COMMON ASSUMPTION OF RATIONALITY *

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Abstract

We build on the work of Brandenburger, Friedenberg, and Keisler (2008, BFK) by showing that *rationality and common assumption of rationality (RCAR)* is possible in complete lexicographic type structures and characterizes iterated admissibility—i.e., iterated elimination of weakly dominated strategies. Our result is unexpected in light of BFK's result proving the impossibility of RCAR in continuous complete lexicographic type structures. We reconcile the two by showing that continuous complete lexicographic type structures differ from complete lexicographic type structures that admit RCAR only in how much caution—in a sense that we formalize—is required to assume events.

KEYWORDS: Epistemic game theory, admissibility, iterated weak dominance, assumption, best-rationalization principle

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1 INTRODUCTION

Brandenburger, Friedenberg, and Keisler (2008, henceforth BFK) solved a long-standing puzzle in game theory that had been pointed out in Samuelson (1992): How does one model a player's belief that simultaneously *includes* every strategy of the opponent and *excludes* every weakly dominated strategy of the opponent? BFK cut this Gordian knot by modeling beliefs as lexicographic probability systems (Blume et al., 1991a), which generalize standard probability measures in ways that eliminate this tension.

By taking advantage of this elegant solution, BFK obtained two results that came close to providing the epistemic foundations of iterated admissibility (IA)[1] using the epistemic conditions *rationality and common assumption of rationality (RCAR)* and *rationality and m-th order assumption of rationality (RmAR)*. These can be worded as follows:

- For any given game, there exists some lexicographic type structure such that the set of strategies predicted by RCAR is exactly the IA set.
- For any given game and any *complete* lexicographic type structure, there is some m such that the set of strategies predicted by RmAR is exactly the IA set.

These results fall short of the epistemic foundations that BFK sought, which they described as follows:

> One might hope to characterize the IA set as the projection of a set of states which is constructed in a uniform way in all complete lexicographic type structures. One would expect the RCAR set to be a natural candidate for this set of states.

BFK proved instead a discouraging result, hereafter called the *impossibility theorem*, saying that RCAR is impossible in *any* continuous complete lexicographic type structure.[2] In light of this, BFK posed the following question:

(I) For every finite game, does there exist a (necessarily discontinuous) complete lexicographic type structure in which RCAR is possible?

In this paper, we answer this question affirmatively with a *possibility theorem* (Theorem 3.4). Furthermore, we provide an epistemic foundation for IA by showing that the IA set is exactly the set of strategies that players choose when RCAR holds in complete lexicographic type structures (Theorem 3.5):

[1] IA is iterated *maximal* elimination of inadmissible (i.e., weakly dominated) strategies. It is a procedure that eliminates *all* weakly dominated strategies of *all* players in each round. It is well-known that other orders of elimination may yield different results.

[2] The impossibility theorem has motivated several recent papers that have sought foundations of IA outside of BFK's framework. See Halpern and Pass (2009); Heifetz et al. (2010); Barelli and Galanis (2013); Catonini and De Vito (2014); Yang (2015); Lee (2016); Catonini and De Vito (2022).

(II) For every complete lexicographic type structure in which RCAR is possible, the IA set is exactly the projection of the RCAR set.

An informal interpretation of these results is that the line of reasoning given below is possible (by the affirmative answer to (I)) and that its predictions for the game are exactly the IA strategies (by (II)).

a1: Ann is rational i.e., she chooses optimally after considering all possibilities about Bob	b1: Bob is rational i.e., he chooses optimally after considering all possibilities about Ann
a2: a1 and Ann assumes b1	b2: b1 and Bob assumes a1
a3: a2 and Ann assumes b2	b3: b2 and Bob assumes a2
... and so on	... and so on

Nevertheless, our positive results also fall short of BFK's original hope of finding a game-independent (i.e., "a uniform") epistemic characterization of IA, because the complete lexicographic type structure we construct depends on the game in a very subtle way through topology. To better understand why, we explore the connection between our positive results and BFK's impossibility theorem. An early interpretation of the impossibility theorem was that RCAR is incompatible with continuous complete lexicographic type structures because they induce too many belief hierarchies. We cast doubt on that interpretation by proving the following result (Corollary 4.12).

(III) For any given complete lexicographic type structure in which RCAR is possible, there is a continuous complete lexicographic type structure that induces the same belief hierarchies.

To put our results relating to continuity/discontinuity into context, it should be noted that this condition plays no role in other well-known epistemic characterizations. For example, take *rationality and common belief of rationality (RCBR)*, which is used to characterize rationalizability. Continuity truly is a technical condition with respect to RCBR. For any type structure based on standard—i.e., non-lexicographic—probabilities, rearranging the topology on types in an arbitrary way to make the type structure discontinuous does not impact the set of states that satisfy RCBR as long as the new topology generates the same (Borel) measurable sets. Similar analogues hold for *rationality and common strong belief of rationality* (Battigalli and Siniscalchi, 2002) and *cautious rationality and common cautious belief of cautious rationality* (Catonini and De Vito, 2022). This is immediate because these other epistemic conditions depend on topology only to the extent that they affect the measurable sets on which the relevant belief operators are defined. Thus, exploring the role of discontinuity with respect to the possibility of RCAR may help clarify a few of the many ways in which RCAR differs from other epistemic conditions.

The remainder of our paper is organized as follows. In Section 2, we review the underlying framework from BFK. In Section 3, we review some results from BFK and state our affirmative answer to question (I) along with result (II). Section 4 deals with result (III), which reconciles our possibility result with the impossibility theorem. Section 5 connects BFK to other closely related papers.

2 UNDERLYING FRAMEWORK

Throughout this paper, we fix a two-player[3] game $G = \langle S^a, S^b, \pi^a, \pi^b \rangle$ with finite strategy sets S^a, S^b. To avoid the trivial cases, we require that at least one of S^a, S^b has cardinality greater than 1 (so $|S^a \times S^b| > 1$). The indices a and b respectively stand for Ann and Bob. At times, we use c to denote a generic player (either a or b), and d to denote the other player. π^c is c's utility (payoff) function on $S^a \times S^b$. A player c is said to be **indifferent** in the game G if $\pi^c(r^c, s^d) = \pi^c(s^c, s^d)$ for all r^c, s^c, s^d. We let Ω denote a non-empty Polish space.[4] By an **event** (in Ω), we mean a Borel subset U of Ω.

2.1 Lexicographic probability system

$\mathscr{M}(\Omega)$ denotes the Polish space of all Borel probability measures on Ω with the topology of weak* convergence. A **lexicographic probability system (LPS)** of length n on Ω is an n-tuple of probability measures $\mu = (\mu_0, \ldots, \mu_{n-1})$ on Ω. $\mathscr{N}_n(\Omega) \equiv \prod_{k=1}^n \mathscr{M}(\Omega)$ denotes the space of length-n LPSs on Ω with the product topology. $\mathscr{N}(\Omega) \equiv \bigcup_{n \geq 1} \mathscr{N}_n(\Omega)$ denotes the space of LPSs on Ω with the union topology. Both of these spaces are Polish.

For any $\mu, \nu \in \mathscr{M}(\Omega)$, μ and ν are **mutually singular** (written $\mu \perp \nu$) if there exist disjoint events U, V such that $\mu(U) = 1 = \nu(V)$. A **lexicographic conditional probability system (LCPS)** is an LPS $(\mu_0, \ldots, \mu_{n-1})$ such that $\mu_j \perp \mu_k$ for all $j \neq k$.[5] $\mathscr{L}_n(\Omega)$ and $\mathscr{L}(\Omega)$ respectively denote the space of length-n LCPS's on Ω and the space of LCPS's on Ω. Both spaces are endowed with the subspace topology relative to $\mathscr{N}(\Omega)$.

The **support** of an LPS $\mu = (\mu_0, \ldots, \mu_{n-1})$ is denoted by $\operatorname{supp} \mu \equiv \bigcup_{j<n} \operatorname{supp} \mu_j$. The LPS μ has **full support** on Ω if $\operatorname{supp} \mu = \Omega$. $\mathscr{N}^+(\Omega)$ denotes the space of full-support LPSs on Ω. $\mathscr{M}^+(\Omega)$, $\mathscr{N}_n^+(\Omega)$, $\mathscr{L}^+(\Omega)$, and $\mathscr{L}_n^+(\Omega)$ are defined analogously. Note that $\mathscr{L}^+(\Omega) \subseteq \mathscr{L}(\Omega) \subseteq \mathscr{N}(\Omega)$ and $\mathscr{N}^+(\Omega) \subseteq \mathscr{N}(\Omega)$.

The **concatenation** of LPSs μ and ν, which is denoted by $\mu\nu$, is a longer LPS formed by appending ν to the end of μ, e.g., if $\mu = (\mu_0, \mu_1, \mu_2)$ and $\nu = (\nu_0, \nu_1)$, then $\mu\nu = (\mu_0, \mu_1, \mu_2, \nu_0, \nu_1)$.

[3] The restriction to two players simplifies the presentation and proofs. Generalizing to environments with any finite number of players is straightforward.
[4] A Polish space is a topological space that is separable and completely metrizable.
[5] BFK used the term LPS for LCPS. We revert to the terminology of Blume et al. (1991a) in this regard.

2.2 Assumption

BFK defined when a decision maker whose preferences are given by $\succsim$ **assumes** an event E. Instead of restating that definition, we rely on their precise characterization of the conditions under which a belief μ represents the preferences of a decision maker who assumes E.

Proposition 2.1 (Proposition 5.1 in BFK)**.** *Fix an event $E \subseteq \Omega$ and a full-support LCPS $\mu = (\mu_0, \ldots, \mu_{n-1}) \in \mathcal{N}^+(\Omega)$. E is **assumed under** μ at level j if and only if*

(i) $\mu_i(E) = 1$ *for all* $i \leq j$*;*
(ii) $\mu_i(E) = 0$ *for all* $i > j$*;*
(iii) if an event U is open with $U \cap E \neq \varnothing$ then $\mu_i(U \cap E) > 0$ for some i.

E is assumed under μ if and only if E is assumed under μ at some level j.

For brevity, we write "μ assumes E" to mean that E is assumed under μ. Assumption is not **monotonic**, that is, μ does not necessarily assume F even if μ assumes $E \subseteq F$. However, it does have the following properties (note that, for each event $E \subseteq \Omega$, $\overline{E}$ denotes its **closure** in Ω).

Proposition 2.2. *For each full-support LCPS μ on Ω, we have:*

(i) If μ assumes both E and F at level j, then μ assumes any event H such that $E \cap F \subseteq H \subseteq E \cup F$ at level j.
(ii) If μ assumes both E and F at level j, then $\overline{E} = \overline{F}$.
(iii) If μ assumes E_m at level j for each $m \in \mathbb{N}$, then μ assumes $\bigcap_m E_m$ at level j.

Proof. Parts (i) and (ii) are Properties 6.1 and 6.2 in BFK. Part (iii) follows from the proof of Property 6.3 in BFK, which is (iii) with the references to level j removed. □

2.3 Admissibility

The usual lexicographic order $\geq^L$ on $\mathcal{N}_k(S^d)$ is defined as follows. For any $u = (u_0, \ldots, u_{k-1}), v = (v_0, \ldots, v_{k-1}) \in \mathbb{R}^k$, we write $u >^{\mathrm{L}} v$ if there is some $n < k$ such that $(u_0, \ldots, u_{n-1}) = (v_0, \ldots, v_{n-1})$ and $u_n > v_n$. We write $u \geq^{\mathrm{L}} v$ whenever $u >^{\mathrm{L}} v$ or $u = v$.

Given any $\mu = (\mu_0, \ldots, \mu_{k-1}) \in \mathcal{N}(S^d)$, and strategy $s^c \in S^c$, the **lexicographic expected utility** for s^c under μ is the k-tuple

$$\mathsf{LEU}(s, \mu) \equiv \left(\sum_{s^d \in S^d} \mu_j(s^d) \pi^c(s^c, s^d) \right)_{j=0}^{k-1}.$$

For strategies $r, s \in S^c$ write $s \succ_\mu r$ if $\mathsf{LEU}(s, \mu) \geq^L \mathsf{LEU}(r, \mu)$. For $\mu, \nu \in \mathcal{N}(S^d)$, write $\mu \sim \nu$ if the relations $\succ_\mu$ and $\succ_\nu$ on S^c are the same.

We extend the notion of the (pure strategy) **best-response (BR)** set as follows. Given any $\mu = (\mu_0, \ldots, \mu_{k-1}) \in \mathcal{N}(S^d)$, we define the best-response set $\mathbf{BR}^c(\mu)$ as the set of c's pure strategies that maximize lexicographic expected utility under μ in the game G. Formally, $\mathbf{BR}^c(\mu)$ is the set of $s^c \in S^c$ such that for all $x^c \in S^c$ we have

$$\mathsf{LEU}(s^c, \mu) \geq^{\mathsf{L}} \mathsf{LEU}(x^c, \mu)$$

Note that for every μ, the set $\mathbf{BR}^c(\mu)$ is a non-empty subset of S^c. Since $\sim$ is an equivalence relation, $\mu \sim \nu$ implies $\mathbf{BR}^c(\mu) = \mathbf{BR}^c(\nu)$. Furthermore, if $\mu \sim \mu'$ and $\nu \sim \nu'$ then $\mu\nu \sim \mu'\nu'$. Given a set $X \subseteq \mathcal{N}(S^d)$, we define $\mathbf{BR}^c(X) \equiv \{\mathbf{BR}^c(\mu) \mid \mu \in X\}$.

A strategy s^c is **admissible**—i.e., not weakly dominated—in the game G if s^c is a best response under some full-support LPS on S^d. Let $S^c_0 \equiv S^c$. For each $m \geq 1$, S^c_{m+1} denotes c's admissible strategies in the reduced game $G_m \equiv \langle S^a_m, S^b_m, \pi^a, \pi^b \rangle$.[6] Strategy s^c is m-**admissible** if it is in S^c_m. It is **iteratively admissible (IA)** if it is in $S^c_\infty \equiv \bigcap_{m=0}^{\infty} S^c_m$.

