal numbers: the Extension Principle, the Transfer Principle, and the Standard Part Principle. The first two principles are presented in this section, and the third principle is in the next section.

We begin with the Extension Principle, which gives us new numbers called hyperreal numbers and extends all real functions to these numbers. The Extension Principle will deal with *hyperreal functions* as well as real functions. Our discussion of real functions in Section 1.2 can readily be carried over to hyperreal functions. Recall that for each real number a, a real function f of one variable either associates another real number $b = f(a)$ or is undefined. Now, for each hyperreal number H, a hyperreal function F of one variable either associates another hyperreal number $K = F(H)$ or is undefined. For each pair of hyperreal numbers H and J, a hyperreal function G of two variables either associates another hyperreal number $K = G(H, J)$ or is undefined. Hyperreal functions of three or more variables are defined in a similar way.

I. THE EXTENSION PRINCIPLE

(a) *The real numbers form a subset of the hyperreal numbers, and the order relation $x < y$ for the real numbers is a subset of the order relation for the hyperreal numbers.*

(b) *There is a hyperreal number that is greater than zero but less than every positive real number.*
(c) *For every real function f of one or more variables we are given a corresponding hyperreal function f^* of the same number of variables. f^* is called the natural extension of f.*

Part (a) of the Extension Principle says that the real line is a part of the hyperreal line. To explain part (b) of the Extension Principle, we give a careful definition of an infinitesimal.

DEFINITION

A hyperreal number b is said to be:

positive infinitesimal *if b is positive but less than every positive real number.*

negative infinitesimal *if b is negative but greater than every negative real number.*

infinitesimal *if b is either positive infinitesimal, negative infinitesimal, or zero.*

With this definition, part (b) of the Extension Principle says that there is at least one positive infinitesimal. We shall see later that there are infinitely many positive infinitesimals. A positive infinitesimal is a hyperreal number but cannot be a real number, so part (b) ensures that there are hyperreal numbers that are not real numbers.

Part (c) of the Extension Principle allows us to apply real functions to hyperreal numbers. Since the addition function $+$ is a real function of two variables, its natural extension $+^*$ is a hyperreal function of two variables. If x and y are hyperreal numbers, the sum of x and y is the number $x +^* y$ formed by using the natural extension of $+$. Similarly, the product of x and y is the number $x \cdot^* y$ formed by using the natural extension of the product function $\cdot$. To make things easier to read, we shall drop the asterisks and write simply $x + y$ and $x \cdot y$ for the sum and product of two hyperreal numbers x and y. Using the natural extensions of the sum and product functions, we will be able to develop algebra for hyperreal numbers. Part (c) of the Extension Principle also allows us to work with expressions such as $\cos(x)$ or $\sin(x + \cos(y))$, which involve one or more real functions. We call such expressions *real expressions*. These expressions can be used even when x and y are hyperreal numbers instead of real numbers. For example, when x and y are hyperreal, $\sin(x + \cos(y))$ will mean $\sin^*(x + \cos^*(y))$, where $\sin^*$ and $\cos^*$ are the natural extensions of sin and cos. The asterisks are dropped as before.

We now state the Transfer Principle, which allows us to carry out computations with the hyperreal numbers in the same way as we do for real numbers. Intuitively, the Transfer Principle says that the natural extension of each real function has the same properties as the original function.

II. TRANSFER PRINCIPLE

Every real statement that holds for one or more particular real functions holds for the hyperreal natural extensions of these functions.

Here are seven examples that illustrate what we mean by a *real statement*. In general, by a real statement we mean a combination of equations or inequalities about real expressions, and statements specifying whether a real expression is defined or undefined. A real statement will involve real variables and particular real functions.

(1) Closure law for addition: for any x and y, the sum $x + y$ is defined.
(2) Commutative law for addition: $x + y = y + x$.
(3) A rule for order: If $0 < x < y$, then $0 < 1/y < 1/x$.
(4) Division by zero is never allowed: $x/0$ is undefined.
(5) An algebraic identity: $(x - y)^2 = x^2 - 2xy + y^2$.
(6) A trigonometric identity: $\sin^2 x + \cos^2 x = 1$.
(7) A rule for logarithms: If $x > 0$ and $y > 0$, then $\log_{10}(xy) = \log_{10} x + \log_{10} y$.

Each example has two variables, x and y, and holds true whenever x and y are real numbers. The Transfer Principle tells us that each example also holds whenever x and y are hyperreal numbers. For instance, by Example (4), $x/0$ is undefined, even for hyperreal x. By Example (6), $\sin^2 x + \cos^2 x = 1$, even for hyperreal x.

Notice that the first five examples involve only the sum, difference, product, and quotient functions. However, the last two examples are real statements involving the transcendental functions sin, cos, and $\log_{10}$. The Transfer Principle extends all the familiar rules of trigonometry, exponents, and logarithms to the hyperreal numbers.

In calculus we frequently make a computation involving one or more unknown real numbers. The Transfer Principle allows us to compute in exactly the same way with hyperreal numbers. It "transfers" facts about the real numbers to facts about the hyperreal numbers. In particular, the Transfer Principle implies that a real function and its natural extension always give the same value when applied to a real number. This is why we are usually able to drop the asterisks when computing with hyperreal numbers.

A real statement is often used to define a new real function from old real functions. By the Transfer Principle, whenever a real statement defines a real function, the same real statement also defines the hyperreal natural extension function. Here are three more examples.

