11

PARTIAL DIFFERENTIATION

11.1 SURFACES

The rectangular coordinate axes in (x, y, z) space are drawn as in Figure 11.1.1. Points in real space are identified with triples (x, y, z) of real numbers, and points in hyperreal space with triples (x, y, z) of hyperreal numbers. The set of all points for which an equation is true is called the *graph*, or *locus*, of the equation. The graph of an equation in the three variables x, y, and z is a surface in space. We have seen in the last chapter that the graph of a linear equation

$$ax + by + cz = d$$

is a plane. The graphs of other equations are often curved surfaces. The simplest planes are:

The vertical planes $x = x_0$ perpendicular to the x-axis. The plane $x = 0$ is called the *(y, z) plane.*

The vertical planes $y = y_0$ perpendicular to the y-axis. The plane $y = 0$ is called the *(x, z) plane.*

The horizontal planes $z = z_0$ perpendicular to the z-axis. The plane $z = 0$ is called the *(x, y) plane.*

Examples of the planes $x = x_0$, $y = y_0$ and $z = z_0$ are pictured in Figure 11.1.2.

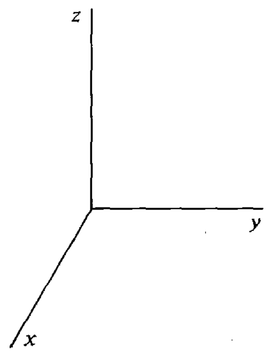


Figure 11.1.1

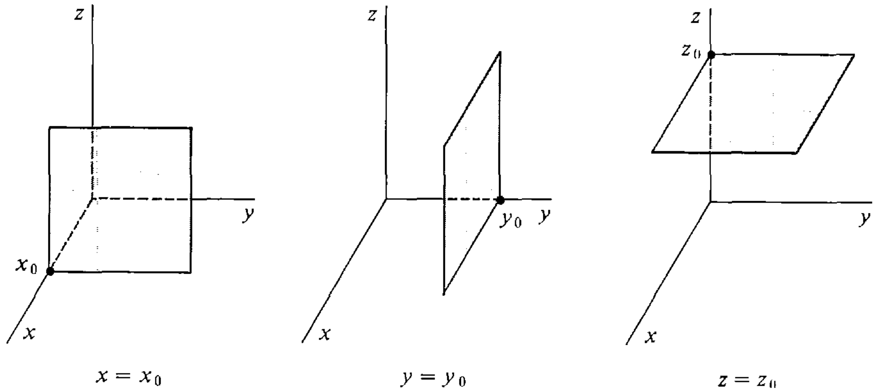


Figure 11.1.2

By the *graph* of a function f of two variables we mean the graph of the equation $z = f(x, y)$. Recall that a real function of two variables is a set of ordered triples (x, y, z) such that for each (x, y) there is at most one z with $z = f(x, y)$. Geometrically this means that the graph of a function intersects each vertical line through (x, y) in at most one point (x, y, z). The value of z is the height of the surface above (x, y). Figure 11.1.3 shows part of a surface $z = f(x, y)$.

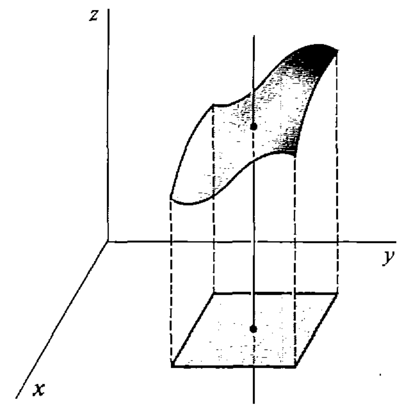


Figure 11.1.3

Whenever one quantity depends on two others we have a function of two variables. The height of a surface above (x, y) is one example. A few other examples are: the density of a plane object at (x, y), the area of a rectangle of length x and width y, the size of a wheat crop in a season with rainfall r and average temperature t, the number of items which can be sold if the price is p and the advertising budget is a, and the force of the sun's gravity on an object of mass m at distance d.

A rough sketch of the graph can be very helpful in understanding a function of two variables or an equation in three variables. In this section we do two things. First we describe a class of surfaces whose equations are simple and easily recognized, the quadric surfaces. After that we shall give a general method for sketching the graph of an equation. Graph paper with lines in the x, y, and z directions is available in many bookstores.

The graph of a second degree equation in x, y, and z is called a *quadric surface*. These surfaces correspond to the conic sections in the plane. There are several types of quadric surfaces. We shall present each of them in its simplest form.

Quadric Cylinders If z does not appear in an equation, its graph will be a cylinder parallel to the z-axis. The cylinder is generated by a line parallel to the z-axis moving along a curve in the plane $z = 0$.

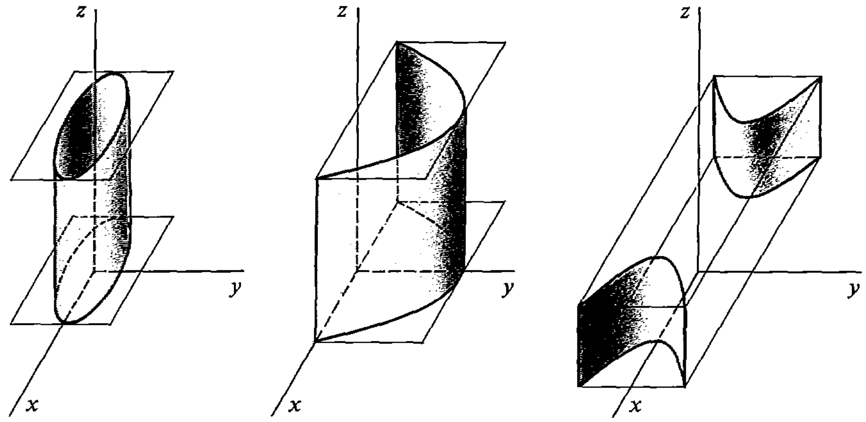
The graph (in space) of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an *elliptic cylinder*.

It intersects any horizontal plane $z = z_0$ in an ellipse.

The graph of $y = ax^2 + bx + c$ is a *parabolic cylinder*.

The graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = c$ is a *hyperbolic cylinder*.

Cylinders parallel to other axes are similar. The three types of quadric cylinders are shown in Figure 11.1.4.



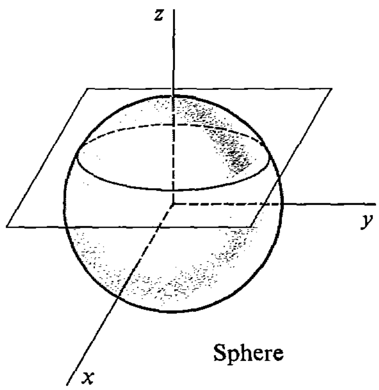
(a) Elliptic cylinder (b) Parabolic cylinder (c) Hyperbolic cylinder

Figure 11.1.4

The Sphere The sphere of radius r and center $P(a, b, c)$ has the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

It is the set of all points at distance r from P (Figure 11.1.5). A sphere intersects any plane in a circle (possibly a single point or no intersection).



Sphere

Figure 11.1.5

The Ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

This egg-shaped surface intersects a plane perpendicular to any coordinate axis in an ellipse (Figure 11.1.6). It is inscribed in the rectangular solid

$$-a \le x \le a, \qquad -b \le y \le b, \qquad -c \le z \le c.$$

The Elliptic Cone $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}.$

This surface intersects a horizontal plane $z = z_0$ in an ellipse, and the vertical planes $x = 0$ and $y = 0$ in two intersecting lines (Figure 11.1.7).

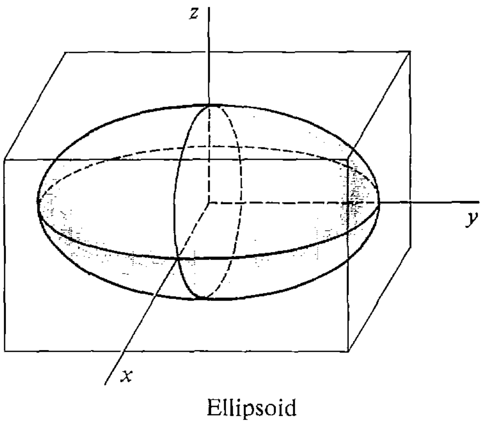


Figure 11.1.6

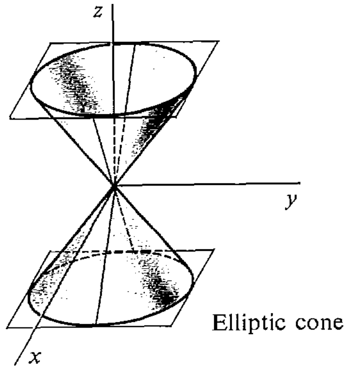


Figure 11.1.7

The Elliptic Paraboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}.$

This surface intersects a horizontal plane $z = z_0$ in an ellipse and a vertical plane $x = x_0$ or $y = y_0$ in a parabola. It is shaped like a bowl if c is positive and a mound if c is negative (Figure 11.1.8).

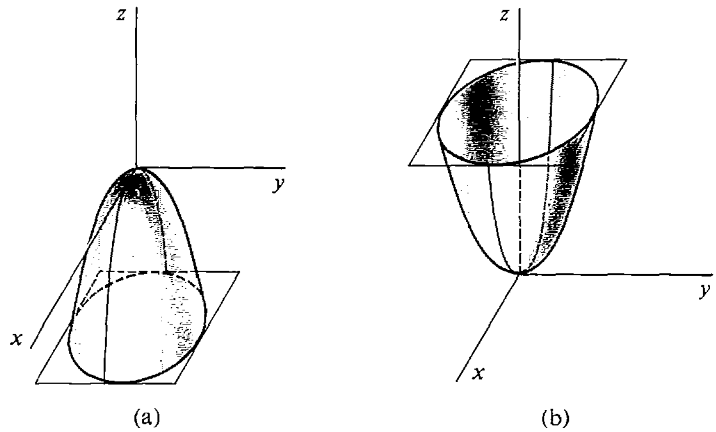


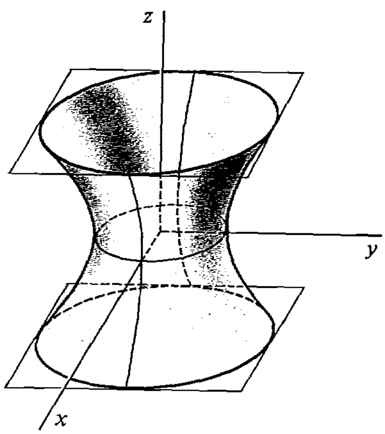
Figure 11.1.8

The Hyperboloid of One Sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

The intersection of this surface with a horizontal plane $z = z_0$ is an ellipse. The intersection with a vertical plane $x = x_0$ or $y = y_0$ is a hyperbola (Figure 11.1.9).

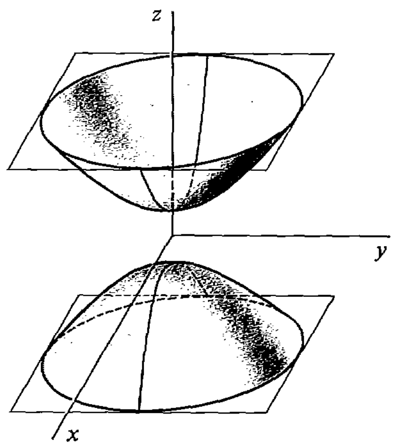
The Hyperboloid of Two Sheets $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

The surface has an upper sheet with $z \geq c$ and a lower sheet with $z \leq -c$. It intersects a horizontal plane $z = z_0$ in an ellipse if $|z_0| > c$. It intersects a vertical plane $x = x_0$ or $y = y_0$ in a hyperbola (Figure 11.1.10).



Hyperboloid of one sheet

Figure 11.1.9



Hyperboloid of two sheets

Figure 11.1.10

The Hyperbolic Paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}.$

This surface has the shape of a saddle. It intersects a horizontal plane $z = z_0$ in a hyperbola, and a vertical plane $x = x_0$ or $y = y_0$ in a parabola (Figure 11.1.11).

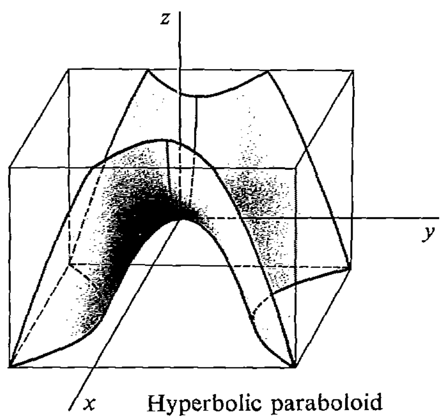


Figure 11.1.11 Hyperbolic paraboloid

We shall describe a method for sketching cylinders and then other graphs in space. We concentrate on a finite portion of (x, y, z) space.

EXAMPLE 1 Sketch the portion of the cylinder $x^2 + y^2 = 1$ where $1 \le z \le 2$ (Figure 11.1.12).

Step 1 Draw the curve $x^2 + y^2 = 1$ in the (x, y) plane. The curve is a circle of radius one.

Step 2 Draw the three coordinate axes and the horizontal planes $z = 1$, $z = 2$.

Step 3 Draw the circles $x^2 + y^2 = 1$ where the surface intersects the two planes $z = 1, z = 2$.

Step 4 Complete the sketch by drawing heavy lines for all edges which would be visible on an "opaque" model of the given surface. This surface is called a *circular cylinder*.

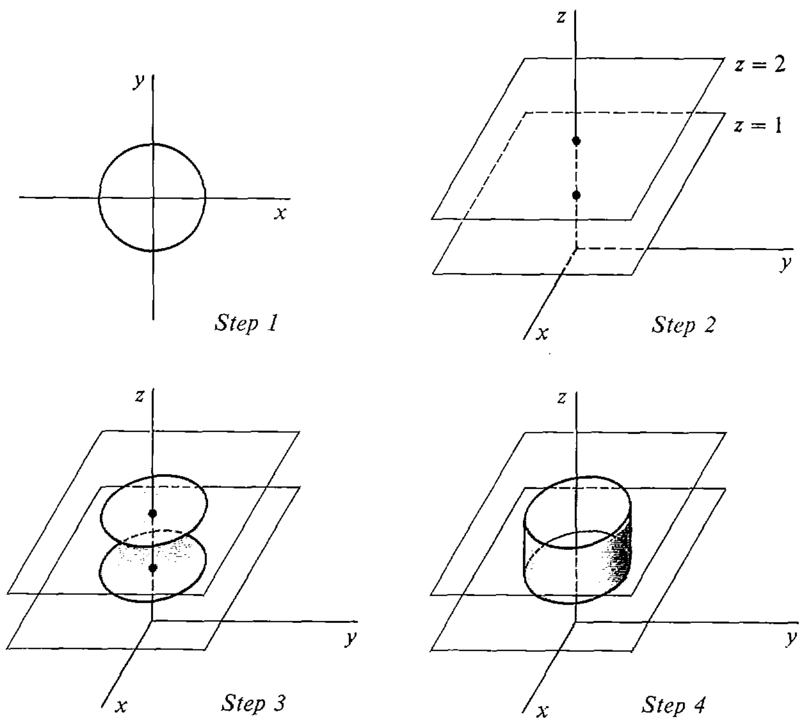


Figure 11.1.12

EXAMPLE 2 Sketch the part of the cylinder $z = x^2$ where $0 \le y \le 2$, $0 \le z \le 1$. This is a parabolic cylinder parallel to the y-axis, because y does not appear in the equation. The four steps are shown in Figure 11.1.13.

For sketching the graph of a function $z = f(x, y)$, a *topographic map*, or *contour map*, can often be used as a first step. It is a method of representing a surface which is often found in atlases. In a topographic map, the curves $f(x, y) = z_0$ are

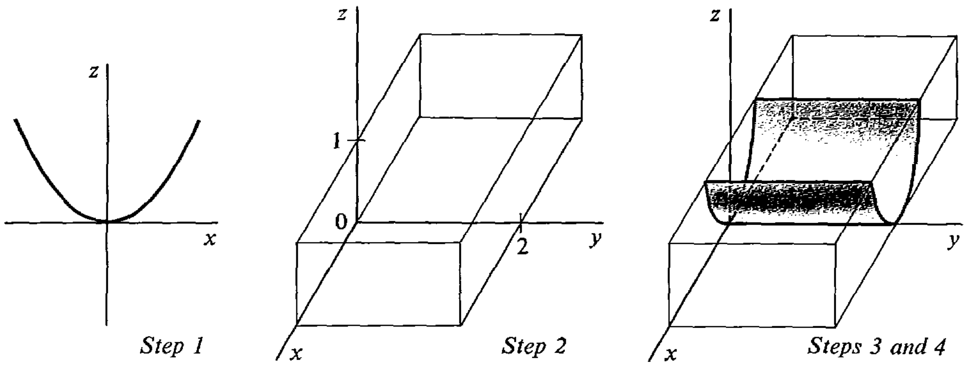


Figure 11.1.13

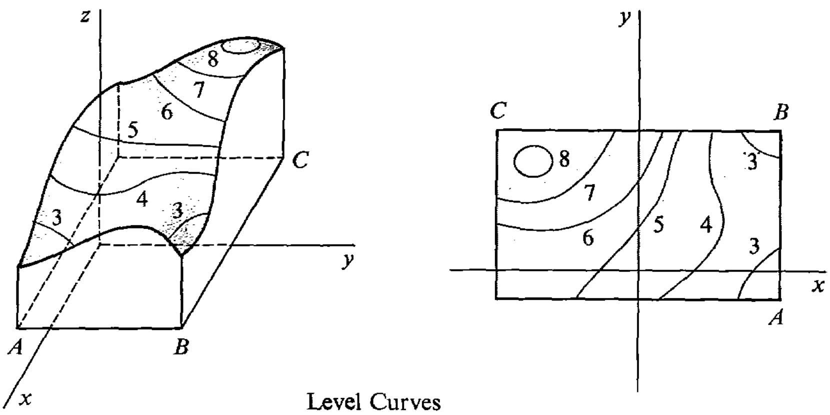


Figure 11.1.14

sketched in the (x, y) plane for several different constants z_0, and each curve is labeled (Figure 11.1.14). These curves are called *level curves*, or *contours*.

EXAMPLE 3 Sketch the part of the surface $z = x^2 + y^2$ where $-1 \le z \le 1$. This is an elliptic paraboloid (Figure 11.1.15).

Step 1 Draw the topographic map. The level curves are circles.

Step 2 Draw the axes and the planes $z = -1$, $z = 1$.

Step 3 Draw the intersections of the surface with the planes $z = -1, z = 1$ and also the planes $x = 0$ and $y = 0$.

$z = -1$:	No intersection.
$z = 1$:	The circle $x^2 + y^2 = 1$.
$x = 0$:	The parabola $z = y^2$.
$y = 0$:	The parabola $z = x^2$.

Step 4 Complete the figure with heavy lines for visible edges.

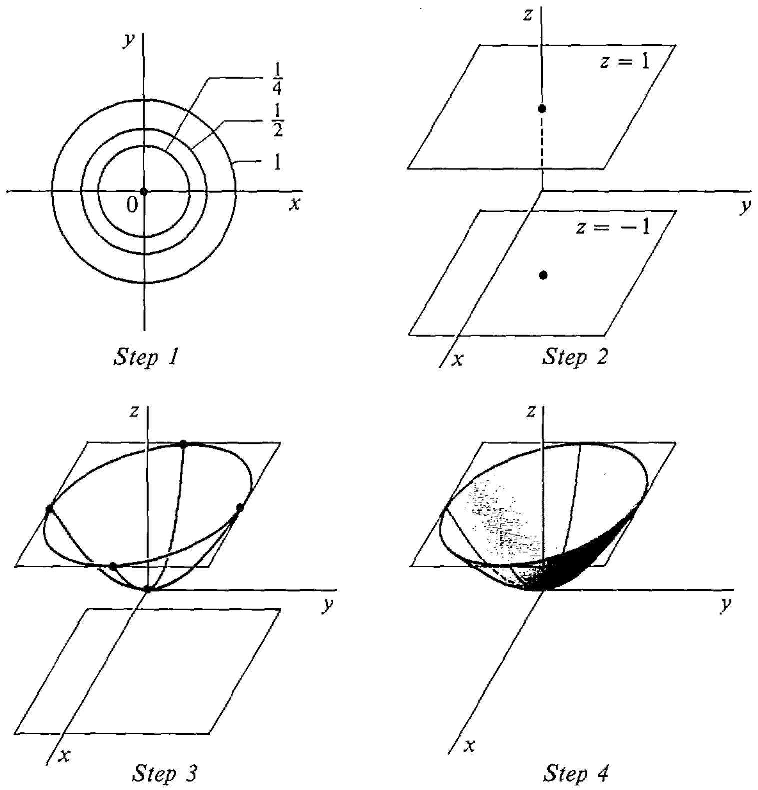


Figure 11.1.15

EXAMPLE 4 Graph the function

$$\frac{x^2}{4} - y^2 = z,$$

where $-3 \le x \le 3$, $-2 \le y \le 2$, $-1 \le z \le 1$. This is a hyperbolic paraboloid (Figure 11.1.16).

Step 1 Draw a topographic map. The level curves are hyperbolas.

Step 2 Draw the axes and rectangular solid.

Step 3 Draw the curves where the surface intersects the faces and also the planes $x = 0$, $y = 0$. The topographic map gives the curves on $z = -1$, $z = 0$, and $z = 1$. The curves on $x = 0$ and $y = 0$ are parabolas.

Step 4 Complete Figure 11.1.16.

EXAMPLE 5 Sketch the surface

$$-x^2 - \frac{y^2}{4} + z^2 = 1$$

where $-2 \le z \le 2$. This is a hyperboloid of two sheets (Figure 11.1.17).

