WHY SATURATED PROBABILITY SPACES ARE NECESSARY

H. JEROME KEISLER AND YENENG SUN

Abstract. An atomless probability space $(\Omega, \mathcal{A}, P)$ is said to have the saturation property for a probability measure μ on a product of Polish spaces $X \times Y$ if for every random element f of X whose law is $marg_X(\mu)$, there is a random element g of Y such that the law of (f, g) is μ. $(\Omega, \mathcal{A}, P)$ is said to be saturated if it has the saturation property for every such μ. We show each of a number of desirable properties holds for every saturated probability space and fails for every non-saturated probability space. These include distributional properties of correspondences, such as convexity, closedness, compactness and preservation of upper semi-continuity, and the existence of pure strategy equilibria in games with many players. We also show that any probability space which has the saturation property for just one "good enough" measure, or which satisfies just one "good enough" instance of the desirable properties, must already be saturated.

Our underlying themes are: (1) There are many desirable properties that hold for all saturated probability spaces but fail everywhere else; (2) Any probability space that out-performs the Lebesgue unit interval in almost any way at all is already saturated.

1. Introduction

Atomless probability spaces are widely used in mathematics and its applications. However, it has been found in [8], [14], [15] and [24] that the typical atomless probability space, the Lebesgue unit interval, does not have a number of desirable properties. Properties that fail on the Lebesgue unit interval include saturation properties for random elements and stochastic processes, existence of solutions of stochastic integral equations, regularity properties for distributions of correspondences such as convexity, closedness, compactness and preservation of upper semi-continuity, and the existence of pure strategy equilibria in games with many players.

In the papers [8], [15] and [24] it was shown that there do exist atomless probability spaces with these desirable regularity properties. These earlier papers used methods from nonstandard analysis. In this paper, using ordinary standard methods, we find exactly which probability spaces have these properties—each of these properties holds for every saturated probability space, and fails for every non-saturated atomless probability space. This improves the earlier results.

Formally, an atomless probability space $(\Omega, \mathcal{A}, P)$ is said to have the saturation property for a probability measure μ on a product of Polish spaces $X \times Y$ if for every random element f of X whose law is $marg_X(\mu)$, there is a random element g of Y such that the law of (f, g) is μ. $(\Omega, \mathcal{A}, P)$ is said to be saturated if it has the saturation property for every such μ.

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Saturated probability spaces were introduced in [8]. It was shown there that a probability space $(\Omega, \mathcal{A}, P)$ is saturated if and only if it satisfies the following condition: *The measure P restricted to a set of positive measure is never countably generated modulo the null sets.*[1] By Maharam's theorem, a probability space satisfies this condition if and only if its measure algebra is a finite or countable convex combination of measure algebras of uncountable powers of $[0, 1]$.

We will first prove local saturation results showing that if $(\Omega, \mathcal{A}, P)$ has the saturation property for just one "good enough" measure on a product of Polish spaces then Ω must already be saturated. We then will apply the saturation property directly to give characterizations of saturated probability spaces by distributional properties of correspondences, and by the existence of pure strategy equilibria in games with many players. We will then apply the local saturation results to show that one "good enough" instance of one of these properties already implies that the space is saturated.

Our underlying themes are: (1) There are many desirable properties that hold for all saturated probability spaces but fail everywhere else; (2) Any probability space that out-performs the Lebesgue unit interval in almost any way at all is already saturated.

The paper is organized as follows. Fundamental results about saturated probability spaces are presented in Section 2, including a brief review of global conditions for saturation and new local conditions for saturation. Section 3 deals with various distributional properties of correspondences by saturation, while Section 4 concerns the existence of pure strategy equilibria in games with many players. In Section 5, we point out that if these results on the distributional properties of correspondences and the existence of pure strategy equilibria in large games are established for one particular saturated probability space, then it follows easily that they hold for any other saturated probability space.

While the notions and methods here are entirely standard, this paper owes a great deal to earlier work using nonstandard methods, in particular using the Loeb probability spaces introduced in [17]. The papers [15] and [24] showed that atomless Loeb spaces have the desired properties for correspondences and large games. One realized that the atomless Loeb spaces are very rich in the sense that they have many more measurable sets than the Lebesgue unit interval. The richness is fully captured by the standard notion of a saturated probability space—it was shown in [8] that every atomless Loeb space is saturated. This gave a hint that these properties might hold for all saturated probability spaces. The earlier work with Loeb spaces was the motivation for Section 5 in this paper. The known facts that the desired properties hold for atomless Loeb spaces and that atomless Loeb spaces are saturated, combined with Section 5 here, give an alternative proof that the desired properties hold for all saturated probability spaces.

The positive direction of our results here can be compared with results in the literature on stochastic integral equations. In the papers [9], [10], and [16], it was shown that adapted Loeb probability spaces have the desirable property that every stochastic integral equation with random continuous coefficients has a strong solution. In the paper [8] it was shown by standard methods that every saturated adapted probability space has this desirable property. In the papers [3] and [4],

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[1] This condition is called "$\aleph_1$-atomless" in [8], "nowhere countably generated" in [18], "rich" in [21] and our earlier draft [13], and "super-atomless" in [22].

a variety of other existence and optimization results for stochastic integrals and controls were obtained for all saturated adapted probability spaces (using [11]).

In this paper we work with ordinary probability spaces, while the earlier work on stochastic integral equations involved adapted probability spaces. The notion of a saturated adapted probability space was introduced in [8], where it was shown that every adapted Loeb probability space is saturated, and that every saturated adapted probability space is saturated as a probability space. An important difference between this paper and the earlier work on stochastic integral equations is that here we also get converse results showing that the desired properties fail on every non-saturated probability space, and we get even stronger local converse results.

2. Saturated Probability Spaces

In what follows, $X, Y, Z, \ldots$ denote Polish spaces (complete metrizable topological spaces), and $\mathcal{M}(X)$ is the space of all Borel probability measures on X with the Prohorov metric ρ. We recall that $\mathcal{M}(X)$ is again a Polish space, $\mathcal{M}(X)$ has the topology of weak convergence, and if X is compact then so is $\mathcal{M}(X)$. For each $\mu \in \mathcal{M}(X \times Y)$, let $marg_X(\mu)$ be the marginal of μ in $\mathcal{M}(X)$; thus $marg_X : \mathcal{M}(X \times Y) \to \mathcal{M}(X)$ is a continuous surjection. Throughout this paper, **probability space** means complete countably additive probability space. The triples $(\Omega, \mathcal{A}, P)$, $(\Gamma, \mathcal{C}, Q)$ will denote atomless probability spaces. $(T, \mathcal{L}, \lambda)$ is the usual Lebesgue probability space on the unit interval $T = [0, 1]$. $L^0(\Omega, X)$ is the space of all random elements of X (measurable functions $f : \Omega \to X$) with the metric of convergence in probability. The **law** (or **distribution**) function $law : L^0(\Omega, X) \to \mathcal{M}(X)$ is defined by $law(f)(U) = P(f^{-1}(U))$ for each Borel set U.

If $\mathcal{A}$ is a σ-algebra and $S \in \mathcal{A}$, we let $\mathcal{A}^S = \{B \in \mathcal{A} : B \subseteq S\}$. The set of all P-null sets is denoted by $\mathcal{N}$, or $\mathcal{N}(P)$. We use f, g to denote random elements of X, Y respectively, on some probability space $(\Omega, \mathcal{A}, P)$. Given a random element f, $\sigma(f)$ is the smallest σ-algebra that contains $\{f^{-1}(U) : U \text{ Borel}\} \cup \mathcal{N}$.

We will often use the following well-known facts about the *law* mappings on an arbitrary probability space $(\Omega, \mathcal{A}, P)$.

Lemma 2.1. *(i) $law : L^0(\Omega, X) \to \mathcal{M}(X)$ is continuous.*

(ii) If $(\Omega, \mathcal{A}, P)$ is atomless, then $law : L^0(\Omega, X) \to \mathcal{M}(X)$ is surjective.

(iii) Suppose $law(f_n)$ weakly converges to ν and $law(g_n)$ weakly converges to π as $n \to \infty$. Then some subsequence of $law(f_n, g_n)$ weakly converges to a measure $\mu \in \mathcal{M}(X \times Y)$ such that $marg_X(\mu) = \nu$ and $marg_Y(\mu) = \pi$.

Proof. We prove (iii). The sequence $law(f_n)$ is relatively compact, since it is contained in the compact set $\{law(f_n) : N \in \mathbb{N}\} \cup \{\nu\}$. By the converse Prohorov's theorem (see [1]), $law(f_n)$ is tight, that is, for each $\varepsilon > 0$ there is a compact set J_ε in X such that $P(f_n(\omega) \in J_\varepsilon) \geq 1 - \varepsilon$ for each n. Similarly, there is a compact set K_ε in Y such that $P(g_n(\omega) \in K_\varepsilon) \geq 1 - \varepsilon$ for each n. Then $J_\varepsilon \times K_\varepsilon$ is compact in $X \times Y$, and $P((f_n, g_n)(\omega) \in J_\varepsilon \times K_\varepsilon) \geq 1 - 2\varepsilon$ for each n. Thus the sequence $law(f_n, g_n)$ is tight. By the direct Prohorov theorem, the sequence $\{law(f_n, g_n) : n \in \mathbb{N}\}$ is contained in a compact set $C \subseteq \mathcal{M}(X \times Y)$. Therefore some subsequence of $law(f_n, g_n)$ weakly converges to a measure $\mu \in \mathcal{M}(X \times Y)$. Since the functions $marg_X$ and $marg_Y$ are continuous, it follows that $marg_X(\mu) = \nu$ and $marg_Y(\mu) = \pi$. ∎

Definition 2.2. *(i)* $(\Omega, \mathcal{A}, P)$ *is said to satisfy the* ***saturation property*** *for a measure* $\mu \in \mathcal{M}(X \times Y)$ *if for every* $f \in L^0(\Omega, X)$ *with* $law(f) = marg_X(\mu)$*, there exists* $g \in L^0(\Omega, Y)$ *such that* $law(f, g) = \mu$.

(ii) A probability space $(\Omega, \mathcal{A}, P)$ *is* ***saturated*** *(or has full saturation) if* $(\Omega, \mathcal{A}, P)$ *is atomless, and for every* X, Y, $(\Omega, \mathcal{A}, P)$ *satisfies the saturation property for every* $\mu \in \mathcal{M}(X \times Y)$.

Note that for a pair of random elements (f', g') on some probability space, the space $(\Omega, \mathcal{A}, P)$ satisfies the saturation property for $law(f', g')$ if and only if for every $f \in L^0(\Omega, X)$ with $law(f) = law(f')$, there exists $g \in L^0(\Omega, Y)$ with $law(f, g) = law(f', g')$.

As shown in [8], typical examples of saturated probability spaces are atomless Loeb probability spaces, and product spaces of the form $\{0, 1\}^\kappa$ and $[0, 1]^\kappa$, where κ is an uncountable cardinal, $\{0, 1\}$ has the uniform measure, and $[0, 1]$ has the Lebesgue measure. The Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ and the probability spaces $[0, 1]^\mathbb{N}$ and $\{0, 1\}^\mathbb{N}$ are examples of atomless probability spaces that are not saturated.