(8) The square root function is defined by the real statement $y = \sqrt{x}$ if, and only if, $y^2 = x$ and $y \geq 0$.
(9) The absolute value function is defined by the real statement $y = |x|$ if, and only if, $y = \sqrt{x^2}$.
(10) The common logarithm function is defined by the real statement $y = \log_{10} x$ if, and only if, $10^y = x$.

In each case, the hyperreal natural extension is the function defined by the given real statement when x and y vary over the hyperreal numbers. For example, the hyperreal natural extension of the square root function, $\sqrt{\ }^*$, is defined by Example (8) when x and y are hyperreal.

An important use of the Transfer Principle is to carry out computations with infinitesimals. For example, a computation with infinitesimals was used in the slope calculation in Section 1.4. The Extension Principle tells us that there is at least one positive infinitesimal hyperreal number, say ε. Starting from ε, we can use the Transfer Principle to construct infinitely many other positive infinitesimals. For example, ε^2 is a positive infinitesimal that is smaller than ε, $0 < \varepsilon^2 < \varepsilon$. (This follows from the Transfer Principle because $0 < x^2 < x$ for all real x between 0 and 1.) Here are several positive infinitesimals, listed in increasing order:

$$\varepsilon^3,\ \varepsilon^2,\ \varepsilon/100,\ \varepsilon,\ 75\varepsilon,\ \sqrt{\varepsilon},\ \varepsilon + \sqrt{\varepsilon}.$$

We can also construct negative infinitesimals, such as $-\varepsilon$ and $-\varepsilon^2$, and other hyperreal numbers such as $1 + \sqrt{\varepsilon}$, $(10 - \varepsilon)^2$, and $1/\varepsilon$.

We shall now give a list of rules for deciding whether a given hyperreal number is infinitesimal, finite, or infinite. All these rules follow from the Transfer Principle alone. First, look at Figure 1.5.1, illustrating the hyperreal line.

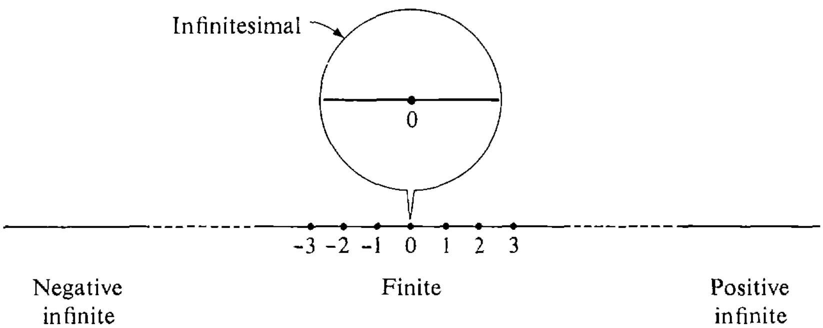

Figure 1.5.1

DEFINITION

A hyperreal number b is said to be:

finite *if b is between two real numbers.*

positive infinite *if b is greater than every real number.*

negative infinite *if b is less than every real number.*

Notice that each infinitesimal number is finite. Before going through the whole list of rules, let us take a close look at two of them.

If ε is infinitesimal and a is finite, then the product $a \cdot \varepsilon$ is infinitesimal. For example, $\frac{1}{2}\varepsilon$, -6ε, 1000ε, $(5 - \varepsilon)\varepsilon$ are infinitesimal. This can be seen intuitively from Figure 1.5.2; an infinitely thin rectangle of length a has infinitesimal area.

If ε is positive infinitesimal, then $1/\varepsilon$ is positive infinite. From experience we know that reciprocals of small numbers are large, so we intuitively expect $1/\varepsilon$ to be positive infinite. We can use the Transfer Principle to prove $1/\varepsilon$ is positive infinite. Let r be any positive real number. Since ε is positive infinitesimal, $0 < \varepsilon < 1/r$. Applying the Transfer Principle, $1/\varepsilon > r > 0$. Therefore, $1/\varepsilon$ is positive infinite.

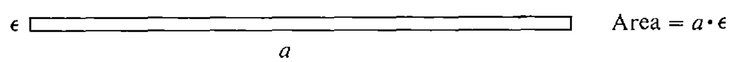

Figure 1.5.2

RULES FOR INFINITESIMAL, FINITE, AND INFINITE NUMBERS *Assume that ε, δ are infinitesimals; b, c are hyperreal numbers that are finite but not infinitesimal; and H, K are infinite hyperreal numbers.*

(i) ***Real numbers:***
The only infinitesimal real number is 0.
Every real number is finite.

(ii) ***Negatives:***
$-\varepsilon$ is infinitesimal.

$-b$ is finite but not infinitesimal.
$-H$ is infinite.

(iii) ***Reciprocals:***
If $\varepsilon \neq 0$, $1/\varepsilon$ is infinite.
$1/b$ is finite but not infinitesimal.
$1/H$ is infinitesimal.

(iv) ***Sums:***
$\varepsilon + \delta$ is infinitesimal.
$b + \varepsilon$ is finite but not infinitesimal.
$b + c$ is finite (possibly infinitesimal).
$H + \varepsilon$ and $H + b$ are infinite.