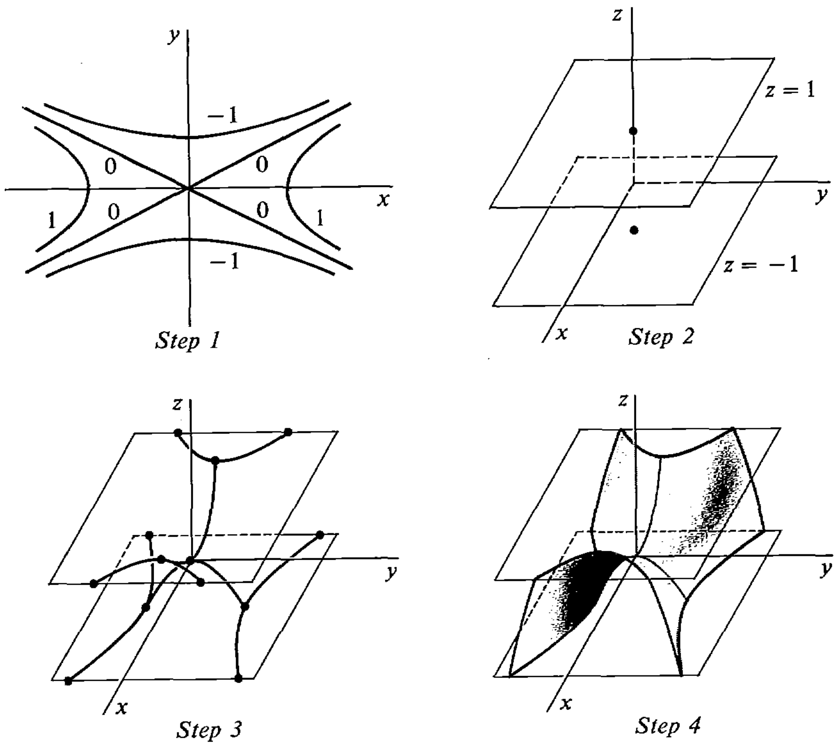


Figure 11.1.16

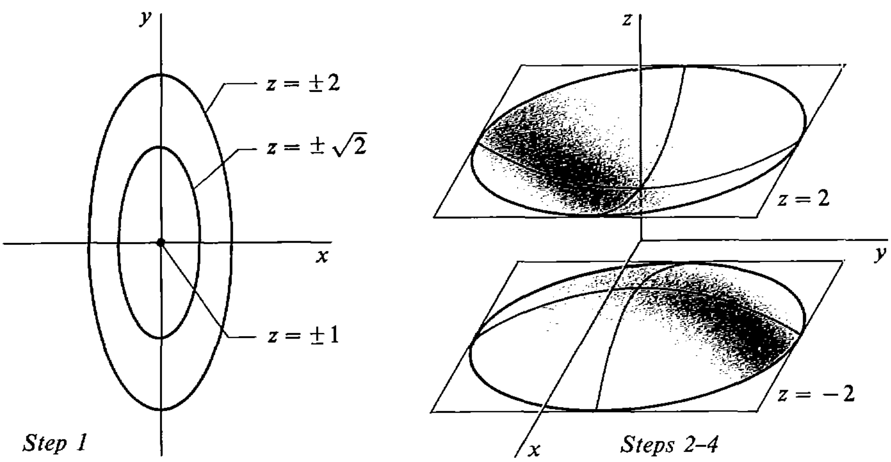


Figure 11.1.17

Although it is not a function, it can be broken up into two functions

$$z = \sqrt{1 + x^2 + \frac{y^2}{4}}, \qquad z = -\sqrt{1 + x^2 + \frac{y^2}{4}}.$$

Step 1 Draw topographic maps for $z = \sqrt{1 + x^2 + y^2/4}$ and $z = -\sqrt{1 + x^2 + y^2/4}$. The level curves are ellipses.

Step 2 Draw the axes and the planes $z = 2$, $z = -2$.

Step 3 Draw the intersections of the surface with the planes

$$z = -2, \qquad z = 2, \qquad x = 0, \qquad y = 0.$$

The surface intersects $x = 0$ and $y = 0$ in the hyperbolas

$$-\tfrac{1}{4}y^2 + z^2 = 1, \qquad -x^2 + z^2 = 1.$$

Step 4 Complete Figure 11.1.17.

EXAMPLE 6 Graph the *sum function* $z = x + y$. The graph is a plane. A topographic map and sketch of the surface are shown in Figure 11.1.18.

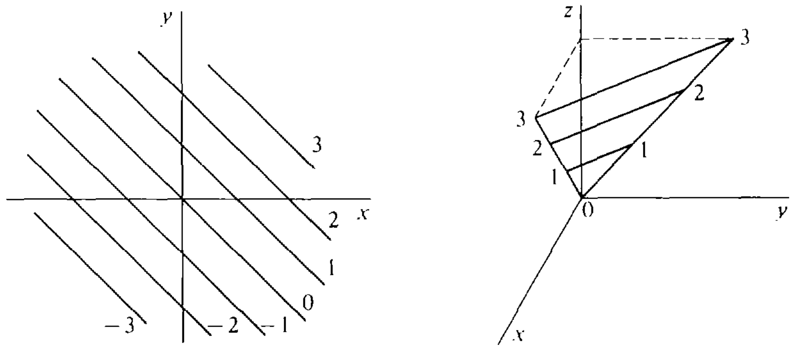


Figure 11.1.18

EXAMPLE 7 Sketch the graph of the *product function* $z = xy$, where

$$-2 \le x \le 2, \qquad -2 \le y \le 2, \qquad -1 \le z \le 1.$$

The surface is saddle shaped. It intersects the horizontal plane $z = z_0$ in the curve $y = z_0/x$. It intersects the vertical planes $x = x_0$ and $y = y_0$ in the lines $z = x_0 y$ and $z = xy_0$. The surface is shown in Figure 11.1.19.

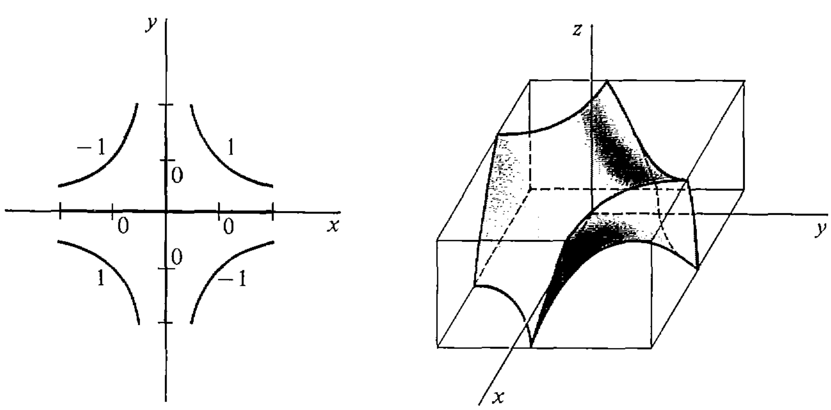


Figure 11.1.19

EXAMPLE 8 Graph the function $z = \sqrt{x} + y^2$ where

$$0 \le x \le 1, \qquad -1 \le y \le 1, \qquad 0 \le z \le 1.$$

Step 1 The topographic map has level curves

$$\sqrt{x} + y^2 = c, \qquad x = (c - y^2)^2 \quad \text{with} \quad y^2 \le c.$$

The derivative $dx/dy = 4y(c - y^2)$ has zeros at $y = 0$ and $y = \pm\sqrt{c}$. The table shows that the curves are bell shaped.

y	x	dx/dy	
$-\sqrt{c}$	0	0	Min
0	c^2	0	Max
$\sqrt{c}$	0	0	Min

Step 2 Draw the rectangular solid.

Step 3 The surface intersects the plane $x = 0$ in the parabola $z = y^2$, and intersects the plane $y = 0$ in the curve $z = \sqrt{x}$. It intersects the plane $z = 1$ in the curve $x = (1 - y^2)^2$.

Step 4 The surface, shown in Figure 11.1.20, is shaped like a beaker spout.

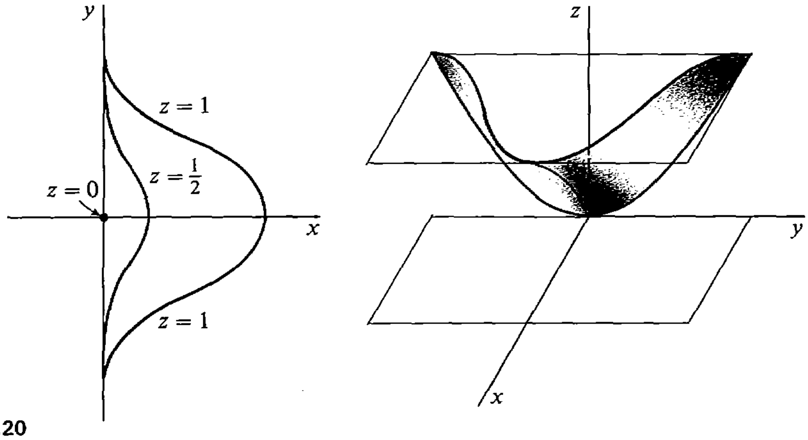


Figure 11.1.20

PROBLEMS FOR SECTION 11.1

Sketch the following graphs in (x, y, z) space.

1 $x^2 + y^2 = 4, \quad -1 \le z \le 1$

2 $x^2 + z^2 = 1, \quad 0 \le y \le 2$

3 $(x - 2)^2 + (y - 1)^2 = 1, \quad -1 \le z \le 1$

4 $(y - 1)^2 + (z + 1)^2 = 1, \quad 0 \le x \le 3$

5 $y^2 + z = 1, \quad 0 \le z, \quad 0 \le x \le 2$

6 $y = x^2 - x, \quad y \le 0, \quad 0 \le z \le 2$

7 $x^2 + \frac{1}{4}y^2 = 1, \quad 0 \le z \le 3$

8 $y^2 - x^2 = 1, \quad -2 \le x \le 2, \quad 0 \le z \le 4$

9 $x = \sin y, \quad 0 \le y \le \pi, \quad 0 \le z \le 2$

10 $z = e^{-x}, \quad 0 \le x \le 2, \quad 0 \le y \le 2$

11 $x^2 + y^2 + z^2 = 4$

12 $(x - 1)^2 + y^2 + (z + 1)^2 = 1$

13 $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 9$

14 $x^2 + (y + 4)^2 + (z - 2)^2 = 4$

15 $x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 = 1$

16 $\frac{1}{4}x^2 + \frac{1}{25}y^2 + z^2 = 1$

Make contour maps and sketch the following surfaces.

17 $x^2 + y^2 = z^2, \quad -4 \le z \le 4$

18 $x^2 + y^2 = 4z^2, \quad -1 \le z \le 1$

19 $x^2 + \frac{1}{4}y^2 = z^2, \quad -4 \le z \le 4$

20 $z = \frac{1}{9}x^2 + y^2, \quad -4 \le z \le 4$

21 $z = -x^2 - y^2, \quad -4 \le z \le 4$

22 $2z = -x^2 - \frac{1}{4}y^2, \quad -4 \le z \le 4$

23 $x^2 + \frac{1}{4}y^2 - z^2 = 1, \quad -4 \le z \le 4$

24 $x^2 + y^2 - 9z^2 = 1, \quad -2 \le z \le 2$

25 $-x^2 - y^2 + z^2 = 1, \quad -4 \le z \le 4$

26 $-4x^2 - y^2 + 4z^2 = 1, \quad -2 \le z \le 2$

27 $z = x^2 - y^2, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

28 $z = y^2 - x^2, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

Make contour maps of the following surfaces.

29 $z = x - y$

30 $z = y - 2x$

31 $z = (x^2 + y^2 + 1)^{-1}, \quad -4 \le x \le 4, \quad -4 \le y \le 4$

32 $z = \dfrac{x^2 + y^2}{x^2 + y^2 + 1}, \quad -4 \le x \le 4, \quad -4 \le y \le 4$

33 $z = x + y^2, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

34 $z = xy^2, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

35 $z = x\sqrt{y}, \quad -2 \le x \le 2, \quad 0 \le y \le 4$

36 $z = \sqrt{x} + \sqrt{y}, \quad 0 \le x \le 4, \quad 0 \le y \le 4$

37 $z = \dfrac{x}{y}, \quad -2 \le x \le 2, \quad -2 \le y \le 2, \quad -4 \le z \le 4$

38 $z = (x + y)^{-1}, \quad -2 \le x \le 2, \quad -2 \le y \le 2, \quad -4 \le z \le 4$

39 $z = \cos x + \sin y, \quad -\pi/2 \le x \le \pi/2, \quad 0 \le y \le \pi$

40 $z = \cos x \cdot \sin y, \quad -\pi/2 \le x \le \pi/2, \quad 0 \le y \le \pi$

41 $z = e^{x+y}, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

42 $z = e^{-x^2-y^2}, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

43 $z = x^y, \quad 0 < x \le 4, \quad -2 \le y \le 2, \quad -4 \le z \le 4$

44 $z = \log_x y, \quad 0 < x \le 4, \quad 0 < y \le 4, \quad -4 \le z \le 4$

11.2 CONTINUOUS FUNCTIONS OF TWO OR MORE VARIABLES

Two points (x_1, y_1) and (x_2, y_2) in the hyperreal plane are said to be *infinitely close*, $(x_1, y_1) \approx (x_2, y_2)$, if both $x_1 \approx x_2$ and $y_1 \approx y_2$. If

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1,$$

then the distance between (x_1, y_1) and (x_2, y_2) is

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}.$$

LEMMA 1

Two points are infinitely close to each other if and only if the distance between them is infinitesimal.

This lemma can be seen from Figure 11.2.1. (An easy proof of the lemma in terms of vectors was given in Section 10.8.)

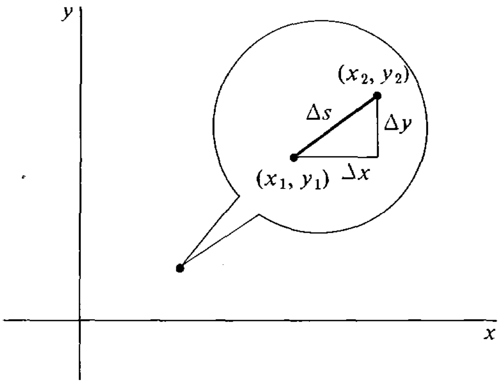


Figure 11.2.1

The definition of a continuous function in two variables is similar to the definition in one variable.

DEFINITION

A real function $f(x, y)$ is said to be ***continuous*** *at a real point (a, b) if whenever (x, y) is infinitely close to (a, b), $f(x, y)$ is infinitely close to $f(a, b)$. In other words,*

$$\text{if } st(x) = a \text{ and } st(y) = b, \text{ then } st(f(x, y)) = f(a, b).$$

Figure 11.2.2 shows (a, b) and $f(a, b)$ under the microscope.

Remark It follows from the definition that if $f(x, y)$ is continuous at (a, b), then $f(x, y)$ is defined at every hyperreal point infinitely close to (a, b). In fact, it can even be proved that $f(x, y)$ is defined at every point in some real rectangle $a_1 < x < a_2, b_1 < y < b_2$ containing (a, b).

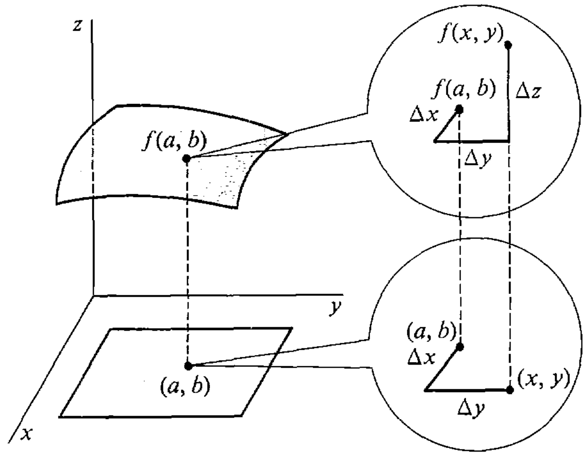


Figure 11.2.2

EXAMPLE 1 Show that $f(x, y) = 2x + xy^2$ is continuous for all (a, b). Let $st(x) = a$ and $st(y) = b$. Then

$$st(2x + xy^2) = st(2x) + st(xy^2) = 2st(x) + st(x)st(y^2) = 2a + ab^2.$$

Here is a list of important continuous functions of two variables.

THEOREM 1

The following are continuous at all real points (x, y) as indicated.

(i) The Sum Function $f(x, y) = x + y$.

(ii) The Difference Function $f(x, y) = x - y$.

(iii) The Product Function $f(x, y) = xy$.

(iv) The Quotient Function $f(x, y) = x/y$, $(y \neq 0)$.

(v) The Exponential Function $f(x, y) = x^y$, $(x > 0)$.

(i)–(iv) follow at once from the corresponding rules for standard parts,

$$\begin{aligned} st(x + y) &= st(x) + st(y), \\ st(x - y) &= st(x) - st(y), \\ st(xy) &= st(x)st(y), \\ st\left(\frac{x}{y}\right) &= \frac{st(x)}{st(y)} \qquad \text{if } st(y) \neq 0. \end{aligned}$$

(v) is equivalent to the new standard parts rule

$$st(x^y) = st(x)^{st(y)} \qquad \text{if } st(x) > 0.$$

We prove this rule using the fact that e^u and $\ln u$ are continuous functions of one variable.

$$st(x^y) = st(e^{y \ln x}) = e^{st(y \ln x)} = e^{st(y)st(\ln x)} = e^{st(y) \ln st(x)} = st(x)^{st(y)}.$$

The next theorem shows that most functions we deal with are continuous.

THEOREM 2

(i) *If $f(x, y)$ is continuous at (a, b) and $g(u)$ is continuous at $f(a, b)$, then*

$$h(x, y) = g(f(x, y))$$

is continuous at (a, b).

(ii) *Sums, differences, products, quotients, and exponents of continuous functions are continuous.*

PROOF (i) If $(x, y) \approx (a, b)$ then $f(x, y) \approx f(a, b)$, hence $g(f(x, y)) \approx g(f(a, b))$, and thus $h(x, y) \approx h(a, b)$.

(ii) Let $f(x, y)$ and $g(x, y)$ be continuous at (a, b). As an illustration we show that if $f(x, y) > 0$ then

$$h(x, y) = f(x, y)^{g(x,y)}$$

is continuous at (a, b). Let $(x, y) \approx (a, b)$. Then

$$st(h(x, y)) = st(f(x, y)^{g(x,y)}) = st(f(x, y))^{st(g(x,y))} = f(a, b)^{g(a,b)} = h(a, b).$$

EXAMPLE 2 By (i), $h(x, y) = \sin(x + y)$ is continuous for all (x, y).

EXAMPLE 3 By (ii), $h(x, y) = \sin x \cos y$ is continuous for all (x, y).

A function is said to be *continuous on a set S* of points in the plane if it is continuous at every point in S. Thus the quotient function $f(x, y) = x/y$ is continuous on the set of all (x, y) such that $y \neq 0$. The function $f(x, y) = x^y$ is continuous on the set of all (x, y) such that $x > 0$.

EXAMPLE 4 Find a set on which $h(x, y) = \ln(x + y)$ is continuous.

By Theorems 1 and 2,

$x + y$ is continuous for all (x, y),
$\ln u$ is continuous for $u > 0$,
$\ln(x + y)$ is continuous for $x + y > 0$.

Answer $\ln(x + y)$ is continuous on the set of all (x, y) such that $x + y > 0$, shown in Figure 11.2.3.

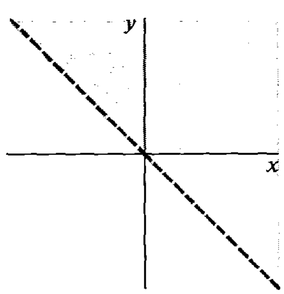


Figure 11.2.3

EXAMPLE 5 Find a set on which $h(x, y) = x^y + \cos\sqrt{x^2 - y}$ is continuous.

x^y is continuous for $x > 0$.
x^2 is continuous for all x.
$x^2 - y$ is continuous for all (x, y).

$\sqrt{x^2 - y}$ is continuous for $x^2 - y > 0$.

$\cos\sqrt{x^2 - y}$ is continuous for $x^2 - y > 0$.

$x^y + \cos\sqrt{x^2 - y}$ is continuous for $x > 0$ and $x^2 - y > 0$.

Answer $h(x, y)$ is continuous on the set of all (x, y) such that $x > 0$ and $x^2 - y > 0$. The set is shown in Figure 11.2.4.

EXAMPLE 6 Find a set on which $h(x, y) = \log_x y$ is continuous.

We use the identity $$\log_x y = \frac{\ln y}{\ln x}.$$

$\ln y$ is continuous for $y > 0$.
$\ln x$ is continuous for $x > 0$.
$\ln y/\ln x$ is continuous for $x > 0,\quad \ln x \neq 0, y > 0$,
that is, $x > 0,\quad x \neq 1,\quad y > 0$.
$\log_x y$ is continuous for $x > 0,\quad x \neq 1,\quad y > 0$.

Answer $\log_x y$ is continuous on the set of all (x, y) such that $x > 0$, $x \neq 1$, $y > 0$ (Figure 11.2.5).

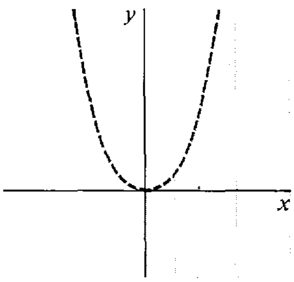


Figure 11.2.4

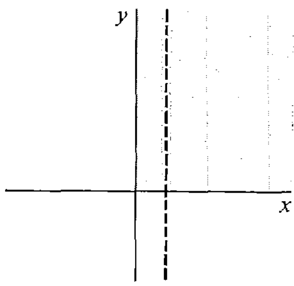


Figure 11.2.5

Continuous functions of three or more variables are defined in the natural way, and Theorem 2 holds for such functions.

EXAMPLE 7 Find a set where the function

$$h(x, y, z) = \frac{x^2 y}{x + y + z}$$

is continuous.

x^2 is continuous for all x.
$x^2 y$ is continuous for all (x, y).

$x + y$ is continuous for all (x, y).

$(x + y) + z$ is continuous for all (x, y, z).

$\dfrac{x^2 y}{x + y + z}$ is continuous for $x + y + z \neq 0$.

Answer $h(x, y, z)$ is continuous on the set of all (x, y, z) such that $x + y + z \neq 0$.

PROBLEMS FOR SECTION 11.2

Find the largest set you can in which the following functions are continuous.

1 $f(x, y) = 2x - 3y$

2 $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$

3 $f(x, y) = e^{x^2 - y}$

4 $f(x, y) = \dfrac{1}{2 + \sin(xy)}$

5 $f(x, y) = \dfrac{xy}{x + y}$

6 $f(x, y) = \dfrac{1}{x^2 + y^2}$

7 $f(x, y) = \dfrac{x^3}{y + 2}$

8 $f(x, y) = \dfrac{x + y}{xy}$

9 $f(x, y) = \dfrac{1}{(x - 2)(y + 1)}$

10 $f(x, y) = \sqrt{x} + \sqrt{y}$

11 $f(x, y) = \sqrt{x + y}$

12 $f(x, y) = \sqrt{x^2 + y^2}$

13 $f(x, y) = \sqrt{x - y}$

14 $f(x, y) = \dfrac{\sqrt{y}}{\sqrt{x + 2y}}$

15 $f(x, y) = x^{x + y}$

16 $f(x, y) = y^{\sin x}$

17 $f(x, y) = (x^2 - y)^x$

18 $f(x, y) = y^{1/x}$

19 $f(x, y) = x^{y^x}$

20 $f(x, y) = \dfrac{1}{1 - x^y}$

21 $f(x, y) = \ln(x^2 - y)$

22 $f(x, y) = \ln(xy)$

23 $f(x, y) = \dfrac{1}{\ln x + \ln y}$

24 $f(x, y) = \ln(\ln(x - y))$

25 $f(x, y) = \log_{x+y}(xy)$

26 $f(x, y) = \log_{2x-y}(x + 3y)$

27 $\dfrac{\sqrt{y^2 - x}}{x - 4y}$

28 $\dfrac{1}{\sin x \cos y}$

29 $\sqrt{\cos x + y}$

30 $\sqrt{|x| + |y|}$

31 $\ln|x - y|$

32 $x^y + y^z$

33 $\dfrac{\sqrt{x - y}}{y - z}$

34 $\dfrac{1}{x + 2y + 3z}$

35 $\dfrac{1}{x^2 + y^2 + z^2}$

36 $\log_x(y + z)$

37 $(x + y)^{1/z}$

□ 38 Let

$$f(x, y) = \begin{cases} 0 & \text{if } xy = 0, \\ 1 & \text{if } xy \neq 0. \end{cases}$$

Show that f is not continuous at $(0, 0)$.