The following two propositions deal with the trivial case that every atomless probability space has the saturation property for $law(f, g)$.

Proposition 2.3. *If* f *has countable range, then every atomless probability space has the saturation property for* $law(f, g)$.

Proof. Let $law(f') = law(f)$. By modifying f' on a null set, we may assume that f' has the same range as f. If f is a constant function, so is f', and Lemma 2.1 gives us a g' such that $law(f', g') = law(f, g)$. In the general case, f' is the union of countably many constant functions, and we obtain a g' with $law(f', g') = law(f, g)$ by taking a countable union of functions that work on each constant part of f'. ∎

In view of Proposition 2.3, in the rest of this section we will concentrate on the case that $law(f)$ is an atomless measure.

Proposition 2.4. *Suppose* $law(f)$ *is atomless. The following are equivalent:*

(i) The Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ *has the saturation property for* $law(f, g)$.

(ii) Every atomless probability space has the saturation property for $law(f, g)$.

(iii) g *is* $\sigma(f)$*-measurable.*

Proof. It is clear that (iii) implies (ii) and that (ii) implies (i). We assume (i) and prove (iii). Since $law(f)$ is atomless and $[0, 1]$ is separable, there is an $f' \in L^0([0, 1], X)$ such that $law(f') = law(f)$ and $\sigma(f')$ is the set of all Borel subsets of $[0, 1]$. By saturation there exists $g' \in L^0([0, 1], Y)$ such that $law(f', g') = law(f, g)$. Then g' must be $\sigma(f')$-measurable, so g is $\sigma(f)$-measurable. ∎

2.1. **Global Conditions for Saturation.** In this subsection we give some global necessary and sufficient conditions for a probability space to be saturated. We first list some results from [8] and [19].

Fact 2.5. *For each atomless probability space* $(\Omega, \mathcal{A}, P)$*, the following are equivalent:*

(i) $(\Omega, \mathcal{A}, P)$ *is saturated.*

(ii) There is no set $S \in \mathcal{A}$ *such that* $P(S) > 0$ *and* $\mathcal{A}^S$ *is countably generated modulo the null sets.*

(iii) The measure algebra of $(\Omega, \mathcal{A}, P)$ *is a finite or countable convex combination of measure algebras of the form* $[0,1]^\kappa$ *where* κ *is an uncountable cardinal.*

(iv) If $f \in L^0(\Omega, X)$, $g_n \in L^0(\Omega, Y)$ *for each* $n \in \mathbb{N}$, *and* $law(f, g_n)$ *converges weakly to a measure* μ, *there exists* $g \in L^0(\Omega, Y)$ *such that* $law(f, g) = \mu$.

Proof. The equivalence of (i) and (ii) is proved in [8], Corollary 4.5. The equivalence of (ii) and (iii) follows from Maharam's Theorem ([19]). A direct proof that (i) is equivalent to (iii) is also given in [5], Theorem 3B.7. The equivalence of (i) and (iv) is a special case of Theorem 5.2 in [8], and is proved directly in [12], Proposition 2.3. ■

We will often use condition (iv). This condition says that if a law problem has approximate solutions, then it has an exact solution. Some additional global conditions for full saturation are given in [11] (Theorem 4.8), and in [22]. The papers [3], [4], [8], [11], and the monographs [5] and [10], studied the more complicated saturated adapted probability spaces, as well as saturated probability spaces.

Here is a global characterization of full saturation that appears to be new.

Proposition 2.6. *An atomless probability space* $(\Omega, \mathcal{A}, P)$ *is saturated if and only if:*

(v) For each X, Y, *compact set* $C \subseteq \mathcal{M}(X \times Y)$, *and* $f \in L^0(\Omega, X)$, *the set*

$$\{law(g) : g \in L^0(\Omega, Y) \text{ and } law(f, g) \in C\}$$

is compact.

Proof. It is shown in [12], Proposition 4.9, that condition (iv) of Fact 2.5 implies (v).

We prove that (v) implies condition (iv) of Fact 2.5. If Y has only one point, then so does $L^0(\Omega, Y)$, and (iv) is trivially true. Assume that Y has at least two points, and (v) holds. Suppose that $law(f, g_n)$ converges weakly to μ in $\mathcal{M}(X \times Y)$. Let $\nu = marg_Y(\mu) \in \mathcal{M}(Y)$. Since $(\Omega, \mathcal{A}, P)$ is atomless and Y has more than one point, we can perturb g_n to a sequence h_n in $L^0(\Omega, Y)$ such that $law(f, h_n)$ converges weakly to μ and in addition that $law(h_n) \neq \nu$ for each n. Let

$$C = \{law(f, h_n) : n \in \mathbb{N}\} \cup \{\mu\}.$$

Then C is compact in $\mathcal{M}(X \times Y)$. By (v) the set

$$D = \{law(g) : g \in L^0(\Omega, Y) \text{ and } law(f, g) \in C\}$$

is compact in $\mathcal{M}(Y)$. For each n, $law(f, h_n) \in C$ and hence $law(h_n) \in D$. Moreover, $law(h_n) = marg_Y(law(f, h_n))$, and since $marg_Y$ is continuous, $law(h_n)$ converges weakly to ν. Therefore $\nu \in D$, and thus there exists $g \in L^0(\Omega, Y)$ such that $law(f, g) \in C$ and $law(g) = \nu$. Since $law(h_n) \neq law(g)$, we have $law(f, g) \neq law(f, h_n)$ for each n. Hence $law(f, g) = \mu$. This proves (iv). ■

2.2. **Local Conditions for Saturation.** We will now begin to address our underlying theme (2), that any probability space that out-performs the Lebesgue unit interval in almost any way is saturated.

The following theorem shows that one particular non-trivial instance of the saturation property already implies full saturation. This gives a condition for full saturation that is local in the sense that it involves one particular measure on the product $X \times Y$.

Theorem 2.7. *Let f, g be random elements of X, Y on some probability space and assume that $law(f)$ is atomless. Suppose $(\Omega, \mathcal{A}, P)$ has the saturation property for $law(f, g)$ but the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ does not. Then $(\Omega, \mathcal{A}, P)$ is saturated.*

Proof. We assume that $(\Omega, \mathcal{A}, P)$ is not saturated and arrive at a contradiction.

We may take f, g to be random elements on $(\Omega, \mathcal{A}, P)$. Since $(T, \mathcal{L}, \lambda)$ does not have the saturation property for $law(f, g)$, g is not $\sigma(f)$-measurable. By Fact 2.5 (ii), there is an $n \in \mathbb{N}$ and a set $S \in \mathcal{A}$ of measure $P(S) = 1/n$ such that $\mathcal{A}^S$ is countably generated (modulo the null sets). Using the fact that $(\Omega, \mathcal{A}, P)$ is atomless, there is a countably generated σ-algebra $\mathcal{A}_0$ such that $\mathcal{A}^S \subseteq \mathcal{A}_0 \subseteq \mathcal{A}$ and $\mathcal{A}_0$ is atomless with respect to P. Since $law(f)$ is atomless, we may partition Ω into n $\sigma(f)$-measurable subsets $U_1, \ldots, U_n$ of measure $1/n$. Because g is not $\sigma(f)$-measurable, there is at least one of the sets $U = U_k$ such that the restriction of g to U is not $\sigma(f)$-measurable.

By Maharam's theorem [19], any two countably generated atomless measure algebras are isomorphic. Since $P(S) = P(U)$, it follows that there is a measure algebra isomorphism ψ from the measure algebra of $(\Omega, \sigma(f), P)$ to the measure algebra of $(\Omega, \mathcal{A}_0, P)$ such that $\psi(U/\mathcal{N}) = S/\mathcal{N}$. By [6], Theorem 4.12 on page 937, ψ is induced by a measurable mapping $h : \Omega \to \Omega$. Then $f'(\omega) = f(h(\omega))$ is measurable. We have $\sigma(f') = \mathcal{A}_0$ and $law(f', 1_S) = law(f, 1_U)$, where 1_S and 1_U are the respective indicator functions of S and U. By the saturation property for $law(f, g)$, there exists g' on $(\Omega, \mathcal{A}, P)$ such that $law(f', g') = law(f, g)$. Since U is $\sigma(f)$-measurable, and $law(f', 1_S) = law(f, 1_U)$, there is a Borel measurable function φ such that $1_U = \varphi(f)$ and $1_S = \varphi(f')$. Hence, we have $law(f', g', 1_S) = law(f, g, 1_U)$. The restriction of g to U is not $\sigma(f)$-measurable, but the restriction of g' to S is $\sigma(f')$-measurable because $\mathcal{A}^S \subseteq \sigma(f')$. This contradicts the fact that $law(f, g) = law(f', g')$. Therefore $(\Omega, \mathcal{A}, P)$ is saturated after all. ∎

As a corollary, we see that only trivial cases of the saturation property hold for a probability space that is not saturated. This corollary is a generalization of Proposition 2.4

Corollary 2.8. *Suppose $(\Omega, \mathcal{A}, P)$ is atomless but not saturated, and let f, g be random elements of X, Y on some probability space such that $law(f)$ is atomless. Then $(\Omega, \mathcal{A}, P)$ has the saturation property for $law(f, g)$ if and only if g is $\sigma(f)$-measurable.*

Proof. By Proposition 2.4 and Theorem 2.7. ∎

When the space $(\Omega, \mathcal{A}, P)$ is clear from the context, $law^{-1}(\nu)$ will denote the set of all $f \in L^0(\Omega, X)$ such that $law(f) = \nu$. Given a set $C \subseteq \mathcal{M}(X \times Y)$ and a random element $f \in L^0(\Omega, X)$, let

$$C(f) = C \cap \{law(f, g) : g \in L^0(\Omega, Y)\}.$$

By Fact 2.5 (iv), on a saturated probability space the set $C(f)$ is closed for every closed set C and random element f of X.

Theorem 2.9. *Assume that $C \subseteq \mathcal{M}(X \times Y)$ and $\nu = marg_X\, \mu$ for some $\mu \in C$. Suppose that $C(f)$ is closed for every $f \in law^{-1}(\nu)$ on $(\Omega, \mathcal{A}, P)$, but there is an $f' \in law^{-1}(\nu)$ on the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ such that $C(f')$ is not closed and $\sigma(f') = \mathcal{L}$. Then $(\Omega, \mathcal{A}, P)$ is saturated.*

Proof. Since $\sigma(f') = \mathcal{L}$, ν is an atomless measure. Let $\mu \in cl(C(f')) \setminus C(f')$. Then there is a sequence $g'_n \in L^0(T, Y)$ such that $law(f', g'_n) = \mu_n \in C$ and μ_n converges weakly to μ. Since $\sigma(f') = \mathcal{L}$, each g'_n is $\sigma(f')$ measurable, and hence there is a Borel function $\psi_n : X \to Y$ such that $g'_n = \psi_n(f')$ a.s. Let $f \in law^{-1}(\nu)$ on $(\Omega, \mathcal{A}, P)$, and let $g_n = \psi_n(f)$. Then $law(f, g_n) = law(f', g'_n) = \mu_n$, so $\mu_n \in C(f)$. Since $C(f)$ is closed, $\mu \in C(f)$. Hence there exists $g \in L^0(\Omega, Y)$ with $law(f, g) = \mu$. It follows that $(\Omega, \mathcal{A}, P)$ has the saturation property for μ but $(T, \mathcal{L}, \lambda)$ does not. By Theorem 2.7, $(\Omega, \mathcal{A}, P)$ is saturated. ∎

The following lemma is a sort of approximate saturation that holds for every atomless probability space.