We now introduce three pieces of auxiliary notation based on the m-admissible sets that will be useful for writing proofs later on.[7] Firstly, $\mathscr{P}^d_m(G)$ is defined to be the set of all length-$(m+1)$ LPSs $\mu = (\mu_0, \ldots, \mu_m)$ on S^d such that

$$\operatorname{supp} \mu_0 = S^d_m, \ \operatorname{supp} \mu_1 = S^d_{m-1}, \ldots, \text{ and } \operatorname{supp} \mu_m = S^d_0.$$

Secondly, $\mathscr{B}^c_m(G)$ is defined to be the family of best response sets for G of elements of $\mathscr{P}^d_m$, i.e.,

$$\mathscr{B}^c_m(G) \equiv \mathbf{BR}^c(\mathscr{P}^d_m) = \{\mathbf{BR}^c(\mu) \mid \mu \in \mathscr{P}^d_m\}.$$

We will sometimes omit the game G from $\mathscr{P}^d_m(G)$ and $\mathscr{B}^c_m(G)$ and simply write $\mathscr{P}^d_m$ and $\mathscr{B}^c_m$ when doing so poses no danger of confusion. These sets depend on the game G only to the extent that the m-admissible sets depend on the game G. Note that $\varnothing \neq \mathscr{B}^c_m(G)$ and each $X \in \mathscr{B}^c_m(G)$ is a non-empty subset of $S^c = S^c_0$. We also let $\mathscr{B}^c_\infty(G) \equiv \bigcap_{m \in \mathbb{N}} \mathscr{B}^c_m(G)$. Thirdly, $M(G)$ is defined to be the least $M \in \mathbb{N} \cup \{\infty\}$ such that $M > 0$ and $\mathscr{B}^c_k(G) = \mathscr{B}^c_M(G)$ for each $c \in \{a, b\}$ and all $k \geq M$.

2.4 Lexicographic type structures

An (S^a, S^b)-based **lexicographic type structure** for G is a tuple

$$\mathbf{T} = \langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$$

[6] Here we slightly abuse notation by letting the restriction of π^c to $S^a_m \times S^b_m$ be denoted by π^c.

[7] They are introduced here even though they are only used in the appendix because they depend directly on the m-admissible sets defined here.

such that, for each player c,

(i) c's **type space** T^c is non-empty and Polish;
(ii) c's **type-belief map** $\lambda^c : T^c \to \mathscr{L}(S^d \times T^d)$ is Borel.

Elements of $S^a \times T^a \times S^b \times T^b$ are called **states of the world**. $\lambda^c(t^c)$ is the **lexicographic belief** of the **type** $t^c \in T^c$. We frequently omit the adjective "lexicographic" for the sake of brevity in obvious contexts. We say that $\mathbf{T}$ is **continuous** if λ^a and λ^b are continuous and that $\mathbf{T}$ is **one-to-one** if λ^a and λ^b are one-to-one maps.

We say that $\mathbf{T}$ is **complete** if $\lambda^c(T^c) = \mathscr{L}(S^d \times T^d)$ for each player c.[8] Our terminology differs slightly from that in BFK, where $\mathbf{T}$ is called a complete lexicographic type structure if $\lambda^c(T^c) \supsetneq \mathscr{L}^+(S^d \times T^d)$ for each player c. Thus, *complete* in our sense implies *complete* in the sense of BFK. This is a harmless change that does not affect the interpretation of the results, because it is relatively straightforward to modify the proofs in either paper to work under both definitions.[9] Furthermore, the requirement here that type-belief maps are surjective more naturally reflects the word "complete", which has been used that way elsewhere in the literature.[10]

Lemma 2.3. *If* $\mathbf{T}$ *is a complete lexicographic type structure for* G*, then the Polish spaces* T^a, T^b *are both uncountable.*

Proof. We have required that at least one of the strategy sets, say S^a, has cardinality greater than 1. Then $S^a \times T^a$ has cardinality greater than 1, so $\mathscr{L}(S^a \times T^a)$ is uncountable. By completeness, λ^b maps T^b onto $\mathscr{L}(S^a \times T^a)$, so T^b is uncountable. Therefore $S^b \times T^b$ has cardinality greater than 1, so by the previous argument, T^a is also uncountable. □

For brevity, from now on "type structure" will always mean "lexicographic type structure".

2.5 Rationality

Fix a type structure $\mathbf{T} = \langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$.

We say that player c is **rational** for G at the pair (s^c, t^c), and at the state (s^a, t^a, s^b, t^b), if

(i) c's type maps to a full-support belief—i.e., $\lambda^c(t^c) \in \mathscr{L}^+(S^d \times T^d)$;
(ii) c's strategy is a best response under this belief—i.e., $s^c \in \mathbf{BR}^c(\mathrm{marg}_{S^d} \lambda^c(t^c))$.

[8] Dekel et al. (2016) showed that BFK's impossibility theorem holds even when the notions of completeness and assumption are extended to type structures in which beliefs are not required to be LCPSs (i.e., not mutually singular). Our results can also be demonstrated to hold under such modifications after appropriate changes to some proofs. In some cases, like Theorem 4.10, no modification of the proof is needed.

[9] For example, Theorem 3.1 can be proved for our definition of completeness by modifying BFK's proof so that Theorem 13.7 (Kechris, 1995) is used instead of Theorem 7.9 (Kechris, 1995). Theorem 3.2 is immediate because our notion of completeness is stronger.

[10] e.g., Friedenberg (2010), also "belief-complete" in Battigalli and Siniscalchi (2002)

Player c **assumes** the event $E \subseteq S^d \times T^d$ at the state of the world (s^a, t^a, s^b, t^b) if E is assumed under $\lambda^c(t^c)$. Define

$$\mathbf{A}^c(E) \equiv \{t^c \in T^c \mid E \text{ is assumed under } \lambda^c(t^c)\}. \tag{1}$$

Note that the set $\mathbf{A}^c(E)$ depends on the type structure $\mathbf{T}$ but does not depend on the game G.

The iterated rationality set R^c_m is defined inductively for all $m \in \mathbb{N}$ as follows. Let $R^c_0 = S^c \times T^c$. Let R^c_1 denote the set of strategy-type pairs $(s^c, t^c) \in S^c \times T^c$ at which c is rational. For each $m > 0$ we let

$$R^c_{m+1} \equiv R^c_m \cap \left(S^c \times \mathbf{A}^c(R^d_m)\right) \qquad\qquad R^c_\infty \equiv \bigcap_{m \geq 1} R^c_m \tag{2}$$

The set R^c_m depends on both the type structure $\mathbf{T}$ and the game G. If a state (s^a, t^a, s^b, t^b) belongs to $R^a_{m+1} \times R^b_{m+1}$, then we say that it satisfies **rationality and m-th order assumption of rationality (RmAR)**. If (s^a, t^a, s^b, t^b) belongs to $R^a_\infty \times R^b_\infty$, then we say that it satisfies **rationality and common assumption of rationality (RCAR)**.

3 EPISTEMIC JUSTIFICATION FOR ITERATED ADMISSIBILITY

In this section we state three results from BFK and then present our main results (Theorems 3.4 and 3.5).

Recall the line of reasoning (taken from BFK), which was also mentioned in the Introduction.

a1: Ann is rational i.e., she chooses optimally after considering all possibilities about Bob	b1: Bob is rational i.e., he chooses optimally after considering all possibilities about Ann
a2: a1 and Ann assumes b1	b2: b1 and Bob assumes a1
a3: a2 and Ann assumes b2	b3: b2 and Bob assumes a2
… and so on	… and so on

BFK showed two results that, when taken together, give an epistemic justification of m-admissible strategies related to this line of reasoning.

Proposition 3.1 (Proposition 7.2 in BFK)**.** *There exists a continuous complete type structure.*

Theorem 3.2 (Theorem 9.1 in BFK)**.** *Fix a complete type structure* $\mathbf{T}$. *Then*

$$\forall m \in \mathbb{N} \quad \text{proj}_{S^a \times S^b}(R^a_m \times R^b_m) = S^a_m \times S^b_m.$$

The fact that there is a continuous complete type structure such that $R^a_m \times R^b_m \neq \emptyset$ is informally interpreted as saying that it is possible for the players to reason in ways described by statements am and bm. Because the m-admissible set is exactly the projection of $R^a_m \times R^b_m$ in any such type structure, the m-admissible set is exactly the set of predictions for the game when players reason as described by am and bm. Given this interpretation, the following *impossibility theorem* of BFK immediately casts serious doubt on whether the *full* line of reasoning described above (which includes am and bm for *all* m) is possible.

We say that a type structure $\mathbf{T}$ **admits** RCAR if $R^a_\infty \times R^b_\infty \neq \emptyset$.

Theorem 3.3 (Impossibility Theorem, 10.1 in BFK)**.** *Fix a continuous complete type structure* $\mathbf{T}$*. Unless both players are indifferent,* $\mathbf{T}$ *does not admit RCAR.*[11]

Theorem 3.4 below is our affirmative answer to question (I) that was left open in BFK. That answer is surprising, because Theorem 3.4 and the impossibility theorem intuitively point in opposite directions. We will reconcile the two in Section 4 through a notion that we call "degree of caution". Taken together, the impossibility theorem and Theorem 3.4 will tell us that for every non-trivial game, no **continuous** complete type structure admits RCAR, but some **non-continuous** complete type structure does admit RCAR.

Theorem 3.4. *For all uncountable Polish spaces T^a, T^b, there exists a complete type structure*

$$\mathbf{T} = \langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$$

that admits RCAR.[12]

Part (ii) of the following theorem is result (II) in the Introduction.

Theorem 3.5. *Fix a complete type structure* $\mathbf{T}$ *that admits RCAR. Then*

(i) There exists $M \in \mathbb{N}$ such that $\overline{R^c_m} = \overline{R^c_M}$ for all $m \geq M$.
(ii) $\operatorname{proj}_{S^a \times S^b}(R^a_\infty \times R^b_\infty) = S^a_\infty \times S^b_\infty$.

Proof. Fix some $(s^a, t^a, s^b, t^b) \in R^a_\infty \times R^b_\infty$. Then $\sigma = \lambda^a(t^a)$ is a full-support LCPS that assumes every event in the sequence $(R^b_1, R^b_2, \ldots)$. Since σ has only finitely many levels, there exists a level k at which σ assumes R^b_m for infinitely many $m \in \mathbb{N}$. By Proposition 2.2(i), it follows that there exist some k and smallest M such that σ assumes R^b_m at level k for all $m \geq M$. By Proposition 2.2(iii),

[11] The statement in BFK had the hypothesis that some player is not indifferent, which is equivalent to saying "unless both players are indifferent". Every type structure that is complete in our sense is complete in the sense of BFK, so the result as stated in BFK implies the result as stated here.
[12] The complete type structure we will construct to prove Theorem 3.4 depends on the fixed game G.

σ assumes R^b_∞ at level k. By Proposition 2.2(ii), whenever $M \le m \in \mathbb{N}$ we have $\overline{R^b_\infty} = \overline{R^b_m} = \overline{R^b_M}$. Thus condition (i) above is proved.

Since $\{s^b\} \times T^b$ is open for all $s^b \in S^b$, $\text{proj}_{S^b} R^b_\infty = \text{proj}_{S^b} R^b_m$. By Theorem 3.2, $\text{proj}_{S^b} R^b_m = S^b_m$. Therefore $\text{proj}_{S^b} R^b_\infty = S^b_m = S^b_\infty$ for all $m \ge M$. Analogously, $\text{proj}_{S^a} R^a_\infty = S^a_\infty$. This proves (ii). □

The preceding two theorems say that the reasoning described by the full list of statements at the beginning of this section is an epistemic justification of IA— the reasoning is possible (Theorem 3.4) and its predictions are exactly the IA strategies (Theorem 3.5). Notice that, while the complete type structures that admit RCAR (as in Theorem 3.4) may not be the same for every game, the line of reasoning about beliefs and rationality that is the epistemic justification of IA is the same for every game. Nevertheless, it is natural to ask whether Theorem 3.4 can be strengthened in the following way.

Question 3.6. *Given a pair of finite strategy sets S^a, S^b, does there exist a complete type structure that admits RCAR for every game with these strategy sets?*

We leave this as an open question. If such type a structure exists, one hopes that it might be canonical in a sense that makes it the right one to use for the epistemic analysis of all games.

4 CAUTION, CONTINUITY, AND RCAR

In the epistemic game theory literature, a richness of beliefs, most typically the generation of all belief hierarchies (e.g., Battigalli and Siniscalchi, 2002), henceforth B-S, has been important for the game-independent analysis of epistemic conditions other than RCAR. BFK introduced completeness as a richness condition that is important for the study of the epistemic condition RCAR. Catonini and De Vito (2014) and Yang (2015) have constructed type structures that admit a weaker property than RCAR for every game, but being continuous, these do not admit RCAR. Does that mean that analysis of RCAR cannot be done in a game-independent way and therefore must always be dependent on context? We do not rule out that possibility. However, we would argue that richness (in the form of completeness) should not be the sole desideratum.

If we start with a game G and pair of uncountable Polish spaces T^a, T^b, then Theorem 3.4 says that there exist maps λ^a, λ^b such that $\mathbf{T}$ is a complete type structure that admits RCAR for G. But if we start with a game G and a complete type structure $\mathbf{T}$, there will be many complete type structures $\mathbf{U}$ that are Borel-equivalent to $\mathbf{T}$ (Definition 4.4 below). Those type structures have the same Borel sets and type-belief maps but different topologies. We will see in this section that some type structures Borel-equivalent to $\mathbf{T}$ will admit RCAR and some will not. So a complete type structure that admits RCAR does so not because of its richness alone but also because of a kind of balance between its topology and its family of Borel sets.

4.1 Continuity and richness of beliefs

As mentioned in the Introduction, an early interpretation of BFK's impossibility theorem was that a continuous complete type structure cannot admit RCAR because it generates too many belief hierarchies. Intuitively, if the set of belief hierarchies generated by a type structure is small, then the players can be said to know a lot about each other since they rule out all hierarchies that it does not generate. Similarly, if the set of belief hierarchies generated by a type structure is very large, then the players can be said to know very little about each other. The early interpretation was therefore based on the idea that RCAR is possible only when players are familiar with each other in the sense that we have just described. Indeed, BFK showed that it is easy to construct *incomplete* type structures that admit RCAR.

Unlike B-S, BFK do not construct a complete type structure by first defining all hierarchies of beliefs. However, the usual constructions of "universal" type structures satisfy analogs of both continuity and completeness.[13] Furthermore, Friedenberg (2010) had previously shown that—when beliefs are *standard probability measures*—a complete[14] type structure generates all belief hierarchies if the type spaces are compact and the type-belief maps are continuous.

This preexisting association between continuity and richness, albeit in non-lexicographic frameworks, gave intuitive appeal to the idea that continuous complete type structures were "too rich" to admit RCAR, and hence to the early interpretation of the impossibility theorem. In this section, we develop a notion of *caution required to assume events* and use it to question that interpretation.

