

Math 341

Some useful things

Symbols:

\mathbb{R} : The field of all real numbers.

\mathbb{C} : The field of all complex numbers, i.e. $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$, where $i^2 = -1$

\mathbb{Q} : The field of all rational numbers.

\mathbb{Z} : The set of all integers (not a field, but a different kind of structure called a *ring*, which is studied in Math 441 or 541)

\mathbb{N} : The set of all natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, 4, 5 \dots\}$.

F^n : The vector space of all n -tuples of elements from the field F .

$M_{m \times n}(F)$: The vector space (but it's more than that, as we'll see) of $m \times n$ matrices with entries in the field F .

$\text{tr}(A)$: The trace of the $n \times n$ square matrix M : $\text{tr}(M) = M_{11} + M_{22} + \dots + M_{nn}$.

$\dim(V)$: The dimension of the vector space V , which is the cardinality (size) of any basis. e.g. $\dim(F^n) = n$, and $\dim(M_{m \times n}(F)) = mn$.

$\text{Span}(S)$: The span of S , or the subspace of all (finite) linear combinations of vectors from the subset S of some vector space.

$A \cap B$: The intersection of the sets A and B : $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

$A \cup B$: The union of the sets A and B : $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

$A \subseteq B$: The statement that A is a subset of B – i.e. every element of A is an element of B .

Proof FAQ:

The most important skill when beginning proof-writing is the skill of analyzing the statement that you wish to prove, and the given information, and understanding what everything MEANS. If the word “subspace” is in the statement, you have to know what this means. If you are trying to prove that two sets are equal, you have to know what it means for two sets to be equal (see below).

Q: How do I prove an assertion of the form “If [statement 1], then [statement 2]?”

A: You begin by assuming statement 1 is true, and you apply definitions, given information, and facts you already know (like theorems proved in the book or in lecture) in order to conclude that statement 2 is true. You should AVOID writing statement 2 at the beginning. For example, if you want to prove that $x + y = 0$, then you should AVOID starting your proof by writing $x + y = 0$. If you are stuck, you might try proving the contrapositive (see below).

Q: How do I prove an assertion of the form [statement 1] if and only if [statement 2]?

A: You must prove that if statement 1 is true, then statement 2 is true, *and* prove that if statement 2 is true, then statement one is true. In other words, you must prove a certain assertion and its converse (see below).

Q: How do I prove that $A \subseteq B$ (“ A is a subset of B ”)?

A: By definition, $A \subseteq B$ means that every element of A is an element of B , so you must prove that if $a \in A$, then $a \in B$.

Q: How do I prove that $A = B$ (when A and B are sets)?

A: Prove that $A \subseteq B$ and $B \subseteq A$ (see above). This is what it means for two sets to be equal.

Q: How do I prove an assertion of the form [statement 1] if and only if [statement 2]?

A: You must prove that if statement 1 is true, then statement 2 is true, *and* prove that if statement 2 is true, then statement one is true.

Q: What is a contrapositive?

A: The assertion

“If [statement 1] is true, then [statement 2] is true.”

is logically equivalent to the assertion

“If [statement 2] is false, then [statement 1] is false.”

The second statement is the *contrapositive* of the first. For example, the assertion

“If it is raining, then the ground is wet.”

is equivalent to the assertion

“If the ground is dry, then it is not raining.”

Another example: the assertion

“If A is a skew-symmetric matrix, then all of its diagonal entries are zero.”

is logically equivalent to the assertion

“If one of the diagonal entries of a matrix A is not zero, then A is not skew-symmetric.”

Q: What is a converse?

A: The *converse* of the assertion

“If [statement 1] is true, then [statement 2] is true.”

is the assertion

“If [statement 2] is true, then [statement 1] is true.”

A statement is not equivalent to its converse! For example, if a and b are real numbers, the assertion “If $a = 0$ then $ab = 0$ ” is NOT equivalent to its converse, which is the assertion “If $ab = 0$, then $a = 0$.” In fact, one of these is true, and the other is false!

Q: How do I prove that a subset W of a vector space V is a subspace?

A: You apply Theorem 1.3 from our text: prove that the zero vector is in W , and prove that W is closed under both addition and scalar multiplication. (What is the zero vector? What is the rule for adding vectors? What is the rule for scalar multiplication? If you are proving W is a subspace of V , you already know V is a vector space, which means you should already know the answers to these questions.)

Q: How do I prove that a set S of vectors is linearly independent?

A: You prove that the only finite linear combination of vectors from S which yields the zero vector is the trivial combination, i.e. the one where each coefficient is zero.

Q: How do I prove that a set S of vectors is linearly dependent?

A: You show how to write the zero vector as a nontrivial (i.e. with not all coefficients equal to zero) linear combination of finitely many vectors in S .

Q: How do I prove that a set S of vectors spans a vector space V ?

A: You prove that any vector in V can be written as a finite linear combination of vectors from S , typically by showing explicitly how to make such a linear combination. So, start with an arbitrary vector in V (what does an arbitrary vector look like? It depends on what space you are in!), and show how to write it as a linear combination of vectors from S .

Notice that these last two answers amount to simply understanding the relevant definitions. You might try extending this list of FAQ on your own as you encounter new problems. For example, if you have a problem where you want to prove something is a basis for some vector space, ask yourself, how do I prove something is a basis? (Look at the definition, or any relevant theorems. Depending on the situation, there could be many ways to prove something is a basis. You could apply the definition directly, or you could possibly be able to use a theorem to make it easier on you (see the theorems from Section 1.6).