Lemma 2.10. *Suppose $\mu \in \mathcal{M}(X \times Y)$, $\nu = marg_X\, \mu$, and ν is atomless. Then for each $f \in law^{-1}(\nu)$ there is a sequence of $\sigma(f)$-measurable random elements g_n of Y such that $law(f, g_n)$ converges weakly to μ.*

Proof. Take $(f', g') \in law^{-1}(\mu)$. There is a sequence of $\sigma(f')$-measurable simple functions f'_n in $L^0(\Omega, X)$ such that f'_n converges to f' a.s. Since $law(f')$ is atomless, for each n there is a $\sigma(f')$-measurable random element g'_n of Y such that $law(f'_n, g'_n) = law(f'_n, g')$. Since $f'_n \to f'$ a.s., $law(f'_n, g')$ converges weakly to $law(f', g') = \mu$. Because each g'_n is $\sigma(f')$-measurable, there are $\sigma(f)$-measurable g_n such that $law(f, g_n) = law(f', g'_n)$, and $law(f, g_n)$ converges weakly to μ. ∎

The next theorem is a consequence of Theorem 2.9 and Lemma 2.10.

Theorem 2.11. *Suppose ν is an atomless measure in $\mathcal{M}(X)$, Y has cardinality greater than 1, and $C = \mathcal{M}(X \times Y)$. Then $(\Omega, \mathcal{A}, P)$ is saturated if and only if for each $f \in law^{-1}(\nu)$, the set $C(f)$ is closed in $\mathcal{M}(X \times Y)$.*

Proof. We prove the non-trivial direction. Suppose $C(f)$ is closed for every $f \in law^{-1}(\nu)$. Take two distinct points y_0, y_1 in Y. There is a random element (f, g) of $X \times Y$ such that $f \in law^{-1}(\nu)$, $g \in L^0(\Omega, \{y_0, y_1\})$, $P(g^{-1}(\{y_1\}) = 1/2$, and g is independent of $\sigma(f)$. Then g is not $\sigma(f)$-measurable. Let $\mu = law(f, g)$. Since ν is atomless there is an $f' \in L^0(T, X)$ such that $law(f') = \nu$ and $\sigma(f') = \mathcal{L}$. Then $\mu \notin C(f')$. By Lemma 2.10, μ belongs to the closure of $C(f')$. Hence $C(f')$ is not closed. By Theorem 2.9, $(\Omega, \mathcal{A}, P)$ is saturated. ∎

3. Distribution of Correspondences on Saturated Probability Spaces

Measurable correspondences and their selections are important in many areas of mathematics, including optimization, control theory, pattern analysis, stochastic analysis, and mathematical economics. The paper [24] developed a theory of distribution of correspondences on probability spaces, and proved that Loeb probability spaces have several desirable regularity properties that fail for the more familiar probability spaces such as the Lebesgue unit interval. We'll call these properties $P1$–$P6$. The proofs of $P1$–$P6$ for Loeb spaces in [24] made substantial use of methods from nonstandard analysis. In this section we will prove, without using nonstandard methods, that each saturated probability space has each property $P1$–$P6$, and for each non-saturated probability space, each property $P1$–$P6$ fails.

In Subsection 3.2 we apply the full saturation condition directly to prove that every saturated probability space has properties $P1$–$P6$. In Subsection 3.3 we prove global converse results, showing that each of the properties $P1$–$P6$ fails for every non-saturated probability space. These two subsections address our underlying theme (1), that many desirable properties hold for all saturated probability spaces but fail everywhere else. In Subsection 3.4 we use Theorem 2.7 to prove stronger local converse results, which show that a single "good enough" instance of one of the properties $P1$–$P4$ or $P6$ already implies full saturation. This addresses our underlying theme (2), that any probability space that out-performs the Lebesgue unit interval in almost any way at all is already saturated.

3.1. **Background.** We refer to [24], Section 2, for some basic standard notions and results on correspondences. Here is a brief summary.

By definition, every Polish space is separable and admits a complete metric. Since a Polish space is compact if and only if it admits a complete totally bounded metric, and every Polish space is embeddable in the compact Polish space $[0, 1]^{\mathbb{N}}$, every Polish space admits a (not necessarily complete) totally bounded metric (see pages 217–219 of [1]).

Throughout this section we let X be a Polish space and let d be a totally bounded metric on X.

A **correspondence** from Y to Z is a mapping from Y to the family of nonempty subsets of Z. Let G be a correspondence from a probability space $(\Omega, \mathcal{A}, P)$ to X. A measurable mapping $g : \Omega \to X$ is called a **measurable selection** of G if $g(\omega) \in G(\omega)$ for P-almost all $\omega \in \Omega$. The correspondence G is said to be **measurable** if its graph

$$\{(\omega, x) \in \Omega \times X : x \in G(\omega)\}$$

belongs to the product σ-algebra $\mathcal{A} \otimes \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the Borel σ-algebra on X.

The correspondence G is said to be **closed (compact) valued** if $G(\omega)$ is a closed (compact) subset of X for all $\omega \in \Omega$. For each set $B \subseteq X$, define

$$G^{-1}(B) = \{\omega : G(\omega) \cap B \neq \emptyset\}.$$

If G is closed valued, then G is a measurable correspondence if and only if $G^{-1}(O)$ is measurable for every open set O in X.

For a point $x \in X$ and a nonempty subset B of X, let the distance $d(x, B)$ from the point x to the set B be $\inf\{d(x, y) : y \in B\}$. For nonempty subsets A and B of X, the corresponding **Hausdorff distance** $\rho_d(A, B)$ between the sets A and B is defined by

$$\rho_d(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}.$$

Let $\mathcal{F}_X$ be the **hyperspace** of nonempty closed subsets of X, equipped with the metric ρ_d.

We will need the following elementary facts from [24], which allow us to treat measurable correspondences as mappings into a new Polish space and to apply full saturation to such mappings.

Fact 3.1. *(i) The hyperspace $\mathcal{F}_X$ with the Hausdorff distance ρ_d is still a Polish space.*

(ii) Let G be a closed valued measurable correspondence from a probability space $(\Omega, \mathcal{A}, P)$ to X. Then the induced mapping $\omega \mapsto G(\omega)$ is a measurable function from $(\Omega, \mathcal{A}, P)$ to $\mathcal{M}(\mathcal{F}_X)$.

Proof. See [24], Lemma 2.2 and Proposition 2.3. ■

Definition 3.2. *Let G be a correspondence from $(\Omega, \mathcal{A}, P)$ to X.*
(i) The ***distribution of*** *G is the set*

$$\mathcal{D}_G = \{law(g) : g \text{ is a measurable selection of } G\}.$$

(ii) Suppose G is a closed valued measurable correspondence. The ***law of*** *G is the induced measure $law(G) \in \mathcal{M}(\mathcal{F}_X)$ defined by $law(G)(U) = P(G^{-1}(U))$.*

In the above definition, the topology on $\mathcal{F}_X$ and the law of a correspondence depend on the metric d on X, not just on the topology of X.

Definition 3.3. *A correspondence G from a topological space Y to another topological space Z is said to be* ***upper semi-continuous*** *at $y_0 \in Y$ if for any open set U that contains $G(y_0)$, there exists a neighborhood V of y_0 such that $y \in V$ implies that $G(y) \subseteq U$. G is upper semi-continuous if it is upper semi-continuous at every point $y \in Y$.*

For a compact valued correspondence, upper semi-continuity has the following characterization by sequences (see [7], page 24).

Fact 3.4. *Let G be a compact valued correspondence from a Polish space Y to a Polish space Z. G is upper semi-continuous at a point $y \in Y$ if and only if whenever y_n converges to y in Y and $z_n \in G(y_n)$ for each n, z_n has a subsequence that converges to a point $z \in G(y)$.*

We need the following approximate version of Proposition 3.5 of [24]. This approximate version holds for all atomless probability spaces and has an elementary proof.

Proposition 3.5. *Let F be a closed valued measurable correspondence from an atomless probability space $(\Omega, \mathcal{A}, P)$ to a Polish space X, and let μ be a Borel probability measure on X. The following are equivalent.*
(i) μ belongs to the closure of $\mathcal{D}_F$.
(ii) For every open set O in X, $\mu(O) \leq P(F^{-1}(O))$.

Proof. (i) ⇒ (ii): Assume (i). There is a sequence f_n of measurable selections of F such that $\mu_n = law(f_n)$ weakly converges to μ. For each $n \in \mathbb{N}$ and open set O in X, $f_n^{-1}(O) \subseteq F^{-1}(O)$, so $\mu_n(O) \leq P(F^{-1}(O))$. Hence $\mu(O) \leq \liminf_{n\to\infty} \mu_n(O) \leq P(F^{-1}(O))$.

(ii) ⇒ (i): The proof is the same as the proof of the implication (iv) ⇒ (i) in Proposition 3.5 of [24], but it uses an arbitrary atomless probability space instead of a Loeb space. ■

3.2. **Regularity Properties for Distribution of Correspondences.** In this subsection we generalize the main theorems of [24] to all saturated probability spaces. We let X, Y be Polish spaces.

Given a closed valued measurable correspondence F from $(\Omega, \mathcal{A}, P)$ to X, we say that $\mathcal{D}_F$ is **maximal** if we have $\mathcal{D}_F \supseteq \mathcal{D}_G$ for every closed valued measurable correspondence G that has the same law as F.

Theorem 3.6. *Let $(\Omega, \mathcal{A}, P)$ be a saturated probability space. Then we have the following.*

P1: For each closed valued measurable correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F$ is maximal.

P2: For any correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F$ is convex.

P3: For any closed valued correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F$ is closed.

P4: For any compact valued correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F$ is compact.

P5: Let F be a compact valued correspondence from $(\Omega, \mathcal{A}, P)$ to X. Suppose that Y is a metric space and G is a closed valued correspondence from $\Omega \times Y$ to X such that:
(a) For all $(\omega, y) \in \Omega \times Y$, $G(\omega, y) \subseteq F(\omega)$.
(b) For each fixed $y \in Y$, $G(\cdot, y)$ (denoted by G_y) is a measurable correspondence from $(\Omega, \mathcal{A}, P)$ to X.
(c) For each fixed $\omega \in \Omega$, $G(\omega, \cdot)$ is upper semi-continuous from Y to X.
Then the correspondence $H(y) = \mathcal{D}_{G_y}$ is upper semi-continuous from Y to $\mathcal{M}(X)$.