4.2 Definition of hierarchies

Before we proceed, we need to be more precise about what we mean by "belief hierarchies", so that our statements about them are unambiguous.

Definition 4.1. *Let A and B be non-empty Polish spaces and $f : A \to B$ be a Borel map. The* ***push-forward*** *map $\widehat{f} : \mathcal{N}(A) \to \mathcal{N}(B)$ is defined as follows. For all $\sigma = (\mu_0, \dots, \mu_{n-1}) \in \mathcal{N}(A)$, $\widehat{f}(\sigma) = (\widehat{\mu}_0, \dots, \widehat{\mu}_{n-1})$, where $\widehat{\mu}_j(E) = \mu_j \circ f^{-1}(E)$ for all Borel $E \subseteq B$ and $0 \le j \le n-1$.*

Definition 4.2. *Let $\Xi^c_0 \equiv S^c$ and inductively define the sequence $(\Xi^c_m)_{m=0}^{\infty}$ of Polish spaces by letting $\Xi^c_{m+1} \equiv \Xi^c_m \times \mathcal{N}(\Xi^d_m)$ for all $m \ge 0$. Player c's* ***(belief) hierarchy*** *is a sequence $h^c = (h^c_{m+1})_{m=0}^{\infty} \in \prod_{m=0}^{\infty} \mathcal{N}(\Xi^d_m)$ of LPSs, where h^c_m is called c's m-**th order belief**.*

In the rest of this section,

$$\mathbf{T} = \langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle \quad \text{and} \quad \mathbf{U} = \langle S^a, S^b, U^a, U^b, \lambda^a, \lambda^b \rangle$$

[13] e.g., Mertens and Zamir (1985), Brandenburger and Dekel (1993), and B-S.

[14] Friedenberg (2010)'s definition of complete type structures accordingly differs from ours in only one way: $\lambda^c(T^c) = \mathcal{M}(S^d \times T^d)$ instead of $\lambda^c(T^c) = \mathcal{L}(S^d \times T^d)$.

will denote type structures. **T** generates hierarchies by using the type-belief maps. As an intermediate step we need to inductively define the Borel map $\xi^c_m : S^c \times T^c \to \Xi^c_m$ for $m = 0, 1, \ldots$ by letting, for all $(s^c, t^c) \in S^c \times T^c$,

$$\xi^c_0(s^c, t^c) \equiv s^c \quad \text{and} \quad \xi^c_{m+1}(s^c, t^c) \equiv (\xi^c_m(s^c, t^c), \widehat{\xi}^d_m \circ \lambda^c(t^c)) \quad \text{for all } m \geq 0.$$

Definition 4.3. *The **hierarchy generated by type** $t^c \in T^c$ is the sequence $(\widehat{\xi}^d_m \circ \lambda^c(t^c))_{m=0}^{\infty}$.*

Note that the hierarchy in Definition 4.3 does not depend on a game G.

4.3 Caution required to assume events

Definition 4.4. *We say that two Polish spaces X and Y are **Borel equivalent** if they have the same points and the same Borel sets. We say that* **T**, **U** *are **Borel-equivalent** if* **T** *and* **U** *have the same strategies, types, and type-belief maps, and T^c, U^c are Borel-equivalent for each player c.*

Two Borel-equivalent type structures may differ only in their respective topologies (i.e., which sets of types are considered open). The next remark follows easily from the definitions.

Remark 4.5. *Suppose* **T** *and* **U** *are Borel-equivalent. Then the following hold.*

(a) **T** *and* **U** *have the same LCPSs.*
(b) **U** *is complete if and only if* **T** *is complete.*
(c) **T** *and* **U** *generate the same belief hierarchies.*

However, the set of LPCS's that assume a given event in **T** depends on the topology of **T**, and the rationality set R^c_1 depends on both the topology of **T** and the game G.

Definition 4.6. *A **Borel-equivalent refinement** of a Polish space X is a Polish space Y such that Y has the same points as X, and every open set in X is open in Y. We say that* **U** *is a **Borel-equivalent refinement** of* **T** *and write* $\mathbf{U} \geq^{\text{ref}} \mathbf{T}$ *if* **T** *and* **U** *have the same strategies, types, and type-belief maps, and U^c is a Borel-equivalent refinement of T^c for each player c.*

To sum up, $\mathbf{U} \geq^{\text{ref}} \mathbf{T}$ means that **U** is obtained from **T** by only adding open sets to the Polish spaces T^c. The relation $\geq^{\text{ref}}$ is a partial order on the class of all type structures. Hereafter, whenever we speak of any type structures labeled **U** and **T**, we let $U_m = U^a_m \times U^b_m$ and $R_m = R^a_m \times R^b_m$ denote their respective RmAR sets. In view of the next proposition, when $\mathbf{U} \geq^{\text{ref}} \mathbf{T}$, we will also say that **U** requires a greater **degree of caution** (in assuming events) than **T** does.

Proposition 4.7. *Suppose* $\mathbf{U} \geq^{\text{ref}} \mathbf{T}$. *Then the following hold.*

(i) **T** *and* **U** *are Borel-equivalent (and hence satisfy Remark 4.5 (a)–(c)).*
(ii) Every LCPS that assumes an event E in **U** *assumes E in* **T**.

(iii) *For all $m \geq 0$, if $U_m = R_m$, then $U_{m+1} \subseteq R_{m+1}$. In particular, $U_1 \subseteq R_1$.*

Proof. By hypothesis, T^c and U^c have the same points, and every open set in T^c is open in U^c. Then by Exercise 15.4 in Kechris (1995), T^c and U^c have exactly the same Borel sets, so T^c is Borel-equivalent to U^c. Therefore (i) holds. By Remark 4.5 (a), **T** and **U** have the same LCPSs. (ii) follows from this and Proposition 2.1. Part (iii) follows from part (ii). □

We wish to emphasize that all of the following may change in the *Borel-equivalent refinement of a type structure:* the rationality sets, the set of LCPSs that assume a given event, and the events that are to be assumed by players if RCAR is to hold. Because $U_1 \subseteq R_1$, one might guess that $U_m \subseteq R_m$ for all m, and hence that **U** has fewer RCAR states than **T**. This would be correct if assumption were monotonic. But assumption is *not* monotonic, and the above guess turns out to be incorrect for complete type structures.

Proposition 4.8. *Suppose* **T** *and* **U** *are complete and Borel-equivalent. Then either $U_m = R_m$ for all m, or there are only finitely many m such that $U_m \subseteq R_m$.*

Proof. We show even more: If $U_m \neq R_m$, then U_n and R_n are incomparable for all $n > m$.

Suppose that $U^c_m \neq R^c_m$. It is obvious that $m > 0$. It suffices to show that $U^d_{m+1} \nsubseteq R^d_{m+1}$, because by symmetry it would then follow that $R^d_{m+1} \nsubseteq U^d_{m+1}$. Since **T** and **U** are complete, it is enough to find an LCPS σ that has full support in U^c, and assumes U^c_k for all $k \leq m$ but does not assume R^c_k for all $k \leq m$.

By Lemma E.3 of BFK, for each $k \in \mathbb{N}$, $R^c_{k+1} \subsetneq R^c_k$ and $U^c_{k+1} \subsetneq U^c_k$. Let $\mathscr{A}$ be the finite Boolean algebra generated by the sets R^c_k and U^c_k for $k \leq m$. Let C be a countable subset of $S^c \times T^c$ such that for every set $B \in \mathscr{A}$, $C \cap B$ is dense in B with respect to the topology of $S^c \times U^c$. Let μ be a probability measure on $S^c \times T^c$ such that $\mu(C) = 1$ and $\mu(\{c\}) > 0$ for each $c \in C$. For each $k < m$, let $\mu_k(E) = \mu(E \mid U^c_k \setminus U^c_{k+1})$. Let $\mu_m(E) = \mu(E \mid U_m)$. Then $\sigma = (\mu_m, \mu_{m-1}, \ldots, \mu_0)$ is an LCPS that has full support in U^c, and assumes U^c_k at level $m - k$ for each $k \leq m$.

Suppose σ assumes R^c_k for all $k \leq m$. Since C meets $R^c_k \setminus R^c_{k+1}$ for each $k < m$, σ cannot assume more than one of these sets at the same level, so σ assumes R^c_m at level 0. Then $\mu_m(R^c_m) = 1$, so $U^c_m \subseteq R^c_m$, and $\mu_k(R^c_m) = 0$ for all $k < m$, so $R^c_m \subseteq U^c_m$. This contradicts $U^c_m \neq R^c_m$, and completes the proof. □

One might also expect that a Borel-equivalent refinement of a continuous type structure is also continuous, but this is not necessarily so because adding more open sets to the type space T^c will also add open sets to the space $\mathscr{L}(S^d \times T^d)$. Nevertheless, we show in Proposition 4.9 that RCAR is unattainable in any type structure that Borel-refines (and therefore requires a higher degree of caution than) a continuous complete type structure.

Proposition 4.9. *Suppose neither player is indifferent in G,* **T** *is a continuous complete type structure, and* **U** *is a Borel-equivalent refinement of* **T**. *Then* **U** *does not admit RCAR.*

Proof. In this proof, for any subset E of $S^c \times T^c$, $\overline{E}$ will always denote the closure of E in the sense of $\mathbf{T}$ rather than of $\mathbf{U}$. It suffices to show that in $\mathbf{U}$, no type whose image under λ^c has length less than m can belong to U^c_m.

Lemmas F.1 and F.2 in BFK show that, for each m, the set $R^c_m \setminus \overline{R^c_{m+1}}$ is uncountable. Their proof also shows that $U^c_m \setminus \overline{U^c_{m+1}}$ is uncountable. Therefore, the sets $U^c_m, m \in \mathbb{N}$ all have different closures in the sense of $\mathbf{T}$. Every closed set in the sense of $\mathbf{T}$ is also closed in the sense of $\mathbf{U}$. Therefore, by Proposition 2.2(ii), no type in $\mathbf{U}$ can assume more than one of the sets U^c_m at the same level. The result follows. □

In light of the above discussion, it may be surprising that there is a continuous complete type structure that is a Borel-equivalent refinement of a complete type structure that admits RCAR. This will be shown in Corollary 4.11 of Theorem 4.10:

Theorem 4.10. *Every type structure* $\mathbf{T}$ *has a continuous Borel-equivalent refinement* $\mathbf{U}$.[15]

Proof. We will use Theorem 13.11 in Kechris (1995): Whenever f is a Borel map from a Polish space X to a second countable space Z, X has a Borel-equivalent refinement Y such that $f : Y \to Z$ is continuous. Let $T^c_0 = T^c$. For each n, we inductively obtain a Borel-equivalent refinement T^c_{n+1} of T^c_n such that λ^c is continuous from T^c_{n+1} to $\mathscr{L}(S^d \times T^d_n)$ by directly applying Theorem 13.11 of Kechris (1995). Let $\mathscr{T}^c_n$ denote the topology of T^c_n and let $\mathscr{T}^c_\infty$ denote the topology generated by $\bigcup_n \mathscr{T}_n$. Let U^c be the topological space that has the same points as T^c and the topology $\mathscr{T}^c_\infty$. Kechris (1995, 13.3) directly shows that U^c is a Borel-equivalent refinement of T^c_n for all n. It follows that λ^c is continuous from U^c to $\mathscr{L}(S^d \times T^d_n)$ for each n.

Let $u^c_k \to u^c$ in U^c. Then $\lambda^c(u^c_k) \to \lambda^c(u^c)$ in $\mathscr{L}(S^d \times T^d_n)$ for all n. Let M denote the length of $\lambda^c(u^c)$—i.e., $\lambda^c(u^c) \in \mathscr{L}_M(S^d \times U^d)$. Then there is some K such that $\lambda^c(u^c_k) \in \mathscr{L}_M(S^d \times U^d)$ for all $k \geq K$. Without loss of generality, we assume that $K = 0$ in the remainder of this proof.

Let $\lambda^c(u^c_k) = (\mu^k_0, \dots, \mu^k_{M-1})$ and $\lambda^c(u^c_k) = (\mu_0, \dots, \mu_{M-1})$. Therefore $\mu^k_m \to \mu_m$ in $\mathscr{M}(S^d \times T^d_n)$ for all (m, n). By Kechris (1995, Theorem 17.20),[16] $\liminf_k \mu^k_m(O) \geq \mu_m(O)$ for each (m, n, O) such that O is open in $S^d \times T^d_n$. By the definition of U^d, the open sets of $S^d \times T^d_n$ across all n jointly form an open basis for $S^d \times U^d$. It follows from (the other direction of) Kechris (1995, Theorem 17.20) that $\mu^k_m \to \mu_m$ in $\mathscr{M}(S^d \times U^d)$ for all m. Therefore $\lambda^c(u^c_k) \to \lambda^c(u^c)$ in $\mathscr{L}(S^d \times U^d)$, which shows that λ^c is continuous from U^c to $\mathscr{L}(S^d \times U^d)$. □

The next two corollaries are immediate consequences of Theorems 3.4 and 4.10.

Corollary 4.11. *There exists a continuous complete type structure that is a Borel-equivalent refinement of a complete type structure that admits RCAR.*

[15] Note that Proposition 3.1 is an immediate consequence of Theorems 3.4 and 4.10.

[16] Kechris (1995, Theorem 17.20): *Let* X *be a Polish space and let* $\mathscr{O}$ *be an open basis for* X. *A sequence* μ_k *weakly converges to* μ *in* $\mathscr{M}(X)$ *if and only if* $\liminf_k \mu_k(O) \geq \mu(O)$ *for every* $O \in \mathscr{O}$. This result is stated for all open sets in Kechris (1995), but the version stated here with an open basis follows from his proof.

Corollary 4.12. *There exists a continuous complete type structure that generates the same hierarchies as some complete type structure that admits RCAR.*

Corollary 4.12 is the result (III) stated in the Introduction, which casts doubt on the early interpretation of the impossibility theorem saying that continuous complete type structures are incompatible with RCAR because they generate too many hierarchies of belief.

The results in this section shift our focus from the belief hierarchies generated by continuous complete type structures to the degree of caution they require for the assumption of events. Corollary 4.12 shows that there are structures **T** and **U** that generate the same hierarchies such that **T** admits RCAR but **U** does not! With regard to beliefs, the *only* fundamental difference between **U** and **T** is that players must be more cautious in assuming an event in **U** than they are in assuming the same event in **T**. This suggests that the collection of open sets of **T**, at least in the role it plays in the definition of assumption, should be viewed as a primitive of a model whose type-belief maps are Borel but unrelated to the closeness/convergence of types.

