Math 341
Group Worksheet #3: spans and linear transformations
Friday, February 6, 2015

1. As we have explained in lecture, if V is a vector space of dimension n, a collection of fewer than n vectors cannot be a spanning set for V. For example, we know that the vectors $(1, 2, 3)$ and $(4, 5, 6)$ cannot possibly span $\mathbb{R}^3$, because $\mathbb{R}^3$ is 3-dimensional.

   How do we determine the span of $\{(1, 2, 3), (4, 5, 6)\}$? Here is how you do it. Let (x, y, z) be an arbitrary vector in $\mathbb{R}^3$, and assume that (x, y, z) is in the span of $(1, 2, 3)$ and $(4, 5, 6)$, which means (x, y, z) is a linear combination of these two vectors. By manipulating the resulting system of equations, you will determine a *relationship* between the coordinates x, y, and z. This relationship is in the form of one or more linear equations relating the three coordinates. If you have one equation, it determines a plane. If you have two equations, they determine a line (the intersection of two planes).

   Carry out this procedure to find a nice description of the span of $\{(1, 2, 3), (4, 5, 6)\}$. (Since the two vectors are linearly independent, you know the answer will be a 2-dimensional subspace of $\mathbb{R}^3$, a.k.a. a plane passing through the origin – you are to find the equation of this plane). Remember, if you want to prove that $\mathrm{Span}(\{(1, 2, 3), (4, 5, 6)\})$ is equal to some plane P, you should prove that $\mathrm{Span}(\{(1, 2, 3), (4, 5, 6)\}) \subseteq P$ and $P \subseteq \mathrm{Span}(\{(1, 2, 3), (4, 5, 6)\})$

2. Recall the definition of a linear transformation:

   Definition 1. Let V and W be vector spaces over a field F. A function $T : V \to W$ is called a *linear transformation* if

   (i) for any $x_1, x_2 \in V$, we have $T(x_1 + x_1) = T(x_1) + T(x_2)$, and
   (ii) for any $x \in V$, $c \in F$, we have $T(cx) = c \cdot T(x)$.

   Recall that the kernel (a.k.a. null space) and image (a.k.a. range) of a linear transformation $T : V \to W$ are defined by

   $$\ker(T) = \{x \in V \mid T(x) = 0\}, \text{ and}$$

   $$\mathrm{im}(T) = \{y \in W \mid \text{there is some } x \in V \text{ such that } T(x) = y\}.$$

   Let V and W be vector spaces over a field F, and let $T : V \to W$ be a linear transformation.

   (a) Prove that $\ker(T)$ is a subspace of V.
   (b) Prove that $\mathrm{im}(T)$ is a subspace of W.
   (c) If $\{v_1, \ldots, v_n\}$ is a basis for V, then prove that $\{T(v_1), \ldots, T(v_n)\}$ is a spanning set for W.