P6: Let $\mathcal{G}$ be a measurable mapping from $(\Omega, \mathcal{A}, P)$ to the space $\mathcal{M}(X)$ of probability measures on X. Then there is a measurable mapping f from $(\Omega, \mathcal{A}, P)$ to X such that:
(a) for every Borel set B in X, $law(f)(B) = \int_\Omega \mathcal{G}(\omega)(B)\, dP$;
(b) for each $\omega \in \Omega$, $f(\omega) \in supp\, \mathcal{G}(\omega)$, where $supp\, \mathcal{G}(\omega)$ is the support of the probability measure $\mathcal{G}(\omega)$ on X.

Proof. $P1$: Suppose $law(G) = law(F)$ and g is a measurable selection of G. Let

$$\mathcal{H} = \{(x, A) \in X \times \mathcal{F}_X : x \in A\}.$$

Since g is a measurable selection of G, we have $law(g, G)(\mathcal{H}) = 1$. By full saturation there is an $f \in L^0(\Omega, X)$ such that $law(f, F) = law(g, G)$. Then $law(f, F)(\mathcal{H}) = 1$, which means that f is a measurable selection of F. Therefore, $\mu = law(f) = law(g) \in \mathcal{D}_F$. This shows that $\mathcal{D}_G \subseteq \mathcal{D}_F$.

$P2$: Let $\mu, \nu \in \mathcal{D}_F$. Then there are measurable selections f, g of F such that $law(f) = \mu$ and $law(g) = \nu$. Let $G(\omega) = \{f(\omega), g(\omega)\}$. Then $G \subseteq F$ and G is a closed valued measurable correspondence from $(\Omega, \mathcal{A}, P)$ to X. Let $\alpha \in (0, 1)$. We show that there is a measurable selection h of G such that

$$law(h) = \alpha\mu + (1 - \alpha)\nu.$$

Choose sequences of simple functions $f_n, g_n : \Omega \to X$ such that $f_n(\omega)$ is within 2^{-n} of $f(\omega)$, and $g_n(\omega)$ is within 2^{-n} of $g(\omega)$, with probability at least $1 - 2^{-n}$. For each n there is a finite measurable partition $\mathcal{P}_n$ of Ω such that:
(a) For each $S \in \mathcal{P}_n$, f_n and g_n are constant on S;
(b) The union of the sets $S \in \mathcal{P}_n$ on which f_n is everywhere within 2^{-n} of f and g_n is everywhere within 2^{-n} of g has probability at least $1 - 2 \cdot 2^{-n}$.
Since $(\Omega, \mathcal{A}, P)$ is atomless, for each $S \in \mathcal{P}_n$ there is a measurable set $S_0 \subseteq S$ such that $P(S_0) = \alpha P(S)$. Let $h_n : \Omega \to X$ be the simple function such that on

each partition set $S \in \mathcal{P}_n$, $h_n(\omega) = f_n(\omega)$ for any $\omega \in S_0$, and $h_n(\omega) = g_n(\omega)$ for any $\omega \in S \setminus S_0$. Then

$$law(h_n) = \alpha law(f_n) + (1-\alpha) law(g_n).$$

Moreover, $h_n(\omega)$ is within 2^{-n} of $G(\omega)$ with probability at least $1 - 2 \cdot 2^{-n}$, and $law(h_n)$ converges weakly to $\alpha\mu + (1-\alpha)\nu$. Since the law function is continuous, $law(f_n, g_n)$ also converges weakly to $law(f, g)$. By Lemma 2.1, some subsequence of $law(f_n, g_n, h_n, G)$ converges weakly to a measure π in $\mathcal{M}(X \times X \times X \times \mathcal{F}_X)$ whose marginals on the three copies of X are respectively μ, ν, and $\alpha\mu + (1-\alpha)\nu$. Without loss of generality, we assume that the sequence $law(f_n, g_n, h_n, G)$ converges weakly to π. Let τ be the marginal measure of π on the third copy of X with $\mathcal{F}_X$. Then, $law(h_n, G)$ converges weakly to τ. For each $k \geq 1$ let $\mathcal{H}^k = \{(x, A) \in X \times \mathcal{F}_X : d(x, A) \leq 1/k\}$. Then

$$\tau(\mathcal{H}^k) \geq \limsup_{n\to\infty} law(h_n, G)(\mathcal{H}^k) = 1,$$

which implies that $\tau(\mathcal{H}^k) = 1$. Therefore, $\tau(\mathcal{H}) = 1$. Since $(\Omega, \mathcal{A}, P)$ is saturated, there is an $h \in L^0(\Omega, X)$ such that $law(f, g, h, G) = \pi$. Then $law(h) = \alpha\mu + (1-\alpha)\nu$. It follows from $\tau(\mathcal{H}) = law(h, G) = 1$ that h is a measurable selection of G as required.

$P3$: Suppose $\mu_n \in \mathcal{D}_F$ for each $n \in \mathbb{N}$, and μ_n converges weakly to some $\mu \in \mathcal{M}(X)$. Choose a measurable selection f_n of F such that $law(f_n) = \mu_n$. Let G be the new correspondence from Ω to X such that for each ω, $G(\omega)$ is the closure of $\{f_n(\omega) : n \in \mathbb{N}\}$. By Theorem III.30 in [2], G is a closed valued measurable correspondence. Since $law(f_n)$ converges weakly to μ, Lemma 2.1 implies that the sequence $law(f_n, G)$ has a subsequence that converges weakly to some measure ν such that $marg_X(\nu) = \mu$. Without loss of generality, we assume that the sequence $law(f_n, G)$ converges weakly to ν. Hence, $\nu(\mathcal{H}) \geq \limsup_{n\to\infty} law(f_n, G)(\mathcal{H}) = 1$, which implies that $\mu(\mathcal{H}) = 1$. Because $(\Omega, \mathcal{A}, P)$ is saturated, there exists $f \in L^0(\Omega, X)$ such that $law(f, G) = \nu$. It follows that $law(f) = \mu$, and $d(f(\omega), G(\omega)) = 0$ P-almost surely. Since F is closed valued, we have $G(\omega) \subseteq F(\omega)$ for each ω. Therefore f is a measurable selection of F. Thus $\mu = law(f) \in \mathcal{D}_F$, and hence $\mathcal{D}_F$ is closed.

$P4$: This follows from $P3$ and the first two paragraphs of the proof of Theorem 4 in [24].

$P5$: We show that every atomless probability space that satisfies $P4$ satisfies $P5$. By $P4$, $\mathcal{D}_F$ is compact. By the assumptions that G is closed valued and F dominates G_y for each $y \in Y$, we know that $H(y) = \mathcal{D}_{G_y}$ is a closed subset of $\mathcal{D}_F$. Therefore H is a compact valued correspondence from Y to $\mathcal{M}(X)$. To show that H is upper semi-continuous, let y_n converge to $y \in Y$, and for each n let $\mu_n \in H(y_n) = \mathcal{D}_{G_{y_n}}$ and let f_n be a measurable selection of G_{y_n} such that $law(f_n) = \mu_n$ with $\lim_{n\to\infty} \mu_n = \mu$. Define a new correspondence J from $(\Omega, \mathcal{A}, P)$ to $X \times Y$ by

$$J(\omega) = cl\{(f_n(\omega), y_n) : n \in \mathbb{N}\}.$$

By Theorem III.30 in [2], J is a closed valued measurable correspondence. Let Y_0 be the compact set $\{y\} \cup \{y_n : n \in \mathbb{N}\}$. Since $J(\omega) \subseteq F(\omega) \times Y_0$ for each ω, J is compact valued. By $P4$, $\mathcal{D}_J$ is compact.

It is clear that $law(f_n, y_n)$ converges weakly to $\mu \otimes \delta_y$, where δ_y is the Dirac measure at y. Since $law(f_n, y_n)$ belongs to $\mathcal{D}_J$ and $\mathcal{D}_J$ is compact, we know that $\mu \otimes \delta_y \in \mathcal{D}_J$. Hence, there is a measurable selection (f, y) of J such that $law(f, y) = \mu \otimes \delta_y$. By (c), $\mu \in \mathcal{D}_{G_y} = H(y)$. By Fact 3.4, H is upper semi-continuous.

$P6$: Since $\mathcal{G}$ is measurable, it follows that the function $\omega \mapsto \mathcal{G}(\omega)(B)$ is measurable for each Borel set B in X. Let F be the correspondence from Ω to X such that $F(\omega) = \operatorname{supp} \mathcal{G}(\omega)$. It is easily checked that F is a closed valued, measurable correspondence. Note that $\mathcal{G}(\omega)(F(\omega)) = 1$ for all ω. Let μ be the probability measure on X such that $\mu(B) = \int_\Omega \mathcal{G}(\omega)(B)\, dP$ for each Borel set B in X. We must find an $f \in L^0(\Omega, X)$ such that $law(f) = \mu$, which gives (a), and f is a measurable selection of F, which gives (b).

For every open subset O of X, we have

$$F^{-1}(O) = \{\omega : F(\omega) \cap O \neq \emptyset\} = \{\omega : \mathcal{G}(\omega)(O) > 0\}.$$

Thus $F^{-1}(O)$ is measurable, and

$$\begin{aligned} \mu(O) = \int_\Omega \mathcal{G}(\omega)(O)\, dP &= \int_{\mathcal{G}(\omega)(O)>0} \mathcal{G}(\omega)(O)\, dP \\ &\leq P(\mathcal{G}(\omega)(O) > 0) = P(F^{-1}(O)). \end{aligned}$$

So for every open set O in X we have

$$\mu(O) \leq P(F^{-1}(O)).$$

Then by Proposition 3.5, μ belongs to the closure of $\mathcal{D}_F$. By $P3$, $\mathcal{D}_F$ is closed, so $\mu \in \mathcal{D}_F$ and there is a measurable selection f of F such that $law(f) = \mu$. ∎

3.3. **Global Converse Results.** In [24] it was shown that each of the properties $P1$–$P6$ in Theorem 3.6 fails for the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$. That paper gave a correspondence G from $(T, \mathcal{L}, \lambda)$ to $[-1, 1]$ that is a counterexample to each of $P1$–$P4$, and also gave counterexamples to $P5$ and $P6$ on the space $(T, \mathcal{L}, \lambda)$. In the following, we adapt these counterexamples to show that each part of Theorem 3.6 fails for *every* non-saturated atomless probability space.

Theorem 3.7. *Each of the properties $P1$–$P6$ in Theorem 3.6 fails for every atomless probability space that is not saturated.*

Proof. Let $T = [0, 1]$, let $(T, \mathcal{L}, \lambda)$ be the Lebesgue unit interval, and let G be the correspondence from T to the closed interval $[-1, 1]$ such that $G(t) = \{t, -t\}$ for all $t \in T$. Then, $\mathcal{D}_G$ is neither closed nor convex, and the uniform distribution μ on $[-1, 1]$ is not in $\mathcal{D}_G$ (see Example 1 of [24]). There is another correspondence G' on $(T, \mathcal{L}, \lambda)$ such that $law(G) = law(G')$ but $\mu \in \mathcal{D}_{G'}$, so $\mathcal{D}(G)$ is not maximal. Moreover, there is a sequence of finite valued correspondences G_n such that $\mu \in \mathcal{D}_{G_n}$ and $G_n(t) \to G(t)$ for each t, so property $P5$ fails for $(T, \mathcal{L}, \lambda)$ (see Example 3 of [24]).

