Cooperative Games on Infinite Trees*

Adam Brandenburger[†], H. Jerome Keisler[‡], and Paula Miret[§]

December 31, 2018

Abstract

We consider cooperative games with infinitely many players, where: (i) the players occupy different nodes in a tree; (ii) superadditivity holds; and (iii) for each finite set of nodes, additivity holds on connected components. We show that, under some regularity conditions on the characteristic function, such games have non-empty cores. Examples are given to show that none of our conditions can be dropped without losing non-emptiness.

1 Introduction

The core is a fundamental solution concept in cooperative game theory (Owen [1995]). In a game in which the core is non-empty, it identifies possible allocations of value among the players which can arise under a free-form bargaining process. In games in which the core is empty, this bargaining process has no stable outcome. An important question for both the understanding and the design of cooperative games is to identify families of games for which the core is always non-empty.

We move to some formal definitions. A *cooperative game* is a pair $G = (N, \nu)$ such that $\nu\colon (2^N \setminus \{\emptyset\}) \to \mathbb{R}$. To simplify notation, we also define $\nu(\emptyset) = 0$, so ν has domain 2^N. The set N comprises the *players* in the game, and the map ν is called the *characteristic function* of the game. Note that the set of players might be finite or infinite. The *core* of G is the set of all functions $x\colon N \to \mathbb{R}$ such that $\sum_{i\in N} x_i = \nu(N)$ and $\sum_{i\in S} x_i \geq \nu(S)$ for all $\emptyset \neq S \subseteq N$.

*Financial support from the NYU Stern School of Business, NYU Shanghai, and J.P. Valles is gratefully acknowledged.

†NYU Stern School of Business, NYU Tandon School of Engineering, NYU Shanghai, New York University, New York, NY 10012, U.S.A., adam.brandenburger@nyu.edu, http://www.adambrandenburger.com

‡Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, U.S.A., keisler@math.wisc.edu, http://www.math.wisc.edu/~keisler

§pmiretf@gmail.com

A cooperative game will be called *superadditive* if for all finite $S, T \subseteq N$ such that $S \cap T = \emptyset$ we have $\nu(S \cup T) \geq \nu(S) + \nu(T)$. An undirected graph is called a *forest* if it has no cycles of length greater than two. A *tree* is a connected forest.

We are interested in cooperative games where the players occupy different nodes in a tree. Our result is that such games have non-empty cores, provided that superadditivity holds, additivity holds on connected components for each finite set of nodes, and the characteristic function satisfies some regularity conditions. We go on to provide examples showing that none of our conditions can be dropped without losing non-emptiness. For games with a finite number of players, non-emptiness of the core was already established by Demange (2004). Related analyses on finite games are Herrings et al. (2010) and Igarashi and Yamamoto (2012).

2 The Result

We first prove a general result which we use to prove our result about infinite trees. We need an additional definition. If $G = (N, \nu)$ is a cooperative game and $M \subseteq N$, the *subgame* $G \restriction M$ is the cooperative game $H = (M, \mu)$ such that $\mu(S) = \nu(S)$ for each $S \subseteq M$.

Theorem 1. *Suppose $G = (N, \nu)$ is a cooperative game. Assume that:*

- (Superadditivity) For all finite $S, T \subseteq N$ such that $S \cap T = \emptyset$, we have $\nu(S \cup T) \geq \nu(S) + \nu(T)$.
- (Non-negativity) $\nu(S) \geq 0$ for every finite $S \subseteq N$.
- (Continuity) If $S_0 \subseteq S_1 \subseteq \cdots$ is a countable increasing chain of finite subsets of N, then $\nu(\bigcup_n S_n) = \lim_{n\to\infty} \nu(S_n)$.
- (Countable Character) There is a countable set $C \subseteq N$ such that $\nu(S) = \nu(S \cap C)$ for every set $S \subseteq N$.
- (Finite Subgames) For every finite set $A \subseteq N$ there is a finite subset B such that $A \subseteq B \subseteq N$ and the subgame $G \restriction B$ has non-empty core.