□ **39** Suppose $f(x, y)$ is continuous at (a, b). Prove that $g(x) = f(x, b)$ is continuous at $x = a$.

□ **40** Prove that if $f(x)$ and $g(x)$ are continuous at $x = a$ and if $h(u, v)$ is continuous at $(f(a), g(a))$, then

$$k(x) = h(f(x), g(x))$$

is continuous at $x = a$.

□ **41** Prove that if $f(x, y)$ and $g(x, y)$ are both continuous at (a, b) and if $h(u, v)$ is continuous at $(f(a, b), g(a, b))$, then

$$k(x, y) = h(f(x, y), g(x, y))$$

is continuous at (a, b).

The notation
$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$
means that whenever (x, y) is infinitely close to but not equal to (a, b), $f(x, y)$ is infinitely close to L.

□ **42** Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{|x| + |y|}$.

□ **43** Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} (1 + x^2 + y^2)^{1/(x^2+y^2)}$.

□ **44** Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$.

□ **45** Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{x^2 + y^2}}$.

□ **46** Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2 + y^2}}$ does not exist.

11.3 PARTIAL DERIVATIVES

Partial derivatives are used to study the rates of change of functions of two or more variables. In general the rate of change of $z = f(x, y)$ will depend both on the rate of change of x and the rate of change of y. Partial derivatives deal with the simplest case, where only one of the independent variables is changing and the other is held constant.

Given a function $z = f(x, y)$, if we hold y fixed at some constant value b we obtain a function

$$g(x) = f(x, b)$$

of x only. Geometrically the curve $z = g(x)$ is the intersection of the surface $z = f(x, y)$ with the vertical plane $y = b$. The rate of change of z with respect to x with y held constant is the slope of the curve $z = g(x)$. This slope is called the partial derivative of $f(x, y)$ with respect to x (Figure 11.3.1(a)). There is also a partial derivative with respect to y (Figure 11.3.1(b)).

Here is a precise definition.

DEFINITION

*The **partial derivatives** of $f(x, y)$ at the point (a, b) are the limits*

$$f_x(a, b) = \lim_{\Delta x\to 0} \frac{f(a + \Delta x, b) - f(a, b)}{\Delta x},$$

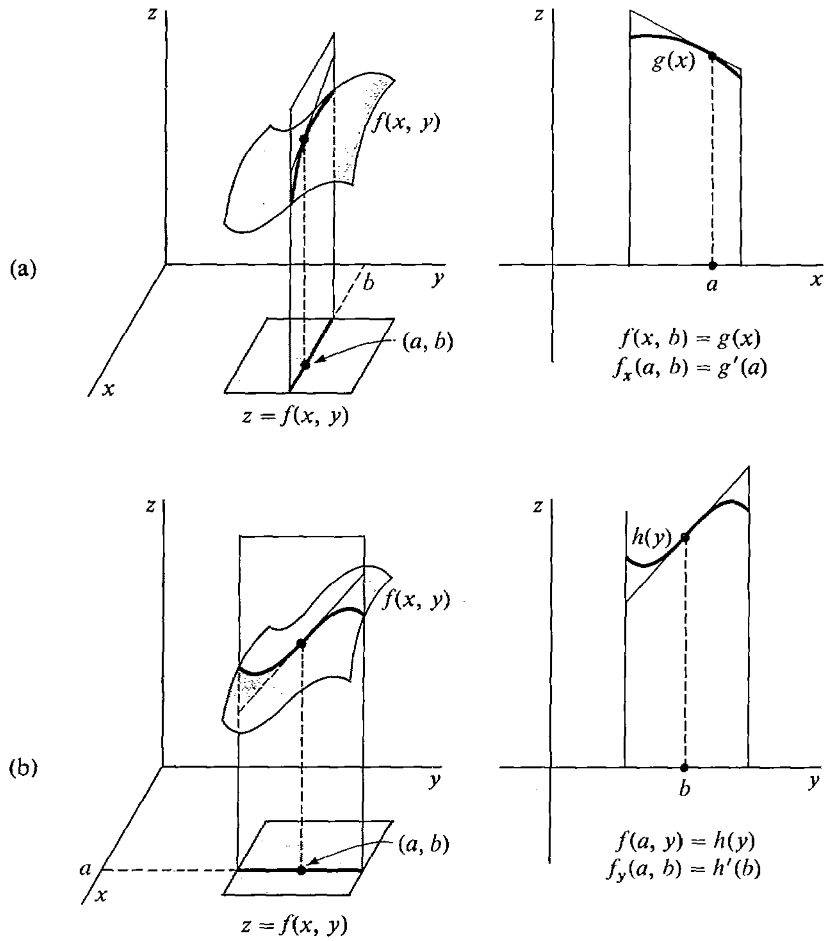


Figure 11.3.1 Partial Derivatives

$$f_y(a, b) = \lim_{\Delta y \to 0} \frac{f(a, b + \Delta y) - f(a, b)}{\Delta y}.$$

A partial derivative is undefined if the limit does not exist.

When $f_x(a, b)$ exists, it is equal to the standard part

$$f_x(a, b) = st\left(\frac{f(a + \Delta x, b) - f(a, b)}{\Delta x}\right)$$

for any nonzero infinitesimal Δx. Similarly when $f_y(a, b)$ exists,

$$f_y(a, b) = st\left(\frac{f(a, b + \Delta y) - f(a, b)}{\Delta y}\right)$$

for any nonzero infinitesimal Δy.

Just as the one-variable derivative $f'(x)$ is a function of x, the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ are again functions of x and y. At each point (x, y), the partial derivative $f_x(x, y)$ either has exactly one value or is undefined.

Another convenient notation for the partial derivatives uses the Cyrillic lower case D, ∂, called a "round d". If $z = f(x, y)$, we use:

$$\frac{\partial z}{\partial x}(x, y), \quad \frac{\partial z}{\partial x}, \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{for } f_x(x, y),$$

$$\frac{\partial z}{\partial y}(x, y), \quad \frac{\partial z}{\partial y}, \quad \text{or} \quad \frac{\partial f}{\partial y} \quad \text{for } f_y(x, y).$$

Partial derivatives, like ordinary derivatives, may be represented as quotients of infinitesimals.

In $\partial z/\partial x$, ∂x means Δx and ∂z means $f_x(x, y)\,\Delta x$.

In $\partial z/\partial y$, ∂y means Δy and ∂z means $f_y(x, y)\,\Delta y$.

Notice that ∂z has a different meaning in $\partial z/\partial x$ than it has in $\partial z/\partial y$. For this reason we shall avoid using the symbol ∂z alone.

Partial derivatives are easily computed using the ordinary rules of differentiation with all but one variable treated as a constant.

EXAMPLE 1 Find the partial derivatives of the function

$$f(x, y) = x^2 + 3xy - 8y$$

at the point $(2, -1)$.

To find $f_x(x, y)$, we treat y as a constant,

$$f_x(x, y) = 2x + 3y.$$

To find $f_y(x, y)$, we treat x as a constant,

$$f_y(x, y) = 3x - 8.$$

Thus $\quad f_x(2, -1) = 2 \cdot 2 + 3(-1) = 1, \qquad f_y(2, -1) = 3 \cdot 2 - 8 = -2.$

Figure 11.3.2 shows the surface $z = f(x, y)$ and the tangent lines at the point $(2, -1)$.

EXAMPLE 2 A point $P(x, y)$ has distance $z = \sqrt{x^2 + y^2}$ from the origin (Figure 11.3.3). Find the rate of change of z at $P(3, 4)$ if:

(a) P moves at unit speed in the x direction.
(b) P moves at unit speed in the y direction.

In this problem the round d notation is convenient.

(a) $$\frac{\partial z}{\partial x}(x, y) = \frac{x}{\sqrt{x^2 + y^2}},$$

$$\frac{\partial z}{\partial x}(3, 4) = \frac{3}{\sqrt{3^2 + 4^2}} = \frac{3}{5}.$$

(b) $$\frac{\partial z}{\partial y}(x, y) = \frac{y}{\sqrt{x^2 + y^2}},$$

$$\frac{\partial z}{\partial y}(3, 4) = \frac{4}{\sqrt{3^2 + 4^2}} = \frac{4}{5}.$$

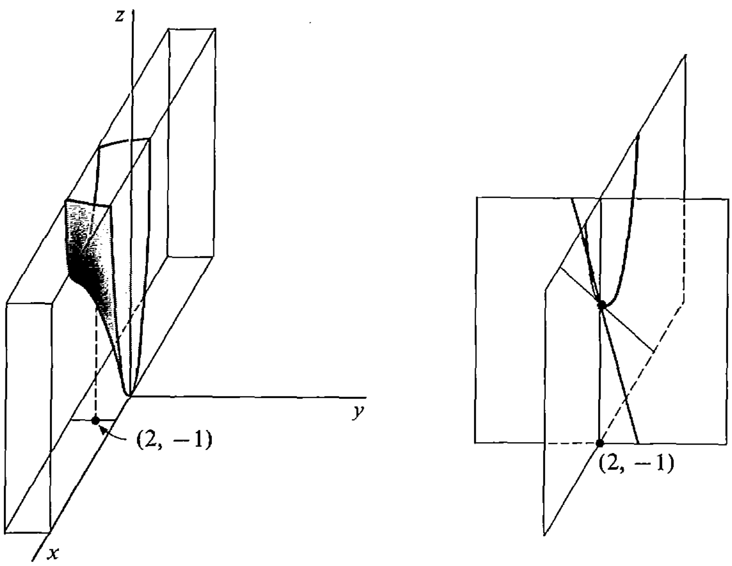


Figure 11.3.2

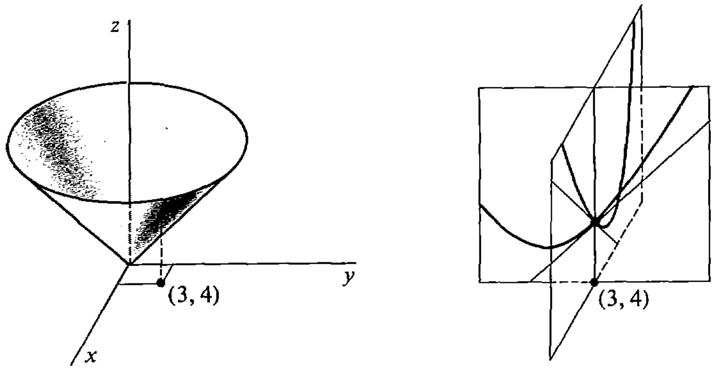


Figure 11.3.3

Functions of three or more variables cannot easily be represented graphically. However, they can be given other physical interpretations. For example, $w = f(x, y, t)$ may be pictured as a moving surface in (x, y, w) space where t is time. Alternatively, $w = f(x, y, z)$ may be thought of as assigning a number to each point of (x, y, z) space where it is defined; for example, w could be the density of a three-dimensional object at the point (x, y, z).

Partial derivatives of functions of three or more variables are defined in a manner analogous to the two-variable case.

DEFINITION

The partial derivatives of $f(x, y, z)$ at the point (a, b, c) are the limits

$$f_x(a, b, c) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x, b, c) - f(a, b, c)}{\Delta x},$$

$$f_y(a, b, c) = \lim_{\Delta y \to 0} \frac{f(a, b + \Delta y, c) - f(a, b, c)}{\Delta y},$$

$$f_z(a, b, c) = \lim_{\Delta z \to 0} \frac{f(a, b, c + \Delta z) - f(a, b, c)}{\Delta z}.$$

A partial derivative is undefined if the limit does not exist.

When $f_x(a, b, c)$ exists we have

$$f_x(a, b, c) = st\left(\frac{f(a + \Delta x, b, c) - f(a, b, c)}{\Delta x}\right)$$

for nonzero infinitesimal Δx.

Thus $f_x(x, y, z)$ is the rate of change of $f(x, y, z)$ with respect to x when y and z are held constant.

We also use the round d notation. If $w = f(x, y, z)$, we use:

$$\frac{\partial w}{\partial x}(x, y, z), \quad \frac{\partial w}{\partial x}, \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{for } f_x(x, y, z),$$

$$\frac{\partial w}{\partial y}(x, y, z), \quad \frac{\partial w}{\partial y}, \quad \text{or} \quad \frac{\partial f}{\partial y} \quad \text{for } f_y(x, y, z),$$

$$\frac{\partial w}{\partial z}(x, y, z), \quad \frac{\partial w}{\partial z}, \quad \text{or} \quad \frac{\partial f}{\partial z} \quad \text{for } f_z(x, y, z).$$

EXAMPLE 3 Find the partial derivatives of

$$f(x, y, z) = \sin(x^2 y - z)$$

at the point (1, 0, 0).

To find $f_x(x, y, z)$ we treat y and z as constants.

$$f_x(x, y, z) = 2xy\cos(x^2 y - z).$$
$$f_y(x, y, z) = x^2\cos(x^2 y - z).$$
$$f_z(x, y, z) = -\cos(x^2 y - z).$$

Thus

$$f_x(1, 0, 0) = 2 \cdot 1 \cdot 0 \cos(1^2 \cdot 0 - 0) = 0.$$
$$f_y(1, 0, 0) = 1^2 \cos(1^2 \cdot 0 - 0) = 1.$$
$$f_z(1, 0, 0) = -\cos(1^2 \cdot 0 - 0) = -1.$$

PROBLEMS FOR SECTION 11.3

In Problems 1–28, find the partial derivatives.

1 $z = 4x - 3y$

2 $z = 1 + 3x + 5y$

3 $z = xy^2 + x^3 y$

4 $z = x^3 y^2$

5 $z = \dfrac{1}{x^2 + y^2}$

6 $z = \dfrac{1}{xy + 1}$

7 $f(x, y) = xy$

8 $f(x, y) = x/y$

9 $f(x, y) = ax + by$

10 $f(x, y) = e^{ax+by}$

11 $f(x, y) = e^{x^2-y^2}$

12 $f(x, y) = \sin x \cos y$

13 $f(x, y) = \sqrt{x + 2y}$

14 $f(x, y) = \sqrt{xy} + \sqrt{x} + \sqrt{y}$

15 $z = x^y$

16 $z = x^{1/y}$

17 $z = \ln(xy)$

18 $z = \ln(ax + by)$

19 $z = \log_x y$

20 $z = \tan x \arctan y$

21 $z = \arcsin(x^2 y)$

22 $w = xyz$

23 $w = \sqrt{x^2 + y^2 + z^2}$

24 $f(x, y, z) = xe^{y-z}$

25 $f(x, y, z) = ax + by + cz$

26 $f(x, y, z) = x^a y^b z^c$

27 $w = z \cos x + z \sin y$

28 $w = z \cosh x + z \sinh y$

In Problems 29–40 find the partial derivatives at the given point.

29 $f(x, y) = xy^2, \quad x = 1, \quad y = 2$

30 $f(x, y) = x\sqrt{y}, \quad x = 2, \quad y = 4$

31 $f(x, y) = 1/xy, \quad x = -1, \quad y = 1$

32 $f(x, y) = \frac{1}{x} + \frac{1}{y}, \quad x = 3, \quad y = 4$

33 $z = e^{xy}, \quad x = 0, \quad y = 2$

34 $z = e^{x+y}, \quad x = 0, \quad y = 2$

35 $z = e^x \cos y, \quad x = 1, \quad y = 0$

36 $z = e^x \sin y, \quad x = 1, \quad y = 0$

37 $z = \frac{1}{x^2 + y^3}, \quad x = 2, \quad y = 3$

38 $z = \sqrt{x^2 + xy + 2y^2}, \quad x = 1, \quad y = 1$

39 $f(x, y, z) = x^2 + y^2 + z^2, \quad x = 1, \quad y = 2, \quad z = 3$

40 $f(x, y, z) = \frac{x}{y} - \frac{x}{z}, \quad x = 1, \quad y = 1, \quad z = 1$

41 A point $P(x, y)$ at $(1, 2)$ is moving at unit speed in the x direction. Find the rate of change of the distance from P to the origin.

42 A point $P(x, y)$ at $(1, 2)$ is moving at unit speed in the y direction. Find the rate of change of the distance from P to the point $(5, -1)$.

43 A point $P(x, y, z)$ is moving at unit speed in the x direction. Find the rate of change of the distance from P to the origin when P is at $(1, 2, 2)$.

44 A point $P(x, y, z)$ is moving at unit speed in the z direction. Find the rate of change of the distance from P to the origin when P is at $(3, \sqrt{3}, 2)$.

45 Find b and c if for all x and y,

$$z = x^2 + bxy + cy^2 \quad \text{and} \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}.$$

46 Find b if for all x and y

$$z = \sin x \sin y + b \cos x \cos y \quad \text{and} \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}.$$

47 It is found that the cost of producing x units of commodity one and y units of commodity two is

$$C(x, y) = 100 + 3x + 4y - \sqrt{xy}.$$

Find the partial marginal costs with respect to x and y, $\partial C/\partial x$ and $\partial C/\partial y$.

48 When a certain three commodities are produced in quantities x, y, and z respectively, it is found that they can be sold at a profit of

$$P(x, y, z) = 100x + 100y + yz - xy - z^2.$$

Find the marginal profits with respect to x, y and z; i.e., $\partial P/\partial x$, $\partial P/\partial y$, and $\partial P/\partial z$.

11.4 TOTAL DIFFERENTIALS AND TANGENT PLANES

Most of the functions we encounter have continuous partial derivatives. To keep our theory simple we shall concentrate on such functions in this chapter.

DEFINITION

*A function $f(x, y)$ is said to be **smooth** at (a, b) if both of its partial derivatives exist and are continuous at (a, b).*

The definition for three or more variables is similar.

The Increment Theorem for a differentiable function of one variable shows that the increment Δz is very close to the differential dz, and leads to the notion of a tangent line. In this section we introduce the increment and total differential for a function of two variables. Then we state an Increment Theorem for a smooth function of two variables, which leads to the notion of a tangent plane.

Let z depend on the two independent variables x and y, $z = f(x, y)$. Let Δx and Δy be two new independent variables, called the *increments* of x and y. Usually Δx and Δy are taken to be infinitesimals.

We now introduce two new dependent variables, the increment Δz and the total differential dz.

DEFINITION

*When $z = f(x, y)$, the **increment** of z is the dependent variable Δz given by*

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

The increment Δz depends on the four independent variables x, y, Δx, Δy, and is equal to the change in z as x changes by Δx and y changes by Δy. Thus

$$\Delta z = \Delta f(x, y, \Delta x, \Delta y),$$

where Δf is the function

$$\Delta f(x, y, \Delta x, \Delta y) = f(x + \Delta x, y + \Delta y) - f(x, y).$$

DEFINITION

*When $z = f(x, y)$, the **total differential** of z is the dependent variable dz given by*

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy,$$

or equivalently $$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

When x and y are independent variables, dx and dy are the same as Δx and Δy. The total differential dz depends on the four independent variables x, y, dx, and dy. Thus

$$dz = df(x, y, dx, dy),$$

where df is the function

$$df(x, y, dx, dy) = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

Figure 11.4.1 shows Δz under the microscope.

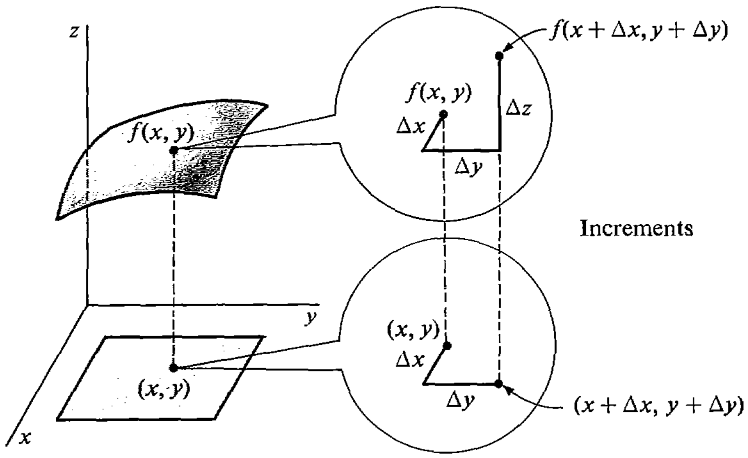


Figure 11.4.1

EXAMPLE 1 Find the increment and total differential of the product function $z = xy$ (Figure 11.4.2).

Increment: $\Delta z = (x + \Delta x)(y + \Delta y) - xy = y\,\Delta x + x\,\Delta y + \Delta x\,\Delta y.$

Total differential: $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = y\,dx + x\,dy.$

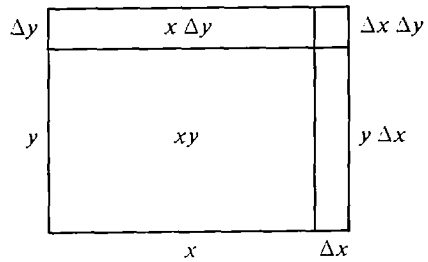


Figure 11.4.2

EXAMPLE 2 Find the increment and total differential of $z = x^2 - 3xy^2$.

Increment:

$$\begin{aligned}\Delta z &= [(x + \Delta x)^2 - 3(x + \Delta x)(y + \Delta y)^2] - [x^2 - 3xy^2] \\ &= [x^2 + 2x\,\Delta x + \Delta x^2 - 3xy^2 - 6xy\,\Delta y - 3x\,\Delta y^2 - 3\,\Delta x y^2 \\ &\quad - 6\,\Delta x y\,\Delta y - 3\,\Delta x\,\Delta y^2] - [x^2 - 3xy^2]\end{aligned}$$

$$= 2x\,\Delta x + \Delta x^2 - 6xy\,\Delta y - 3x\,\Delta y^2 - 3\,\Delta xy^2 - 6\,\Delta xy\,\Delta y - 3\,\Delta x\,\Delta y^2$$
$$= (2x - 3y^2)\,\Delta x - 6xy\,\Delta y + \Delta x^2 - 3x\,\Delta y^2 - 6y\,\Delta x\,\Delta y - 3\,\Delta x\,\Delta y^2.$$

Total differential:

$$\frac{\partial z}{\partial x} = 2x - 3y^2, \qquad \frac{\partial z}{\partial y} = -6xy.$$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = (2x - 3y^2)\,dx - 6xy\,dy.$$

We shall now state the Increment Theorem. It shows that Δz is very close to dz.

INCREMENT THEOREM FOR TWO VARIABLES

Suppose $z = f(x, y)$ is smooth at (a, b). Let Δx and Δy be infinitesimal. Then

$$\Delta z = dz + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

for some infinitesimals ε_1 and ε_2 which depend on Δx and Δy.

Before proving the Increment Theorem, let us check it for Examples 1 and 2.