Suppose that $(\Omega, \mathcal{A}, P)$ is an atomless probability space that is not saturated. By Fact 2.5 there is a set $S \in \mathcal{A}$ such that $P(S) > 0$ and $\mathcal{A}^S$ is countably generated (modulo sets of measure 0). Let P^S be the probability measure on $(S, \mathcal{A}^S)$ rescaled from P. As in the proof of Theorem 2.7, there is a measurable mapping h from S to T such that h induces an isomorphism between the corresponding measure algebras of $(S, \mathcal{A}^S, P^S)$ and the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$. Thus, one can convert a counterexample on the unit Lebesgue interval to a counterexample on the non-saturated probability space $(\Omega, \mathcal{A}, P)$ through h.

Let $\mathcal{B}$ denote the Borel subsets of $[-1, 1]$. Let F be the correspondence from Ω to $[-1, 1]$ defined by setting $F(\omega) = G(h(\omega))$ for $\omega \in S$ and $F(\omega) = \{0\}$ for $\omega \notin S$. For any measurable selection f of F, $f(\omega) = 0$ for $\omega \notin S$, and there is a Borel measurable mapping φ from T to $[-1, 1]$ such that $f(\omega) = \varphi(h(\omega))$ for $\omega \in S$. It is clear that φ is a measurable selection of G. Let δ_0 be the probability measure on $[-1, 1]$ such that $\delta_0(\{0\}) = 1$. It is then straightforward to check that $\mathcal{D}_F = (1-P(S))\{\delta_0\}+P(S)\mathcal{D}_G$. Hence, $\mathcal{D}_F$ is neither maximal, closed, nor convex. Therefore, properties $P1$–$P4$ fail for the non-saturated probability space $(\Omega, \mathcal{A}, P)$. Moreover, putting $F_n(\omega) = G_n(h(\omega))$ on S and $F_n(\omega) = 0$ for $\omega \notin S$, we see that property $P5$ also fails for $(\Omega, \mathcal{A}, P)$.

Let $\mathcal{H}$ be the $\mathcal{A}$-measurable mapping from Ω to $\mathcal{M}([-1, 1])$ defined by setting $\mathcal{H}(\omega) = (\delta_{h(\omega)} + \delta_{-h(\omega)})/2$ for $\omega \in S$ and $\mathcal{H}(\omega) = \delta_0$ for $\omega \notin S$. Suppose that there is an $\mathcal{A}$-measurable mapping f from Ω to $[-1, 1]$ such that f is a measurable selection of F (which is $supp\,\mathcal{H}$), and for every Borel set B in $[-1, 1]$, $law(f)(B) = \int_\Omega \mathcal{H}(\omega)(B)dP$. Thus, $law(f) = (1 - P(S))\delta_0 + P(S)\mu$, which implies that $\mu \in \mathcal{D}_G$. This is a contradiction. Hence property $P6$ fails for the non-saturated probability space $(\Omega, \mathcal{A}, P)$. ∎

3.4. **Local Converse Results.** We will use Theorem 2.7 on local saturation to prove local converses for each of the parts $P1$–$P4$ and $P6$ of Theorem 3.6. These improve the global converse results in Theorem 3.7, and are proved by a different method. We do not have a nice local converse for $P5$, because $P5$ involves an infinite family of correspondences. In this section it will always be understood that F, G are closed valued measurable correspondences to a Polish space X. For properties $P1$–$P4$, we will focus on the set of correspondences F that have a particular law $\nu \in \mathcal{M}(\mathcal{F}_X)$.

To get local converse results for a particular law ν, we need ν to be sufficiently powerful—we need ν to be atomless, and to be traceable as in Definition 3.8 below. Traceable correspondences are powerful in the sense that the original correspondence can be recovered from any selection. Our local converse result will show that $(\Omega, \mathcal{A}, P)$ must already be saturated if there is just one atomless traceable measure ν and one of the properties $P1$–$P4$ that holds for all $F \in law^{-1}(\nu)$ on $(\Omega, \mathcal{A}, P)$ but fails for some $G \in law^{-1}(\nu)$ on the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$.

Definition 3.8. *A Borel set $S \subseteq \mathcal{F}_X$ is* ***traceable*** *if there is a Borel function $\psi : X \to \mathcal{F}_X$ such that $\psi(x) = U$ whenever $x \in U \in S$. A correspondence F is* ***traceable*** *if there is a traceable set S such that $F(\omega) \in S$ a.s. We also say that F is* ***traced by*** *ψ.*

It is easy to see that if F is traced by ψ and $law(F) = law(G)$, then G is traced by ψ. So if F is traced by ψ and $\nu = law(F)$, we may say without ambiguity that ν is traceable, and that ν is traced by ψ.

Example 3.9. *(i) The correspondence G from the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ to $[-1, 1]$ used in the proof of Theorem 3.7 is traced by the Borel function $\psi(s) = \{s, -s\}$. Recall that each of the properties $P1$–$P4$ fail for G. We also note that the correspondence G is such that $\sigma(G) = \mathcal{L}$, and that $law(G)$ is atomless.*

(ii) For any G, we can build a new correspondence $\widehat{G}$ to $\mathcal{F}_X \times X$ that "carries along" G, by defining $\widehat{G}(\cdot) = \{G(\cdot)\} \times G(\cdot)$. $\widehat{G}$ is closed-valued and measurable,

and is traced by the Borel function $\psi(U,x) = \{U\} \times U$. Note that if any one of the properties P1–P4 fails for G, then that property also fails for $\widehat{G}$.

Part (ii) of the example shows that every correspondence can be upgraded in a canonical way to a traceable correspondence. Here are our local converses for $P1$–$P4$. Example 3.9 shows that these results are not vacuous.

Theorem 3.10. *Let $\nu \in \mathcal{M}(\mathcal{F}_X)$ be atomless and traceable. Then each of the following holds.*

$P1(\nu)$: Suppose that $(\Omega, \mathcal{A}, P)$ has the property that $\mathcal{D}_F$ is maximal whenever $law(F) = \nu$, but the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ does not have this property. Then $(\Omega, \mathcal{A}, P)$ is saturated.

$P2(\nu)$: Suppose that $(\Omega, \mathcal{A}, P)$ has the property that $\mathcal{D}_F$ is convex whenever $law(F) = \nu$, but this property fails for $(T, \mathcal{L}, \lambda)$ with a counterexample G such that $\sigma(G) = \mathcal{L}$. Then $(\Omega, \mathcal{A}, P)$ is saturated.

$P3(\nu)$: Suppose that $(\Omega, \mathcal{A}, P)$ has the property that $\mathcal{D}_F$ is closed whenever $law(F) = \nu$, but $(T, \mathcal{L}, \lambda)$ does not have this property. Then $(\Omega, \mathcal{A}, P)$ is saturated.

$P4(\nu)$: Assume that $\nu\{Z \in \mathcal{F}_X : Z \text{ compact}\} = 1$. Suppose $(\Omega, \mathcal{A}, P)$ has the property that $\mathcal{D}_F$ is compact whenever $law(F) = \nu$, but $(T, \mathcal{L}, \lambda)$ does not have this property. Then $(\Omega, \mathcal{A}, P)$ is saturated.

Before beginning the proof, we make some observations about traceable correspondences.

Suppose $\psi : X \to Y$ is Borel. For any random element f of X, let $f_\psi(\omega) = (\psi(f(\omega)), f(\omega))$. Note that if $law(f) = law(g)$ then $law(f_\psi) = law(g_\psi)$, so if $\mu = law(f)$ we may unambiguously define $\mu_\psi = law(f_\psi)$.

If ν is traced by ψ, then whenever $law(F) = \nu$ and f is a measurable selection of F with $law(f) = \mu$, we have $f_\psi(\omega) = (F(\omega), f(\omega))$ a.s. and hence $law(F, f) = \mu_\psi$ and $marg_{\mathcal{F}_X}\, \mu_\psi = \nu$.

Lemma 3.11. *Suppose that ν is traced by ψ, and that $\mu \in \mathcal{D}_H$ for some H such that $law(H) = \nu$. Then $(\Omega, \mathcal{A}, P)$ has the saturation property for μ_ψ if and only if for every correspondence $F \in law^{-1}(\nu)$ on $(\Omega, \mathcal{A}, P)$, we have $\mu \in \mathcal{D}_F$.*

Proof. Suppose that for every correspondence $F \in law^{-1}(\nu)$ on $(\Omega, \mathcal{A}, P)$, we have $\mu \in \mathcal{D}_F$. Then every $F \in law^{-1}(\nu)$ has a measurable selection f with $law(f) = \mu$, and so $law(F, f) = \mu_\psi$. This shows that $(\Omega, \mathcal{A}, P)$ has the saturation property for μ_ψ.

Now suppose $(\Omega, \mathcal{A}, P)$ has the saturation property for μ_ψ and let $F \in law^{-1}(\nu)$. By hypothesis, there is a $H \in law^{-1}(\nu)$ that has a measurable selection h such that $law(h) = \mu$. Then $law(H, h) = \mu_\psi$. By the saturation property for μ_ψ, there exists f such that $law(F, f) = \mu_\psi$. Then $law(f) = \mu$. Since h is a measurable selection of H, it follows from the proof of P1 in Theorem 3.6 that f is a measurable selection of F, and hence $\mu \in \mathcal{D}_F$. ∎

We now prove Theorem 3.10.

Proof. In this proof we will deal with the two probability spaces $(\Omega, \mathcal{A}, P)$ and the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$. When we use the law^{-1} notation, it will always be understood to be with respect to $(\Omega, \mathcal{A}, P)$.

By Theorem 3.6, there is an H on a saturated probability space such that $law(H) = \nu$ and $\mathcal{D}_H$ is maximal. We let $D = \mathcal{D}_H$. Then for any F with $law(F) = \nu$, $\mathcal{D}_F$ is maximal if and only if $\mathcal{D}_F = D$. Moreover, by Theorem 3.6, D is convex and closed.

By hypothesis, ν is traced by the Borel function $\psi : X \to \mathcal{F}_X$. Note that since ν is atomless, μ_ψ is atomless for every $\mu \in D$.

$P1(\nu)$: By hypothesis there is a correspondence G on $(T, \mathcal{L}, \lambda)$ such that $law(G) = \nu$ and $\mathcal{D}_G$ is not maximal, so there is a measure $\mu \in D \setminus \mathcal{D}_G$. Therefore by Lemma 3.11, $(T, \mathcal{L}, \lambda)$ does not have the saturation property for μ_ψ. But for each $F \in law^{-1}(\nu)$, $\mathcal{D}_F$ is maximal, so $\mu \in \mathcal{D}_F$. By Lemma 3.11, $(\Omega, \mathcal{A}, P)$ does has the saturation property for μ_ψ. By Theorem 2.7, $(\Omega, \mathcal{A}, P)$ is saturated.