(v) ***Products:***
$\delta \cdot \varepsilon$ and $b \cdot \varepsilon$ are infinitesimal.
$b \cdot c$ is finite but not infinitesimal.
$H \cdot b$ and $H \cdot K$ are infinite.

(vi) ***Quotients:***
ε/b, ε/H, and b/H are infinitesimal.
b/c is finite but not infinitesimal.
b/ε, H/ε, and H/b are infinite, provided that $\varepsilon \neq 0$.

(vii) ***Roots:***
If $\varepsilon > 0$, $\sqrt[n]{\varepsilon}$ is infinitesimal.
If $b > 0$, $\sqrt[n]{b}$ is finite but not infinitesimal.
If $H > 0$, $\sqrt[n]{H}$ is infinite.

Notice that we have given no rule for the following combinations:

ε/δ, the quotient of two infinitesimals.
H/K, the quotient of two infinite numbers.
$H\varepsilon$, the product of an infinite number and an infinitesimal.
$H + K$, the sum of two infinite numbers.

Each of these can be either infinitesimal, finite but not infinitesimal, or infinite, depending on what ε, δ, H, and K are. For this reason, they are called *indeterminate forms.*

Here are three very different quotients of infinitesimals.

$\frac{\varepsilon^2}{\varepsilon}$ is infinitesimal (equal to ε).

$\frac{\varepsilon}{\varepsilon}$ is finite but not infinitesimal (equal to 1).

$\frac{\varepsilon}{\varepsilon^2}$ is infinite $\left(\text{equal to } \frac{1}{\varepsilon}\right)$.

Table 1.5.1 on the following page shows the three possibilities for each indeterminate form. Here are some examples which show how to use our rules.

EXAMPLE 1 Consider $(b - 3\varepsilon)/(c + 2\delta)$. ε is infinitesimal, so -3ε is infinitesimal, and $b - 3\varepsilon$ is finite but not infinitesimal. Similarly, $c + 2\delta$ is finite but not infinitesimal. Therefore the quotient

$$\frac{b - 3\varepsilon}{c + 2\delta}$$

is finite but not infinitesimal.

Table 1.5.1

indeterminate form	Examples: infinitesimal	finite (equal to 1)	infinite
$\frac{\varepsilon}{\delta}$	$\frac{\varepsilon^2}{\varepsilon}$	$\frac{\varepsilon}{\varepsilon}$	$\frac{\varepsilon}{\varepsilon^2}$
$\frac{H}{K}$	$\frac{H}{H^2}$	$\frac{H}{H}$	$\frac{H^2}{H}$
$H\varepsilon$	$H \cdot \frac{1}{H^2}$	$H \cdot \frac{1}{H}$	$H^2 \cdot \frac{1}{H}$
$H + K$	$H + (-H)$	$(H + 1) + (-H)$	$H + H$

The next three examples are quotients of infinitesimals.

EXAMPLE 2 The quotient

$$\frac{5\varepsilon^4 - 8\varepsilon^3 + \varepsilon^2}{3\varepsilon}$$

is infinitesimal, provided $\varepsilon \neq 0$.

The given number is equal to

(1) $$\tfrac{5}{3}\varepsilon^3 - \tfrac{8}{3}\varepsilon^2 + \tfrac{1}{3}\varepsilon.$$

We see in turn that $\varepsilon, \varepsilon^2, \varepsilon^3, \frac{1}{3}\varepsilon, -\frac{8}{3}\varepsilon^2, \frac{5}{3}\varepsilon^3$ are infinitesimal; hence the sum (1) is infinitesimal.

EXAMPLE 3 If $\varepsilon \neq 0$, the quotient

$$\frac{3\varepsilon^3 + \varepsilon^2 - 6\varepsilon}{2\varepsilon^2 + \varepsilon}$$

is finite but not infinitesimal.

Cancelling an ε from numerator and denominator, we get

(2) $$\frac{3\varepsilon^2 + \varepsilon - 6}{2\varepsilon + 1}.$$

Since $3\varepsilon^2 + \varepsilon$ is infinitesimal while -6 is finite but not infinitesimal, the numerator

$$3\varepsilon^2 + \varepsilon - 6$$

is finite but not infinitesimal. Similarly, the denominator $2\varepsilon + 1$, and hence the quotient (2) is finite but not infinitesimal.

EXAMPLE 4 If $\varepsilon \neq 0$, the quotient

$$\frac{\varepsilon^4 - \varepsilon^3 + 2\varepsilon^2}{5\varepsilon^4 + \varepsilon^3}$$

is infinite.

We first note that the denominator $5\varepsilon^4 + \varepsilon^3$ is not zero because it can be written as a product of nonzero factors,

$$5\varepsilon^4 + \varepsilon^3 = \varepsilon \cdot \varepsilon \cdot \varepsilon \cdot (5\varepsilon + 1).$$

When we cancel ε^2 from the numerator and denominator we get

$$\frac{\varepsilon^2 - \varepsilon + 2}{5\varepsilon^2 + \varepsilon}.$$

We see in turn that:

$\varepsilon^2 - \varepsilon + 2$ is finite but not infinitesimal,

$5\varepsilon^2 + \varepsilon$ is infinitesimal,

$\dfrac{\varepsilon^2 - \varepsilon + 2}{5\varepsilon^2 + \varepsilon}$ is infinite.

EXAMPLE 5 $\dfrac{2H^2 + H}{H^2 - H + 2}$ is finite but not infinitesimal.