Then G has a non-empty core.

Proof. We first prove the result in the case that N is countable. By Superadditivity and Non-negativity, if $S \subseteq T \subseteq N$ then $\nu(S) \leq \nu(T)$. Then by Finite Subgames, there is an increasing chain $C_0 \subseteq C_1 \subseteq \cdots$ of finite subsets of N such that $N = \bigcup_n C_n$ and for each n, the finite subgame $G \restriction C_n$ has a non-empty core. Hence for each n, $0 \leq \nu(C_n) \leq \nu(N)$, and there is a function $x_n\colon C_n \to \mathbb{R}$ that belongs to the core of $G \restriction C_n$. Then for each n and $S \subseteq C_n$ we have $\sum_{b\in C_n} x_n(b) = \nu(C_n)$, and $\sum_{b\in S} x_n(b) \geq \nu(S)$. It follows that each finite subset of the following set of inequalities has a solution $\{y(b)\colon b \in N\}$:

$$\sum_{b\in S} y(b) \geq \nu(S) \text{ for each } n \in \mathbb{N} \text{ and } S \subseteq C_n, \tag{1}$$

$$\sum_{b\in C_n} y(b) \leq \nu(N) \text{ for each } n \in \mathbb{N}. \tag{2}$$

Note that (1) and (2) imply that $0 \leq y(b) \leq \nu(N)$ for each $b \in N$.

Claim 1: There is a function x from N into $[0, \nu(N)]$ such that (1) and (2) hold with $x(\cdot)$ in place of $y(\cdot)$. The proof of this claim uses the compactness theorem in first order logic, and is given in an appendix.

By Continuity and (1) for $x(\cdot)$, for each $S \subseteq N$ we have

$$\sum_{b\in S} x(b) = \lim_{n\to\infty} \sum_{b\in S\cap C_n} x(b) \geq \lim_{n\to\infty} \nu(S \cap C_n) = \nu(S). \tag{3}$$

Then by (2) and (3), x belongs to the core of G.

In the general case where N is not necessarily countable, we use Countable Character to find a countable set $C \subseteq N$ such that $\nu(S) = \nu(S \cap C)$ for all $S \subseteq N$. It follows that the countable subgame $G \restriction C$ satisfies all the hypotheses, so by the above paragraph, $G \restriction C$ has an element x in its core. Let $z\colon N \to \mathbb{R}$ be the function that agrees with x on C and has value 0 on $N \setminus C$. Then

$$\sum_{i\in N} z_i = \sum_{i\in C} x_i = \nu(C) = \nu(N \cap C) = \nu(N),$$

and for each $S \subseteq N$,

$$\sum_{i\in S} z_i = \sum_{i\in S\cap C} x_i \geq \nu(S \cap C) = \nu(S).$$

Therefore z belongs to the core of G. □

We use Theorem 1 to prove the following result for cooperative games on trees.

Theorem 2. *Suppose $G = (N, \nu)$ is a cooperative game and (N, E) is an undirected graph. Assume Superadditivity, Continuity, Countable Character, and:*

- (Forest) (N, E) has no cycles of length greater than two.

- (Additivity on Components) For every finite $S \subseteq N$, if $S = \bigcup_{m<n} T_m$ is the unique decomposition of S into connected components, then

$$\nu(S) = \sum_{m<n} \nu(T_m).$$

Then G has a non-empty core.

Proof. As mentioned in the Introduction, this result is known in the case that N is finite. We will use Theorem 1 above to get the general case. Assume first that G has the following additional properties:

- Non-negativity.
- (Connectivity) Any two nodes $i, j \in N$ are connected by a finite path in (N, E).