The notions of Borel equivalence and Borel-equivalent refinement allow the separation of the degree of caution from continuity within BFK's framework. An example is provided by Theorems 3.4 and 4.10, which show that one can start with a complete type structure **T** admitting RCAR, and then take a Borel-equivalent refinement **U** that is continuous and complete. In such an analysis, one is defining rationality and assumption with respect to the coarse topology on T^c, while defining continuity will be with respect to the fine topology on U^c. We leave open the following natural question about whether we can go in the other direction:

Question 4.13. *Is every continuous complete type structure a Borel-equivalent refinement of some type structure that admits RCAR?*

An affirmative answer to this question would imply that any complete type structure can be modified only in its degree of caution (i.e., its topology on types) so that it admits RCAR. Because the modification would leave the type-belief maps unchanged, this would say that every complete type structure that does not admit RCAR generates the same hierarchies as some that do admit RCAR. Recall that Corollary 4.12 says that every complete type structure that admits RCAR generates the same hierarchies as some that do not admit RCAR.

5 RELATED LITERATURE

The two papers (other than BFK) that are most closely related to the present paper are Yang (2015)[17] and Lee (2016). Yang (2015) used lexicographic type structures as in BFK and it is therefore easier to identify parallels between his approach and ours. As we mentioned in the

[17] The model in Yang (2015) closely resembles that in Catonini and De Vito (2014), and either paper could fit the comparison here, but we specifically refer to Yang (2015) here for the sake of brevity.

Introduction, Yang (2015) introduced a condition called *RCWAR (rationality and common weak assumption of rationality)*—which is defined by replacing all instances of *assumption* in RCAR with *weak assumption*—and showed that it characterizes IA in a lexicographic type structure that is continuous and complete.

BFK's *assumption* is a natural generalization of Blume et al. (1991a)'s notion of "infinitely more likely" from finite spaces to infinite ones. It is a purely decision-theoretic notion that does not refer to game-specific objects such as strategy sets and type spaces. *Weak assumption* weakens assumption by changing the part of assumption that references open sets. Consider the following template statement: *Ann considers every "part of the event E" to be infinitely more likely than the complement of E*. If "part of the event E" is defined to be any nonempty intersection of E with an *open set*, we get "Ann assumes E". If "part of the event E" is defined to be any nonempty intersection of E with a *strategy cylinder*, we get "Ann weakly assumes E".[18]

On the other hand, in this paper we maintain BFK's definition of assumption but carefully choose which sets are open in our construction. Our Theorem 3.4 shows that RCAR is possible in a complete lexicographic type structure, while Yang (2015) shows that RCWAR is possible in a continuous complete lexicographic type structure. Theorem 3.4 is game-specific in a subtle way since the sets we choose to be open depend on the game. It is a nontrivial exercise in its own right to choose which sets are open in a way that preserves the definitions of all objects as given in BFK.[19]

An alternative approach that maintains BFK's notion of assumption in RCAR without changing the topology can be found in Lee (2016). By working directly with hierarchies of beliefs rather than lexicographic type structures, including hierarchies that cannot be represented in any lexicographic type structure, it shows that RCAR characterizes iterated admissibility. More specifically, the hierarchies that commonly assume rationality in Lee (2016)'s model cannot be represented in lexicographic type structures. This suggests that the impossibility theorem is not easily interpreted in terms of hierarchies. Lee (2016)'s result is driven by hierarchies that cannot be expressed in lexicographic type structures.

Our results in Section 4 of this paper shed light on the connection between admitting RCAR and continuity, completeness, and hierarchies of beliefs. They show that adding too many open sets to a model that admits RCAR can lead to a model that generates the same hierarchies but no longer admits RCAR. (If $\mathbf{T}$ is a complete lexicographic type structure that admits RCAR, then by Proposition 4.7 and Theorem 4.10 there is a continuous complete lexicographic type structure $\mathbf{W} \geq^{\text{ref}} \mathbf{T}$. By Propositions 4.7 and 4.9, every lexicographic type structure $\mathbf{U} \geq^{\text{ref}} \mathbf{W}$ generates the same hierarchies as $\mathbf{T}$ but does not admit RCAR.)

[18] A strategy cylinder is the product of a *subset* of a player's strategies with the set of *all* her types.

[19] For example, we could give each type space the trivial topology, but then it would not be Polish as required by BFK.

The epistemic characterizations in both Yang (2015) and Lee (2016) are game-independent in the sense that a single model is given for all games with the same strategy set. But the two papers depart from BFK in different ways: Yang (2015) by weakening assumption to depend on a finite number of open sets (i.e., the strategy cylinders) instead of all open sets; and Lee (2016) by using more hierarchies than are possible in BFK.

Yet another conceptually related paper is B-S, which used non-lexicographic type structures that model beliefs as conditional probability systems instead of lexicographic probability systems. B-S showed that the extensive form rationalizable (EFR) strategies are exactly the strategies played in states at which there is *rationality and common strong belief of rationality (RCSBR)* in the continuous complete non-lexicographic type structure that canonically generates all belief hierarchies. A discussion of the relationship between strong belief and assumption can be found in BFK and Brandenburger et al. (2007). Assumption, weak assumption, and strong belief are non-monotonic notions of belief that share some important similarities that relate to the solutions they are used to characterize.

Each of the papers BFK, B-S, and Yang (2015) is motivated by the *Best Rationalization Principle*, which is stated in Battigalli (1996) as follows:

> A player should always believe that her opponents are implementing one of the "most rational" (or "least irrational") strategy profiles which are consistent with her information.

In B-S, "most rational" is interpreted as "highest degree of strategic sophistication", where degree of strategic sophistication m means *rationality and m-th order strong belief of rationality* (RmSBR). The analogous notions of strategic sophistication in the frameworks of Yang (2015) and BFK are summarized in the following table.

Strategic Sophistication	B-S	Yang requires consistency with	BFK
$\geq m+1$	RmSBR	RmWAR	RmAR
$= \infty$	RCSBR	RCWAR	RCAR

Table 1: Strategic sophistication in B-S, Yang (2015), and BFK

Section 4 introduced the idea that the open sets of type spaces encode the degree of caution required to assume events. In the online supplement to BFK, it is pointed out that the family of open sets in BFK plays the same role as the family of conditioning events in B-S. In Yang (2015), the family of strategy cylinders plays that role. In light of these similarities, it is an interesting open question whether it is possible to provide a unified treatment of all three epistemic conditions that sheds light on their relationship to each other. As a starting point, we explore some apparent parallels in Appendix D using the notion of *best rationalization systems*.

More recently, Catonini and De Vito (2022) defined and used *cautious belief* to provide epistemic characterizations of IA. While both cautious belief and weak assumption depend on the family of strategy cylinders, Catonini and De Vito (2022) departs from earlier papers[20] by basing cautious belief on Lo (1999)'s comparatively less restrictive notion of "infinitely more likely" instead of that which is found in Blume et al. (1991a). Hence, less is required for cautious belief than is for weak assumption. Nevertheless, the finiteness of strategy cylinders remains crucial to their positive result.

The weakening of "infinitely likely" in Catonini and De Vito (2022) has further notable implications. It is known via the proofs and discussions in BFK and Yang (2015) that epistemic characterizations of m-admissibility (and therefore also of IA) in those papers are incompatible with transparency of rational types. This is true of our paper as well since we use BFK's framework. Catonini and De Vito (2022) are able to provide an alternative epistemic characterization that is compatible with transparency of rational types precisely because of the specific way in which the notion of "infinitely likely" is weakened.

APPENDIX A PROOF OF THEOREM 3.4

In this appendix, we will introduce the notion of a *best rationalization system* and auxiliary results related to it. These will help clarify the structure of the constructive parts of our proof. By the end, we will have reduced the proof of Theorem 3.4 down to a near immediate consequence of two theorems (A.11 and A.13) relating to best rationalization systems.

Definition A.1. *A* ***best rationalization system*** *(BRS) over Ω is a triple*

$$\mathbf{B} = (\Omega, \langle Q_m, m \in \mathbb{N}\rangle, \mathscr{C})$$

where Ω is a Polish space, $\langle Q_m, m \in \mathbb{N}\rangle$ is a decreasing chain of non-empty Borel subsets of Ω such that $Q_0 = \Omega$, and $\mathscr{C}$ is a family of non-empty Borel subsets of Ω.

We define $Q_\infty \equiv \bigcap_{m\in\mathbb{N}} Q_m$. We say that a set $C \in \mathscr{C}$ is ***best-rationalized at degree*** *m in $\mathbf{B}$ if m is the greatest natural number such that C meets Q_m (that is, $C \cap Q_m$ is non-empty). We say that an integer M is a* ***finite bound*** *for $\mathbf{B}$ if every $C \in \mathscr{C}$ that is best-rationalized at some finite degree is best-rationalized at some degree $< M$.*

In a BRS $\mathbf{B}$, we interpret m as the degree of strategic sophistication, and we interpret $\mathscr{C}$ as a set of conditioning events. Note that for each $C \in \mathscr{C}$ there is at most one m such that C is best-rationalized at degree m in $\mathbf{B}$. If C meets Q_∞ then there is no such m.

In this appendix, we will consider the BRS generated by a game, type structure, and player. In Appendix D, we will consider BRSs generated by other structures from the literature.

[20] including Catonini and De Vito (2014)

Definition A.2. *The BRS **generated** by a game G, type structure $\mathbf{T}$ for G, and player c is the BRS*

$$\mathbf{B}^c \equiv (S^c \times T^c, \langle R^c_m, m \in \mathbb{N}\rangle, \mathscr{C}^c)$$

where $\mathscr{C}^c$ is the family of non-empty open subsets of $S^c \times T^c$. We call $(\mathbf{B}^a, \mathbf{B}^b)$ the BRS pair generated by G and $\mathbf{T}$.

Note that every BRS $\mathbf{B}^c$ over $S^c \times T^c$ has the form

$$\mathbf{B}^c = (S^c \times T^c, \langle Q^c_m, m \in \mathbb{N}\rangle, \mathscr{C}^c). \tag{3}$$

A BRS pair $(\mathbf{B}^a, \mathbf{B}^b)$ specifies the Polish spaces T^c and the sets Q^c_m for each player c, but does not specify the mappings λ^c.

When $(\mathbf{B}^a, \mathbf{B}^b)$ is generated by G and $\mathbf{T}$, Q^c_m is the set R^c_m of strategy-type pairs where player c is rational at level m. And $Q^a_\infty \times Q^b_\infty$ is the set $R^a_\infty \times R^b_\infty$ of states of the world in which RCAR holds. Intuitively, the sets Q^c_m and Q^c_∞ correspond to properties of strategy-type pairs that may occur in a line of reasoning used by either player.

The next result shows that every game G and type structure $\mathbf{T}$ for G generates a BRS pair, whether or not $\mathbf{T}$ is complete or admits RCAR.

Proposition A.3. *For every finite game G and type structure $\mathbf{T}$ over G, there is a unique BRS pair $(\mathbf{B}^a, \mathbf{B}^b)$ that is generated by G and $\mathbf{T}$.*

Proof. Let
$$\mathbf{B}^c = (S^c \times T^c, \langle R^c_m, m \in \mathbb{N}\rangle, \mathscr{C}^c).$$
By definition, T^c is a Polish space, $R^c_0 = S^c \times T^c$, and $R^c_{m+1} \subseteq R^c_m$ for each $m \in \mathbb{N}$. By Lemma C.4 in BFK, R^c_m is Borel for each $m \in \mathbb{N}$. Therefore $(\mathbf{B}^a, \mathbf{B}^b)$ is a BRS pair. □

Theorem A.11 below will answer the following question:

Question A.4. *For a given game G, which BRS pairs $(\mathbf{B}^a, \mathbf{B}^b)$ are generated by G and some complete type structure that admits RCAR?*

Given a game G with strategy sets S^a, S^b and a BRS pair $(\mathbf{B}^a, \mathbf{B}^b)$ of the form (3), Question A.4 asks when there exist Borel mappings λ^c such that $\mathbf{T} = (S^a, S^b, T^a, T^b, \lambda^a, \lambda^b)$ is a complete type structure that admits RCAR with $R^c_m = Q^c_m$ for each c and m.

An obvious necessary condition for an affirmative answer to Question A.4 is that $Q^a_\infty \times Q^b_\infty \neq \emptyset$. By Lemma 2.3, another necessary condition is that T^a and T^b are uncountable Polish spaces. Theorem A.11 below will give a necessary and sufficient condition. That condition will involve two things:

- For each set $Q \subseteq S^c \times T^c$, a partition

$$T^c = \bigcup \{\Gamma(X, Q) \mid X \subseteq S^c\} \tag{4}$$

  of T^c into at most $2^{|S^c|}$ pairwise disjoint sets $\Gamma(X, Q)$.
- A family $\mathscr{B}^c_m(G)$ of subsets of S^c that depends only on m and the game G, which was defined in Section 2.3.

Definition A.5. *Let S^c, T^c be sets with S^c finite. For each $Q \subseteq S^c \times T^c$ and each $X \subseteq S^c$, let*

$$\Gamma(X,Q) \equiv \{t^c \in T^c \mid X = \{s^c \mid (s^c,t^c) \in Q\}\}.$$

Thus $\Gamma(X,Q)$ is the set of all $t^c \in T^c$ such that the section of Q at t^c is X. It is possible that $\Gamma(X,Q)$ is empty. Since S^c is finite, (4) is a partition of T^c, and

$$Q = \bigcup \{X \times \Gamma(X,Q) \mid X \subseteq S^c\} \tag{5}$$

is a partition of Q into at most $2^{|S^c|}$ pairwise disjoint sets.

Informally, we may think about and $Q \subseteq S^c \times T^c$ as a *theory* about both player c's behavior (i.e., strategy) and belief (i.e., type). For a given type t^c, call the set $X = \{s^c \mid (s^c,t^c) \in Q\}$ of strategies for t^c that are possible under the theory Q the *behavioral implication* of t^c under Q. Then $\Gamma(X,Q) \subseteq T^c$ is the collection of all types whose behavioral implication under theory Q is exactly X. Thus, the partitioning of Q in (5) represents a division of the theory Q into mutually exclusive sub-theories $\Gamma(X,Q)$ based on the type's behavioral implication under Q. We say that a type t^c is *consistent* with Q if some strategy for t^c is possible under Q, that is, the behavioral implication is not empty. We say that a type t^c is *ruled out* by Q if t^c is not consistent with Q, that is,

$$t^c \in \{t^c \in T^c \mid (s^c,t^c) \notin Q\} = \{t^c \in T^c \mid \varnothing = \{s^c \mid (s^c,t^c) \in Q\}\} = \Gamma(\varnothing,Q). \tag{6}$$

Thus, the partitioning of T^c in (4) covers both the types that are consistent with and are ruled out by Q.