EXAMPLE 1 (Continued) The product function $z = xy$ is smooth for all (x, y). Express Δz in the form

$$\Delta z = dz + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y.$$

We have
$$\Delta z = y\,\Delta x + x\,\Delta y + \Delta x\,\Delta y,$$
$$dz = y\,\Delta x + x\,\Delta y.$$

Thus
$$\Delta z = dz + \Delta x \cdot \Delta y.$$

The problem has more than one correct answer. One answer is $\varepsilon_1 = 0$ and $\varepsilon_2 = \Delta x$, so that

$$\Delta z = dz + 0 \cdot \Delta x + \Delta x \cdot \Delta y = dz + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y.$$

Another answer is $\varepsilon_1 = \Delta y$ and $\varepsilon_2 = 0$, so that

$$\Delta z = dz + \Delta y \cdot \Delta x + 0 \cdot \Delta y = dz + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y.$$

EXAMPLE 2 (Continued) The function $z = x^2 - 3xy^3$ is smooth for all (x, y). Express Δz in the form

$$\Delta z = dz + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

at an arbitrary point (x, y) and at the point $(5, 4)$. We have

$$\Delta z = (2x - 3y^2)\,\Delta x - 6xy\,\Delta y + \Delta x^2 - 3x\,\Delta y^2 - 6y\,\Delta x\,\Delta y - 3\,\Delta x\,\Delta y^2,$$
$$dz = (2x - 3y^2)\,\Delta x - 6xy\,\Delta y.$$

Then
$$\Delta z = dz + \Delta x^2 - 3x\,\Delta y^2 - 6y\,\Delta x\,\Delta y - 3\,\Delta x\,\Delta y^2.$$

Each term after the dz has either a Δx or a Δy or both. Factor Δx from all the terms where Δx appears and Δy from the remaining terms.

$$\Delta z = dz + (\Delta x - 6y\,\Delta y - 3\,\Delta y^2)\,\Delta x + (-3x\,\Delta y)\,\Delta y.$$

Then $$\Delta z = dz + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y,$$

where $$\varepsilon_1 = \Delta x - 6y\,\Delta y - 3\,\Delta y^2, \qquad \varepsilon_2 = -3x\,\Delta y.$$

At the point (5, 4),

$$\Delta z = dz + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y,$$

where $$\varepsilon_1 = \Delta x - 24\,\Delta y - 3\,\Delta y^2, \qquad \varepsilon_2 = -15\,\Delta y.$$

PROOF OF THE INCREMENT THEOREM We break Δz into two parts by going first from (a, b) to $(a + \Delta x, b)$ and then from $(a + \Delta x, b)$ to $(a + \Delta x, b + \Delta y)$, as shown in Figure 11.4.3,

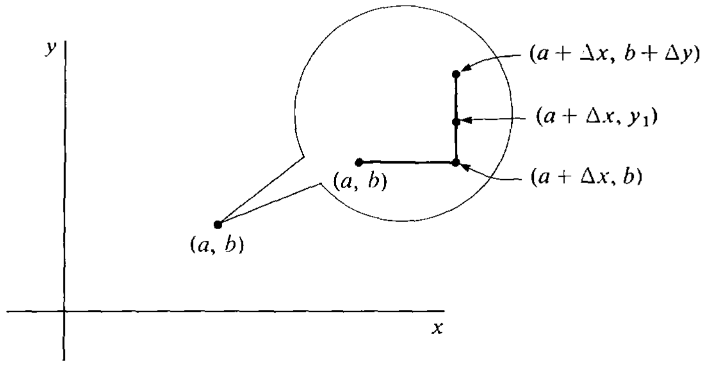


Figure 11.4.3

$$\Delta z = [f(a + \Delta x, b + \Delta y) - f(a + \Delta x, b)] + [f(a + \Delta x, b) - f(a, b)].$$

Our plan is as follows. First, we regard $f(a, b)$ as a one-variable function of a and show that

(1) $$f(a + \Delta x, b) - f(a, b) = f_x(a, b)\,\Delta x + \varepsilon_1\,\Delta x \quad \text{for some infinitesimal } \varepsilon_1.$$

Second, we regard $f(a + \Delta x, b)$ as a one-variable function of b and show that

(2) $$f(a + \Delta x, b + \Delta y) - f(a + \Delta x, b) = f_y(a, b)\,\Delta y + \varepsilon_2\,\Delta y \quad \text{for some infinitesimal } \varepsilon_2.$$

Once Equations 1 and 2 are established the proof will be complete because by adding Equations 1 and 2 we get the desired result

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y = dz + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y.$$

Equation 1 follows at once from the one-variable Increment Theorem since $f_x(a, b)$ exists.

We now prove Equation 2. We regard $f(a + \Delta x, y)$ as a one-variable function of y. For all y between b and $b + \Delta y$, the point $(a + \Delta x, y)$ is infinitely close to (a, b), so $f_y(a + \Delta x, y)$ is defined. By the one-variable Mean Value Theorem on the interval $[b, b + \Delta y]$, there is a y_1 between b and $b + \Delta y$ such that

$$f_y(a + \Delta x, y_1) = \frac{f(a + \Delta x, b + \Delta y) - f(a + \Delta x, b)}{\Delta y}.$$

Since f_y is continuous at (a, b),

$$f_y(a + \Delta x, y_1) = f_y(a, b) + \varepsilon_2,$$

where ε_2 is infinitesimal. Then

$$\frac{f(a + \Delta x, b + \Delta y) - f(a + \Delta x, b)}{\Delta y} = f_y(a, b) + \varepsilon_2,$$

and Equation 2 follows.

The following corollary is analogous to the theorem that a differentiable function of one variable is continuous.

COROLLARY 1

If a function $z = f(x, y)$ is smooth at (a, b) then it is continuous at (a, b).

PROOF Let (x, y) be infinitely close to (a, b) and let

$$\Delta x = x - a, \qquad \Delta y = y - b.$$

Then

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$= \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.$$

Since Δx and Δy are infinitesimal, Δz is infinitesimal, so $f(x, y) \approx f(a, b)$.

Some examples of what can happen when the function is not smooth are given in the problem set.

If a function $z = f(x, y)$ is smooth at (a, b), the curve $z = f(x, b)$ has a tangent line L_1 on the plane $y = b$, and the curve $z = f(a, y)$ has a tangent line L_2 on the plane $x = a$.

L_1 has the equation $\qquad z - f(a, b) = f_x(a, b)(x - a)$

and L_2 has the equation $\qquad z - f(a, b) = f_y(a, b)(y - b).$

The plane determined by the lines L_1 and L_2 is called the *tangent plane*. It has the equation

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

because the graph p of this equation is a plane and intersects the plane $y = b$ in L_1 and the plane $x = a$ in L_2 (Figure 11.4.4).

DEFINITION

The ***tangent plane*** *of a smooth function $z = f(x, y)$ at (a, b) is the plane with the equation*

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

If we set $x = a$ and $y = b$ in this equation we get $z = f(a, b)$. If we set $x - a = dx$ and $y - b = dy$ we get $z - f(a, b) = dz$. Therefore:

The tangent plane touches the surface at (a, b).

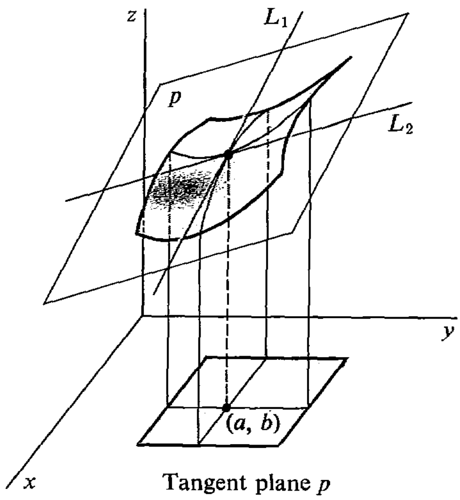
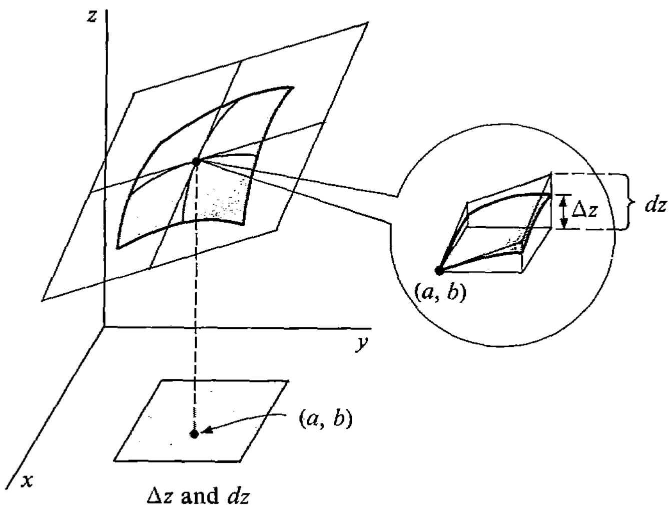


Figure 11.4.4 Tangent plane p

Δz = *change in z on the surface.*
dz = *change in z on the tangent plane.*

Figure 11.4.5 shows Δz and dz.



Δz and dz

Figure 11.4.5

Our second corollary to the Increment Theorem shows that the tangent plane closely follows the surface.

COROLLARY 2

Suppose $z = f(x, y)$ is smooth at (a, b). Then for every point (x, y) at an infinitesimal distance

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$$

from (a, b), the change in z on the tangent plane is infinitely close to the change in z along the surface compared to Δs, *i.e.,*

$$\frac{\Delta z}{\Delta s} \approx \frac{dz}{\Delta s}.$$

PROOF We have $\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$. Both $\Delta x/\Delta s$ and $\Delta y/\Delta s$ are finite, so

$$\frac{\Delta z}{\Delta s} = \frac{dz}{\Delta s} + \varepsilon_1 \frac{\Delta x}{\Delta s} + \varepsilon_2 \frac{\Delta y}{\Delta s},$$

$$\frac{\Delta z}{\Delta s} - \frac{dz}{\Delta s} = \varepsilon_1 \frac{\Delta x}{\Delta s} + \varepsilon_2 \frac{\Delta y}{\Delta s} \approx 0.$$

In Figure 11.4.6, we see that the piece of the surface seen through an infinitesimal microscope aimed at $(a, b, f(a, b))$ is infinitely close to a piece of the tangent plane, compared to the field of view of the microscope.

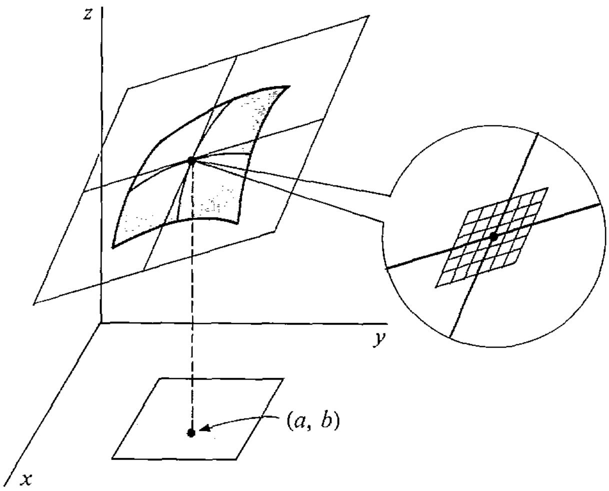


Figure 11.4.6

EXAMPLE 3 Find the equation of the tangent plane to

$$z = 1 + \sin(2x + 3y)$$

at the point (0, 0).

We have

$$\frac{\partial z}{\partial x}(x, y) = 2\cos(2x + 3y), \qquad \frac{\partial z}{\partial y}(x, y) = 3\cos(2x + 3y).$$

At the point (0, 0), $\quad z = 1 + \sin(0 + 0) = 1,$

$$\frac{\partial z}{\partial x}(0, 0) = 2\cos(0 + 0) = 2, \qquad \frac{\partial z}{\partial y}(0, 0) = 3\cos(0 + 0) = 3.$$

The equation of the tangent plane is $z - 1 = 2(x - 0) + 3(y - 0)$, or $z = 2x + 3y + 1$.

EXAMPLE 4 Find the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 14$$

at the point (1, 2, 3) (Figure 11.4.7).

The top hemisphere has the equation $z = \sqrt{14 - x^2 - y^2}$.

Then
$$\frac{\partial z}{\partial x}(x, y) = -\frac{x}{\sqrt{14 - x^2 - y^2}} = -\frac{x}{z},$$
$$\frac{\partial z}{\partial y}(x, y) = -\frac{y}{\sqrt{14 - x^2 - y^2}} = -\frac{y}{z}.$$

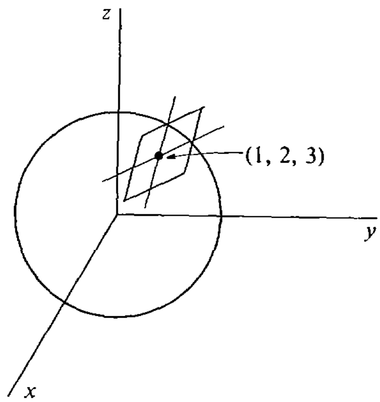


Figure 11.4.7

At (1, 2), $\quad z = 3, \quad \frac{\partial z}{\partial x}(1, 2) = -\frac{1}{3}, \quad \frac{\partial z}{\partial y}(1, 2) = -\frac{2}{3}.$

Then the tangent plane has the equation

$$z - 3 = \frac{\partial z}{\partial x}(x - 1) + \frac{\partial z}{\partial y}(y - 2),$$
$$z - 3 = -\frac{1}{3}(x - 1) + \left(-\frac{2}{3}\right)(y - 2),$$

or
$$x + 2y + 3z = 14.$$

The *total differential* of a function $w = f(x, y, z)$ of three variables is the dependent variable dw given by

$$dw = f_x(x, y, z)\,dx + f_y(x, y, z)\,dy + f_z(x, y, z)\,dz,$$

or equivalently
$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz.$$

The following Increment Theorem has a proof like the Increment Theorem for two variables.

INCREMENT THEOREM FOR THREE VARIABLES

Suppose $w = f(x, y, z)$ is smooth at (a, b, c). Let Δx, Δy, and Δz be infinitesimal. Then the increment Δw is equal to

$$\Delta w = dw + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z$$

for some infinitesimals $\varepsilon_1, \varepsilon_2, \varepsilon_3$ which depend on Δx, Δy, and Δz.

EXAMPLE 5 Given $w = xyz$, express the increment Δw in the form

$$\Delta w = dw + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z.$$

We first find Δw and dw,

$$\Delta w = (x + \Delta x)(y + \Delta y)(z + \Delta z) - xyz$$
$$= yz\,\Delta x + xz\,\Delta y + xy\,\Delta z + x\,\Delta y\,\Delta z + y\,\Delta x\,\Delta z + z\,\Delta x\,\Delta y + \Delta x\,\Delta y\,\Delta z$$

$$\frac{\partial w}{\partial x} = yz, \qquad \frac{\partial w}{\partial y} = xz, \qquad \frac{\partial w}{\partial z} = xy.$$

$$dw = yz\,\Delta x + xz\,\Delta y + xy\,\Delta z.$$

Thus $\Delta w = dw + (y\,\Delta z + z\,\Delta y)\,\Delta x + (x\,\Delta z)\,\Delta y + (\Delta x\,\Delta y)\,\Delta z.$

Figure 11.4.8 pictures dw and Δw.

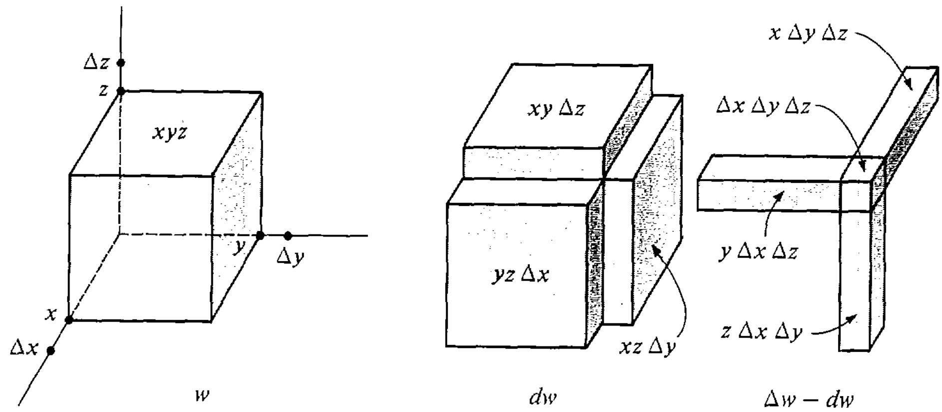


Figure 11.4.8

PROBLEMS FOR SECTION 11.4

In Problems 1–16, find the increment and total differential.

1	$z = 1 + 3x - 2y$	2	$z = x^2 - y^2$
3	$z = x^2y^2$	4	$z = x^3y$
5	$z = 1/xy$	6	$z = e^{x+y}$
7	$z = e^{3x-4y}$	8	$z = \cos x + \sin y$
9	$z = \cos x \sin y$	10	$z = \ln(x + 2y)$
11	$z = x \ln y$	12	$z = \sqrt{xy}$
13	$w = x + 2y + 3z$	14	$w = x^2 + y^2 + z^2$
15	$w = xy + yz$	16	$w = \sqrt{x} + \sqrt{y} + \sqrt{z}$

In Problems 17–22, express Δz in the form $\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$.

17 $z = x^2 + y^2$

18 $z = x^3 + y^3$

19 $z = x^2 y$

20 $z = 3xy - 2x^2 + y^2$

21 $z = \dfrac{x}{y}$

22 $z = y\sqrt{x}$

In Problems 23–40 find the tangent plane at the given point.

23 $z = 2x^2 + y^2$ at $(1, 2)$

24 $z = x^2 - 4y^2$ at $(-2, 1)$

25 $z = 2x^2y + y^2 + 3$ at $(1, 1)$

26 $z = x^2y^2 + xy^3 + 2$ at $(-1, 2)$

27 $z = \sqrt{xy} + 1$ at $(1, 1)$

28 $z = \sqrt{x} - 2\sqrt{y}$ at $(4, 1)$

29 $z = e^{x^2y}$ at $(1, 3)$

30 $z = e^{x^2+y^3}$ at $(-1, -1)$

31 $z = \sin x \sin y$ at $(\pi/3, \pi/4)$

32 $z = \tan(xy)$ at $(\pi, 1/4)$

33 $z = xy^2 - 2$ at $(0, 1)$

34 $z = x^2y^2 + 2$ at $(0, 0)$

35 $z = \cos x \cos y$ at $(0, 0)$

36 $z = \arctan(2x - y)$ at $(1, 4)$

37 $x^2 + y^2 + z^2 = 9$ at $(1, -2, 2)$

38 $x^2 + 2y^2 + 3z^2 = 6$ at $(-1, 1, -1)$

39 $x^2 + y^2 - z^2 = 1$ at $(1, 1, 1)$

40 $-x^2 - y^2 + z^2 = 1$ at $(2, -2, 3)$

☐ **41** Show that if z is a linear function of x and y, $z = ax + by + c$, then $\Delta z = dz$ at every point (x, y).

☐ **42** Let $f(x, y) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{if } xy \neq 0. \end{cases}$

Show that at $(0, 0)$
(a) $f(x, y)$ is not continuous;
(b) $f_x(0, 0)$ and $f_y(0, 0)$ exist;
(c) $f(x, y)$ is not smooth.

☐ **43** Let $f(x, y) = \sqrt{xy}$. Prove that at the point $(0, 0)$,
(a) $f(x, y)$ is continuous;
(b) $f_x(0, 0)$ and $f_y(0, 0)$ exist;
(c) $f(x, y)$ is not smooth;
(d) Δz is not infinitely close to dz compared to $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$.

☐ **44** Let $f(x, y) = |xy|$. Show that at $(0, 0)$,
(a) $f(x, y)$ is continuous;
(b) $f_x(0, 0)$ and $f_y(0, 0)$ exist;
(c) $f(x, y)$ is not smooth;
(d) Δz is infinitely close to dz compared to Δs.

☐ **45** Let $f(x, y) = |x| + |y|$. Show that at $(0, 0)$,
(a) $f(x, y)$ is continuous;
(b) $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

11.5 CHAIN RULE

The Chain Rule is useful when several variables depend on each other. A typical case is where z depends on x and y, while x and y depend on another variable t. We shall call t the *independent variable*, x and y the *intermediate variables*, and z the *dependent variable*. Figure 11.5.1 shows which variables depend on which.

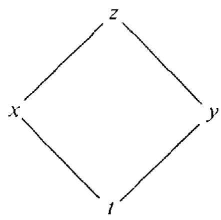


Figure 11.5.1

CHAIN RULE

If z is a smooth function of x and y, while x and y are differentiable functions of t, then dz/dt exists and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Discussion If $z = F(x, y)$ and $x = g(t)$, $y = h(t)$, then z as a function of t is

$$z = f(t) = F(g(t), h(t)).$$

We can give a more precise statement of the Chain Rule using functional notation:

If $g(t)$ and $h(t)$ are differentiable at t_0, and $F(x, y)$ is smooth at (x_0, y_0) where $x_0 = g(t_0)$ and $y_0 = h(t_0)$, then $f'(t_0)$ exists and

$$f'(t_0) = F_x(x_0, y_0)g'(t_0) + F_y(x_0, y_0)h'(t_0).$$

We shall give some examples and then prove the Chain Rule.

EXAMPLE 1 A particle moves in such a way that

$$\frac{dx}{dt} = 6, \qquad \frac{dy}{dt} = -2.$$

Find the rate of change of the distance from the particle to the origin when the particle is at the point (3, −4) (Figure 11.5.2).

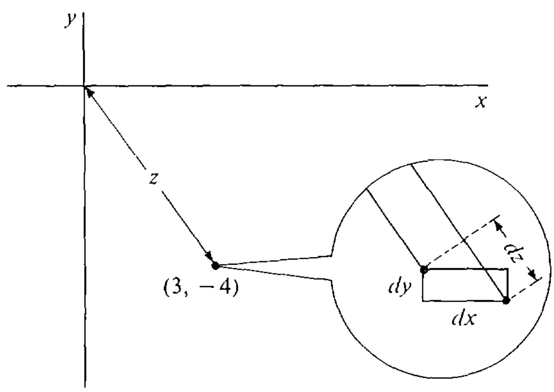


Figure 11.5.2

$$z = \sqrt{x^2 + y^2},$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{6x}{\sqrt{x^2 + y^2}} - \frac{2y}{\sqrt{x^2 + y^2}}$$

$$= \frac{6 \cdot 3}{\sqrt{3^2 + 4^2}} - \frac{2 \cdot (-4)}{\sqrt{3^2 + 4^2}} = \frac{26}{5}.$$

EXAMPLE 2 Find the derivative of $z = \sqrt[t]{\sin t}$, using the Chain Rule. (This can also be done by logarithmic differentiation.)