$P2(\nu)$: By hypothesis, there is a correspondence G on $(T, \mathcal{L}, \lambda)$, measures $\mu_1, \mu_2 \in \mathcal{D}_G$, and $\alpha \in (0, 1)$ such that $law(G) = \nu$, $\sigma(G) = \mathcal{L}$, and

$$\mu = \alpha\mu_1 + (1 - \alpha)\mu_2 \notin \mathcal{D}_G.$$

Since $\mathcal{D}_G \subseteq D$ and D is convex, $\mu \in D$. By Lemma 3.11, $(T, \mathcal{L}, \lambda)$ does not have the saturation property for μ_ψ. We have $\mu_1 = law(g_1)$ and $\mu_2 = law(g_2)$ for some measurable selections g_1, g_2 of G. Since $\sigma(G) = \mathcal{L}$, g_1 and g_2 are $\sigma(G)$-measurable, so there are Borel functions ψ_1, ψ_2 such that $g_1 = \psi_1(G), g_2 = \psi_2(G)$.

It follows that for each $F \in law^{-1}(\nu)$, $\mu_1 = law(\psi_1(F))$ and $\mu_2 = law(\psi_2(F))$ belong to $\mathcal{D}_F$. By hypothesis, $\mathcal{D}_F$ is convex, so μ belongs to $\mathcal{D}_F$. By Lemma 3.11, $(\Omega, \mathcal{A}, P)$ has the saturation property for μ_ψ. By Theorem 2.7, $(\Omega, \mathcal{A}, P)$ is saturated.

$P3(\nu)$: By hypothesis, there is a correspondence G on $(T, \mathcal{L}, \lambda)$ such that $law(G) = \nu$ and $\mathcal{D}_G$ is not closed. Fix μ in $cl(\mathcal{D}_G) \setminus \mathcal{D}_G$. Since $\mathcal{D}_G \subseteq D$ and D is closed, $\mu \in D$. By Lemma 3.11, $(T, \mathcal{L}, \lambda)$ does not have the saturation property for μ_ψ.

Take any $F \in law^{-1}(\nu)$ on $(\Omega, \mathcal{A}, P)$. By Proposition 3.5, for every open set O in X we have $\mu(O) \leq \lambda(G^{-1}(O)) = P(F^{-1}(O))$, and hence $\mu \in cl(\mathcal{D}_F)$. Since $\mathcal{D}_F$ is closed, $\mu \in \mathcal{D}_F$. By Lemma 3.11, $(\Omega, \mathcal{A}, P)$ has the saturation property for μ_ψ. By Theorem 2.7, $(\Omega, \mathcal{A}, P)$ is saturated.

$P4(\nu)$: By hypothesis, there is a compact valued correspondence G on $(T, \mathcal{L}, \lambda)$ such that $law(G) = \nu$ and $\mathcal{D}_G$ is not compact. By Theorem 3.6, $\mathcal{D}_G$ is contained in a compact set. Therefore $\mathcal{D}_G$ is not closed. The result now follows from $P3(\nu)$. ∎

To get a local converse to $P6$, we need a notion of a traceable measurable mapping from Ω to $\mathcal{M}(X)$ that is analogous to the notion for correspondences.

Definition 3.12. *A Borel set $S \subseteq \mathcal{M}(X)$ is **traceable** if there is a Borel function $\psi : X \to \mathcal{M}(X)$ such that $\psi(x) = \tau$ whenever $\tau \in S$ and $x \in supp\,\tau$. A measurable mapping $\mathcal{G}$ from Ω to $\mathcal{M}(X)$ is **traceable** if there is a traceable set S such that $\mathcal{G}(\omega) \in S$ a.s. We also say that $\mathcal{G}$ is **traced by** ψ.*

Note that if $\mathcal{G}$ is traced by ψ and $law(\mathcal{G}') = law(\mathcal{G})$, then $\mathcal{G}'$ is traced by ψ. If $\mathcal{G}$ is traced by ψ and $\nu = law(\mathcal{G})$, we say that ν is traceable, and that ν is traced by ψ.

As in the case of correspondences, if ν is traced by ψ, $law(\mathcal{G}) = \nu$, $F(\omega) = supp\,\mathcal{G}(\omega)$, and f is a measurable selection of F with $\mu = law(f)$, then $f_\psi(\omega) = (\mathcal{G}(\omega), f(\omega))$ a.s., $law(\mathcal{G}, f) = \mu_\psi$, and $marg_{\mathcal{M}(X)}\, \mu_\psi = \nu$.

Example 3.13. *(i) The measurable mapping $\mathcal{G}(t) = (\delta_t + \delta_{-t})/2$ on the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ is traceable. This mapping was used in the proof of Theorem 3.7 above, and it was shown in* [24] *that property $P6$ fails for $\mathcal{G}$. We also note that $law(\mathcal{G})$ is atomless.*

(ii) Let $\mathcal{G}$ be a measurable mapping from $(\Omega, \mathcal{A}, P)$ to $\mathcal{M}(X)$. Let $\widehat{\mathcal{G}}$ be the measurable mapping from $(\Omega, \mathcal{A}, P)$ to $\mathcal{M}(\mathcal{M}(X) \times X)$ such that for each $\omega \in \Omega$, $\widehat{\mathcal{G}}(\omega) = \delta_{\mathcal{G}(\omega)} \otimes \mathcal{G}(\omega)$ where δ_y is the Dirac measure at y and $\otimes$ is the independent product operation. Note that

$$supp\, \widehat{\mathcal{G}}(\omega) = \{\mathcal{G}(\omega)\} \times supp\, \mathcal{G}(\omega).$$

Therefore $\widehat{\mathcal{G}}$ is traced by the Borel function $\psi(\tau, x) = \delta_\tau \otimes \tau$. We note that if property $P6$ fails for $\mathcal{G}$, then it fails for $\widehat{\mathcal{G}}$.

Here is our local converse result for $P6$. Again, Example 3.13 shows that the result is not vacuous.

Theorem 3.14. *Let $\nu \in \mathcal{M}(\mathcal{M}(X))$ be atomless and traceable. Suppose that $(\Omega, \mathcal{A}, P)$ has the property that $P6$ holds for $\mathcal{G}$ whenever $law(\mathcal{G}) = \nu$, but the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ does not have this property. Then $(\Omega, \mathcal{A}, P)$ is saturated.*

Proof. Let ν be traced by the Borel function $\psi : X \to \mathcal{M}(X)$. There is a unique measure $\mu \in \mathcal{M}(X)$ depending only on ν such that whenever $law(\mathcal{G}) = \nu$ on some probability space $(\Gamma, \mathcal{C}, Q)$ and B is Borel in $\mathcal{M}(X) \times X$,

$$\mu(B) = \int_\Gamma \mathcal{G}(\gamma)(B)\, dQ.$$

Since ν is traced by ψ, $marg_{\mathcal{M}(X)}\, \mu_\psi = \nu$.

Let $\mathcal{G} \in law^{-1}(\nu)$ on $(\Omega, \mathcal{A}, P)$. By hypothesis, there is a random element f of X that satisfies conditions (a) and (b) of $P6$. By (a), $law(f) = \mu$. By (b), f is a measurable selection of F where $F(\omega) = supp\, \mathcal{G}(\omega)$. Since ν is traced by ψ, $law(\mathcal{G}, f) = \mu_\psi$. This shows that $(\Omega, \mathcal{A}, P)$ has the saturation property for μ_ψ.

By hypothesis, on the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ there is a random element $\mathcal{G}'$ of $\mathcal{M}(X)$ such that $law(\mathcal{G}') = \nu$ but $P6$ fails for $\mathcal{G}'$. We will prove that there is no h with $law(\mathcal{G}', h) = \mu_\psi$. This will show that $(T, \mathcal{L}, \lambda)$ does not have the saturation property for μ_ψ, and by Theorem 2.7 it will follow that $(\Omega, \mathcal{A}, P)$ is saturated. Assume to the contrary that there is an h with $law(\mathcal{G}', h) = \mu_\psi$. Then $law(h) = \mu$, so $(\mathcal{G}', h)$ satisfies condition (a). Let f be the random element on $(\Omega, \mathcal{A}, P)$ from the preceding paragraph, with $law(\mathcal{G}, f) = \mu_\psi$. Since $law(\mathcal{G}', h) = \mu_\psi = law(\mathcal{G}, f)$ and f is a measurable selection of $supp\, \mathcal{G}(\cdot)$, h is a measurable selection of $supp\, \mathcal{G}'(\cdot)$. Thus $(\mathcal{G}', h)$ satisfies condition (b) as well, contradicting our hypothesis that property $P6$ fails for $\mathcal{G}'$. This proves that there is no h with $law(\mathcal{G}', h) = \mu_\psi$, as required. ■

4. Large Games on Saturated Probability Spaces

In this section we further develop our underlying themes in the context of games. We show that saturated spaces have the desirable property that every game with a large number of players has a Nash equilibrium, and that no other probability spaces have this property. We also show that saturated spaces have the property that the set of laws of Nash equilibria for each game is closed in the weak topology, and we use Theorem 2.9 to get a strong local converse for that fact.

We shall first give a formal definition of a game based on a probability space of players $(\Omega, \mathcal{A}, P)$. Let A be a compact metric space, and let $\mathcal{U}_A$ be the space of real-valued continuous functions on $A \times \mathcal{M}(A)$ endowed with the sup-norm topology. By a **game** $\mathcal{G}$ with player space Ω and action space A we will mean a random element of $\mathcal{U}_A$ on $(\Omega, \mathcal{A}, P)$. Thus, a game simply associates each player $\omega \in \Omega$ with a payoff function $\mathcal{G}(\omega)(a, \tau)$ that depends on the player's own action a and the distribution τ of actions by all the players. To improve readability, we also use $\mathcal{G}_\omega$ to denote $\mathcal{G}(\omega)$.

We will concentrate on the case that $(\Omega, \mathcal{A}, P)$ is atomless. We say that such a game is "large" because it has at least continuum many players.

Definition 4.1. *A **Nash equilibrium** of a game $\mathcal{G}$ is a random element g of A such that for P-almost all $\omega \in \Omega$,*

$$\mathcal{G}_\omega(g(\omega), law(g)) \geq \mathcal{G}_\omega(a, law(g))$$

for all $a \in A$.

Thus, if g is a Nash equilibrium, then the distribution of actions by all the players is $law(g)$ and every player chooses her optimal action $g(\omega)$ under this societal distribution. Note that we only consider pure-strategy Nash equilibria here.

Mas-Colell [20] introduced a corresponding notion of a measure game and Nash equilibrium distribution. A measure game with action space A is a probability measure $\nu \in \mathcal{M}(\mathcal{U}_A)$.