Lemma A.6. *If S^c is a finite set, T^c is a Polish space, $Q \subseteq S^c \times T^c$ is Borel, and $X \subseteq S^c$, then $\Gamma(X,Q)$ is Borel.*

Proof. For each $s^c \in S^c$, the set $\{s^c\} \times T^c$ is Borel, so the set $(\{s^c\} \times T^c) \cap Q$ is Borel. The mapping $(s^c,t^c) \mapsto t^c$ is a Borel injection from $(\{s^c\} \times T^c) \cap Q$ onto the subset

$$Y(s^c) \equiv \{t^c \in T^c \mid (s^c,t^c) \in Q\}$$

of T^c. By Theorem 15.1 in (Kechris, 1995), $Y(s^c)$ is Borel. We note that for each $X \subseteq S^c$ and $t^c \in T^c$, we have $t^c \in \Gamma(X,Q)$ if and only if $t^c \in Y(s^c)$ for all $s^c \in X$ and $t^c \notin Y(s^c)$ for all $s^c \notin X$. □

It follows that (4) is a finite partition of T^c into Borel sets, and (5) is a finite partition of Q into Borel sets.

Remark A.7. *In a type structure* **T** *for a finite game G, whenever $0 < h \in \mathbb{N}$ and $\varnothing \neq X \subseteq S^c$, $\Gamma(X,R^c_{h+1})$ is a subset of $\Gamma(X,R^c_h)$.*

Proof. Suppose $t^c \in \Gamma(X,R^c_{h+1})$. Then

$$X = \{s^c \mid (s^c,t^c) \in R^c_{h+1}\} = \bigcap_{m \le h} \{s^c \mid (s^c,t^c) \in R^c_m \wedge t^c \in \mathbf{A}^c(R^d_m)\}.$$

Since X is non-empty, $t^c \in \bigcap_{m=1}^h \mathbf{A}^c(R^d_m)$. Therefore

$$X = \bigcap_{m=1}^{h} \{s^c \mid (s^c,t^c) \in R^c_m\} = \{s^c \mid (s^c,t^c) \in R^c_h\},$$

so $t^c \in \Gamma(X, R^c_h)$. □

Recall that $\mathscr{P}^d_m$ is the set of all LPSs $\mu = (\mu_0, \dots, \mu_m)$ on S^d such that

$$\operatorname{supp}\mu_0 = S^d_m,\ \operatorname{supp}\mu_1 = S^d_{m-1}, \dots, \text{ and } \operatorname{supp}\mu_m = S^d_0.$$

In the remainder, we make use of the fact that, by Proposition 1 in Blume et al. (1991b), for each $m \in \mathbb{N}$ and $\nu \in \mathcal{N}^+(S^d_m)$, there is a measure $\rho \in \mathcal{M}^+(S^d_m)$ such that $(\rho) \sim \nu$. The following lemma shows that for each type t^c that is consistent with R^c_m, there is an LPS in $\mathscr{P}^d_m$ that ranks strategies in the same order that the marginal of $\lambda^c(t^c)$ does.

Lemma A.8. *Suppose* **T** *be a complete type structure for G,* $0 < m \in \mathbb{N}$, *and* $(s^c, t^c) \in R^c_m$. *There exists* $\rho \in \mathscr{P}^d_m$ *such that* $\rho \sim \operatorname{marg}_{S^d} \lambda^c(t^c)$.

Proof. Let $\lambda^c(t^c) = \mu = (\mu_0, \dots, \mu_{n-1})$. For each $k < m$, let $a(k)$ be the level at which R^d_k is assumed under μ. Then $n-1 = a(0) \ge \cdots \ge a(m-1) \ge 0$. For each $k < m-1$ let $\nu^k = \operatorname{marg}_{S^d}(\mu_0, \dots, \mu_{a(k)})$. Note that

$$\nu^0 = \operatorname{marg}_{S^d}(\mu_0, \dots, \mu_{n-1}) = \operatorname{marg}_{S^d}\mu = \operatorname{marg}_{S^d}\lambda^c(t^c),$$

and for each $k < m$, ν^{k+1} is an initial segment of ν^k. Therefore $\nu^0 \cdots \nu^{m-1} \sim \nu^0$. Since R^d_k is assumed under μ at level $a(k)$, we see from Proposition 2.1 and Theorem 3.2 that $\operatorname{supp}\nu^k = \operatorname{proj}_{S^d} R^d_k = S^d_k$. Therefore $\nu^k \in \mathcal{N}^+(S^d_k)$, so there is a measure $\rho^k \in \mathcal{M}^+(S^d_k)$ such that $(\rho^k) \sim \nu^k$ by Proposition 1 in Blume et al. (1991b). Then

$$\rho \equiv (\rho^0, \dots, \rho^{m-1}) \in \mathscr{P}^d_m, \quad \rho \sim \nu^0 \cdots \nu^{m-1} \sim \operatorname{marg}_{S^d}\lambda^c(t^c).$$ □

Recall the definition $\mathscr{B}^c_m(G) \equiv \mathbf{BR}^c(\mathscr{P}^d_m)$ from Section 2.3. An immediate and useful implication of Lemma A.8 is that

$$\mathscr{B}^c_m(G) = \{\mathbf{BR}^c(\lambda^c(t^c)) \mid t^c \in \operatorname{proj}_{T^c} R^c_m\}.$$

We will also need the following lemma.

Lemma A.9. *For each* $m \in \mathbb{N}$ *we have* $\mathscr{B}^c_{m+1}(G) \subseteq \mathscr{B}^c_m(G)$.

Proof. Suppose $m > 0$ and $X \in \mathscr{B}^c_m(G)$. We show that $X \in \mathscr{B}^c_{m-1}(G)$. Because $X \in \mathscr{B}^c_m(G)$, there is some $\nu = (\nu_0, \dots, \nu_m) \in \mathscr{P}^d_m$ such that $X = \mathbf{BR}^c(\nu)$. Then $\nu_i \in \mathcal{M}^+(S^d_{m-i})$ and $(\nu_0, \dots, \nu_i) \in \mathcal{N}^+(S^d_{m-i})$ for each $i \le m$. Hence for each $i \le m$ there is a $\mu_i \in \mathcal{M}^+(S^d_{m-i})$ such that $(\mu_i) \sim (\nu_0, \dots, \nu_i)$. Then for each $i < m$, $(\mu_i, \nu_{i+1}) \sim (\nu_0, \dots, \nu_{i+1}) \sim (\mu_{i+1})$.

Now let $i < m$ and consider any $r, s \in S^c$. If $s \succ_{(\mu_i)} r$ then $s \succ_{(\mu_i, \mu_{i+1})} r$, $s \succ_{(\mu_i, \nu_{i+1})} r$, and hence $s \succ_{(\mu_{i+1})} r$. So if $s \succ_{(\mu_i)} r$ then $s \succ_{(\mu_i, \mu_{i+1})} r$ iff $s \succ_{(\mu_{i+1})} r$. Similarly, if $r \succ_{(\mu_i)} s$ then $s \succ_{(\mu_i, \mu_{i+1})} r$ iff $s \succ_{(\mu_{i+1})} r$. If neither $s \succ_{(\mu_i)} r$ nor $r \succ_{(\mu_i)} s$, then again $s \succ_{(\mu_i, \mu_{i+1})} r$ iff $s \succ_{(\mu_{i+1})} r$. Therefore $(\mu_i, \mu_{i+1}) \sim (\mu_{i+1})$, and hence $(\mu_i, \mu_{i+1}, \dots, \mu_m) \sim (\mu_{i+1}, \dots, \mu_m)$.

μ_m was chosen so that $(\mu_m) \sim \nu$. By the above, $(\mu_{i+1}, \dots, \mu_m) \sim \nu$ implies $(\mu_i, \mu_{i+1}, \dots, \mu_m) \sim \nu$. So by induction we have $(\mu_1, \dots, \mu_m) \sim \nu$. Therefore $\mathbf{BR}^c((\mu_1, \dots, \mu_m)) = \mathbf{BR}^c(\nu) = X$. Since $\mu_i \in \mathcal{M}^+(S^d_{m-i})$ for each i, we have $(\mu_1, \dots, \mu_m) \in \mathscr{P}^d_{m-1}$. Thus $X \in \mathbf{BR}^c(\mathscr{P}^d_{m-1}) = \mathscr{B}^c_{m-1}(G)$. □

Corollary A.10. *Recall from Section 2.3 that* $M(G)$ *is the least* $M \in \mathbb{N} \cup \{\infty\}$ *such that* $M > 0$ *and* $\mathscr{B}^c_k(G) = \mathscr{B}^c_M(G)$ *for each* $c \in \{a, b\}$ *and all* $k \ge M$. *Then* $M(G)$ *is finite and it immediately follows that* $\mathscr{B}^c_\infty(G) = \mathscr{B}^c_{M(G)}(G) \neq \varnothing$.

Proof. By Lemma A.9 and the fact that $\mathscr{B}^c_m(G)$ is finite and non-empty for each $m \in \mathbb{N}$. □

Here is our answer to Question A.4.

Theorem A.11. *Let G be a game with strategy sets S^c and $(\mathbf{B}^a, \mathbf{B}^b)$ be a BRS pair of the form (3). Then the following are equivalent.*

(i) *$(\mathbf{B}^a, \mathbf{B}^b)$ is generated by G and some complete type structure $\mathbf{T}$ that admits RCAR.*
(ii) *$(\mathbf{B}^a, \mathbf{B}^b)$ is generated by G and some complete one-to-one type structure $\mathbf{T}$ that admits RCAR.*
(iii) *For each player c, we have:*
  (iii.a) *There exists $M \in \mathbb{N}$ such that $\overline{Q^c_h} = \overline{Q^c_M}$ whenever $M \le h \in \mathbb{N}$.*
  (iii.b) *If $0 < h \in \mathbb{N}$ and $X \in \mathscr{B}^c_h(G)$ then $\Gamma(X, Q^c_h) \setminus \Gamma(X, Q^c_{h+1})$ is uncountable.*
  (iii.c) *If $0 < h \in \mathbb{N}$, $Y \subseteq S^c$, $Y \neq \varnothing$, and $Y \notin \mathscr{B}^c_h(G)$, then $\Gamma(Y, Q^c_h) = \varnothing$.*
  (iii.d) *$\Gamma(\varnothing, Q^c_1)$ is uncountable.*

The proof of Theorem A.11 is in Appendix B. Note that the implication (ii) ⇒ (i) is trivial. The idea of the proof that (iii) ⇒ (ii) is to construct a one-to-one type structure by using the Borel Isomorphism Theorem (15.6 in (Kechris, 1995)), which provides a surjective one-to-one Borel mapping between any two Borel sets of the same cardinality. Theorem 3.5(i) shows that (i) ⇒ (iii.a). In Appendix B, we show that if $(\mathbf{B}^a, \mathbf{B}^b)$ is generated by G and some complete type structure (which does not necessarily admit RCAR), then (iii.b)–(iii.d) hold.

We do not know of an economic interpretation of the property that the type-belief maps are one-to-one. But type structures with no redundant types are necessarily one-to-one. Moreover, one-to-one is a very useful technical property for Borel maps between Polish spaces because it guarantees that images of Borel sets are Borel sets.

Recall that in a topological space, a set is **nowhere dense** if its interior is empty, and is **meager** if it is the union of countably many nowhere dense sets. Intuitively, meager sets are small in a topological sense. When X, Y are sets in a topological space and $Y \subseteq X$, we say that Y is nowhere dense, or meager, **in** X if Y is nowhere dense, or meager, in the relative topology on X.

Theorem A.11 has the following consequence.

Corollary A.12. *Suppose $(\mathbf{B}^a, \mathbf{B}^b)$ is a BRS pair of the form (3) that is generated by G and some complete type structure $\mathbf{T}$ that admits RCAR. Then there exists $M \in \mathbb{N}$ such that for each player c:*

(i) *For each finite $k \ge M$, $Q^c_M \setminus Q^c_k$ is nowhere dense in Q^c_M.*
(ii) *$Q^c_M \setminus Q^c_\infty$ is meager in Q^c_M.*
(iii) *M is a finite bound for $\mathbf{B}^c$.*

Proof. Since G and $\mathbf{T}$ generate $(\mathbf{B}^a, \mathbf{B}^b)$, $Q^c_k = R^c_k$ for each c and $k \in \mathbb{N}$, and $Q^c_\infty = R^c_\infty$. By Theorem A.11, conditions (iii.a)–(iii.d) hold. Let M be as in (iii.a).

(i): Suppose $Q^c_M \setminus Q^c_k$ is not nowhere dense in Q^c_M. Then there is an open set O that meets Q^c_M but does not meet Q^c_k. Therefore $\overline{Q^c_k} \neq \overline{Q^c_k}$, so $k < M$.

(ii) follows from (i) since $Q^d_M \setminus Q^d_\infty = \bigcup_{k \ge M} (Q^d_M \setminus Q^d_k)$.

(iii): By (i), whenever $k > M$, each open set O that meets Q^c_M also meets Q^c_k. Therefore no open set can be best-rationalized in $\mathbf{B}^c$ at a degree $k > M$, so M is a finite bound for $\mathbf{B}^c$. □

Theorem A.13. *Let G be a finite game with strategy sets S^a, S^b, and T^a, T^b be uncountable Polish spaces. There is a BRS pair $(\mathbf{B}^a, \mathbf{B}^b)$ of the form (3) that satisfies conditions (iii.a)–(iii.d) of Theorem A.11 with $M = M(G)$.*

The proof of Theorem A.13 is in Appendix C.

Proof of Theorem 3.4. By Theorem A.13, there is a BRS pair $(\mathbf{B}^a, \mathbf{B}^b)$ of the form (3) that satisfies conditions (iii.a)–(iii.d). By Theorem A.11, there exists a complete type structure that admits RCAR and generates $(\mathbf{B}^a, \mathbf{B}^b)$ with G, and hence has the required type spaces T^a, T^b. □

APPENDIX B PROOF OF THEOREM A.11

We will need two more lemmas.

