Math 341 – Homework #8
Due Wednesday, March 25, 2015

For this assignment, you should refer to the definitions and theorems of Sections 1.2-1.6, 2.1-2.5, and 3.1-3.4 of our textbook, as well as notes from lecture.

1. For any positive integer n and field F, we denote by $P_n(F)$ the vector space of polynomials with coefficients in F and degree at most n. The *standard ordered basis* for P_n is the set $\{1, x, x^2, \ldots, x^n\}$. It is easily seen that $P_n(F)$ is $(n+1)$-dimensional. Define a linear transformation $D : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ by

$$(Dp)(x) = p'(x),$$

that is, D takes in a polynomial p and gives as output the derivative of p. For example, if $p(x) = 1 + x + x^3$, then $(Dp)(x) = 1 + 3x^2$. If p has degree at most 3, it's clear that the derivative has degree at most 2. Basic properties of derivatives imply that D is a linear transformation.

   (a) Let $\mathcal{B}$ denote the standard ordered basis $\{1, x, x^2, x^3\}$ for $P_3(\mathbb{R})$, and let $\mathcal{C}$ denote the standard ordered basis $\{1, x, x^2\}$ for $P_2(\mathbb{R})$. Write out $[D]_{\mathcal{B}}^{\mathcal{C}}$, the matrix representation of D with respect to $\mathcal{B}$ and $\mathcal{C}$.

   (b) Let $\mathcal{B}' = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$; notice that $\mathcal{B}'$ is a different ordered basis for $P_3(\mathbb{R})$. Write out $[D]_{\mathcal{B}'}^{\mathcal{C}}$, the matrix representation of D with respect to $\mathcal{B}'$ and $\mathcal{C}$.

   (c) Show how you can compute the derivative of $f(x) = x^3$ in a really inefficient way: by multiplying $[D]_{\mathcal{B}'}^{\mathcal{C}}$ times $[f]_{\mathcal{B}'}$.

2. Define a linear transformation $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $T(A) = A^t$. That is, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

   Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$; this is an ordered basis for $M_{2\times 2}(\mathbb{R})$. Write out $[T]_{\mathcal{B}}^{\mathcal{B}}$ the matrix representation of T with respect to the same ordered bases $\mathcal{B}$ and $\mathcal{B}$.

3. Let $H = \begin{pmatrix} 2 & 3 \\ -5 & 1 \end{pmatrix}$, and consider the associated linear transformation $L_H : \mathbb{R}^2 \to \mathbb{R}^2$. We understand this linear transformation as $L_H(x) = Hx$, where the vector x is written as a 2×1 column vector with respect to the standard basis. Here are some ordered bases for $\mathbb{R}^2$:

$$\mathcal{A} = \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$
$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$$
$$\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

(a) Write out the matrices $[L_H]_{\mathcal{E}}^{\mathcal{E}}$, $[L_H]_{\mathcal{E}}^{\mathcal{B}}$, $[L_H]_{\mathcal{B}}^{\mathcal{B}}$, and $[L_H]_{\mathcal{A}}^{\mathcal{B}}$. Use the definition of the matrix representation associated to a linear transformation (bottom of p. 80, or see your lecture notes from 3/18/15) to write each matrix.

(b) Let $I_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the identity transformation, i.e. $I_{\mathbb{R}^2}(x) = x$ for all $x \in \mathbb{R}^2$. Let $P = [I_{\mathbb{R}^2}]_{\mathcal{B}}^{\mathcal{E}}$, and let $Q = [I_{\mathbb{R}^2}]_{\mathcal{E}}^{\mathcal{A}}$. (So, P is the matrix that changes $\mathcal{B}$-coordinates to $\mathcal{E}$-coordinates, etc.) Write out what P, P^{-1}, Q, and Q^{-1} are.

Tip: You can use the following formula to simplify your life:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

for example

$$\begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix}^{-1} = \frac{1}{13} \begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} & \frac{-1}{13} \\ \frac{3}{13} & \frac{2}{13} \end{pmatrix}.$$

(c) Looking at the next page: for each matrix in the Column 1, identify which matrix in Column 2 is equal to it. For the first four matrices in Column 1 (the ones you wrote out in part (a)), verify directly that the are equal to the corresponding entry you choose from Column 2. For the rest, just say what the answer is – you don't need to explain yourself.

4. We say that two matrices A and B in $M_{n\times n}(F)$ are *similar* if there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that $B = Q^{-1}AQ$. We'll use the notation $A \sim B$ to denote that A is similar to B. Prove that $\sim$ is an equivalence relation on $M_{n\times n}(F)$. That is, prove the following:

(i) for any $A \in M_{n\times n}(F)$, we have $A \sim A$ (*reflexivity*),

(ii) for any $A, B \in M_{n\times n}(F)$, if $A \sim B$, then $B \sim A$ (*symmetry*), and

(iii) for any $A, B, C \in M_{n\times n}(F)$, if $A \sim B$ and $B \sim C$, then $A \sim C$ (transitivity).

Table for Problem 3, part (c).

Column 1	Column 2
$[L_H]_{\mathcal{E}}^{\mathcal{E}}$	$P^{-1}HP$
$[L_H]_{\mathcal{E}}^{\mathcal{B}}$	PHP^{-1}
$[L_H]_{\mathcal{B}}^{\mathcal{B}}$	H
$[L_H]_{\mathcal{A}}^{\mathcal{B}}$	$P^{-1}H$
$[L_H]_{\mathcal{B}}^{\mathcal{E}}$	PH
$[L_H]_{\mathcal{B}}^{\mathcal{A}}$	HP
$[L_H]_{\mathcal{E}}^{\mathcal{A}}$	HP^{-1}
$[L_H]_{\mathcal{A}}^{\mathcal{A}}$	$Q^{-1}HQ$
$[L_H]_{\mathcal{A}}^{\mathcal{E}}$	QHQ^{-1}
	$Q^{-1}H$
	QH
	HQ
	HQ^{-1}
	PHQ^{-1}
	P^-1HQ
	QHP^{-1}
	$Q^{-1}HP$
	QHP
	PHQ
	$P^{-1}HQ^{-1}$
	$Q^{-1}HP^{-1}$