Let $$x = \sin t, \qquad y = \frac{1}{t}.$$

Then $$z = x^y.$$

$$\frac{\partial z}{\partial x} = yx^{y-1}, \qquad \frac{\partial z}{\partial y} = (\ln x)x^y,$$

$$\frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = -\frac{1}{t^2}.$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= yx^{y-1}\cos t + (\ln x)x^y\left(-\frac{1}{t^2}\right)$$

$$= \frac{\sqrt[t]{\sin t}\cos t}{t \sin t} - \frac{\ln(\sin t)\sqrt[t]{\sin t}}{t^2}.$$

EXAMPLE 3 Suppose the price z of steel is proportional to the population x divided by the supply y,

$$z = \frac{cx}{y}.$$

x and y depend on time in such a way that

$$\frac{dx}{dt} = 0.01x, \qquad \frac{dy}{dt} = -\sqrt{x}.$$

Find the rate of increase in the price z when $x = 1{,}000{,}000$, $y = 10{,}000$.

$$\frac{\partial z}{\partial x} = \frac{c}{y}, \qquad \frac{\partial z}{\partial y} = -\frac{cx}{y^2}.$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{c}{y}(0.01x) + \left(-\frac{cx}{y^2}\right)(-\sqrt{x})$$

$$= c \cdot 10^{-4} \cdot 10^{-2} \cdot 10^6 + c \cdot 10^6 \cdot (10^{-4})^2 \cdot (10^6)^{1/2}$$

$$= c(1 + 10) = 11c.$$

PROOF OF THE CHAIN RULE We use the Increment Theorem. Let Δt be a nonzero infinitesimal, and let Δx, Δy, and Δz be the corresponding increments of x, y and z. Then Δx and Δy are infinitesimal, and

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where ε_1 and ε_2 are infinitesimal. Dividing by Δt,

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}.$$

Taking standard parts, we see that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

There is a Chain Rule for any number of independent and intermediate variables. We state the simplest cases here.

The Chain Rules for two or more independent variables follow from the Chain Rules for one independent variable.

If z depends on x and x depends on s and t, we have the diagram in Figure 11.5.3. The Chain Rule for this case is:

If z is a differentiable function of x and x is a smooth function of s and t, then

$$\frac{\partial z}{\partial s} = \frac{dz}{dx}\frac{\partial x}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{dz}{dx}\frac{\partial x}{\partial t}.$$

This follows from the ordinary Chain Rule in Chapter 2 by holding s or t constant.

If z depends on x and y while x and y depend on s and t, we have the diagram in Figure 11.5.4. The Chain Rule for this case is:

If z is a smooth function of x and y while x and y are smooth functions of s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

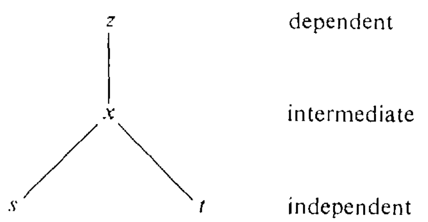


Figure 11.5.3

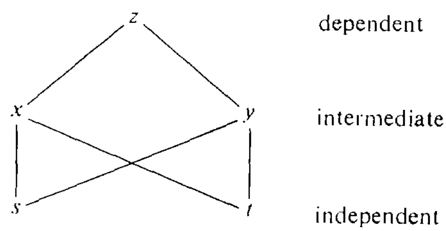


Figure 11.5.4

The Chain Rule for three intermediate variables is proved like the Chain Rule for two intermediate variables. We have the diagram in Figure 11.5.5.

If w is a smooth function of x, y, and z, which are in turn differentiable functions of t, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

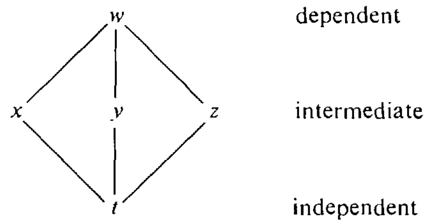


Figure 11.5.5

EXAMPLE 4 Use the Chain Rule to compute $\partial z/\partial s$ and $\partial z/\partial t$ where

$$z = \frac{x^2}{y}, \qquad x = st, \qquad y = s^2 - t^2.$$

$$\frac{\partial z}{\partial x} = \frac{2x}{y}, \qquad \frac{\partial z}{\partial y} = -\frac{x^2}{y^2}.$$

$$\frac{\partial x}{\partial s} = t, \qquad \frac{\partial y}{\partial s} = 2s.$$

$$\frac{\partial x}{\partial t} = s, \qquad \frac{\partial y}{\partial t} = -2t.$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{2x}{y}t - \frac{x^2}{y^2}2s$$

$$= \frac{2st^2}{s^2 - t^2} - \frac{2s^3t^2}{(s^2 - t^2)^2}.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{2x}{y}s - \frac{x^2}{y^2}(-2t)$$

$$= \frac{2s^2t}{s^2 - t^2} + \frac{2s^2t^3}{(s^2 - t^2)^2}.$$

As a check, we compute $\partial z/\partial s$ and $\partial z/\partial t$ directly without the Chain Rule.

$$z = \frac{x^2}{y} = \frac{s^2t^2}{s^2 - t^2}.$$

$$\frac{\partial z}{\partial s} = \frac{(s^2 - t^2)2st^2 - (2s)s^2t^2}{(s^2 - t^2)^2} = \frac{2st^2}{s^2 - t^2} - \frac{2s^3t^2}{(s^2 - t^2)^2}.$$

$$\frac{\partial z}{\partial t} = \frac{(s^2 - t^2)(2s^2t) - (-2t)s^2t^2}{(s^2 - t^2)^2} = \frac{2s^2t}{s^2 - t^2} + \frac{2s^2t^3}{(s^2 - t^2)^2}.$$

EXAMPLE 5 Let z depend on x and y and let $x = r\cos\theta$, $y = r\sin\theta$. Use the Chain Rule to obtain formulas for $\partial z/\partial r$ and $\partial z/\partial\theta$.

$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial y}{\partial r} = \sin\theta.$$

$$\frac{\partial x}{\partial\theta} = -r\sin\theta, \qquad \frac{\partial y}{\partial\theta} = r\cos\theta.$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta.$$

$$\frac{\partial z}{\partial\theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial\theta} = -\frac{\partial z}{\partial x}r\sin\theta + \frac{\partial z}{\partial y}r\cos\theta.$$

EXAMPLE 6 A rectangular solid has sides x, y, and z. Find the rate of change of the volume $V = xyz$ if

$$x = 1, \qquad y = 2, \qquad z = 3 \qquad \text{(in feet)},$$

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = -5, \qquad \frac{dz}{dt} = 2 \qquad \text{(in feet per second)}.$$

We have
$$\frac{\partial V}{\partial x} = yz, \qquad \frac{\partial V}{\partial y} = xz, \qquad \frac{\partial V}{\partial z} = xy,$$

so
$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$$

$$= 2\cdot 3\frac{dx}{dt} + 1\cdot 3\frac{dy}{dt} + 1\cdot 2\frac{dz}{dt}$$

$$= 2\cdot 3\cdot 1 + 1\cdot 3\cdot(-5) + 1\cdot 2\cdot 2 = -5.$$

Thus the volume is decreasing at -5 cubic feet per second (Figure 11.5.6).

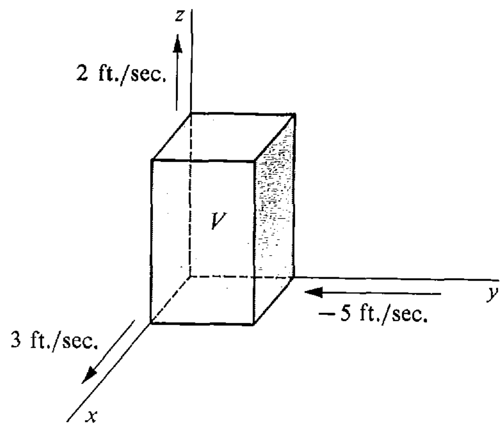


Figure 11.5.6

PROBLEMS FOR SECTION 11.5

In Problems 1–6, calculate dz/dt by the Chain Rule and check by a direct calculation.

1 $z = x^2 - y^2, \quad x = e^t, \quad y = e^{-t}$

2 $z = x^2y^2, \quad x = \cos t, \quad y = \sin t$

3 $z = \dfrac{1}{ax + by}, \quad x = \sin\left(\dfrac{t}{a}\right), \quad y = \sin\left(\dfrac{t}{b}\right)$

4 $z = e^{ax+by}, \quad x = \sqrt{t}, \quad y = \dfrac{1}{\sqrt{t}}$

5 $z = \dfrac{\ln x}{\ln y}, \quad x = \cosh(2t), \quad y = \sinh(2t)$

6 $z = \ln x \ln y, \quad x = \tan(3t), \quad y = \sec(3t)$

In Problems 7–14, calculate dz/dt by the Chain Rule.

7 $z = (t + 1)^{1/t}$

8 $z = \left(1 + \dfrac{1}{t}\right)^t$

9 $z = \sin t^{\cos t}$

10 $z = \sqrt{t}^{\sqrt{t}}$

11 $z = \log_{(t^2+1)}(t^2 - 1)$

12 $z = \log_{\sin t}(\cos t)$

13 $z = 3x - 2y, \quad \dfrac{dx}{dt} = \sqrt{1 - t^4}, \quad \dfrac{dy}{dt} = \sqrt{1 - t^3}$

14 $z = x + 2y + 3, \quad \dfrac{dx}{dt} = \cos\dfrac{1}{t}, \quad \dfrac{dy}{dt} = \sin\dfrac{1}{t}$

In Problems 15–20, find $\partial z/\partial s$ and $\partial z/\partial t$ by the Chain Rule.

15 $z = y^3, \quad y = s\cos t$

16 $z = \sin y, \quad y = st^2$

17 $z = \ln x, \quad x = s^2 - t^2$

18 $z = e^x, \quad x = \cos(2s) + \sin(3t)$

19 $z = ax + by, \quad x = \dfrac{1}{s + t}, \quad y = \dfrac{1}{s - t}$

20 $z = x^2 - y^2, \quad x = s\cos t, \quad y = s\sin t$

21 If $z = f(ax + by)$ and f is differentiable, show that $b\dfrac{\partial z}{\partial x} = a\dfrac{\partial z}{\partial y}$.

22 If $z = f(x + at, y + bt)$ and f is smooth, show that

$$\frac{\partial z}{\partial t} = a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y}.$$

23 If $z = f(x, y), \quad x = r\cos\theta, \quad y = r\sin\theta$, and f is smooth, show that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

24 If $z = f\left(\dfrac{xy}{x^2 + y^2}\right)$, where f is differentiable, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

25 Find dw/dt where $w = x\cos z + y\sin z$, $x = e^t$, $y = e^{-t}$, $z = \sqrt{t}$.

26 Find dw/dt where $w = xy^2z^3$, $x = 2t + 1$, $y = 3t - 2$, $z = 1 - 4t$.

In Problems 27–30, find formulas for dz/dt.

27 $z = \sqrt{x^2 + y^2}, \quad x = f(t), \quad y = g(t)$

28 $z = x^a y^b, \quad x = f(t), \quad y = g(t)$

29 $z = x^y, \quad x = f(t), \quad y = g(t)$

30 $z = \log_x y, \quad x = f(t), \quad y = g(t)$

In Problems 31–36, find formulas for $\partial z/\partial s$ and $\partial z/\partial t$.

31 $z = f(u), \quad u = as + bt$

32 $z = f(u), \quad u = st$

33 $z = e^u, \quad u = f(s, t)$

34 $z = f(u), \quad u = g(s) + h(t)$

35 $z = g(s)h(t)$

36 $z = f(x, y), \quad x = g(s), \quad y = h(t)$

37 A particle moves in the (x, y) plane so that $dx/dt = 2$, $dy/dt = -4$. Find dz/dt, where z is the distance from the origin, when the particle is at the point (3, 4).

38 A particle moves in the (x, y) plane so that

$$\frac{dx}{dt} = \frac{1}{x} + \frac{1}{y}, \qquad \frac{dy}{dt} = 2x + y.$$

Find dz/dt, where z is the distance of the particle from the point (1, 2), when the particle is at (2, 3).

39 A particle moves in space so that

$$\frac{dx}{dt} = 3, \qquad \frac{dy}{dt} = 4, \qquad \frac{dz}{dt} = -2.$$

Find the rate of change of the distance from the origin when $x = 1$, $y = -2$, $z = 2$.

40 Find the rate of change of the area of a rectangle when the sides have lengths $x = 5$ and $y = 6$ and are changing at rates $dx/dt = 3$, $dy/dt = -4$.

41 Find the rate of change of the perimeter of a rectangle when the sides are $x = 2$, $y = 4$ and are changing at the rates $dx/dt = -2$, $dy/dt = 3$.

42 The per capita income of a country is equal to the national income x divided by the population y. Find the rate of change in per capita income when $x = \$10$ billion, $y = 10$ million, $dx/dt = \$10$ million per year, $dy/dt = 50{,}000$ people per year.

43 The profit of a manufacturer is equal to the total revenue x minus the total cost y. As the number of items produced, u, is increased, the revenue and cost increase at the rates $dx/du = 500/u$ and $dy/du = 1/\sqrt{u}$. Find the rate of increase of profit with respect to u when $u = 10{,}000$.

44 When commodities one and two have prices p and q respectively, their respective demands are $D_1(p, q)$ and $D_2(p, q)$. The revenue at prices p and q is the quantity

$$R(p, q) = pD_1(p, q) + qD_2(p, q),$$

since a quantity $D_1(p, q)$ can be sold at price p and a quantity $D_2(p, q)$ at price q. Find formulas for the partial marginal revenues with respect to price, $\partial R/\partial p$ and $\partial R/\partial q$.

11.6 IMPLICIT FUNCTIONS

In many applications of the Chain Rule, one or more of the independent variables is also used as an intermediate variable. The simplest case where this occurs is when z depends on x and y while y depends on x,

$$y = g(x), \qquad z = F(x, y).$$

Figure 11.6.1 shows which variables depend on which.

Assuming $F(x, y)$ is smooth and $dy/dx = g'(x)$ exists, the Chain Rule gives

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}, \quad \text{or} \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

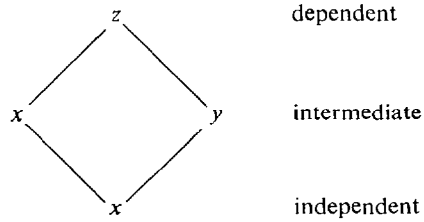


Figure 11.6.1

Here dz/dx stands for $f'(x)$ where $z = f(x) = F(x, g(x))$, and $\partial z/\partial x$ stands for $F_x(x, y)$. The round d, ∂, is useful in telling the two apart.

EXAMPLE 1 If

$$z = 2x + 3y, \qquad y = \sin x,$$

find $\partial z/\partial x$ and dz/dx when $x = 0$.

$$\frac{\partial z}{\partial x} = 2.$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 2 + 3\cos x.$$

When $x = 0$,

$$\frac{dz}{dx} = 2 + 3\cos 0 = 5.$$

As a check, we find dz/dx directly.

$$z = 2x + 3y = 2x + 3\sin x.$$

$$\frac{dz}{dx} = 2 + 3\cos x.$$

When $x = 0$,

$$\frac{dz}{dx} = 2 + 3\cos 0 = 5.$$

EXAMPLE 2 Use the Chain Rule to obtain a formula for dz/dx where $z = x^y$ and y depends on x.

$$\frac{\partial z}{\partial x} = yx^{y-1}, \qquad \frac{\partial z}{\partial y} = (\ln x)x^y.$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = yx^{y-1} + (\ln x)x^y\frac{dy}{dx}.$$

The Chain Rule can also be used in problems where dz/dx is known and dy/dx is to be found.

In many problems we are given a relationship between x and y which can be expressed by an equation of the form $F(x, y) = 0$, and we wish to find dy/dx. The graph of $F(x, y) = 0$ is usually a curve in the (x, y) plane. If we put $z = F(x, y) = 0$, then $dz/dx = 0$ while dy/dx is the slope of the curve. Ordinarily such a curve can be divided into finitely many pieces each of which is the graph of a function $y = g(x)$.

For example, the top and bottom halves of the circle

$$x^2 + y^2 - 1 = 0$$

are the functions

$$y = \sqrt{1 - x^2}, \qquad y = -\sqrt{1 - x^2}$$

shown in Figure 11.6.2.

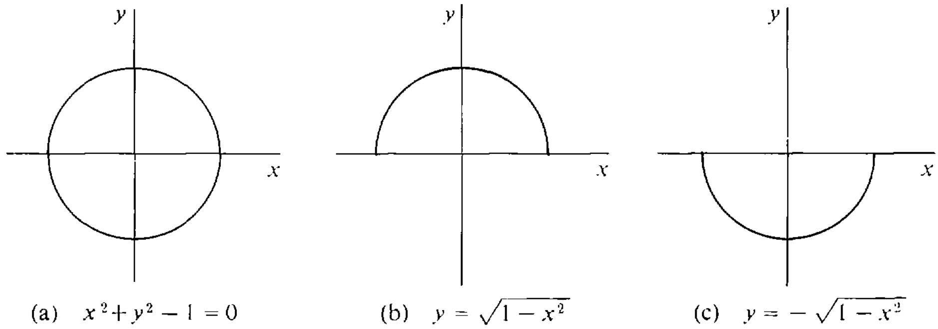


Figure 11.6.2

DEFINITION

*An **implicit function** of the curve $F(x, y) = 0$ at (a, b) is a function $y = g(x)$ such that:*

(i) $g(a) = b$;

(ii) *The domain of $g(x)$ is an open interval containing a;*

(iii) *The graph of $y = g(x)$ is a subset of the graph of $F(x, y) = 0$.*

*If every implicit function of $F(x, y) = 0$ has the same slope S at (a, b), we call S the **slope** of the curve.*

Figure 11.6.3 shows an implicit function $y = g(x)$ of a curve $F(x, y) = 0$. It is often hard or impossible to express an implicit function in terms of known (or elementary) functions. However, the next theorem gives an easy test for showing that there is an implicit function and finding its slope.

IMPLICIT FUNCTION THEOREM

Suppose that at the point (a, b), $z = F(x, y)$ is smooth, $F(a, b) = 0$, and $\partial z/\partial y \neq 0$.

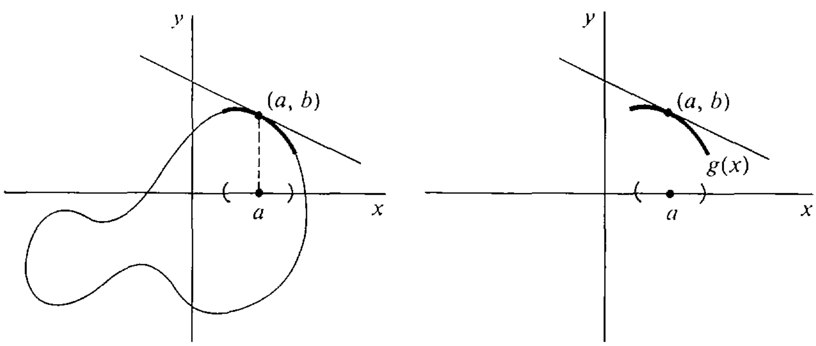


Figure 11.6.3

Then the curve $F(x, y) = 0$ at (a, b) has an implicit function and the slope

$$\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}.$$

There are three things to prove:

(1) There exists an implicit function $y = g(x)$ at (a, b).
(2) The slope $dy/dx = g'(a)$ exists.
(3) dy/dx has the required value.

Instead of proving the whole theorem, we give an intuitive argument for (1) and (2) and then prove (3). The surface $z = F(x, y)$ has a tangent plane at $(a, b, 0)$. If we intersect the surface and tangent plane with the plane $z = 0$ we get the curve $0 = F(x, y)$ and a line L. Through an infinitesimal microscope aimed at the point (a, b), the curve looks like the graph of a function $y = g(x)$ which has the tangent line L and thus has a slope at (a, b) (Figure 11.6.4).

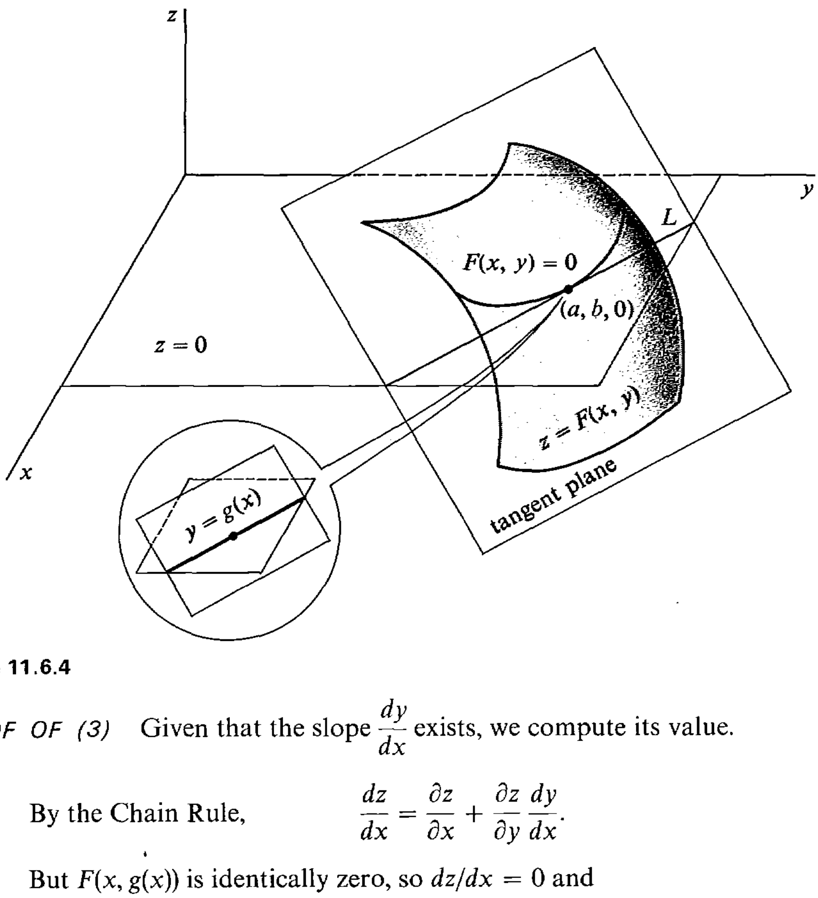


Figure 11.6.4

PROOF OF (3) Given that the slope $\frac{dy}{dx}$ exists, we compute its value.

By the Chain Rule, $$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

But $F(x, g(x))$ is identically zero, so $dz/dx = 0$ and

$$0 = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

Since $\frac{\partial z}{\partial y} \neq 0$, $$\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}.$$

The best way to remember the minus sign in the above equation is to derive the equation yourself. Start with the Chain Rule for $dz/dx = 0$ and solve for dy/dx. One way to understand the minus sign is as follows: if $\partial z/\partial x$ and $\partial z/\partial y$ are positive, an increase in x must be offset by a decrease in y to keep z constant, so dy/dx should be negative.

Warning: The two ∂z's have different meanings and cannot be cancelled.

EXAMPLE 3 Find the slope dy/dx of the circle

$$x^2 + y^2 - 4 = 0$$

at the point $(1, \sqrt{3})$ (see Figure 11.6.5).