Lemma B.1. *Let G be a game with strategy sets S^c and $(\mathbf{B}^a, \mathbf{B}^b)$ be a BRS pair of the form (3). Suppose $h \in \mathbb{N}$ and $\mu = (\mu_0, \ldots, \mu_h) \in \mathscr{P}_h^d$. Then there are $2^{\aleph_0}$ different $\sigma = (\sigma_0, \ldots, \sigma_h) \in \mathscr{L}^+(S^d \times T^d)$ such that:*

(i) $\operatorname{marg}_{S^d} \sigma = \mu$.
(ii) *Q_ℓ^d is assumed under σ whenever $\ell \leq h$.*
(iii) *Q_{h+1}^d is not assumed under σ.*

Proof. Note that X is a non-empty subset of S^c. The sequence of measures μ depends only on G, X and h. By the definition of $\mathscr{P}_h^d$, for each $j \leq h$ we have $\sum_{s^d \in S^d} \mu_j(\{s^d\}) = 1$. For each $m \in \mathbb{N}$, let $\Delta Q_m^d = Q_m^d \setminus Q_{m+1}^d$. By Theorem 3.2, $\operatorname{proj}_{S^d} Q_{h-j}^d = S_{h-j}^d$ for each $j \leq h$. Therefore $Q_{h-j}^d \cap (\{s^d\} \times T^d)$ is non-empty iff $s^d \in S_{h-j}^d$. By Lemma E.3 in BFK, $\Delta Q_{h-j}^d \cap (\{s^d\} \times T^d)$ is also non-empty whenever $s^d \in S_{h-j}^d$.

Let $0 < \alpha < 1$. We show that there exists $\sigma = (\sigma_0, \ldots, \sigma_h) \in \mathscr{L}^+(S^d \times T^d)$ such that: $\operatorname{marg}_{S^d} \sigma = \mu$ and

(a1) If $0 < j \leq h$ and $s^d \in S^d$, then $\sigma_j(\Delta Q_{h-j}^d \cap (\{s^d\} \times T^d)) = \mu_j(\{s^d\})$.
(a2) If $0 < j \leq h$ and O is open and meets ΔQ_{h-j}^d, then $\sigma_j(O \cap \Delta Q_{h-j}^d) > 0$.
(a3) $\sigma_0(Q_h^d \cap (\{s^d\} \times T^d)) = \mu_0(\{s^d\})$ for each $s^d \in S^d$.
(a4) $\sigma_0(Q_{h+1}^d \cap (\{s^d\} \times T^d)) = \alpha \cdot \mu_0(\{s^d\})$ for each $s^d \in S_{h+1}^d$.
(a5) If O is open and meets Q_h^d, then $\sigma_0(O \cap Q_h^d) > 0$.

By Lemma E.2 in BFK, for each $s^d \in S_{h-j}^d$ the set $\Delta Q_{h-j}^d \cap (\{s^d\} \times T^d)$ is uncountable. Since T^d is Polish, $\Delta Q_{h-j}^d \cap (\{s^d\} \times T^d)$ has a dense subset $D(j, s^d)$ of cardinality $\aleph_0$, and there is a probability measure σ_j such that for each $s^d \in S_{h-j}^d$, $\sigma_j(D(j, s^d)) = \mu_j(\{s^d\})$ and $\sigma_j(\{x\}) > 0$ for each $x \in D(j, s^d)$. Therefore σ_j satisfies (a1) and (a2), and $\operatorname{marg}_{S^d} \sigma_j = \mu_j$.

By a similar argument, there is a probability measure σ_0 that satisfies (a3), (a4), and (a5), and $\operatorname{marg}_{S^d} \sigma_0 = \mu_0$. The sets Q_h^d and ΔQ_{h-j}^d, $j \leq h$ are pairwise disjoint. $\sigma_0(Q_h^d) = 1$ and $\sigma_j(\Delta Q_{h-j}^d) = 1$ for each $j \leq h$. Hence the measures $\sigma_0, \ldots, \sigma_h$ are mutually singular, so $\sigma \in \mathscr{L}(S^d \times T^d)$ and $\operatorname{marg}_{S^d} \sigma = \mu$. By (a2) and (a5), $\sigma \in \mathscr{L}^+(S^d \times T^d)$. Thus $\sigma = (\sigma_0, \ldots, \sigma_h)$ satisfies (a1)–(a5).

By (a1), whenever $0 < j \leq h$ we have $\sigma_j(\Delta Q_{h-j}^d) = 1$. By (a3), $\sigma_0(Q_h^d) = 1$. It follows that for all $j, \ell \in \{0, \ldots, h\}$, $\sigma_j(Q_\ell^d) = 1$ when $j \leq h - \ell$ and $\sigma_j(Q_\ell^d) = 0$ when $j > h - \ell$. By Proposition 2.1, (a2), and (a5), Q_ℓ^d is assumed under σ whenever $\ell \leq h$.

By (a3), (a4), and (a5),

$$\sigma_0(R_{h+1}^d \cap (\{s^d\} \times T^d)) \leq \sigma_0(R_h^d \cap (\{s^d\} \times T^d))$$

for each $s^d \in S^d$, and

$$\sigma_0(R^d_{h+1} \cap (\{s^d\} \times T^d)) < \sigma_0(R^d_h \cap (\{s^d\} \times T^d))$$

for each $s^d \in S^d_{h+1}$. Therefore

$$0 < \sigma_0(R^d_{h+1}) < \sigma_0(R^d_h) = 1.$$

Hence by Proposition 2.1, Q^d_{h+1} is not assumed under σ. We have shown $\sigma \in \mathscr{L}^+(S^d \times T^d)$ and σ satisfies (i)–(iii).

Since there are $2^{\aleph_0}$ different values $0 < \alpha < 1$ and different values of α lead to different values of σ, there are $2^{\aleph_0}$ different $\sigma \in \mathscr{L}^+(S^d \times T^d)$ that satisfy (i)–(iii). □

Lemma B.2. *Let G be a game with strategy sets S^c and $(\mathbf{B}^a, \mathbf{B}^b)$ be a BRS pair of the form (3).*

Suppose $M = M(G)$, Q^d_∞ is an uncountable dense subset of Q^d_M, and $\mu = (\mu_0, \ldots, \mu_{M-1}) \in \mathscr{P}^d_{M-1}$. Then there are $2^{\aleph_0}$ different $\sigma = (\sigma_0, \ldots, \sigma_{M-1}) \in \mathscr{L}^+(S^d \times T^d)$ such that:

(i) $\operatorname{marg}_{S^d} \sigma = \mu$.
(ii) *Q^d_ℓ is assumed under σ for every $\ell \le \infty$.*

Proof. Let $h = M - 1$. We argue as in the proof of Lemma B.1 to show that there exists $\sigma = (\sigma_0, \ldots, \sigma_{M-1}) \in \mathscr{L}^+(S^d \times T^d)$ that satisfies (i) and (a1)–(a3) and

(a4') $\sigma_0(Q^d_\infty \cap (\{s^d\} \times T^d) = \mu_0(\{s^d\})$ for each $s^d \in S^d_\infty$.
(a5') If O is open and meets Q^d_∞, then $\sigma_0(O \cap Q^d_\infty) > 0$.

Using (a1)–(a5'), a proof that is similar to but simpler than the proof of Lemma B.1 shows that σ satisfies (ii). Since Q^d_∞ is uncountable and Borel, it has cardinality $2^{\aleph_0}$. It follows that there are $2^{\aleph_0}$ ways to choose σ_0 so that σ satisfies (i) and (ii). □

Theorem A.11 (repeated). *Let G be a game with strategy sets S^c and $(\mathbf{B}^a, \mathbf{B}^b)$ be a BRS pair of the form (3). Then the following are equivalent.*

(i) *$(\mathbf{B}^a, \mathbf{B}^b)$ is generated by G and some complete type structure $\mathbf{T}$ that admits RCAR.*
(ii) *$(\mathbf{B}^a, \mathbf{B}^b)$ is generated by G and some complete one-to-one type structure $\mathbf{T}$ that admits RCAR.*
(iii) *For each player c, we have:*
  (iii.a) *There exists $M \in \mathbb{N}$ such that $\overline{Q^c_h} = \overline{Q^c_M}$ whenever $M \le h \in \mathbb{N}$.*
  (iii.b) *If $0 < h \in \mathbb{N}$ and $X \in \mathscr{B}^c_h(G)$ then $\Gamma(X, Q^c_h) \setminus \Gamma(X, Q^c_{h+1})$ is uncountable.*
  (iii.c) *If $0 < h \in \mathbb{N}$, $Y \subseteq S^c$, $Y \neq \emptyset$, and $Y \notin \mathscr{B}^c_h(G)$, then $\Gamma(Y, Q^c_h) = \emptyset$.*
  (iii.d) *$\Gamma(\emptyset, Q^c_1)$ is uncountable.*

It is trivial that Theorem A.11 (ii) implies Theorem A.11 (i).

Proof that Theorem A.11 (i) implies Theorem A.11 (iii). Let $\mathbf{T}$ be a complete type structure such that $(\mathbf{B}^a, \mathbf{B}^b)$ is generated by G and $\mathbf{T}$. We must show that if $\mathbf{T}$ admits RCAR then conditions (iii.a)–(iii.d) hold. Theorem 3.5 shows that if $\mathbf{T}$ admits RCAR then (iii.a) holds. We will now show, even without assuming that $\mathbf{T}$ admits RCAR, that conditions (iii.b)–(iii.d) hold.

Let R^c_h be the rationality set at level h for the complete type structure $\mathbf{T}$. We have $Q^c_\infty = R^c_\infty$ and $Q^c_h = R^c_h$ for each $h \in \mathbb{N}$. By Lemma C.4 in BFK, each R^c_h is Borel. Since S^c is finite, each $\Gamma(X, R^c_h)$ is also Borel. By Lemma 2.3, T^a and T^b are uncountable.

Proof of (iii.b): Since $\mathbf{T}$ is complete, it follows from Lemma E.2 of BFK that if $\Gamma(X,R^c_h)\setminus\Gamma(X,R^c_{h+1})$ is non-empty then it is uncountable. So to prove (iii.b) it suffices to show that whenever $0<h\in\mathbb{N}$,

$$\text{For every } X\in\mathscr{B}^c_h(G) \text{ there exists } t^c_h\in\Gamma(X,R^c_h)\setminus\Gamma(X,R^c_{h+1}). \tag{7}$$

We now prove (7). Let $0<h\in\mathbb{N}$, $X\in\mathscr{B}^c_h(G)$, and $X=\mathbf{BR}^c(\mu)$ where $\mu\in\mathscr{P}^d_h$. By Lemma B.1 there exists $\sigma\in\mathscr{L}^+(S^d\times T^d)$ such that $\text{marg}_{S^d}\sigma=\mu$, and σ assumes R^d_ℓ for each $\ell\le h$ but does not assume R^d_{h+1}. Since $\mathbf{T}$ is complete, there exists $t^c_h\in T^c$ such that $\lambda^c(t^c_h)=\sigma$. Then

$$\text{marg}_{S^d}\lambda^c(t^c_h)\in\mathscr{L}^+(S^d\times T^d),\quad X=\mathbf{BR}^c(\text{marg}_{S^d}\lambda^c(t^c_h)) \tag{8}$$

and

$$t^c_h\in\bigcap_{\ell\le h}\mathbf{A}^c(R^d_\ell),\quad t^c_h\notin\mathbf{A}^c(R^d_{h+1}). \tag{9}$$

By Lemma C.4 in BFK,

$$R^c_0=S^c\times T^c,\quad R^c_h=R^c_1\cap(S^c\times\bigcap_{\ell<h}\mathbf{A}^c(R^d_\ell)) \text{ when } h>0. \tag{10}$$

By (8)–(10) and the definition of R^c_1, for each $s^c\in S^s$ the following are equivalent:

- $(s^c,t^c_h)\in R^c_h$
- $(s^c,t^c_h)\in R^c_1$ and $t^c_h\in\bigcap_{\ell<h}\mathbf{A}^c(R^d_\ell)$
- $(s^c,t^c_h)\in R^c_1$
- $\lambda^c(t^c_h)\in\mathscr{L}^+(S^d\times T^d)$ and $s^c\in\mathbf{BR}^c(\text{marg}_{S^d}\lambda^c(t^c_h))$
- $s^c\in X$.

Recall that

$$\Gamma(X,Q)\equiv\{t^c\in T^c\mid X=\{s^c\mid(s^c,t^c)\in Q\}\}.$$

Therefore

$$X=\{s^c\mid(s^c,t^c_h)\in R^c_h\},\quad t^c_h\in\Gamma(X,R^c_h).$$

But by (9) and by (10) for $h+1$,

$$\{s^c\mid(s^c,t^c_h)\in R^c_{h+1}\}=\varnothing\ne X,\quad t^c_h\notin\Gamma(X,R^c_{h+1}).$$

This completes the proof of (7), and thus (iii.b) is proved.

Proof of (iii.c): Suppose $0<h\in\mathbb{N}$, $Y\subseteq S^c$, $s^c\in Y$, and $t^c\in\Gamma(Y,Q^c_h)$. To prove (iii.c), it is enough to show that $Y\in\mathscr{B}^c_h(G)$. By $Q^c_h=R^c_h$ and (5), $(s^c,t^c)\in R^c_h$. Let $X=\mathbf{BR}^c(\text{marg}_{s^d}\lambda^c(t^c))$. (s^c,t^c) is rational, so $s^c\in X$ and $(s^c,t^c)\in X\times R^c_h$. Since (5) is a partition of T^c, $X\times R^c_h=Y\times R^c_h$, and hence $X=Y$. By Lemma A.8, there exists $\rho\in\mathscr{P}^d_m$ such that $\rho\sim\text{marg}_{S^d}\lambda^c(t^c)$. Therefore $Y=X=\mathbf{BR}^c(\rho)$, so $Y\in\mathscr{B}^c_h(G)$, as required.

Proof of (iii.d): Since T^d is uncountable,

$$\mathscr{L}(S^d\times T^d)\setminus\mathscr{L}^+(S^d\times T^d)$$

is uncountable. Since $\mathbf{T}$ is complete, there are uncountably many $t^c\in T^c$ such that

$$\lambda^c(t^c)\in\mathscr{L}(S^d\times T^d)\setminus\mathscr{L}^+(S^d\times T^d).$$

For any such t^c, $\{s^c \mid (s^c, t^c) \in R_1^c\} = \emptyset$, so $t^c \in \Gamma(\emptyset, R_1^c)$. Therefore (iii.d) holds.