Definition 4.2. *A **Nash equilibrium distribution** of a measure game ν is a probability measure $\mu \in \mathcal{M}(\mathcal{U}_A \times A)$ such that $marg_{\mathcal{U}_A}\, \mu = \nu$, and*

$$\mu\{(u, x) : (\forall a \in A) u(x, marg_A\, \mu) \geq u(a, marg_A\, \mu)\} = 1.$$

It is easy to see that for any game $\mathcal{G}$ with action space A and any random element g of A, g is a Nash equilibrium of $\mathcal{G}$ if and only if $law(\mathcal{G}, g)$ is a Nash equilibrium distribution of $law(\mathcal{G})$. Given a Nash equilibrium distribution μ of a measure game ν, we say that a probability space $(\Omega, \mathcal{A}, P)$ **realizes** μ if every game $\mathcal{G} \in law^{-1}(\nu)$ on that space has a Nash equilibrium g such that $law(\mathcal{G}, g) = \mu$.

As a special case of Theorem 2.7, we immediately get a local characterization of saturated spaces in terms of Nash equilibria.

Corollary 4.3. *Let ν be a measure game that is atomless as a measure, and μ be a Nash equilibrium distribution for ν. Suppose the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ does not realize μ. Then an atomless probability space $(\Omega, \mathcal{A}, P)$ is saturated if and only if it realizes μ.*

Proof. We see from the definitions that a probability space realizes a Nash equilibrium distribution μ if and only if it has the saturation property for μ. ∎

Corollary 4.4. *Let A be a compact metric space and let ν be an atomless measure on $\mathcal{U}_A$ that has a unique Nash equilibrium distribution. Suppose that $(\Omega, \mathcal{A}, P)$ has the property that every $\mathcal{G} \in law^{-1}(\nu)$ has a Nash equilibrium, but the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ does not. Then $(\Omega, \mathcal{A}, P)$ is saturated.*

Proof. Let μ be the unique Nash equilibrium distribution of ν. Then μ is atomless. Since μ is unique, whenever $law(\mathcal{G}) = \nu$ and g is a Nash equilibrium of $\mathcal{G}$ we have $law(\mathcal{G}, g) = \mu$. It follows that $(\Omega, \mathcal{A}, P)$ realizes μ but $(T, \mathcal{L}, \lambda)$ does not. By Corollary 4.3, $(\Omega, \mathcal{A}, P)$ is saturated. ∎

We now turn to the question of the existence of Nash equilibria. We will use Mas-Colell's existence theorem for Nash equilibrium distributions.

Fact 4.5. *(Mas-Colell* [20]*). Every measure game with compact metric action space A has a Nash equilibrium distribution.*

Our next theorem gives a global characterization of saturated probability spaces by the existence of Nash equilibria.

Theorem 4.6. *Let $(\Omega, \mathcal{A}, P)$ be an atomless probability space, and A an uncountable compact metric space. Then $(\Omega, \mathcal{A}, P)$ is saturated if and only if every game $\mathcal{G}$ with player space $(\Omega, \mathcal{A}, P)$ and action space A has a Nash equilibrium.*

Proof. Suppose that $(\Omega, \mathcal{A}, P)$ is saturated. By Fact 4.5, the measure game $\nu = law(\mathcal{G})$ has a Nash equilibrium distribution μ. By full saturation, there is a random element g of A such that $law(\mathcal{G}, g) = \mu$. Therefore g is a Nash equilibrium of $\mathcal{G}$.

Instead of using Fact 4.5, one can also get a Nash equilibrium of $\mathcal{G}$ by using the proof of Theorem 1 of [15]. One only needs to use the distributional properties of correspondences on saturated probability spaces (instead of Loeb spaces) such as the convexity, compactness, and preservation of upper semi-continuity.[2]

For the converse, we first consider the case that the action space A is the interval $[-1, 1]$. By Corollary 4.4 it suffices to find a game $\mathcal{G}$ on the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ such that $\mathcal{G}$ has no Nash equilibrium and $law(\mathcal{G})$ is atomless and has a unique Nash equilibrium distribution. Let $\mathcal{G}$ be the game defined in Section 2 of [14], and Example 3 in [23], where $\mathcal{G}$ is a one-to-one continuous mapping from T to $\mathcal{U}_A$. Let U be the image of $\mathcal{G}$, which is a compact set in $\mathcal{U}_A$. It is shown in [14] that $\mathcal{G}$ has no Nash equilibrium. It is clear from the definition in [14] that $\nu = law(\mathcal{G})$ is atomless. The uniqueness of the Nash equilibrium distribution for the measure game $law(\mathcal{G})$ is implicit in the proof in [14] and [23], and can be seen as follows.

Let μ be any Nash equilibrium distribution for the measure ν. Let $\mathcal{G}'$ be a game on $(\Omega, \mathcal{A}, P)$ with a Nash equilibrium f and $law(\mathcal{G}', f) = \mu$. By modifying the definition of $\mathcal{G}'$ on a null set, we may assume that $\mathcal{G}'$ takes values in the set U. As shown in [14] and [23], $law(f)$ is the uniform distribution on $[-1, 1]$, and when $\mathcal{G}'(\omega) = \mathcal{G}(t)$, the best response is t or $-t$. Fix any $\omega \in \Omega$ such that $f(\omega)$ is a best response; note that such elements $\omega \in \Omega$ form a set of P-full measure. Then, there is a unique $t \in T$ such that $\mathcal{G}'(\omega) = \mathcal{G}(t)$, which implies that $f(\omega)$ must be t or $-t$. Hence, we have $\mathcal{G}'(\omega) = \mathcal{G}(|f(\omega)|)$. Therefore, $\mu = law(\mathcal{G}', f) = law(\mathcal{G}(|f(\cdot)|), f)$ is the unique Nash equilibrium distribution for ν.

Finally, we prove the converse for the general case that A is any fixed uncountable compact metric space. Suppose that $(\Omega, \mathcal{A}, P)$ is not saturated. By the converse for the case that $[-1, 1]$ is the action space, there is a game $\mathcal{G}^1$ with player space $(\Omega, \mathcal{A}, P)$ and action space $[-1, 1]$ but without a Nash equilibrium.

[2] In fact, the full statements of Theorem 1 for large games and Theorem 3 for finite-player games with incomplete information in [15] can be restated on saturated probability spaces instead of Loeb spaces; exactly the same proofs work in this more general situation. The same thing also works for Theorem 2 of [15] by using integration of correspondences on saturated probability spaces.

It is noted in [23], page 339, that there exists a continuous surjective mapping F from A to $[-1,1]$ and a continuous injective mapping $\mathcal{F}^0$ from $\mathcal{U}_{[-1,1]}$ to $\mathcal{U}_A$ such that $\mathcal{F}^0(u)(x,y) = u(F(x), yF^{-1})$ whenever $u \in \mathcal{U}_A, x \in A, y \in \mathcal{M}(A)$. We can now define a new game $\mathcal{G}^2$ with player space $(\Omega, \mathcal{A}, P)$ and action space A by using the composition mapping $\mathcal{G}^2 = \mathcal{F}^0 \circ \mathcal{G}^1$. Suppose f^2 is a Nash equilibrium for the new game $\mathcal{G}^2$. Then it can be easily checked that the composition mapping $f^1 = F \circ f^2$ is a Nash equilibrium for the game $\mathcal{G}^1$. This is a contradiction. Hence the set of Nash equilibria of $\mathcal{G}^2$ is empty, and the converse for the general case is shown. ∎

For each game $\mathcal{G}$, let

$$\mathcal{E}_{\mathcal{G}} = \{law(\mathcal{G}, g) : g \text{ is a Nash equilibrium of } \mathcal{G}\}.$$

Note that every measure $\mu \in \mathcal{E}_{\mathcal{G}}$ is a Nash equilibrium distribution of $law(\mathcal{G})$. We now prove that saturated probability spaces have a closure property for Nash equilibria, and then use Theorem 2.9 to get a local converse.

Theorem 4.7. *Suppose $(\Omega, \mathcal{A}, P)$ is saturated. Then for every game $\mathcal{G}$ on Ω with compact metric action space A, the set $\mathcal{E}_{\mathcal{G}}$ is closed in $\mathcal{M}(\mathcal{U}_A \times A)$.*

Proof. Suppose $\mu_n \in \mathcal{E}_{\mathcal{G}}$ and μ_n converges weakly to μ. Take a Nash equilibrium g_n of $\mathcal{G}$ such that $law(\mathcal{G}, g_n) = \mu_n$. By full saturation, there is a $g \in L^0(\Omega, A)$ with $law(\mathcal{G}, g) = \mu$. Let $\nu_n = law(g_n)$ and $\nu = law(g)$. Then ν_n converges weakly to ν. Since A is separable it has a countable dense subset A_0.

Fix any $a \in A_0$. We have

$$\mathcal{G}_\omega(g_n(\omega), \nu_n) \geq \mathcal{G}_\omega(a, \nu_n) \ a.s.$$

By [1], Theorem 4.4 on page 27, $law(\mathcal{G}, g_n, \nu_n)$ converges weakly to $law(\mathcal{G}, g, \nu)$ in $\mathcal{M}(\mathcal{U}_A \times A \times \mathcal{M}(A))$. It follows that $h_n(\omega) = \mathcal{G}_\omega(g_n(\omega), \nu_n) - \mathcal{G}_\omega(a, \nu_n)$ converges weakly to $h(\omega) = \mathcal{G}_\omega(g(\omega), \nu) - \mathcal{G}_\omega(a, \nu)$. Since $law(h_n)([0,\infty)) = 1$ for each n, we have $law(h)([0,\infty)) \geq \limsup_{n\to\infty} law(h_n)([0,\infty)) = 1$. Hence,

$$\mathcal{G}_\omega(g(\omega), \nu) \geq \mathcal{G}_\omega(a, \nu) \ a.s.$$

By grouping countably many null sets together, we see that for P-almost all $\omega \in \Omega$,

$$(\forall a \in A)\, \mathcal{G}_\omega(g(\omega), \nu) \geq \mathcal{G}_\omega(a, \nu),$$

which means that g is a Nash equilibrium of $\mathcal{G}$, so $\mu = law(\mathcal{G}, g) \in \mathcal{E}_{\mathcal{G}}$. ∎

Corollary 4.8. *For each measure game ν with compact metric action space A, the set of all Nash equilibrium distributions for ν is closed in $\mathcal{M}(\mathcal{U}_A \times A)$.*

Here is our local converse for Theorem 4.7.

Corollary 4.9. *Let ν be a measure game with compact metric action set A. Suppose that on $(\Omega, \mathcal{A}, P)$, $\mathcal{E}_{\mathcal{G}}$ is closed for every game $\mathcal{G} \in law^{-1}(\nu)$, but on the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ there is a game $\mathcal{G}' \in law^{-1}(\nu)$ such that $\sigma(\mathcal{G}') = \mathcal{L}$ and $\mathcal{E}_{\mathcal{G}'}$ is not closed. Then $(\Omega, \mathcal{A}, P)$ is saturated.*

Proof. Let C be the set of Nash equilibrium distributions for ν. For each game $\mathcal{G}$ with $law(\mathcal{G}) = \nu$, $\mathcal{E}_{\mathcal{G}} = C(\mathcal{G})$. C is non-empty by Fact 4.5, so $\nu = marg_{\mathcal{U}_A}\, \mu$ for some $\mu \in C$. Then by Theorem 2.9, $(\Omega, \mathcal{A}, P)$ is saturated. ∎

The following simple example shows that Corollary 4.9 is not vacuous.