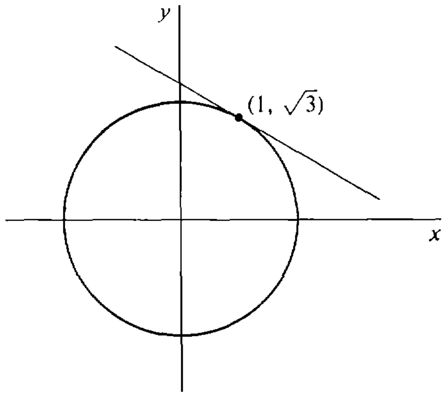


Figure 11.6.5

Put $$z = x^2 + y^2 - 4 = 0.$$

At a point (x, y),

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y, \quad \frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y} = -\frac{x}{y}.$$

At the given point $(1, \sqrt{3})$, $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{3}}.$

In this problem we can solve for y as a function of x and check the answer directly.

$$y = \sqrt{4 - x^2}.$$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{4 - x^2}} = \frac{-2}{2\sqrt{4 - 1}} = -\frac{1}{\sqrt{3}}.$$

The Implicit Function Theorem gives us a convenient equation for the tangent line to the curve $F(x, y) = 0$ at (a, b).

$$y - b = \frac{dy}{dx}(x - a),$$

$$y - b = -\frac{\partial z/\partial x}{\partial z/\partial y}(x - a),$$

and finally

Tangent Line: $$\frac{\partial z}{\partial x}(x - a) + \frac{\partial z}{\partial y}(y - b) = 0.$$

EXAMPLE 3 (Continued) Find the equation for the tangent line in Example 3.

At the point $(1, \sqrt{3})$,

$$\frac{\partial z}{\partial x} = 2x = 2, \qquad \frac{\partial z}{\partial y} = 2y = 2\sqrt{3},$$

and the tangent line is

$$2(x - 1) + 2\sqrt{3}(y - \sqrt{3}) = 0.$$

EXAMPLE 4 Find the tangent line and slope of the curve

$$y + \ln y + x^3 = 0$$

at the point $(-1, 1)$ (Figure 11.6.6).

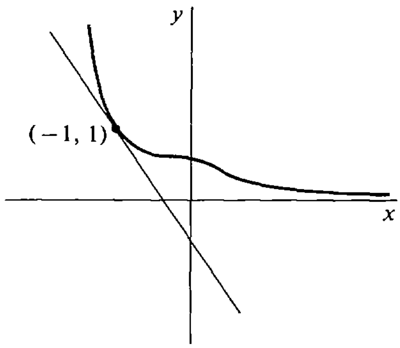


Figure 11.6.6

Put $$z = y + \ln y + x^3.$$

Then $$\frac{\partial z}{\partial x} = 3x^2, \qquad \frac{\partial z}{\partial y} = 1 + \frac{1}{y}.$$

At $(-1, 1)$, $$\frac{\partial z}{\partial x} = 3, \qquad \frac{\partial z}{\partial y} = 2.$$

Tangent Line: $$3(x + 1) + 2(y - 1) = 0.$$

Slope: $$\frac{dy}{dx} = -\frac{3}{2}.$$

EXAMPLE 5 Find the tangent line and slope of the level curve of the hyperbolic paraboloid

$$z = x^2 - y^2$$

at the point (a, b) (where $b \neq 0$) (Figure 11.6.7).

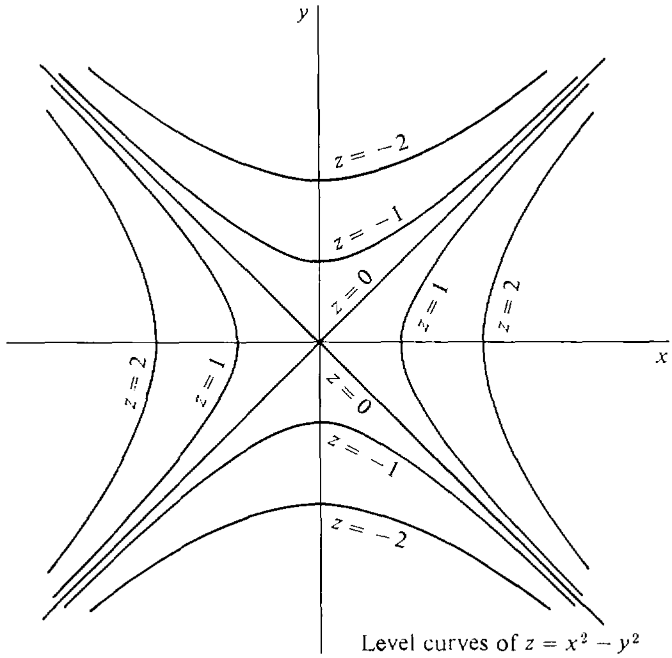


Figure 11.6.7

Level curves of $z = x^2 - y^2$

The level curve has the equation

$$x^2 - y^2 = a^2 - b^2,$$

$$x^2 - y^2 - (a^2 - b^2) = 0.$$

Put $$w = x^2 - y^2 - (a^2 - b^2) = 0.$$

Then $$\frac{\partial w}{\partial x} = 2x, \qquad \frac{\partial w}{\partial y} = -2y.$$

At (a, b), $$\frac{\partial w}{\partial x} = 2a, \qquad \frac{\partial w}{\partial y} = -2b.$$

Tangent Line: $$2a(x - a) - 2b(y - b) = 0.$$

Slope: $$\frac{dy}{dx} = -\frac{2a}{-2b} = \frac{a}{b}.$$

Let us next consider the case where w depends on x, y, and z, while z depends on x and y,

$$w = F(x, y, z), \qquad z = g(x, y).$$

Figure 11.6.8 shows which variables depend on which.

If $F(x, y, z)$ is smooth and $\partial z/\partial x$, $\partial z/\partial y$ exist, the Chain Rule gives

$$\frac{\partial w}{\partial x}(x, y) = \frac{\partial w}{\partial x}(x, y, z)\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}(x, y, z)\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}(x, y, z)\frac{\partial z}{\partial x},$$

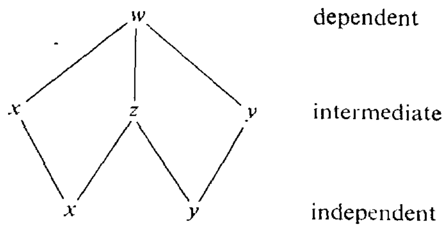


Figure 11.6.8

or $$\frac{\partial w}{\partial x}(x, y) = \frac{\partial w}{\partial x}(x, y, z) + \frac{\partial w}{\partial z}(x, y, z)\frac{\partial z}{\partial x}.$$

Similarly, $$\frac{\partial w}{\partial y}(x, y) = \frac{\partial w}{\partial y}(x, y, z) + \frac{\partial w}{\partial z}(x, y, z)\frac{\partial z}{\partial y}.$$

We used the fact that for the independent variables x and y,

$$\frac{\partial x}{\partial x} = \frac{\partial y}{\partial y} = 1, \qquad \frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0.$$

Notice that in this case $\partial w/\partial x$ alone is ambiguous so we had to use the more complete notation

$$\frac{\partial w}{\partial x}(x, y, z) \qquad \text{for } F_x(x, y, z),$$

$$\frac{\partial w}{\partial x}(x, y) \qquad \text{for } f_x(x, y), \quad \text{where } f(x, y) = F(x, y, g(x, y)).$$

EXAMPLE 6 Find $\frac{\partial w}{\partial x}(x, y)$ and $\frac{\partial w}{\partial y}(x, y)$ where

$$w = x^2 + 2y^2 + 3z^2, \qquad z = e^{5x+y}.$$

$$\frac{\partial w}{\partial x}(x, y, z) = 2x, \qquad \frac{\partial w}{\partial y}(x, y, z) = 4y, \qquad \frac{\partial w}{\partial z}(x, y, z) = 6z.$$

$$\frac{\partial z}{\partial x} = 5e^{5x+y}, \qquad \frac{\partial z}{\partial y} = e^{5x+y}.$$

Then $$\frac{\partial w}{\partial x}(x, y) = 2x + 6z \cdot 5e^{5x+y} = 2x + 30ze^{5x+y}$$

$$= 2x + 30e^{2(5x+y)}.$$

$$\frac{\partial w}{\partial y}(x, y) = 4y + 6z \cdot e^{5x+y} = 4y + 6e^{2(5x+y)}.$$

The graph of an equation

$$F(x, y, z) = 0$$

is a surface in space. The Implicit Function Theorem can be generalized to this case.

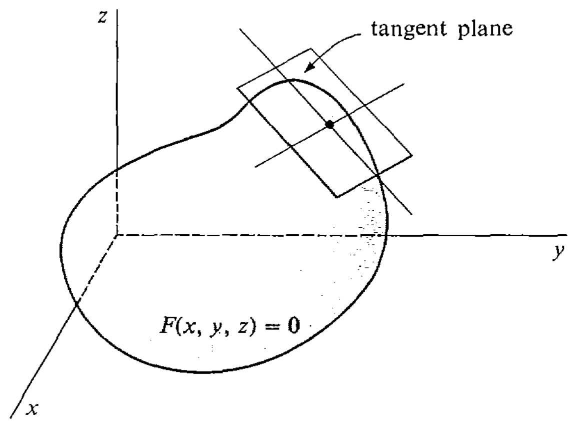


Figure 11.6.9

We shall skip the details, but the end result is an equation for the tangent plane of the surface, pictured in Figure 11.6.9.

THEOREM

Suppose the function $w = F(x, y, z)$ is smooth at the point (a, b, c), and $F_z(a, b, c) \neq 0$. Then the implicit surface $F(x, y, z) = 0$ has the partial derivatives

$$\frac{\partial z}{\partial x} = -\frac{F_x(a, b, c)}{F_z(a, b, c)}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y(a, b, c)}{F_z(a, b, c)}$$

and the tangent plane

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

The equation for the tangent plane is obtained as follows.

$$z - c = \frac{\partial z}{\partial x}(x - a) + \frac{\partial z}{\partial y}(y - b),$$

$$z - c = -\frac{F_x(a, b, c)}{F_z(a, b, c)}(x - a) - \frac{F_y(a, b, c)}{F_z(a, b, c)}(y - b),$$

and finally $F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$.

EXAMPLE 7 Find the tangent plane to the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 6$$

at the point (1, 1, 1) (see Figure 11.6.10).

Put $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 6.$

Then $F_x(x, y, z) = 2x, \quad F_y(x, y, z) = 4y, \quad F_z(x, y, z) = 6z.$

$F_x(1, 1, 1) = 2, \quad F_y(1, 1, 1) = 4, \quad F_z(1, 1, 1) = 6.$

The tangent plane has the equation

$$2(x - 1) + 4(y - 1) + 6(z - 1) = 0.$$

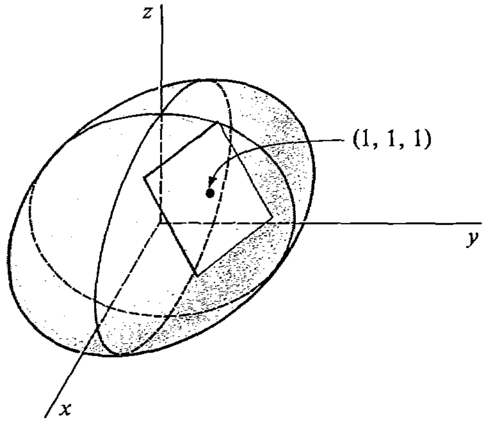


Figure 11.6.10

Find $\partial z/\partial x$ and $\partial z/\partial y$ at $(1, 1, 1)$.

$$\frac{\partial z}{\partial x} = -\frac{F_x(1,1,1)}{F_z(1,1,1)} = -\frac{2x}{6z} = -\frac{x}{3z},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(1,1,1)}{F_z(1,1,1)} = -\frac{4y}{6z} = -\frac{2y}{3z}.$$

At $(1, 1, 1)$,
$$\frac{\partial z}{\partial x} = -\frac{1}{3}, \quad \frac{\partial y}{\partial y} = -\frac{2}{3}.$$

PROBLEMS FOR SECTION 11.6

In Problems 1–8, find $\partial z/\partial x$ and dz/dx by the Chain Rule.

1 $z = 3x - 4y, \quad y = e^x$

2 $z = xy, \quad y = \ln x$

3 $z = \cos x + \sin y, \quad y = 3x$

4 $z = \dfrac{1}{2x + 3y}, \quad y = \sqrt{x}$

5 $z = x^y, \quad y = x$

6 $z = x^y, \quad y = \sqrt{x}$

7 $z = \arctan(xy), \quad y = e^{-x}$

8 $z = \sin x \sin y, \quad y = 2x$

In Problems 9–14, find dy/dx.

9 $x^2 + 2xy - y^2 = 2$

10 $\sqrt{x} + \sqrt{xy} + \sqrt{y} = 1$

11 $x^2 + 2xy^3 + y = 2$

12 $e^{xy} + 3x + 2y^2 = 1$

13 $\sin xy + x + 2 = 0$

14 $\ln x + 2\ln y + xy = 1$

In Problems 15–22, find the tangent line and the slope of the curve at the given point.

15 $(x + 1)^2 + (y + 2)^2 = 25$ at $(2, 2)$

16 $x^2 + 4y^2 = 4$ at $(\sqrt{3}, \frac{1}{2})$

17 $x^2 - 3xy - y^2 = 3$ at $(1, -1)$

18 $\sqrt{x} + \sqrt{y} = 2$ at $(1, 1)$

19 $x^3 + y^3 = 2$ at $(1, 1)$

20 $x + \sqrt{xy} - 2y = 8$ at $(8, 2)$

21 $\cos x \sin y = \frac{1}{2}$ at $(\pi/4, \pi/4)$

22 $y + e^x \ln y = 1$ at $(2, 1)$

In Problems 23–26 find $\frac{\partial w}{\partial x}(x, y)$ and $\frac{\partial w}{\partial y}(x, y)$.

23 $w = 3x - 4y + 6z, \quad z = 2x - 5y$

24 $w = z\cos x + z\sin y, \quad z = \sqrt{x^2 + y^2}$

25 $w = \sqrt{x^2 + y^2 + z^2}, \quad z = -3x + 2y$

26 $w = z/xy, \quad z = y\ln x$

In Problems 27–32, find the tangent plane to the surface at the given point.

27 $3x^2 + 5y^2 + 4z^2 = 21$ at $(-2, 1, -1)$

28 $2x^2 - 4y^2 + z^2 = 2$ at $(1, 1, -2)$

29 $xyz + x^2 + y^2 + z^2 = 4$ at $(1, 1, 1)$

30 $xy + xz + yz = 3$ at $(1, 1, 1)$

31 $xe^y + ye^z + ze^x = 0$ at $(0, 0, 0)$

32 $\sin x \cos y \tan z = 1$ at $(\pi/2, 0, \pi/4)$

In Problems 33–38, find $\partial z/\partial x$ and $\partial z/\partial y$.

33 $x^2y + z^2 = 1$

34 $x^2 + 2y^2 - 3z^2 = 4$

35 $\sin xy + \cos yz = 1$

36 $e^x + e^y + e^z = 1$

37 $xy^2z^3 + 2 = 0$

38 $x^2 + y^3 + \ln z = 2$

39 Suppose that x items can be bought at a price of y dollars per item, where y depends on x in such a way that $dy/dx = -1/(1 + \sqrt{x})$. Find the rate of change of the total cost $z = xy$ with respect to x.

40 A point moves along the parabola $y = x^2$. Find the rate of change with respect to x of the distance from the origin.

41 Suppose w depends on x, y, and z, and both y and z depend on x. Find a formula for dw/dx using the Chain Rule.

42 Suppose z depends on x and y, while y depends on x and t. Use the Chain Rule to find a formula for $\frac{\partial z}{\partial x}(x, t)$.

11.7 MAXIMA AND MINIMA

The theory of maxima and minima for functions of two variables is similar to the theory for one variable. The student should review the one-variable case at this time.

DEFINITION

*Let $z = f(x, y)$ be a function with domain D. f is said to have a **maximum** at a point (x_0, y_0) in D if*

$$f(x_0, y_0) \geq f(x, y)$$

*for all (x, y) in D. The value $f(x_0, y_0)$ is called the **maximum value** of f.*

*A **minimum** and the **minimum value** of f are defined analogously.*

We shall first study functions defined on closed regions, which correspond to closed intervals. By a *closed region* in the plane we mean a set D defined by inequalities

$$a \le x \le b, \qquad f(x) \le y \le g(x),$$

where f and g are continuous and $f(x) \le g(x)$ on $[a, b]$. D is called *the region between* $f(x)$ *and* $g(x)$ *for* $a \le x \le b$ (Figure 11.7.1).

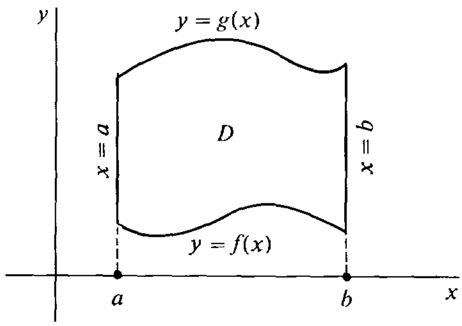


Figure 11.7.1 A closed region

The points of D on the four curves

$$x = a, \qquad x = b, \qquad y = f(x), \qquad y = g(x)$$

are called *boundary points.* All other points of D are called *interior points.*

EXTREME VALUE THEOREM

Suppose $z = f(x, y)$ is continuous at every point of a closed region D. Then the function f with its domain restricted to D has a maximum and a minimum.

The proof is similar to the corresponding proof for one variable.

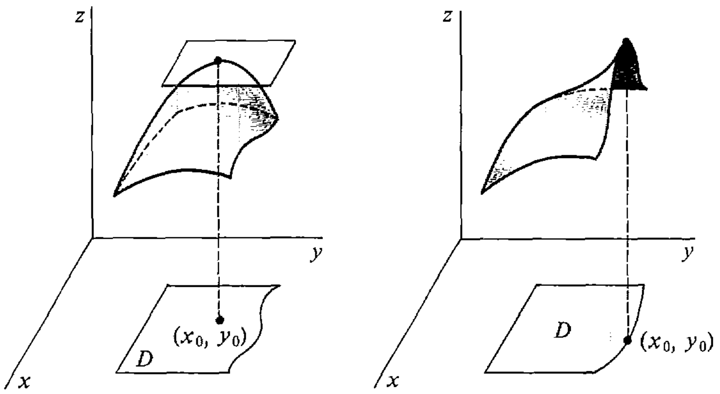
CRITICAL POINT THEOREM

Suppose the domain of $z = f(x, y)$ is a closed region D and f is smooth at every interior point of D. If f has a maximum or minimum at (x_0, y_0), then either

(i) $f_x(x_0, y_0) = 0$ *and* $f_y(x_0, y_0) = 0$, *or*

(ii) (x_0, y_0) *is a boundary point of D.*

Figure 11.7.2 illustrates the two cases when f has a maximum at (x_0, y_0). An interior point where both partial derivatives are zero is called a *critical point.* Thus a critical point is a point where the tangent plane is horizontal. On the graph of a surface, an interior point looks like a mountain summit if it is a maximum and a valley bottom if it is a minimum. The theorem states that every interior maximum or minimum is a critical point. An interesting kind of critical point which is neither



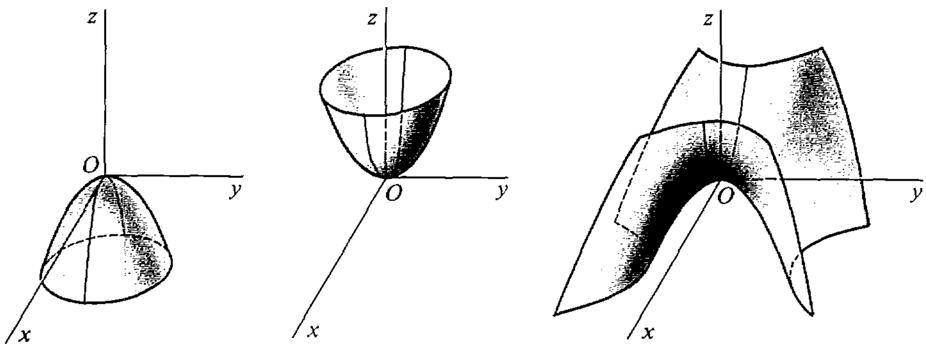
Case (*i*) Interior Maximum *Case* (*ii*) Boundary Maximum

Figure 11.7.2 Critical Point Theorem

a maximum nor a minimum is a *saddle point*, which looks like the summit of a pass between two mountains. Table 11.7.1 gives three simple examples of critical points, one maximum, one minimum, and one saddle point. They are illustrated in Figure 11.7.3.

Table 11.7.1

Function	Partials	Critical Point	Type
$z = -(x^2 + y^2)$	$\frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y$	(0, 0)	Maximum
$z = x^2 + y^2$	$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y$	(0, 0)	Minimum
$z = x^2 - y^2$	$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = -2y$	(0, 0)	Saddle Point



(a) Maximum (b) Minimum (c) Saddle point

Figure 11.7.3

PROOF OF THE CRITICAL POINT THEOREM Suppose f has a maximum at an interior point (x_0, y_0) of D. (x_0, y_0) is not a boundary point so we must prove (i). The function

$$g(x) = f(x, y_0)$$

is differentiable and has a maximum at x_0. By the Critical Point Theorem for one variable, $g'(x_0) = f_x(x_0, y_0) = 0$. Similarly $f_y(x_0, y_0) = 0$.

METHOD FOR FINDING MAXIMA AND MINIMA ON A CLOSED REGION

When to Use *$z = f(x, y)$ is continuous on a closed region D and smooth on the interior of D.*

Step 1 *Set the problem up and sketch D.*

Step 2 *Compute $\partial z/\partial x$ and $\partial z/\partial y$.*

Step 3 *Find the critical points of f, if any, and the value of f at each critical point.*

Step 4 *Find the maximum and minimum of f on the boundary of D. This can be done by solving for z as a function of x or y alone and using the method for one variable.*

CONCLUSION *The largest of the values from Steps 3 and 4 is the maximum value, and the smallest is the minimum value.*

It is convenient to record the results of Steps 3 and 4 on the sketch of D.

EXAMPLE 1 Find the maximum and minimum of $z = x^2 + y^2 - xy - x$ on the closed rectangle $0 \le x \le 1, 0 \le y \le 1$.

Step 1 The region D is sketched in Figure 11.7.4.

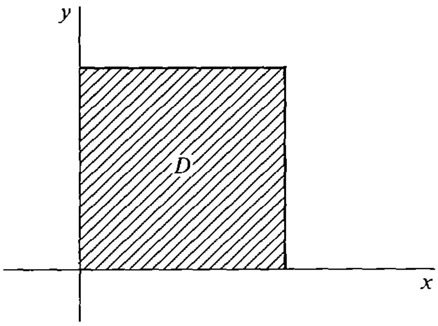


Figure 11.7.4

Step 2 $\frac{\partial z}{\partial x} = 2x - y - 1, \qquad \frac{\partial z}{\partial y} = 2y - x.$

Step 3 $2x - y - 1 = 0, \qquad 2y - x = 0.$

Solving for x and y we get one critical point

$$y = \tfrac{1}{3}, \qquad x = \tfrac{2}{3}.$$

The value of z at that point is

$$z = (\tfrac{2}{3})^2 + (\tfrac{1}{3})^2 - \tfrac{2}{3}\cdot\tfrac{1}{3} - \tfrac{2}{3} = -\tfrac{1}{3}.$$

Step 4 We make a table.