In this example the trick is to multiply both numerator and denominator by $1/H^2$. We get

$$\frac{2 + 1/H}{1 - 1/H + 2/H^2}.$$

Now $1/H$ and $1/H^2$ are infinitesimal. Therefore both the numerator and denominator are finite but not infinitesimal, and so is the quotient.

In the next theorem we list facts about the ordering of the hyperreals.

THEOREM 1

(i) *Every hyperreal number which is between two infinitesimals is infinitesimal.*

(ii) *Every hyperreal number which is between two finite hyperreal numbers is finite.*

(iii) *Every hyperreal number which is greater than some positive infinite number is positive infinite.*

(iv) *Every hyperreal number which is less than some negative infinite number is negative infinite.*

All the proofs are easy. We prove (iii), which is especially useful. Assume H is positive infinite and $H < K$. Then for any real number r, $r < H < K$. Therefore, $r < K$ and K is positive infinite.

EXAMPLE 6 If H and K are positive infinite hyperreal numbers, then $H + K$ is positive infinite. This is true because $H + K$ is greater than H.

Our last example concerns square roots.

EXAMPLE 7 If H is positive infinite then, surprisingly,

$$\sqrt{H+1} - \sqrt{H-1}$$

is infinitesimal.

This is shown using an algebraic trick.

$$\sqrt{H+1} - \sqrt{H-1} = \frac{(\sqrt{H+1} - \sqrt{H-1})(\sqrt{H+1} + \sqrt{H-1})}{\sqrt{H+1} + \sqrt{H-1}}$$

$$= \frac{(H+1) - (H-1)}{\sqrt{H+1} + \sqrt{H-1}} = \frac{2}{\sqrt{H+1} + \sqrt{H-1}}.$$

The numbers $H+1$, $H-1$, and their square roots are positive infinite, and thus the sum $\sqrt{H+1} + \sqrt{H-1}$ is positive infinite. Therefore the quotient

$$\sqrt{H+1} - \sqrt{H-1} = \frac{2}{\sqrt{H+1} + \sqrt{H-1}},$$

a finite number divided by an infinite number, is infinitesimal.

PROBLEMS FOR SECTION 1.5

In Problems 1–40, assume that: ε, δ are positive infinitesimal, H, K are positive infinite. Determine whether the given expression is infinitesimal, finite but not infinitesimal, or infinite.

1 $76{,}000{,}000\varepsilon$

2 $3\varepsilon + 4\delta$

3 $1 + 1/\varepsilon$

4 $3\varepsilon^3 - 2\varepsilon^2 + \varepsilon + 1$

5 $1/\sqrt{\varepsilon}$

6 ε/H

7 $H/1{,}000{,}000$

8 $(3+\varepsilon)^2 - 9$

9 $(3+\varepsilon)(4+\delta) - 12$

10 $\dfrac{1 + \varepsilon + 3\varepsilon^2}{2 - \varepsilon - 8\varepsilon^3}$

11 $\dfrac{2\varepsilon^3 - \varepsilon^4}{4\varepsilon - \varepsilon^2 + \varepsilon^3}$

12 $\dfrac{2\varepsilon^3 - \varepsilon^4}{4\varepsilon^3 + \varepsilon^4}$

13 $\dfrac{3\varepsilon - 4\varepsilon^2}{\varepsilon^2 + 5\varepsilon^3}$

14 $\dfrac{\sqrt{\varepsilon} + \varepsilon}{\sqrt{\varepsilon} + 1}$

15 $\dfrac{1}{\sqrt{\varepsilon} - \varepsilon}$

16 $\dfrac{1}{\varepsilon} \cdot \sqrt{\varepsilon}$

17 $\dfrac{1}{\varepsilon} \cdot 5\varepsilon$

18 $\dfrac{1}{\varepsilon} \cdot \varepsilon^3$

19 $\dfrac{1}{\varepsilon}\left(\dfrac{1}{3+\varepsilon} - \dfrac{1}{3}\right)$

20 $\dfrac{2H+1}{3H+2}$

21 $\dfrac{2H^4 + 3H - 6}{4H^3 + 5}$

22 $\dfrac{H + 4 + \varepsilon}{H^2 + 2\varepsilon}$

23 $\dfrac{H+K}{HK}$

24 $\dfrac{H-K}{H^2+K^2}$

25 $H^2 - H$

26 $\sqrt{H+1} - \sqrt{H}$

27 $\left(H + \dfrac{1}{H}\right)^2 - \left(H - \dfrac{1}{H}\right)^2$

28 $\left(H + \dfrac{\varepsilon}{H}\right)^2 - \left(H - \dfrac{\varepsilon}{H}\right)^2$

29 $\dfrac{\sqrt{4+\varepsilon}-2}{\varepsilon}$

30 $\dfrac{1}{\varepsilon}\left(1-\dfrac{1}{\sqrt{1+\varepsilon}}\right)$

31 $H\left(\sqrt{3+\dfrac{1}{H}}-\sqrt{3}\right)$

32 $\dfrac{\sqrt{H}}{\sqrt{H+1}+\sqrt{H+2}}$

33 $H(\sqrt{H+2}-\sqrt{H})$

34 $\dfrac{1-\sqrt[3]{1+\varepsilon}}{\varepsilon}$

35 $\sqrt[3]{H}-\sqrt[3]{H+1}$

36 $H-\sqrt{H+1}\sqrt{H+2}$

37 $\dfrac{(3+\varepsilon)(4+\delta)-12}{\varepsilon\delta}$

38 $\dfrac{5+\varepsilon}{7+\delta}-\dfrac{5}{7}$

39 $\dfrac{\varepsilon+\delta}{\sqrt{\varepsilon^2+\delta^2}}$
(*Hint:* Assume $\varepsilon \geq \delta$ and divide through by ε.)

