Math 341 – Homework #6
Due Wednesday, March 11, 2015

For this assignment, you should refer to the definitions and theorems of Sections 1.2-1.6, 2.1, and 3.4 of our textbook, as well as notes from lecture.

1. We have seen augmented matrices of the form $(P|b)$, where P is a matrix and b is a column vector. We can use similar notation for two matrices concatenated: if A is an $n \times p$ matrix and B is an $n \times q$ matrix, then we write $(A|B)$ for the $n \times (p+q)$ matrix whose first p columns are the columns of A, and last q columns are the columns of B, with a bar in between. The bar is meaningless when we multiply the matrix $(A|B)$ times something else. For example, if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 & 5 \\ 7 & 9 & 7 \\ 5 & 3 & 2 \end{pmatrix}$, then $(A|B) = \left(\begin{array}{cc|ccc} 1 & 2 & 2 & 3 & 5 \\ 3 & 4 & 7 & 9 & 7 \\ 5 & 6 & 5 & 3 & 2 \end{array}\right)$, where the vertical bar doesn't really mean anything.

   Let M be an $m \times n$ matrix, let A be an $n \times p$ matrix, and let B be $n \times q$ matrix. Prove that $M(A|B) = (MA|MB)$.

2. (a) Over the field $\mathbb{R}$, let $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. Consider the augmented matrix

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

   Use Gaussian elimination on $(A|I_3)$ to obtain a matrix in reduced row echelon form. The matrix you end up with should be of the form $(I_3|B)$, where B is a new 3×3 matrix. Verify that $AB = I_3$. (This is the procedure for finding the inverse of a matrix, which we will discuss in class soon, and we will explain why the procedure works.)

   (b) Try to repeat part (a) with the matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 7 \\ 1 & 2 & 8 \end{pmatrix}$. Your RREF matrix will not have I_3 as the first block in this example. Prove that A is not invertible.

   Hint: In class we proved that, if A is invertible, then L_A is one-to-one and onto.

3. Let A be an $m \times n$ matrix, and let D and E be $q \times m$ matrices. Prove that

$$(D+E)A = DA + EA.$$

   Hint: you are proving that two matrices are equal. The most direct way to do this is to show that each entry of one matrix is equal to the corresponding entry of the other matrix: to prove (matrix 1) = (matrix 2), you show that $(\text{matrix }1)_{ij} = (\text{matrix }2)_{ij}$ for an arbitrary i and j. This problem is part of Theorem 2.12 (a) from our text. You may want to look at their proof of the other part for comparison.

4. Recall that the transpose of an $m \times n$ matrix A is the $n \times m$ matrix, denoted A^t, such that $(A^t)_{ij} = A_{ji}$. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

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(a) If A is an $m \times n$ matrix and B is an $n \times p$ matrix, prove that $(AB)^t = B^t A^t$.

(b) Recall that we say an $n \times n$ matrix A is invertible if there is another $n \times n$ matrix B such that $AB = I_n$, where I_n is the $n \times n$ identity matrix. We call the matrix B the inverse of A, and we write $B = A^{-1}$. Prove that if A is an invertible $n \times n$ matrix, then A^t is also invertible. What is the inverse of A^t (in terms of the inverse of A)?

Hint: in class we proved that, if $AB = I_n$, then $BA = I_n$. You may use this.