It is clear that every finite subgame of G satisfies all the hypotheses except perhaps Connectivity. Since each path connecting two nodes is finite, every finite set of nodes $A \subseteq N$ is contained in a finite connected set $B \subseteq N$. Then $G \restriction B$ satisfies Connectivity. Since the result holds when N is finite, the core of $G \restriction B$ is non-empty, so G has the Finite Subgames property. Then by Theorem 1, the core of G is non-empty.

We now drop the Non-negativity assumption, but still assume Connectivity. Choose an element $r \in N$, which will play the role of the root of the tree. We call a node $i \in N$ *even* if the length of the path from r to i is even, and *odd* if the length of the path from r to i is odd. We call a node $i \in N$ *positive* if $\nu(\{i\}) > 0$, and *negative* if $\nu(\{i\}) < 0$. Let U_e^+ be the set of even positive nodes in N, and define U_o^+, U_e^-, and U_o^- analogously. Then the sets U_e^+, U_o^+, U_e^-, U_o^- are pairwise disjoint. Now let U be one of these four sets. Then each connected component of U is a singleton. By Countable Character, there is a countable set $C \subseteq N$ such that $\nu(S) = \nu(S \cap C)$ for all $S \subseteq N$. Then U is contained in C, and hence U is countable. By Continuity and Additivity on Components, we have $\nu(U) = \sum_{i \in U} \nu(\{i\})$.

We now define a new cooperative game $H = (N, \mu)$ on the same tree (N, E). For each $i \in N$, let u_i be the absolute value of $\nu(\{i\})$. Note that $u_i \geq 0$ for all $i \in N$. For each $S \subseteq N$, define $\mu(S) = \nu(S) + \sum_{i \in S} u_i$. Since $\nu(\{i\})$ has the same sign for all $i \in U$, we have $\sum_{i \in U} u_i = |\nu(U)|$. Therefore $\sum_{i \in N} u_i$ is finite, so μ maps 2^N into $\mathbb{R}$ and H is a cooperative game. It is easily checked that H satisfies all the hypotheses of the theorem and also satisfies Non-negativity and Connectivity. Hence there is a function y that belongs to the core of H. It follows that the function $x_i = y_i - u_i$ belongs to the core of G.

Finally, we drop the Connectivity assumption and prove that G still has a non-empty core. For each connected component T of (N, E), choose an element $r_T \in T$. Let (N^+, E^+)

be the undirected graph formed by adding a new node r to N, and adding the new edge (r, r_T) to E for each connected component T of (N, E). The idea is that each connected component T of (N, E) is a tree with root r_T, and N^+ is a tree with root r. Now let $G^+ = (N^+, \nu^+)$ be the cooperative game such that $\nu^+(S) = \nu(S \cap N)$ for each set $S \subseteq N^+$. In particular, $\nu^+(N) = \nu(N) = \nu^+(N^+)$, and $\nu^+(\{r\}) = \nu(\emptyset) = 0$. Hence G^+ satisfies all the hypotheses of the theorem and also satisfies Connectivity. So there is a function $y \colon N^+ \to \mathbb{R}$ in the core of G^+. Moreover, $y_r \geq \nu^+(\{r\}) = 0$.

Let x be the restriction of y to N. Then

$$\sum_{i \in N} x_i = \sum_{i \in N} y_i \geq \nu^+(N) = \nu(N) = \nu^+(N^+) = \sum_{i \in N^+} y_i \geq \sum_{i \in N} y_i,$$

and for each $S \subseteq N$,

$$\sum_{i \in S} x_i = \sum_{i \in S} y_i \geq \nu^+(S) = \nu(S).$$

Therefore x belongs to the core of G. □

3 Examples

Each of the following is a cooperative game with an empty core that satisfies all but one of the hypotheses of Theorem 2, and also satisfies Monotonicity, Non-negativity, and Connectivity.

- Only Superadditivity fails: $N = \{a, b, c\}$, $E = \{(a, b), (a, c)\}$, $\nu(a) = \nu(b) = \nu(c) = \nu(a, b) = \nu(a, c) = 1$, $\nu(b, c) = \nu(a, b, c) = 2$.