40 $\dfrac{H+K}{\sqrt{H^2+K^2}}$

41 In (a)–(f) below, determine which of the two numbers is greater.

(a) ε or ε^2 (b) $\dfrac{1}{\varepsilon^3}$ or $\dfrac{1}{\varepsilon^4}$ (c) H or H^2

(d) ε or $\sqrt{\varepsilon}$ (e) H or $\sqrt{H}$ (f) $\sqrt{H}$ or $\sqrt[3]{H}$

□ 42 Let x, y be positive hyperreal numbers. Can $\dfrac{x}{y}+\dfrac{y}{x}$ be infinite? Finite? Infinitesimal?

□ 43 Let a and b be real. When is $(3\varepsilon^2-\varepsilon+a)/(4\varepsilon^2+2\varepsilon+b)$
(a) infinitesimal?
(b) finite but not infinitesimal?
(c) infinite?

□ 44 Let a and b be real. When is $(aH^2-2H+5)/(bH^2+H-2)$
(a) infinitesimal?
(b) finite but not infinitesimal?
(c) infinite?

1.6 STANDARD PARTS

In this section we shall develop a method that will enable us to compute the slope of a curve by means of infinitesimals. We shall use the method to find slopes of curves in Chapter 2 and to find areas in Chapter 4. The key step will be to find the standard part of a given hyperreal number, that is, the real number that is infinitely close to it.

DEFINITION

*Two hyperreal numbers b and c are said to be **infinitely close** to each other, in symbols $b \approx c$, if their difference $b - c$ is infinitesimal. $b \not\approx c$ means that b is not infinitely close to c.*

Here are three simple remarks.

(1) *If ε is infinitesimal, then $b \approx b + \varepsilon$.* This is true because the difference, $b-(b+\varepsilon)=-\varepsilon$, is infinitesimal.

(2) *b is infinitesimal if and only if $b \approx 0$.* The formula $b \approx 0$ will be used as a short way of writing "b is infinitesimal."

(3) *If b and c are real and b is infinitely close to c, then b equals c.* $b - c$ is real and infinitesimal, hence zero; so $b = c$.

The relation $\approx$ between hyperreal numbers behaves somewhat like equality, but, of course, is not the same as equality. Here are three basic properties of $\approx$.

THEOREM 1

Let a, b and c be hyperreal numbers.

(i) $a \approx a$.
(ii) *If $a \approx b$, then $b \approx a$.*
(iii) *If $a \approx b$ and $b \approx c$, then $a \approx c$.*

These properties are useful when we wish to show that two numbers are infinitely close to each other.

The reason for (i) is that $a - a$ is an infinitesimal, namely zero. For (ii), we note that if $a - b$ is an infinitesimal ε, then $b - a = -\varepsilon$, which is also infinitesimal. Finally, (iii) is true because $a - c$ is the sum of two infinitesimals, namely $a - b$ and $b - c$.

THEOREM 2

Assume $a \approx b$. Then

(i) *If a is infinitesimal, so is b.*
(ii) *If a is finite, so is b.*
(iii) *If a is infinite, so is b.*

The real numbers are sometimes called "standard" numbers, while the hyperreal numbers that are not real are called "nonstandard" numbers. For this reason, the real number that is infinitely close to b is called the "standard part" of b. An infinite number cannot have a standard part, because it can't be infinitely close to a finite number (Theorem 2). Our third principle (stated next) on hyperreal numbers is that every finite number has a standard part.

III. STANDARD PART PRINCIPLE

Every finite hyperreal number is infinitely close to exactly one real number.

DEFINITION

Let b be a finite hyperreal number. The ***standard part*** *of b, denoted by $st(b)$, is the real number which is infinitely close to b. Infinite hyperreal numbers do not have standard parts.*

Here are some facts that follow at once from the definition.

Let b be a finite hyperreal number.

(1) $st(b)$ *is a real number.*
(2) $b = st(b) + \varepsilon$ *for some infinitesimal* ε.
(3) *If b is real, then* $b = st(b)$.

Our next aim is to develop some skill in computing standard parts. This will be one of the basic methods throughout the Calculus course. The next theorem is the principal tool.

THEOREM 3

Let a and b be finite hyperreal numbers. Then

(i) $st(-a) = -st(a)$.
(ii) $st(a + b) = st(a) + st(b)$.
(iii) $st(a - b) = st(a) - st(b)$.
(iv) $st(ab) = st(a) \cdot st(b)$.
(v) *If* $st(b) \neq 0$, *then* $st(a/b) = st(a)/st(b)$.
(vi) $st(a^n) = (st(a))^n$.
(vii) *If* $a \geq 0$, *then* $st(\sqrt[n]{a}) = \sqrt[n]{st(a)}$.
(viii) *If* $a \leq b$, *then* $st(a) \leq st(b)$.

This theorem gives formulas for the standard parts of the simplest expressions.