Boundary Line	z	Maximum	Minimum
$x = 0, \quad 0 \le y \le 1$	y^2	1 at (0, 1)	0 at (0, 0)
$x = 1, \quad 0 \le y \le 1$	$y^2 - y$	0 at corners	$-\frac{1}{4}$ at $(1, \frac{1}{2})$
$y = 0, \quad 0 \le x \le 1$	$x^2 - x$	0 at corners	$-\frac{1}{4}$ at $(\frac{1}{2}, 0)$
$y = 1, \quad 0 \le x \le 1$	$x^2 + 1 - 2x$	1 at (0, 1)	0 at (1, 1)

The values from Steps 3 and 4 are also shown on the sketch of D in Figure 11.7.5.

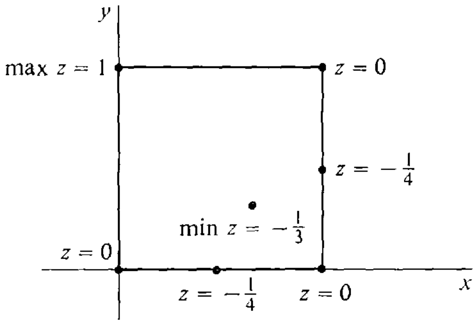


Figure 11.7.5

CONCLUSION

Maximum: $z = 1$ at (0, 1)
Minimum: $z = -\frac{1}{3}$ at $(\frac{2}{3}, \frac{1}{3})$.

The maximum is at a boundary point and the minimum at an interior point.

In many problems we are to maximize a function of three variables which are related by a *side condition*. We wish to find the maximum or minimum of

$$w = F(x, y, z)$$

given the side condition

$$g(x, y, z) = 0.$$

To work a problem of this type we use the side condition to get w as a function of just two independent variables and then proceed as before.

EXAMPLE 2 For a package to be mailed in the United States by parcel post, its length plus its girth (perimeter of cross section) must be at most 84 inches. Find the dimensions of the rectangular box of maximum volume which can be mailed by parcel post.

Step 1 Let x, y, and z be the dimensions of the box, with z the length. We wish to find the maximum of the volume

$$V = xyz$$

given the side condition

$$\text{length} + \text{girth} = z + 2x + 2y = 84.$$

We eliminate z using the side condition and express V as a function of x and y.

$$z = 84 - 2x - 2y,$$
$$V = xy(84 - 2x - 2y).$$

Since x, y, and z cannot be negative the domain is the closed triangle

$$0 \le x, \qquad 0 \le y, \qquad 0 \le 84 - 2x - 2y.$$

This is the same as the closed region

$$0 \le x \le 42, \qquad 0 \le y \le 42 - x.$$

The region is sketched in Figure 11.7.6.

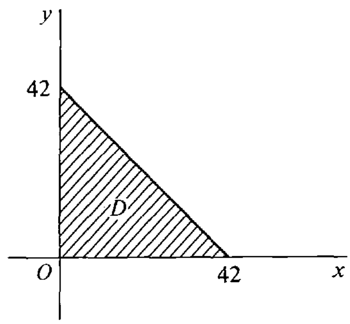


Figure 11.7.6

Step 2 $\dfrac{\partial V}{\partial x} = 84y - 4xy - 2y^2,$

$\dfrac{\partial V}{\partial y} = 84x - 2x^2 - 4xy.$

Step 3 $84y - 4xy - 2y^2 = 0,$

$84x - 2x^2 - 4xy = 0.$

Since $x > 0$ and $y > 0$ at all interior points, we have

$$84 - 4x - 2y = 0,$$
$$84 - 2x - 4y = 0.$$

There is one critical point

$$x = 14, \qquad y = 14,$$
$$V = (84 - 28 - 28) \cdot 14 \cdot 14 = 2(14)^3.$$

Step 4 On all three of the boundary lines

$$x = 0, \qquad y = 0, \qquad 84 - 2x - 2y = 0$$

we have $$V = (84 - 2x - 2y)xy = 0.$$

Therefore the maximum value of V on the boundary of D is 0.

CONCLUSION The maximum of V is at $x = 14$, $y = 14$, where $V = 2(14)^3$ (Figure 11.7.7). The box has dimensions

$$x = 14, \qquad y = 14, \qquad z = 28.$$

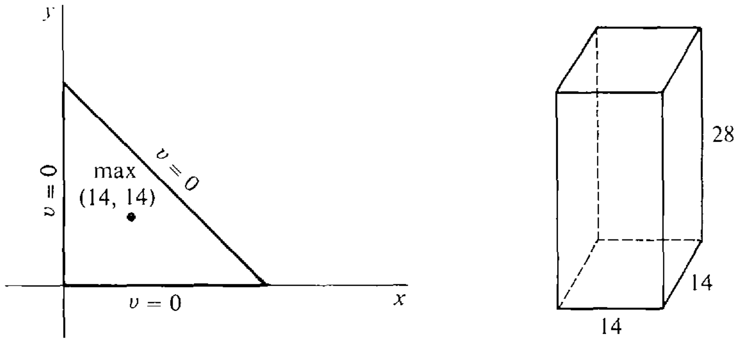


Figure 11.7.7 (a) (b)

We shall now develop a method for finding maxima and minima of functions defined on open regions.

A *bounded open region* D is a set of points given by strict inequalities

$$a < x < b, \qquad f(x) < y < g(x)$$

where f and g are continuous and $f(x) < g(x)$ on (a, b). A closed region with its boundary removed is a bounded open region.

We shall also consider *unbounded open regions*, which are given by strict inequalities where one or more of $a, b, f(x), g(x)$ are replaced by infinity symbols. For example, the following are unbounded open regions:

(1) $-\infty < x < \infty, \qquad f(x) < y < g(x)$.
(2) $0 < x < \infty, \qquad 0 < y < \infty$.
(3) The whole plane $-\infty < x < \infty, \qquad -\infty < y < \infty$.

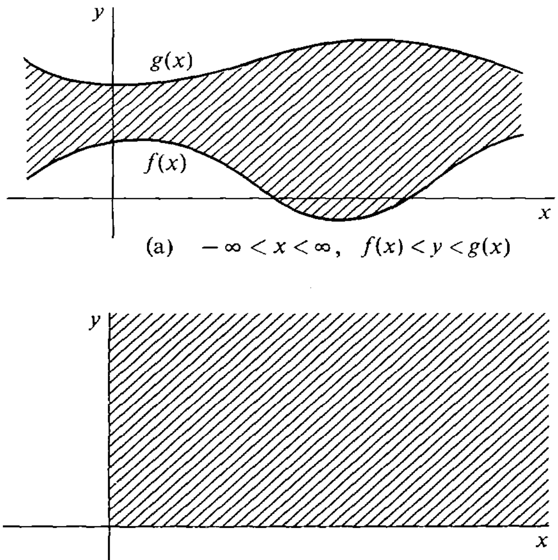
Unbounded open regions are pictured in Figure 11.7.8.

A smooth function whose domain is an open region may or may not have a maximum or minimum. Many problems have at most one critical point, and we shall concentrate on that case. The method can readily be extended to the case of two or more critical points. The Critical Point Theorem holds for open regions as well as closed regions. The corollary below shows how it can be used in maximum or minimum problems.

COROLLARY

Suppose the domain of the function $z = f(x, y)$ is an open region D, and f is smooth on D.

(i) *If f has no critical points it has no maximum or minimum.*



(a) $-\infty < x < \infty, \quad f(x) < y < g(x)$

(b) $0 < x < \infty, \quad 0 < y < \infty$

Figure 11.7.8 Unbounded Open Regions

(ii) *Let f have exactly one critical point (x_0, y_0). If f has a maximum or minimum, it occurs at (x_0, y_0).*

This corollary can be used to show certain functions do not have a maximum or minimum. If we are sure a function has a maximum or minimum, the corollary can be used to find it.

EXAMPLE 3 Show that the function $z = e^x \ln y$ has no maximum or minimum.

The domain is the open region

$$-\infty < x < \infty, \qquad 0 < y < \infty.$$

The partial derivatives are

$$\frac{\partial z}{\partial x} = e^x \ln y, \qquad \frac{\partial z}{\partial y} = \frac{e^x}{y}.$$

There are no critical points because $\partial z/\partial y$ is never zero. Therefore there is no maximum or minimum.

EXAMPLE 4 Show that the function $z = x^2 + 2y^2$ has no maximum.

The domain is the whole plane.

We have

$$\frac{\partial z}{\partial x} = 2x, \qquad \frac{\partial z}{\partial y} = 4y.$$

There is one critical point at (0, 0). At this point, $z = 0$. This is not a maximum because, for example, $z = 3$ at (1, 1). Hence z has no maximum.

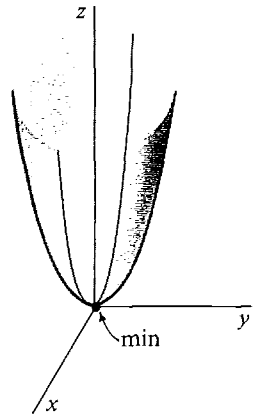


Figure 11.7.9

Notice that z has a minimum at $(0, 0)$ because $x^2 + 2y^2$ is always ≥ 0 (Figure 11.7.9).

EXAMPLE 5 Find the point on the plane $4x - 6y + 2z = 7$ which is nearest to the origin.

Step 1 The distance from the origin to (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$. It is easier to work with the square of the distance, which has a minimum at the same point that the distance does. So we wish to find the minimum of

$$w = x^2 + y^2 + z^2$$

given that

$$4x - 6y + 2z = 7.$$

We eliminate z using the plane equation.

$$z = \tfrac{1}{2}(7 - 4x + 6y),$$
$$w = x^2 + y^2 + \tfrac{1}{4}(7 - 4x + 6y)^2.$$

The domain is the whole (x, y) plane.

Step 2
$$\frac{\partial w}{\partial x} = 2x + 2 \cdot \frac{1}{4}(-4)(7 - 4x + 6y) = -14 + 10x - 12y,$$

$$\frac{\partial w}{\partial y} = 2y + 2 \cdot \frac{1}{4} \cdot 6(7 - 4x + 6y) = 21 - 12x + 20y.$$

Step 3
$$-14 + 10x - 12y = 0,$$
$$21 - 12x + 20y = 0.$$

Solving for x and y we get one critical point

$$x = \tfrac{1}{2}, \qquad y = -\tfrac{3}{4}.$$

CONCLUSION We know from geometry that there is a point on the plane which is closest to the origin (the point where a perpendicular line from the origin meets the plane). Therefore w has a minimum and it must be at the critical point

$$x = \tfrac{1}{2}, \qquad y = -\tfrac{3}{4}.$$

The value of z at this point is

$$z = \tfrac{1}{2}(7 - 4x + 6y) = \tfrac{1}{4}.$$

The answer is $(\frac{1}{2}, -\frac{3}{4}, \frac{1}{4})$. The plane is shown in Figure 11.7.10.

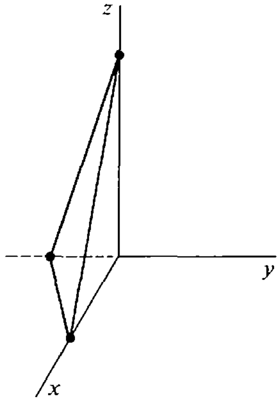


Figure 11.7.10

If we know a function has a maximum or minimum, we can find it simply by finding the critical point. But usually we are not sure whether a function has a maximum or minimum. Here is a method that can be used when a function has a unique critical point in an open region. It is based on the fact that the Extreme Value Theorem holds for closed regions of the hyperreal plane as well as the real plane (because of the Transfer Principle).

Given a real open region D we can find a hyperreal closed region E which contains the same real points as D (Figure 11.7.11).

For example, if D is the real region

$$a < x < b, \qquad f(x) < y < g(x),$$

we can take for E the hyperreal region

$$a + \varepsilon \leq x \leq b - \varepsilon, \qquad f(x) + \varepsilon \leq y \leq g(x) - \varepsilon$$

where ε is positive infinitesimal.

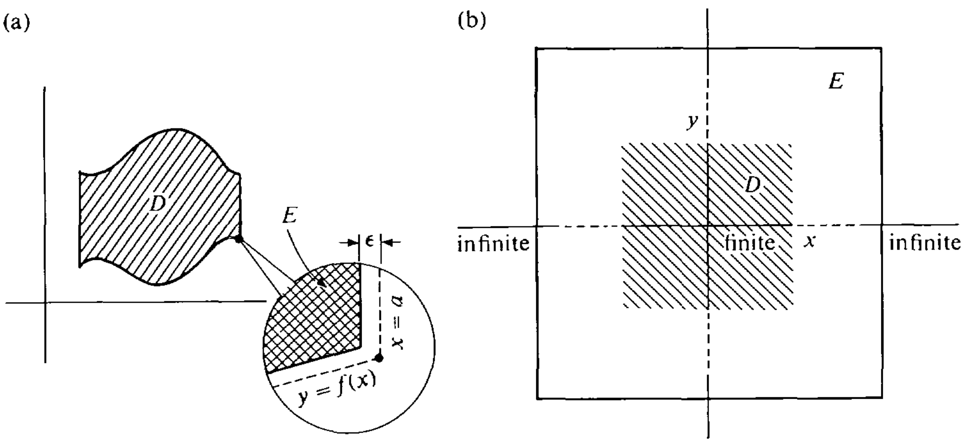


Figure 11.7.11 Hyperreal Closed Regions

If D is the whole real plane we can take for E the hyperreal region

$$-H \le x \le H, \qquad -H \le y \le H$$

where H is positive infinite.

METHOD FOR FINDING MAXIMA AND MINIMA ON AN OPEN REGION

When to Use *$z = f(x, y)$ is a smooth function whose domain is an open region D, and f has exactly one critical point.*

Step 1 *Set up the problem and sketch D if necessary.*

Step 2 *Compute $\partial z/\partial x$ and $\partial z/\partial y$.*

Step 3 *Find the critical point (x_0, y_0) and the value $f(x_0, y_0)$. If we already know there is a maximum (or minimum), it must be (x_0, y_0) and we can stop here.*

Step 4 *Find a hyperreal closed region E with the same real points as D.*

Step 5 *Compare $f(x_0, y_0)$ with the values of f on the boundary of E.*

CONCLUSION *f has a maximum at (x_0, y_0) if $f(x_0, y_0) \ge f(x, y)$ for every boundary point (x, y) of E. Otherwise f has no maximum.*

A similar rule holds for the minimum.

EXAMPLE 6 Find the maximum and minimum, if any, of the function

$$z = \frac{1}{(x + y)^2 + (x + 1)^2 + y^2}.$$

Step 1 The domain is the whole (x, y) plane because the denominator is always positive.

Step 2
$$\frac{\partial z}{\partial x} = -[2(x + y) + 2(x + 1)][(x + y)^2 + (x + 1)^2 + y^2]^{-2},$$
$$\frac{\partial z}{\partial y} = -[2(x + y) + 2y][(x + y)^2 + (x + 1)^2 + y^2]^{-2}.$$

Step 3 The partial derivatives are zero when

$$2(x + y) + 2(x + 1) = 0, \qquad 2(x + y) + 2y = 0,$$

or

$$2x + y + 1 = 0, \qquad x + 2y = 0.$$

The critical point is

$$x = -\tfrac{2}{3}, \qquad y = \tfrac{1}{3}, \quad \text{and} \quad z = 3.$$

Step 4 Let E be the hyperreal region

$$-H \le x \le H, \qquad -H \le y \le H$$

where H is positive infinite.

Step 5 At a boundary point of E where $x = \pm H$, $(x + 1)^2$ is infinite so z is infinitesimal. At a boundary point where $y = \pm H$, y^2 is infinite so again z is infinitesimal.

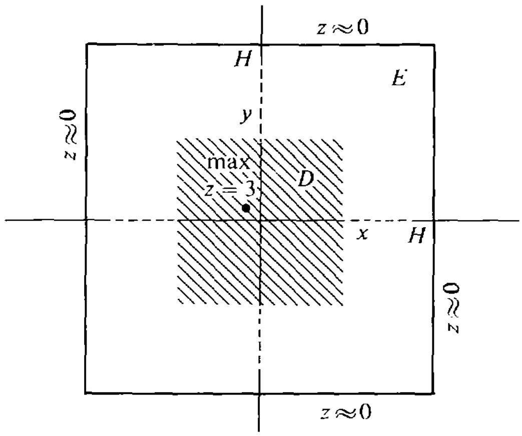


Figure 11.7.12

CONCLUSION z has a maximum of 3 at the critical point $(-\frac{2}{3}, \frac{1}{3})$. z has no minimum. The region E is sketched in Figure 11.7.12.

EXAMPLE 7 Find the dimensions of the box of volume one without a top which has the smallest area (if there is one). The box is sketched in Figure 11.7.13.

Step 1 Let x, y, and z be the dimensions of the box, with z the height. We want the minimum of the area

$$A = xy + 2xz + 2yz$$

given that

$$xyz = 1.$$

Eliminating z, we have

$$z = \frac{1}{xy},$$

$$A = xy + \frac{2}{y} + \frac{2}{x}.$$

The domain is the open region $x > 0, y > 0$ (see Figure 11.7.14).

Step 2 $\frac{\partial A}{\partial x} = y - \frac{2}{x^2}, \quad \frac{\partial A}{\partial y} = x - \frac{2}{y^2}.$

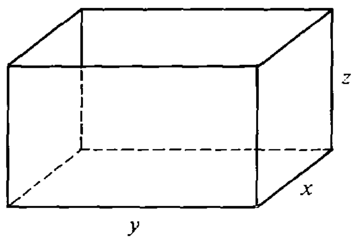


Figure 11.7.13

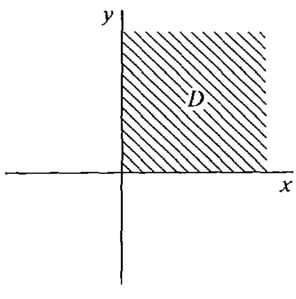


Figure 11.7.14

Step 3 $y - \dfrac{2}{x^2} = 0, \qquad x - \dfrac{2}{y^2} = 0.$

The critical point is $\qquad x = \sqrt[3]{2},\ y = \sqrt[3]{2},$

where $\qquad A = 2^{2/3} + 2 \cdot 2^{-1/3} + 2 \cdot 2^{-1/3} = 2^{2/3} + 2^{5/3}.$

Step 4 Take for E the hyperreal region $\varepsilon \le x \le H$, $\varepsilon \le y \le H$ where ε is positive infinitesimal and H is positive infinite.

Step 5 Let (x, y) be a boundary point of E. As we can see from Figure 11.7.15, there are four possible cases.

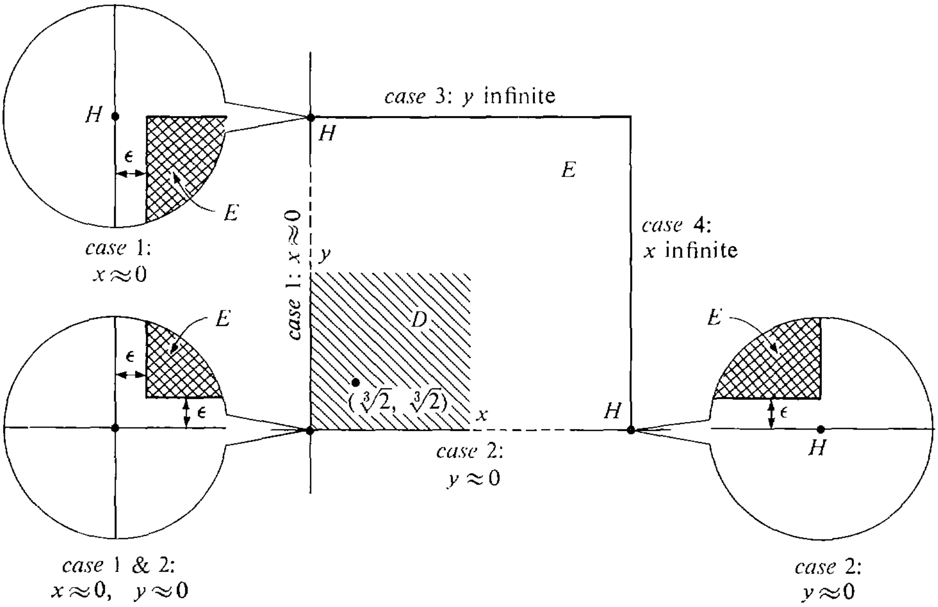


Figure 11.7.15

Case 1 x is infinitesimal. Then A is infinite because $2/x$ is.

Case 2 y is infinitesimal. A is infinite because $2/y$ is.

Case 3 x is not infinitesimal and y is infinite. A is infinite because xy is.

Case 4 y is not infinitesimal and x is infinite. A is infinite because xy is.

CONCLUSION A is infinite and hence greater than $2^{2/3} + 2^{5/3}$ on the boundary of E. Therefore A has a minimum at the critical point

$$x = \sqrt[3]{2}, \qquad y = \sqrt[3]{2}.$$

The box has dimensions

$$x = \sqrt[3]{2}, \qquad y = \sqrt[3]{2}, \qquad z = \frac{1}{xy} = \frac{1}{\sqrt[3]{4}}.$$

PROBLEMS FOR SECTION 11.7

In Problems 1–10, find the maxima and minima.

1 $x^2 + xy + y^2, \quad -1 \le x \le 1, \quad -1 \le y \le 1$

2 $-x^2 - 2y^2 + x - y + 2, \quad -1 \le x \le 1, \quad -1 \le y \le 1$

3 $x^2 + 2y^2 - 2x + 8y + 3, \quad -3 \le x \le 3, \quad -3 \le y \le x$

4 $x - xy + 2y, \quad -4 \le x \le 4, \quad -4 \le y \le x$

5 $x^2 - y^2 - 2x + 2y + 3, \quad 0 \le x \le 2, \quad 0 \le y \le 2x$

6 $xy + \frac{1}{x} + \frac{8}{y}, \quad \frac{1}{4} \le x \le 4, \quad 1 \le y \le 8$

7 $\sin x + \sin y, \quad 0 \le x \le \pi, \quad 0 \le y \le \pi$

8 $\sin x \sin y, \quad 0 \le x \le \pi, \quad 0 \le y \le \pi$

9 $x^2 + y^2 - y, \quad -1 \le x \le 1, \quad x^2 \le y \le 1$

10 $4 - x^2 - y^2, \quad -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$

In Problems 11–16, find the maximum and minimum subject to the given side conditions.

11 $z(x - y), \quad x + y + z = 1, \quad 0 \le x \le 1, \quad 0 \le y \le 1$

12 $xyz, \quad z = x + y, \quad -1 \le x \le 1, \quad -1 \le y \le 1$

13 $x + y + z, \quad z = x^2 + y^2, \quad z \le 1$

14 $x + y + z, \quad z = \sqrt{3 - x^2 - y^2}$

15 $x^2 + y^2 + z^2, \quad z = xy, \quad -1 \le x \le 1, \quad -1 \le y \le 1$

16 $xy + yz + xz, \quad xyz = 1, \quad \frac{1}{4} \le x \le 4, \quad \frac{1}{4} \le y \le 4$

In Problems 17–26, determine whether the maxima and minima exist, and if so, find them.