This completes the proof that Theorem A.11 (i) implies Theorem A.11 (iii). □

Proof that Theorem A.11 (iii) implies Theorem A.11 (ii). We are given a game G and a BRS pair $(\mathbf{B}^a, \mathbf{B}^b)$ of the form (3) that satisfies condition (iii), and must find Borel mappings λ^c such that $\mathbf{T} = (S^a, S^b, T^a, T^b, \lambda^a, \lambda^b)$ is a complete type structure that admits RCAR with $R_m^c = Q_m^c$ for each c and m.

Partition of types. For each non-empty set $X \subseteq S^c$, let $b^c(X)$ be the greatest $h \in \mathbb{N} \cup \{\infty\}$ such that $X \in \mathscr{B}_h^c(G)$. And let $b^c(\emptyset) = 0$. Note that by (iii.c), if $b^c(X) = 0$ then $X \times \Gamma(X, Q_1^c) = \emptyset$. Also, $b^c(X) > 0$ if and only if $X \in \mathscr{B}_1^c(G)$. And $b^c(X) = \infty$ if and only if $X \in \mathscr{B}_\infty^c(G)$. By Corollary A.10, if $X \in \mathscr{B}_{M(G)}^c$ then $b^c(X) = \infty$, so $b^c(X)$ always exists.

The BRS $\mathbf{B}^c$ gives us the following partition of T^c into countably many pairwise disjoint Borel sets:

(T1) $\Gamma(\emptyset, Q_1^c)$.
(T2) $\Gamma(X, Q_m^c) \setminus \Gamma(X, Q_{m+1}^c)$ for each $X \in \mathscr{B}_1^c(G)$ and $0 < m < b^c(X)$.
(T3) $\Gamma(X, Q_{b^c(X)}^c)$ for each $X \in \mathscr{B}_1^c(G)$.

By Lemma A.6, each of the sets in the partition (T1)–(T3) is Borel. We now show that each of the sets in the partition (T1)–(T3) is uncountable. By hypotheses (iii.d) and (iii.b), each of the sets (T1) and (T2) is uncountable, and the sets (T3) are uncountable when $b^c(X) < \infty$.

Suppose $b^c(X) = \infty$. Then for each $(s^c, t^c) \in S^c \times T^c$, $(s^c, t^c) \in Q_\infty^c$ if and only if $(s^c, t^c) \in Q_m^c$ for all $m \in \mathbb{N}$. Hence $\Gamma(X, Q_\infty^c) = \bigcap_{m \in \mathbb{N}} \Gamma(X, Q_m^c)$. By (T2), $\Gamma(X, Q_M^c)$ is uncountable. By hypothesis (iii.a), $\Gamma(X, Q_M^c) \setminus \Gamma(X, Q_h^c)$ is nowhere dense in $\Gamma(X, Q_M^c)$ whenever $M < h < \infty$. Therefore by the Baire Category Theorem, $\Gamma(X, Q_\infty^c)$ is an uncountable Borel set. Thus every set in the partition (T1)–(T3) is uncountable. By Corollary 6.5 in Kechris (1995), every uncountable Borel set in a Polish space has cardinality $2^{\aleph_0}$. So every set in the partition (T1)–(T3) has cardinality $2^{\aleph_0}$.

Partition of beliefs. For each $X \in \mathscr{B}_1^c(G)$ and $h \in \mathbb{N} \cup \{\infty\}$ let

$$\Lambda(X, h) = \{\sigma \in \mathscr{L}^+(S^d \times T^d) \mid X = \mathbf{BR}^c(\mathrm{marg}_{S^d} \sigma) \wedge (\forall k < h)\ \sigma \text{ assumes } Q_k^d\}.$$

And let

$$\Lambda(\emptyset, 1) = \mathscr{L}(S^d \times T^d) \setminus \mathscr{L}^+(S^d \times T^d).$$

The following partition of $\mathscr{L}(S^d \times T^d)$ "mirrors" the partition (T1)–(T3) of T^c.

(B1) $\Lambda(\emptyset, 1)$.
(B2) $\Lambda(X, m) \setminus \Lambda(X, m+1)$ for each $X \in \mathscr{B}_1^c(G)$ and $0 < m < b^c(X)$.
(B3) $\Lambda(X, b^c(X))$ for each $X \in \mathscr{B}_1^c(G)$.

It is clear that the sets in the family (B1)–(B3) are pairwise disjoint. If $\sigma \in \mathscr{L}^+(S^d \times T^d)$ and $X = \mathbf{BR}^c(\mathrm{marg}_{S^d} \sigma)$ then X is non-empty and $X \in \mathscr{B}_1^c(G)$ by Lemma A.9, so $b^c(X) > 0$. It follows that the union of the sets in (B2)–(B3) is $\mathscr{L}^+(S^d \times T^d)$, and the union of the sets in (B1)–(B3) is $\mathscr{L}(S^d \times T^d)$. By Lemmas C2–C3 of BFK, each set in (B1)–(B3) is Borel. We will show that each set in (B1)–(B3) has cardinality $2^{\aleph_0}$.

Since T^d is uncountable, $\Lambda(\emptyset, 1)$ is uncountable. If $0 < m < b^c(X)$, then $X \in \mathscr{B}_{b^c(X)}^c(G)$, so $X \in \mathscr{B}_m^c(G)$ by Lemma A.9. Then by Lemma B.1, $\Lambda(X, m) \setminus \Lambda(X, m+1)$ has cardinality $2^{\aleph_0}$. Similarly, if $0 < b^c(X) < \infty$, then $X \in \mathscr{B}^c_{b^c(X)}(G)$, and by Lemma B.1, $\Lambda(X, b^c(X))$ has cardinality $2^{\aleph_0}$. Finally, suppose $b^c(X) = \infty$. By hypothesis (iii.a), $Q^d_M \setminus Q^d_\infty$ is meager in Q^d_M, so by the Baire Category Theorem, Q^d_∞ is an uncountable dense subset of Q^d_M. Therefore by Lemma B.2, the set $\Lambda(X, \infty)$ has cardinality $2^{\aleph_0}$. Thus every set in the partition (B1)–(B3) has cardinality $2^{\aleph_0}$.

Type-belief mappings To construct the mapping $\lambda^c : T^c \to \mathscr{L}(S^d \times T^d)$ we will use the following result from the literature.

Theorem (Borel Isomorphism Theorem). *(Theorem 15.6 in (Kechris, 1995).) Let A, B be Borel subspaces of Polish spaces. If A and B have the same cardinality, then there is a one-to-one Borel mapping from A onto B.*[21]

By the Borel Isomorphism Theorem, there is a one-to-one mapping $\lambda^c : T^c \to \mathscr{L}(S^d \times T^d)$ that sends each set in the partition (T1)–(T3) onto the corresponding set in the partition (B1)–(B3). That is,

- The restriction of λ^c to $\Gamma(\emptyset, Q^c_1)$ is Borel and sends $\Gamma(\emptyset, Q^c_1)$ onto $\Lambda(\emptyset, 1)$.
- Whenever $X \in \mathscr{B}^c_1(G)$ and $0 < m < b^c(X)$, the restriction of λ^c to $\Gamma(X, Q^c_m) \setminus \Gamma(X, Q^c_{m+1})$ is Borel and sends λ^c to $\Gamma(X, Q^c_m) \setminus \Gamma(X, Q^c_{m+1})$ onto $\Lambda(X, m) \setminus \Lambda(X, m+1)$.
- Whenever $X \in \mathscr{B}^c_1(G)$, the restriction of λ^c to $\Gamma(X, Q^c_{b^c(X)})$ is Borel and sends $\Gamma(X, Q^c_{b^c(X)})$ onto $\Lambda(X, b^c(X))$.

It follows that whenever $X \in \mathscr{B}^c_1(G)$ and $0 < m \leq b^c(X)$, λ^c sends $\Gamma(X, Q^c_m)$ onto $\Lambda(X, m)$, and that λ^c itself is a one-to-one Borel mapping from T^c onto $\mathscr{L}(S^d \times T^d)$. Therefore

$$\mathbf{T} = (S^a, S^b, T^a, T^b, \lambda^a, \lambda^b)$$

is a complete one-to-one type structure.

Proof that $(\mathbf{B}^a, \mathbf{B}^b)$ *is generated by G and* $\mathbf{T}$. Let R^c_k be the rationality sets for $\mathbf{T}$. We must show that whenever $0 < m \in \mathbb{N}$ and $c \in \{a, b\}$.

$$R^c_m = Q^c_m \text{ in the type structure } \mathbf{T}. \tag{11}$$

Note that by (5) and hypothesis (iii.c), whenever $0 < m \in \mathbb{N}$ we have

$$Q^c_m = \bigcup_{X \in \mathscr{B}^c_m(G)} X \times \Gamma(X, Q^c_m).$$

We first show that $R^c_1 = Q^c_1$. By hypothesis, condition (iii.c) holds for $(\mathbf{B}^a, \mathbf{B}^b)$, and we have shown above that (iii.c) also holds for the BRS generated by G and $\mathbf{T}$. Therefore

$$\begin{aligned}
R^c_1 &= \{(s^c, t^c) \in S^c \times T^c \mid s^c \in \mathbf{BR}^c(\mathrm{marg}_{S^d} \lambda^c(t^c)) \ \wedge \ \lambda^c(t^c) \in \mathscr{L}^+(S^d \times T^d)\} \\
&= \bigcup_{X \in \mathscr{B}^c_1(G)} \{(s^c, t^c) \in S^c \times T^c \mid s^c \in X = \mathbf{BR}^c(\mathrm{marg}_{S^d} \lambda^c(t^c)) \ \wedge \ \lambda^c(t^c) \in \Lambda(X, 1)\} \\
&= \bigcup_{X \in \mathscr{B}^c_1(G)} \{(s^c, t^c) \in S^c \times T^c \mid s^c \in X = \mathbf{BR}^c(\mathrm{marg}_{S^d} \lambda^c(t^c)) \ \wedge \ t^c \in \Gamma(X, Q^c_1)\}
\end{aligned}$$

[21] In (Kechris, 1995), this result is stated in terms of standard Borel spaces, which are the measurable spaces associated with Borel subspaces of Polish spaces.

$$= \bigcup \{X \times \Gamma(X, Q_1^c) \mid X \in \mathscr{B}_1^c(G)\} = Q_1^c \qquad \square$$

Thus $R_1^c = Q_1^c$.

We now prove (11) by induction on m. Suppose $0 < k \in \mathbb{N}$ and (11) holds whenever $0 < m \leq k$. We will prove that $R_{k+1}^c = Q_{k+1}^c$. For each $X \in \mathscr{B}_1^c(G)$ we have

$$\begin{aligned}\Lambda(X, k+1) &= \{\sigma \in \mathscr{L}^+(S^d \times T^d) \mid X = \mathbf{BR}^c(\mathrm{marg}_{S^d}\,\sigma) \wedge (\forall m \leq k)\ \sigma \text{ assumes } Q_m^d\} \\ &= \{\sigma \in \mathscr{L}^+(S^d \times T^d) \mid X = \mathbf{BR}^c(\mathrm{marg}_{S^d}\,\sigma) \wedge (\forall m \leq k)\ \sigma \text{ assumes } R_m^d\}.\end{aligned}$$

Hence λ^c sends $\Gamma(X, Q_{k+1}^c)$ onto $\Lambda(X, k+1)$, and by (10), λ^c sends $\Gamma(X, R_{k+1}^c)$ onto $\Lambda(X, k+1)$. Therefore $\Gamma(X, Q_{k+1}^c) = \Gamma(X, R_{k+1}^c)$. By (5) it follows that $R_{k+1}^c = Q_{k+1}^c$.

Proof that $\mathbf{T}$ *admits RCAR.* We have shown above that Q_∞^c has cardinality $2^{\aleph_0}$ and hence is non-empty. Since $R_m^c = Q_m^c$ for all $m \in \mathbb{N}$, $R_\infty^c = Q_\infty^c$, so R_∞^c is non-empty and $\mathbf{T}$ admits RCAR.

We have shown that Theorem A.11 (iii) implies Theorem A.11 (ii). The proof of Theorem A.11 is complete.

APPENDIX C PROOF OF THEOREM A.13

Theorem A.13 (restated) *Let G be a finite game with strategy sets S^a, S^b, and T^a, T^b be uncountable Polish spaces. There is a BRS pair $(\mathbf{B}^a, \mathbf{B}^b)$ of the form (3) that satisfies conditions (iii.a)–(iii.d) of Theorem A.11 with $M = M(G)$.*

Proof of Theorem A.13. $M(G)$ is the least $M \geq 1$ such that $\mathscr{B}_M^a(G) = \mathscr{B}_\infty^a(G)$ and $\mathscr{B}_M^b(G) = \mathscr{B}_\infty^b(G)$. As before, for each non-empty $X \subseteq S^c$, $b^c(X)$ is the greatest $h \in \mathbb{N} \cup \{\infty\}$ such that $X \in \mathscr{B}_h^c(G)$, and $b^c(\varnothing) = 0$. Therefore for each $X \subseteq S^c$, $b^c(X) \geq M$ if and only if $b^c(X) = \infty$.

We partition each T^c into countably many Borel sets in the following way. By Corollary 6.5 in Kechris (1995), every infinite Polish space T^c contains a homeomorphic copy of the Cantor space $\mathbf{C} = \{0, 1\}^{\mathbb{N}}$, which is in turn homeomorphic to $\mathbf{C} \times \mathbf{C}$. We may assume that $\mathbf{C} \times \mathbf{C}$ is a subspace of T^c. Each non-empty open subset of $\mathbf{C}$ has the cardinality of the continuum and hence is uncountable.

First pick a partition $\{O^c(X) \mid X \subseteq S^c\}$ of $\mathbf{C}$, where $O^c(X)$ is uncountable for all X and open for all $X \neq \varnothing$. Let $\{P^c(X) \mid X \subseteq S^c\}$ be the unique partition of T^c such that $P^c(X) = O^c(X) \times \mathbf{C}$ for each $X \subseteq S^c$. Then for each $t^c \in T^c$, there is a unique $X(t^c) \subseteq S^c$ such that $t^c \in P^c(X(t^c))$. Next, pick countably many distinct points $u_0, u_1, u_2, \ldots$ in $\mathbf{C}$. Finally, for each non-empty X and each $m \in \mathbb{N} \cup \{\infty\}$, let $P_m^c(X)$ be the Borel subset of $P^c(X)$ defined using the points $u_0, u_1, u_2, \ldots$ as shown below:

$$P_m^c(X) \equiv \begin{cases} O^c(X) \times \{u_m\} & \text{if } m < b^c(X) \\ O^c(X) \times (\mathbf{C} \setminus \{u_k \mid k < m\}) & \text{if } m = b^c(X) \\ \varnothing & \text{if } m > b^c(X) \end{cases}$$

Note that the family $\{P_m^c(X) \mid m \leq b^c(X)\}$ is a partition of $P^c(X)$. For each $m < b^c(X)$, $P_m^c(X)$ is meager (thus intuitively topologically small) in $P^c(X)$. Meanwhile, the difference $P^c(X) \setminus P_{b^c(X)}^c(X)$ is meager (so intuitively, $P_{b^c(X)}^c(X)$ has the same topological size as $P^c(X)$).