All of the rules in Theorem 3 follow from our three principles for hyperreal numbers. As an illustration, let us prove the formula (iv) for $st(ab)$. Let r be the standard part of a and s the standard part of b, so that

$$a = r + \varepsilon, \qquad b = s + \delta,$$

where ε and δ are infinitesimal. Then

$$\begin{aligned} ab &= (r + \varepsilon)(s + \delta) \\ &= rs + r\delta + s\varepsilon + \varepsilon\delta \approx rs. \end{aligned}$$

Therefore $$st(ab) = rs = st(a) \cdot st(b).$$

Often the symbols Δx, Δy, etc. are used for infinitesimals. In the following examples we use the rules in Theorem 3 as a starting point for computing standard parts of more complicated expressions.

EXAMPLE 1 When Δx is an infinitesimal and x is real, compute the standard part of

$$3x^2 + 3x\,\Delta x + (\Delta x)^2.$$

Using the rules in Theorem 3, we can write

$$\begin{aligned} st(3x^2 + 3x\,\Delta x + (\Delta x)^2) &= st(3x^2) + st(3x\,\Delta x) + st((\Delta x)^2) \\ &= 3x^2 + st(3x) \cdot st(\Delta x) + st(\Delta x)^2 \\ &= 3x^2 + 3x \cdot 0 + 0^2 = 3x^2. \end{aligned}$$

EXAMPLE 2 If $st(c) = 4$ and $c \neq 4$, find

$$st\left(\frac{c^2 + 2c - 24}{c^2 - 16}\right).$$

We note that the denominator has standard part 0,

$$st(c^2 - 16) = st(c)^2 - 16 = 4^2 - 16 = 0.$$

However, since $c \neq 4$ the fraction is defined, and it can be simplified by factoring the numerator and denominator,

$$\frac{c^2 + 2c - 24}{c^2 - 16} = \frac{(c + 6)(c - 4)}{(c + 4)(c - 4)} = \frac{c + 6}{c + 4}.$$

Then

$$st\left(\frac{c^2 + 2c - 24}{c^2 - 16}\right) = st\left(\frac{c + 6}{c + 4}\right) = \frac{st(c + 6)}{st(c + 4)}$$

$$= \frac{st(c) + 6}{st(c) + 4} = \frac{4 + 6}{4 + 4} = \frac{10}{8}.$$

We now have three kinds of computation available to us. First, there are computations involving hyperreal numbers. In Example 2, the two steps giving

$$\frac{c^2 + 2c - 24}{c^2 - 16} = \frac{c + 6}{c + 4}$$

are computations of this kind. The computations of this first kind are justified by the Transfer Principle.

Second, we have computations which involve standard parts. In Example 2, the three steps giving

$$st\frac{c^2 + 2c - 24}{c^2 - 16} = \frac{st(c) + 6}{st(c) + 4}$$

are of this kind. This second kind of computation depends on Theorem 3.

Third there are computations with ordinary real numbers. Sometimes the real numbers will appear as standard parts. In Example 2, the last two steps which give

$$\frac{st(c) + 6}{st(c) + 4} = \frac{10}{8}$$

are computations with ordinary real numbers.

Usually, in computing the standard part of a hyperreal number, we use the first kind of computation, then the second kind, and then the third kind, in that order. We shall give two more somewhat different examples and pick out these three stages in the computations.

EXAMPLE 3 If H is a positive infinite hyperreal number, compute the standard part of

$$c = \frac{2H^3 + 5H^2 - 3H}{7H^3 - 2H^2 + 4H}.$$

In this example both the numerator and denominator are infinite, and we have to use the first type of computation to get the equation into a different form before we can take standard parts.

First stage

$$c = \frac{2H^3 + 5H^2 - 3H}{7H^3 - 2H^2 + 4H} = \frac{H^{-3}\cdot(2H^3 + 5H^2 - 3H)}{H^{-3}\cdot(7H^3 - 2H^2 + 4H)} = \frac{2 + 5H^{-1} - 3H^{-2}}{7 - 2H^{-1} + 4H^{-2}}.$$

Second stage H^{-1} and H^{-2} are infinitesimal, so

$$\begin{aligned} st(c) &= st\left(\frac{2 + 5H^{-1} - 3H^{-2}}{7 - 2H^{-1} + 4H^{-2}}\right) = \frac{st(2 + 5H^{-1} - 3H^{-2})}{st(7 - 2H^{-1} + 4H^{-2})} \\ &= \frac{st(2) + st(5H^{-1}) - st(3H^{-2})}{st(7) - st(2H^{-1}) + st(4H^{-2})} = \frac{2 + 0 - 0}{7 - 0 + 0}. \end{aligned}$$

Third stage

$$st(c) = \frac{2 + 0 - 0}{7 - 0 + 0} = \frac{2}{7}.$$

EXAMPLE 4 If ε is infinitesimal but not zero, find the standard part of

$$b = \frac{\varepsilon}{5 - \sqrt{25 + \varepsilon}}.$$

Both the numerator and denominator are nonzero infinitesimals.

