Math 341 – Homework #7
Due Wednesday, March 18, 2015

For this assignment, you should refer to the definitions and theorems of Sections 1.2-1.6, 2.1, 2.2, and 3.1-3.4 of our textbook, as well as notes from lecture.

Let's establish some very general notation and terminology. For a set S, the *identity function* on S is the function $id_S : S \to S$ defined by $id_S(x) = x$ for all $x \in S$. For example, as a function from $F^n \to F^n$, the identity matrix I_n corresponds to a linear transformation which is the identity function on F_n, since for each $x \in F^n$, we have

$$x = I_n x = id_{F^n}(x).$$

If $f : S \to S'$ is a function from a set S to another set S', we say that f is *invertible* if there is a function $g : S' \to S$ such that

$$g(f(x)) = x, \text{ for all } x \in S, \tag{1}$$

and

$$f(g(y)) = y, \text{ for all } y \in S'. \tag{2}$$

Statement (1) can be restated as saying that $g \circ f = id_S$, while statement (2) can be restated as saying that $f \circ g = id_{S'}$. We write $f^{-1} = g$.

Just to review, a function $f : S \to S'$ is called *one-to-one* (or *injective*) if it has only one input corresponding to each output, i.e. if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. We say f is *onto* (or *surjective*) if every element of the codomain is an output, i.e. if for every y in S', there is some x in S such that $f(x) = y$. A function which is both injective and surjective is said to be *bijective*. (These words have noun forms, too: we can say f is an *injection*, a *surjection*, or a *bijection*.)

1. Prove that a function is invertible if and only if it is one-to-one and onto.

2. Let $T : V \to W$ be a linear transformation from a vector space V to another vector space W, and assume that, as a function, T is invertible. Prove that T^{-1} is also a linear transformation.*

3. Let $B = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & 2 \end{pmatrix}$.

   (a) Find a 3×2 matrix C such that

   $$BC = I_2. \tag{3}$$

   Such a matrix C is called a *right inverse* of B. There's not a special notation for this, because right inverses aren't necessarily unique.

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*You're proving that if a function has a certain property and is invertible, then the inverse also enjoys that property – in this case, linearity. Not all properties of functions work this way! For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is invertible, and it's also differentiable. However, its inverse is *not* differentiable! (It has a vertical tangent line at one point.) So differentiability doesn't play nice with inverses the way linearity does.

(b) Is it possible to find a left inverse of B? That is, can you find a 3×2 matrix A such that

$$AB = I_3? \tag{4}$$

Why or why not?

(c) If B is an $m \times n$ matrix, how can you tell, based on m, n and the rank of B, whether or not B has a right inverse? A left inverse?

4. Let V be an n-dimensional vector space over a field F, and let $\mathcal{B}$ be an ordered basis for V. Define a function

$$\phi_{\mathcal{B}} : V \to F^n$$

by

$$\phi_{\mathcal{B}}(x) = [x]_{\mathcal{B}}.$$

(See page 80 in our text or your lecture notes to recall what this notation means).

(a) Prove that $\phi_{\mathcal{B}}$ is linear.

(b) Prove that $\phi_{\mathcal{B}}$ is one-to-one.

(c) Prove that $\phi_{\mathcal{B}}$ is onto.