17 $x^2 + 4x + y^2$

18 $-x^2 - y^2 + 2x - 4y$

19 $1/xy, \quad 0 < x, \quad 0 < y$

20 $x^3 + 2x + y^3 - y^2$

21 $xy + \frac{1}{x} + \frac{8}{y}, \quad 0 < x, \quad 0 < y$

22 $x + 4y + \frac{1}{x} + \frac{1}{y}, \quad 0 < x, \quad 0 < y$

23 $\frac{1}{x^2 + y^2 + 1}$

24 $\frac{1}{\sqrt{1 - x^2 - y^2}}, \quad x^2 + y^2 < 1$

25 $x^2 - 4y^2$

26 $x^y, \quad 0 < x$

27 Find three positive numbers x, y, and z such that $x + y + z = 8$ and x^2yz is a maximum.

28 Find three positive numbers x, y, and z such that $x + y + z = 100$ and x^2y^2z is a maximum.

29 A package can be sent overseas by the air mail small packet rate if its length plus girth is at most 36 inches. Find the dimensions of the rectangular solid of maximum volume which can be sent by the small packet rate.

30 Find the volume of the largest rectangular solid which can be inscribed in a sphere of radius one.

31 Find the volume of the largest rectangular solid with faces parallel to the coordinate planes which can be inscribed in the ellipsoid $x^2/4 + y^2 + z^2/9 = 1$.

32 A triangle with sides a, b, c and perimeter $p = a + b + c$ has area

$$A = \sqrt{2p(2p - a)(2p - b)(2p - c)}.$$

Find the triangle of maximum area with perimeter $p = 1$.

33 Find the point on the plane $x + 2y - z = 10$ which is nearest to the origin.

34 Find the point on the plane $x + y + z = 0$ which is nearest to the point $(1, 2, 3)$.

35 Find the points on the surface $xyz = 1$ which are nearest to the origin.

36 Find the point on the surface $z = xy + 1$ which is nearest to the origin.

37 Show that the rectangular solid with volume one and minimum surface area is the unit cube.

38 Show that the rectangular solid with surface area six and maximum volume is the unit cube.

39 A rectangular box with volume V in.^{3} is to be built with the sides and bottom made of material costing one cent per square inch, and the top costing two cents per square inch. Find the shape with the minimum cost.

40 A firm can produce and sell x units of one commodity and y units of another commodity for a profit of

$$P(x, y) = 100x + 200y - 10xy - x^2 - 500.$$

Due to limitations on plant capacity, $x \le 10$ and $y \le 5$. Find the values of x and y where the profit is a maximum.

41 x units of commodity one and y units of commodity two can be produced and sold at a profit of

$$P(x, y) = 400x + 500y - x^2 - y^2 - xy - 20000.$$

Find the values of x and y where the profit is a maximum.

42 x units of commodity one can be produced at a cost of

$$C_1(x) = 1000 + 5x,$$

and y units of commodity two can be produced at a cost of

$$C_2(y) = 2000 + 8y.$$

Moreover, x units of one and y units of two can be sold for a total revenue of

$$R(x, y) = 100\sqrt{x} + 200\sqrt{y} + 10\sqrt{xy}.$$

Find the values of x and y where the profit is a maximum.

43 Suppose that with x man hours of labor and y units of capital, $z = f(x, y)$ units of a commodity can be produced. The ratio z/x is called the average production per man hour. Show that $\partial z/\partial x = z/x$ when the average production per man hour is a maximum.

□ 44 (*Method of Least Squares*) A straight line is to be fit as closely as possible to the set of three experimentally observed points $(1, 6)$, $(2, 9)$, and $(3, 10)$. The line which *best fits* these points is the line $y = mx + b$ for which the sum of the squares of the errors,

$$E = [(m \cdot 1 + b) - 6]^2 + [(m \cdot 2 + b) - 9]^2 + [(m \cdot 3 + b) - 10]^2,$$

is a minimum. Find m and b such that E is a minimum.

11.8 HIGHER PARTIAL DERIVATIVES

Given a function $z = f(x, y)$ of two variables, the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ may themselves be differentiated with respect to either x or y. Thus there are four possible second partial derivatives. Here they are.

Twice with respect to x: f_{xx}, *or* $\dfrac{\partial^2 z}{\partial x^2}$.

Twice with respect to y: f_{yy}, *or* $\dfrac{\partial^2 z}{\partial y^2}$.

First with respect to x and then with respect to y:

$$(f_x)_y = f_{xy}, \quad or \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x}.$$

First with respect to y and then with respect to x:

$$(f_y)_x = f_{yx}, \quad or \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y}.$$

Similar notation is used for three or more variables and for higher partial derivatives.

EXAMPLE 1 Find the four second partial derivatives of

$$z = e^x \sin y + xy^2.$$

$$\frac{\partial z}{\partial x} = e^x \sin y + y^2, \qquad \frac{\partial z}{\partial y} = e^x \cos y + 2xy.$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(e^x \sin y + y^2) = e^x \sin y,$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(e^x \cos y + 2xy) = -e^x \sin y + 2x.$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}(e^x \sin y + y^2) = e^x \cos y + 2y,$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}(e^x \cos y + 2xy) = e^x \cos y + 2y.$$

Notice that in this example the two mixed second partials $\partial^2 z/\partial y\,\partial x$ and $\partial^2 z/\partial x\,\partial y$ are equal. The following theorem shows that it is not just a coincidence.

THEOREM 1 (Equality of Mixed Partials)

Suppose that the first and second partial derivatives of $z = f(x, y)$ are continuous at (a, b). Then at (a, b),

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}.$$

Discussion This is a surprising theorem. $\partial^2 z/\partial y\,\partial x$ is the rate of change with respect to y of the slope $\partial z/\partial x$, while $\partial^2 z/\partial x\,\partial y$ is the rate of change with respect to x of the slope $\partial z/\partial y$. There is no simple intuitive way to see that these should be equal.

As a matter of fact, there *are* functions $f(x, y)$ whose mixed second partial derivatives exist but are not equal. One such example is the function

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

We have left the computation of the second partials of $f(x, y)$ as a problem. It turns out that at (0, 0),

$$\partial^2 f/\partial x\, \partial y = 1, \qquad \partial^2 f/\partial y\, \partial x = -1.$$

How can this be in view of Theorem 1? The answer is that in this example the second partial derivatives exist but are not continuous at (0, 0), so the theorem does not apply. We shall only rarely encounter functions whose second partial derivatives are not continuous, so in all ordinary problems it is true that the mixed partials are equal. We shall prove the theorem later. We now turn to some applications. Our first application concerns mixed third partial derivatives.

If the third partial derivatives of $z = f(x, y)$ are continuous, then

$$\frac{\partial^3 z}{\partial x\, \partial x\, \partial y} = \frac{\partial^3 z}{\partial x\, \partial y\, \partial x} = \frac{\partial^3 z}{\partial y\, \partial x\, \partial x},$$

so we write $\dfrac{\partial^3 z}{\partial x^2\, \partial y}$ for each of them. Similarly,

$$\frac{\partial^3 z}{\partial x\, \partial y\, \partial y} = \frac{\partial^3 z}{\partial y\, \partial x\, \partial y} = \frac{\partial^3 z}{\partial y\, \partial y\, \partial x}$$

and we write $\dfrac{\partial^3 z}{\partial x\, \partial y^2}$ for each of them.

We prove the first equation as an illustration.

$$\frac{\partial^3 z}{\partial x\, \partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x\, \partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y\, \partial x}\right) = \frac{\partial^3 z}{\partial x\, \partial y\, \partial x}.$$

EXAMPLE 2 Find the third partial derivatives of $z = e^{2x} \sin y$.

$$\frac{\partial z}{\partial x} = 2e^{2x} \sin y, \qquad \frac{\partial z}{\partial y} = e^{2x} \cos y,$$

$$\frac{\partial^2 z}{\partial x^2} = 4e^{2x} \sin y, \qquad \frac{\partial^2 z}{\partial y^2} = -e^{2x} \sin y,$$

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y} = 2e^{2x} \cos y,$$

$$\frac{\partial^3 z}{\partial x^3} = 8e^{2x} \sin y, \qquad \frac{\partial^3 z}{\partial y^3} = -e^{2x} \cos y,$$

$$\frac{\partial^3 z}{\partial x^2\, \partial y} = 4e^{2x} \cos y, \qquad \frac{\partial^3 z}{\partial x\, \partial y^2} = -2e^{2x} \sin y.$$

If a function has continuous second partial derivatives we may apply the Chain Rule to the first partial derivatives. For one independent variable,

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{dy}{dt},$$
$$\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}.$$

EXAMPLE 3 If $z = f(x, y)$ has continuous second partials, $x = r\cos\theta$, and $y = r\sin\theta$, find $\partial^2 z/\partial r^2$.

We use the Chain Rule three times.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta.$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial r}\left(\sin\theta\frac{\partial z}{\partial y}\right)\\
&= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\\
&= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial r}\right)\cos\theta + \left(\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right)\sin\theta\\
&= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta\\
&= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta.
\end{aligned}$$

By holding one variable fixed in Theorem 1, we get equalities of mixed partials for functions of three or more variables.

COROLLARY (Equality of Mixed Partials, Three Variables)

Suppose that the first and second partial derivatives of $w = f(x, y, z)$ are continuous at (a, b, c). Then at (a, b, c),

$$\frac{\partial^2 w}{\partial y\,\partial x} = \frac{\partial^2 w}{\partial x\,\partial y}, \qquad \frac{\partial^2 w}{\partial z\,\partial x} = \frac{\partial^2 w}{\partial x\,\partial z}, \qquad \frac{\partial^2 w}{\partial z\,\partial y} = \frac{\partial^2 w}{\partial y\,\partial z}.$$

PROOF OF THEOREM 1 The plan is to prove a corresponding result for average slopes and then use the Mean Value Theorem, which states that the average slope of a function on an interval is equal to the slope at some point in the interval.

Let Δx and Δy be positive infinitesimals. We hold Δx and Δy fixed. The first and second partial derivatives of $f(x, y)$ exist for (x, y) in the rectangle

$$a \le x \le a + \Delta x, \qquad b \le y \le b + \Delta y.$$

We shall use the following notation for average slopes in the x and y directions:

$$g(y) = \frac{f(a + \Delta x, y) - f(a, y)}{\Delta x}, \qquad h(x) = \frac{f(x, b + \Delta y) - f(x, b)}{\Delta y}.$$

Label the corners of the rectangle A, B, C, and D as in Figure 11.8.1.

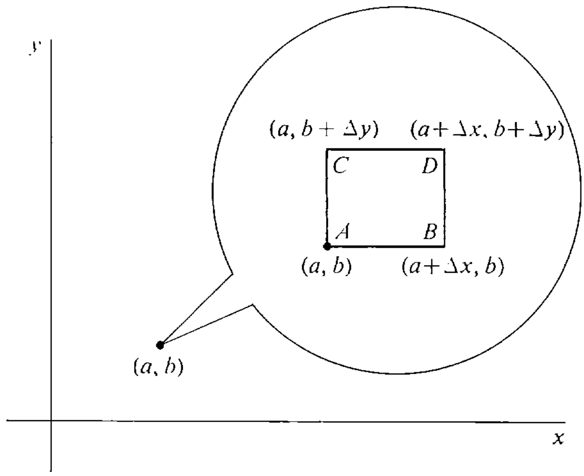


Figure 11.8.1

We first show that the following two quantities are equal:

$$\frac{\Delta^2 f}{\Delta y\,\Delta x} = \frac{g(b+\Delta y) - g(b)}{\Delta y}, \qquad \frac{\Delta^2 f}{\Delta x\,\Delta y} = \frac{h(a+\Delta x) - h(a)}{\Delta x}.$$

$\Delta^2 f/\Delta y\,\Delta x$ is the average slope in the y direction of the average slope in the x direction of f.

$$\frac{\Delta^2 f}{\Delta y\,\Delta x} = \frac{g(b+\Delta y) - g(b)}{\Delta y} = \frac{\dfrac{f(D) - f(C)}{\Delta x} - \dfrac{f(B) - f(A)}{\Delta x}}{\Delta y}$$

$$= \frac{f(D) - f(C) - f(B) + f(A)}{\Delta x\,\Delta y} = \frac{\dfrac{f(D) - f(B)}{\Delta y} - \dfrac{f(C) - f(A)}{\Delta y}}{\Delta x}$$

$$= \frac{h(a+\Delta x) - h(a)}{\Delta x} = \frac{\Delta^2 f}{\Delta x\,\Delta y}.$$

By the Mean Value Theorem,

$$\frac{\Delta^2 f}{\Delta y\,\Delta x} = \frac{g(b+\Delta y) - g(b)}{\Delta y} = g'(y_1),$$

where $b < y_1 < b + \Delta y$. Using the Mean Value Theorem again,

$$g'(y_1) = \frac{\partial\left(\dfrac{f(a+\Delta x, y) - f(a, y)}{\Delta x}\right)}{\partial y}(a, y_1)$$

$$= \frac{\dfrac{\partial f}{\partial y}(a+\Delta x, y_1) - \dfrac{\partial f}{\partial y}(a, y_1)}{\Delta x}$$

$$= \frac{\partial^2 f}{\partial x\,\partial y}(x_1, y_1),$$

where $a < x_1 < a + \Delta x$. Since $\partial^2 f/\partial x\, \partial y$ is continuous at (a, b),

$$st\left(\frac{\Delta^2 f}{\Delta y\, \Delta x}\right) = \frac{\partial^2 f}{\partial x\, \partial y}(a, b).$$

A similar computation gives

$$st\left(\frac{\Delta^2 f}{\Delta x\, \Delta y}\right) = \frac{\partial^2 f}{\partial y\, \partial x}(a, b).$$

Therefore

$$\frac{\partial^2 f}{\partial x\, \partial y}(a, b) = \frac{\partial^2 f}{\partial y\, \partial x}(a, b).$$

We conclude this section by stating a Second Derivative Test for maxima and minima of functions of two variables. In practice the test often fails except on small regions D. We therefore have emphasized the tests in the preceding section rather than the Second Derivative Test.

SECOND DERIVATIVE TEST

Suppose $z = f(x, y)$ has continuous first and second partial derivatives on a rectangle D, and (a, b) is a critical point of f in D.

(i) *f has a minimum at (a, b) if*

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 > 0, \qquad \frac{\partial^2 z}{\partial x^2} > 0$$

at every point of D.

(ii) *f has a maximum at (a, b) if*

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 > 0, \qquad \frac{\partial^2 z}{\partial x^2} < 0$$

at every point of D.

(iii) *f has a saddle point at (a, b) if*

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 < 0 \qquad \text{at } (a, b).$$

For an indication of the proof of the Second Derivative Test, see Extra Problem 55 at the end of this chapter.

PROBLEMS FOR SECTION 11.8

In Problems 1–12, find all the second partial derivatives.

1 $z = x^2 + 2y^2$

2 $z = -3xy$

3 $z = ax^2 + bxy + cy^2$

4 $z = (ax + by + c)^n$

5 $z = xe^{x+y}$

6 $z = \cos(x + y) + \sin(x - y)$

7 $z = \ln(ax + by)$

8 $z = \sqrt{x^2 + y^2}$

9 $z = x^a y^b$

10 $w = xyz$

11 $w = \sqrt{x + y + z}$

12 $w = z \cos x + z \sin y$

In Problems 13–16, find all the third partials.

13 $z = 2x^3y^2 - 6x^2y^3$

14 $z = \sqrt{xy}$

15 $z = e^{ax+by}$

16 $z = \cos x \sin y$

17 If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, find $\partial^2 z/\partial\theta^2$.

18 If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, find $\partial^2 z/\partial\theta\,\partial r$.

In Problems 19–24, find $\partial^2 z/\partial x^2$, $\partial^2 z/\partial y^2$, and $\partial^2 z/\partial x\,\partial y$.

19 $z = f(u), \quad u = ax + by$

20 $z = f(u), \quad u = xy$

21 $z = g(x) + h(y)$

22 $z = g(x)h(y)$

23 $z = u^n, \quad u = f(x, y)$

24 $z = e^u, \quad u = f(x, y)$

In Problems 25–28 find $\partial^2 z/\partial s^2$, $\partial^2 z/\partial t^2$, and $\partial^2 z/\partial s\,\partial t$.

25 $z = ax + by, \quad x = f(s, t), \quad y = g(s, t)$

26 $z = xy, \quad x = f(s, t), \quad y = g(s, t)$

27 $z = f(x), \quad x = g(s) + h(t)$

28 $z = f(x, y), \quad x = g(s), \quad y = h(t)$

29 Suppose $z = f(x + at) + g(x - at)$ where f and g have continuous second derivatives. Show that z satisfies the *wave equation*

$$a^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}.$$

30 Show that if

$$z = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

then all the second partial derivatives of z are constant.

□ **31** Let $f(x, y)$ be the function

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Find the first and second partial derivatives of f. Show that

(a) $\partial^2 f/\partial x\,\partial y \neq \partial^2 f/\partial y\,\partial x$ at $(0, 0)$,

(b) $\partial^2 f/\partial x\,\partial y$ is not continuous at $(0, 0)$.

Hint: All the derivatives must be computed separately for the cases $(x, y) = (0, 0)$ and $(x, y) \neq (0, 0)$.

EXTRA PROBLEMS FOR CHAPTER 11

In Problems 1–4, make a contour map and sketch the surface.

1 $\frac{1}{9}x^2 + y^2 = z^2, \; -4 \le z \le 4$

2 $z = x^2 + \frac{1}{4}y^2, \; -4 \le z \le 4$

3 $z = x - y^2, \; -2 \le x \le 2, \; -2 \le y \le 2$

4 $z = \sqrt{xy}, \; -4 \le x \le 4, \; -4 \le y \le 4$

5 Find the largest set you can on which $f(x, y) = y + 1/x^2$ is continuous.

6 Find the largest set you can on which $f(x, y) = \sqrt{x^2 - y}/x$ is continuous.

7 Find the largest set you can on which $f(x, y) = \ln(1/x + 1/y)$ is continuous.

8 Find the largest set you can on which $f(x, y, z) = (\ln(x + y))/z$ is continuous.

9 Find the partial derivatives of $f(x, y) = ax - by$.

10 Find the partial derivatives of $f(x, y) = a_1 \sin(b_1 x) + a_2 \cos(b_2 y)$.

11 Find the partial derivatives of $z = \ln x/\ln y$.

12 Find the partial derivatives of $w = (x - y)e^z$.

13 Find the increment and total differential of $z = 1/x + 2/y$.

14 Find the increment and total differential of $z = \sqrt{x + y}$.

15 Find the tangent plane of $z = x^3 y + 4$ at $(2, 0)$.

16 Find the tangent plane of $z = \arcsin(xy)$ at $(3, \frac{1}{4})$.

17 Find dz/dt by the Chain Rule where $z = \log_{(2t+1)}(3t + 2)$.

18 Find $\partial z/\partial s$ and $\partial z/\partial t$ where $z = x/y, x = e^{s+t}, y = as + bt$.

19 A particle moves in space so that $dx/dt = z\cos x$, $dy/dt = z \sin y$, $dz/dt = 1$. Find the rate of change of the distance from the origin when $x = 0, y = 0, z = 1$.

20 A company finds that it can produce x units of item 1 at a total cost of $x + 100\sqrt{x}$ dollars, and y units of item 2 at a total cost of $20y - \sqrt{y}$ dollars. Moreover, x units of item 1 and y units of item 2 can be sold for a total revenue of $10x + 30y - xy/100$ dollars. If z is the total profit (revenue minus cost), find $\partial z/\partial x$ and $\partial z/\partial y$, the partial marginal profit with respect to items 1 and 2.

21 Find the tangent line and slope of $x^4 + y^4 = 17$ at $(2, 1)$.

22 Find the tangent plane to the surface $x^4 + y^4 + z^2 = 18$ at $(1, 2, 1)$.

23 Find the maxima and minima of

$$z = x^2 + y^2 - 2x - 4y + 4, \qquad 0 \le x \le 3, \qquad x \le y \le 3.$$

24 Find the maxima and minima of

$$z = x + 4y + \frac{1}{x} + \frac{1}{y}, \qquad \frac{1}{4} \le x \le 4, \qquad \frac{1}{4} \le y \le 4.$$

25 Determine whether the surface $z = \log_x y, x > 1, y > 0$ has any maxima or minima.

26 Find the dimensions of the rectangular box of maximum volume such that the sum of the areas of the bottom and sides is one.

27 Find all second partial derivatives of $z = \arctan(xy)$.

28 Find all second partial derivatives of $w = (x^2 - y^2)z$.

29 Find $\partial^2 z/\partial r^2$ if $z = f(x, y)$, $x = r\cosh\theta$, $y = r\sinh\theta$.

□ **30** Let $f(x)$ be continuous for $a < x < b$. Prove that the function $F(u, v) = \int_u^v f(x)\,dx$ is continuous whenever u and v are in (a, b).

□ **31** Prove that $f(x, y)$ is continuous at (a, b) if and only if the following ε, δ condition holds. For every real $\varepsilon > 0$ there is a real $\delta > 0$ such that whenever (x, y) is within δ of (a, b), $f(x, y)$ is within ε of $f(a, b)$.

□ **32** Let
$$f(x, y) = \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are rational,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is discontinuous at every point.

□ **33** Prove that

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if and only if for every real $\varepsilon > 0$ there is a real $\delta > 0$ such that whenever (x, y) is different from but within δ of (a, b), $f(x, y)$ is within ε of L. (See Problems for Section 11.2.)

□ **34** Prove that the following are equivalent.

(a) $f_x(x, y) = 0$ for all (x, y).

(b) The value of $f(x, y)$ depends only on y.

□ 35 Prove that the following are equivalent.
(a) $f_x(x, y) = 0$ and $f_y(x, y) = 0$ for all (x, y).
(b) f is a constant function.

□ 36 A function $z = f(x, y)$ is said to be *differentiable* at (x, y) if it satisfies the conclusion of the Increment Theorem. That is, whenever Δx and Δy are infinitesimal,

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

for some infinitesimals ε_1 and ε_2 which depend on Δx and Δy. Prove the Chain Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

assuming only that the functions $z = f(x, y)$, $x = g(t)$, and $y = h(t)$ are differentiable.

□ 37 Prove that the function $f(x, y) = |xy|$ is differentiable but not smooth at $(0, 0)$.

□ 38 A smooth function $z = f(x, y)$ is said to be *homogeneous of degree n* if

(1) $$f(tx, ty) = t^n f(x, y)$$

for all x, y, and t. Prove that if $z = f(x, y)$ is homogeneous of degree n then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

Hint: Differentiate Equation 1 with respect to t and set $t = 1$.

□ 39 Suppose $f(x, y)$ has continuous second partial derivatives and that $\partial^2 f/\partial x\,\partial y$ is identically zero (i.e., zero at every point (x, y)). Prove that $f(x, y) = g(x) + h(y)$ for some functions g and h.

□ 40 Find all functions $f(x, y)$ all of whose second partial derivatives are identically zero.