First stage We multiply both numerator and denominator by $5 + \sqrt{25 + \varepsilon}$.

$$\begin{aligned} b &= \frac{\varepsilon}{5 - \sqrt{25 + \varepsilon}} = \frac{\varepsilon(5 + \sqrt{25 + \varepsilon})}{(5 - \sqrt{25 + \varepsilon})(5 + \sqrt{25 + \varepsilon})} \\ &= \frac{\varepsilon(5 + \sqrt{25 + \varepsilon})}{25 - (25 + \varepsilon)} = \frac{\varepsilon(5 + \sqrt{25 + \varepsilon})}{-\varepsilon} \\ &= -5 - \sqrt{25 + \varepsilon}. \end{aligned}$$

Second stage

$$\begin{aligned} st(b) &= st(-5 - \sqrt{25 + \varepsilon}) = st(-5) - st(\sqrt{25 + \varepsilon}) \\ &= -5 - \sqrt{st(25 + \varepsilon)} = -5 - \sqrt{25}. \end{aligned}$$

Third stage

$$st(b) = -5 - \sqrt{25} = -10.$$

EXAMPLE 5 Remember that infinite hyperreal numbers do not have standard parts. Consider the infinite hyperreal number

$$\frac{3 + \varepsilon}{4\varepsilon + \varepsilon^2},$$

where ε is a nonzero infinitesimal. The numerator and denominator have standard parts

$$st(3 + \varepsilon) = 3, \qquad st(4\varepsilon + \varepsilon^2) = 0.$$

However, the quotient has no standard part. In other words,

$$st\left(\frac{3 + \varepsilon}{4\varepsilon + \varepsilon^2}\right) \text{ is undefined.}$$

PROBLEMS FOR SECTION 1.6

Compute the standard parts of the following.

1 $\quad 2 + \varepsilon + 3\varepsilon^2$, $\quad\varepsilon$ infinitesimal

2 $b + 2\varepsilon - \varepsilon^2$, $st(b) = 5$, ε infinitesimal

3 $\dfrac{2 - 3\varepsilon}{5 + 4\varepsilon}$, ε infinitesimal

4 $y^4 + 2y^2 \Delta y + \Delta y^3$, y real, Δy infinitesimal

5 $(x^2 + 3x\,\Delta x + \Delta x^2)^6$, x real, Δx infinitesimal

6 $\sqrt{x + \Delta x} + \sqrt{x - \Delta x}$, x positive real, Δx infinitesimal

7 $\dfrac{\varepsilon^3 - \varepsilon^2 + 4\varepsilon}{3\varepsilon^2 + 2\varepsilon - 3}$, ε infinitesimal

8 $\dfrac{\varepsilon^4 - \varepsilon^3 + \varepsilon^2}{2\varepsilon^2}$, $\varepsilon \neq 0$ infinitesimal

9 $\dfrac{4\varepsilon^4 - 3\varepsilon^3 + 2\varepsilon^2}{3\varepsilon^4 - 2\varepsilon^3 + \varepsilon^2}$, $\varepsilon \neq 0$ infinitesimal

10 $(2 + \varepsilon + \delta)(3 - \varepsilon\delta)$, ε, δ infinitesimal

11 $\sqrt{a + \varepsilon}\sqrt{a + \delta}$, $st(a) = 3$, ε, δ infinitesimal

12 $\dfrac{2H + 4}{3H - 6}$, H infinite

13 $\dfrac{6H - 7}{H^2 + 2}$, H infinite

14 $\dfrac{3H^2 - 5H + 2}{H^2 + 1}$, H infinite

15 $\dfrac{H + 1 + \varepsilon}{2H - 1 + 3\varepsilon}$, H infinite, ε infinitesimal

16 $\dfrac{H^4 + 3H^2 + 1}{4H^4 + 2H^2 - 1}$, H infinite

17 $\dfrac{2b^2 + c + 1}{3c^2 + 6b + 1}$, $st(b) = 2$, $st(c) = -1$

18 $\sqrt{b^2 + bc + b - c}$, $st(b) = 3$, $st(c) = 2$

19 $\dfrac{(x + \varepsilon)(y + \varepsilon) - xy}{\varepsilon}$, x, y real, $\varepsilon \neq 0$ infinitesimal

20 $\dfrac{(x + \Delta x)^2 - x^2}{\Delta x}$, x real, $\Delta x \neq 0$ infinitesimal

21 $\dfrac{(x + \Delta x)^3 - x^3}{\Delta x}$, x real, $\Delta x \neq 0$ infinitesimal

22 $\dfrac{1/(a + \varepsilon) - 1/a}{\varepsilon}$, $a \neq 0$ real, $\varepsilon \neq 0$ infinitesimal

23 $\dfrac{b^2 - 25}{b - 5}$, $b \neq 5$ and $st(b) = 5$

24 $\dfrac{4 - a}{2 - \sqrt{a}}$, $a \neq 4$ and $st(a) = 4$

25 $\dfrac{3 - \sqrt{c + 2}}{c - 7}$, $c \neq 7$ and $st(c) = 7$

26 $\dfrac{3 - \sqrt{c + 2}}{c - 7}$, $st(c) = 5$

27 $\dfrac{a^2 - 5a + 6}{a - 3}$, $a \neq 3$ and $st(a) = 3$

28 $\dfrac{2b^2 - b - 6}{b^2 - 3b + 2}$, $b \neq 2$ and $st(b) = 2$

29 $\dfrac{c^2 + 5c + 6}{c^2 + 4c + 3}$, $c \neq -3$ and $st(c) = -3$

30 $\dfrac{\sqrt{25 - \varepsilon} - 5}{\varepsilon}$, $\varepsilon \neq 0$ and ε infinitesimal

31 $\dfrac{1}{\varepsilon}\left(\dfrac{1}{\sqrt{4 + \varepsilon}} - \dfrac{1}{2}\right)$, $\varepsilon \neq 0$ and ε infinitesimal

32 $2H\left(\sqrt{1 + \dfrac{1}{H}} - 1\right)$, H positive infinite

33 $\dfrac{\sqrt{H + 1}}{\sqrt{2H} + \sqrt{H - 1}}$, H positive infinite

34 $\sqrt{H^2 + H + 1} - H$, H positive infinite

In the following problems let a, b, a_1, b_1 be hyperreal numbers with $a \approx a_1, b \approx b_1$.