Example 4.10. *Let $\mathcal{G}'$ be the game on the Lebesgue unit interval $(T, \mathcal{L}, \lambda)$ with action space $A = [-1, 1]$ and the payoff function $\mathcal{G}'_t(a, \tau) = -|t - |a||$. Then $\sigma(\mathcal{G}') = \mathcal{L}$. There is a game $\mathcal{G}$ with law$(\mathcal{G})$ = law$(\mathcal{G}')$ that has a Nash equilibrium g such that law(g) is the uniform probability measure on A. Then law$(\mathcal{G}, g)$ belongs to the closure of $\mathcal{E}_{\mathcal{G}'}$ but not to $\mathcal{E}_{\mathcal{G}'}$.*

5. From One Saturated Probability Space to Another

In this section, we show that if the properties $P1$–$P6$ in Theorem 3.6 are established for one saturated probability space, then they hold for every other saturated probability space. The key is the choice of appropriate mappings for applying full saturation. This result combined with the known theorems in [24] that properties $P1$–$P6$ hold for atomless Loeb probability spaces, and the fact that atomless Loeb probability spaces are saturated, gives an alternative proof of Theorem 3.6. We will also prove analogous results for Theorems 4.6 and 4.7 concerning Nash equilibria in large games.

Of course, each of the properties under discussion was already proved outright for all saturated probability spaces in Theorem 3.6 and in Section 4. Our point here is that if we add the hypothesis that the properties hold for some particular saturated probability space, then there is a very simple proof that the properties hold for all saturated probability spaces.

This demonstrates a general technique for extending certain types of results from atomless Loeb probability spaces (or even the simplest hyperfinite Loeb counting spaces) to all saturated probability spaces. Thus, hyperfinite Loeb counting spaces can play a prototype role in the class of all saturated probability spaces.

Lemma 5.1. *Let $(\Omega, \mathcal{A}, P)$ and $(\Gamma, \mathcal{C}, Q)$ be two saturated probability spaces. For any closed measurable correspondences F on $(\Omega, \mathcal{A}, P)$ and F' on $(\Gamma, \mathcal{C}, Q)$, if law(F) = law(F') then $\mathcal{D}_F = \mathcal{D}_{F'}$.*

Proof. Let $\mu \in \mathcal{D}_F$. We have $\mu =$ *law*(f) for some measurable selection f of F. Since $(\Gamma, \mathcal{C}, Q)$ is saturated, there is a random element f' of X such that *law*(F', f') = *law*(F, f). Because f is a selection of F and *law*(F', f') = *law*(F, f), it follows from the proof of P1 in Theorem 3.6 that f' is a selection of F'. Therefore *law*$(f') \in \mathcal{D}_{F'}$. This shows that $\mathcal{D}_{f'} \supseteq \mathcal{F}_F$. The other inclusion follows by symmetry. ∎

The following proposition shows that if all the regularity properties $P1$–$P6$ for distribution of correspondences hold for one particular saturated probability space, then they hold for any other saturated probability space.

Proposition 5.2. *Let $(\Omega, \mathcal{A}, P)$ and $(\Gamma, \mathcal{C}, Q)$ be two saturated probability spaces. Assume that each of the properties $P1$–$P6$ hold for $(\Omega, \mathcal{A}, P)$. Then each of the properties $P1$–$P6$ hold for $(\Gamma, \mathcal{C}, Q)$.*

Proof. $P1$: Let F' be a closed valued measurable correspondence from $(\Gamma, \mathcal{C}, Q)$ to X. Since $(\Omega, \mathcal{A}, P)$ is atomless, there is a measurable mapping F from $(\Omega, \mathcal{A}, P)$ to $\mathcal{F}_X$ such that *law*(F) = *law*(F'). It follows that F is a closed valued, measurable correspondence from $(\Omega, \mathcal{A}, P)$ to X. Lemma 5.1 shows that $\mathcal{D}_F = \mathcal{D}_{F'}$. By property $P1$ for $(\Omega, \mathcal{A}, P)$, $\mathcal{D}_F$ is maximal, so $\mathcal{D}_{F'}$ is also maximal.

$P2$: The convexity of the set of laws of measurable selections of an arbitrary correspondence follows from the case of a correspondence consisting of two measurable functions. Without loss of generality, assume that F' is a closed valued measurable correspondence on $(\Gamma, \mathcal{C}, Q)$.

As in the proof for $P1$, there exists a closed valued, measurable correspondence from $(\Omega, \mathcal{A}, P)$ to X such that $law(F) = law(F')$. By Lemma 5.1, $\mathcal{D}_{F'} = \mathcal{D}_F$. Since $\mathcal{D}_F$ is convex, so is $\mathcal{D}_{F'}$.

$P3$: We can assume without loss of generality that F' is a closed valued measurable correspondence on $(\Gamma, \mathcal{C}, Q)$. The second paragraph in the proof of $P2$ above shows that $\mathcal{D}_{F'} = \mathcal{D}_F$ for some closed valued, measurable correspondence from $(\Omega, \mathcal{A}, P)$ to X. Since $\mathcal{D}_F$ is closed, so is $\mathcal{D}_{F'}$.

$P4$: Let $\mathcal{C}_X$ be the space of compact subsets of X endowed with the Hausdorff metric, which is a complete separable metric. We can assume without loss of generality that F' is a compact valued measurable correspondence on $(\Gamma, \mathcal{C}, Q)$. The second paragraph in the proof of $P2$ shows that $\mathcal{D}_{F'} = \mathcal{D}_F$ for some closed valued, measurable correspondence F from $(\Omega, \mathcal{A}, P)$ to X with $law(F) = law(F')$. Since F' is compact valued, $law(F')(\mathcal{C}_X) = 1$, and hence $law(F)(\mathcal{C}_X) = 1$. This means that one can take F to be compact valued. Since $\mathcal{D}_F$ is compact, so is $\mathcal{D}_{F'}$.

$P5$: Since property $P4$ holds for $(\Omega, \mathcal{A}, P)$, the preceding paragraph shows that $P4$ holds for $(\Gamma, \mathcal{C}, Q)$. The proof of Theorem 3.6 shows that any atomless probability space that has property $P4$ has property $P5$. Thus $P5$ holds for $(\Gamma, \mathcal{C}, Q)$.

$P6$: Let $\mathcal{G}'$ be a measurable mapping from $(\Gamma, \mathcal{C}, Q)$ to the space $\mathcal{M}(X)$ of probability measures on X, and F' a correspondence from $(\Gamma, \mathcal{C}, Q)$ to X such that $F'(\gamma) = \operatorname{supp} \mathcal{G}'(\gamma)$ for each $\gamma \in \Gamma$.

As in the proof of $P2$, let $\mathcal{G}$ and F be measurable mappings from $(\Omega, \mathcal{A}, P)$ to $\mathcal{M}(X)$ and $\mathcal{F}_X$ respectively such that $law(\mathcal{G}, F) = law(\mathcal{G}', F')$. It follows that F is a closed valued, measurable correspondence from $(\Omega, \mathcal{A}, P)$ to X. Since $\mathcal{G}'(\gamma)(F'(\gamma)) = 1$ for all $\gamma \in \Gamma$, we know that $G(\omega)(F(\omega)) = 1$ for almost all ω.

There is a measurable mapping f from $(\Omega, \mathcal{A}, P)$ to X such that (i) for every Borel set B in X, $law(f)(B) = \int_\Omega \mathcal{G}(\omega)(B)dP$; (ii) for each $\omega \in \Omega$, $f(\omega) \in \operatorname{supp} \mathcal{G}(\omega) \subseteq F(\omega)$. By full saturation, there is a measurable mapping f' from $(\Gamma, \mathcal{C}, Q)$ to X such that $law(\mathcal{G}, F, f) = law(\mathcal{G}', F', f')$. This f' will have the desired property. ∎

Note that for each of the properties $P1$–$P4$ and $P6$, the above proof of that property for $(\Gamma, \mathcal{C}, Q)$ used only the assumption that the same property holds for $(\Omega, \mathcal{A}, P)$; it did not use any of the results from Section 3. To prove that $P5$ holds for $(\Gamma, \mathcal{C}, Q)$, we used the assumption that $P4$ holds for $(\Omega, \mathcal{A}, P)$ and a fact from Subsection 3.1.

The following proposition shows (without using Mas-Colell's result stated in Fact 4.5) that if the existence result for large games as stated in Theorem 4.6 holds for one particular saturated probability space as the player space, then it holds for any other saturated probability space as the player space. A similar statement holds for the closure result in Theorem 4.7.

Proposition 5.3. *Let $(\Omega, \mathcal{A}, P)$ and $(\Gamma, \mathcal{C}, Q)$ be two saturated probability spaces, and let A be a compact metric space.*

(i) If every game with player space $(\Omega, \mathcal{A}, P)$ and compact metric action space A has a Nash equilibrium, then every game with player space $(\Gamma, \mathcal{C}, Q)$ and the same action space A has a Nash equilibrium.

(ii) If for every game with player space $(\Omega, \mathcal{A}, P)$ and compact metric action space A, $\mathcal{E}_{\mathcal{G}}$ is closed, then for every game $\mathcal{G}'$ with player space $(\Gamma, \mathcal{C}, Q)$ and the same action space A, $\mathcal{E}_{\mathcal{G}'}$ is closed.

Proof. (i) Suppose that every game $\mathcal{G}$ with player space $(\Omega, \mathcal{A}, P)$ and action space A has a Nash equilibrium. Let $\mathcal{G}'$ be a game with player space $(\Gamma, \mathcal{C}, Q)$ and the same action space A. By Lemma 2.1 (ii) there is a random element $\mathcal{G}$ of $\mathcal{U}_A$ on $(\Omega, \mathcal{A}, P)$ such that $law(\mathcal{G}) = law(\mathcal{G}')$. The game $\mathcal{G}$ has a Nash equilibrium $g \in L^0(\Omega, A)$, so $law(\mathcal{G}, g)$ is a Nash equilibrium distribution for $law(\mathcal{G})$. By full saturation there is a random element g' of A on $(\Gamma, \mathcal{C}, Q)$ such that $law(\mathcal{G}', g') = law(\mathcal{G}, g)$. Hence g' is a Nash equilibrium for $\mathcal{G}'$.

(ii) An argument like the proof of part (i) shows that for any games $\mathcal{G}$ on $(\Omega, \mathcal{A}, P)$ and $\mathcal{G}'$ on $(\Gamma, \mathcal{C}, Q)$ with $law(\mathcal{G}) = law(\mathcal{G}')$, we have $\mathcal{E}_{\mathcal{G}} = \mathcal{E}_{\mathcal{G}'}$. Since $\mathcal{E}_{\mathcal{G}}$ is closed, it follows that $\mathcal{E}_{\mathcal{G}'}$ is closed as well. ∎

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Department of Mathematics, University of Wisconsin,, 480 Lincoln Drive, Madison WI 53706, U.S.A.
E-mail address: keisler@math.wisc.edu

Department of Mathematics, National University of Singapore,, 2 Science Drive 2, Singapore 117543, Republic of Singapore
E-mail address: matsuny@nus.edu.sg