- Only Additivity on Components fails: $N = \{a, b, c\}$, $E = \{(a, b), (a, c)\}$, $\nu(a) = \nu(b) = \nu(c) = 0$, $\nu(a, b) = \nu(a, c) = \nu(b, c) = 3$, $\nu(a, b, c) = 4$.

- Only Forest fails: $N = \{a, b, c\}$, $E = \{(a, b), (a, c), (b, c)\}$, $\nu(a) = \nu(b) = \nu(c) = 0$, $\nu(a, b) = \nu(a, c) = \nu(b, c) = 3$, $\nu(a, b, c) = 4$.

- Only Continuity fails: (N, E) is a countable tree, U is a non-principal ultrafilter over N, and $\nu(S) = 1$ if S is in U, and 0 otherwise.

- Only Countable Character fails, and Superadditivity holds for all, even infinite, subsets of N: (N, E) is a tree such that $N = [0, 1]$, and $\nu(S)$ is the Lebesgue inner measure of S.

- Only Countable Character fails, and Continuity holds for all subsets of N: (N, E) is a tree such that $N = [0, 1]$, and $\nu(S)$ is the Lebesgue outer measure of S.

- Only Countable Character fails, and all the other hypotheses of Theorem 2 hold for all subsets of N: (N, E) is a tree and ν is a probability measure such that every subset of N is measurable and every finite set has measure 0. (This can happen if there is either a measurable cardinal or a real-valued measurable cardinal less than or equal to the cardinality of N. (See, for example, Jech [2002, Section 10].))

References

Chang, C., and H. J. Keisler. *Model Theory*, 3rd edition, Dover, 2012.

Demange, G., "On Group Stability in Hierarchies and Networks," *Journal of Political Economy*, 112, 2004, 754-778.

Herings, P., G. van der Laan, A. Talman, and Z. Yang, "The Average Tree Solution for Cooperative Games with Communication Structure," *Games and Economic Behavior*, 68, 2010, 626-633.

Igarashi, A., and Y. Yamamoto, "Average Tree Solution and Core for Cooperative Games with Graph Structure," working paper, 2012, available at http://infoshako.sk.tsukuba.ac.jp/~yamamoto/Recent_papers/files/dp1293revised.pdf.

Jech., T., *Set Theory*, 4th edition, Springer, 2002.

Owen, G., *Game Theory*, 3rd edition, Academic Press, 1995.

Appendix: Proof of Claim 1

The compactness theorem in first order logic says the following (see Chang and Keisler [2012, Section 2.1]). *Let K be a set of sentences in first order logic. If every finite subset of K has a model, then K has a model.* We now prove Claim 1. Let V be the vocabulary consisting of the symbols for an ordered field, together with a constant symbol for each particular real number. Let J be the set of all sentences in the vocabulary V that are true in the ordered field of real numbers. Let W be the vocabulary V with additional constant symbols $\{y(b) \colon b \in N\}$. Let K be the union of J and the sets of sentences (1) and (2), where for each $S \subseteq N$, $\nu(S)$ is understood to be the constant symbol in V for the real number $\nu(S)$ in the given game G. From the proof of Theorem 1, each finite subset of K has a model consisting of the field of real numbers and interpretations for the constant symbols of W. Then by the compactness theorem, the whole set K has a model F, which will consist of a real closed ordered field with interpretations of the constant symbols of W. Consider an element $b \in N$ and the constant $y(b)$ in F. Since (1) and (2) hold, we have $0 \leq y(b) \leq \nu(N)$ in F. Therefore there exists a real number $x(b) = \sup\{s \in \mathbb{R} \colon s \leq y(b)\}$ (called the standard part of $y(b)$). So in the real numbers, the set of sentences K is satisfied with $x(\cdot)$ in place of $y(\cdot)$, as required.