□ 35 Show that $a + b \approx a_1 + b_1$.
Hint: Put $a_1 = a + \varepsilon$, $b_1 = b + \delta$, and compute the difference $(a_1 + b_1) - (a + b)$.

□ 36 Show that if a, b are finite, then $ab \approx a_1 b_1$.

□ 37 Show that if $a = b = H$, $a_1 = b_1 = H + 1/H$, then $ab \not\approx a_1 b_1$. (H positive infinite).

EXTRA PROBLEMS FOR CHAPTER 1

1 Find the distance between the points $P(2, 7)$ and $Q(1, -4)$.

2 Find the slope of the line through the points $P(2, -6)$ and $Q(3, 4)$.

3 Find the slope of the line through $P(3, 5)$ and $Q(6, 0)$.

4 Find the equation of the line through $P(4, 4)$ and $Q(5, 9)$.

5 Find the equation of the line through $P(4, 5)$ with slope $m = -2$.

6 Find the velocity and equation of motion of a particle which moves with constant velocity and has positions $y = 2$ at $t = 0$, $y = 5$ at $t = 2$.

7 Find the equation of the circle with radius $\sqrt{5}$ and center at $(1, 3)$.

8 Find the equation of the circle that has center $(1, 0)$ and passes through the point $(0, 1)$.

Let ε be positive infinitesimal. Determine whether the following are infinitesimal, finite but not infinitesimal, or infinite.

9 $(4\varepsilon + 5)(2\varepsilon + 6)$

10 $(4\varepsilon + 5)(\varepsilon^2 - \varepsilon)$

11 $1/\varepsilon - 2/\varepsilon^2$

12 $1 - \sqrt{1 - \varepsilon}$

Let H be positive infinite. Determine whether the following are infinitesimal, finite but not infinitesimal, or infinite.

13 $(H - 2)(2H + 5)$

14 $\dfrac{H - 2}{2H + 5}$

15 $\dfrac{H + 6}{H^2 - 2}$

16 $\sqrt{H^2 + 1} - H$

Compute the standard parts in Problems 17–22.

17 $(b + 2\varepsilon)(3b - 4\varepsilon)$, $st(b) = 4$, ε infinitesimal

18 $\sqrt{2 + \varepsilon + 3\varepsilon^2}$, ε infinitesimal

19 $\dfrac{4 + 5\varepsilon}{7 - 3\varepsilon^2}$, ε infinitesimal

20 $\left(\dfrac{1}{3} - \dfrac{1}{3 + \Delta x}\right)\left(\dfrac{1}{\Delta x}\right)$, $0 \neq \Delta x$ infinitesimal

21 $\dfrac{(3H + 4)(5K + 6)}{(H + 1)(1 - 4K)}$, H, K positive infinite

22 $(\sqrt{H^2 + 4} - H)H$, H positive infinite

23 If $f(x) = 1/\sqrt{x}$, find $f(x + \Delta x) - f(x)$.

24 What is the domain of the function $f(x) = \dfrac{1}{x(x + 1)(x + 2)}$?

□ 25 Show that if $a < b$, then $(a + b)/2$ is between a and b; that is, $a < (a + b)/2 < b$.

□ 26 Show that every open interval has infinitely many points.

□ 27 The *union* of two sets X and Y, $X \cup Y$, is the set of all x such that x is either in X or Y or both. Prove that the union of two bounded sets is bounded.

28 The *intersection* of X and Y, $X \cap Y$, is the set of all x such that x is in both X and Y. Prove that the intersection of two closed intervals is either empty or is a closed interval.

□ 29 Prove that the intersection of two open intervals is either empty or is an open interval.

□ 30 Prove that two (real) straight lines with different slopes intersect.

□ 31 Prove that if H is infinite, then $1/H$ is infinitesimal.

□ 32 Prove that if H is infinite and b is finite, then $H + b$ is infinite.

□ 33 Prove that if ε is positive infinitesimal, so is $\sqrt[n]{\varepsilon}$.

□ 34 Prove that if a, b are not infinitesimal and $a \approx b$, then $1/a \approx 1/b$.

□ 35 Prove that if a is finite, then $st(|a|) = |st(a)|$.

□ 36 Suppose a is finite, r is real, and $st(a) < r$. Prove that $a < r$.

□ 37 Suppose a and b are finite hyperreal numbers with $st(a) < st(b)$. Prove that there is a real number r with $a < r < b$.

□ 38 Suppose that f is a real function.

Show that the set of real solutions of the equation $f(x) = 0$ is bounded if and only if every hyperreal solution of $f^*(x) = 0$ is finite.