Math 341
Group Worksheet #1: vector spaces and subspaces
Monday, January 26, 2015

Recall from Theorem 1.3 of our text that a subset W of a vector space V is a subspace if and only if

(a) W contains the zero vector,

(b) W is closed under addition, and

(c) W is closed under scalar multiplication.

Problems:

1. For an $n \times n$ matrix A, we define the *trace* of A, denoted $\operatorname{tr}(A)$ or $\operatorname{tr} A$, to be the sum of the entries on the diagonal, so if the entry in row i and column j of A is denoted A_{ij}, we have

$$\operatorname{tr} A = \sum_{i=1}^{n} A_{ii} = A_{11} + A_{22} + \cdots + A_{nn}. \tag{1}$$

For example, in $M_{3\times 3}(\mathbb{R})$ we have

$$\operatorname{tr} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 2 & 3 & -7 \end{pmatrix} = 1 + 1 - 7 = -5.$$

   (a) Prove that for any $A, B \in M_{n\times n}(F)$ and any $c \in F$, we have

      (i) $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$, and

      (ii) $\operatorname{tr}(cA) = c(\operatorname{tr} A)$.

   (b) Prove that the set of *trace-zero matrices*

$$\{A \in M_{n\times n}(F) \mid \operatorname{tr} A = 0\} \tag{2}$$

   is a subspace of $M_{n\times n}(F)$.

2. Let V be a vector space over a field F and let W_1 and W_2 be subspaces of V.

   (a) Prove that $W_1 \cap W_2$ is also a subspace of V.

   (b) Give an example that shows that $W_1 \cup W_2$ may not be a subspace of V.

   (c) Prove that $W_1 \cup W_2$ is a subspace of V *if and only if* either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. (To prove a statement of the form "[statement A] if and only if [statement B]," you must prove two things: if statement A is true, then statement B is true, and if statement B is true, then statement A is true.)

   (d) We define

$$W_1 + W_2 = \{x + y \mid x \in W_1,\ y \in W_2\}. \tag{3}$$

   Prove that $W_1 + W_2$ is a subspace of V.