We are now ready to build a BRS pair with the required properties by defining the sets Q^c_m. For each $m \in \mathbb{N}$ we define

$$Q^c_m \equiv \bigcup_{X \in \mathscr{B}^c_m(G)} \left(X \times \bigcup_{k \geq m} P^c_k(X) \right). \tag{12}$$

Using Lemma A.9, it is easily seen that:

(i) $Q_0 = S^c \times T^c$.
(ii) $Q^c_{m+1} \subseteq Q^c_m$ for each $m \in \mathbb{N}$.
(iii) $Q^c_\infty = \bigcup_{X \in \mathscr{B}^c_\infty(G)} (X \times P^c_\infty(X))$.
(iv) Q^c_m is Borel for each $m \in \mathbb{N} \cup \{\infty\}$.

By (i) and (ii), $\mathbf{B}^c$ is a BRS over $S^c \times T^c$. It remains to prove that $(\mathbf{B}^a, \mathbf{B}^b)$ satisfies conditions (iii.a)–(iii.d) of Theorem A.11.

Proof of (iii.a) for $\mathbf{B}^c$. The Cantor set $\mathbf{C}$ is closed and has no isolated points. Therefore the set

$$\mathbf{D} \equiv \mathbf{C} \setminus \{u_m \mid m \in \mathbb{N}\}$$

is dense in $\mathbf{C}$, so $\overline{\mathbf{D}} = \mathbf{C}$. Let $M \leq h \in \mathbb{N}$ and $X \subseteq S^c$. From the definition of $P^c_m(X)$ we have:
If $X \notin \mathscr{B}^c_\infty(G)$, then $b^c(X) < M$ and $\bigcup_{k \geq h} P^c_k(X) = \varnothing$.
If $X \in \mathscr{B}^c_\infty(G)$, then

$$O(X) \times \mathbf{C} \supseteq \bigcup_{k \geq h} P^c_k(X) \supseteq O^c(X) \times \mathbf{D}.$$

Therefore

$$\overline{Q^c_M} = \bigcup_{X \in \mathscr{B}^c_\infty(G)} \left(X \times \overline{O(X))} \times \mathbf{C} \right) = \overline{Q^c_h},$$

so (iii.a) holds for M. □

Proof of (iii.b) for $\mathbf{B}^c$. Suppose $0 < h \in \mathbb{N}$, X is non-empty, and $h \leq b^c(X)$. From the definition of $P^c_m(X)$, $P_h(X)$ is uncountable and

$$\bigcup_{k \geq h} P^c_k(X) \setminus \bigcup_{k \geq h+1} P^c_k(X) = P_h(X).$$

By Lemma A.9, $X \in \mathscr{B}_h(G)$, $h \leq b^c(X)$, and $P^c_h(X) \neq \varnothing$ are equivalent. By the definition of Q^c_m in (12), if follows that

$$Q^c_h \setminus Q^c_{h+1} = \bigcup_{X \subseteq S^c} X \times P_h(X) = \bigcup_{X \in \mathscr{B}^c_h(G)} X \times P_h(X). \tag{13}$$

Now suppose $0 < h \in \mathbb{N}$, $X \in \mathscr{B}_h(G)$ and $t^c \in P^c_h(X)$. To prove (iii.b) it suffices to show that $t^c \in \Gamma(X, Q^c_h) \setminus \Gamma(X, Q^c_{h+1})$.

Since $X \in \mathscr{B}_h(G)$ we have $h \leq b^c(X)$. If $s^c \in X$ then $(s^c, t^c) \in X \times \bigcup_{k \geq h} P^c_k(X)$ because $t^c \in P^c_h(X)$, so $(s^c, t^c) \in Q^c_h$ by (13). Then by (12) there is a set $Y \subseteq^c$ such that $s^c \in Y$ and $t^c \in \bigcup_{k \geq h} P^c_k(Y)$, so $t^c \in P^c(Y)$ and hence $Y = X$ and $s^c \in X$. Therefore $X = \{s^c \mid (s^c, t^c) \in Q^c_h\}$, so $t^c \in \Gamma(X, Q^c_h)$.

If $h < b^c(X)$, then $t^c \in O^c(X) \times \{u_h\}$, so $t^c \notin \bigcup_{k \geq h+1} P^c_k(X)$. If $h = b^c(X)$, then $\bigcup_{k \geq h+1} P^c_k(X) = \varnothing$, so again $t^c \notin \bigcup_{k \geq h+1} P^c_k(X)$. In either case, the set $\{s^c \mid (s^c, t^c) \in Q^c_{h+1}\}$ is empty, so $t^c \notin \Gamma(X, Q^c_{h+1})$. Therefore $t^c \in \Gamma(X, Q^c_h) \setminus \Gamma(X, Q^c_{h+1})$ and (iii.b) is proved. □

Proof of (iii.c) for $\mathbf{B}^c$. Let $0 < h \in \mathbb{N}$, $Y \subseteq S^c$, $Y \neq \emptyset$, and $t^c \in \Gamma(Y, Q_h^c)$. To prove (iii.c) we show that $Y \in \mathscr{B}_h^c(G)$. We have $Y = \{s^c \mid (s^c, t^c) \in Q_h^c\}$. Let X be the unique subset of S^c such that $t^c \in P^c(X) = O(X) \times \mathbf{C}$. For each $s^c \in Y$ we have $(s^c, t^c) \in Q_h^c$, so by (12) there is a set $Z \in \mathscr{B}_h^c(G)$ such that

$$(s^c, t^c) \in Z \times \bigcup_{k \geq h} P_k^c(Z) \subseteq Q_h^c.$$

Then $s^c \in Z$ so $Y \subseteq Z$, and $t^c \in \bigcup_{k \geq h} P_k^c(Z) \subseteq P^c(Z)$. Hence $t^c \in P^c(Z) \cap P^c(X)$, so $Z = X$.

Since $Y \neq \emptyset$, we have $X \in \mathscr{B}_h^c(G)$, $Y \subseteq X$, and $t^c \in \bigcup_{k \geq h} P_k(X)$. If $s^c \in X$, then by (12) again,

$$(s^c, t^c) \in X \times \bigcup_{k \geq h} P_k(X) \subseteq Q_h^c,$$

so $s^c \in Y$. Therefore $Y = X$ and hence $Y \in \mathscr{B}_h^c(G)$, as required. □

Proof of (iii.d) for $\mathbf{B}^c$. By (12),

$$Q_1^c = \bigcup_{X \in \mathscr{B}_1^c(G)} (X \times \bigcup_{k \geq j1} P_k^c(X)) \subseteq S^c \times (\mathbf{C} \setminus O^c(\emptyset)) \times \mathbf{C},$$

so Q_1^c is disjoint from the set $S^c \times O^c(\emptyset) \times \mathbf{C}$. Therefore whenever $t^c \in O^c(\emptyset) \times \mathbf{C}$ we have $\{s^c \mid (s^c, t^c) \in Q_1^c\} = \emptyset$, so $\Gamma(\emptyset, Q_1^c)$ contains the uncountable set $O^c(\emptyset) \times \mathbf{C}$. □

This completes the proof of Theorem A.13. □

APPENDIX D PARALLELS BETWEEN RELATED EPISTEMIC CONDITIONS

In this appendix, we attempt to draw parallels between *assumption and open sets*, *weak assumption and strategy cylinders*, and *strong belief and conditioning events*. This will also give another explanation of the fact that RCAR is impossible in all continuous complete lexicographic type structures, but possible in some complete lexicographic type structures.

We will use the notion of a BRS

$$\mathbf{B} = (\Omega, \langle Q_m, m \in \mathbb{N} \rangle, \mathscr{C})$$

to give a common treatment of all three settings. In each setting, Ω will be the set of a fixed player's strategy-type pairs, $\mathscr{C}$ will be the set of conditioning event analogues (e.g., open sets, conditioning events, or strategy cylinders), and Q_m will be the set of strategy-type pairs of the same player that exhibit m-th degree strategic sophistication. We will consider the property of having a finite bound in different settings. Note that if the set $\mathscr{C}$ is finite, then the existence of a finite bound is trivial.

Proposition D.1. *Suppose that* $\mathbf{T}$ *is a complete lexicographic type structure, c is a player, and* $\mathbf{B}$ *is a BRS where* $\Omega = S^c \times T^c$, *and* $\mathscr{C}$ *is finite. Then* $\mathbf{B}$ *has a finite bound.*

For finite games, the set of strategy cylinders is finite. If $\mathbf{T}$ is a complete lexicographic type structure, $\Omega = S^a \times T^a \times S^b \times T^b$, $R_m^a \times R_m^b$ is the RmWAR set as in Yang (2015), and $\mathscr{C}$ is the set of strategy cylinders, then $\mathbf{B}$ is a BRS. So by Proposition D.1, $\mathbf{B}$ has a finite bound.

Given a Polish space Ω and a family $\mathscr{C}$ of non-empty Borel sets in Ω, Rényi (1955) introduced the notion of a conditional probability system (CPS) on $(\Omega, \mathscr{C})$. For the case where $\mathscr{C}$ is finite, B-S introduced the notion of a CPS *strongly believing* an event. In B-S, $\mathscr{C}$ was generated from the information sets of dynamic games. In the online supplement to BFK, the notion of strong belief was generalized to the case where the family $\mathscr{C}$ is infinite. We will not need the definition of strong belief here, but we will need the following result about strong belief. Theorem D.2 below is Theorem S.1 in the online supplement to BFK, but reformulated in a way that uses the above notion of a BRS. Whenever we mention a BRS $\mathbf{B}$, we will use the notation

$$\mathbf{B} = (\Omega, \langle Q_m, m \in \mathbb{N}\rangle, \mathscr{C}).$$

Theorem D.2. *For every BRS* $\mathbf{B}$*, the following are equivalent:*

(i) For each $C \in \mathscr{C}$*, either* C *meets* $\bigcap_{m\in\mathbb{N}} Q_m$*, or there is a greatest integer* M *such that* C *meets* Q_M*.*
(ii) There exists a CPS on $(\Omega, \mathscr{C})$ *that strongly believes each event* Q_m*.*

This has a consequence concerning finite bounds in the strong belief setting, even in the case where the family $\mathscr{C}$ is infinite.

Proposition D.3. *Suppose* $\mathbf{B}$ *is a BRS such that* $\mathscr{C}$ *is closed under countable unions. If there exists a CPS on* $(\Omega, \mathscr{C})$ *that strongly believes each event* Q_m*, then* $\mathbf{B}$ *has a finite bound.*

Proof. By hypothesis, Theorem D.2(ii) holds, and therefore Theorem D.2(i) holds. Suppose by way of contradiction that $\mathbf{B}$ does not have a finite bound. Then there is an infinite subset $A \subseteq \mathbb{N}$ such that for each $m \in A$ there exists $C_m \in \mathscr{C}$ that is best-rationalized at degree m. Then $C \equiv \bigcup_{m\in A} C_m$ belongs to $\mathscr{C}$, and C meets Q_m for each $m \in A$, so by Theorem D.2(i), C meets $\bigcap_{n\in A} Q_n$. Therefore there exists $m \in A$ such that C_m meets $\bigcap_{n\in A} Q_n$, so C_m meets Q_n for some $n > m$, contradicting the fact that C_m is best-rationalized at degree m. □

Note that the above proof did not use the definition of a CPS or strong belief at all. It only used Theorem D.2. Proposition D.3 has the following analogue in the assumption setting.

Proposition D.4. *Suppose* $\mathbf{B}$ *is a BRS such that* $\mathscr{C}$ *is the family of all non-empty open subsets of* Ω*. If there exists an LCPS* σ *on* Ω *that assumes each event* Q_m*, then* $\mathbf{B}$ *has a finite bound.*

Proof. σ has finite length ℓ. By Proposition 2.1, there exists $j \le \ell$ such that σ assumes Q_m at level j for infinitely many m. Let M be the least m such that σ assumes Q_m at level j. The sets Q_m form a decreasing chain, so by Proposition 2.2 (i), σ assumes Q_m at level j for all $m \ge M$. Then by Proposition 2.2 (ii), we have $\overline{Q_m} = \overline{Q_M}$ for all $m \ge M$. Therefore $Q_M \setminus Q_m$ is nowhere dense in Q_m for all $m \ge M$. This means that whenever $m > M$, each open set $O \in \mathscr{C}$ that meets Q_M also meets Q_m. Therefore no open set can be best-rationalized at a degree $m > M$, so M is a finite bound for $\mathbf{B}$. □

RCSBR requires that each player strongly believes a decreasing sequence of events—the RmSBR sets—representing *increasing* strategic sophistication of the other players. Similarly, RCAR requires that each player assumes a decreasing sequence of events—the RmAR sets—representing *increasing* strategic sophistication of the other players. And RWCAR requires that each player weakly assumes a decreasing sequence of events—the RmWAR sets—also representing *increasing* strategic sophistication of the other players. In light of these requirements, Propositions D.1, D.3, and D.4, lead to the same principle in three different frameworks:

> In order for the limit of strategic sophistication (RCSBR/RCAR/RCWAR) to be attainable, it is necessary that the associated BRS has a finite bound.

Thus there is a finite M such that, for every cautioning event, if there is a highest finite degree m of the opponent's strategic sophistication that is consistent with that event, then $m \leq M$. An increase in strategic sophistication beyond M will have no additional implications for the players' strategy choices.

In BFK, open sets play the role of conditioning events that must be best-rationalized. In a *continuous* complete lexicographic type structure, for every finite m, there is some open set such that m is the highest degree of the opponent's strategic sophistication that is consistent with that event, so RCAR is not attainable by the above principle.

The complete (but not continuous) lexicographic type structure that we construct in Theorem 3.4 avoids the pitfall because the construction ensures that there is an integer M such that the difference sets $R^c_M \setminus R^c_m$ are topologically small for all $m > M$. There is no open set that is best-rationalized at a degree $m > M$, and RCAR is attained.

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